MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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AN ELEMENTARY ANALYSIS OF THE ADAPTATION OF THE BEAMS' EXPERIMENTAL CONCEPT TO A $\Delta G/G$ DETECTOR TO BE USED IN SPACE

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ABSTRACT

This research note presents an elementary analysis for an experimental apparatus. In order to detect changes in the gravitational attraction between two or more test bodies, the experiment uses inertial angular acceleration as a balance torque (in a D'Alembertian sense). The Dicke-Brans theory General Relativity predicts a different value for the Newtonian Gravitational Constant, G, in regions of different gravitational potential. This difference is very small for gravity fields of objects in the solar system. For an initially remote observor approaching the earth, the maximum difference is

$$\frac{\Delta G}{G} \sim - 7 \times 10^{-11}$$

This note contains a discussion of the problems associated with adapting an experimental concept of J.W. Beams to an apparatus that can be used in a spacecraft. The experiment will detect $\Delta G/G$ as the craft orbits to regions of different gravitational potential. The chief results of the analysis are

1. Establishment of an idea of the size (and mass) of the experiment

2. Isolation of problems to be considered in subsequent analyses.

ACKNOWLEDGEMENT

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NOTE

This research note is one of a series of reports published by the MIT Measurement Systems Laboratory. The "RN" designation indicates a semi-formal presentation of research results. These presentations represent work in progress so the results are not necessarily complete or fully refined.

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I. Detection of ΔG/G

The Dicke-Brans scalar-interaction theory in General Relativity predicts a different value for the Newtonian Gravitational Constant, G, in regions of different gravitational potential. This effect is very small for gravity fields of objects in the solar system. For an initially remote observer approaching the earth, the maximum difference is

$$\frac{\Delta G}{G} \sim -7 \times 10^{-11}.$$

To make a measurement of ΔG/G, we require instrumentation of unprecedented sensitivity. The Measurement Systems Laboratory (MSL) has been working on the design of an apparatus that can be used in a spacecraft to detect ΔG/G as the craft orbits to regions of different gravitational potential.

In general the experimental apparatus, as presently conceived (at MSL and elsewhere), will involve a set of precisely known test-masses disposed in a precisely known geometry. In one version, gravitational forces between the test masses are exactly balanced by some other calibrated forces† (presumably not subject to change with gravitational potential). These calibrated forces can be adjusted to account for Dicke-Brans changes in the gravitational forces. The balance is detected by observing the relative displacements of the masses. In a second version, the masses are arranged to permit relative periodic motion, either libration or rotation.[(1),(2)*] With

*Superscript numbers refer to the list of references

†We include here and throughout this note the notion of inertial reaction as a force in the D'Alembertian sense.

gravitational attraction as the restoring force (alone or in combination with some other calibrated force[3]), the period of the motion is related to G.

This note is concerned with a few aspects of the force-balance version. It seems reasonable to assume that the most precise force balance is that in which we counteract gravitational attraction with an inertial force.* Since inertial forces can be determined by the direct measurements of mass, length, and time, we then have the possibility of making an absolute determination (in terms of present mass, length, and time standards) of G (at a given point) in the process of detecting $\Delta G/G$.

Each of the three inertial forces that arise in rotational motion has been suggested for a balance force. Centrifugal force as a balance is the design basis for a device consisting (in part) of a massive sphere of uniform density, ρ, and a small test mass, m, free to move without friction in a radial tunnel.[4] The sphere is given an inertial angular velocity, ϕ, such that

$$m\,\phi^2 r = \frac{4}{3}\,\pi\,\rho\,G\,r\,m$$

and then we can get

$$G \propto \phi^2\,\frac{1}{\rho}\,.$$

Coriolis force (in the form of a gyroscopic torque) also has been considered, as in the use of a Pendulous Integrating Gyro Accelerometer (PIGA)† mounted on a massive sphere[5].

* The use of inertial forces is in contradistinction to establishing the force balance with electromagnetic, elastic, or other physical forces.

† An accelerometer used in inertial guidance and navigation systems.

Here the balance equation is

$$H \dot{\psi} = K G$$

in which

H = a constant gyro angular momentum

$\dot{\psi}$ = an inertial angular velocity, applied transverse to H, to provide the torque balance

K = a constant determined by the mass of the sphere, the pendulosity and geometric factors.

The third inertial force (torque) reaction to angular acceleration, has been used by J.W. Beams and others (all at the University of Virginia) in an experiment to measure the absolute value of G.[6] The force balance equation is

$$I \ddot{\phi} = - \frac{\partial (P.E.)}{\partial \theta}$$

in which

I = moment of inertia of a pivoted test body

θ = relative angle between the test body and a set of known attracting masses

P.E. = potential energy (proportional to G) of the test body in the field of the attracting masses

$\ddot{\phi}$ = an inertial angular acceleration imparted to the whole apparatus to effect a torque balance.

In this note we are specifically concerned with some preliminary thoughts on adapting the Beams' method to a space experiment for the detection of $\Delta G/G$.

II. Beams' Experimental Concept

Newton's Gravitational Constant, G, is measured in the Beams' experiment by balancing a gravitational torque acting on a pivoted test body with an inertial angular acceleration. As shown in Fig. I, the test body is torqued by the gravitational attraction of two sphere, while an equal and opposite torque is provided by accelerating the table that holds the apparatus so that the relative displacement, θ, remains constant.*

The theoretical gravitational torque can be calculated to great precision (except for the constant factor G) since spheres of uniform density are used and the dimensions of the experiment are determined accurately. The dimensions are held to their measured values by running the experiment in a temperature controlled environment. The angular acceleration of the table is proportional to G, so the data taken are the time increments for successive rotations. These time increments are then used to calculate the acceleration.

The test body is, of course, subject to gravitational torques due to other objects than the spheres. However, by the conservative nature of gravitational fields, external (to the table) stationary masses have effects that are averaged to zero for complete (relative to the laboratory) revolutions of the table. The data taken are then the time increments for complete revolutions. Torques arising from masses (other than the spheres) on the table, as well as the effects of the fibre suspension are

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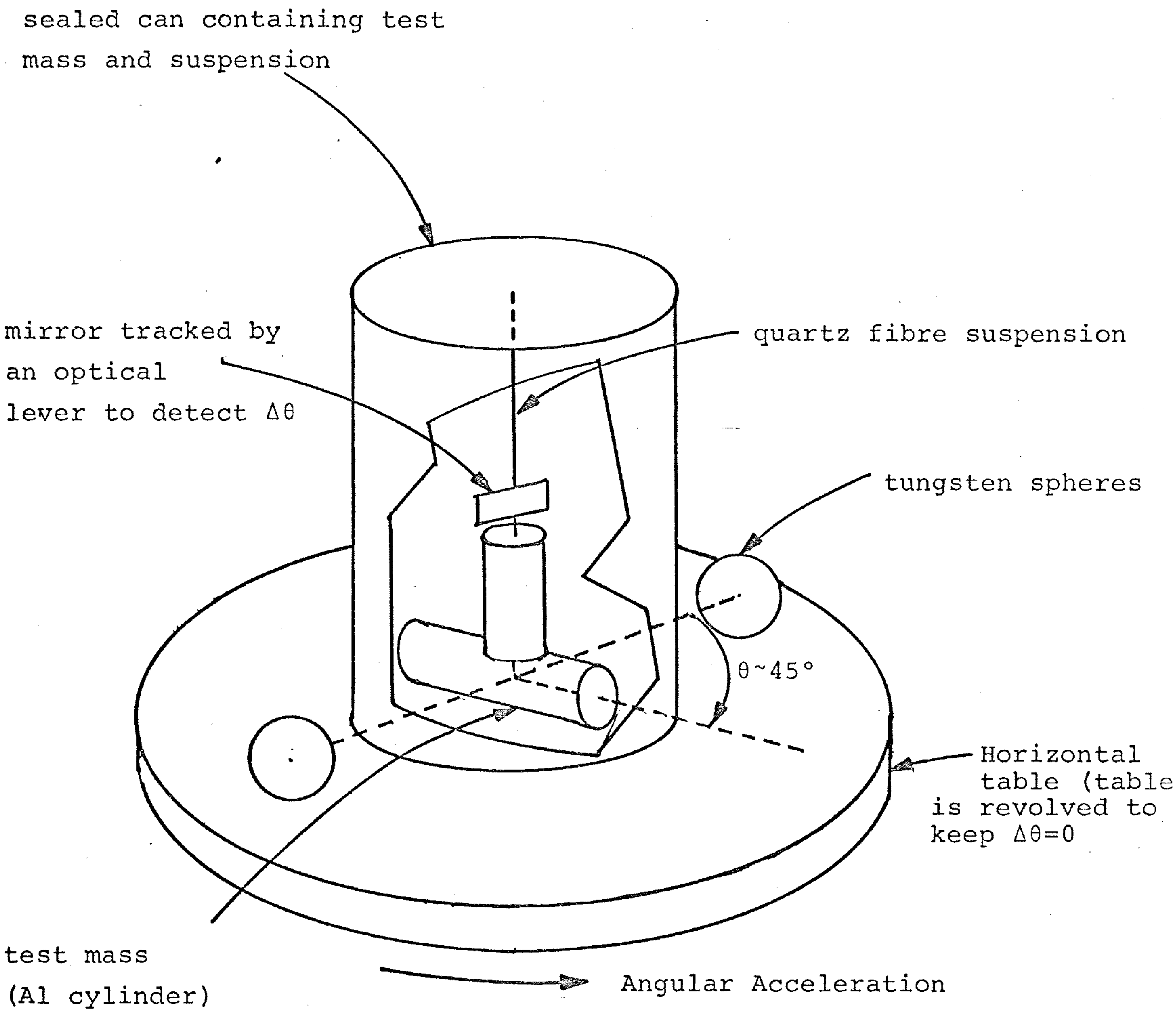
* The situation is analogous to the case of linear motion when we have a test mass, m "freely falling" in the field of another body, M. The relative separation between m and M is maintained constant by accelerating M so that, in effect, m is "chasing" M.



Note: Not to scale. Spheres would be 3".

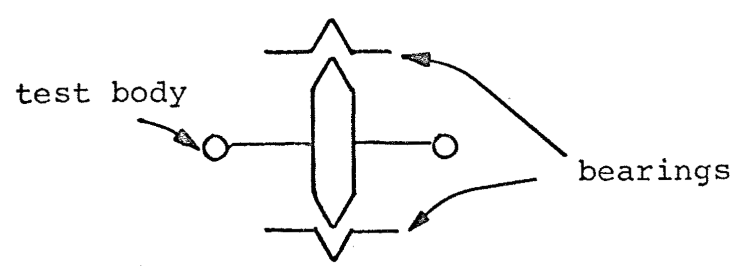
FIG. I. BEAMS' EXPERIMENT

calibrated out by running the experiment with and without the spheres on the table.

The Beams' experiment has been built and operated at the University of Virginia. Experimental results have confirmed the presently accepted value of G (known to one part in 500).[6] Further results show that the apparatus gives consistent measurements to one part in 34,000 or about 3 parts in 10^{-5}.*

III. Adaptation of the Beams' Concept to a Space Experiment

For operation in a spacecraft, the most obvious change in the Beams' device is the suspension of the test body. In view of the single-degree-of-freedom nature of the experiment, we require a suspension that permits the test body to rotate in a set of bearings something like a watch balance wheel.



Since θ is to be held constant during the experiment, the bearing need only be "frictionless" over a small range around the operating value, θ_p.†

---

*Verbal communication with J.W. Beams.

†Suspensions that can be considered are

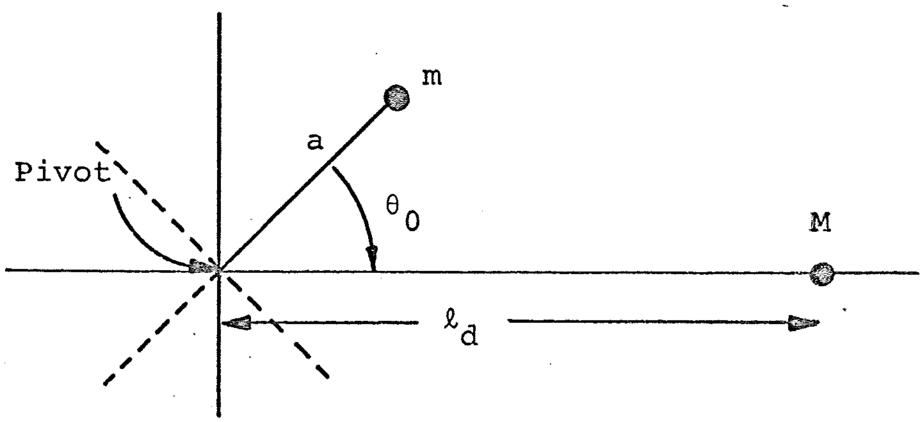
1. Double torsion fibre
2. Magnetic - Servo controlled electromagnets[7]
3. Magnetic - Diamagnetic substance at room temperature[8]
4. Magnetic - Meissner effect at cryogenic temperatures[9][1]
5. Electrostatic.

Of these, 3 and 4 seem the most promising, at least for an initial analysis.

Before the selection and design of a suspension system, we need to answer some questions about the size, mass, and geometry of the experiment. Most of the remainder of this note is concerned with these preliminary design questions.

IV. Preliminary Design Considerations

One of the compelling features of the Beams' concept is that external stationary masses have effects, which average to zero for complete revolutions of the experiment. In a spacecraft, however, we will have the motion of astronauts and the changing mass of the craft as fuel is expended, as well as a changing position in the gravity gradient field of the orbited body. To minimize the effects of these moving objects, we can consider making the test body of a number of symmetrically arranged arms. To show this, we make some calculations based on the idealized representation of the arms by massless rods of length, a, with point masses, m, at the ends. In the sketch we show a disturbing mass, M, (taken



as a point or spherical mass for simplicity) at a distance ℓ_d from the pivot. The potential energy of m in the field of M is

$$\text{P.E.} = \frac{m\ M\ G}{\sqrt{a^2+\ell_d^{\ 2}-2a\ell_d\cos\theta_0}} \qquad (1)$$

For the case of n symmetrical arms, we have

$$\text{P.E.} = \sum_{i=1}^{n} mMG \frac{1}{\ell_d} (1+h^2-2h \cos\theta_i)^{-1/2} \qquad (2)$$

in which

$$h \equiv \frac{a}{\ell_d}$$

and

$$\theta_i \equiv \theta_0 + \frac{i}{n} (360) \text{ degrees.}$$

We can expand the radical in (2) in terms of Legendre polynomials

$$(1+h^2-2h \cos\theta_i)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} (h)^k P_K(\cos\theta_i).$$

Equation (2) becomes

$$\text{P.E.} = \frac{mMG}{\ell_d} \sum_{i=1}^{n} \sum_{k=0}^{\infty} (h)^k P_k(\cos\theta_i). \qquad (3)$$

The first few polynomials are

$$P_0(\cos\theta_i) = 1$$

$$P_1(\cos\theta_i) = \cos\theta_i$$

$$P_2(\cos\theta_i) = \frac{1}{2} (\cos^2\theta_i-1)$$

$$P_3(\cos\theta_i) = \frac{1}{2} (5 \cos^3\theta_i-3 \cos\theta_i)$$

$$P_4(\cos\theta_i) = \frac{1}{8}(35 \cos^4\theta_i - 30 \cos^2\theta_i + 3) .$$

Presuming that the disturbing mass is at some greater distance than the arm length or $h < 1$, we write (3) out to the precision of h^4.

$$P.E. = \frac{mMG}{\ell_d}\Big[\sum_{i=1}^{n} P_0(\cos\theta_i) + h \sum_{i=1}^{n} P_1(\cos\theta_i) + h^2 \sum_{i=1}^{n} P_2(\cos\theta_i)$$

$$+ h^3 \sum_{i=1}^{n} P_3(\cos\theta_i) + h^4 \sum_{i=1}^{n} P_4(\cos\theta_i)\Big] .$$

Substituting the expressions for the P_k's and rearranging the terms gives us

$$P.E. = \frac{mMG}{\ell_d}\Big[n\left(1 - \frac{h^2}{2} + \frac{3h^4}{8}\right) + \sum_{i=1}^{n} \cos\theta_i\left(h - \frac{3}{2}h^3\right)$$

$$+ \sum_{i=1}^{n} \cos^2\theta_i\left(\frac{3}{2}h^2 - \frac{30}{8}h^4\right)$$

$$+ \sum_{i=1}^{n} \cos^3\theta_i\left(\frac{5}{2}h^3\right) + \sum_{i=1}^{n} \cos4\theta_i\left(\frac{35}{8}h^4\right)\Big] .$$

For n=3, the summations give us*

---

*We ignore n=1 and n=2, since in the first case M gives a direct torque and in the second case the gradient of M's field will torque the arms.

$$P.E. = \frac{mMG}{\ell_d}\left[3\left(1-\frac{h^2}{2}+\frac{3h^4}{8}\right)+\frac{3}{2}\left(\frac{3}{2}h^2-\frac{30}{8}h^4\right)\right.$$

$$\left.+\frac{3}{4}\cos 3\theta_0\left(\frac{5}{2}h^3\right)+\frac{9}{8}\left(\frac{35}{8}h^4\right)\right]$$

and for n = 4,

$$P.E. = \frac{mMG}{\ell_d}\left[4\left(1-\frac{h^2}{2}+3\frac{h^4}{8}\right)+2\left(\frac{3}{2}h^2-\frac{30}{8}h^4\right)\right.$$

$$\left.+\left(\frac{12}{8}+\frac{1}{2}\cos 4\theta_0\right)\left(\frac{35}{8}h^4\right)\right].$$

To determine the torque exerted by M's field, we have

$$T \equiv \text{Torque} \equiv \frac{\partial P.E.}{\partial\theta_0} .$$

For n=3,

$$T_{n=3} = \frac{-mMG}{\ell_d}\ \frac{45}{8}h^3 \sin 3\theta_0 \tag{4}$$

and for n=4,

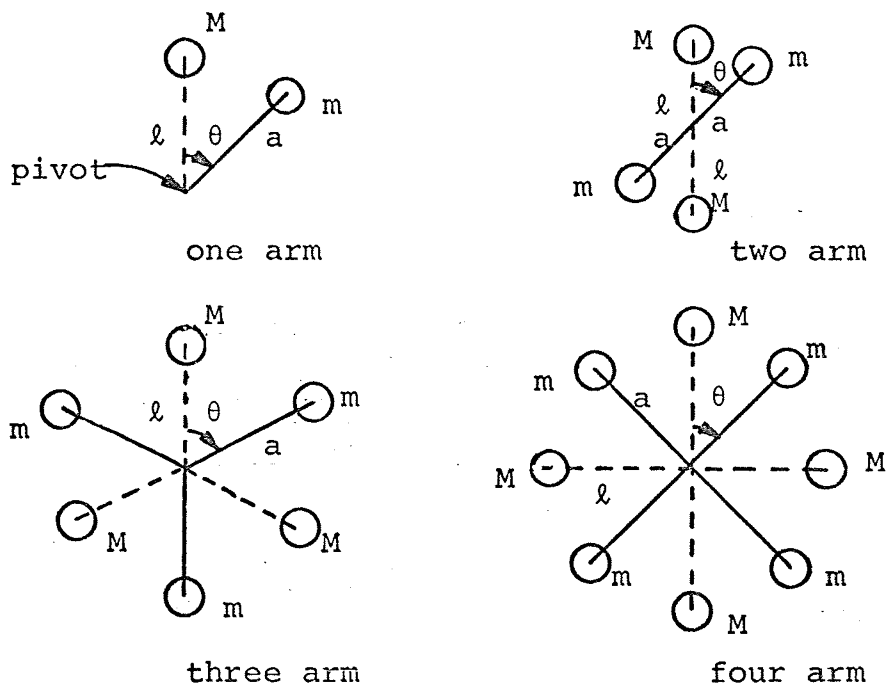
$$T_{n=4} = \frac{-mMG}{\ell_d}\ \frac{35}{4}h^4 \sin 4\theta_0 . \tag{5}$$

With the disturbing mass, M, at a distance, ℓ_d, we see that the peak torque on the 4-arm device is about 3/2 h of that on the 3-arm device. We conclude that the 4-arm device has a distinct advantage (if h < 2/3) especially in the changing mass environment of a manned spacecraft.

V. Geometric Design Considerations

Increasing the number of arms reduces the sensitivity to external fields -- yet we can not increase the number of arms indefinitely because we must still apply a torque by placing our calibrated masses in some sort of known juxtaposition to the moving arms.*

For our simple analysis, a pivoted device consisting of n symmetrically disposed massless rods tipped with point (spherical) masses, m, are attracted by n symmetrically disposed fixed spheres, M. The sketch shows the configurations to be analyzed. We can determine the torque expression for these configurations by differentiating (2) with respect to θ and multiplying by n



*In the absurd extreme we could increase the number of arms until we had a wheel. Then no torques could be applied, disturbance or otherwise. We obviously have a design "trade-off" between sensitivity to external masses and the efficient use of the calibrated masses.

(for the n M's). Equation (2) (with $\ell_d \rightarrow \ell$) gives

$$T = n \frac{mMG}{\ell} \sum_{i=1}^{n} \frac{h \sin (\theta + \frac{i}{n} 360)}{[1+h^2-2h \cos(\theta + \frac{i}{n} 360)]^{3/2}} \tag{6}$$

in which

$h = \frac{a}{\ell}$, a design parameter

a = length of the pivoted arm

ℓ = distance from the pivot to the calibrated attracting mass.

For given values of our design parameter, h, we can calculate an operating angle, θ_p, for maximum torque. Setting

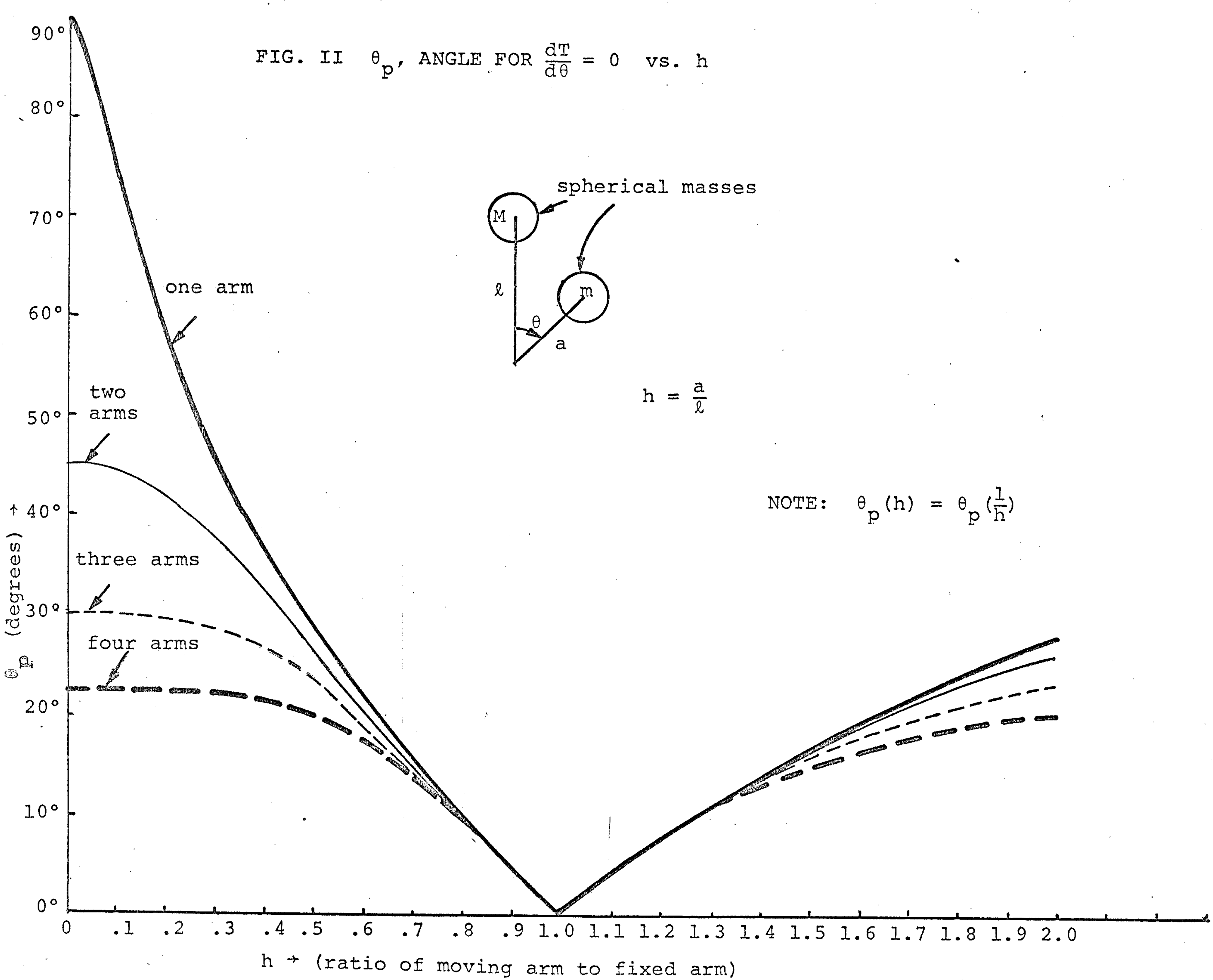
$$\frac{dT}{d\theta} = 0$$

we obtain for n=1

$$\cos\theta_p = \frac{1}{2} \left[- \frac{1+h^2}{h} + \sqrt{\left(\frac{1+h^2}{h}\right)^2 + 12} \right]. \tag{7}$$

For other values of n, we must calculate $\cos\theta_p$ by numerical methods. Fig. II shows the results of these calculations. Fig. III gives the same information in a more instructive manner. If M, the attracting mass, lies as shown, then the moving mass, m, must lie on the curve as shown if $dt/d\theta = 0$.

FIG. II θ_p, ANGLE FOR $\frac{dT}{d\theta} = 0$ vs. h



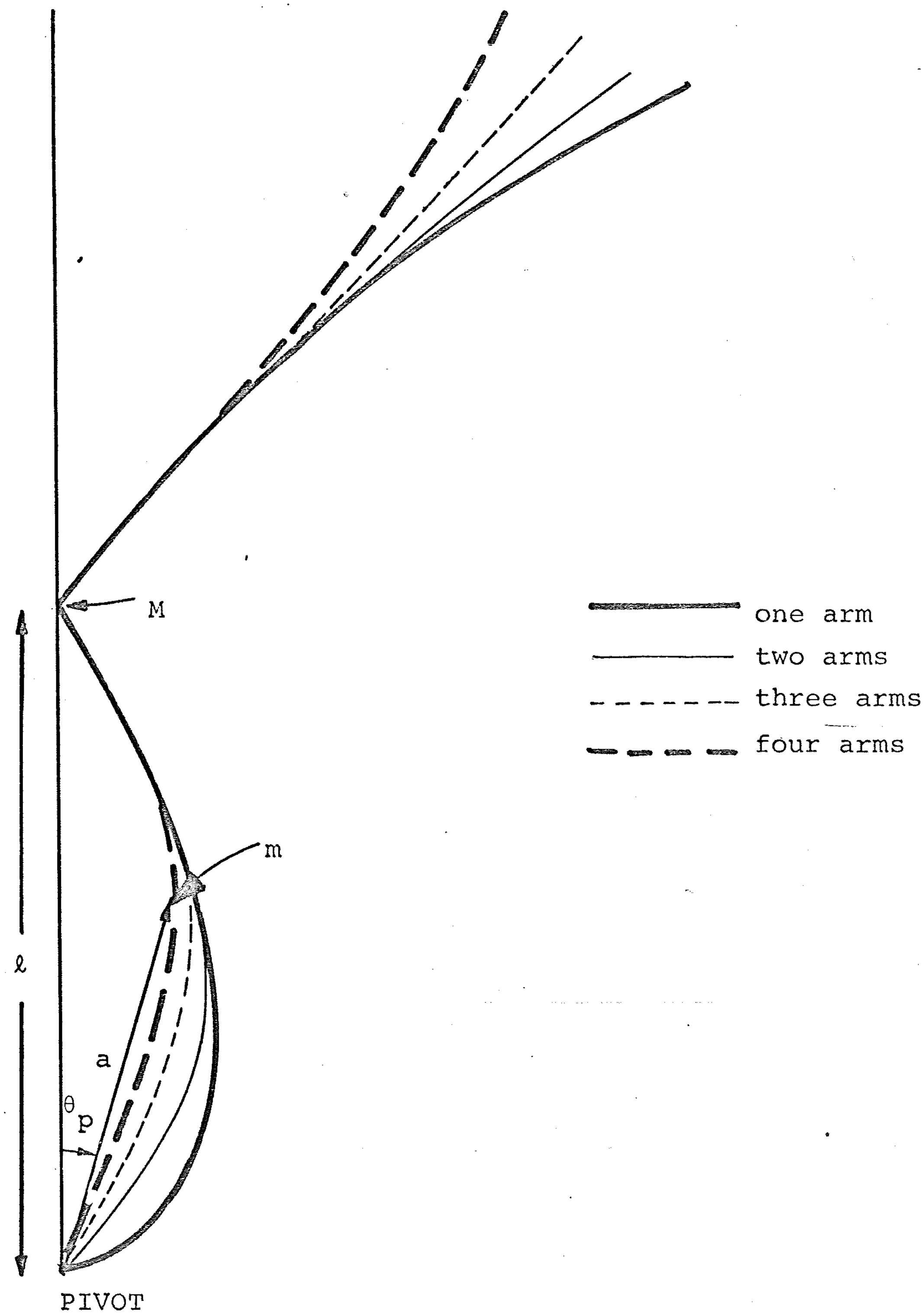


FIG. III. CURVES ON WHICH m MUST LIE FOR $\frac{d\ddot{T}}{d\theta} = 0$

In addition to the desirability of placing m so that the torque is a maximum (for a given h), it is also important to operate at an angle θ_p, the angle at which the magnitude of the torque is relatively insensitive to changes in θ. In general, $\Delta\theta$ will be the error signal to the servo that accelerates the experiment.* Operating at θ_p, we can make an estimate of the change in torque, ΔT, with $\Delta\theta$, by expanding the torque expression in a Taylor series around $\theta = \theta_p$ to get

$$T(\theta_p+\Delta\theta) = T(\theta_p) + T''(\theta_p)\,\frac{\Delta\theta^2}{2} + \ldots .$$

in which we have used

$$T'(\theta_p) \equiv 0.$$

Now we have

$$\frac{\Delta T}{T} = \frac{T(\theta_p+\Delta\theta) - T(\theta_p)}{T(\theta_p)} = \frac{T''(\theta_p)\,\frac{\Delta\theta^2}{2}}{T(\theta_p)} .$$

Differentiating (6) twice and using $T'(\theta_p) \equiv 0$, we get

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*It can be argued that $\Delta\theta$ could be monitored and then applied with a correction factor in the data reduction. This is true for operation at any θ; however, the sensitivity of our result to errors in $\Delta\theta$ and the correction factor will be reduced if θ_p is used.

$$\frac{\Delta T}{T} = \frac{T''(\theta_p)\ \frac{\Delta\theta^2}{2}}{T(\theta_p)} = \frac{\displaystyle\sum_{i=1}^{n} \frac{-\sin(\theta_p + \frac{i}{n}\,360)\left[2h\cos(\theta_p + \frac{i}{n}\,360) + (1+h^2)\right]}{\left[1+h^2-2h\cos(\theta_p+\frac{i}{n}\,360)\right]^{5/2}}}{\displaystyle\sum_{i=1}^{n} \frac{\sin(\theta_p+\frac{i}{n}\,360)}{\left[1+h^2-2h\cos(\theta_p+\frac{i}{n}\,360)\right]^{3/2}}}\ \frac{\Delta\theta^2}{2}$$

Using the values of θ_p shown in Fig. II, we can evaluate this equation for the different values of n. The results are shown in Fig. IV. A value for $\Delta\theta$ of one arc second was used. This value was selected as a reasonable tracking error for instrument servomechanisms. From the curves in Fig. IV, we see that for high precision ($\Delta T/T < 1.5\ 10^{-10}$)

$$h > 1.7 \text{ or } h < 0.5$$

at least for the simple model used in these calculations.

The significance of operating at θ_p can be illustrated by an example. Using the simple model, we will assume a two-arm device with $h = 0.15$ and an operating angle, $\theta = 45°$.* (From Fig. II, we note that for $h = 0.15$, $\theta_p = 43.2°$.) Using (6), we get for n=2

$$\frac{dT/d\theta}{T} = \frac{\left[(1+h^2+2h\cos\theta)^{-5/2}(h\cos^2\theta-(1+h^2)\cos\theta-3h)\right]+\left[(1+h^2-2h\cos\theta)^{-5/2}\left[(h\cos^2\theta+(1+h^2)\cos\theta-3h)\right]\right]}{\sin\theta\left[(1+h^2-2h\cos\theta)^{-3/2} - (1+h^2+2h\cos\theta)^{-3/2}\right]}$$

* These are very nearly the conditions for the Beams' experiment, if we approximate the moving cylinder (see Fig. I) by a dumbbell of equal mass and moment of inertia.

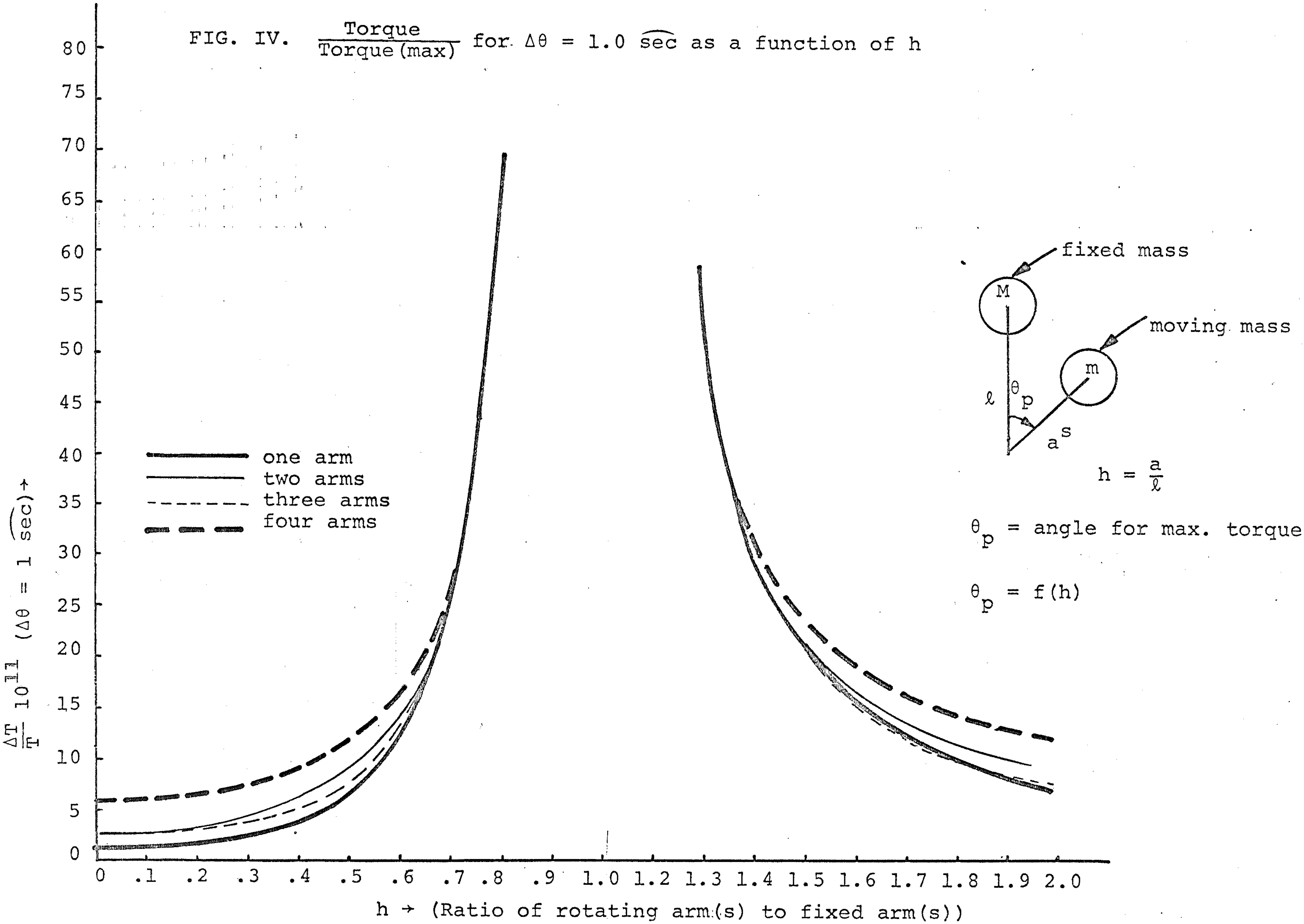


FIG. IV. $\frac{\text{Torque}}{\text{Torque (max)}}$ for $\Delta\theta = 1.0\ \widehat{sec}$ as a function of h

Evaluating this expression for h = 0.15 and $\theta = 45°$, we get

$$\frac{dT/d\theta}{T} = 0.13.$$

For a servo error angle, $\Delta\theta$ = 20 seconds of arc we have

$$\frac{\Delta T}{T} = \frac{dT/d\theta}{T} \Delta\theta \approx (0.13)10^{-4} = 1.3\ 10^{-5}.$$

In this simple example, servo error angles of 20 seconds of arc cause variations in our assumed value of T to a part in 10^{-5}.* For contrast we see from Fig. IV that operation at $\theta = \theta_p = 43.2°$ would give

$$\frac{\Delta T}{T}\ (\Delta\theta = 20 \text{ sec. of arc}) \rightarrow 3\ 10^{-11}(400)$$

$$\rightarrow 1.2\ 10^{-8},$$

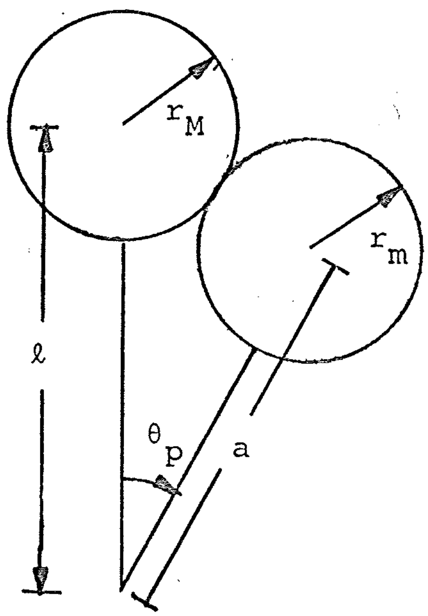
an improvement approaching 3 orders of magnitude.

VI. Size of the Experiment

To get an idea of the magnitudes of the torques and accelerations as a function of the size of the experiment, we proceed by assuming

*Reference 6 on the Beams' experiment reports tracking errors as large as 20 seconds of arc for short periods of time; however, in general they assumed the tracking error to be less than 0.2 seconds of arc. For the latter error angle, operation at $\theta = 45°$ would not contribute an appreciable error to their present results. However, for a refined version of the Beams' experiment, it would seem prudent to operate at $\theta = \theta_p$ as calculated for the particular mass configuration used.

spherical masses of maximum size for a given θ_p.* Further, to maximize the torque, we will apportion the total mass of the experiment equally between the fixed and moving arms. From the sketch we see the maximum diameter spheres that we can fit in for a given θ_p.



$$r_M = r_m$$

$$h = \frac{a}{\ell}$$

$$\theta_p \equiv F(h)$$

$$m = M = \frac{4}{3} \pi \rho r_m^3$$

ρ = density of the spheres

The maximum radii are

$$r_m = r_M = \frac{\ell}{2} \sqrt{1+h^2-2h \cos\theta_p} \quad . \tag{8}$$

Equation (6) now becomes

$$T = n \left[\frac{\pi\rho}{6}\right]^2 G(1+h^2-2h \cos\theta_p)^3 \ell^6 \frac{h}{\ell} \sum_{i=1}^{n} \frac{\sin(\theta+ \frac{i}{n} 360)}{[1+h^2-2h \cos(\theta+ \frac{i}{n}360)]^{3/2}} \quad . \tag{9}$$

* We will assume that the experiment will operate at $\theta = \theta_p$ in this analysis.

Taking

$$\rho = 21.45 \text{ grams/cm}^3 \quad \text{(platinum)}$$

and

$$G = 6.675\ 10^{-8}$$

we get

$$T = n\ 8.07\ 10^{-6} \ell^5 h(1+h^2-2h\cos\theta_p)^3$$

$$\sum_{i=1}^{n} \frac{\sin(\theta + \frac{i}{n} 360)}{[1+h^2-2h\cos(\theta+\frac{i}{n} 360)]^{3/2}} \quad . \qquad (10)$$

We see from the above sketch that the attractive force between the spheres creates a torque about the pivot that is the same in magnitude whether M is the fixed mass and m moves or m is fixed and M moves. For convenience, we introduce

$$h' = \frac{\text{length of shorter arm}}{\text{length of longer arm}}$$

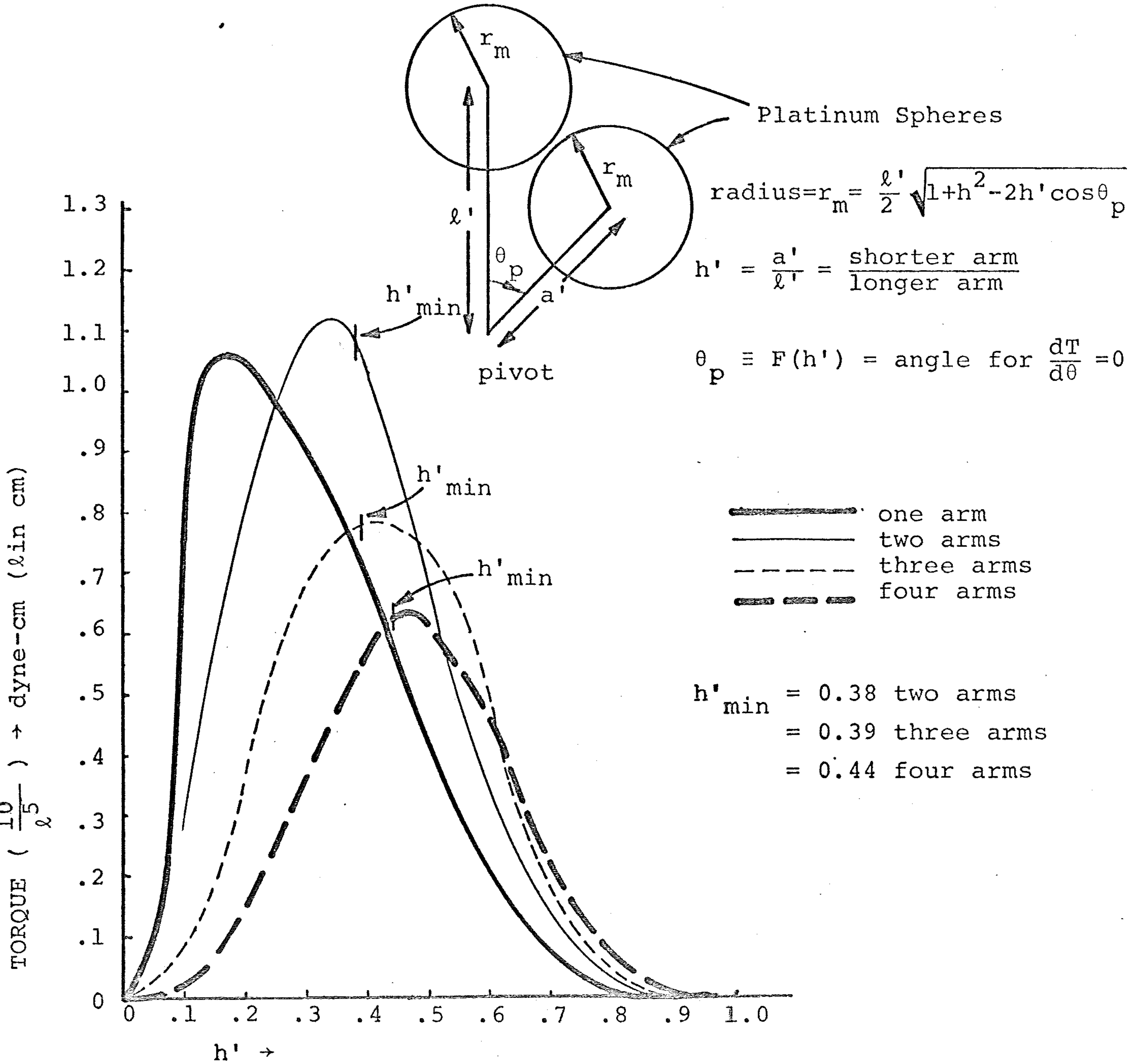
$$\ell' = \text{length of longer arm.}$$

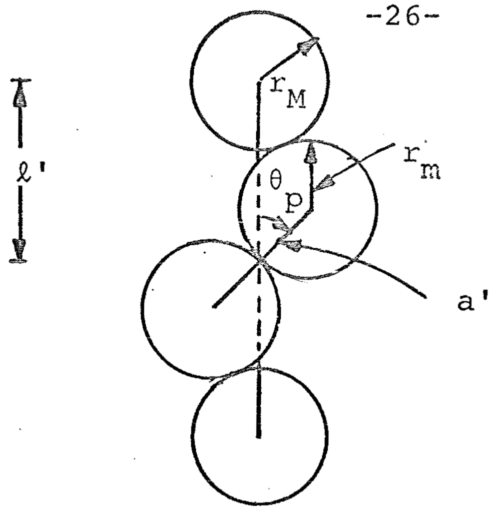
Equation (9) remains the same with $h \rightarrow h'$ and $\ell \rightarrow \ell'$. Torque normalized to $(\ell')^5$ is given as a function of h' in Fig. V. Before discussing the curves, we note that there are minimum values for h' established by mechanical interference, as illustrated in the following sketch.*

---

* This interference arises because our model calls for spheres that have diameters which are functions of h' and $\theta_p = F(h')$.

FIG. V. MAXIMUM GRAVITATIONAL TORQUE NORMALIZED TO $(\ell')^5$





Mechanical interference occurs when the two inner spheres become tangent. At this point

$$r_m = a'$$

and using (8)

$$r_m = \frac{\ell'}{2} \sqrt{1+h'^2-2h'\cos\theta_p} = a'.$$

Then

$$\frac{a'}{\ell'} = h' = \frac{1}{2} \sqrt{1+h'^2-2h'\cos\theta_p} \quad .$$

Using (7) for $\cos\theta_p$ and solving for h', we get

$$h'_{min} = 0.38 \text{ for two arms}$$

Similar arguments give

$$h'_{min} = 0.39 \text{ for three arms}$$

and

$$h'_{min} = 0.44 \text{ for four arms.}$$

From Fig. V, we note that for any ℓ', maximum torque occurs at a particular h'. For the 4-arm device, optimum torque occurs at $h' \sim 0.46$ and with $\ell' = 16$ cm, for example, the torque is 0.64 dyne-cm.

This optimization is important for keeping the experiment size small and yet getting the maximum torque for mass (spheres in this model) used. Continuing on this line of thought, we can divide (9) by $n\ a^2 m$ to get

$$\frac{T}{na^2m} = \left[\pi \frac{\rho}{6}\right] G \frac{1}{h} (1+h^2-2h \cos\theta_p)^{3/2} \sum_{i=1}^{n} \frac{\sin(\theta_p + \frac{i}{n} 360)}{[1+h^2-2h \cos(\theta_p + \frac{i}{n} 360)]^{3/2}} \tag{11}$$

This expression is plotted in Fig. VI. Using (3) we can calculate the maximum disturbance torque caused by a 150 lb. astronaut (taken as a spherical object) at 2 meters from a 4-arm device. This disturbance torque is

$$T_d = ma^4(.124\ 10^{-12}) \text{ dyne-cm}$$

or

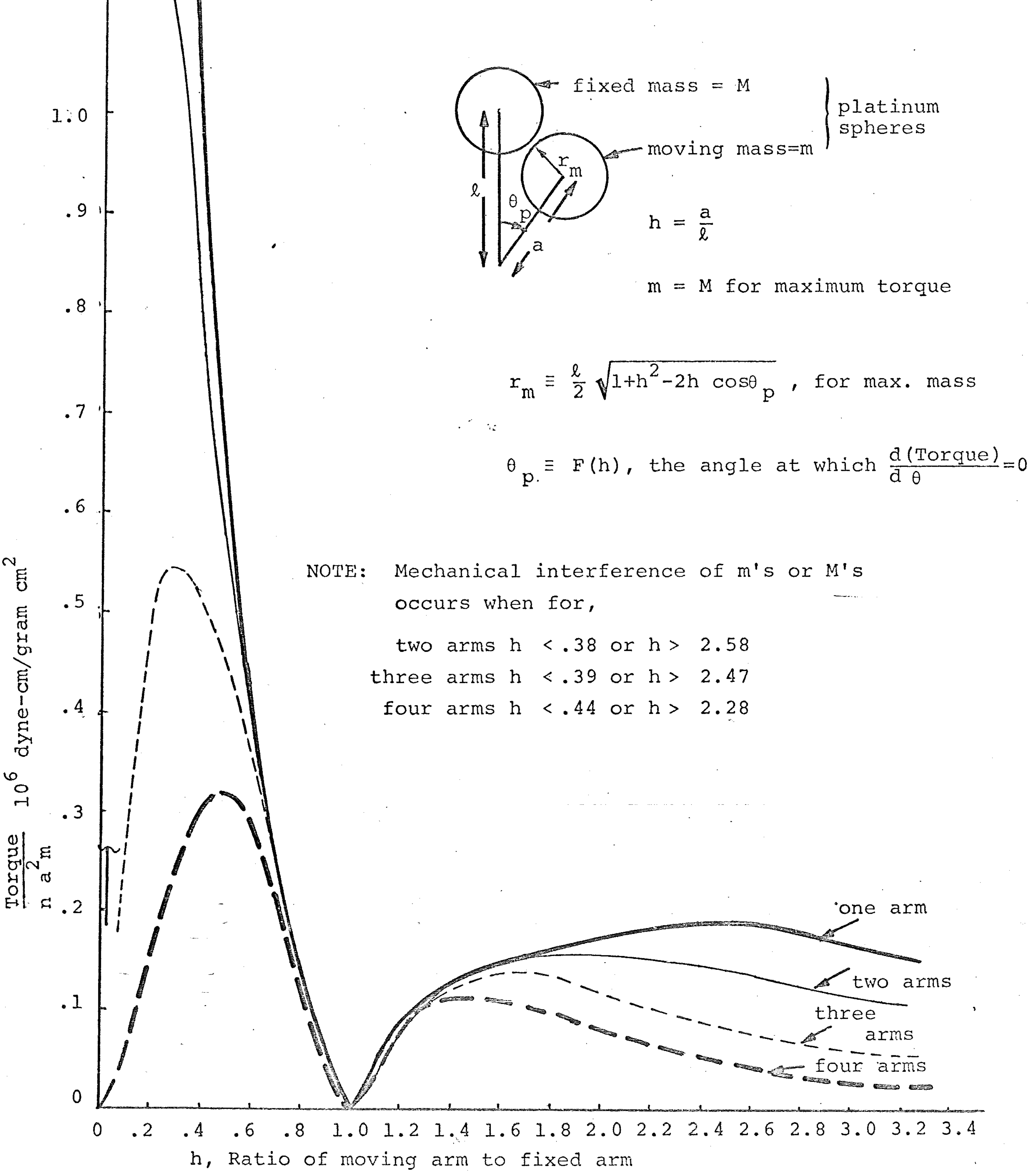
$$\frac{T_d}{ma^4} = .124\ 10^{-12}. \tag{12}$$

We can divide (12) by (11) to get an expression for a^2 in terms of T_d/T. For example, say we wish to limit

$$\frac{T_d}{T} < 10^{-6}$$

then from Fig. VI (4-arm) we have the maximum $T/4a^2m = 0.32\ 10^{-6}$ at $h \sim 0.5$. Dividing (12) by this gives

FIG. VI TORQUE AS A FUNCTION OF h NORMALIZED TO (n a^2m)



$$a^2 = 10.3$$

or

$$a = 3.2 \text{ cm (4-arm device)}.$$

This is the maximum size for "a" if we want to limit the effect of the astronaut to 1 part in 10^6. If we want to put the same limit on this disturbance torque's effect on a 3-arm device, we get

$$a_{max} = 0.13 \text{ cm} \quad \text{(3 arms)}$$

For this simple model, the 4-arm device has a significant advantage in terms of size vs. effect of disturbance masses.

Using Fig. 6, we can select the value of h that gives the maximum torque for given values of a and m*. For the four arm device, maximum torque occurs at $h \sim 0.5$.

Up to this point, we have only considered the gravitational torque acting on the arms; now we can calculate the angular acceleration needed to balance this torque by dividing (9) by the moment of inertia of the moving arms. For the model of spheres on massless rods, the moment of inertia about the pivot is

$$I = n[\frac{2}{5} m \, r_m^2 + ma^2] .$$

Using (8) gives

$$r_m^2 = \frac{a^2}{4}\left[\frac{1}{h^2} + 1 - \frac{2}{h}\cos\theta_p\right]$$

---

*We recall that the total mass of the 4-arm experiment is 8 m.

and then

$$I = n\,m\,a^2[1.1 + \frac{.1}{h^2} - \frac{.2}{h}\cos\theta_p]. \quad (13)$$

Dividing (9) by (13) gives the acceleration

$$\frac{T}{I} = \left[\frac{\pi\rho}{6}\right] G \frac{(1+h^2-2h\cos\theta_p)}{h(1.1+\frac{.1}{h^2} - \frac{.2}{h}\cos\theta_p)} \sum_{i=1}^{n} \frac{\sin(\theta_p + \frac{i}{n}360)}{[1+h^2-2h\cos(\theta_p+\frac{i}{n}360)]^{3/2}}. \quad (14)$$

This expression for n=4 is plotted in Fig. VII. The peak acceleration occurs near $h \sim 0.5$. From the results shown in Figs. V, VI, and VII it appears that h = 0.5 would be a near optimum choice for the design of the 4-arm device.*

Fig. VIII is a full-scale sketch of a 4-arm device that has h = 0.5 and a peak gravitational torque of 0.01 dyne-cm. For comparison purposes, Fig. IX shows a full-scale sketch of a 4-arm device that has h = 0.5 and a peak gravitational torque of 0.001 dyne-cm. If we use platinum spheres to make a 4-arm device (h = 0.5), the total mass of the experiment $\approx 8\ 10^4 T^{3/5}$ grams.† Fig.X shows the torque vs. the total mass and size

*Some caution must be used in interpreting Figs. V, VI and VII, since the results shown represent variations with design parameters. The curves do not represent the operation of a particular device. For example, the variation of torque with h (Figs. V and VI) can not be used (at least directly) in a temperature-sensitivity analysis because we are also changing the mass with h in these figures.

†Here we mean, of course, only the mass that is active in the gravitational torque equations

FIG. VII Angular Acceleration of a Four-Armed Device

Platinum Spheres, $r_m = \frac{\ell}{2}\sqrt{1+h^2-2h\cos\theta_p}$

$h = \frac{a}{\ell}$

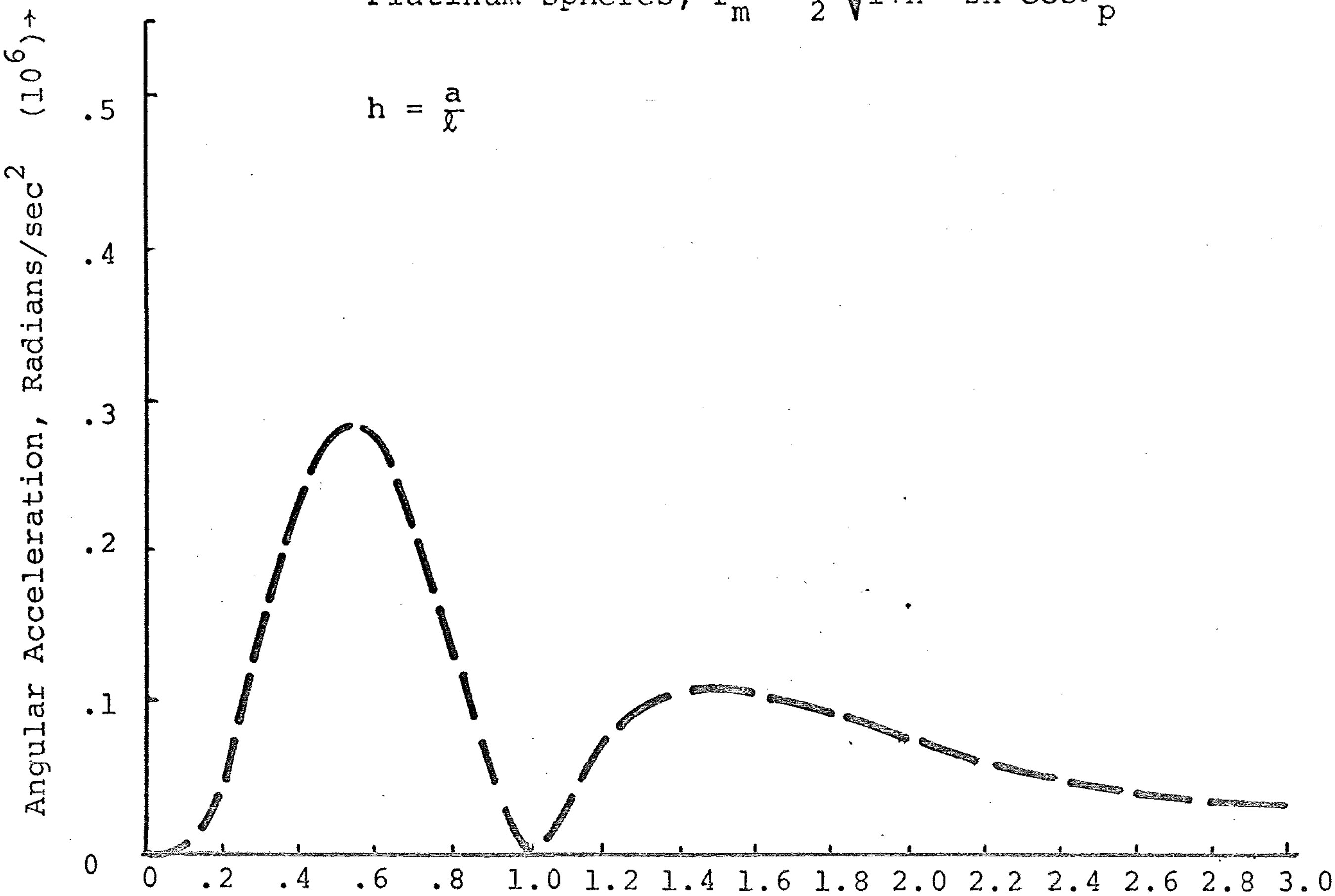
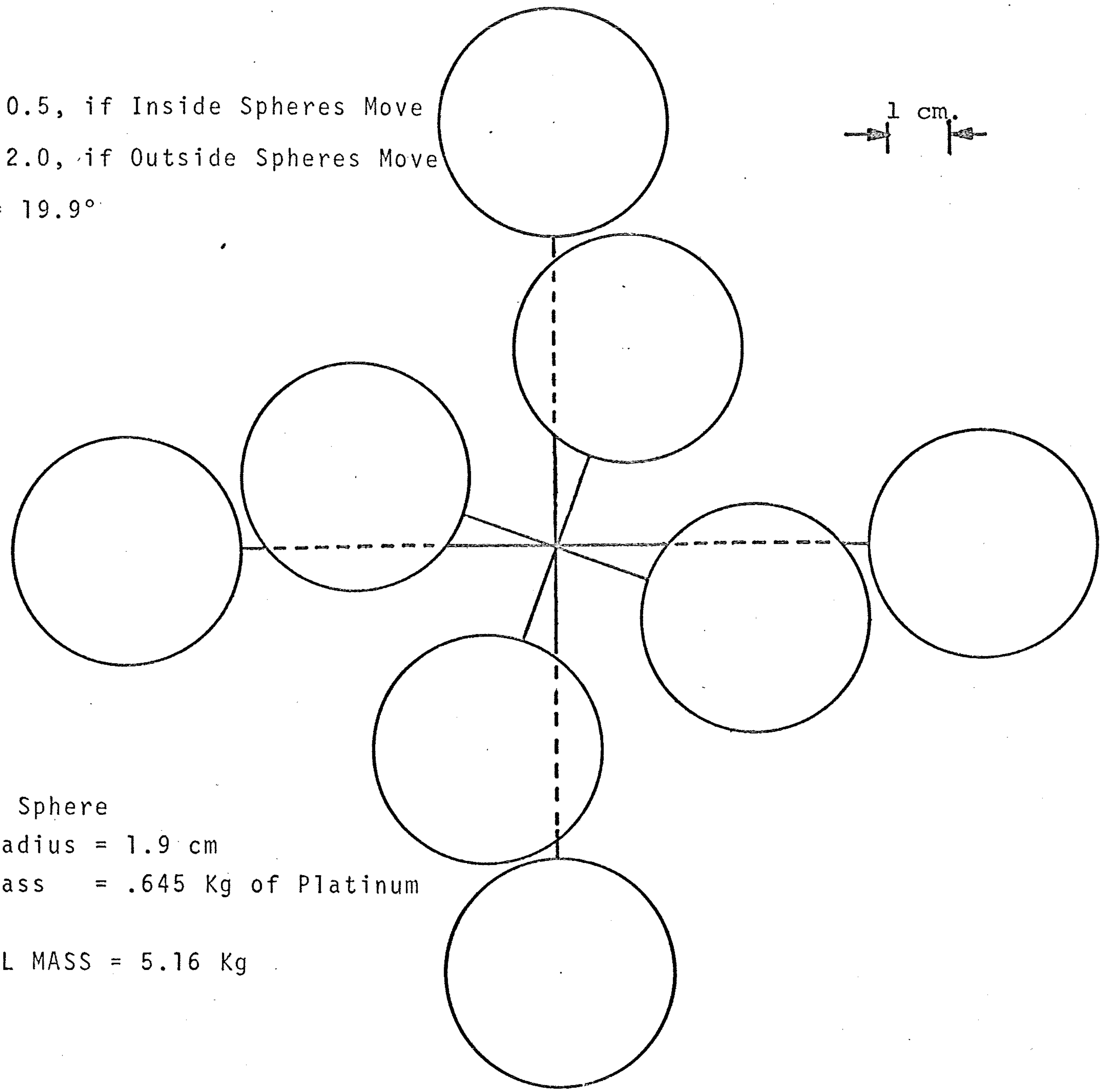


FIG. VIII - FULL-SCALE SKETCH OF A FOUR-ARMED DEVICE
(Gravitational Torque = 0.01 DYNE-CM)

$h = 0.5$, if Inside Spheres Move

$h = 2.0$, if Outside Spheres Move

$\theta_p = 19.9°$



Each Sphere
Radius = 1.9 cm
Mass = .645 Kg of Platinum

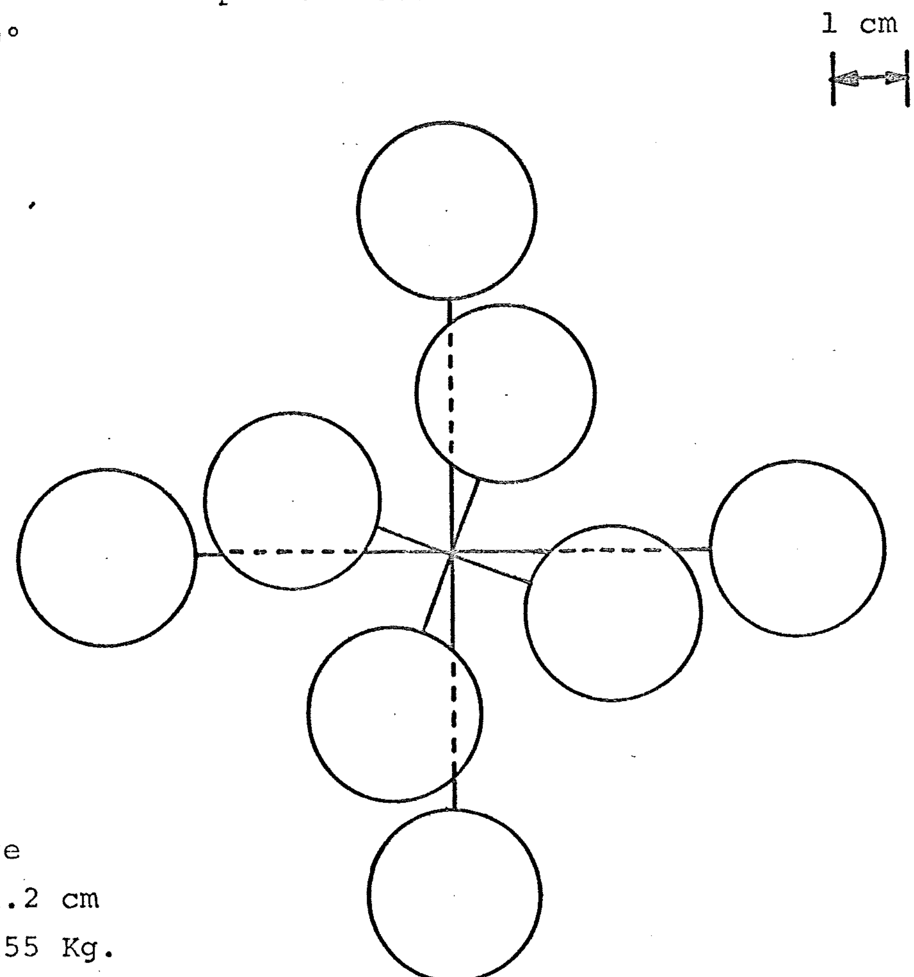
TOTAL MASS = 5.16 Kg

For $h = 0.5$, Angular Acceleration = $.284\ 10^{-6}$ rd/sec^2

$h = 2.0$, Angular Acceleration = $.076\ 10^{-6}$ rd/sec^2

FIG. IX FULL-SCALE SKETCH OF A FOUR ARMED DEVICE
(GRAVITATIONAL TORQUE = 0.001 dyne-cm)

$h = 0.5$ if inside spheres move
$h = 2.0$ if outside spheres move
$\theta_p = 19.9°$



Each Sphere
Radius = 1.2 cm
Mass = 0.155 Kg.
Total Mass = 1.24 Kg.

For $h = 0.5$ Angular Acceleration = $0.284\ 10^{-6}$ rd/sec^2
$h = 2.0$ Angular Acceleration = $0.076\ 10^{-6}$ rd/sec^2

(ℓ) of a 4-arm (h = 0.5) experiment with platinum spheres. In a rough way, we can use Fig. X to estimate the size and mass needed to get a certain accuracy in the face of suspension and other uncertainty torques. We note in this connection that the torque level in the Beams' experiment was about $0.2\ 10^{-4}$ dyne-cm.

VII. Further Analysis

In the foregoing analysis, we used an idealized test body made of spheres and massless rods. This made the analysis tractable for slide-rule calculations and also made the results easy to visualize. These results do have practical validity for the classical configurations of the Cavendish experiment. Traditionally, spheres and cylinders have been employed as test masses in experiments on gravitational attraction. Beyond the obvious analytical advantages, spheres (of small size) and cylinders are practical objects to fabricate with precise dimensions and uniform density. A.H. Cook, in a contemporary Cavendish experiment, has found it expedient to use cylindrical test masses to avoid fabrication difficulties.[10]* For the fixed attracting masses, Cook uses cylinders of radius "a" and length $2\sqrt{3}$ a. With these dimensions and the addition of some small cylindrical end caps, the composite object (see sketch)

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*In a description of a new Cavendish Experiment A.H. Cook says, "The masses attached to the pendulum will be in the form of spheres, since it is not difficult to make spheres of about 10 Kg with high accuracy and with reasonable assurance that the density is uniform..... The stationary attracting masses are to be made much larger, 500 Kg and cannot be spheres, both because of the difficulty of handling them and because of the difficulty of ensuring that the density is indeed uniform."[10]

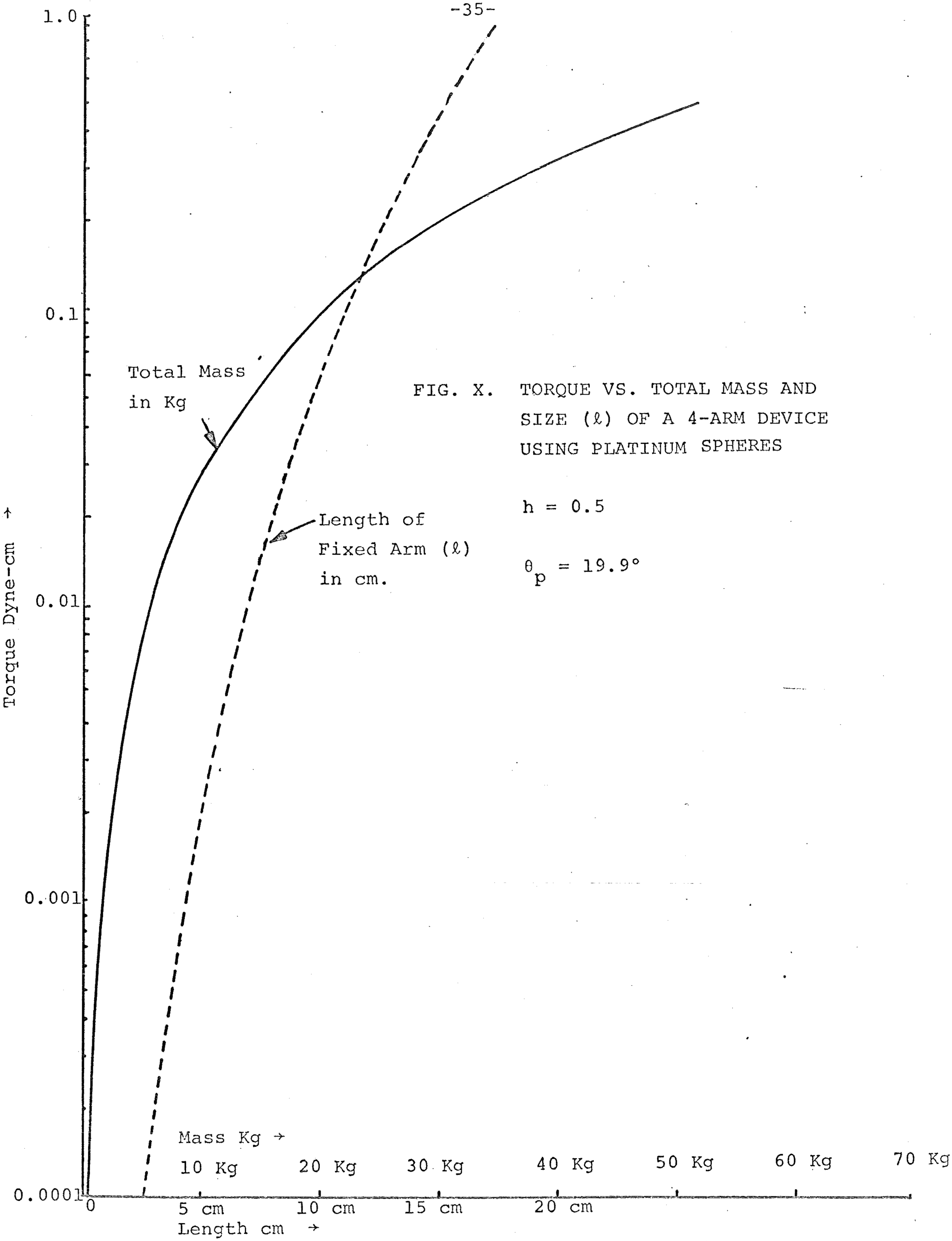
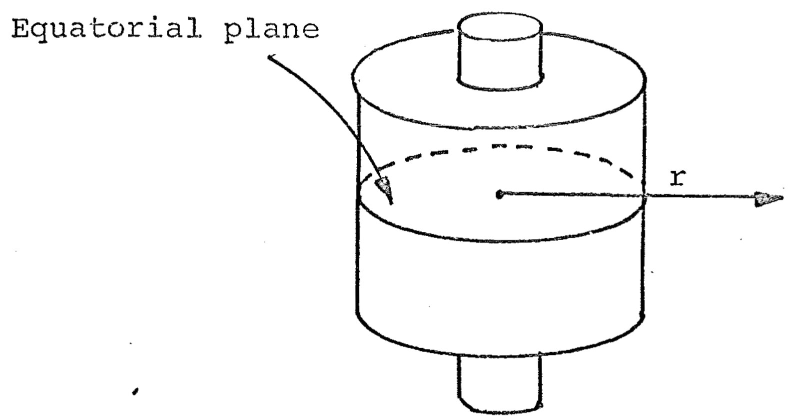


FIG. X. TORQUE VS. TOTAL MASS AND SIZE (ℓ) OF A 4-ARM DEVICE USING PLATINUM SPHERES



has a field in its equatorial plane equivalent to that of a sphere (at least up to correcting terms proportional to r^{-9} or less).

However, we can use any object as a test mass if it can be fabricated (or measured) to the required dimensional and density tolerances. With a digital computer, we can easily overcome the analytical difficulties of gravitational-field calculations. Considering the stringent size and mass limits on space experiments, it would be useful to examine test-mass shapes that give optimum torque levels for the amount of mass used.* Also, in this analysis of optimum configurations, it would be useful to consider the criteria of the foregoing analysis (e.g., operating at a θ_p, $dT/d\theta = 0$) as well as other criteria which would reduce the sensitivity of the apparatus to dimensional changes (e.g., temperature effects).

In addition to the study of optimum configurations, we must do an analysis of the dynamics of the Beams' experimental concept for operation in a spacecraft. Since a spacecraft is generally in accelerated motion with respect to inertial space, we need to make certain corrections in the experimental measurements. Spacecraft angular acceleration is an interfering quantity that adds (or subtracts) directly in the force-balance equation of

*In these examinations, we would also consider the effects of test-body density variations, surface roughness, and other effects such as Van der Waal force.

the experiment. We can remove this interference by measuring the angular acceleration of the experiment with respect to an inertial reference device. This reference device could be a gyroscopic stable-element or a set of star trackers.

There is a way, however, to avoid the use of an inertial reference device. Consider two separate experimental setups mounted (close together) such that the angular accelerations (needed for force balance) are colinear but in opposite directions. For one of the setups, the spacecraft angular acceleration will add in the force balance and in the second, it will subtract. By combining the data (obtained over the same time intervals) from both setups, we can (in concept, at least) remove the effect of spacecraft angular acceleration. One of the difficulties with this technique is that the two setups will experience different (integrated) effects from external fields, since their rotational periods will necessarily be different. This difficulty needs to be analyzed in terms of experiment size and the expected spectrum of spacecraft motions.

Finally it would be useful to extend our analysis to the design of a laboratory prototype of a space experiment using Beams' concept. This design would be based on a detailed consideration of the suspension system. For this prototype it seems prudent (from cost and ease of construction considerations) to base the initial design on a simple mass configuration and a simple suspension scheme.

VIII. List of References

1. Blitch, M.G., "The Feasibility of a Gravitational Clock to Test the General Theory of Relativity," S.M. Thesis in Physics, M.I.T., August 1969.

2. Chapman, P.K., "A Cavendish Experiment in Earth Orbit," unpublished note, December 1969.

3. Douglas, D.H. Jr., "Two Gravity Experiments: Time Variation of G and Gravitational Wave Detectors at Low Frequencies," A paper presented at the Third Cambridge Conference on Relativity, New York, June 1970.

4. Lee, W.N., "A Space Experiment to Determine the Constancy of the Gravitational Constant G," Measurement Systems Laboratory Report PR-7, "Review of NASA-Sponsored Research at the Measurement Systems Laboratory," May 1970.

5. Hovorka, J., "A Force-Measurement in Orbit to Check Scalar Gravitational Theory," Measurement Systems Laboratory Report RN-57, August, 1969.

6. Rose, D.R., "A New Method for Determination of Newton's Gravitational Constant," D.Sc. Dissertation, University of Virginia, August, 1969.

7. Beams, J.W., "Magnetic Bearings", A paper presented to the Automotive Engineering Congress, Detroit, Mich., January 1964.

8. Fischback, D.B., "Diamagnetic Susceptibility of Pyrolytic Graphite," Physical Review, Vol. 123, No. 5, September 1961.

9. Ezekiel, S., Chapman, P.K., and Egan, J.T., "Analytical and Experimental Investigations of Low-Level Acceleration Measurement Techniques," M.I.T. Experimental Astronomy Laboratory Report, RE-23, April, 1966.

10. Cook, A.H., "A New Determination of the Constant of Gravitation," Contemp. Phys., 1968, Vol. 9, No. 3, 227-238.