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A VARIATIONAL TECHNIQUE AND ITS APPLICATION TO MODEL-REFERENCE CONTROL

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July 1970

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This work has been sponsored in part by the National Aeronautics and Space Administration Research Grant NGL 07-002-002

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The University of Connecticut, 1966, 1967

When it is desired to control a plant in a particular way, an accurate description of the plant is necessary. When there is some lack of knowledge about the plant, adaptive control can be used to achieve the original control objective despite this ignorance.

The model-reference adaptive system presented here is based on knowing the structure of the plant but not knowing some or all of the parameter values. In addition to this uncertainty, the analysis also accounts for plant parameters which are both unknown and vary in an unknown manner.

The model-reference system uses a model subject to the same input as the plant. Since the model and plant structure are the same, any difference between the states of the two (tracking error) is due to differences in the parameter values of each. The model parameters are chosen to achieve the desired control objective. Hence when adaptive action reduces the error, the plant behaves in the desired manner despite ignorance of its parameters.

In contrast to previous work, variational techniques are employed in obtaining two algorithms for effecting this adaptive control. The first is a parameter adjustment scheme where the algorithm defines how controller parameters should be adjusted to reduce the tracking error. The second method utilizes a control signal input to achieve the same

objectives. Both techniques are implemented continuously in real time, and afford uncomplicated analysis of the closed-loop system in the absence of disturbances.

The more important results are: 1 Either algorithm guarantees stability of the error when the plant parameters are stationary. 2. In the presence of unknown parameter variations a bound on the error can be guaranteed. This bound can be reduced by increasing adaptive loop gains. 3. Even if all unknown parameters are not compensated for, an error bound is still guaranteed. This results in a greatly simplified configuration for the parameter adjustment scheme. 4. The parameter adjustment development is formulated so that it is directly applicable to plant parameter identification, as well as adaptation.

Simulation results are presented illustrating application of the parameter adjustment algorithm to pitch control of an aircraft. In addition other non-trivial simulations are presented which demonstrate the effectiveness of both methods.

Acknowledgement

The author acknowledges the technical and editorial assistance of his advisory committee, Professors Ralph Kochenburger, David Lindorff and Charles Knapp.

He also acknowledges the understanding, encouragement, assistance and patience of his wife Lorraine who in no small way contributed much of the inspiration to carry this effort through.

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INTRODUCTION

A system has been defined as adaptive if a means is provided to monitor its performance and modify the control action accordingly in an attempt to make it an acceptably performing system (4). Since this work is concerned with only model reference adaptive systems, an acceptably performing system is one whose performance is closely enough slaved to some reference performance so that the plant dynamic response to a particular input is closely described by that of the model. In this context the model is the physical implementation of the designer's concept of an acceptably performing system.

A strong parallel exists between model reference systems and conventional control systems. In both cases desired performance can be achieved through external compensation, prefilters and feedback elements. In the conventional feedback controller the concept of what an acceptably performing system is need not be implemented. Since the structure of the plant and the value of all plant parameters are known, the desired performance can be obtained using fixed compensation. If the plant parameters vary, the system with fixed compensation may no longer behave acceptably. If the parameter variations were known, the fixed scheme could be replaced by more complex time variable compensation. The behavior of this scheme would be exactly that which compensated for plant parameter variations to produce desired system performance.

The more realistic problem, and the one with which this thesis deals, is where plant parameters vary in an unknown manner. The cost of compensating for these variations is a more complex controller configuration. In particular, it now becomes necessary to know how plant performance compares to desired performance. The adaptive controller function is to analyze this relative performance and act in a way to keep the plant performing satisfactorily. The controller must therefore measure performance and for every measurement have a strategy which is designed to improve the plant response. The basis for a particular strategy as well as the choice of a performance measure to a large extent define the adaptive controller.

The ultimate factor in determining a design must be the particular application for which adaptive control is required. Some plants may vary at rates which prohibit acceptable response from parameter adjustment techniques. There may also be a fundamental limitation imposed by having inaccessible plant states. Other techniques designed to work for linear plants may fail to work with nonlinear plants.

This thesis reports an approach to both the parameter adjustment and the signal augmentation model reference adaptive control systems. What is believed new in this research is the basic analytical approach-based on variational reasoning- one which yields adaptive algorithms which are simple and which afford uncomplicated analysis of the closed loop system in the absence of noise. The limitations of the method become clear, and to some extent, can be minimized by appropriate design.

The most important techniques used for model reference adaptive control are highlighted in Chapter I. The differences between plant identification and model adaptation are discussed.

Chapter II formulates the control problem in mathematical terms.

A method for adjusting controller parameters is derived under ideal conditions. These ideal conditions require that the controller parameters are the only factors determining adaptive response. The adjustment technique is then extended to allow for synthesizing time varying input signals to the plant.

In Chapter III the assumptions on ideal conditions are relaxed. The adaptive system response will depend on both command input signal variation and the effects introduced by unknown parameter variation. By choosing the adaptive loop gains large the worst case effects of these variations can be minimized. Simple examples illustrate the validity of the suggested design technique.

In Chapter IV specific forms for the adaptive controller are examined. Since the parameter adjustment controller is quite complex a simpler form is investigated for specific systems. Digital and Analog computer simulation results are presented.

In order to implement a signal augmentation controller when the plant has a zero requires exact knowledge of the zero. When this is not known, the controller must be implemented using the model zero. The effects of this approximation on adaptive behavior are studied.

Chapter V outlines some of the more important areas which require closer examination.

I. PROBLEM DEFINITION AND BACKGROUND

Figure 1.1 is fundamental to many model reference control systems. The plant structure is known but its parameters may be unknown or unknown and time varying. The model provides a real time representation of desired plant performance for the given input r(t). By appropriately adjusting controller parameters the tracking error e(t), is to be made small. The prefilter and feedback compensation elements contain the controllable parameters. The adaptive analyzer measures plant and model states and plant inputs, and generates the appropriate control signals. In the parameter adjustment system these signals provide the proper variations to the controllable parameters.

If the parameter adjustments are to be used for identifying plant parameters, the adjustable or controllable parameters would be in the model. The model parameters would be adjusted to reduce the tracking error. When the error is nulled the model parameters have the same value as the plant parameters for an arbitrary input r(t).

The adaptive analyzer for the control signal augmentation technique provides a continuous time function which is applied to the plant to keep the tracking error small. Regardless of which class of controller is to be used, the system design reduces to one of specifying the adaptive analyzer. This thesis presents a method useful for designing this analyzer to perform any of the aforementioned tasks. Before discussing the details it is instructive to summarize some of the more important earlier work.

The study of adaptive control was motivated by the practical

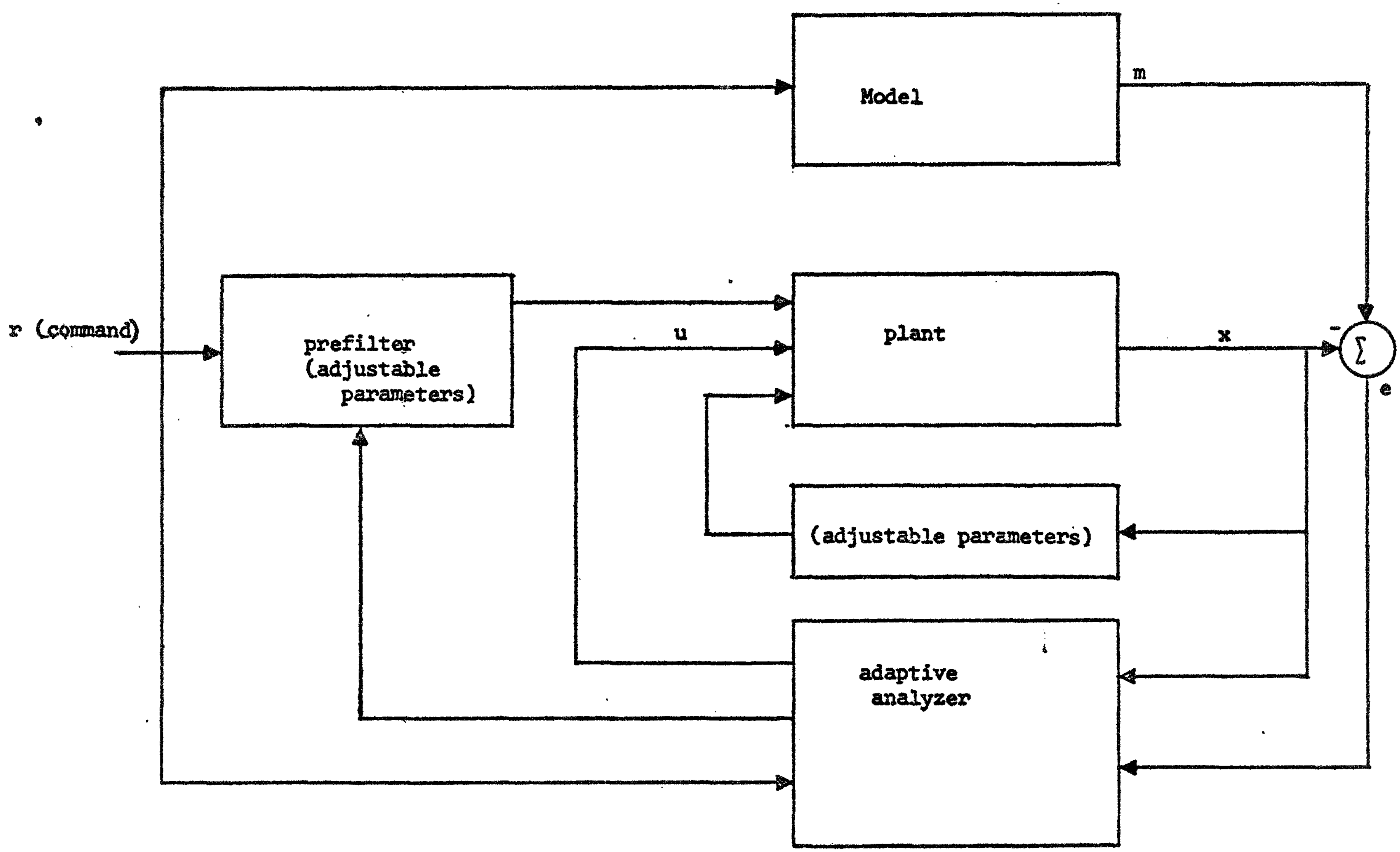

Model
m
r (command)
prefilter
(adjustable
parameters)
u
plant
x
-
∑
e
(adjustable parameters)
adaptive
analyzer

Figure 1.1

challenges of the early to mid 1950's. As aircraft were designed to fly higher and faster the environmental effects on flight characteristics became increasingly troublesome. The fixed compensation schemes of the past and even scheduled parameter adjustment techniques became unsatisfactory. The scheduled gain changing systems require that the plant be identified. To do this usually required an air data computer to analyze flight parameters and deduce from these the plant dynamics (1). This method at first seems appealing since high speed calculations can keep the phase lag due to measurement small. The complexities of actually handling the enormous store of data needed often proved to be too complicated a task. In addition, measurement of environmental conditions was not always possible (1).

An alternative to using air data measurements to identify the plant is to adjust model parameters on line, relying only on present values of the plant states. A technique employing on line parameter identification for use in an adaptive autopilot was reported in 1968 (19) The theoretical basis for parameter adjustments in the model is given in (18). A fault of this method is the requirement that for an n^{th} order plant the derivative of the n^{th} state variable is needed in the adjustment algorithm.

Another interesting on line identification technique uses a Taylor series expansion of the error between plant and model to deduce parameter adjustment algorithms (9). By continuously changing the model parameters until the error is zero, at least for the input being applied, the plant and model are made to behave dynamically the same. As with any identification technique, if the particular input is not sufficiently broad band complete identification cannot be guaranteed.

A drawback to any on line identification scheme compared to adaption is that identification does not alter or improve the plant performance. Use must be made of the parameter information to accomplish the control objective. These techniques take considerable time to converge to accurate parameter values (9). The question of what should be done during the identification delay is not easy to answer. Since the plant is responding continuously, it must be controlled. But to control it properly requires knowledge of the plant - information not available until identification is complete. The reason for this complication is that most on-line techniques for identification are based on reducing the square of the tracking error. The error is not a measure of the system performance however. A more meaningful error measure for identification is the square of the difference between model and plant parameters. But to actually implement such a scheme would require knowledge of the plant parameters. An adaptive system based on minimizing tracking error does not present this disadvantage. By continually reducing the error, performance is improved directly. Although adaptation and identification are conceptually the same process, for the reasons cited above, when model-reference control is the objective, adaptation is the more direct solution.

The term "model-reference adaptive system" first appeared as a description of adaptive control work done at M.I.T. in the late 1950's. This research was aimed at controlling yaw, pitch and roll loops under varying environmental conditions for a particular aircraft. An adaptive scheme suitable for this task was designed which performed satisfactorily in both simulation and limited flight testing (20). In addition to this important first work in adaptive control, a basic groundwork was laid for the design of adaptive systems. These five steps summarize the important considerations:

1) Design of a model to meet the system specifications.
2) Selection of the control system loop configuration.
3) Determination of which parameters should be varied and how they affect system response.
4) Determination of error criteria which will adjust parameters.
5) Analysis and simulation to determine the convergence times and dynamic operating performance of the system.

Step one is perhaps the most important. Care must be taken to guarantee that performance, as specified by the model, can actually be achieved. For example, the requirements placed on the system must be consistent with any constraints on available control effort. This restriction can usually be satisfied by not requiring performance from the plant for which the plant is not designed. The purpose of model reference control is then to keep the plant responding as it was designed to despite variation of its parameters.

Regardless of what error measure is used, the adaptive analyzer must be physically implemented. The ability to do this will depend to a great extent on which parameters are to be varied to achieve adaptation. Although the development presented in this thesis is complete in that it defines how each parameter of the plant could be compensated, the resultant adaptive loop is greatly simplified by using only some, or perhaps only one, parameter in the compensation. The validity of such a simplification can most reliably be tested by simulation.

Other than making a proper or meaningful choice of error measure, the qualitative structure of the error measure will be important. How much each state is weighed in measuring error will be important in terms of the dynamic operating characteristics of the system. Once the error measure has been chosen and the adaptive loop structure has been determined, the final design would be "fine-tuned" as regards the error weighting. Simulation studies would be used to a large degree in this stage of design.

Work undertaken at U.C.L.A. at about the same time as that referenced above attempted to generalize on possible adaptive techniques (13). This work was not constrained to the design of a particular system and consequently some rather broad fundamental questions were posed on the philosophy of adaptive control. The adaptive problem as viewed then was essentially one of identifying the plant and updating controller parameters. The identification was to be carried out by adjusting model parameters using system input and output information. As mentioned earlier, this approach is the dual of the adaptation procedure. Hence the questions raised, and paraphrased below, are pertinent also to the design of adaptive systems.

1) What are the ways in which the model can have its parameters adjusted?
2) Is it stable? Will the model have its parameters adjusted so that they are a satisfactory representation of plant dynamics?
3) What is the dynamic performance of the model? How fast does it respond to a change in the plant dynamics?

A great deal of the effort in model reference systems has been directed towards answering these questions. The problem of adaptive loop stability was particularly troublesome, and limited to analysis (5). It was in fact the analysis of the stability properties of adaptive systems which led Butchard and Shackcloth to present the first synthesis technique which guaranteed stability (3). This technique was popularized by Parks (17) and has subsequently received considerable attention in the literature.

Included in the category of dynamic performance is the adaptive behavior to changing plant parameters. A technique based on the assumption of slowly varying plant parameters may be unable to perform satisfactorily if the parameters vary rapidly. Although an adaptive controller using control signal augmentation has been designed which guarantees stability of the adaptive process despite rapid parameter variations (7), there is no reported technique employing parameter adjustments which guarantees that rapidly varying plant parameters can be tracked. In general, even for stationary plants the error will not be zeroed since imperfections or simplifications in plant modeling will prohibit the plant and model from behaving exactly alike. This type of bounded error response might aptly be called "practical adaptive stability" after the practical stability definition of LaSalle (12).

The work reported here falls into this category since the scheme can be designed to guarantee practical stability in the presence of varying parameters even though the basic structure is derived assuming stationary plant parameters.

The alternative to parameter adjustments is to provide a control signal to the plant which will keep the tracking error small. This technique is referred to as a signal synthesis approach or control signal augmentation (4). The adaptive controller is designed to produce a control input which causes the plant output to track the model output. This is generally simpler to implement than the parameter adjustment scheme since only one control parameter (the control signal) is being changed for a single input plant. Most of the work in this area has been aimed towards generating a control which guaranteed the existence of a Lyapunov function of the tracking error. Grayson first introduced the concept in 1961 (7). Since then the method has been refined to make it of practical engineering importance (14).

This survey of methods is not intended to be complete, but rather intended to reference some of the previous work in parameter adjustment and control signal augmentation techniques. Many other philosophies exist for designing adaptive systems. Among these are the parameter perturbation schemes, the high gain schemes, and techniques employing test signals for excitation in addition to normal plant inputs (6).

II DESIGN BASED ON IDEAL PLANT

2.1 PLANT DEFINITION

The first step in the design process is to define a measure of the tracking error. The influence of all controllable parameters as well as all control inputs is then used to advantage to reduce this error measure. As a result, algorithms defining parameter adjustments are derived which act to decrease the difference between plant and model. Similarly, for the control augmentation technique, a method is deduced which defines a control input useful in reducing the error measure.

Because of the duality between parameter adjustments for adaptive control and for identification, the algorithm will be developed with the former in mind, any differences inherent to the identification scheme will be brought out as the development progresses. Regardless of the actual structure for implementation, be it identification or adaptation, the design problem can be stated as follows: Given the differential equation representation of the plant it is desired to control the plant to perform in a specified manner despite a lack of knowledge of the plant parameters. The model is used to describe this reference performance.

The structure of the plant and model are the same, the only difference is the parameter values of each. The function of the adaptive controller is to reduce the difference between plant and model via parameter adjustments or control augmentation. The plant is described by a vector differential equation

$$\dot{x} = F(x,u,r,p,t) \tag{2.1.1}$$

where x_i are the plant states, u_j are the control inputs and r_j are the plant commands. The plant parameters are contained in the vector p. There are N plant states, K plant parameters, J command and control inputs. Any or all elements of p may be unknown or unknown and varying in an unknown manner with time.

The model is structurally the same as the plant

$$\dot{m} = F\ (m,r,a,t) \tag{2.1.2}$$

where m_i is one of N model states, a_k is one of K model parameters and the command vector r is the same as applied to the plant. The tracking error is the difference between model and plant state vectors

$$e = m - x \tag{2.1.3}$$

The dynamic behavior or the error is given by

$$\dot{e} = g'(m,x,r,p,a,t) \tag{2.1.4}$$

and if the model structure is chosen properly may be expressed as

$$\dot{e} = g(e,x,r,u,\delta,t) \tag{2.1.5}$$

where δ = p-a is the N dimensional parameter difference vector.

The goal of parameter adjustment adaptation is one of adjusting δ to zero despite ignorance of p. Although in the development of the final design p is allowed to be time varying, it will be pointed out that zero tracking error cannot be guaranteed under these conditions. Rather, the error will be kept small despite these variations.

The algorithm is based on the ability to change δ. In particular, rates of change of the components of δ are deduced which are sufficient to continually decrease the error measure.

2.2 DEVELOPMENT OF THE ADJUSTMENT ALGORITHM

Since the object is to minimize the tracking error between plant and model, a measure of this error with a meaningful minimum is required. Let this function be L(e) and require it to be non-negative for all possible values of e. With this requirement, the smaller the value of L(e) the closer the plant states will be to the model states.

The only means available to effect a controlled change in L are thru δ and u. L(e) could be minimized by making adjustments in the negative gradient direction of L. However, because there is no direct way of changing e, it is necessary to make changes in L thru changes in δ or u. If L is a function of δ then to be meaningful it must be a non-negative function of δ. With this restriction, the gradient of L would also be a function of δ and since δ is unknown the gradient could not be evaluated. Several of the techniques developed previously have circumvented this problem by defining L to be a function only of e and then approximating the partial derivatives of the error with respect to δ and making adjustments accordingly.(4) In addition to the inaccuracies introduced in approximation, these methods have the disadvantage of being overly complex.

If an integral criterion were to be minimized L(e) could be considered the rate of cost and $\int_{t_0}^{t_f} L(e)dt$ the cost associated with the system (2.1.5). The adaptive problem could then be stated as follows: Given the system governed by (2.1,5) it is desired to choose δ,u to minimize

$$I = \int_{t_0}^{t_f} L(e)dt \tag{2.2.1}$$

Since (2.2.1) represents a functional to be minimized subject to constraint (2.1.5) consider instead minimizing

$$I = \int_{t_0}^{t_f} [L(e) + \sum_{i=1}^{N} \alpha_i(\dot{e}_i - g_i)]\, dt \qquad (2.2.2)$$

The elements of the N dimensional vector α are Lagrange multipliers. By multiplying each of the N differential constraints by a corresponding α_i, the minimization of (2.2.2) may procede with no additional constraints (2). The necessary conditions for the $\delta(t)$ which minimizes (2.2.2) can be derived using variational calculus but the result is meaningless since the value of the parameter difference vector is unknown. Because of this it is impossible to set the value of δ at any instant. If this could be done then δ would be analogous to a control input and could be synthezized to minimize I.

Rather than finding these conditions, since the resultant scheme could not be implemented, conditions will be found describing appropriate changes in δ which guarantee a reduction in I.

Consider two subintervals comprising $[t_0, t_f]$ namely $[t_0, t_g]$ and $(t_g, t_f]$.* Let the nominal values of the variables on $[t_0, t_f]$ be represented using circumflex notation. Then the independent variables are $\hat{a}, \hat{\delta}, \hat{r}$ and the dependent variables are $\hat{\alpha}, \hat{x}, \hat{m}, \hat{g}, \hat{L}$. Express (2.2.2) as the sum of two integrals

$$I = \int_{t_0}^{t_g} [\hat{L} + \sum_{i=1}^{N} \hat{\alpha}_i(\hat{\dot{e}}_i - \hat{g}_i)]dt + \int_{t_g}^{t_f} [\hat{L} + \sum_{i=1}^{N} \hat{\alpha}_i(\hat{\dot{e}}_i - \hat{g}_i)]dt = I_1 + I_2 \qquad (2.2.3)$$

It will be shown that the system (2.1.5) can always be adjusted on

* The square bracket denotes a closed interval, the smooth bracket an open interval

$(t_g,t_f]$ in such a way that the total cost on $[t_0,t_f]$ can be made equal to or less than the cost on $[t_0,t_f]$ if no adjustments were made on $(t_g, t_f]$. The induction is clear; if these changes are initiated repeatedly over successive intervals, the cost I is continually reduced from what it would be if no change were made.

It is assumed that:

1) Any change in I over $(t_g,t_f]$ is completely caused by changes in δ,u over this same interval. This says that the effect on e, and hence on I due to the input r changing on $(t_g,t_f]$ is negligible.

2) Any change in x and m is equivalent to a change in e.

3) Any change in δ is a controlled change. The value of δ is unknown but there can be no variation in δ not caused by the adaptive action.

The effects of relaxing these assumptions will be investigated in Chapter III.

Recall that δ has K components, u has J components and e has N components. If δ,u are changed incrementally on $(t_g,t_f]$:

$$\delta_k = \hat{\delta}_k + \Delta\delta_k \qquad k=1,2,\ldots K \tag{2.2.4a}$$

$$u_j = \hat{u}_j + \Delta u_j \qquad j=1,2,\ldots J \tag{2.2.4b}$$

the error components will change incrementally, and the error on $(t_g,t_f]$ is:

$$e_i = \hat{e}_i + \Delta e_i \qquad i=1,2,\ldots N \tag{2.2.4c}$$

The changes (2.2.4) also cause changes in g,L on $(t_g,t_f]$. The α_i's are chosen to be independent of any change occuring on this interval. The value of α is completely determined by the system behavior on $[t_0,t_g]$ with no change on $(t_g,t_f]$ despite changes in the control, the parameters and the error. This is no restriction since (2.2.2) is equivalent to (2.2.1) for any choice of α. The changes in g,L are:

$$g_i = \hat{g}_i + \Delta g_i \qquad i=1,2,\ldots N \tag{2.2.5a}$$

$$L = \hat{L} + \Delta L \tag{2.2.5b}$$

which can be expressed as

$$\Delta g_i = \sum_{j=1}^{N} \frac{\partial \hat{g}_i}{\partial e_j} \Delta e_j + \sum_{k=1}^{K} \frac{\partial \hat{g}_i}{\partial \delta_k} \Delta\delta_k + \sum_{j=1}^{J} \frac{\partial \hat{g}_i}{\partial u_j} \Delta u_j \qquad i=1,2,\ldots N \tag{2.2.6a}$$

$$\Delta L = \sum_{j=1}^{N} \frac{\partial \hat{L}}{\partial e_j} \Delta e_j \tag{2.2.6b}$$

where the circumflex indicates the partial derivatives are evaluated about their nominal or unperturbed values. The nominal values are the values solely determined by the functional behavior on $[t_0, t_g]$.

Substituting into (2.2.3) gives:

$$I = \int_{t_0}^{t_f} [\hat{L} + \sum_{i=1}^{N} \hat{\alpha}_i[\hat{g}_i - \hat{\dot{e}}_i]]dt + \int_{t_g}^{t_f} [\hat{L} + \Delta L + \sum_{i=1}^{N} \hat{\alpha}_i[\hat{g}_i + \Delta g_i - \hat{\dot{e}}_i - \Delta\dot{e}_i]]dt \tag{2.2.7}$$

Using I_1 as defined in (2.23) and combining (2.27) with (2.2.6):

$$I = I_1 + \int_{t_g}^{t_f} [\hat{L} + \sum_{j=1}^{N} \frac{\partial \hat{L}}{\partial e_j} \Delta e_j + \sum_{i=1}^{N} \hat{\alpha}_i[\hat{g}_i - \hat{\dot{e}}_i - \Delta\dot{e}_i + \sum_{j=1}^{N} \frac{\partial \hat{g}_i}{\partial e_j} \Delta e_j + \sum_{k=1}^{K} \frac{\partial \hat{g}_i}{\partial \delta_k} \Delta\delta_k$$

$$+ \sum_{j=1}^{J} \frac{\partial \hat{g}_i}{\partial u_j} \Delta u_j]]dt \tag{2.2.8}$$

$$= I_1 + \int_{t_g}^{t_f} \sum_{i=1}^{N} [\hat{g}_i - \hat{\dot{e}}_i]\, \hat{\alpha}_i \, dt + \int_{t_g}^{t_f} \left[\sum_{j=1}^{N} \Delta e_j \left[\frac{\partial \hat{L}}{\partial e_j} + \sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial \hat{g}_i}{\partial e_j} \right] \right] dt$$

$$+ \int_{t_g}^{t_f} \left[\sum_{k=1}^{K} \Delta\delta_k \right] \left[\sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial \hat{g}_i}{\partial \delta_k} \right] dt + \int_{t_g}^{t_f} \left[\sum_{j=1}^{J} \Delta u_j \right] \left[\sum_{i=1}^{N} \hat{\alpha}_i \frac{\partial \hat{g}_i}{\partial u_j} \right] dt$$

$$+ \int_{t_g}^{t_f} \hat{L} \, dt - \int_{t_g}^{t_f} \sum_{i=1}^{N} \hat{\alpha}_i \cdot \Delta \dot{e}_i \, dt + \text{H.O.T.} \qquad (2.2.9)$$

H.O.T. represents all higher order terms required to make (2.2.9) an exact equality. Expression (2.2.9) can be simplified by appropriately choosing the $\hat{\alpha}_i$'s. If these are chosen as the co-state variables associated with the unperturbed system they must satisfy the necessary conditions of the Maximum Principle: Given (2.1.5) and $e(t_0)$, the value of δ, u to minimize (2.2.1) must minimize for all time the Hamiltonian defined as

$$H = L + \sum_{i=1}^{N} \lambda_i g_i \qquad (2.2.10)$$

where the λ_i's are co-state variables defined by

$$\dot{\lambda}_i = -\frac{\partial H}{\partial e_i} \quad i=1,2,\ldots N, \quad \lambda_i(t_f)=0 \qquad (2.2.11)$$

By associating the λ_i's in (2.2.9) with the unperturbed system, they are independent of any changes on $(t_g, t_f]$. These are exactly the conditions required for the $\hat{\alpha}_i$'s, and choosing $\hat{\alpha}=\lambda$ validates the development to this point.

From (2.2.10) the following identities hold:

$$\hat{\dot{\alpha}}_j = \hat{\dot{\lambda}}_j = -\frac{\partial \hat{L}}{\partial e_j} - \sum_{i=1}^{N} \hat{\lambda}_i \frac{\partial \hat{g}_i}{\partial e_j} \qquad (2.2.12a)$$

$$\frac{\partial \hat{H}}{\partial \delta_k} = \sum_{i=1}^{N} \hat{\lambda}_i \frac{\partial g_i}{\partial \delta_k} \qquad (2.2.12b)$$

$$\frac{\partial \hat{H}}{\partial u_j} = \sum_{i-1}^{J} \hat{\lambda}_i \frac{\partial \hat{g}_i}{\partial u_j} \qquad (2.2.12c)$$

where H has been denoted as $\hat{H}$ to indicate it defines the Hamiltonian for the unperturbed system. Using these identities in (2.2.12) and noting that $\hat{\dot{e}}_i = \hat{g}_i$ results in

$$I = I_1 + \int_{t_g}^{t_f} \sum_{j=1}^{N} \Delta e_j \hat{\dot{\lambda}}_f dt - \int_{t_g}^{t_f} \sum_{j=1}^{N} \hat{\lambda}_f \Delta \dot{e}_f dt + \int_{t_g}^{t_f} \hat{L}\, dt$$

$$+ \int_{t_g}^{t_f} \sum_{k=1}^{K} \Delta\delta_k \frac{\partial \hat{H}}{\partial \delta_k} dt + \int_{t_g}^{t_f} \sum_{j=1}^{J} \Delta u_j \frac{\partial \hat{H}}{\partial u_j} dt + \text{H.O.T.} \qquad (2.2.13)$$

Since $\sum_{j=1}^{N} (\Delta e_j \hat{\dot{\lambda}}_j + \hat{\lambda}_j \Delta \dot{e}_j)$ is the differential of $\sum_{j=1}^{N} \Delta e_j \hat{\lambda}_j$, (2.2.13) can be written as (16)

$$I = \int_{t_g}^{t_f} \hat{L} dt - \sum_{j=1}^{N} \Delta e_j(t_f)\hat{\lambda}_j(t_f) + \Delta e_j(t_g)\hat{\lambda}_j(t_g) + \int_{t_g}^{t_f} \sum_{k=1}^{K} \Delta\delta_k \frac{\partial \hat{H}}{\partial \delta_k} dt$$

$$I_1 + \int_{t_g}^{t_f} \sum_{j=1}^{J} \Delta u_j \frac{\partial \hat{H}}{\partial u_j} dt + \text{H.O.T.} \qquad (2.2.14a)$$

$$= \int_{t_0}^{t_f} \hat{L} dt + \int_{t_g}^{t_f} \sum_{k=1}^{K} \Delta\delta_k \frac{\partial \hat{H}}{\partial \delta_k} dt + \int_{t_g}^{t_f} \sum_{j=1}^{N} \Delta u_j \frac{\partial \hat{H}}{\partial u_j} dt + \text{H.O.T.} \qquad (2.2.14b)$$

The term $\Delta e_j(t_g)$ represents the change in the jth error component at the beginning of the interval $(t_g,t_f]$. This change is measured relative to the nominal value $\hat{e}_j$ which is determined solely by $\hat{\delta}$, $\hat{u}$ on $[t_0,t_g]$. If the partial derivatives

$$\frac{\partial e_j}{\partial u_m}, \quad \frac{\partial e_j}{\partial \delta_k} \quad \forall\, j, k, m$$

exist then e_j cannot change discontinuously. The value of e_j at the beginning of $(t_g,t_f]$ must be exactly that at the end of the interval $[t_0,t_g]$. The term $\Delta e_j(t_g)$ must then be zero. Because the α_j's were chosen as the co-state variables, from (2.2.11) $\lambda_j(t_f)=0$.

From (2.2.14b) the total cost, I, on $[t_0,t_f]$ is the sum of four terms. The first term represents the cost associated with the system using $\hat{\delta}$, $\hat{u}$. The second and third terms indicate that the total cost can be changed if δ or u or both are changed on $(t_g,t_f]$. Because of assumptions (1) and (2) the change in I is completely controlled by the changes in δ, u. The fourth term represents all higher order terms resulting from any change on $(t_g,t_f]$. Assuming the system equations are sufficiently smooth, these terms are negligible for sufficiently small changes in δ, u.

The second term in (2.2.14) can be made negative and the total cost on $[t_0,t_f]$ can be reduced by changing δ as

$$\Delta\delta_k = -\beta_k \frac{\partial \hat{H}}{\partial \delta_k} \quad k=1,2,\ldots K \tag{2.2.15}$$

where β_k, k=1,2,...N are positive constraints whose value depends upon the value of $\partial\hat{H}/\partial\delta_k$ along the interval $(t_g,t_f]$ and must be chosen small enough to allow higher order terms to be neglected.

Using (2.2.15) in (2.2.14) it is clear that a reduction in cost can always be realized providing that all partial derivatives of H with respect to δ are non-zero. In the case where the $GRAD_\delta$ H is identically zero, the cost cannot be reduced by changing an δ_k. The adaptive problem is solved once $\delta=0$. But, from (2.2.15) the condition is imposed that δ should be changed until $GRAD_\delta$ H=0. It does not necessarily follow that δ=o under these conditions. (2.2.15) is sufficient to reduce the cost while $GRAD_\delta$ H $\neq$ 0 and it also shows that once it is zero, system performance cannot be improved by adjusting δ further. This result is only valid for the particular choice of L and the particular plant input under consideration. It may be that $\delta \neq 0$ and $GRAD_\delta$ H = 0 for one choice of L or r and for another choice this will not be the case. If L could be chosen to be a function of δ as well, then it could be guaranteed that when $GRAD_\delta$ H =0 δ must be zero. As pointed out earlier, L cannot be so chosen since implementation of (2.2.15) would then require knowledge of δ.

There are two situations where difficulties might arise when $GRAD_\delta$ H is zero. The first is when the parameter adjustment scheme stops at a solution but the error measure is non-zero or even increasing with time. The other is where L will be zero but some of the components of δ will be non-zero. In the first case the problem can be eliminated if the error system is represented in phase variable form. The reason for this will become obvious as the adjustment algorithn is more fully developed.

The second contingency is not a serious one in adaptive control, since it results in L=0, at least for the input being applied. The goal of adaptation is achieved because the plant and model behave dynamically the same subject to the same input. The fault lies with the fact that for the particular input the system error is insensitive to certain of the parameters. Put in another way, the input is not sufficiently broad band to excite all modes of the error system. If the parameter adjustment scheme was used off-line for plant identification the system inputs could be chosen to eliminate this problem. If it were used on-line, random fluctuations and input disturbances might serve to keep all modes excited. If control signal augmentation were used in addition to parameter adjustments the excitation caused by the control signal could be sufficient to maintain dynamic response of all modes.

The adjustment algorithm can be more fully developed by considering L to be a symmetric quadratic form with $\ell_{ij}=\ell_{ji}$.

$$L = \sum_{i,j=1}^{N} \ell_{ij}\, e_i\, e_j \tag{2.2.16}$$

With the system equation (2.1.5), the Hamiltonian is formed as

$$H = \sum_{i,j=1}^{N} \ell_{ij}\, \hat{e}_i\, \hat{e}_j + \sum_{j=1}^{N} \hat{\lambda}_j\, \hat{g}_j \tag{2.2.17}$$

with

$$\left.\begin{aligned} \dot{\hat{\lambda}}_j &= -\frac{\partial H}{\partial e_j} \\ \hat{\lambda}_j\,(t_f) &= 0 \end{aligned}\right\}\; j=1,2,\ldots N \tag{2.2.18}$$

and

$$\dot{\hat{e}}_i = \frac{\partial H}{\partial \lambda_j} = \hat{g}_i \qquad i=1,2,\ldots N \tag{2.2.19}$$

The parameters should be adjusted on $(t_g,t_f]$ according to:

$$\Delta\delta_k = -\beta_k \sum_{i=1}^{N} \hat{\lambda}_i\, \frac{\partial \hat{g}_i}{\partial \delta_k} \qquad k=1,2,\ldots K \tag{2.2.20}$$

with $\hat{\lambda}_i$ given by

$$\dot{\hat{\lambda}}_i = - \sum_{j=1}^{N} \ell_{ij} \hat{e}_j - \sum_{j=1}^{N} \hat{\lambda}_j \frac{\partial \hat{g}_j}{\partial e_i} \quad i,1,2,\ldots N \qquad (2.2.21)$$

$\hat{\lambda}_i$ as a function of time on the interval $(t_g,t_f]$ is

$$\hat{\lambda}_i = \int_{t_f}^{t} \left[\sum_{j=1}^{N} \ell_{ij} \hat{e}_j + \sum_{j=1}^{N} \hat{\lambda}_j \frac{\partial \hat{g}_j}{\partial e_i} \right] dt + \hat{\lambda}_i (t_f) \quad i=1,2,\ldots N \qquad (2.2.22)$$

(2.2.20) indicates that to specify $\Delta\delta_k$ on $(t_g,t_f]$ requires knowing $\hat{\lambda}$ on $(t_g,t_f]$, the result of integrating (2.2.21) backwards over this interval. Since $\hat{e}$ is the error resulting from δ with no change on $(t_g,t_f]$, implementation of (2.2.20) will involve a time delay of t_f-t_g seconds.

The fact that a time delay results from trying to implement (2.2.20) is not surprising. The parameter δ_k was to be changed as a function of time over the interval $(t_g,t_f]$ to reduce the cost over this interval. This then requires knowing the error response on the interval before implementing the parameter change. The net result is that in order to reduce cost on $(t_g, t_f]$ requires waiting until t_f. There would be an additional computational delay associated with integrating the collected information from t_f backwards to specify $\hat{\lambda}$ for t on $(t_g,t_f]$. The conclusion must be that this change designed to reduce the integral on the interval $(t_g,t_f]$ has no effect on system performance on this interval since it must be implemented after the interval of interest.

The development leading to (2.2.20) extends analogously to the case where Δu is to be chosen on $(t_g,t_f]$ to effect a reduction in I. Starting from (2.2.14) it follows that Δu should be chosen as

$$\Delta\mu_j = -\gamma_j \frac{\partial \hat{H}}{\partial \mu_j} = -\gamma_j \sum_{i=1}^{N} \hat{\lambda}_i \frac{\partial \hat{g}_i}{\partial \mu_j} \qquad j=1,2,\ldots J \qquad (2.2.23)$$

Reasoning as in the parameter adjustment case, to implement (2.2.23) requires a time delay of at least $t_f - t_g$ seconds.

2.3 REFINEMENT OF ADJUSTMENT ALGORITHM TO REAL TIME

There are several reasons why the algorithm as developed above is not satisfactory.

1) Storage facilities are required to save the error vector over the interval of minimization and compute (2.2.22).

2) The delay associated with the required computations will have a destablizing effect for rapidly varying dynamics.

3) The means for changing the parameters as functions of time may be difficult to implement.

The development of a real time scheme will necessitate abandoning integral reduction for reduction of an instantaneous error measure. It is felt, however, that since the resultant scheme will not have the faults listed above that it is a more appealing solution to the adaptation problem.

By adjusting δ according to (2.2.20) the integral cost I is reduced over an interval of arbitrary duration. If this interval is reduced to length Δt, t_f becomes $t_g + \Delta t$ and (2.2.22) becomes

$$\hat{\lambda}_i = \int_{t_g+\Delta t}^{t} \left[\sum_{j=1}^{N} \ell_{ij} \hat{e}_j + \sum_{j=1}^{N} \hat{\lambda}_j \frac{\partial \hat{g}_j}{\partial e_i} \right] dt \qquad i=1,2,\ldots N \qquad (2.3.1)$$

From the parameter adjustment algorithm (2.22.20) and (2.3.1)

$$\Delta\delta_k = -\beta_k \sum_{i=1}^{N} \left[\int_{tg+\Delta t}^{t} \left[\sum_{j=1}^{N} \ell_{ij}\, \hat{e}_j + \sum_{j=1}^{N} \hat{\lambda}_j \frac{\partial \hat{g}_j}{\partial e_i} \right] dt \right] \frac{\partial \hat{g}_i}{\partial \delta_k}, \quad k=1,2,\ldots K \tag{2.3.2}$$

By the Mean Value Theorem for integrals (2.3.2) may be approximated as

$$\Delta\delta_k \simeq -\beta_k \sum_{i=1}^{N} \left[\sum_{j=1}^{N} \ell_{ij}\, \hat{e}_j(t_g+\Delta t) + \sum_{j=1}^{N} \hat{\lambda}_j(t_g+\Delta t) \frac{\partial \hat{g}_j}{\partial e_i}(t_g+\Delta t) \right] \Delta t . \frac{\partial \hat{g}_i}{\partial \delta_k},$$

$$K=1,2,\ldots K \tag{2.3.4}$$

For any Δt, $\hat{\lambda}_j(t_g+\Delta t)=0 \; \forall_j$ from (2.2.18). Since (2.3.4) represents the change in δ_k over an interval Δt, to implement it over an arbitrarily small interval

$$\frac{d}{dt}\delta_k = \dot{\delta}_k = -\beta_k(t_g) \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij}\, \hat{e}_j\,(t_g)\, \frac{\partial \hat{g}_i}{\partial \delta_k}(t_g) \quad k=1,2,\ldots K \tag{2.3.5}$$

(2.3.5) is equivalent to (2.3.4) over a small enough interval. That is, implementing rates of change as in (2.3.5) gives exactly the same result as implementing (2.3.4) over this same interval. Unlike (2.2.20), there is no time delay associated with (2.3.5). Also there is no computational delay because there is no integration of stored data needed to evaluate (2.3.5). Hence the method does not have the disadvantages associated with (2.2.20). It should be stressed that (2.3.5) is based on the criterion of reducing an instantaneous error measure whereas (2.2.20) was based on reducing an integral error function.

The subscript denoting particular time t_g can be dropped because (2.3.5) is continuously implemented. Also, reasoning as with (2.2.14), $\hat{e}(t_g) = e(t_g)$; the actual measurable error. The constraints are written as $\beta_k(t_g)$ since, as was pointed in the discussion of (2.2.15) they are positive but may be dependent on $GRAD_\delta H$. It is sufficient to let them be constant but the freedom is there to allow dependence on $GRAD_\delta H$. This result becomes more intuitive if the parameter adjustments are regarded as movements towards a minimum in the K-dimensional parameter space. As is well known, the size of the movement is dependent on the shape of the function. Since (2.3.5) defines movements in this space as a function of time, the location in parameter space is mapped onto the time axis. The result is that it is equivalent to consider β_k as being time dependent or parameter dependent. Equation (2.2.23) can similarly be solved for sufficiently small Δt to yield rates of change of the control vector:

$$\dot{\mu}_k = -\gamma_k \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij}\, e_j \frac{\partial g_i}{\partial \mu_k} \qquad k=1,2,\ldots J \tag{2.3.6}$$

2.4 USE OF PHASE VARIABLES IN THE ADJUSTMENT ALGORITHM

The possibility was raised in connection with (2.2.15) that δ might settle at an equilibrium although the error measure could be $\neq$ 0. At that time it was said that by choosing the error states as phase variables this could be avoided. The following argument supplies the details.

The control exerted by δ on the error system is not direct, but rather constrained by the system equation (2.2.19). This is illustrated in (2.3.5) where the term $\partial\hat{g}_i/\partial\delta_k$ represents the influence δ_k has on the i^{th} error state. It is this coupling between the parameters and the error states which permits the error measure to be reduced through parameter adjustments. For every adjustible parameter it is necessary that the corresponding partial derivative of the form $\partial\hat{g}_i/\partial\delta_k$ be non-zero. As examples are presented it will be seen that these terms are generally either functions of the input or of the plant states. In either case, the assumption that they be non-zero is very weak.

For δ to stay at an equilibrium point requires that $\dot{\delta}=0$. This can only occur if, for every term $\partial\hat{g}_i/\partial\delta_k$ which is non-zero, the corresponding term $\sum_{j=1}^{N} \ell_{ij}\, \hat{e}_j$ be zero. If it is guaranteed that L goes to zero under these conditions then the adaptive goal is achieved.

By choosing the error states in phase variable form, for i=p the requirement is that

$$\sum_{j=1}^{N} \ell_{pj}\, e_j = \sum_{j=1}^{N} (\ell_{pj} s^{j-1})\, e_1 = 0 \qquad (2.4.1)$$

where s is the Laplace operator. If the coefficients ℓ_{pj} are chosen to satisfy the Routh-Hurwitz criterion it is guaranteed that $L\to 0$. This does not necessarily guarantee that the plant parameters are identified. The parameter error vector δ will not tend to zero if the normal plant excitations are not sufficiently broad band to force all modes of the system. This problem can be reduced if the normal plant inputs are augmented by the control signal as given in (2.3.6). The advantage to using control augmentation is that the plant is excited more than it would be if the normal plant inputs were acting alone. The result is that not only will the adaptive process be achieved, but by exciting the plant, the control augmentation will increase the probability that $\delta\to 0$ as $L\to 0$. This is an important consideration since it was assumed that the plant parameters were stationary. If $\delta=0$ then the plant is identified for all future inputs.

Example 2.1

To illustrate some of the preceding ideas consider a plant described in phase variable form as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = p_3 x_3 - p_2 x_2 - p_1 x_1 + K_p (r+u) \qquad \text{(x1.1)}$$

The desired behavior is given by the model equations

$$\dot{m}_1 = m_2$$

$$\dot{m}_2 = m_3$$

$$\dot{m}_3 = -\partial_3 m_3 - \partial_2 m_2 - \partial_1 m_1 + K_m \cdot r \qquad \text{(x1.2)}$$

Because the plant and model are in phase variable form the error states are also phase variables.

$$\dot{e}_1 = e_2 = g_1$$

$$\dot{e}_2 = e_3 = g_2$$

$$\dot{e}_3 = -\partial_3 e_3 - \partial_2 e_2 - \partial_1 e_1 + x_3 \delta_3 + x_2 \delta_2 + x_1 \delta_1 + r\delta_4 - K_p u = g_3 \qquad \text{(x1.3)}$$

where

$$\delta_1 = p_1 - \partial_1$$

$$\delta_2 = p_2 - \partial_2$$

$$\delta_3 = p_3 - \partial_3 \qquad \text{(x1.4)}$$

$$\delta_4 = K_m - K_p$$

The coupling terms are

$$\frac{\partial g_i}{\partial \delta_k} = 0 \quad i=1,2,\ K=1,4$$

$$\frac{\partial g_3}{\partial \delta_k} = x_k \quad k=1,2,3$$

$$\frac{\partial g_3}{\partial \delta_4} = r \qquad \text{(x1.5)}$$

$$\frac{\partial g_3}{\partial \mu} = -K_p$$

Thus for k=1 through 4 there is a non-zero term of the form $\partial g_3/\partial\delta_k$.

Choosing an error measure as in (2.2.16) and applying the parameter adjustment algorithm (2.3.5):

$$\dot{\delta}_1 = -\beta_1 \frac{\partial L}{\partial e_3} x_1$$

$$\dot{\delta}_2 = -\beta_2 \frac{\partial L}{\partial e_3} x_2$$

$$\dot{\delta}_3 = -\beta_3 \frac{\partial L}{\partial e_3} x_3 \qquad \text{(x1.6)}$$

$$\dot{\delta}_4 = -\beta_4 \frac{\partial L}{\partial e_3} r$$

$$\frac{\partial L}{\partial e_3} = (\ell_{31} e_1 + \ell_{32} e_2 + \ell_{33} e_3)$$

From (2.3.6) the control signal is synthesized as

$$\dot{u} = + \frac{\gamma}{K_p} \frac{\partial L}{\partial e_3} \qquad \text{(x1.7)}$$

Because phase variables are used

$$\ell_{31}e_1 + \ell_{32} e_2 + \ell_{33} e_3 = [\ell_{31} + \ell_{32} s + \ell_{33} s^2] e_1 \qquad \text{(x1.8)}$$

and to guarantee $L \to 0$ it is necessary to choose the ℓ's to make (x1.8) stable. Because γ in (2.3.6) can be any positive function, $\frac{\gamma}{Kp}$ is also a positive function (for minumum phase plants) and (x1.7) can be implemented without knowledge of Kp.

A logical question to ask is whether (x1.6,7) could be implemented without using e_3. Specifically,, if $p_3 = a_3 x_3$ is not required in (x1.6). However, if e_3 is not used ℓ_{33} must be zero. This in turn requires that L be independent of e_3; but it is the partial of L with respect to e_3 which appears in (x1.6). Hence the algorithm cannot be used unless the highest plant derivative is used in the error measure.

Example 2.2

To illustrate the mechanics of applying the adjustment algorithm to a non-linear system consider the von der Pol equation with a forcing term:

$$\ddot{x} - (p_2 - p_3 x^2)\,\dot{x} + p_1 x = r + u \qquad \text{(x2.1)}$$

which can be written in phase variable form by letting $x_1 = x$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -p_1 x_1 + (p_2 - p_3 x_1^2)\,x_2 + r + u \qquad \text{(x2.2)}$$

If the desired behavior is given by

$$\dot{m}_1 = m_2$$
$$\dot{m}_2 = -\partial_2 m_2 - \partial_1 m_1 + r \qquad \text{(x2.3)}$$

then by using non-linear feedback $u = c_3 x_1^2 x_2$ in (x2.2)

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = -p_1 x_1 + p_2 x_2 + x_1^2 x_2\,(c_3 - p_3) + r \qquad \text{(x2.4)}$$

and the error equations can be written as

$$\dot{e}_1 = e_2$$
$$\dot{e}_2 = -\partial_2 e_2 - \partial_1 e_1 + x_1\delta_1 - x_2\delta_2 - x_1^2 x_2 \delta_3 \qquad \text{(x2.5)}$$

where

$$\delta_1 = p_1 - \partial_1$$

$$\delta_2 = p_2 + \partial_2 \qquad \text{(x2.6)}$$

$$\delta_3 = \partial_3 - p_3$$

The parameter adjustments then follow from applying (2.3.5)

$$\dot{\delta}_1 = -\beta_1 x_1 (\ell_{21}e_1 + \ell_{22} e_2)$$

$$\dot{\delta}_2 = -\beta_2 x_2 (\ell_{21}e_1 + \ell_{22}e_2)$$

$$\dot{\delta}_3 = -\beta_3 x_1^2 x_2 (\ell_{21}e_1 + \ell_{22}e_2)$$

2.5 STABILITY OF THE ADAPTIVE PROCESS

It was demonstrated in Example 2.1 that if phase variables are used to implement the adaptive system the highest order error state must be included in the error measure. If the plant is third order or greater than some of the phase variables would probably have to be obtained by differentiating lower order plant states. The inherent difficulties arising from such a scheme when noise is an important consideration are well known.

The advantage to using phase variables is that it can be guaranteed that $L \to 0$ when $\dot{\delta}=0$. It is natural to examine the stability of the adjustments when a different set of state variables are used. Instead of examining the adaptive loop stability for a particular state representation, the inherent stability implied by reducing an error measure can be used to formally argue stability.

From (2.2.14)

$$I \triangleq \int_{t_0}^{t_f} L dt = \int_{t_0}^{t_f} \hat{L} dt + \int_{t_g}^{t_f} \sum_{k=1}^{K} \Delta\delta_k \frac{\partial H}{\partial \delta_k} dt + \int_{t_g}^{t_f} \sum_{j=1}^{J} \Delta u_j \frac{\partial H}{\partial u_j} dt \qquad (2.5.1)$$

$$= \int_{t_0}^{t_f} \hat{L} dt + \int_{t_0}^{t_f} (L - \hat{L}) dt$$

$$(2.5.2)$$

$$= \int_{t_0}^{t_f} \hat{L} dt + \int_{t_g}^{t_f} (L - \hat{L}) dt$$

(2.5.2) results from recognizing that $L = \hat{L}$ on $[t_0, t_g]$ since $\hat{L}$ is the loss function for the unperturbed system, and the perturbations occur on $(t_g, t_f]$. On the interval $(t_g, t_f]$ $L - \hat{L}$ is the change in loss resulting from changes of δ, u on $(t_g, t_f]$. Comparing (2.5.2) and (2.5.1) it follows that

$$\int_{t_g}^{t_f} (L-\hat{L}) dt \triangleq \int_{t_g}^{t_f} \Delta L dt = \int_{t_g}^{t_f} \left(\sum_{k=1}^{K} \Delta\delta_k \frac{\partial H}{\partial \delta_k} + \sum_{j=1}^{J} \Delta u_j \frac{\partial H}{\partial u_j} \right) dt \qquad (2.5.3)$$

By choosing $\Delta\delta_k$, Δu_j as in (2.2.20) and (2.2.23) respectively, the right side of (2.5.3) is negative. As Δt shrinks to zero, the left side of (2.5.3) is $\Delta L \Delta t$. The only way for $\Delta L \Delta t$ to be negative over an arbitrarily small interval Δt is if $\frac{d}{dt} L$ is negative. Therefore by choosing L positive definite it must follow that the error response is asymptotically stable. (12) By choosing L positive semidefinite the adjustments can only guarantee that $L \to 0$.

In Appendix A these same results are proved in a more formal way by showing that the adjustments guarantee the existence of a Lyapunov function for the differential equations defining the error system. In particular, if the β_k and γ_j in (2.3.5) and (2.3.6) are constant it is shown that the parameter adjustment algorithm is the same as that proposed by Buchard and Shackcloth (3) and Parks (17).

III ADAPTIVE CONTROL FOR NONSTATIONARY PLANTS

3.1 INTRODUCTION

For the development in Chapter 2 it was necessary to assume

1) Any change in δ was completely controlled. The plant parameters could be unknown but constant.

2) The effects on L caused by plant inputs was negligible compared to the control action on any interval.

3) Any change in the plant state and model state vector was equivalent to a change in the error state vector.

With these assumptions it was shown that δ, u could be changed on $(t_g, t_f]$ to guarantee a reduction in L. In particular, as the interval became infinitesimal, rates of change for δ, u were established which guaranteed L was reduced to a minimum. In this chapter the effects of relaxing these assumptions are investigated.

The need for adaptation arises more from the unknown variation of plant parameters then simply the presence of constant unknown parameters. If the parameters did not vary, off line techniques might be used to identify the plant. This plant description would then be valid for all time and conventional control techniques could be used.

In aircraft control field environmentally sensitive parameters present a real problem. Since the equations of motion are determined largely by the aircraft environment, as higher altitudes and faster speeds are attained the effect is that plant parameters vary widely over a given flight path. It is not surprising that a great deal of the early work in adaptive systems was directed towards these problems (13).

Because of the importance of this problem it has become almost axiomatic that any published report on adaptive techniques contain an example pertaining to aircraft control.

The effects of ageing and breakdown certainly fall into the category of parameter variation. Here the parameters may be constant for a long time and change arbitrarily to a different constant value. Or in the case of ageing, the parameters might be very slowly varying. In either case, the system design should provide for some means of compensation. The classic high gain feedback scheme has been used successfully to offset the effects of parameter variation. Some reports have been published aimed exclusively at using the more modern adaptive techniques to compensate for component failure (11).

3.2 INFLUENCE OF UNCONTROLLED VARIATIONS

By first examining these effects on $(t_g,t_f]$ and then letting this interval shrink to zero their influence on the adaptive process can be determined. On the interval $(t_g,t_f]$,

$$\delta_k = \hat{\delta}_k + \Delta\delta_k + \Delta\tilde{\delta}_k \quad k=1,2,\ldots K \tag{3.2.1}$$

where δ_k is the same as in (2.2.4) except for the $\Delta\tilde{\delta}_k$ term which represents the uncontrolled parameter variation. The input vector changes as

$$r_j = \hat{r}_j + \Delta\tilde{r}_j \qquad j=1,2,\ldots,J \tag{3.2.2}$$

In addition, the control signal changes as

$$u_j = \hat{u}_j + \Delta u_j \qquad j=1,2,\ldots J \tag{3.2.3}$$

As a result of these changes the plant states change incrementally on $(t_g, t_f]$

$$x_i = \hat{x}_i + \Delta x_i + \Delta \hat{x}_i \quad i=1,2,\ldots N \qquad (3.2.4)$$

In Chapter 2 it was assumed that the combined effect of changes in the plant and model states was exactly equivalent to a change in the error state. The effects of relaxing this assumption are examined by including the plant state changes in the analysis.

For notational continuity and simplicity the change in r is written with a superscript tilda and is referred to subsequently as an uncontrolled change. Similarly, the change in x not reflected as an equivalent change in error is referred to as the uncontrolled change and is written with the same super script notation.

As in (2.2.5)

$$g_i = \hat{g}_i + \Delta g_i \qquad i=1,2,\ldots N \qquad (3.2.5a)$$

$$L = \hat{L} + \quad L \qquad (3.2.5b)$$

Because of the changes in r and x and δ the incremental changes in g_i, L are given by

$$\Delta g_i = \sum_{j=1}^{N} \frac{\partial \hat{g}_i}{\partial e_j} \Delta e_j + \sum_{k=1}^{K} \frac{\partial \hat{g}_i}{\partial \delta_k} \Delta \delta_k + \sum_{j=1}^{J} \frac{\partial \hat{g}_i}{\partial u_j} \Delta u_j + \sum_{k=1}^{K} \frac{\partial \hat{g}_i}{\partial \delta_k} \Delta \tilde{\delta}_k + \sum_{j=1}^{J} \frac{\partial \hat{g}_i}{\partial r_j} \Delta \tilde{r}_j$$

$$+ \sum_{j=1}^{N} \frac{\partial \hat{g}_i}{\partial x_j} \Delta \tilde{x}_j \qquad (3.2.6)$$

and

$$\Delta L = \sum_{j=1}^{N} \frac{\partial \hat{L}}{\partial e_j} \Delta e_j \qquad (3.2.7)$$

The effects of these changes on I can be represented analogously to (2.2.8) as

$$I=I_1 + \int_{t_g}^{t_f} [\hat{L} + \sum_{j=1}^{N} \frac{\partial \hat{L}}{\partial e_j} \Delta e_j + \sum_{i=1}^{N} \hat{\lambda}_i [\hat{g}_i - \hat{\dot{e}}_i - \Delta \dot{e}_i + \sum_{j=1}^{N} \frac{\partial \hat{g}_i}{\partial e_j} \Delta e_j + \sum_{k=1}^{K} \frac{\partial \hat{g}_i}{\partial \delta_k} \Delta \delta_k$$

$$+ \sum_{j=1}^{J} \frac{\partial \hat{g}_i}{\partial u_j} \Delta u_j + \sum_{k=1}^{K} \frac{\partial \hat{g}_i}{\partial \delta_k} \Delta \tilde{\delta}_k + \sum_{j=1}^{J} \frac{\partial \hat{g}_i}{\partial r_j} \Delta \tilde{r}_j + \sum_{j=1}^{N} \frac{\partial \hat{g}_i}{\partial x_j} \Delta \tilde{x}_j]]dt + H.O.T.$$

(3.28)

and collecting terms:

$$I=I_1 + \int_{t_g}^{t_f} \sum_{i=1}^{N} [\hat{g}_i = \hat{\dot{e}}_i] \hat{\lambda}_i \, dt + \int_{t_g}^{t_f} \left[\sum_{j=1}^{N} \Delta e_j \left[\frac{\partial \hat{L}}{\partial e_j} + \sum_{i=1}^{N} \hat{\lambda}_i \frac{\partial \hat{g}_i}{\partial e_j} \right] \right] dt$$

$$+ \int_{t_g}^{t_f} \left[\sum_{k=1}^{K} \Delta \delta_k \sum_{i=1}^{N} \hat{\lambda}_i \frac{\partial \hat{g}_i}{\partial \delta_k} \right] dt + \int_{t_g}^{t_f} \left[\sum_{j=1}^{J} \Delta u_j \sum_{i=1}^{N} \hat{\lambda}_i \frac{\partial \hat{g}_i}{\partial u_j} \right] dt$$

$$+ \int_{t_g}^{t_f} \left[\sum_{k=1}^{K} \Delta \tilde{\delta}_k \sum_{i=1}^{N} \hat{\lambda}_i \frac{\partial \hat{g}_i}{\partial \delta_k} \right] dt + \int_{t_g}^{t_f} \left[\sum_{j=1}^{J} \Delta \tilde{r}_j \sum_{i=1}^{N} \hat{\lambda}_i \frac{\partial \hat{g}_i}{\partial r_j} \right] dt$$

$$+ \int_{t_g}^{t_f} \left[\sum_{j=1}^{N} \Delta \tilde{x}_j \sum_{i=1}^{N} \tilde{\lambda}_i \frac{\partial \hat{g}_i}{\partial x_j} \right] dt + \int_{t_g}^{t_f} \tilde{L} dt - \int_{t_g}^{t_f} \sum_{i=1}^{N} \hat{\lambda}_i \Delta \dot{e}_i \, dt + H.O.T.$$

(3.2.9)

Using the relationships for $\hat{\lambda}$ and H in (2.2.17,18) and applying the reasoning leading to (2.2.13), (3.2.9) can be expressed as

$$I = \int_{t_0}^{t_f} \hat{L}dt + \int_{t_g}^{t_f} \sum_{k=1}^{K} \Delta\delta_k \frac{\partial\hat{H}}{\partial\delta_k} dt + \int_{t_g}^{t_f} \sum_{j=1}^{J} \Delta u_j \frac{\partial\hat{H}}{\partial u_j} dt + \int_{t_g}^{t_f} \sum_{k=1}^{K} \Delta\tilde{\delta}_k \frac{\partial\hat{H}}{\partial\delta_k} dt$$

$$+ \int_{t_g}^{t_f} \sum_{j=1}^{J} \Delta\tilde{r}_j \frac{\partial\hat{H}}{\partial r_j} dt + \int_{t_g}^{t_f} \sum_{j=1}^{N} \Delta\tilde{x}_j \frac{\partial\hat{H}}{\partial x_j} dt + \text{H.O.T.} \qquad (3.2.10)$$

Recalling that the circumflect notation refers to unperturbed variables on $[t_0,t_g]$, (3.2.10) shows that the cost can be changed on $[t_0,t_f]$ by changing the integral on $(t_g,t_f]$. In addition to the controlled changes caused by changes in δ, u there are three additional terms which can change the cost. $\Delta\tilde{\delta}_k$ represents the unknown parameter variation, $\Delta\tilde{r}_j$ is the change in plant inputs from nominal values on $[t_0,t_f]$ and $\Delta\tilde{x}_j$ is the plant state variation which is not equivalent to a change in error. It may be that these uncontrolled changes serve to further reduce I. However, if these changes are always such that they tend to increase I then by choosing the controlled changes in (3.2.10) large enough a reduction in I is still possible. Too large a change in these terms would make the higher order terms significant.

As $t_f - t_g \rightarrow \Delta t$ the higher order terms are less important and (3.2.10) becomes

$$I = \int_{t_0}^{t_g+\Delta t} \hat{L}dt + \sum_{k=1}^{K} \Delta\delta_k \frac{\partial\hat{H}}{\partial\delta_k}(t_g)\, \Delta t + \sum_{j=1}^{J} \Delta u_j \frac{\partial\hat{H}}{\partial u_j}(t_g)\Delta t + \sum_{k=1}^{K} \Delta\tilde{\delta}_k \frac{\partial\hat{H}}{\partial\delta_k}(t_g)\Delta t$$

$$+ \sum_{j=1}^{J} \Delta\tilde{r}_j \frac{\partial\hat{H}}{\partial r_j}(t_g)\Delta t + \sum_{j=1}^{N} \Delta\tilde{x}_j \frac{\partial\hat{H}}{\partial x_j}(t_g)\, \Delta t \qquad (3.2.11)$$

A reduction in I can always be guaranteed if the controlled variations in (3.2.11) are sufficiently large to determine the change in I on Δt. This in turn requires that these changes can be made large enough to offset the effects of the worst possible uncontrolled changes.

3.3 ESTABLISHMENT OF A LEAST UPPER BOUND ON L

The worst possible uncontrolled variations in (3.2.11) are

$$\begin{aligned} \Delta\tilde{\delta}_k &= \|D_k\| \operatorname{sgn} \frac{\partial \hat{H}}{\partial \delta_k} \\ \Delta\tilde{r}_j &= \|R_j\| \operatorname{sgn} \frac{\partial \hat{H}}{\partial r_j} \\ \Delta\tilde{x}_j &= \|k_j\| \operatorname{sgn} \frac{\partial \hat{H}}{\partial x_j} \end{aligned} \tag{3.3.1}$$

where $||D_k||, ||R_j||$ and $||x_j||$ are the magnitudes of the largest possible uncontrolled change in δ_k, r, and x, respectively on Δt. If the uncontrolled changes are finite over any interval and have only a finite number of points at which they can be discontinuous (3.3.1) can be represented by derivatives

$$\begin{aligned} ||D_k|| &\cong ||\frac{d}{dt}\tilde{\delta}_k|| \; \Delta t \\ ||R_j|| &\cong ||\frac{d}{dt}\tilde{r}_j|| \; \Delta t \\ ||x_j|| &\cong ||\frac{d}{dt}\tilde{x}_j|| \; \Delta t \end{aligned} \tag{3.3.2}$$

where $||\frac{d}{dt}\tilde{\delta}_k||, ||\frac{d}{dt}\tilde{r}_j||$ and $||\frac{d}{dt}\tilde{x}_j||$ are the maximum uncontrolled rates of change of δ_k, r_j and x_j over the interval.

The controlled variations on $(t_g, t_g + \Delta t]$ are chosen as in (2.2.20) and (2.2.23) as

$$\Delta\delta_k = -\beta_k \frac{\partial \hat{H}}{\partial \delta_k} \tag{3.3.3a}$$

$$\Delta u_j = -\gamma_j \frac{\partial \hat{H}}{\partial u_j} \tag{3.3.3b}$$

The total change in (3.2.11), both controlled and uncontrolled is

$$-\sum_{k=1}^{K} \beta_k \left(\frac{\partial \hat{H}}{\partial \delta_k}\right)^2 \Delta t - \sum_{j=1}^{J} \gamma \left(\frac{\partial \hat{H}}{\partial u_j}\right)^2 \Delta t + \sum_{k=1}^{K} \left|\left|\frac{d}{dt} \tilde{\delta}_k\right|\right| \operatorname{sgn} \frac{\partial \hat{H}}{\partial \delta_k} \cdot \frac{\partial \hat{H}}{\partial \delta_k} (\Delta t)^2$$

$$+ \sum_{j=1}^{J} \left|\left|\frac{d}{dt} \tilde{r}_j\right|\right| \operatorname{sgn} \frac{\partial \hat{H}}{\partial r_j} \cdot \frac{\partial \hat{H}}{\partial r_j} (\Delta t)^2$$

$$+ \sum_{j=1}^{N} \left|\left|\frac{d}{dt} \tilde{x}_j\right|\right| \operatorname{sgn} \frac{\partial \hat{H}}{\partial x_j} \cdot \frac{\partial \hat{H}}{\partial x_j} (\Delta t)^2 \tag{3.3.4}$$

To guarantee a reduction in I it is sufficient to have

$$\sum_{k=1}^{K} \beta_k \left(\frac{\partial \hat{H}}{\partial \delta_k}\right)^2 + \sum_{j=1}^{J} \gamma_j \left(\frac{\partial \hat{H}}{\partial u_j}\right)^2 > \sum_{k=1}^{K} \left|\left|\frac{d}{dt} \tilde{\delta}_k\right|\right| \operatorname{sgn} \frac{\partial \hat{H}}{\partial \delta_k} \cdot \frac{\partial \hat{H}}{\partial \delta_k}$$

$$+ \sum_{j=1}^{J} \left|\left|\frac{d}{dt} \tilde{r}_j\right|\right| \operatorname{sgn} \frac{\partial \hat{H}}{\partial r_j} \cdot \frac{\partial \hat{H}}{\partial r_j}$$

$$+ \sum_{j=1}^{N} \left|\left|\frac{d}{dt} \tilde{x}_j\right|\right| \operatorname{sgn} \frac{\partial \hat{H}}{\partial x_j} \cdot \frac{\partial \hat{H}}{\partial x_j} \tag{3.3.5}$$

(3.3.5) is a conservative criterion because the uncontrolled effects were assumed acting to increase I in the maximum possible way.

Using the Hamiltonian formulation of Chapter 2 and letting L be a symmetric quadratic function of the error as in (2.2.16)

$$\frac{\partial \hat{H}}{\partial \delta_k} = \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t_g+\Delta t) \frac{\partial g_i}{\partial \delta_k} \Delta t \qquad k=1,2,\ldots K$$

$$\frac{\partial \hat{H}}{\partial x_k} = \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t_g+\Delta t) \frac{\partial g_i}{\partial x_k} \Delta t \qquad k=1,2,\ldots N \tag{3.3.6}$$

$$\frac{\partial \hat{H}}{\partial r_k} = \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t_g+\Delta t) \frac{\partial g_i}{\partial r_k} \Delta t \qquad k=1,2,\ldots J$$

$$\frac{\partial \hat{H}}{\partial u_k} = \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t_g+ \Delta t) \frac{\partial g_i}{\partial u_k} \Delta t \qquad k=1,2,\ldots J$$

Substituting (3.3.6) into (3.3.5), dividing both sides by $(\Delta t)^2$ and letting $\Delta t \to 0$: defines a region in error space (with $t_g=t$):

$$\sum_{k=1}^{K} \beta_k \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t) \frac{\partial g_i}{\partial \delta_k} \right|^2 + \sum_{k=1}^{J} \gamma_k \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t) \frac{\partial g_i}{\partial u_k} \right|^2$$

$$> \sum_{k=1}^{K} \left|\left| \frac{d}{dt} \tilde{\delta}_k \right|\right| \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t) \frac{\partial g_i}{\partial \delta_k} \right| \mathrm{sgn} \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t) \frac{\partial g_i}{\partial \delta_k} \right|$$

$$+ \sum_{k=1}^{J} \left|\left| \frac{d}{dt} \tilde{r}_k \right|\right| \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t) \frac{\partial g_i}{\partial r_k} \right| \mathrm{sgn} \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t) \frac{\partial g_i}{\partial r_k} \right|$$

$$+ \sum_{k=1}^{N} \left|\left| \frac{d}{dt} \tilde{x}_k \right|\right| \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t) \frac{\partial g_i}{\partial x_k} \right| \mathrm{sgn} \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j(t) \frac{\partial g_i}{\partial x_k} \right| \tag{3.37}$$

(3.3.7) is a geheral expression for the region in which it can be guaranteed that I is reduced. It is argued below that (3.3.7) implies a least upper bound on L under appropriate conditions. It can thus be argued that the plant states teack the model states with finite error measure. Example 3.1 serves to clarify these results for a particular case.

The implications of (3.3.7) can be illustrated by expressing the integral cost I on the interval Δt as

$$I \triangleq \int_{t_0}^{t_f} L\,dt = \int_{t_0}^{t_f} \hat{L}\,dt + \int_{t_g}^{t_f} \Delta L\,dt \qquad (3.3.8a)$$

$$= \int_{t_0}^{t_f} \hat{L}\,dt + \Delta L\,(t_g)\,\Delta t \qquad (3.3.8b)$$

where (3.3.8b) is exact as $\Delta t \to 0$. The total change in I is equivalent to a change in L. The change in L is comprised of a controlled change and an uncontrolled change:

$$\Delta L(t_g)\Delta t = \Delta L(t)\Delta t = (\Delta L_c + \Delta\tilde{L})\Delta t \qquad (3.3.9)$$

The controlled change is caused by adaptive adjustments of δ, u. The uncontrolled change is caused by the changes in x, r, δ as described previously. An alternate expression for the total change in I is given by (3.3.4). Comparing these two relationships shows that

$$\Delta L_c = -\sum_{k=1}^{K} \beta_k \left(\frac{\partial H}{\partial \delta_k}\right)^2 \Delta t - \sum_{j=1}^{J} \gamma_j \left(\frac{\partial H}{\partial u_j}\right)^2 \Delta t \qquad (3.3.10)$$

$$\Delta\tilde{L} = \sum_{k=1}^{K} \left|\left|\frac{d}{dt}\tilde{\delta}_k\right|\right| \operatorname{sgn} \frac{\partial H}{\partial \delta_k} \cdot \frac{\partial H}{\partial \delta_k}\Delta t + \sum_{j=1}^{J} \left|\left|\frac{d}{dt}\tilde{r}_j\right|\right| \operatorname{sgn} \frac{\partial H}{\partial r_j} \cdot \frac{\partial H}{\partial r_j}\Delta t$$

$$+ \sum_{j=1}^{N} \left|\left|\frac{d}{dt}\tilde{x}_j\right|\right| \operatorname{sgn} \frac{\partial H}{\partial x_j} \cdot \frac{\partial H}{\partial x_j}\Delta t \qquad (3.3.10)$$

By choosing the parameter adjustments and control signal augmentation as in (3.3.3a) and (3.3.3b), ΔL_c is negative. By allowing the worst case variations in the uncontrolled changes $\Delta\tilde{L}$ is always positive. If there is a region in the error space where (3.3.9) is negative under these conditions then it is guaranteed there is a bound on the error under worst case conditions. As $\Delta t \to 0$ the only way (3.3.9) could be negative if $\frac{d}{dt} L$ is negative. Therefore when the adjustments are applied over an infinitesimal Δt I can be reduced if

$$\frac{d}{dt} L_c + \frac{d}{dt} \tilde{L} < 0 \tag{3.3.11}$$

conversely I will increase, under worst case conditions, if the left side of (3.3.9) is positive. That there is an upper bound on e, and L if L is chosen positive definite will be shown below.

The existence of such a region in error space can be argued by setting $\beta_k = \gamma_k \forall k$ and expressing (3.3.7) as

$$\sum_{k=1}^{K}\left[\left[\sum_{i=1}^{N}\sum_{j=1}^{N} \ell_{ij} e_j \frac{\partial g_i}{\partial \delta_k}\right]^2 + \left[\sum_{i=1}^{N}\sum_{j=1}^{N} \ell_{ij} e_j \frac{\partial g_i}{\partial u_k}\right]^2\right]$$

$$> \left(\sum_{k=1}^{K} \left|\left|\frac{d}{dt}\tilde{\delta}_k\right|\right| \left|\sum_{i=1}^{N}\sum_{j=1}^{N} \ell_{ij} e_j \frac{\partial g_i}{\partial \delta_k}\right| + \left|\left|\frac{d}{dt}\tilde{r}_k\right|\right| \cdot \left|\sum_{i=1}^{N}\sum_{j=1}^{N} \ell_{ij} e_j \frac{\partial g_i}{\partial r_k}\right|\right.$$

$$\left. + \left|\left|\frac{d}{dt}\tilde{x}_k\right|\right| \cdot \left|\sum_{i=1}^{N}\sum_{j=1}^{N} \ell_{ij} e_j \frac{\partial g_i}{\partial x_k}\right|\right) \Big/ \beta \tag{3.3.12}$$

Some terms $\frac{\partial g_i}{\partial u_k} = 0$ if $J < K$ and some terms $\left|\left|\frac{d}{dt}\tilde{x}_k\right|\right|$, $\left|\left|\frac{d}{dt}\tilde{r}_k\right|\right| = 0$ for $J, N \times K$. The form (3.3.12 is used only to provide a simplified expression for illustration. The vertical bars on the terms on the right side of (3.3.12) indicate absolute value. These relationships follow from representing a term multiplied by its sign as its absolute value.

For $\beta = 1$ the left side of (3.3.12) will dominate the right side as the error gets large. This is the case since the right side increases as an absolute value and left increases as a square. The region in error space where the right side dominates defines the region where I can increase, or equivalently where $\frac{d}{dt} L_c + \frac{d}{dt} \tilde{L} > 0$. By choosing β large this region can be reduced. When the system error does not satisfy (3.3.12) it is possible for L to increase. Since all the error states are used in implementing the adjustments, as any error component becomes large $\left|\frac{\partial g_i}{\partial \delta_k}\right|$ must increase since these coupling terms are related to the plant states. The left side of (3.3.12) will increase and enter a region where I can be reduced. The following examples illustrate this important result for simple plants. The analytical treatment of these results for more complicated structures would be most untractable.

EXAMPLE 3.1

In Example 1.1 let $\|\frac{d}{dt} \tilde{\delta}_k\|$ k=1,4 be the maximum uncontrolled rates of change of plant parameters. $\|\frac{d}{dt} \tilde{r}\|$ and $\|\frac{d}{dt} \tilde{x}_j\|$ j=1,3 are the bounds on the derivatives of the plant and plant states respectively. The coupling coefficients between g and the parameters, between g and the input and between g and the plant states are given in Example 1.1. Choosing L as a symmetric quadratic form relation (3.3.7) yields

$$(\ell_{31}e_1 + \ell_{32}e_2 + \ell_{33}e_3)^2 \; [\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 r^2 + \gamma K_p^2 \;] >$$

$$\Big[\big|\big|\tfrac{d}{dt}\tilde{\delta}_1\big|\big| \cdot |x_1| + \big|\big|\tfrac{d}{dt}\tilde{\delta}_2\big|\big| \cdot |x_2| + \big|\big|\tfrac{d}{dt}\tilde{\delta}_3\big|\big| \cdot |x_3| + \big|\big|\tfrac{d}{dt}\tilde{\delta}_4\big|\big| \cdot |r|$$

$$+ \big|\big|\tfrac{d}{dt}\tilde{r}\big|\big| \cdot |\delta_4| + \big|\big|\tfrac{d}{dt}\tilde{x}_1\big|\big| \cdot |\delta_1| + \big|\big|\tfrac{d}{dt}\tilde{x}_2\big|\big| \cdot |\delta_2|$$

$$+ \big|\big|\tfrac{d}{dt}\tilde{x}_3\big|\big| \cdot |\delta_3| \Big] \cdot |\ell_{31}e_1 + \ell_{32}e_2 + \ell_{33}e_3| \qquad \text{(x1.1)}$$

which defines the region in error space where I can be reduced under

derivative of L with respect to e_3 defines a region in 3 dimensional space:

$$|\ell_{31}e_1 + \ell_{32}e_2 + \ell_{33}e_3| > \frac{\sum_{j=1}^{3} \big|\big|\frac{d}{dt}\delta_j x_j\big|\big| + \frac{d}{dt}\big|\big|r\delta_4\big|\big|}{\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 r^2 + \gamma K_p^2} \qquad \text{(x1.2)}$$

Since the model will be stable, an increase in the magnitude of error states will be caused by an increase in the magnitude of the plant states. If the system is operating in a region where (x.1.2) is not satisfied the left side may increase. Since the denominator of the right side will increase faster than the numerator the error states will enter a region where (x1.2) is again satisfied. Choosing L to measure all error states means that there must be a least upper bound on L despite worst case parameter variations. By choosing the β's and γ large this least upper bound can be minimized. It is noteworthy that if the restrictions of Chapter 1 are imposed on the right side of (x1.2) is zero.

can be made. This example illustrates how adaptation can be accomplished without identification. If all plant parameters are unknown but constant (x1.2) reduces to

$$|\ell_{31}e_1 + \ell_{32}e_2 + \ell_{33}e_3|$$

$$> \frac{||\frac{d}{dt}\tilde{r}||\cdot|\delta_4| + ||\frac{d}{dt}\tilde{x}_1||\cdot|\delta_1| + ||\frac{d}{dt}\tilde{x}_2||\cdot|\delta_2| + ||\frac{d}{dt}\tilde{x}_3||\cdot|\delta_3|}{[\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 r^2]} \qquad (x1.3)$$

If r is a step input, only the plant output (x_1) will be nonzero. Providing that β_1 or β_4 are non-zero, the least upper bound goes to zero. However, if β_1=0 then δ_1 cannot change the identification is impossible. Similarly if β_4=0 the plant gain does not change and $\delta_4 \neq 0$. The error measure goes to zero but the parameter error vector δ does not. This agrees with intuitition that identification is not generally possible if the input spectrum is not sufficiently broad.

A very important practical consideration is that not all parameters have to be adjusted to keep L small. In particular if only the plant gain is adjusted (x1.2) still represents a bound on L and β_4 can be chosen large to keep this bound small. This bound will not be as small as would result from adjusting all parameters, but the resultant implementation is considerably simplified. Some important design considerations when this simplified scheme is used are:

1) Choose β_4 as large as possible subject to the restriction that β_4 will effect $|\delta_4|$. A trade-off may be necessary between choosing β_4 so large that $|\delta_4|$ dominates the bound and choosing β_4 too small so that the denominator of (x1.2)

dominates.

2) When possible choose the plant input to be large in magnitude and slowly varying.

Because most plants have low pass characteristics the plant derivatives will have low values. This characteristic will tend to reduce the effects of these terms in the numerator of (xl.2). If in addition to the above considerations the plant parameters are slowly varying the least upper bound can be maintained acceptably small using only simple plant gain adaptation. This important simplification is utilized in designing an adaptive controller in Chapter 5.

In view of the preceding discussion it should be pointed that the assumption in Chapter 2 that the input is slowly varying is overly restrictive. If the plant input is sufficiently broad band, and if all the parameters are adjusted, the numerator of (xl.3) will tend to zero as the parameter difference vector δ goes to zero providing the parameters are slowly varying or stationary. Under these conditions L goes to zero regardless of how the plant states vary and regardless of input variations. The singularly important feature which limits adaptive convergence is the variation of plant parameters.

Control signal augmentation can be used to offset the deterioration in performance when a simplified gain adaptation scheme is used. The effect of control augmentation is clear from the denominator of (xl.2). The more effort in this signal, as measured by γ, the larger will be the demonimator. Because no signal multiplication is required to implement control augmentation, the system is not overly complex.

EXAMPLE 3.2

This example is chosen to further illustrate some of the important design considerations:

1) The weighting factors β and the error measure play an important role in defining the adaptive response.
2) It is not necessary to always use all of the error states when the system states are not phase variables.
3) The β's can be chosen to minimize the effects of time variable parameters.
4) If there is freedom in choosing the input signal magnitude this can be used to minimize the effects of parameter variations.

The plant is third order subject to a step reference $r=Ru(t)$.

$$\dot{x}_1 = -\frac{1}{2}x_1 + x_2$$

$$\dot{x}_2 = -x_2 + x_3 \qquad (x2.1)$$

$$\dot{x}_3 = 4x_3 + K_p r$$

The only unknown parameter is the plant gain K_p; enabling the error equations to be written as:

$$\dot{e}_1 = -\frac{1}{2}e_1 + e_2$$

$$\dot{e}_2 = -e_2 + e_3 \qquad (x2.2)$$

$$\dot{e}_3 = -4e_3 + \delta r$$

where $\delta = k_m - k_p$, k_m is the desired gain.

The error measure is chosen as $L = \frac{1}{2}le_3^2$ resulting in

$$\dot{\delta} = -\beta r l e_3 \qquad (x2.3)$$

Provided the error can be quickly reduced to zero the restriction on step inputs is not overly restrictive. From (x2.2) it is obvious that $\delta = 0$ when $e_3 = 0$ for a finite duration; therefore if L can be reduced to zero quickly r need be constant only over a short interval. The error system with adaptive controller is shown in Figure 3.1.

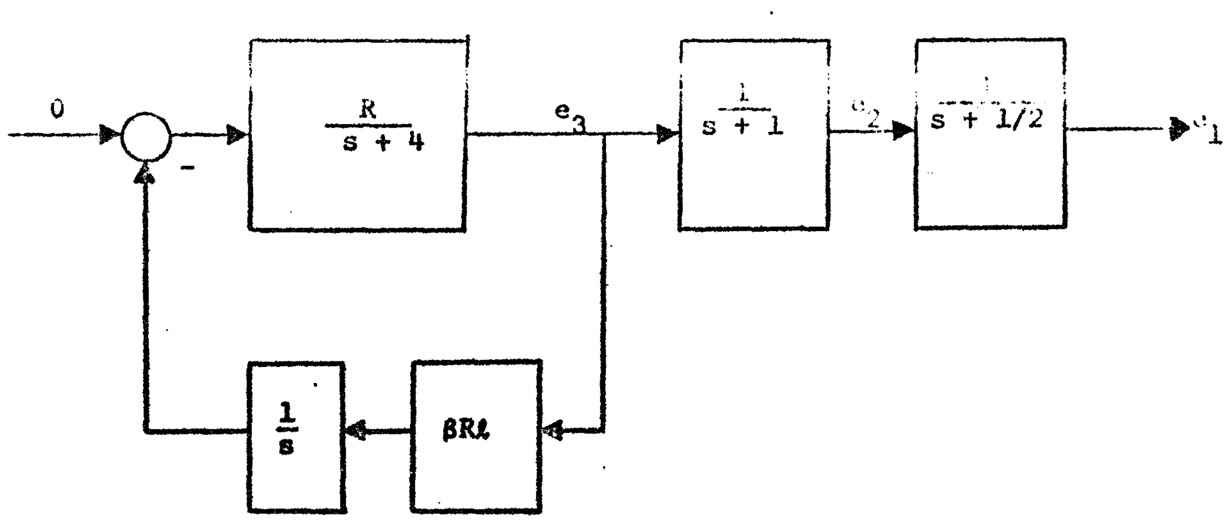


Figure 3.1

After the step reference is applied the dynamic error behavior is governed by the parameter adjustment loop. This error response is

$$e_3(s) = \frac{1}{(s^2 + 4sr\beta R^2\ell)} , \qquad \text{(x2.4)}$$

after normalizing wirh respect to initial conditions at the time adaptation was started. Hence to insure $e_3 \to 0$ it is only necessary to choose $\beta > 0$ and $\ell > 0$. The problem of using all the error states is avoided since the states are not phase variables.

The output e_1 is the solution to

$$e_1(s) = \frac{1}{(s^2 + 4s + \beta R^2\ell)(s+1)(s+ 1/2)} \qquad \text{(x2.5)}$$

by choosing β (or ℓ or both) properly, the transient error response is only limited by the term 1/s+ 1/2.

If the plant gain is time variable then δ has an uncontrolled variation $\tilde{\delta}$ with a bound $||\frac{d}{dt}\tilde{\delta}||$. From (3.3.1) the worst case is $\frac{d}{dt}\tilde{\delta}=||\frac{d}{dt}\tilde{\delta}||\text{sgn}\frac{\partial H}{\partial \delta}=||\frac{d}{dt}\tilde{\delta}||\text{sgn } r$. If R is positive and δ is possitive since $e_3=m_3-x_3$ e_3 will be increasing positively. From (x2.3), $\dot{\delta}$ will be negative. The worst case variation of $\tilde{\delta}$ would be in the positive direction. This same reasoning applies to any combination of ±R, ± δ. For R positive the block diagram for the error response is shown in Figure 3.2

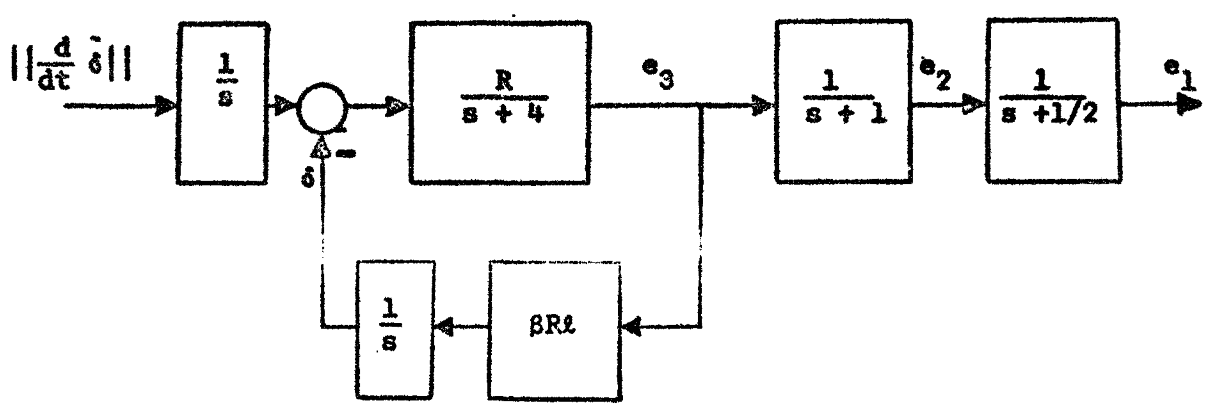


Figure 3.2

The closed loop response is given by

$$\frac{e_3(s)}{||\frac{d}{dt}\tilde{\delta}||(s)} = \frac{R}{s^2+4s+\beta R^2\ell} \qquad \text{(x2.6)}$$

with $\tilde{\delta}$ acting in a worst case way $||\frac{d}{dt}\tilde{\delta}||(s) = ||\frac{d}{dt}\tilde{\delta}||/s$ and

$$e_3(s) = \frac{||\frac{d}{dt}\tilde{\delta}||\cdot R}{s(s^2 + 4s + \beta R^2 \ell)} \qquad (x2.7)$$

As predicted by (3.37) L will not go to zero but rather approach a bound

$$L = \frac{1}{2}\ell e_3^2 \to \frac{||\frac{d}{dt}\tilde{\delta}||^2}{2\beta^2 R^2 \ell} \quad . \qquad (x2.8)$$

From (x2.7) the transient error response is influenced by the choice of β and ℓ. (x2.8) shows that the bound is minimized by choosing $\beta\ell$ large. If the poles of (x2.7) are 0, s_1, s_2 and if the transient response is to be dominated by $\frac{1}{s + 1/2}$, the design must be such that $s_1 s_2$= max subject to $s_1 + s_2 = 4$, $s_1, s_2 > \frac{1}{2}$. As shown in figure 3.3 the "optimum" design is at $s_1 = s_2 = 2$ which results in $\beta R^2 \ell = 4$. If this design results in too large a steady state L, there is one further possibility. Since the adjustment algorithm allowed for β to be time varying, $\beta(t)$ could be chosen to give fast transient response and very small steady state L. For β time varying an analysis similar to the above would be very difficult. Intuitively, however, in a simple case such as this β would be chosen as a slowly increasing function for small steady state error. The transient response would be essentially like that above for slowly varying β.

The advantages to using parameter adjustments with control signal augmentation have been argued. Any analytical treatment when both modes of compensation are used is difficult. Some of the implications of using only control signal augmentation can however be examined by selecting a simple example.

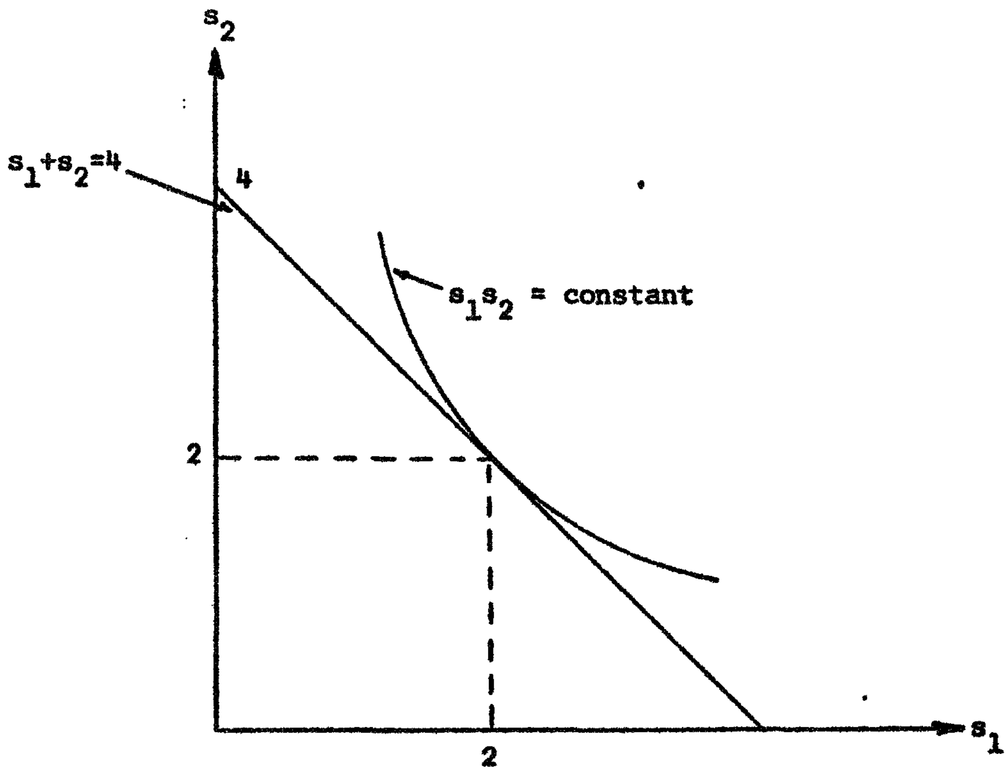


Figure 3.3

EXAMPLE 3.3

Figure 3.4 shows an adaptive system using control signal sugmentation. The plant is known except for a constant gain K_p. In state variable form

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = -(\partial_1 + \partial_2)e_2 - \partial_1\partial_2 e_1 + \delta r - K_p u \qquad \text{(x3.1)}$$

with $\delta = K_m - K_p$

An error measure is chosen as

$$L = \frac{1}{2}(\ell_{11}e_1^2 + 2\ell_{12}e_1e_2 + \ell_{22}e_2^2) \qquad \text{(x3.2)}$$

and applying (2.3.6) the resulting control signal augmentation is

$$\dot{u} = \gamma K_p(\ell_{12}e_1 + \ell_{22}2_2) = \alpha\,(\ell_{12}e_1 + \ell_{22}e_2) \qquad \text{(x3.3)}$$

The error response is

$$e_1(s) = \frac{\delta s - R(s)}{s^3 + s^2(\partial_1+\partial_2) + s(\partial_1\partial_2+\gamma K_p^2\ell_{22})+\gamma K_p^2\ell_{12}} \qquad \text{(x3.4)}$$

It was pointed out in the discussion of Example 3.1 that is all plant parameters were constant, parameter adjustments could be used to force $L\to 0$. This could be guaranteed despite input variation or plant state variation. Using control signal augmentation this is not the case. If r in figure 3.4 is unbounded with time then the error will be unbounded for any finite amount of control effort. This is easily verified by considering worst case conditions. Following the procedure of

space where

$$|\ell_{12}e_1 + \ell_{22}e_2| > \frac{|\delta|\cdot||\frac{d}{dt}r\,||}{\gamma K_p^2} \qquad \text{(x3.5)}$$

If $r(t) = t^2$ (x3.5) is not bounded and control signal augmentation cannot force the plant to adapt. The same conclusion follows from applying $R(s) = 1/s^3$ to (x3.4).

If parameter adjustment was also used in this example $K_m - K_p$ would go to zero and the error bound would be zero regardless of how the plant input varied.

Another disadvantage to using only control signal augmentation is that for certain plant inputs excessive control effort is required to maintain a small bound on L. For a ramp input the bound on L is given from (x3.5) as

$$|\ell_{12}e_1 + \ell_{22}e_2| = \frac{|\delta|}{\gamma K_p^2} \qquad \text{(x3.6)}$$

For $\delta > 0$ e_1 is positive and $\dot{u}$ will remain positive resulting in excessive control effort and plant saturation. This problem also can be avoided by using gain adjustments. Asjusting δ in (x3.6) will force (x3.6) to zero but also will keep δ changing as it approaches zero. As δ changes sign the error will also change sign, resulting in lower levels of control input.

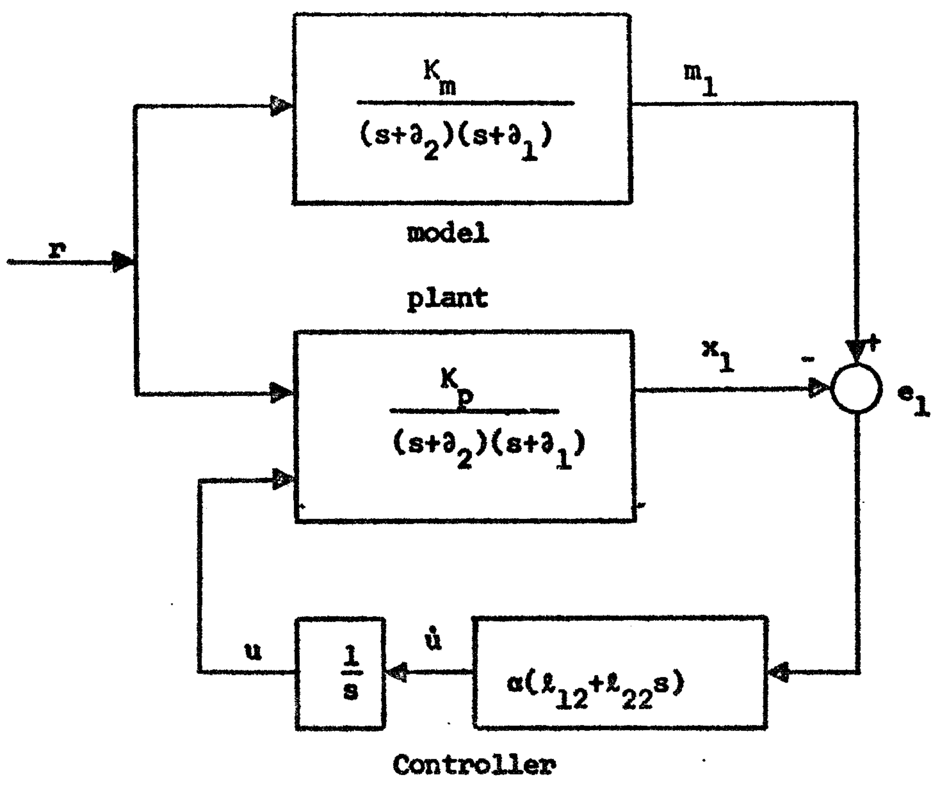


Figure 3.4

IV IMPLEMENTATION OF ADAPTIVE SYSTEM

4.1 Introduction

The preceding chapters have laid the theoretical groundwork for the adaptive system. In this chapter several examples will be presented with computational results.

To this point little has been said about the actual structure of the adaptive controller. The parameter adjustment algorithm for adaptive control presents some problems when implementation is considered. There are however several techniques which can realistically be implemented to provide for the appropriate parameter adjustments. Rather than attempt to generalize a particular scheme for implementation, several different methods will be illustrated by example.

When the algorithm is used for plant parameter identification the implementation problems are not as severe since the model presumably could be designed so that its parameters could be directly manipulated.

Two important design considerations have been discussed in preceding chapters. The first is the importance of the adjustment algorithm gains on the error response. As illustrated in example 1 of chapter 3, these gains have profound influence on the adaptive response. This was discussed by demonstrating that a least upper bound on L could be reduced by properly choosing the adaptive loop gains. The examples in this chapter demonstrate this observation for non trivial cases. As discussed in connection with (2.2.15) these gains serve also as constraints to keep the parameter changes small enough so that higher order terms do not affect the adaptive performance. The examples of chapter 3 illustrate that the least upper bound on L could be minimized by choosing the adaptive gains large. That analysis however assumed that higher order terms were negligible. The effects of making the loop gains so large that these higher order terms are important was not included as part of the analytic treatment. By means of computer simulation the contribution of higher order terms are investigated in this chapter. It will be seen that the gains cannot be chosen arbitrarily large without decreasing performance. The fact remains nevertheless that these gains can be chosen to reduce the worst possible error performance bound.

The second point is that all of the unknown parameters need not be compensated to achieve adaptation. In the parameter adjustment scheme it would be far toocomplex to implement an adaptive loop for each plant parameter. The effect on error response when a simplified parameter adjustment scheme is used is also investigated by means of simulation.

4.2 Simulation of Parameter Adjustment System

This example is taken from an article in (19). The equations represent two-degree of freedom longitudinal airframe dynamics.

$$\ddot{\theta} = - p_1\dot{\theta} - p_2N_p + K_p(r+u)$$
$$N_p = V(\dot{\theta}-\dot{\alpha}) \qquad (4.2.1)$$

The variables θ,α and r+u are aircraft pitch angle, angle of attack and elevator position respectively.

N_p is the normal acceleration at the airframe center of gravity and V is the aircraft velocity. The plant parameters p_1, p_2 and K_p

result from linearizing the true equations of motion. These parameters are functions of dynamic pressure; as aircraft velocity changes the parameter change. From actual flight test data p_1,p_2 and K_p were measured as a function of velocity. Figure 4.1 shows this dependence. The only parameter which varies significantly over the range of velocity shown is K_p. The control objective is to constrain the plant to behave as a stationary model while the aircraft decelerates from 294 ft/sec (200mph) to 147 ft/sec (100mph). As discussed in (19), p_1 is nearly zero over the flight path and is not important enough to compensate for.

The model equations describing desired behavior are

$$\ddot{m} = -a_1\dot{m} - a_2N_m + K_m \cdot r$$
$$N_m = V(\dot{m}-\dot{\alpha}) \qquad (4.2.2)$$

The error is defined as the difference between model and plant pitch angle

$$e = m - \theta \qquad (4.2.3)$$

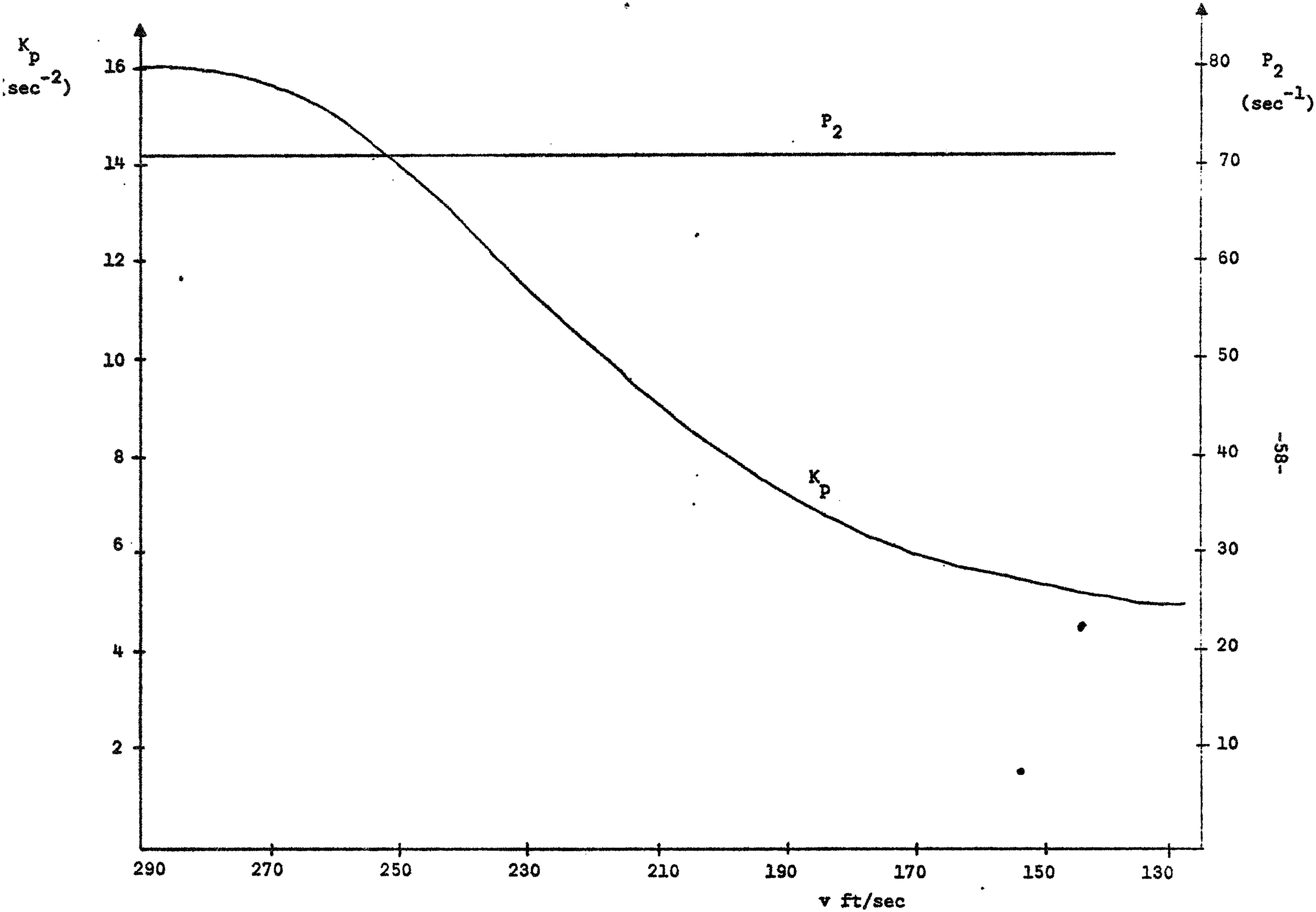
(4.1.1), (4.1.2) are used to obtain

$$\ddot{e}=\ddot{m}-\ddot{\theta}= - a_1(\dot{e}+ \dot{\theta})- a_3V[\dot{e}+ \dot{\theta}-\dot{\alpha}] + K_m \; +p_1\dot{\theta} + p_2N_p - K_pr$$
$$\ddot{e} = -[a_1+a_2V]\dot{e} + [p_2-a_2]N_p + [K_m-K_p]r \qquad (4.2.4)$$

By letting $e_1 = e$, (4.2.4) can be expressed in phase variable form as

$$\dot{e}_1 = e_2$$
$$\dot{e}_2 = -[a_1+a_2V]e_2+[p_2-a_2]N_p+(K_m-K_p)r \qquad (4.2.5)$$

If direct manipulation of p_2 and K_p were possible, (4.2.5) would be



K_p
(sec^{-2})
16
14
12
10
8
6
4
2
P_2
K_p
290
270
250
230
210
190
170
150
130
v ft/sec
80
70
60
50
40
30
20
10
P_2
(sec^{-1})

Figure 4.1

an adequate representation of the error system for implementation purposes. More realistically, it can only be assumed that normal acceleration and pitch rate can be sensed. It is this information with the elevator position that must be used to implement the adaptive controller. By feeding back N_p through a compensator p_c and by varying elevator position by a factor K_c, the adaptive controller can be implemented as in figure 4.2.

The error equations describing figure 4.2 follow directly from (4.1.5)

$$\dot{e}_1 = e_2$$
$$\dot{e}_2 = -[a_1+a_2V]e_2+[p_cK_p+p_2-a_2]N_p+[K_m-K_pK_c]r \qquad (4.2.6)$$

e_1 is the error between model and plant pitch angle and e_2 is the error in pitch rate. For purposes of this example an error measure was chosen as

$$L = 1/2(e_1^2 + 2e_1e_2+e_2^2) \qquad (4.2.7)$$

No investigation was made of the effects on error response caused by different weighting of the error and error rate in (4.2.7)

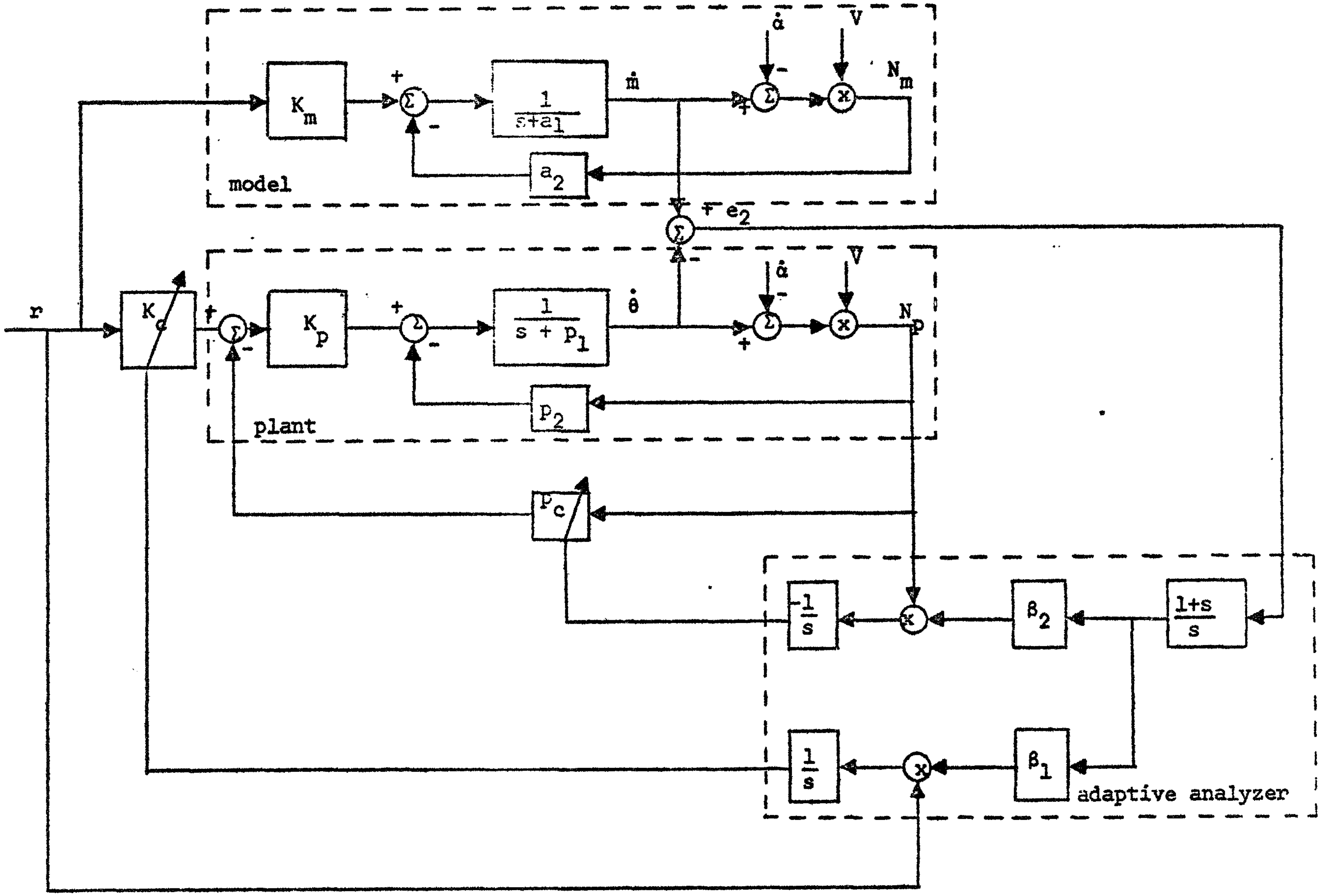
Defining

$$\delta_2 = p_2 - a_2 - p_cK_p$$
$$\Delta = K_m - K_pK_c \qquad (4.2.8)$$

the adjustment algorithm from (2.3.5) and (4.2.8) is

$$\dot{\delta}_2 = - \dot{p}_cK_p - p_c\dot{K}_p + \dot{p}_2 = - \beta_2\cdot N_p\cdot(e_1+e_2) \qquad (4.2.9a)$$

$$\dot{\Delta} = -K_p\dot{K}_c - K_c\dot{K}_p = -\beta_1\cdot r\cdot(e_1+e_2) \qquad (4.2.9b)$$


model
plant
adaptive analyzer
r
K_m
K_c
K_p
$\frac{1}{s+a_1}$
$\frac{1}{s + p_1}$
a_2
p_2
p_c
$\dot{m}$
$\dot{\theta}$
$\dot{\alpha}$
V
N_m
N_p
e_2
β_1
β_2
$\frac{1+s}{s}$
$\frac{1}{s}$
$-\frac{1}{s}$

$\dot{K}_p$ and $\dot{p}_2$ are unknown so implementation of (4.2.9) can only be approximated. It is through simulation that the validity of these assumptions are verified. Since the actual adjustments are $\dot{p}_c$ and $\dot{K}_p$, the adaptive loop gains will be scaled by the terms $1/K_p$. Nevertheless, the choice of β_1, β_2 in (4.2.9) is seen to be very influential on the error response.

The scheme used for achieving adaptation is shown in figure 4.2. By changing K_c and p_c according to (4.2.9), the products K_cK_p and p_cp_2 compensate for changes in the plant parameters. The actual values used in simulation were a_1=.1, a_2=-50, and K_m=10 for the model and p_1=.1, p_2=-67, $\dot{\alpha}$=0* and

$$V = 294 - \int_0^t 10dt \quad \text{ft/sec} \tag{4.2.10}$$

$$K_p = 12 \exp \left(\frac{V-294}{10}\right) + 4$$

for the plant. The input was

$$r = \sin(t) + 5t \tag{4.2.11}$$

To investigate the effects of varying the adaptive gains on the adaptive response a figure of merit was assigned to the system. This was the integral of L over the flight path between speeds of 294 and 147 ft/sec.

$$I = 1/2 \int_{294}^{147} (e_1^2 + 2e_1e_2 + e_2^2)\, dv \tag{4.2.12}$$

To facilitate scaling of figure 4.3, rather than plotting I as a function of the adaptive gains,

$$I' = -1/\log_{10} I \tag{4.2.13}$$

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* As shown in (19) $\dot{\alpha}$ can be synthesized from measurable variables, however for purposes of illustration it is zero for this simulation.

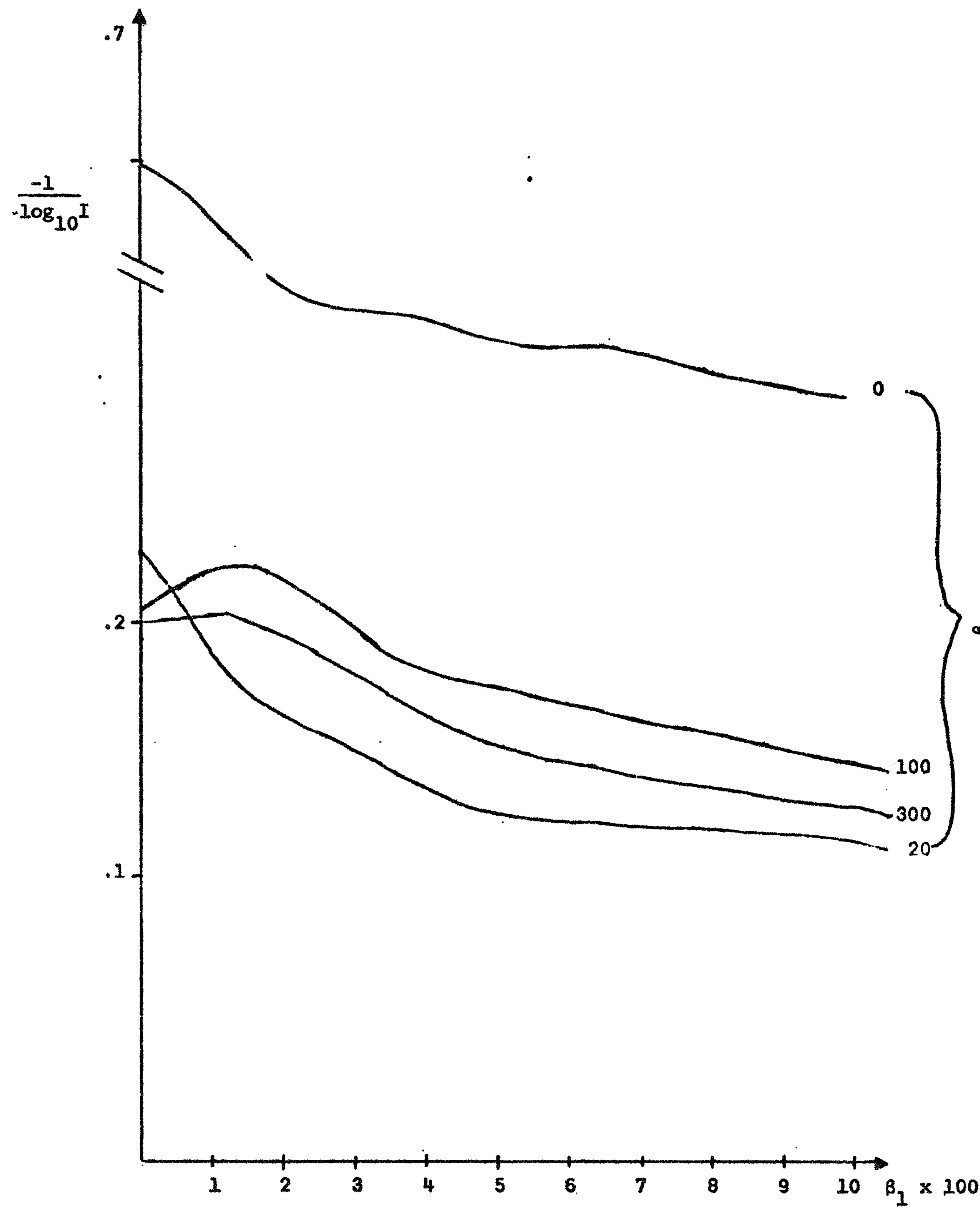


Figure 4.3

was plotted. For the unconpensated system $\beta_1=0$, $\beta_2=0$, $I = 5x10^{-2}$. The best adaptive performance was achieved at $\beta_1=1000$, $\beta_2=20$ resulting in $I = 7.5x10^{-8}$. The analysis of chapter 3 demonstrated that L could be kept very small by properly choosing these gains. It was observed that the larger these gains the smaller the upper bound on L; however, because of higher order effects this result is not completely valid. As shown in figure 4.3 increasing β_2 above 20 caused I to increase for all values of β_1. Figure 4.4 shows L as a function of velocity for two cases. Case I is for $\beta_2=20$, $\beta_1=50$ case II is for $\beta_2=20$, $\beta_1=1000$. The maximum L for case I is 8×10^{-6} whereas the maximum L was held to 7×10^{-8} by increasing β_1. These data were more conveniently scaled by plotting

$$L' = -1/\log_{10}L \qquad (4.2.14)$$

versus velocity in each case.

The effects of adjusting only one parameter were also investigated. Referring to figure 4.3, the minimum I for $\beta_2=0$ is at $\beta_1=1000$. For $\beta_1=0$ the best response was obtained with $\beta_2=300$.

4.3 Example 4.2

This example is the same as example 2 of chapter 1. It is used to illustrate how state variable feedback can be used to implement the adaptive controller. The dynamics of the plant are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -p_1x_1 - p_2x_2 + K_p[r+u] \qquad (4.3.1)$$

The p's and the plant gain are unknown but constant. To provide a means for adjusting parameters state variable feedback is used

$$u = u(x,c) = -[c_1x_1 + c_2x_2] \qquad (4.3.2)$$

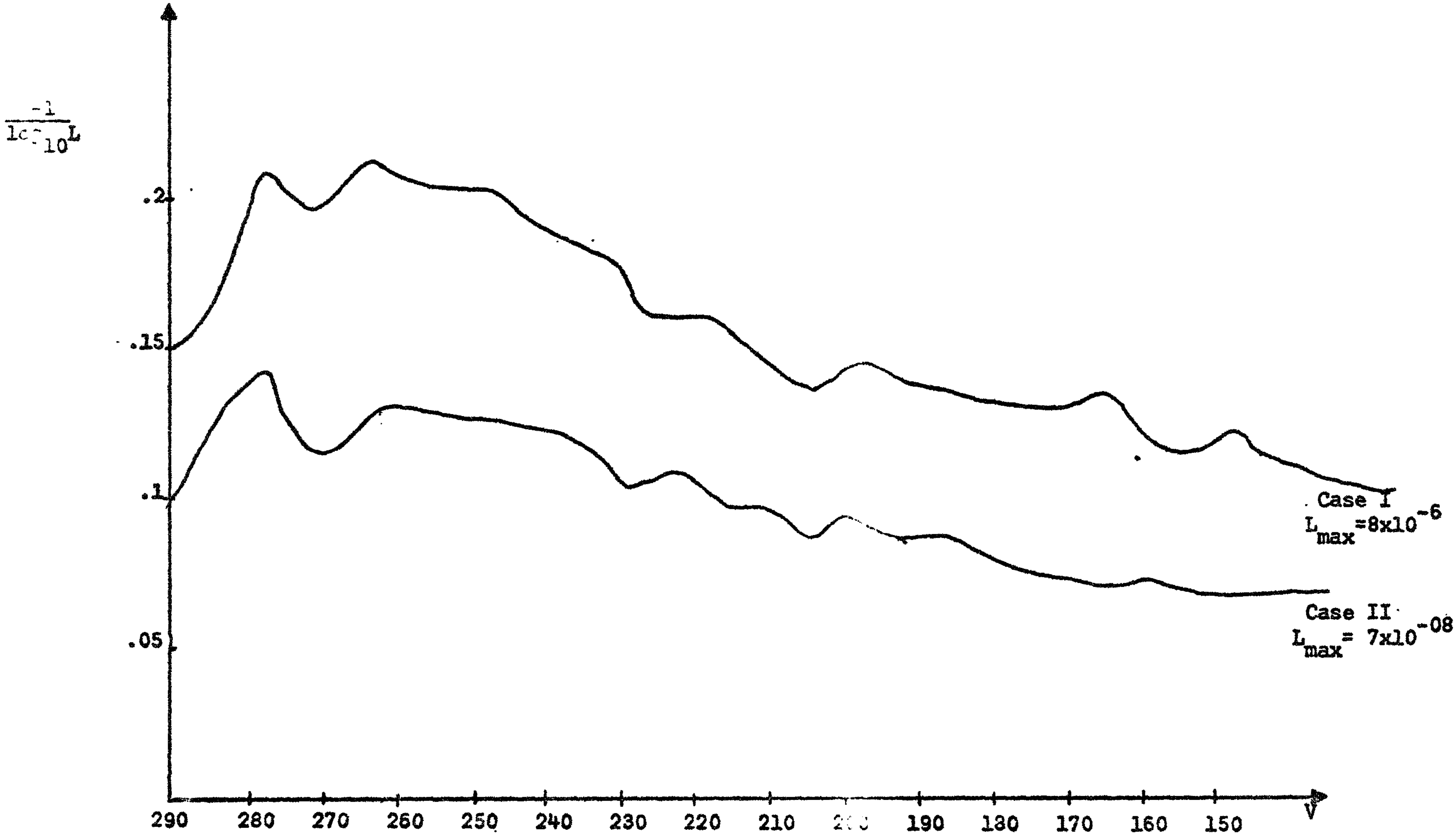

.2
.15
.1
.05
290
280
270
260
250
240
230
220
210
190
180
170
160
150
V
Case I
Lmax=8x10-6
Case II
Lmax= 7x10-08

Figure 4.4

A prefilter K_c is also used to allow for gain compensation. The model equations are

$$\dot{m}_1 = m_2$$
$$\dot{m}_2 = -a_2 m_2 - a_1 m_1 + K_m r \qquad (4.3.3)$$

A block diagram of the system showing how the prefilter and phase variable compensation are used is shown in figure 4.5.

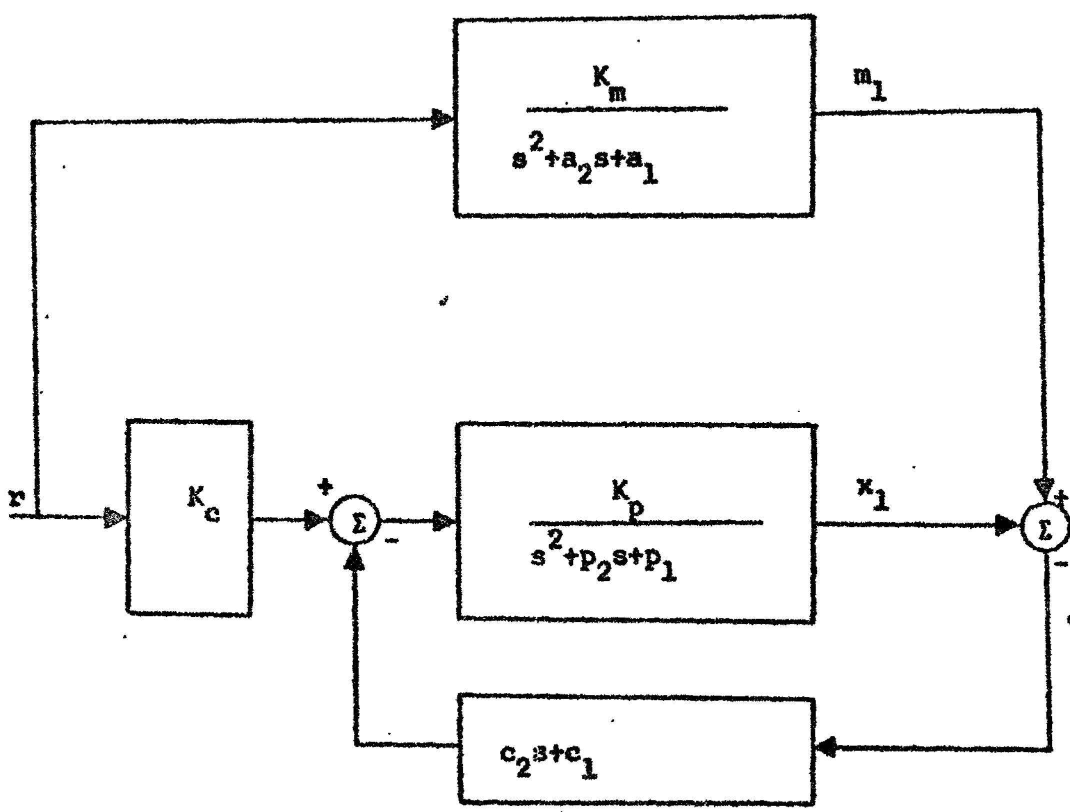


Figure 4.5

Adaptation is achieved by adjusting c_1, c_2 and K_c. The error equations are

$$\dot{e}_1 = e_2$$
$$\dot{e}_2 = -a_2e_2 - a_1e_1 - \delta_1x_1 - \delta_2x_2 + \Delta r \qquad (4.3.4)$$

where

$$\delta_1 = a_1 + p_1 + K_pc_1$$
$$\delta_2 = a_2 + p_2 + K_pc_2 \qquad (4.3.5)$$
$$\Delta = K_m - K_pK_c$$

choosing an error measure

$$L = 1/2\ (e_1^2 + 2e_1e_2 + e_2^2) \qquad (4.3.6)$$

results in the adjustment equations

$$\dot{c}_1 = \frac{\dot{\delta}_1}{K_p} = -\frac{\beta_1}{K_p}\ x_1 \cdot (e_1 + e_2)$$
$$\dot{c}_2 = \dot{\delta}_2/K_p = -(\beta_2/K_p)x_2 \cdot (e_1 + e_2) \qquad (4.3.7)$$
$$\dot{K}_c = \dot{\Delta}/K_p = +\gamma r\ (e_1 + e_2)$$

It is possible to use state variable feedback for compensation only because the plant and model were available in phase variable form. This problem becomes considerable more complicated when the plant states are not measurable. One important situation where this arises is illustrated by the system (4.3.1) where only the plant observation variable

$$y = x_2 + z_1x_1 \qquad (4.3.8)$$

is available. If z_1 is known it is a simple matter to recover x_1, x_2. If z_1 is unknown then the adaptive system cannot be implemented directly because the plant states cannot be measured. As shown in

Appendix B, if the system (4.3.1) (of any order) having unknown but constant parameters is expressed using phase variables when the observation equations can be written as

$$\dot{y} = \dot{y}_1 = y_2$$

$$\dot{y}_2 = -p_2y_2-p_1y_1 + K_pc_1(r+u)+K_p(\dot{u}+\dot{r}) \qquad (4.3.9)$$

A model is then chosen as

$$\dot{m}_1 = m_2$$

$$\dot{m}_2 = -a_2m_2-a_1m_1 + K_md_1r \quad +K_m\dot{r} \qquad (4.3.10)$$

Defining $e_1 = m_1-y_1$, the error equations can be manipulated to give

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = -a_2e_2-a_1e_1 + \delta_2y_2 + \Delta_1[d_1r + \dot{r}] + \Delta_2[rK_p]$$

$$-K_pc_1 \, u-K_p\dot{u} \qquad (4.3.11)$$

with

$$\delta_1 = p_1-a_1$$

$$\delta_2 = p_2-a_2$$

$$\Delta_1 = K_m-K_p \qquad (4.3.12)$$

$$\Delta_2 = d_1-c_1$$

If Γ is chosen as in (4.3.6) the parameter adjustments are

$$\dot{\delta}_1 = -\beta_1y_1(e_1+e_2)$$

$$\dot{\delta}_2 = -\beta_2y_2(e_1 + e_2)$$

$$\dot{\Delta}_1 = -\gamma_1[d_1r+\dot{r}](e_1+e_2) \qquad (4.3.13)$$

$$\dot{\Delta}_2 = -\gamma_2[rK_p](e_1+e_2)$$

Because K_p is unknown in (4.3.13), the effort is to produce some uncertainty in γ_2. Since γ_2 can be arbitrary positive function of time this presents no problem. The adaptive controller implementation for this system using parameter adjustments is shown in figure 4.6.

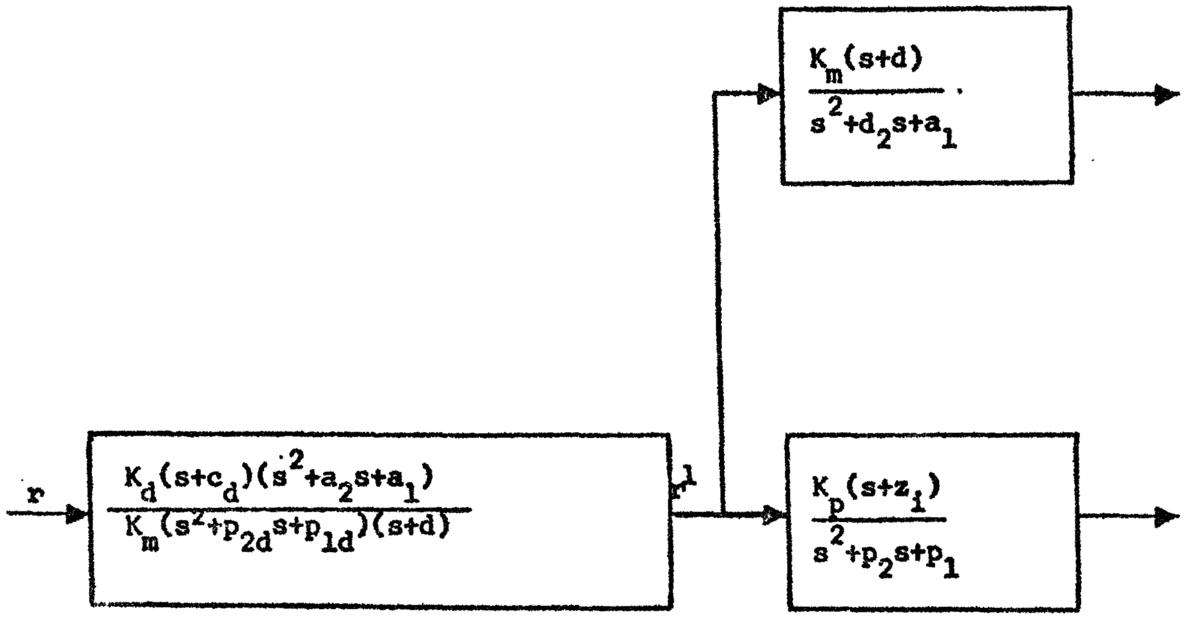


Figure 4.6

The model parameters are adjusted to identify the plant parameters. By adjusting the corresponding model parameters in the prefilter, the complete plant transfer is just the desired transfer

$$\frac{K_d(s+c_d)}{s^2+p_{2d}s+p_{1d}} \tag{4.3.14}$$

From (4.3.13), (4.3.12) it follows that the model parameters should be adjusted as

$$\begin{aligned}
\dot{a}_1 &= +\beta_1 y_1(e_1+e_2) \\
\dot{a}_2 &= +\beta_2 y_2(e_1+e_2) \\
\dot{K}_m &= -\gamma_1[d_1 r+\dot{r}](e_1+e_2) \\
\dot{d}_1 &= -\gamma_2[rK_p](e_1+e_2)
\end{aligned} \qquad (4.3.15)$$

Adjustment of model parameters does not present the problem inherent to the plant parameter adjustment scheme. Although the implementation of figure 4.6 appears complicated, there are no plant parameter adjustments needed. Because of the unknown plant zero state variable feedback cannot be used to implement the parameter adjustments. The system however need not be as complicated as shown. Since the reason adaptation is used is to keep the error small, there is no need to compensate for all the unknown plant parameters. This goal can be achieved by compensating for only the most important plant parameters. The most important ones have the greatest effect on error response. For the system considered here acceptable performance could be achieved by compensating for only zero and gain variations. There is no possibility of identifying plant parameters under these circumstances; however, since the tracking error will be kept small the model must be an adequate representation of the plant for the particular input. This adequate representation is then used in the prefilter to compensate the plant and maintain the overall transfer function close to that desired.

4.4 Control Signal Augmentation for Plants with Zeros

One other alternative to using the parameter adjustment scheme of Figure 4.6 is to use control signal augmentation. The error equations (4.3.11) show that because of the plant zero the error is controlled by the control u and its derivative. Denoting $u=u_1$, $u=u_2$, (4.3.11) can be treated as if there were two control inputs. Applying 2.3.6 will then give

$$\dot{u}_1 = \frac{\gamma_1}{K_p}(e_1 + e_2) \qquad (4.4.1)$$

$$\dot{u}_2 = \frac{\gamma_2}{K_p c_1}(e_1 + \dot{e}_2)$$

Since $\dot{u}_1 = u_2$ both relations cannot be satisfied. This problem is overcome by considering a different control variable,

$$U = c_1 u + \dot{u} \qquad (4.4.2)$$

then application of (2.3.6) results in

$$\dot{U} = \frac{\gamma}{K_p}(e_1 + e_2) \qquad (4.4.3)$$

From (4.4.2), (4.4.3);

$$\ddot{u} + c_1 \dot{u} = \frac{\gamma}{K_p}(e_1 + e_2) \qquad (4.4.4)$$

(4.4.4) cannot be implemented exactly since c_1 is the unknown plant zero. However, by using the model zero in (4.4.4) performance will not be appreciably affected if the model and plant zeros are close.

This observation can be argued heuristically by using a different implementation of (4.4.4). Recall that U was chosen on the basis that it made one term in the expansion of the integral cost function always negative. This same result can be obtained by synthezising u according to

$$\dot{U} = \frac{\gamma}{K_p}\ \mathrm{sgn}(e_1 + e_2) \qquad (4.4.5)$$

This will result in

$$\ddot{u} + d_1 \dot{u} = \frac{\gamma}{K_p}\ \mathrm{sgn}\ (e_1 + e_2) \qquad (4.4.6)$$

as the implementation of the control sugnal augmentation technique where the model zero has been used.

Using (4.4.6) the effects of the model zero in the implementation can be qualitatively studied. It has been argued that the adaptive loop gain can be any positive function of time. For a switch occuring in the control law (4.4.6) at t=ts, the realizable control derivative is given by the solution to (4.4.6). The ideal relation is given by this same solution evaluated for the plant zero. Denoting the realizable control as u_R and the ideal by u_I

$$\dot{u}_R = \frac{\gamma}{K_p d_1}\left[1-e^{-d_1(t-t_s)}\right] + \dot{u}_R(t_s)e^{-d_1(t-t_s)} \tag{4.4.7}$$

$$\dot{u}_I = \frac{\gamma}{K_p c_1}\left[1-e^{-c_1(t-t_s)}\right] + \dot{u}_I(t_s)e^{-c_1(t-t_s)} \tag{4.4.8}$$

If $\dot{u}_R = f(t)\dot{u}_I$ with $f(t) > 0$ then implementing 4.4.7 corresponds to implementing 4.4.8 with γ replaced by some positive function of time. Under these circumstances the adaptive performance should not be significantly affected by using the model zero.

Solving (4.4.7) and (4.4.8) for f(t):

$$f(t) = \frac{\frac{\gamma}{K_p d_1}\left(1-e^{-d_1(t-t_s)}\right) + \dot{u}_R(t_s)e^{-d_1(t-t_s)}}{\frac{\gamma}{K_p c_1}\left(1-e^{-c_1(t-t_s)}\right) + \dot{u}_I(t_s)e^{-c_1(t-t_s)}} \tag{4.4.9}$$

From (4.4.9) it is clear that f(t) cannot be guaranteed always positive. However as $t-t_s$, the time between switching gets large f(t) is positive and approaches a positive constant. If the time between switchings is long the system will operate as if the plant zero were used for implementation. As the error becomes small $\dot{u}_R$ will change sign more rapidly and f(t) may be negative. Under these conditions the error may diverge and as it becomes large the time between switchings will be longer and f(t) will again be positive. This heuristic argument implies a bound on the error caused by using the model zero instead of the plant zero in the adaptive controller.

4.5 Simulation of Adaptive Controller for a Plant with a Zero

This example uses the equations for pitch axis motion of a high altitude aircraft. The plant has been used extensively for illustration purposes in the literature (8,14). As described below it illustrates a simplifed implementation of the parameter adjustment technique using only gain compensation.

The plant is described by

$$G(s) = \frac{x(s)}{R(s)} = K_p \left[\frac{s+z_1}{s^2+p_2 s+p_1}\right] \tag{4.5.1}$$

and an underdamped model is used for purposes of illustration

$$F(s) = \frac{M(s)}{R(s)} = \frac{K_m(s+d_1)}{s^2+a_2 s+a_1} = \frac{5(s+3)}{s^2+s+10} \tag{4.5.2}$$

The variation of plant parameters with Mach Number and altitude are given in (8). For one particular plant state these parameters are $K_p = 14.04$, $z_1=.955$, $p_2=1.886$ and $p_1=6.61$. The configuration is shown in figure 4.7.

K_c is adjusted according to (4.3.12), (4.3.13) as

$$\dot{K}_c = \frac{\gamma_1}{K_p}(d_1 r+\dot{r})(e_1+e_2) \tag{4.5.3}$$

where L is chosen as in (4.3.6).

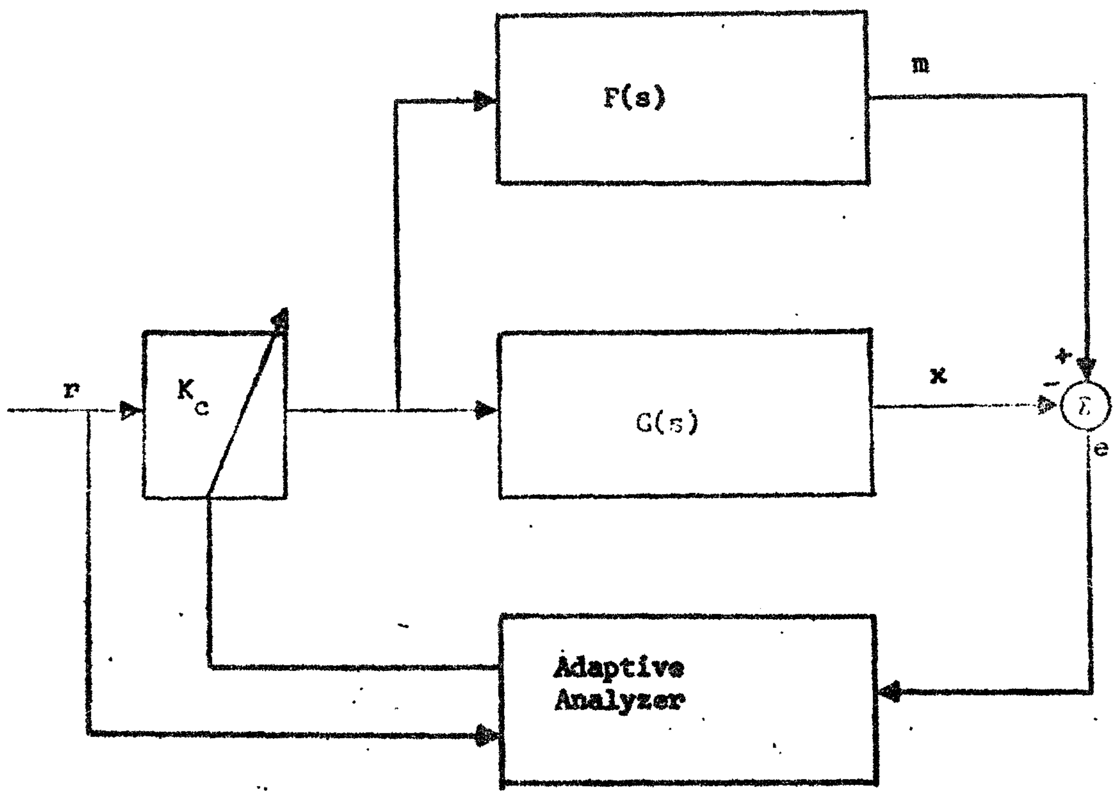
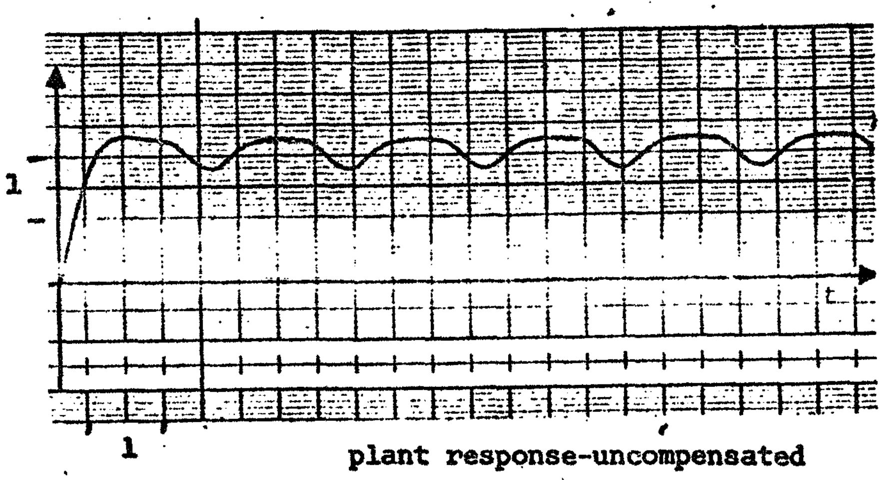


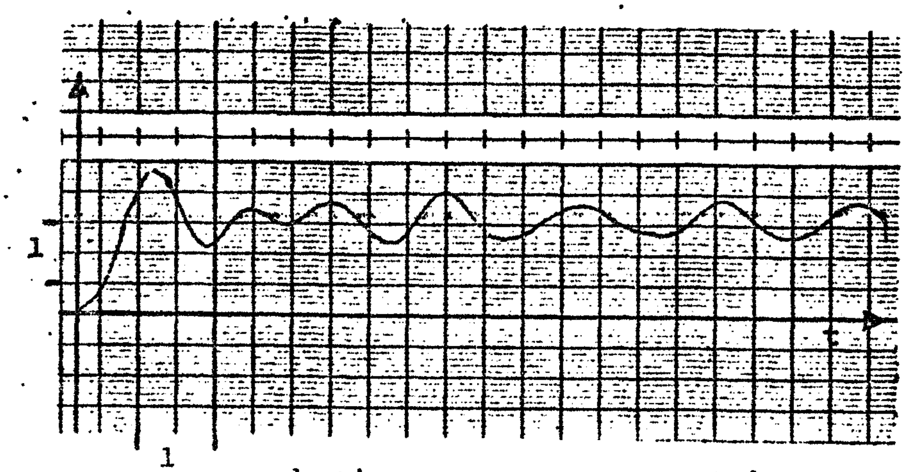
Figure 4.7

The uncontrolled error response and plant output y are shown in figure 4.8. Figure 4.9 shows the adaptive response and error response for the gain compensation scheme. These responses occurred subject to the input shown in Figure 4.10. Additional runs were made for smaller values of γ and the results were as predicated: the error performance could be improved by increasing the adaptive gain. These results were obtained in a PACE 231 R Analog Computer.

The same system was simulated on a Digital Computer (IBM 360) to examine the ideal but unrealistic situation where all model parameters were adjusted and the prefilter contained the compensation of figure 4.6. To evaluate the adaptive error response as separate from the identification error response the system shown in figure 4,11 was

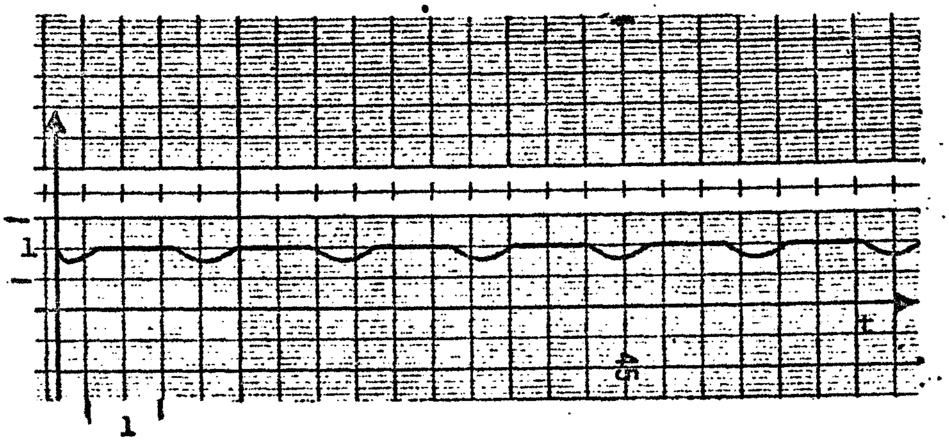


plant response-uncompensated



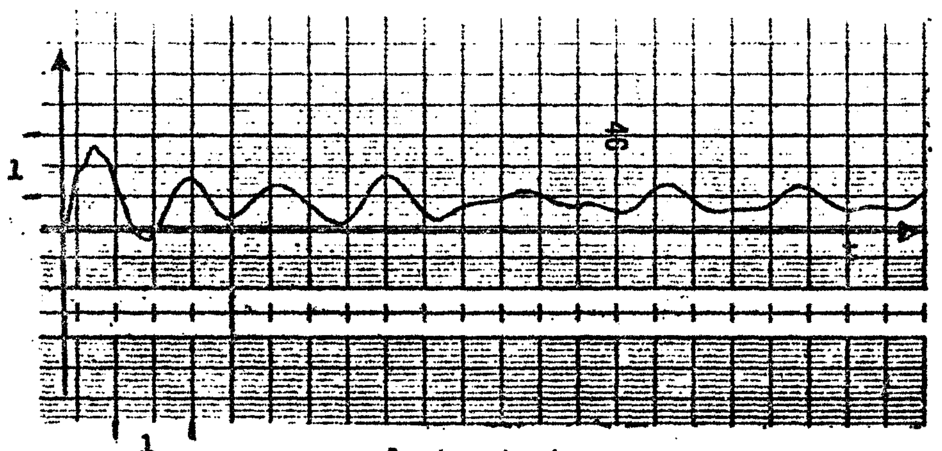
adaptive error - uncompensated

Figure 4.8

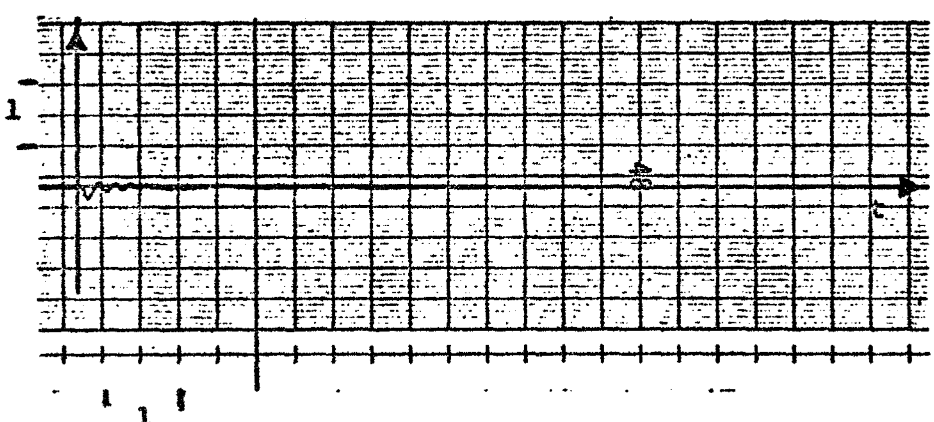


plant input

Figure 4.10



plant output



error response

Figure 4.9

actually used.

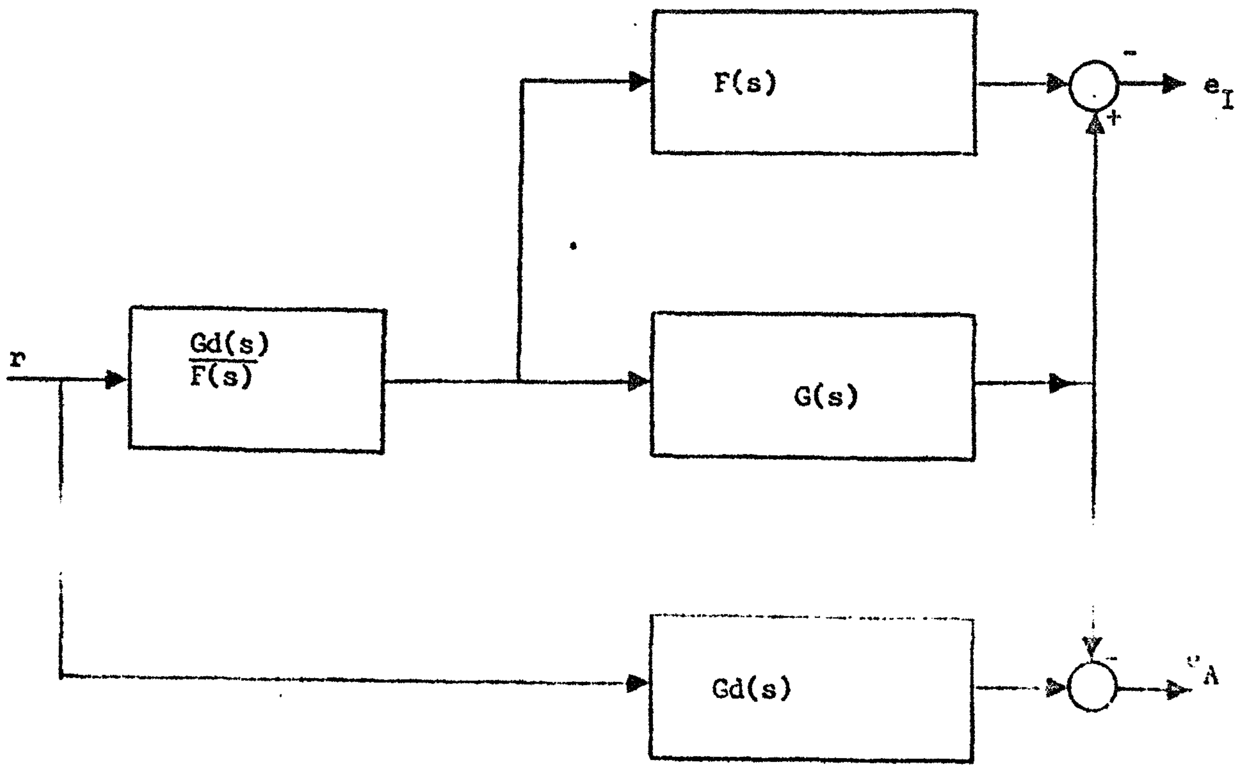


Figure 4.11

The parameters of F(s) in the model and prefilter were adjusted to minimize e_I and it was demonstrated that e_A under these conditions was kept almost identically zero. This was accomplished despite the fact that convergence of the model parameters to the value of the plant parameters was rather slow.

4.6 Simulation of Control Augmentation Technique

This last example was chosen to illustrate an application of the control signal augmentation technique. The simplicity of this configuration makes it particularly appealing. The plant is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - p_2x_2 - p_1x_1 + K_p(r+u) \qquad (4.5.4)$$

$$r = 5 \sin(3t)$$

The plant parameters were chosen for simulation purposes to be

$$p_1 = 2 + e^{-.1t}$$

$$p_2 = 5-2e^{-.2t} \qquad (4.5.5)$$

$$K_p = 5 + e^{-.05t} \sin(t)$$

The control chosen in accordance with (2.3.6) is

$$\dot{u} = \gamma\ (e_1 + e_2) \qquad (4.5.6)$$

but the alternate form as discussed in section 4.5 was found to give slightly better results. The control signal was generated from

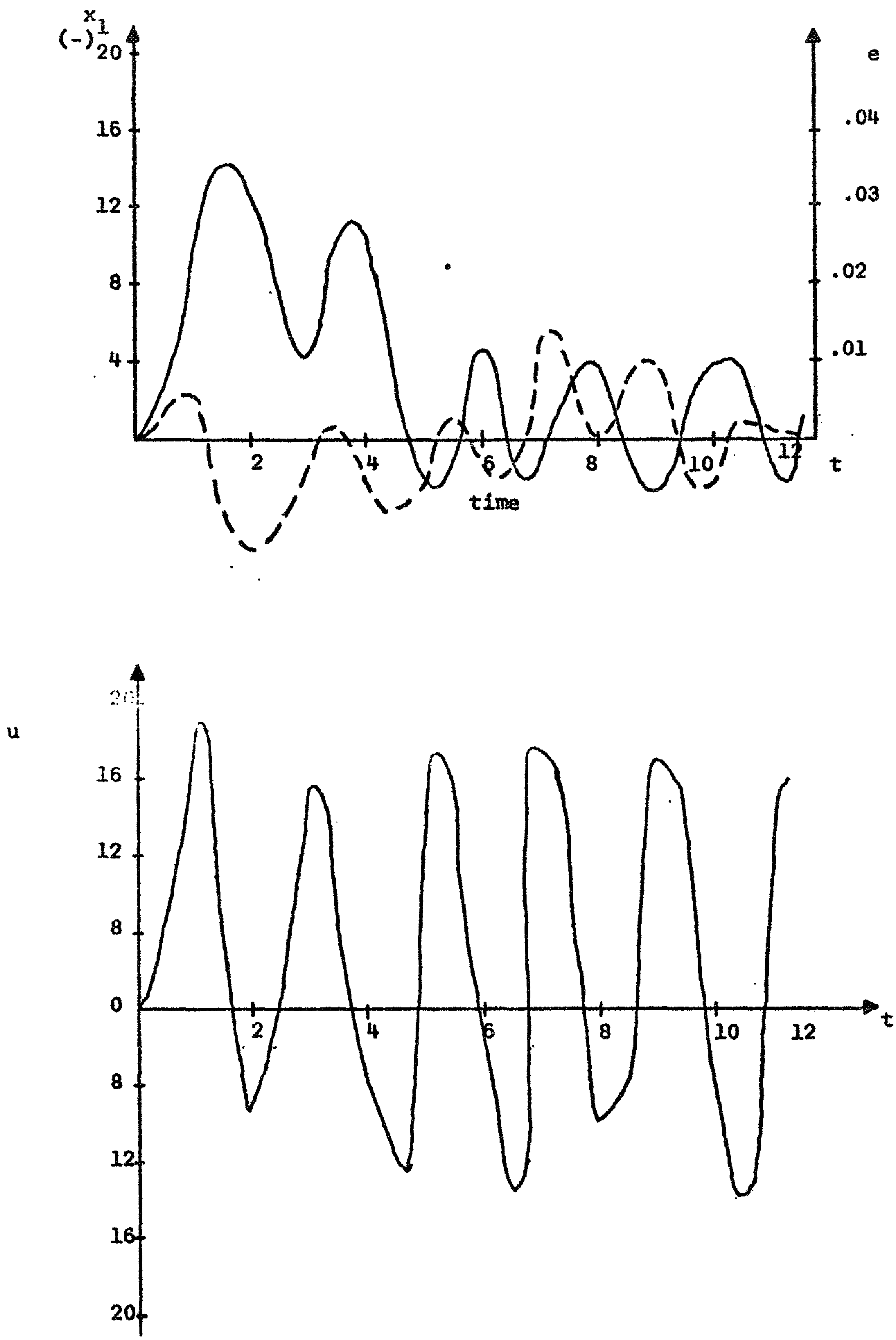
$$\dot{u} = 200 \text{ sgn } (e_1 + e_2) \qquad (4.5.7)$$

A model

$$\dot{m}_1 = m_2$$

$$\dot{m}_2 = - a_2m_2-a_1m_1 + K_mr \qquad (4.5.8)$$

was used with a_2=1, a_1=.5, and K_m = 10. The plant and error response are shown in figure 4.12. The major drawback to this control augmentation scheme is the excessively large control effort required.



x_1
(-)
20
16
12
8
4
e
.04
.03
.02
.01
2
4
6
8
10
12
t
time
u
16
12
8
0
8
12
16
20
2
4
6
8
10
12
t

Figure 4,12

V AREAS FOR FURTHER INVESTIGATION

5.1 Introduction

Perhaps the most important subject for further investigation is the effect of noise on the adaptive process. A general treatment of this important problem was reported in (15) but the results do not apply to the model reference system as developed in this thesis.

Because the model would generally be a computer simulation of desired performance it can be isolated from any noise effects other than input disturbances. The plant is subject to other distrubances, however, state measurements may be noise contaminated. If the model is subject to the same input disturbance as the plant the adaptive system should work as developed since all that is assumed is that the same input is applied to the model and the plant. If the model is chosen to have disturbance rejection properties, and if the adaptive system works in spite of contaminated measurements, the plant will track the model and assume these same rejection properties.

The two important questions are then (1) what effect do noise contaminated plant state measurements have on the adaptive process; (2) what effect does a disturbance not sensed by the model have on the error response. Even for linear plants the parameter adjustment adaptive control algorithm leads to a non linear closed loop system. For this reason an analysis of the scheme in the presence of noise is untractable. The difficulty arises basically because to fully specify the statistical behavior of the error requires knowledge of the error covariance matrix in addition to the mean value. If the error process is not Gaussian this may not even be adequate. For the parameter adjustment scheme the complexity of the analysis is prohibitive.

Some insight into the effects of noise on the control signal augmentation scheme can be gained by making appropriate assumptions. The conclusions reached might then be used as a basis to begin a study of the parameter adjustment scheme in the presence of noise. For illustration a first order plant which has an unknown but constant pole and gain is chosen. The plant state equation is

$$\dot{x} = -px + K_p(r+u) \tag{5.1.1}$$

Desired performance is described by

$$\dot{m} = -a_m + K_m r \tag{5.1.2}$$

As shown in figure 5.1 the plant state is corrupted by noise and the input is the desired input plus noise. It is assumed that both n_1 and n_2 are Gaussian white noise processes with mean values

$$\bar{n}_1 = \bar{n}_2 = 0 \tag{5.1.3}$$

and variance

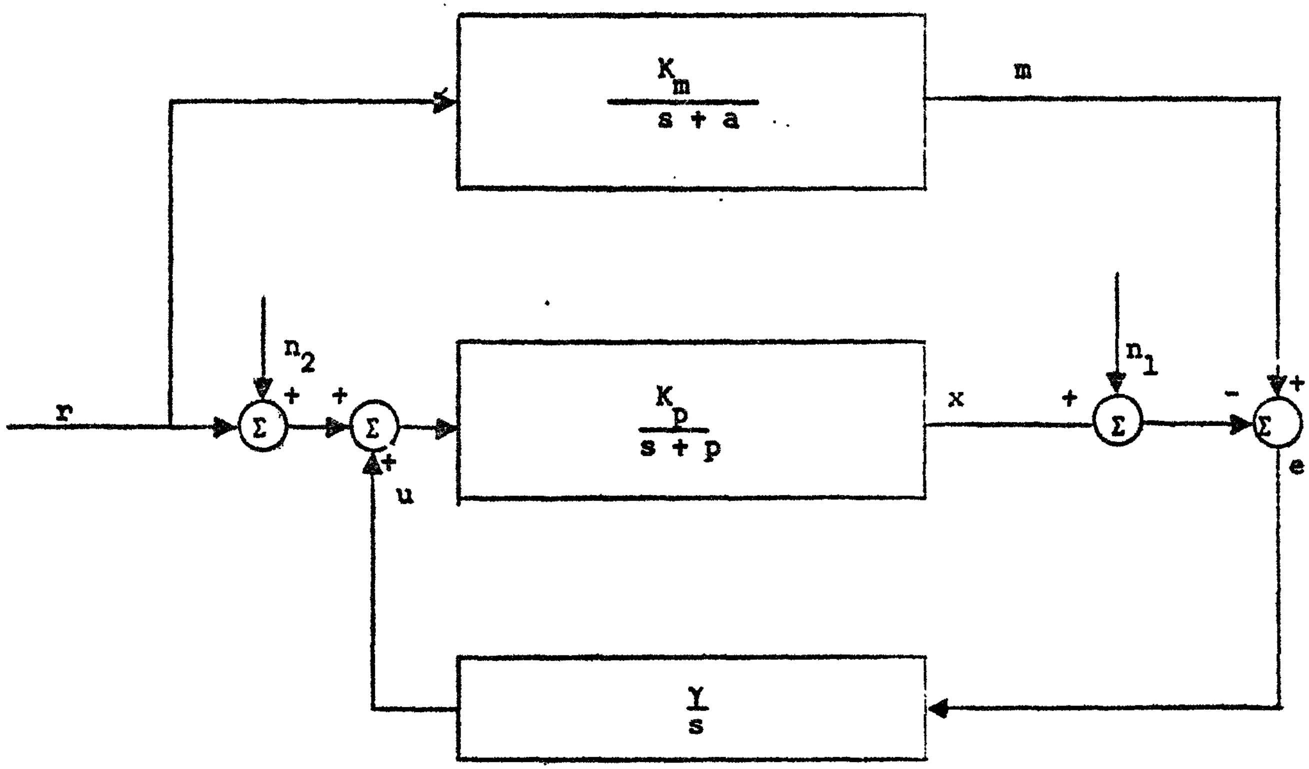
$$\overline{n_i(\alpha)n_j(\beta)} = N_{ij}\delta_{ij}(\alpha-\beta) \tag{5.1.3}$$

δ_{ij} is the delta function

$$\int_{-\infty}^{+\infty} \delta_{ij}(s)ds = \int_{0-\varepsilon}^{0+\varepsilon} \delta_{ij}(s)ds = 1 \tag{5.1.4}$$

The corrupted plant state and the plant input are then

$$x_n = x + n_1$$
$$r_n = r + n_2 \tag{5.1.5}$$



$\frac{K_m}{s + a}$
m
r
n_2
+
+
Σ
Σ
+
u
$\frac{K_p}{s + p}$
x
n_1
+
Σ
-
+
Σ
e
$\frac{\gamma}{s}$

Figure 5.1

The closed loop behavior of x will be defined by the two disturbances, the uncorrupted plant input and the appropriate impulse responses.

$$x(t) = \int_0^t h_1(t,\tau)n_1(\tau)d\tau + \int_0^t h_2(t,\tau)n_2(\tau)d\tau + \int_0^t h_2(t,\tau)r(\tau)d\tau \tag{5.1.6}$$

The impulses responses $h_1(t,)$, $h_2(t,)$ can not be determined exactly because the plant parameters are unknown and may be time varying. Despite this the expected value of x can be found as

$$\tilde{x}(t) = \int_0^t h_2(t,\tau)r(\tau)d\tau \tag{5.1.7}$$

which is the noise free expected value.

With the assumption that the plant output is Gaussian it is only necessary to determine the variance to completely specify the statistics of x. Denoting the expected value operator by Ex:

$$\begin{aligned}
\overline{x(t)x(s)} = \mathrm{Ex}\Big\{ & \int_0^t \int_0^s h_1(t,\tau)h_1(s,\alpha)n_1(\tau)n_1(\alpha)d\tau d\alpha \\
& + \int_0^t \int_0^s h_2(t,\tau)h_2(s,\alpha)n_2(\tau)n_2(\alpha)d\tau d\alpha \\
& + \int_0^t \int_0^s h_2(t,\tau)h_2(s,\alpha)r(\tau)r(\alpha)d\tau d\alpha \\
& + 2\int_0^t \int_0^s h_1(t,\tau)h_2(s,\alpha)n_1(\tau)n_2(\alpha)d\tau d\alpha \\
& + 2\int_0^t \int_0^s h_1(t,\tau)h_2(s,\alpha)n_1(\tau)r(\alpha)d\tau d\alpha \\
& + 2\int_0^t \int_0^s h_2(t,\tau)h_2(s,\alpha)n_2(\tau)r(\alpha)d\tau d\alpha\Big\}
\end{aligned} \tag{5.1.8}$$

Because n_1, n_2 and T are mutually independent (5.1.8) reduces to

$$\overline{x(t)x(s)} = \mathrm{Ex}\left\{ \int_0^t \int_0^s h_1(t,\tau)h_1(s,\alpha)n_1(\tau)n_1(\alpha)d\tau d\alpha \right.$$

$$+ \int_0^t \int_0^s h_2(t,\tau)h_2(s,\alpha)n_2(\tau)n_2(\alpha)d\tau d\alpha \tag{5.1.9}$$

$$\left. + \int_0^t \int_0^s h_2(t,\tau)h_2(s,\alpha)r(\tau)r(\alpha)d\tau d\alpha \right\}$$

By averaging inside the integral the variance can be found as:

$$\overline{x(t)^2} = \int_0^t h_1(t,\tau)^2 N_1^2 d\tau + \int_0^t h_2(t,\tau)^2 N_2^2 d\tau \tag{5.1.10}$$

The last term in (5.1.9) represents the mean of x squared and for this reason has been omitted from the variance expression. The impulse responses $h_1(t,\tau)h_2(t,\tau)$ cannot be found but if the frozen system concept is used these functions can be approximated by the stationary impulse response functions $h_1(t-\tau)h_2(t-\tau)$ associated with the system having the nominal plant parameters. The variance is then

$$\overline{x(t)^2} = N_1^2 \int_0^t h_1(t-\tau)^2 d\tau + N_2^2 \int h_2(t-\tau)^2 d\tau \tag{5.1.11}$$

Making the substitution $t-\tau=\beta$ in (5.1.11) results in

$$\overline{x(t)^2} = N_1^2 \int_0^t h_1(\beta)^2 d\beta + N_2^2 \int_0^t h_2(\beta)^2 d\beta \tag{5.1.12}$$

For the frozen system the plant output variance becomes stationary as $t \to \infty$ and

$$\overline{x(t)^2} = \overline{x^2} = N_1^2 \int_0^\infty h_1(\beta)^2 d\beta + N_2^2 \int_0^\infty h_2(\beta)^2 d\beta \qquad (5.1.13)$$

which is bounded for h_1, h_2 having low pass frequency characteristics.

The error statistics follow directly from the above results. Since $e = m - x$ the contribution due to noise in the error comes exclusively from x. Therefore the expected value of the error caused by noise is the negative of the expected value of x.

$$\bar{e} = \bar{x} = 0 \qquad (5.1.14)$$

Similarly, the contribution of noise to the error variance is the variance of x.

$$\overline{e^2} = \overline{x^2} = N_1^2 \int^\infty h_1(\beta)^2 d\beta + N_2^2 \int^\infty h_2(\beta)^2 d\beta \qquad (5.1.15)$$

The contribution to e not caused by noise is the error response for the uncontaminated system.

For the example considered here it follows from figure 5.1 that

$$h_1(\beta) = \left[\frac{GP}{1+GP}\right] = h_2(\beta) * p(\beta)$$

$$h_2(\beta) = \left[\frac{G}{1+GP}\right] \qquad (5.1.16)$$

where script L denotes the Laplace Transformation, and

$$p(\beta) = \gamma \qquad (5.1.17)$$

From (5.1.16) and (5.1.17) it then follows that

$$h_1(\beta) = \int_0^\infty \gamma h_2(\beta - \phi) d\phi \qquad (5.1.18)$$

Substituting (5.1.18) into (5.1.15):

$$\overline{e^2} = N_1^2\gamma^2 \int_0^\infty \left[\int_0^\infty h_2(\beta-\phi)d\phi \right]^2 d\beta + N_2^2 \int_0^\infty h_2(\beta)^2 d\beta \qquad (5.1.19)$$

For this simple example it is obvious that the adaptive loop gain γ should be chosen as small as possible to reduce the contribution of measurement noise to the error variance. Since the bound on error for the uncontaminated system has been shown to be inversely proportional to γ there will be a trade off required between minimizing the deterministic error bound and minimizing the error variance caused by noise.

5.2 Illustration of Noise Effects on Simple Parameter Adjustment Scheme

Because of the nonlinear nature of the parameter adjustment adaptive system an analysis like that above is not possible. However, some insight into the statistical error behavior can be gained by considering the linearized form of the nonlinear controller.

For the plant in (5.1.1)

$$\dot{x} = p_x + K_p r \qquad (5.2.1)$$

the gain adjustment algorithm can be implemented as shown in figure 5.2. It is assumed that the plant pole is known. The purpose of adaptation is then to compensate for the unknown but constant plant gain K_p. The closed loop plant differential equation is

$$\dot{x} = -p\, x + K_p K_c r \qquad (5.2.2)$$

with

$$K_p \dot{K}_c = \beta r(m-x) \qquad (5.2.3)$$

If r is a step $\tau=R_0u(t)$ then the system in the absence of noise will reach equilibrium at

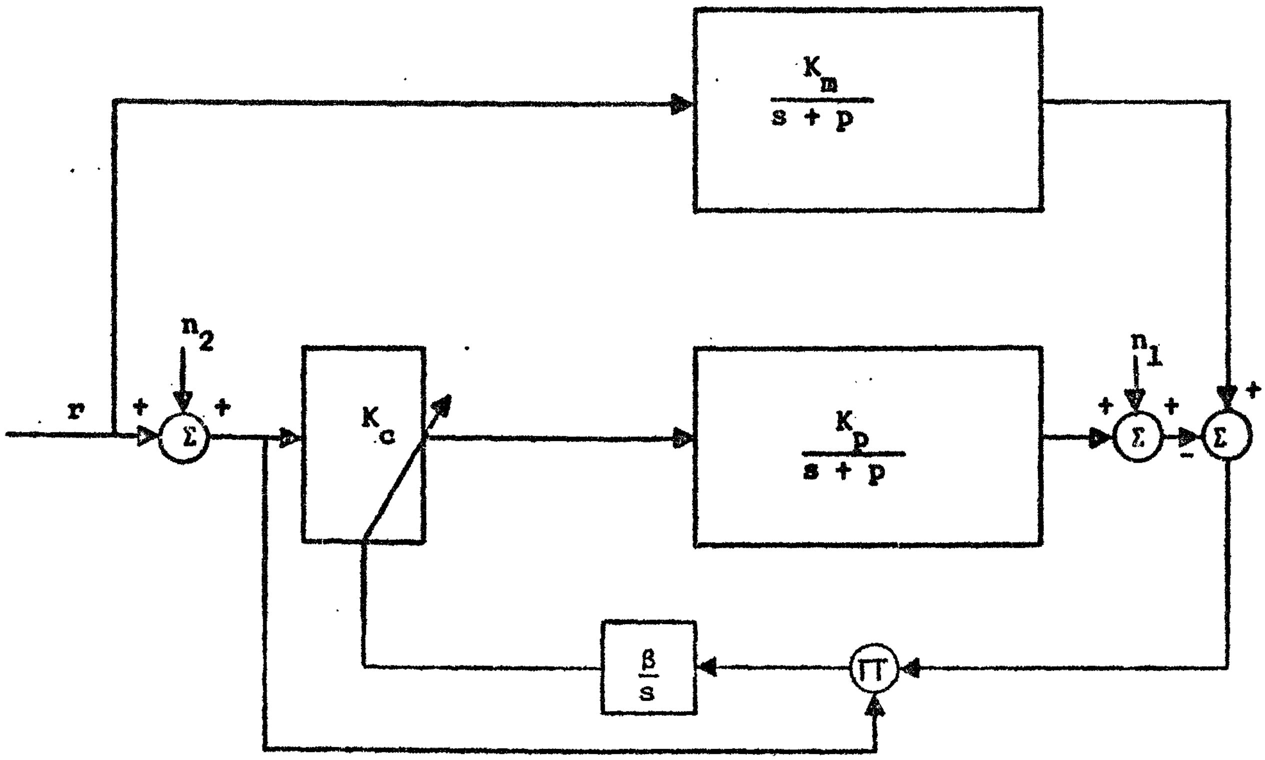


Figure 5.2

$$x_0 = \frac{K_p K_c R_0}{p} \qquad (5.2.4)$$

Any change in x from x_0 will be caused by noise contamination of the plant state and noise affecting the plant but not the model, n_1 and n_2 respectively in figure 5.2. This change in x from x_0 is the solution to the linearized form of (5.2.2).

$$\Delta\dot{x} = -p\Delta x + (K_p K_c)_0 \Delta r + (K_p r)_0 \Delta K_c \qquad (5.2.5)$$

similarly, from (5.2.3), the change $\Delta K_c K_p$ can be expressed as the solution to

$$K_p \Delta\dot{K}_c = -\beta[R_0 \Delta x + x_0 \Delta r] \qquad (5.2.6)$$

Combining (5.2.5), (5.2.6)

$$\Delta\dot{x}(t) = -p\Delta x + K_m \Delta r - \beta K_p R_0 \int_0^t [R_0 \Delta x(\tau) + x_0 \Delta r(\tau)]d\tau \qquad (5.2.7)$$

where $k_m=(K_pK_c)_0$ has been used. The solution to (5.2.7) gives the change in x from the steady state step response, x_0. Any change in r from R_0 is just n_2, and any change in x from x_0 is $\Delta x+n_1$. Replacing Δx by $\Delta x + n_1$ and Δr by n_2 in (5.2.7) and integrating gives

$$\Delta x(t) = \int_0^t g(t-\alpha)\ [K_m n_2(\alpha)-\beta K_p R_0 \int_0^\alpha [R_0[\Delta x(\tau) + n_1(\tau)]+x_0 n_2(\tau)]d\tau]d\alpha \tag{5.2.8}$$

where $g(t-\alpha)$ is the impulse response defined by (5.2.2).

As $t\to\infty$ $x(t)$ becomes stationary and its mean value is

$$\overline{\Delta x(t)} = \overline{\Delta x} = \int_0^\infty g(t-\tau)K_m\overline{n_2(\alpha)}\ d\alpha-\beta K_p R_0 \int_0^\infty \int_0^\infty g(t-\alpha)[R_0[\overline{\Delta x(\tau)}+\overline{n_1(\tau)}) + x_0 n_2(\tau)]d\tau d\alpha \tag{5.2.9}$$

n_1, n_2 are white Gaussian noise processes whose statistics are given in (5.1.3). Since they both have zero mean and with $\overline{\Delta x(t)} = \overline{\Delta x}$ (5.2.9) can be expressed as

$$\overline{\Delta x}\ [1 + \beta K_p R_0^2 \int_0^\infty \int_0^\infty g(t-\alpha)d\alpha d\tau]=0 \tag{5.2.10}$$

From (5.2.10) it follows that

$$\overline{\Delta x} = 0 \tag{5.2.11}$$

The mean value of the perturbation in x caused by noise is zero. Similarily

$$\overline{\Delta e} = 0 \tag{5.2.12}$$

Since the nominal value of error is zero, the error in the presence of noise is zero mean.

The variance could be found starting from (5.2.7). However, it is simpler for this linearlized system to work with the transfer function representations. Taking the Laplace Transform of both sides of (5.2.7):

$$\Delta x(s) = K_m G(s) N_2(s) - \beta K_p R_0 \frac{G(s)}{s} [R_0 \Delta x(s) + R_0 N_1(s) + x_0 N_2(s)] \qquad (5.2.13)$$

G(s) is the transform of g(t) and $N_1(s)$, $N_2(s)$ are transforms of signals entering at n_1, n_2 respectively in figure 5.2. Noting that $G(s) = \frac{1}{s+p}$, (5.2.13) can be expressed as

$$\Delta x(s) = N_2(s) \cdot \frac{[K_m s - \beta K_p R_0 x_0]}{s^2+ps+\beta K_p R_0^2} - N_1(s) \cdot \frac{[\beta K_p R_0^2]}{s^2+ps+\beta K_p R_0^2} \qquad (5.2.14)$$

And from (5.2.4), $x_0 = (K_p K_c)_0 R_0/p = K_m R_0/p$ which gives

$$\Delta x(s) = N_2(s) \frac{K_m[s - \frac{\beta K_p R_0^2}{p}]}{s^2+p2+\beta K_p R_0^2} - N_1(s) \cdot \frac{[\beta K_p R_0^2}{s^2+ps+\beta K_p R_0^2} \qquad (5.2.15)$$

Under the assumption of stationarity, the variance of the change in x can be found using standard techniques. For purposes of illustration $p=R_0=1$ then denoting βK_p as α, (5.2.15) becomes

$$\Delta x(s) = N_2(s) \frac{K_m[s-\alpha]}{s^2+s+\alpha} - \frac{N_1(s)\alpha}{s^2+s+\alpha} \qquad (5.2.16)$$

The variance is the result of the complex integration

$$\overline{\Delta x^2} = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \Delta x(s) \Delta x(-s) ds = J_1 + J_2 \qquad (5.2.17)$$

which can be evaluated using standard tabulated forms (16) as

$$J_1 = \frac{N_2^2 K_m^2}{2}(\alpha+1)$$

$$J_2 = \frac{N_1^2 \alpha}{2} \qquad (5.2.18)$$

Recalling that $\alpha=\beta K_p$, minimizing the variance requires $\beta=0$ in which case the variance is simply

$$\overline{\Delta x^2} = \frac{N_2^2 K_m}{2} \tag{5.2.19}$$

Reasoning as in the control augmentation analysis, the error variance is the same as the plant variance.

To recapitulate, it was assumed that subject to a constant input the adaptive system would null the error caused by different values of model and plant gain. If the deviation from this equilibrium caused by noise is stationary, it will be zero mean with finite variance. The variance can however be minimized by setting the adaptive gain to zero.

To be meaningful the results obtained above must be extrapolated to the situation where adaptation is not achieved before the system is corrupted by noise. If the error is small between plant and model for a step input, then to maintain the error close to zero despite noise, the adaptive loop gain should be as small as possible. Setting the adaptive gain to zero would make adaptation impossible; therefore the gain should be made only as small as results in suitable performance. This reasoning cannot be extended to the more general case where the error between plant and model is large in the presence of noise. To do so would violate the linearizing assumption inherent in (5.2.5).

5.3 Summary of Areas for Additional Investigation

The illustrations in this chapter were not intended to be a conclusive examination of noise effects on system performance. Rather they serve to point up the importance of noise contamination on the adaptive system.

In addition to an extension and generalization of the preceding analysis, fruitful areas for further investigation are

1) Redesign of the adaptive loop to guarantee an error bound as in Chapter 3 but to also afford more control over the noise induced error bound.

2) Investigation of alternative error criterion.

3) Design of the control augmentation technique to limit the plant input. Knowing the saturation limit of the plant, a relay configuration can be used to form the control derivative. Rather than integrating this to obtain the control as done in Example 4.2, a low pass filter could be used which would effectively duplicate the required integration but would at the same time provide a saturation limit on the total control effort.

APPENDIX A

If:

1) The homogeneous error equations are linear and stable in error terms with constant coefficients.
2) The unknown parameter difference vector and the control vector enter linearly into the equations.
3) The scale factors β and γ in (2.3.5) and (2.3.6) are constant.
4) The parameters are constant

then the parameter adjustment algorithm is the same as the method proposed by Butchard and Shackcloth (3) and Parks (17).

To show this the error equations are written as

$$\dot{e}_i = g_i(e,\delta,u,rt) = \sum_{j=1}^{N} a_j e_j + h_i(x,\delta,u,t) \tag{A1}$$

or equivalently as

$$\dot{e} = Ae + h\,(x,\delta,u,t) \tag{A2}$$

in vector form. By choosing a positive definite function

$$V = e^t Le + \delta^t R\delta + \mu^t su \tag{A3}$$

as a possible Lyapunov function, $\dot{V}$ can be formed as

$$\dot{V} = e^t[A^tL+LA]e + 2[h^tLe] + \dot{\delta}^t R\delta + \delta^t R\dot{\delta} + \dot{u}^t su + u^t s\dot{u} \tag{A4}$$

According to (2.3.5) and (2.3.6)

$$\dot{\delta}_k - \beta_k \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j \frac{\partial g_i}{\partial \delta_k} \quad k=1,K \tag{A5}$$

$$\dot{u}_k = -\gamma_k \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j \frac{\partial g_i}{\partial u_k} \quad k=1,J \tag{A6}$$

If R and S are diagonal the (A3) can be expressed as

$$\dot{V} = e^t[A^t L+LA]e+2 \left| \sum_{k=1}^{N} \delta_k \dot{\delta}_k r_{kk} + \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j h_i + \sum_{k=1}^{J} u_k \dot{u}_k s_{kk} \right. \tag{A7}$$

Using assumption 2 above the h's can be written

$$h_i = \sum_{k=1}^{N} h'_{ik} \delta_k + \sum_{k=1}^{J} \overline{h_{ik}} u_k \tag{A8}$$

From (A8) and (A1) it follows that

$$\frac{\partial h_i}{\partial \delta_k} = h'_{ik} = \frac{\partial g_i}{\partial \delta_k} \qquad k=1,2,\ldots K,\ i=1,2,\ldots N$$

$$\frac{\partial h_i}{\partial u_k} = \overline{h_{ik}} = \frac{\partial g_i}{\partial u_k} \quad k,\ 1,2,\ldots J,\ i=1,2,\ldots N \tag{A9}$$

and choosing

$$\beta_k = \frac{1}{r_{kk}} \quad k=1,2,\ldots K \tag{A10}$$

$$\gamma_k = \frac{1}{s_{kk}} \quad k=1,2,\ldots J$$

(A5) and (A6) are

$$\dot{\delta}_k = -\frac{1}{r_{kk}} \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j h'_{ik} \quad k=1,2,\ldots K \tag{A11a}$$

$$\dot{u}_k = -\frac{1}{s_{kk}} \sum_{i=1}^{N} \sum_{j=1}^{N} \ell_{ij} e_j \overline{h_{ik}} \quad k=1,2,\ldots J \tag{A11b}$$

Substituting (A11a) and (A11b) into (A7) results in

$$\dot{V} = e^t Qe \tag{A12}$$

with

$$Q = A^t L + LA \tag{A13}$$

positive definite. (10)

V is a positive definite function of δ and e but $\dot{V}$ is a function of e only and is thus only a positive semidefinite function of e and δ. Whether or not $\dot{V}$ is a Lyapunov function is a moot point. The important fact is that the parameter adjustment algorithm cannot guarantee $\delta \to 0$. This is not surprising since the more straight forward variational approach presented shows that the error measure cannot be a function of δ. Doing so would require the value of δ to implement the adjustment equations; since they would appear in $\partial H/\partial \delta$ in (2.2.15).

Because the Lyapunov method is conservative, the error measure must also satisfy (A13).

APPENDIX B

This Appendix shows that if the only measurable output of a linear system is a linear combination of the states weighted by unknown constants the adjustment algorithms may still be applied using only this output and its derivatives. This result is for plants described in phasevariable form.

For a system in phase variable form with unknown but constant parameters

$$\begin{aligned} \dot{x}_i &= x_{i+1} \quad i=1,2,\ldots N-1 \\ \dot{x}_N &= \sum_{k=1}^{N} p_k x_k + K_p r \end{aligned} \tag{B1}$$

the only measurement available is the observation

$$y = \sum_{j=1}^{N} z_j x_j \tag{B2}$$

with constant but unknown z_j's. The adjustment algorithm as proposed in Chapter 4 can be applied since the state equations describing (B1), (B2) are

$$\begin{aligned} \dot{y} &= \dot{y}_1 = y_2 \\ \dot{y}_i &= y_{i+1} \quad i=2,3,\ldots N-1 \\ \dot{y}_n &= \sum_{k=1}^{N} p_k y_k = K_p \left| \sum_{j=1}^{N} z_j \frac{d^{j-1}}{dt^{j-1}} r \right| \end{aligned} \tag{B3}$$

This follows from letting

$$y_1 = y = \sum_{j=1}^{N} z_j x_j = \sum_{j=1}^{N} z_j \frac{d^{j-1}}{dt^{j-1}} x_1 \tag{B4}$$

then

$$\dot{y}_1 = y_2 = \frac{d}{dt}\sum_{j=1}^{N} z_j x_j = \sum_{j=1}^{N} z_j \frac{d^{j-1}}{dt^{j-1}} x_2$$

in general

$$\dot{y}_i = y_{i+1} = \sum_{J=1}^{N} z_j \frac{d^i}{dt^i} x_j = \sum_{j=1}^{N} z_j \frac{d^{j-1}}{dt^{j-1}} x_{i+1} \qquad \text{(B5)}$$

and

$$\dot{y}_n = \frac{d}{dt} y_n = \sum_{j=1}^{N} z_j \frac{d}{dt} \left| \frac{d^{j-1}}{dt^{j-1}} x_n \right| = \sum_{j=1}^{N} z_j \frac{d^j}{dt^j} x_n \qquad \text{(B6)}$$

From (B1)

$$\frac{d^j}{dt} x_n = \frac{d^{j-1}}{dt} \left| \sum_{k=1}^{N} p_k x_k + K_p \right| \qquad \text{(B7)}$$

Hence

$$\dot{y}_n = \sum_{j=1}^{N} z_j \frac{d^j}{dt^j} x_n = \sum_{j=1}^{N} z_j \left| \sum_{k=1}^{N} p_k \frac{d^{j-1}}{dt^{j-1}} x_k + k_p \frac{d^{j-1}}{dt^{j-1}} r \right| \qquad \text{(B8)}$$

Factoring out p_k:

$$\dot{y}_n = \sum_{k=1}^{N} p_k \left| \sum_{j=1}^{N} \frac{d^{j-1}}{dt^{j-1}} x_k z_j \right| + k_p \sum_{j=1}^{N} z_j \frac{d^{j-1}}{dt^{j-1}} r \qquad \text{(B9)}$$

$$= \sum_{k=1}^{N} p_k y_k + K_p \sum_{j=1}^{N} z_j \frac{d^{j-1}}{dt^{j-1}} r$$

which verifies (B3)

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