Clarkson College of Technology
NGR-33-007-034
Progress Rpt. submitted with
ltr of 5-7-70

N70 32865

NASA CR 110634

EFFECTS ON THE MOTION OF A BODY ATTRACTED BY A ROTATING SOURCE

II. PART

by

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INTRODUCTION

The goal of this paper consists in deriving an iterative method to determine the variation of any order of a planned circular orbit about a rotating gravitational source. The iterative method leads to systems of first order inhomogeneous differential equation such that the corresponding homogeneous equations have the same form. Then linearly independent solutions are listed explicitely.

An application is made in the case of a perturbing axially asymmetric potential to explicit expression for the first variation of the coordinates of a satellite are derived in the case of a synchronized orbit, i. e., of an orbit the period of which equals the period of the rotating source. The coordinates of the satellite are referred to as inertial systems.

I GENERAL CONSIDERATIONS

The equations of motion under the influence of a perturbing force

$$\alpha\ \overline{F},\ \alpha = \text{const.}$$

are

$$\ddot{\overline{x}} = -\frac{\mu}{r^3}\ \overline{x} + \alpha\,\overline{F} \qquad (1)$$

It is most convenient to write this equation in a dimensionless form, above all if we are dealing with quasicircular orbits. As it will be the case here, quasi-circular means that the initial condition are chosen so that the orbit would be circular (radius r_o) if α were o. For this purpose, we substitute

$$\overline{x} = r_o\,\overline{y}$$

$$r = r_o\,\rho,\ \rho^2 = \overline{y}^2$$

$$\mu^{1/2}\,r_o^{-3/2}\ t = \tau \qquad (2)$$

$$\frac{r_o^2}{\mu}\ \alpha = \beta$$

We describe the differentiation with respect to τ by a prime:

$$\frac{d\,f(\tau)}{d\,\tau} = f'$$

The equation (1) assumes the form

$$\overline{y}'' = -\frac{\overline{y}}{\rho^3} + \beta\,\overline{F} \qquad (3)$$

The number $|\beta|$ characterizing a perturbation is small compared with ^{1}.
We now expand the vector $\bar{y}$ into a power series of β

$$\bar{y} = \bar{y}_o + \beta \bar{y}_1 + \beta^2 \bar{y}_2 + \ldots \tag{4}$$

The vector $\bar{y}_i(\tau)$ has the form

$$\bar{y}_o(\tau) = \cos\tau\,\bar{a} + \sin\tau\,\bar{b} \tag{5}$$

where $\bar{a}$ $\bar{b}$ are constant unit vectors, orthogonal to each other. After expressing ρ^{-3} $(\rho^2 = \bar{y}^2)$ as a power series of β, substituting (4) in (3) and comparing the coefficients of β^2 on both sides of equation (3) we obtain

$$\bar{y}_1'' + \bar{y}_1 - 3(\bar{y}_o\bar{y}_1)\bar{y}_o = \bar{F}_o \tag{6}$$

$$\bar{y}_2'' + \bar{y}_2 - 3(\bar{y}_o\bar{y}_2)\bar{y}_o = \bar{F}_1 + 3(\bar{y}_o\bar{y}_1)\bar{y}_1 + \frac{3}{2}\bar{y}_1^{\,2}\bar{y}_o - \frac{15}{2}(\bar{y}_o\bar{y}_1)^2\,\bar{y}_o$$

It is remarkable that the left hand sides of these equations have the same form

$$\bar{y}_i'' + \bar{y}_i - 3(\bar{y}_o\bar{y}_i)\,\bar{y}_o \tag{7}$$

whereas the right hand sides depend on the preceding vectors $\bar{y}_o, \bar{y}_1, \ldots,$ $\bar{y}_{i-1}$ only. All equations (6) can be solved by means of quadratures, using an iterative method. All integrals can be evaluated explicitly in the case of rotating potentials as will be shown later.

If a <u>circular orbit</u> is planned the initial conditions are

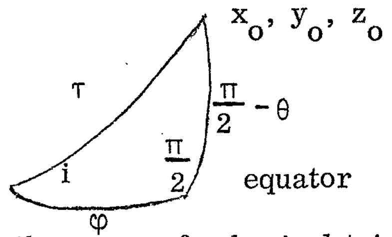
$$\bar{y}_i(0) = 0 \qquad i \neq 0 \tag{8}$$

In order to integrate equations (6) we apply the method of variation of constants, determining first the general solution of the homogeneous equation

$$\bar{x}'' + \bar{x} - 3(\bar{y}_o\bar{x})\,\bar{y}_o = 0 \tag{9}$$

which represents a system of ordinary second order differential equations with nonconstant coefficients.

We now express the components $x_o, y_o z_o$ of $\bar{y}_o$ in terms of τ and the initial inclination angle i (see Figure 1)



Elementary theorems of spherical trigonometry yield

$$x_o = \sin\theta \quad \cos\varphi = \cos\tau$$

$$y_o = \sin\varphi \sin\theta = \sin\tau \cos i \tag{10}$$

$$z_o = \cos\theta = \sin\tau \sin i$$

It is advantageous to introduce two new unit vectors: $\bar{b}$ orthogonal to $\bar{y}_o$ in the plane of $\bar{y}_o(\tau)$ and the constant unit vector normal to $\bar{y}_o$, $\bar{b}$. We can assume

$$\bar{y}_o' = \bar{b} \quad \bar{b}' = -\bar{y}_o \tag{11}$$

The components of $\bar{b}$ and $\bar{n} = \bar{y}_o \times \bar{b}$ are

$$\bar{b} = (-\sin\tau,\ \cos\tau\cos i,\ \cos\tau\sin i)$$

$$\bar{n} = (o,\ -\sin i,\ \cos i) \tag{12}$$

The vectors $\bar{y}_o$, $\bar{b}$, $\bar{n}$ are chosen as direction vectors of the axes of a movable coordinate system. Let λ, μ, σ be the coordinates of $\bar{x}$ with respect to these movable axes

$$\bar{x} = \lambda\bar{y}_o + \mu\bar{b} + \sigma\bar{n} \tag{13}$$

The system (9) then assumes the simple form

$$\begin{aligned} \lambda'' &= 3\lambda + 2\mu' \\ \mu'' &= -2\lambda' \\ \sigma'' &= -\sigma \end{aligned} \tag{14}$$

from which we conclude

$$\begin{aligned} \lambda &= \alpha \cos\tau + \beta \sin\tau + 2C_1 \\ \mu &= -2\alpha \sin\tau + 2\beta\cos\tau - 3C_1\tau + C_2 \\ \sigma &= \gamma\cos\tau + \delta\sin\tau \\ &\alpha, \beta, \gamma, \delta, C_1, C_2 \text{ constants} \end{aligned} \tag{15}$$

Substituting (10) and (12) in (13) we obtain the following 6 linearly independent solution of (9) where the components of the ith solution vector $\overline{x}$ may be denoted by $x^{(i)}$, $y^{(i)}$, $z^{(i)}$

$$\begin{aligned}
x^{(1)} &= 1 + \sin^2\tau, & x^{(2)} &= -\sin\tau\cos\tau \\
y^{(1)} &= -\sin\tau\cos\tau\cos i, & y^{(2)} &= (1+\cos^2\tau)\cos i \\
z^{(1)} &= -\sin\tau\cos\tau\sin i, & z^{(2)} &= (1+\cos^2\tau)\sin i \\
x^{(3)} &= 2\cos\tau + 3\tau\sin\tau & x^{(4)} &= -\sin\tau \\
y^{(3)} &= 2\sin\tau\cos i - 3\tau\cos\tau\cos i & y^{(4)} &= \cos\tau\cos i \\
z^{(3)} &= 2\sin\tau\sin i - 3\tau\cos\tau\sin i & z^{(4)} &= \cos\tau\sin i \\
x^{(5)} &= 0 & x^{(6)} &= 0 \\
y^{(5)} &= -\cos\tau\sin i & y^{(6)} &= -\sin\tau\sin i \\
z^{(5)} &= \cos\tau\cos i & z^{(6)} &= \sin\tau\cos i
\end{aligned} \tag{16}$$

We now return to the system (6) which has the form

$$y_i^{(j)''} = \sum_{k=1}^{3} a_{ik} y_k^{(j)} + f_i^{(j)} \tag{17}$$

where $y_k^{(\ell)}$, $k = 1, 2, 3$ are the components of $\bar{y}^{(\ell)}$.

The sytem (17) is transformed into a first order system by substituting $y_i^{(j)'} = z_i^{(j)}$

$$\begin{aligned} y_i^{(j)'} &= z_i^{(j)} \\ z_i^{(j)'} &= \sum a_{ik} y_k^{(j)} + f_i^{(j)} \end{aligned} \tag{18}$$

In order to make equations (18) more symmetric, we write $y_i^{(j)} = s_i^{(j)}$ $(i = 1, 2, 3)$ and $y_i^{(j)'} = s_{i+3}^{(j)}$. Equations (18) become

$$s_i^{(j)'} = \sum_{k=1}^{6} \alpha_{ik} s_k^{(j)} + g_i^{(j)}, \quad i = 1, 2, \ldots 6 \tag{19}$$

The matrix (α_{ik}) has the form

$$\begin{pmatrix} O & E \\ A & O \end{pmatrix} \tag{20}$$

E is the 3 by 3 unit matrix and A the matrix (a_{ik}). Furthermore,

$$g_1^{(j)} = g_2^{(j)} = g_3^{(j)} = 0, \; g_4^{(j)} = f_1^{(j)}, \; g_5^{(j)} = f_2^{(j)}, \; g_6^{(j)} = f_3^{(j)} \tag{21}$$

In order to solve the inhomogeneous system we first investigate the corresponding homogeneous equations.

$$\sigma_i' = \sum_{k=1}^{6} \alpha_{ik} \sigma_k \tag{22}$$

We interrupt here our studies and mention a general theorem easily to be proved:

Let $x_k^{(i)}$, $i = 1, 2, \ldots$ or, $k = 1, 2, \ldots n$ be n solutions of the system

$$\frac{d x_k}{d \tau} = \sum_{\ell = 1}^{n} h_{k \ell} \, x_{\ell} ,$$

let D be the determinant $| x_k^{(i)} |$, and let $T = \Sigma \, h_{ii}$ be the trace of the matrix (h_{ik}), then

$$\frac{d D}{d \tau} = T D$$

The application of this theorem to (22) yields the result: The determinant Δ of 6 solutions $\sigma_{\alpha i}$ of (22), where the subscript α denotes the α-th solution, is independent of τ because, according to (20) the trace of (α_{ik}) is zero which entails

$$\frac{d \Delta}{d \tau} = 0$$

Six solutions of (22) are immediately derived from equations (16). We have for all variations of the orbit

$$\sigma_{11} = 1 + \sin^2 \tau \qquad \sigma_{14} = \sin 2 \tau$$

$$\sigma_{12} = - \sin \tau \cos \tau \cos i \qquad \sigma_{15} = - \cos 2 \tau \cos i$$

$$\sigma_{13} = - \sin \tau \cos \tau \sin i \qquad \sigma_{16} = - \cos 2 \tau \sin i$$

$$\sigma_{21} = - \frac{1}{2} \sin 2 \tau \qquad \sigma_{22} = (1 + \cos^2 \tau) \cos i$$

$$\sigma_{23} = (1 + \cos^2 \tau) \sin i \qquad \sigma_{24} = - \cos 2 \tau$$

$$\sigma_{25} = -\sin 2\tau \cos i \qquad \sigma_{26} = -\sin 2\tau \sin i$$

$$\sigma_{31} = 2\cos\tau + 3\tau\sin\tau \qquad \sigma_{32} = 2\sin\tau\cos i - 3\tau\cos\tau\cos i$$

$$\sigma_{33} = 2\sin\tau\sin i - 3\tau\cos\tau\sin i \qquad \sigma_{34} = \sin\tau + 3\tau\cos\tau$$

$$\sigma_{35} = -\cos\tau\cos i + 3\tau\sin\tau\cos i \qquad \sigma_{36} = -\cos\tau\sin i + 3\tau\sin\tau\sin i$$

$$\sigma_{41} = -\sin\tau \qquad \sigma_{42} = \cos\tau\cos i$$

$$\sigma_{43} = \cos\tau\sin i \qquad \sigma_{44} = -\cos\tau$$

$$\sigma_{45} = -\sin\tau\cos i \qquad \sigma_{46} = -\sin\tau\sin i \tag{24}$$

$$\sigma_{51} = 0 \qquad \sigma_{52} = -\cos\tau\sin i$$

$$\sigma_{53} = \cos\tau\cos i \qquad \sigma_{54} = 0$$

$$\sigma_{55} = \sin\tau\sin i \qquad \sigma_{56} = -\sin\tau\cos i$$

$$\sigma_{61} = 0 \qquad \sigma_{62} = -\sin\tau\sin i$$

$$\sigma_{63} = \sin\tau\cos i \qquad \sigma_{64} = 0$$

$$\sigma_{65} = -\cos\tau\sin i \qquad \sigma_{66} = \cos\tau\cos i$$

The fact that the determinant $|\sigma_{ik}|$ is independent of τ facilitates the calculation because we can assume $\tau = 0$. The final result is

$$\mathrm{Det}\ |\sigma_{ik}| = 1 \tag{25}$$

For all further calculations concerning any perturbing forces, it is important to list the elements $\bar{\sigma}_{ik}$ of the inverse matrix of (σ_{ik}). However, for our purposes we need only 18 elements, namely those for which the first subscript i equals 4, 5, 6.

$$
\begin{aligned}
\bar{\sigma}_{41} &= \cos\tau \sin\tau & \bar{\sigma}_{51} &= -\cos i\,(1+\cos^2\tau) \\
\bar{\sigma}_{42} &= 1+\sin^2\tau & \bar{\sigma}_{52} &= -\cos i \sin\tau \cos\tau \\
\bar{\sigma}_{43} &= -\sin\tau & \bar{\sigma}_{53} &= +\cos i \cos\tau \\
\bar{\sigma}_{44} &= -2\cos\tau - 3\tau\sin\tau & \bar{\sigma}_{54} &= \cos i\,(3\tau\cos\tau - 2\sin\tau) \\
\bar{\sigma}_{45} &= 0 & \bar{\sigma}_{55} &= \sin i \sin\tau \\
\bar{\sigma}_{46} &= 0 & \bar{\sigma}_{56} &= -\sin i \cos\tau \\
\bar{\sigma}_{61} &= -\sin i\,(1+\cos^2\tau) & \bar{\sigma}_{62} &= -\sin i \sin\tau\cos\tau \\
\bar{\sigma}_{63} &= \sin i \cos\tau & \bar{\sigma}_{64} &= -2\sin i\sin\tau + 3\tau\sin i\cos\tau \\
\bar{\sigma}_{65} &= -\cos i \sin\tau & \bar{\sigma}_{66} &= \cos i\cos\tau
\end{aligned}
\tag{26}
$$

We now solve Equations (19) by writing

$$s_i^{(j)} = \lambda_\alpha^j \sigma_{\alpha i} \tag{27}$$

using Einstein's convention that one has to sum over two indices which occur twice. It follows

$$\lambda_k^{j'} = \bar{\sigma}_{ik}\, g_i^{(j)} \tag{28}$$

or

$$\lambda_k^{\,j} = \int_o^{\tau} \bar{\sigma}_{ik}\, g_i^{(j)}\, d\tau \tag{29}$$

According to (21) Equation (29) can be written

$$\lambda_k^{\,j} = \int_o^{\tau} (\bar{\sigma}_{4k} f_1^{(j)} + \bar{\sigma}_{5k} f_2^{(j)} + \bar{\sigma}_{6k} f_3^{(i)})\, d\tau \tag{30}$$

From this equation we conclude:

In the case of any perturbing potential rotating about an axis at the angular velocity w, the components of all variations $\bar{y}_1$, $\bar{y}_2$, ... can be represented as linear combination of $\tau^n \cos(\ell w + m)\tau$, $\tau^n \sin(\ell w + m)\tau$; ℓ, m positive or negative integers.

Perturbations caused by moon, sun, etc. can easily be included in the terms $f_i^{(j)}$.

After determining the $\lambda_k^{(j)}$, using Equations (30), we find the $s_i^{(j)}$ or $y_i^{(j)}$, $y_i^{(j)'}$ by means of (27) where the $\sigma_{\alpha i}$ are given by Equations (24).

II EXAMPLES

We shall now investigate the influence of an axially asymmetric gravitational potential of a rotating body on the quasicircular orbit of a satellite. According to the previous studies quasicircular means that the initial conditions are chosen so that the orbit would be circular if the acting gravity were spherically symmetric.

We assume the perturbing potential in the dimensionless form

$$\beta \left(\cos 2\omega\tau \frac{x_o y_o}{\rho^5} - \frac{1}{2} \sin 2\omega\tau \frac{x_o^2 - y_o^2}{\rho^5}\right) \tag{31}$$

where, according to (2)

$$\omega \sqrt{\frac{\mu}{r_o^3}} = \text{angular velocity of the rotating source} \tag{32}$$

By means of Equations (10) x_o, y_o, z_o are expressed in terms of τ and i. For the sake of simplicity we substitute

$$\cos i = a, \ \sin i = b, \ \cos \tau = u \ , \ \sin \tau = v$$

and obtain for the first variation

$$f_1^{(1)} = \cos 2\omega\tau\,(a v - 5\,a u^2 v) - \frac{1}{2}\sin 2\omega\tau\,(2u - 5u^3 + 5a^2 u v^2)$$

$$f_2^{(1)} = \cos 2\omega\tau\,(u - 5\,a^2 u\,v^2) - \frac{1}{2}\sin 2\omega\tau\,(-2av -$$

$$5\,a u^2 v + 5\,a^3 v^3) \qquad (33)$$

$$f_3^{(1)} = \cos 2\omega\tau\,(-5\,a b u v^2) - \frac{1}{2}\sin 2\omega\tau\,(-5\,b u^2 v + 5\,a^2 b v^3)$$

It is of particular interest to determine the components of y_1, y_1' in (6) (first variation) after one rotation of the satellite. For this purpose, we have to determine the values $\lambda_k\,(2\pi)$ by substituting (26) and (33) in (30). We list here the result:

$$\lambda_1\,(2\pi) = \sin 4\pi\omega\ \frac{4a\omega + 7 + 7a^2 + 24a^2\omega}{8\,(1 - 4\omega^2)}$$

$$+ \frac{7\sin 4\pi\omega}{8\,(9 - 4\omega^2)}\,(4a\omega + 3 + 3a^2)$$

$$\lambda_2\,(2\pi) = \frac{1 - \cos 4\pi\omega}{8}\ \frac{2a + 32a\omega + 14a^2\omega - 42\omega}{1 - 4\omega^2} \qquad (34)$$

$$\frac{-7\,(1 - \cos 4\pi\omega)}{9 - 4\omega^2}\,(2\omega + 14a^2\omega + 6a)$$

$$\lambda_3\,(2\pi) = -\frac{1 + a^2 + 2a\omega}{4 - 4\omega^2}\ \sin 4\pi\omega$$

$$\lambda_4(2\pi) = -\frac{3(1-\cos 4\pi\omega)\,b^2}{4\omega} - \frac{3\pi\sin 4\pi\omega}{4-4\omega^2}(3+2a+6a^2)$$

$$+\frac{3\,(1-\cos 4\pi\omega)}{(4-4\omega^2)^2}(-1-2a^2-12\,a\,\omega^2+4a)$$

$$+\frac{3\,(1-\cos 4\pi\omega)}{2\,(4-4\omega^2)}(2\omega+2\omega a^2)$$

$$\lambda_5\ (2\pi) = \frac{b\,(1-\cos 4\pi\omega)}{4\omega(1-\omega^2)}(\omega+a)$$

$$(34)$$

$$\lambda_6\ (2\pi) = \frac{\sin 4\pi\omega}{4-4\omega^2}(\omega^2 b+\omega a b+1-\omega^2)$$

III SYNCHRONIZED ORBITS

The formulas developed above become particularly simple if we assume that the rotating source has the same period as the planned circular orbit, in other words, we assume $\omega = \pm 1$. Let us now suppose $\omega = 1$. In this case we have

$$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = \frac{\pi}{2}(1+a^2)$$

$$\lambda_4 = \frac{3\pi}{2}(3+2a+6a^2) - \frac{3\pi^2}{8}(1+2a+2a^2) \qquad (35)$$

$$\lambda_5 = 0,\ \lambda_6 = -\frac{\pi}{2}(b+ab)$$

These values are substituted in (27):

$$s_i^{(1)} = \lambda_\alpha{}'\,\sigma_{\alpha i}(2\pi) \qquad (36)$$

The values of $\sigma_{\alpha i}(2\pi)$ are obtained from (24) by substituting $\tau = 2\pi$. Remembering that $s_i^{(1)} = y_i^{(1)}$ for i = 1, 2, 3; and $S_{i+3}^{(1)} = y_i^{(1)'}$ Equation (36) yields the first variation of $\bar{y}$ after one period: (We recall cos i = a, sin i = b

$$y_1^{(1)} = \pi(1 + a^2)$$

$$y_2^{(1)} = -\frac{3\pi^2}{8}(9 + 2a + 10a^2) + \frac{3\pi}{2}(3a + 2a^2 + 6a^3)$$

$$y_3^{(1)} = -\frac{3\pi^2}{8} b\,(9 + 2a + 10a^2) + \frac{3\pi}{2}(3b + 2ab + 6a^2b)$$

$$y_1^{(1)'} = \frac{3\pi^2}{8}(25 + 2a + 26a^2) - \frac{3\pi}{2}(3 + 2a + 6a^2)$$

$$y_2^{(1)'} = \frac{\pi}{2}(-2a^3 - a + b^2) \tag{37}$$

$$y_3^{(1)'} = -\frac{\pi}{2}(b + ab + 2a^2b)$$