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TECHNICAL MEMORANDUM

ACQUISITION TIME IN A FIRST ORDER PHASE LOCK LOOP

Bellcomm

BELLCOMM, INC.
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ABSTRACT

In this paper the moments of the acquisition time parameter T_A are derived for a first order phase lock loop when the input to the loop is a sine wave in additive Gaussian noise. It is shown that except for differences in boundary values the moments of the acquisition time and time to cycle slip ($\hat{T}$) variables satisfy identical recursive second order differential equations.

It is demonstrated that as the signal to noise ratio (ρ) in the loop increases, both the mean acquisition time $E[T_A]$ and mean time to cycle slip $E[\hat{T}]$ increase.

Thus, as the signal to noise ratio increases it takes longer for the phase error to reach a stable point. But it is also demonstrated that once such a point is reached, it takes a much longer time for the phase error to change by a 2π increment. In addition, it is shown that the ratio $E[\hat{T}]/E[T_A]$ increases as ρ increases.

The recursive second order differential equation, which is satisfied by the acquisition time moments of a second order loop, is given. Unfortunately, computer memory requirements for the solution of this equation is so great that no computed results were obtained.

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SUBJECT: Acquisition Time in a First Order Phase Lock Loop
Case 900

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FROM: L. Schuchman

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TECHNICAL MEMORANDUM

INTRODUCTION

In previous memorandum I studied the problem of cycle slipping in a phase lock loop.[1,2] It was shown that the Nth moment of the time-to-cycle slip parameter ($\hat{T}$) satisfied a recursive differential equation for the nth order phase lock loop. The phenomena of cycle slipping is closely related to the problem of acquisition. It is shown in this memorandum that the Nth moment of the acquisition time parameter (T_a) for a first order loop satisfies the same recursive differential equation that $\hat{T}$ satisfies. The difference in determining the moments of T_a from those of $\hat{T}$ lies in the boundary values that the moments of the two parameters must satisfy.

The phase lock loop is schematically depicted in Figure 1.

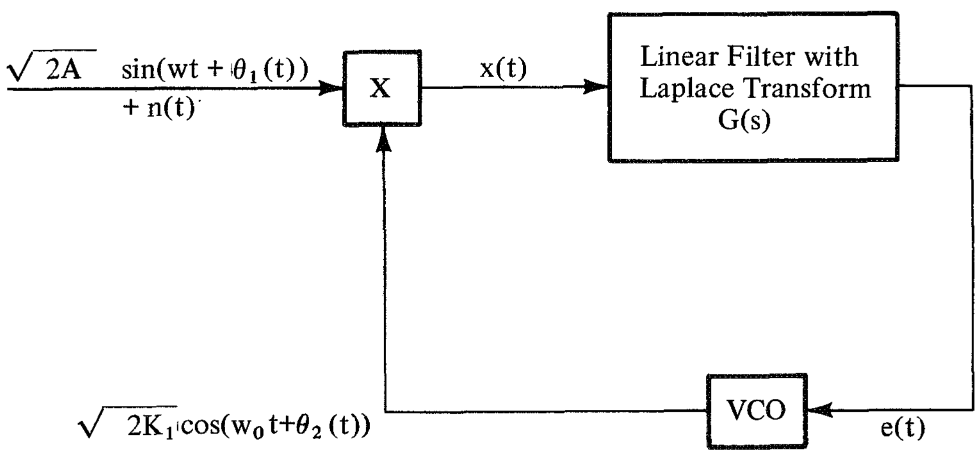

FIGURE 1 – PHASE LOCK LOOP

The loop consists of three components; a multiplier, a time-invariant linear filter g(t) whose Laplace transform is G(s), and a voltage controlled oscillator (VCO). If the loop is operating on a noiseless sine wave, then the steady state output of the VCO is a quadrature replica of the input signal which may differ only in amplitude. The multiplier output x(t), under these conditions, will have only a double frequency term which the linear filter and VCO configuration will not pass. The linear filter has a second and equally important function which is to reduce the effects of noise that normally gets into the loop.

FIRST ORDER PHASE LOCK LOOP (G(s) = 1)

Viterbi[3] has shown that the behavior of a first order phase lock loop in a noiseless environment can be described by the following differential equation

$$\frac{d\phi(t)}{dt} = (\omega-\omega_o) - AK\sin\phi(t) \tag{1}$$

where $\phi(t) = \Theta_1(t) - \Theta_2(t)$

Equation (1) describes how the frequency error changes with phase error. This can be depicted graphically as in Figure 2.

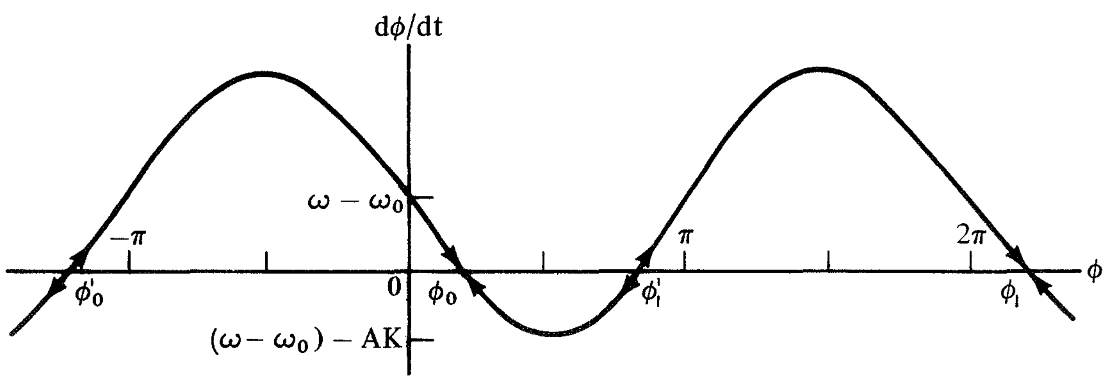

FIGURE 2 – SYSTEM TRAJECTORY FOR FIRST-ORDER LOOP
(SAME AS FIG. 3.2 OF REFERENCE 3)

The phase error will move until the frequency error is reduced to zero. If the frequency error is positive, the phase error will increase with time. On the other hand, if the frequency error is negative, the phase error will decrease with time. Using this fact, the possible trajectories of the phase error are depicted in figure 2. Thus, the values of ϕ for which $\frac{d\phi}{dt} = o$, are given by

$$\phi_n = 2n\pi + \sin^{-1} \frac{\omega-\omega_o}{AK} \qquad n = o, \pm 1, \pm 2, \ . \ . \ . \tag{2}$$

$$\phi'_n = (2n-1)\pi - \sin^{-1} \frac{\omega-\omega_o}{AK}$$

where we note that the ϕ'_n points are unstable and will lead to further migration of the phase error to the nearest stable ϕ_n point.

It is to be noted that equation (2) is valid only when $\omega-\omega_o \leq AK$. Thus, if $\omega-\omega_o \geq AK$ the loop will never attain lock. The time for the phase error to move from an initial setting $\phi(o)$ to a point ϕ' can be determined from equation (1). Thus,

$$t(\phi') = \int_{\phi(o)}^{\phi'} \frac{d\phi}{(\omega-\omega_o) - AK\sin\phi} \tag{3}$$

Equation (3), figure 2, and symmetry considerations allows us to write the expected acquisition time $E[T_A]$ as

$$E[T_A] = \frac{1}{2\pi} \int_{-(\phi_o+\pi)}^{\phi_o} \int_{\phi(o)}^{\phi_o} \frac{d\phi \qquad d\phi(o)}{\omega-\omega_o - AK\sin\phi} \qquad +$$

$$\frac{1}{2\pi} \int_{\phi_o}^{\pi-\phi_o} \int_{\phi(o)}^{\phi_o} \frac{d\phi \qquad d\phi(o)}{(\omega-\omega_o) - AK\sin\phi} \tag{4}$$

In deriving equation (4) we assumed $\phi(o)$ to be uniformly distributed about some ϕ_n value.

The evaluation of equation (4) leads to the following conclusion. In the noiseless case, it takes an infinite amount of time for the loop to acquire lock. Since this result does not agree with our experience with phase lock loops, it is necessary to look at the noisy model in order to obtain results which are more realistic and more meaningful.

In the presence of additive Gaussian noise equation (1) becomes

$$\frac{d\phi(t)}{dt} = (\omega-\omega_o) - AK\sin\phi(t) - n_1(t)\sin\Theta_2(t) + n_2(t)\cos\Theta_2(t) \tag{5}$$

where $n_1(t)$ and $n_2(t)$ are independent Gaussian processes of zero mean and identical low pass spectral densities. The spectral densities of $n_1(t)$ and $n_2(t)$ are assumed flat over a sufficiently wide range so that with respect to the closed loop bandwidth, they can be approximated as white noise with the same magnitude of $N_o/2$ watts/H_z. N_o is the height of the one sided noise spectral density.

It has been shown[1] that the variation of $\phi(t)$ describes a homogeneous first order Markov process so that we may apply the following result.

Define $\phi(t=o) = \phi(o)$ with $a>\phi(t)>b$, then the first passage time $T_{ab}(\phi(o))$ is the random variable defined as the time to reach either boundary a or b the first time, given that $\phi(t)$ starts at ϕ_o.

Mathematically we say

$$T = T_{ab}(\phi(o)) = \sup\{t \,|\, a>\phi(\gamma)>b, \quad o\leq\gamma\leq t\}$$

Let $\phi(t)$ be a homogeneous Markov Process and if $T = T_{ab}(\phi(o))$ is a proper random variable whose moments of order $n \leqq n_o$ exist, then $E\,[T_{ab}{}^n(\phi_o)]$ satisfies

$$\frac{K^2 N_o}{4} \frac{d^2 \left|E[T^n]\right|}{d^2\phi} - \left[AK\sin\phi - \frac{d\theta_1}{dt}\right] \frac{d \left|E[T^n]\right|}{d\phi} = -nE[T^{n-1}] \tag{6}$$

with

$$E[T(a)^n] = E[T(b)^n] = o$$

$$E[T]^o = 1$$

We define the acquisition time random variable given $\phi(o)$, as $T_{a,b}(\phi(o)) = T_A(\phi)$ (where $a = \phi_o$ and $b = 2\pi - \phi_o$). $T_A(\phi)$ satisfies equation (6). To evaluate equation (6) we use the boundary values to determine the initial conditions on $E\left[T_A(\phi)^n\right]$ and $\frac{dE}{d\phi}\left[T_A(\phi)^n\right]$. To do this we note that

$$\frac{dE[T_A{}^n]}{d\phi} = e^{-u(t)} \int_a^t e^{u(z)} g(z)dz + e^{-u(t)} c_1 \tag{7}$$

where

$$u(t) = \int \rho(x)dx$$

and

$$p(x) = -\rho \sin x + \rho \frac{\Omega_o}{4B_L}$$

$$\rho = \frac{A^2}{N_o B_L}, \quad B_L = \frac{AK}{4}, \quad \text{and } \Omega_o = \omega - \omega_o$$

$$g(z) = -\frac{\rho n}{4B_L} E\left[T_A(z)^{n-1}\right]$$

while

$$E[T_A{}^n] = \int_b^x [e^{-u(t)} \int_a^t e^{u(z)} g(z)]\, dz + C_1\, e^{-u(t)}]\, dt + C_2 \tag{8}$$

with

$$E[T_A(a)^n] \quad E[T_A(b)^n] \tag{9}$$

we have

$$C_2 = o$$

and

$$C_1 = - \int_b^a e^{-u(t)} \int_b^t e^{u(z)} g(z)\, dz \Bigg/ \int_b^a e^{-u(t)} dt \tag{10}$$

We are thus in a position to solve equation (6) for the n^{th} moment of the acquisition time, random variable, given $\phi(o)$.

However, one may ask at this time whether $T_A(\phi)$ as defined, is the random variable which determines statistically the acquisition time of an actual first order phase lock loop?

The first possible objection to the definition is that it appears to restrict the possible unknown phase to a 2π interval about ϕ_o. To see that this is not the case one need only substitute ϕ_n for ϕ_o in the definition and note that since $\phi_n = \phi_o \pm 2\pi N$, equation (8) will give identical results for ϕ_n and ϕ_o. This periodicity allows us to restrict our attention to a 2π interval about ϕ_o.

Secondly, we note that T_A is a conditional random variable. Thus, if we were interested in the worst case statistics, we could determine $E[T_A{}^n]$ for all possible values in a 2π interval and select that value of ϕ_o for which $E[T_A{}^n]$ is greatest.

Another approach is to average over all values of ϕ_o. Because of symmetry considerations, assuming $\omega-\omega_o$ either positive or negative leads to identical results. Thus, we need average over only a 2π rather than a 4π interval. We define the moments of the mean acquisition time as

$$E_\phi\left[T_A{}^n\right] = \frac{1}{2\pi}\int_0^{2\pi} E\left[T_A{}^n\right] d\phi_o \qquad (11)$$

We assumed ϕ_o to be uniformly distributed since the receiver has little or no knowledge of the transmitter phase or of the statistics of the transmitter phase.

To compute the moments $E\left[T_A{}^n\right]$ we simulated equation (6) on a digital computer using a program called "Mimic." This is the same procedure that was used to determine the time to cycle slip parameter except in this case C_1 is no longer zero but is given by equation (10).

Our results are summarized in Tables I and II. It can be seen from the data that as the signal to noise ratio increases and as the initial phase offset is increased, the mean acquisition time increases.

However, the mean acquisition time for a phase offset of .875 and a signal to noise ratio of 20, is only 1.82 seconds. It is also interesting to note that one would expect the mean acquisition time to be less than the time to cycle slip since on the average the phase error has to travel a shorter distance to a boundary in the former case. Table III compares the mean acquisition time with the mean time to cycle slip. It can be seen that for a given signal to noise ratio the mean time to cycle slip is significantly greater than $E_{max}\left[T_A\right]$.

TABLE I

Moments of the Acquisition Time Parameter T_A

(ϕ_o = 1/2)

	ρ				
Moment	1/4	1	3	5	10
$E_{max}[T_A]$	.257	.634	.962	1.11	1.31
$E_{\phi}[T_A]$	.168	.389	.548	.615	.733
σ	.194	.398	.473	.491	.494
Skewness	2.104	1.810	1.53	1.45	1.17
Excess	6.48	4.82	3.81	3.76	2.55
Excess Predicted*	6.6	4.86	3.51	3.15	2.05

TABLE II

Moments of the Acquisition Time Parameter T_A

(ϕ_o = .875)

	ρ							
Moment	1/4	2	3	5	10	12	15	20
$E_{max}[T_A]$	.268	.648	.995	1.19	1.48	1.56	1.68	1.82
$E_{\phi}[T_A]$	.176	.410	.607	.710	.860	.90	.951	1.02
σ	.202	.394	.487	.611	.611	.630	.654	.683
Skewness	2.077	1.56	1.118	1.16	1.26	1.30	1.36	1.44
Excess	6.32	3.52	2.32	2.17	3.00	3.35	3.84	4.52
Excess Predicted*	6.47	3.65	1.87	2.02	2.38	2.54	2.77	3.11

*Excess predicted under the assumption that T_A is a Pearson Type III random variable.

TABLE III

Comparison of Mean Time to Cycle Slip ($\hat{T}$) and Maximum Mean Acquisition Time in a First Order Phase Lock Loop

ρ	$E[\hat{T}(\phi_o = o)]$	$E[T_A(\phi_o = 8.75)]$
1/4	1.27	.268
1	7.91	.648
3	352.7	.995
5	18,308.	1.19

In estimating the time to cycle slip probability density distribution, the first three moments of the time to cycle slip parameter were fitted exactly to the first three moments of a Pearson Type III probability function. The Excess was then predicted assuming $\hat{T}$ to be a Pearson Type III random variable, and then this predicted excess was compared to that which was computed using the derived differential equation for the moments of $\hat{T}$. It was noted then that the computed and predicted values of Excess were extremely close in value. As can be seen in Tables I and II the computed and predicted values of the Excess of T_A do not compare as well. The results indicate that as ρ increases, the Pearson Type III assumption becomes less valid.

THE SECOND ORDER PHASE LOCK LOOP

In a second order loop with a perfect integrator in the loop $[G(s) = 1 + \frac{\alpha}{s}]$, Viterbi[3] has shown that in the noiseless case the pull-in range is theoretically infinite. In addition, he has shown that an approximate expression for the change of frequency per cycle for large values of $\dot{\phi}$ is

$$\delta\dot{\phi} \simeq \frac{-\pi}{r\phi^2(o)-1}$$

So that the number of cycles traversed by the phase error in order to obtain lock for an initial frequency offset of Ω_o is

$$\frac{\Omega_o}{\delta\dot{\phi}} = -\frac{\Omega_o}{\pi}\left[r[\dot{\phi}(o)]^2 - 1\right]$$

where

$$r = \frac{AK}{\alpha} \quad \text{and is the damping factor of the loop}$$

$$\dot{\phi}(o) = [\Omega_o - AK\Theta]$$

Θ the phase error of the loop at $t = o$

Assuming a positive frequency offset Ω_o we can write the acquisition time random variable for the noise case as $T_A = T_{a,b}(\Theta)$

where

$$a = +\infty$$

$$b = -\frac{\Omega_o}{\delta\dot{\phi}}$$

This equation holds for values of $\dot{\phi}(o)$ as low as 3 and values of r down to 1.

Using results obtained in reference 1 we find that T_A for a second order loop satisfies equations (6) and (7) with

$$p(x) \doteqdot -\rho\frac{(1+r)}{r}\sin x + \frac{\rho}{r}X + \frac{\Omega_o}{\gamma}$$

Therefore

$$u(x) = \rho\frac{(1+r)}{r}\cos x + \frac{\rho}{2r}x^2 + \frac{\Omega_o}{\alpha}x$$

while

$$g(x) = -\frac{n}{\gamma}E\left[T_A(x)^{n-1}\right]$$

where
$$\gamma = \frac{4B_L}{\rho(1+\frac{1}{r})^2} \quad \text{and } B_L = \frac{AK+\alpha}{4}$$

Unfortunately, numerical results could not be obtained even for $\phi(o)$ as low as 3. To see what the problem is, note that the a and b limits are very large and that u(x) for large values of x is proportional to x^2. Since u(x) is the argument of an exponential in equation 7 we see that the memory requirements for computing $E\left[T_A\right]$ in a second order loop can become extraordinarily large.

2034-LS-ms

Leonard Schuchman

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