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EFFECT OF AXIAL CONSTRAINTS ON BEAMS SUBJECTED TO LATERAL LOADS

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National Aeronautics and Space Administration

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by

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THE CITY COLLEGE RESEARCH FOUNDATION

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NOTATION

A	cross sectional area
EI	flexural rigidity of beam
L	length of beam
$r = \sqrt{I/A}$	radius of gyration of cross sectional area
P_E	Euler buckling load
S	axial constraint force
w	intensity of load/length
x	longitudinal coordinate
y	lateral displacement
$\alpha = L/r$	slenderness ratio
$\rho = S/P_E$	measure of axial constraint force
$\eta = y/L$	non-dimensional lateral displacement
$(\ \ )'$	derivative with respect to argument; e.g. $y' = dy(x)/dx$

1. INTRODUCTION.

The dynamic response of various structural elements - both beams and plates - subjected to sonic boom shock waves, has been the subject of several investigations [1]-[5] in recent years. A problem of considerable practical importance, related to these studies, has been the damage of glass window panes caused by sonic booms. In attempts to describe analytically the response of panes to sonic boom shocks, the dynamic response of rectangular and circular elastic plates as well as prismatic beams under various support conditions has been studied.

In these previous studies, a problem of some difficulty has been encountered in describing accurately and realistically the mounting of window panes by means of suitable models. Indeed, such attempts have been made by considering the application of in-plane compressive loads in addition to lateral loads caused by N-waves of sonic booms [5]; these studies however have necessitated an assumption on the in-plane loads.

It is recognized that the actual mountings of window panes may be of a nature that unknown in-plane tensile constraint forces may arise, thus attenuating the lateral deflection and moments in the member. Where such constraint forces have been neglected, the response, obtained from linear solutions, gives upper bounds to the desired quantities. However, such solutions are sometimes unrealistic since they may predict larger deflections and resulting stresses than may, in reality, exist.

Since the constraint forces, which are naturally unknown quantities, arise due to the stretching of the member, the introduction of such forces gives rise to a non-linear problem which, in general, is of much greater complexity than the corresponding linear problem, particularly in the investigation of dynamic responses.

The present investigation is therefore an attempt to study the effects of in-plane constraints by considering first the simplest corresponding static problem: viz. a simply-supported elastic beam subjected to lateral loads, whose ends are constrained such that no relative longitudinal motion is permitted. It is hoped that the results obtained will give a first quantitative measure to the order of the constraint force and attenuation in the corresponding dynamic case.

It should be noted that several attempts have been made to predict the induced axial forces of constraint in a pin-connected simply-supported beam. Certain expressions, given by Timoshenko [6], have been obtained by approximating the deflection of the beam by either a parabola or the fundamental mode of deflection. However, the results presented are only approximate relations between the constraint force and the maximum deflection of the constrained beam. Furthermore, from the relations, as given, it is apparently impossible to determine the response without knowing one of the desired quantities a priori.

(It should be noted, however, that the results presented in this report are in agreement with the relations given in [6].) Switzky [7] gives more general results for a partially restrained beam but his solution is again obtained by assuming a deflection curve as the first fundamental mode.

It is believed that the study presented herein is the first analysis in which accurate quantitative results are obtained, according to Euler-Bernoulli beam theory without having a need to resort to any approximation of the deflected curve.

The results (which are limited to beams undergoing small rotations) show, as expected, that increasingly large axial constraint forces arise with high load terms, thereby considerably reducing the lateral displacements and moments in the beam. The results, presented by means of universal curves, may be applied to beams consistent with the above theory. The method employed may further be used to obtain the solutions for any lateral loads w(x).

2. AXIALLY CONSTRAINED BEAM SUBJECTED TO LATERAL LOADS.

The problem considered is the static response of a simply-supported elastic beam, pinned at both ends, $x = 0$ and L, and subjected to a uniform lateral load w, [Fig. 1a]. The beam is supported in such a way that no relative longitudinal motion of the ends may occur. Lateral displacements $y(x)$ of the beam due to the application of lateral loads therefore give rise to axial forces $S[y(x)]$, [Fig. 1b].

In the subsequent analysis, the axial force produced is assumed to be constant throughout the beam. The force $S[y]$ may then be considered as a functional of y.

The governing equation for elastic beams, assuming small rotations, is

$$EIy^{IV} - Sy'' = w \tag{2.1}$$

while the associated boundary conditions are

$$y(o) = y''(0) = y(L) = y''(L) = 0 \tag{2.2}$$

Further, the auxiliary boundary condition specifying the constraint against relative longitudinal motion of the ends is

$$S = \frac{AE}{2L} \int_0^L [y'(x)]^2 \, dx. \tag{2.3}$$

The non-linearity in the problem is thus seen to enter through this condition.

Introducing a non-dimensional coordinate $\xi = x/L$, the governing equation and boundary conditions on the non-dimensional displacement $\eta(\xi) = y(x)/L$ may be written as

$$\eta^{IV} - \gamma^2 \eta'' = \frac{wL\alpha^2}{EA} \qquad (2.4)$$

and

$$\eta(0) = \eta''(0) = \eta(1) = \eta''(1) = 0 \qquad (2.5)$$

where

$$\gamma^2 = \frac{\alpha^2 S}{EA} = \frac{\alpha^2}{2}\int_0^1 [\eta'(\xi)]^2 d\xi \qquad (2.6)$$

The quantity $\alpha = L/r$ is defined as the "slenderness ratio" of the beam.

For a given γ, the solution of (2.4) satisfying (2.5), can be written

$$\eta(\xi) = \left(\frac{wL\alpha^2}{EA}\right)\frac{1}{\gamma^4}\left[\cosh\gamma\xi + \left(\frac{1-\cosh\gamma}{\sinh\gamma}\right)\sinh\gamma\xi + \frac{\gamma^2\xi}{2}(1-\xi) - 1\right] \qquad (2.7)$$

Substitution of (2.7) in (2.6) and performing the indicated integration yields the following transcendental equation on the axial force

$$\gamma^2 = \left(\frac{wL\alpha^2}{EA}\right)^2 \frac{\alpha^2}{2\gamma^7}\left[\frac{5\sinh\gamma - \gamma}{\cosh\gamma + 1} + \frac{\gamma^3}{12} - 2\gamma\right] \qquad (2.8)$$

The roots of the above equation may be determined readily by an iteration scheme yielding a sequence of roots $\{\gamma_0, \gamma_1, \gamma_2 \ldots \gamma_n\}$

which, for a wide range of application, converges rapidly. (*)

The above iteration scheme requires a choice of starting values; a natural and convenient choice is the solution obtained for the corresponding unconstrained beam, viz.

$$\eta_0(\xi) = \frac{wL\alpha^2}{24EA}\,\xi(\xi^3 - 2\xi^2 + 1), \quad \gamma_0 = 0 \qquad (2.9a)$$

which upon substitution in (2. 6) gives

$$\gamma_1^2 = \frac{17\alpha^2}{(70)(24)^2}\left(\frac{wL\alpha^2}{EA}\right)^2 \qquad (2.9b)$$

Aside from the axial force and displacement, it is of interest to determine the effect of the constraint upon the maximum moment occuring at the center of the span.

Defining μ as the ratio of the constrained to unconstrained center moment, i. e.

$$\mu = \frac{M(1/2)}{wL^2/8}, \qquad (2.10)$$

and noting that

$$M(\xi) = -\frac{EI}{L}\eta''(\xi), \qquad (2.11)$$

the ratio is written as

$$\mu = -\frac{8}{\gamma^2}\left[\cosh\frac{\gamma}{2} + \left(\frac{1-\cosh\gamma}{\sinh\gamma}\right)\sinh\frac{\gamma}{2} - 1\right] \qquad (2.12)$$

---

* This method is effectively the same as solving (2. 4) for η_i once η_{i-1} is known, according to the iteration scheme

$$\eta_i^{IV} - \frac{\alpha^2}{2}\left(\int_0^1 [\eta_{i-1}']^2 d\xi\right)\eta_i'' = wL\alpha^2/EA$$

Written in this way, it becomes apparent that the convergence improves as the coefficient of η'' becomes smaller.

3. NUMERICAL RESULTS AND CONCLUSIONS.

From the above analysis, it appears that the solution for the induced force of constraint depends upon two parameters: wL/EA, as a measure of the intensity of the load, and $\alpha = L/r$, defining the slenderness of the member. Indeed, the desired quantities may be calculated for various values and the results presented as a family of curves for different values of α .

It is noticed, however, that the product of these two parameters may be expressed in terms of multiples of a single parameter, viz. the Euler buckling load

$$P_E = \pi^2 EI/L^2 \tag{3.1}$$

With this in mind, the following ratios are defined:

$$\beta = wL\alpha/P_E, \qquad \rho = S/P_E \tag{3.2}$$

The resulting equation for the axial constraint force can then be expressed as

$$\rho = \frac{\beta^2}{2\pi^5}\rho^{-7/2}\left[\frac{5\sinh\varphi - \varphi}{\cosh\varphi + 1} + \frac{\varphi^3}{12} - 2\varphi\right] \tag{3.3}$$

where

$$\varphi = \pi\rho^{1/2} \tag{3.4}$$

while the non-dimensional displacements are

$$\alpha\eta(\xi) = \frac{\beta}{\pi^2}\rho^{-2}\left[\cosh\varphi\xi + \left(\frac{1-\cosh\varphi}{\sinh\varphi}\right)\sinh\varphi\xi + \frac{\varphi^2\xi(1-\xi)}{2} - 1\right] \tag{3.5}$$

Although such a parametric representation may not be as physically meaningful as that given in the preceding section, it has the obvious important advantage of permitting representation of results in terms of universal curves rather than by means of families of curves.

Equation (3. 3) was solved by an iteration scheme as discussed in the previous section. For a given convergence criterion

$$\left| \frac{\rho_i - \rho_{i-1}}{\rho_i} \right| < 10^{-4} ,$$

less than four iterations were required for values of β of the order of unity, while for values $\beta > 100$, ten to twelve iterations were required to satisfy the criterion. (According to the note, p. 6, it should be expected that the convergence increases rapidly for smaller values of β).

Results for the axial force of constraint and for center point deflections are presented in Figures 2 and 3 respectively, over an extended range of values. Corresponding deflections for an unconstrained beam allowing relative longitudinal and displacements (obtained from the linear theory) are also shown in Figure 3.

It is apparent that for values of $\beta < 10$, the center span deflections are not affected to any appreciable extent by axial constraints. For values $\beta > 10$, however, the displacements are considerably reduced by the increasing axial constraint force, as may be expected.

The quantity μ, defining the ratio of center moments in the constrained beam to that existing in the equivalent unconstrained beam is presented in Figure 4. For values $\beta > 10$, it is noticed that, due to relatively large axial forces S, the moment is considerably reduced. Indeed, for values $\beta > 200$, the moment is less than 10% of that encountered in unconstrained beams.

For relatively small values of β, i.e. $\beta < 1.0$, values of the axial constraint force and of center point deflections are shown in Figure 5. As in Figure 3, the resulting deflections are seen to coincide with those given by the linear theory. However, the axial constraint force is seen to increasing parabolically with β.

In conclusion, the results presented in this section, by means of universal curves, indicate quantitatively the increasingly important effect of the axial force with higher load terms. For very slender beams subjected to loads of high intensity, the behavior of the member is seen to approach that of a cable.

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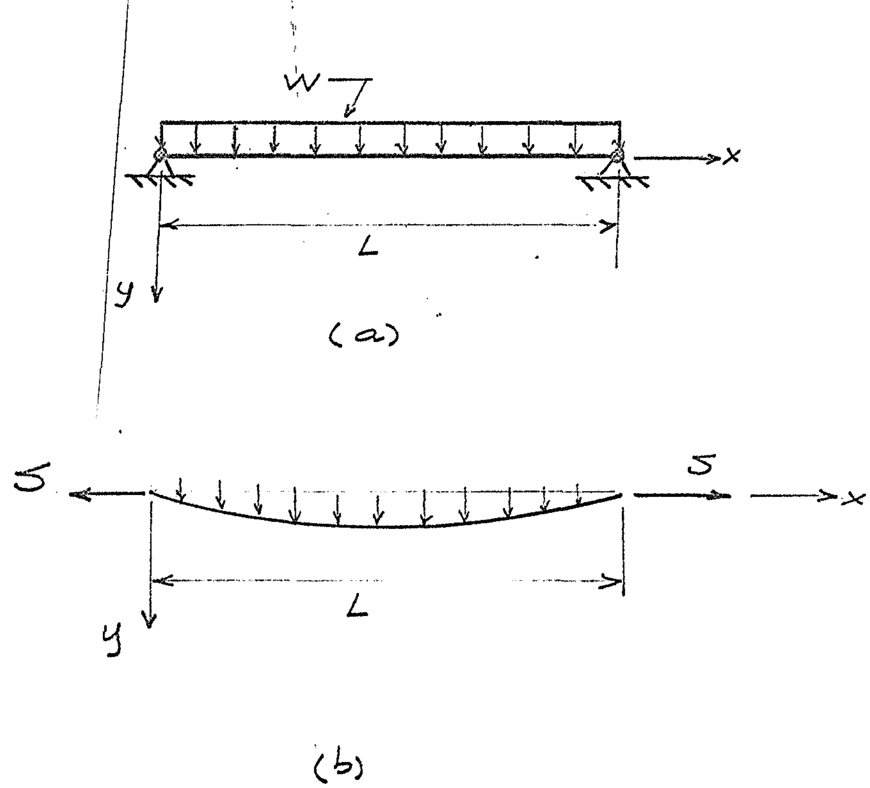

Fig. 1 Geometry of Problem

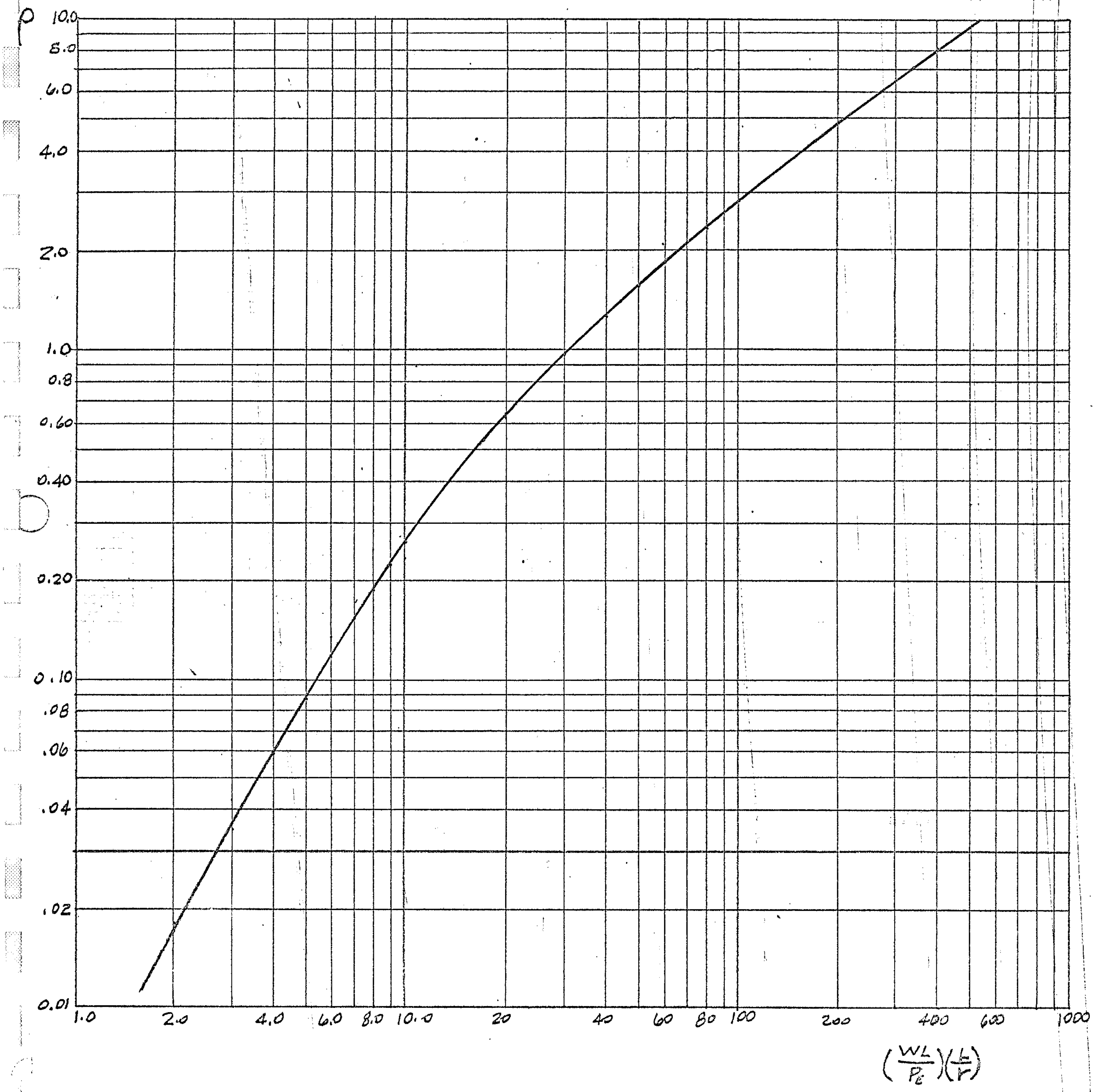

Fig. 2 Axial Constraint Force

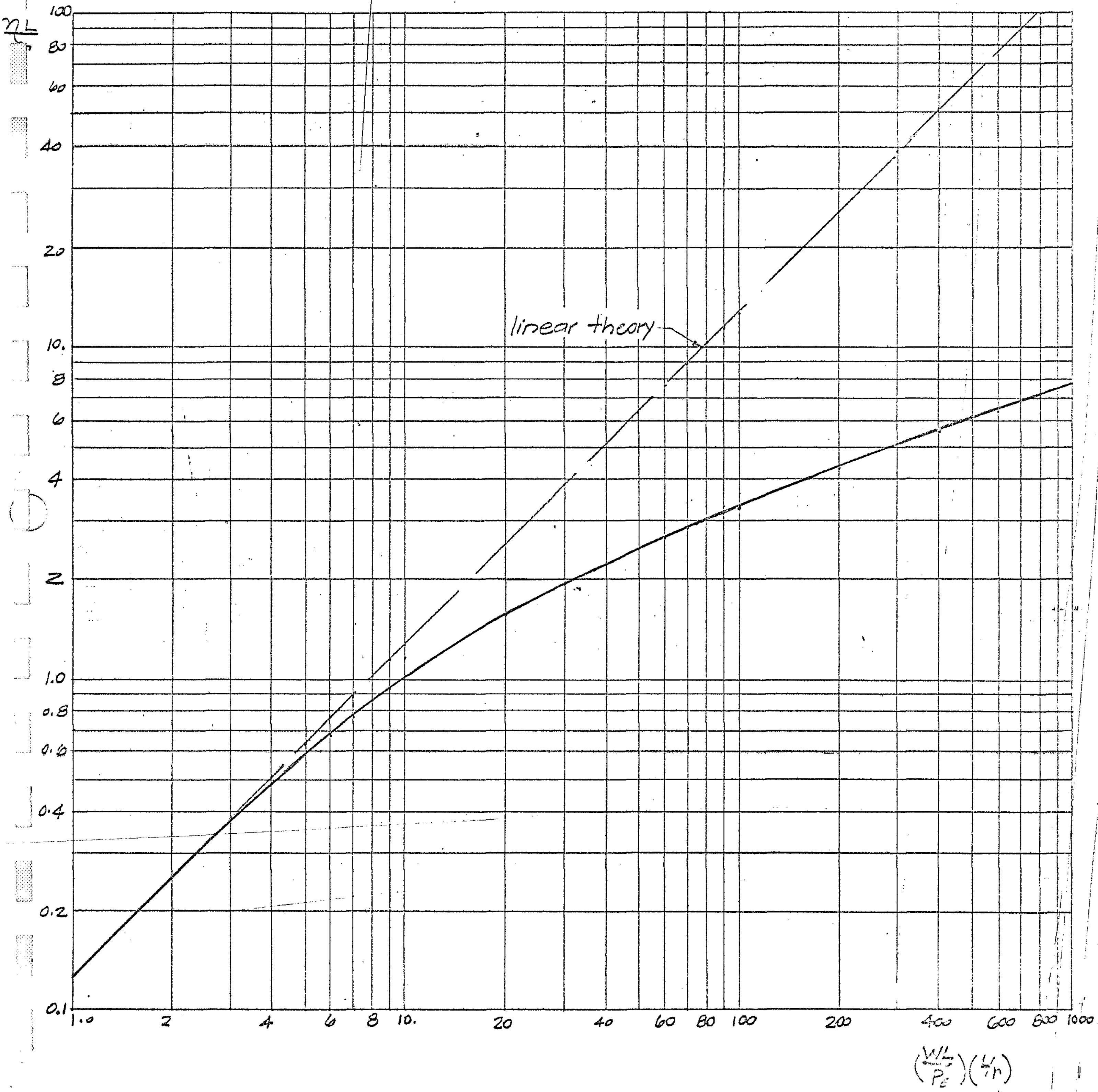

Fig. 3 Mid-Span Deflection

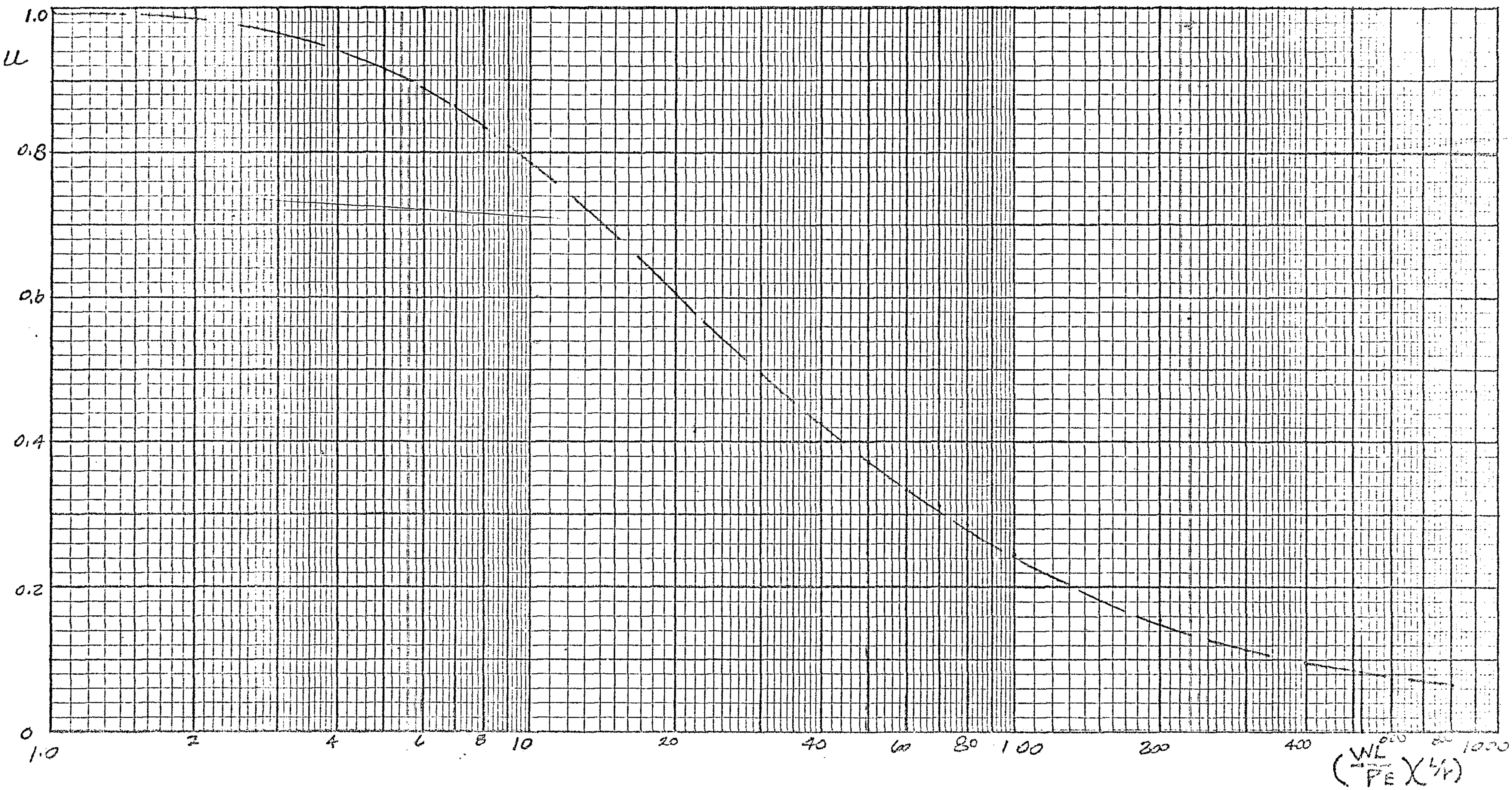

Fig. 4 Constraint Effect on Mid-Span Moment

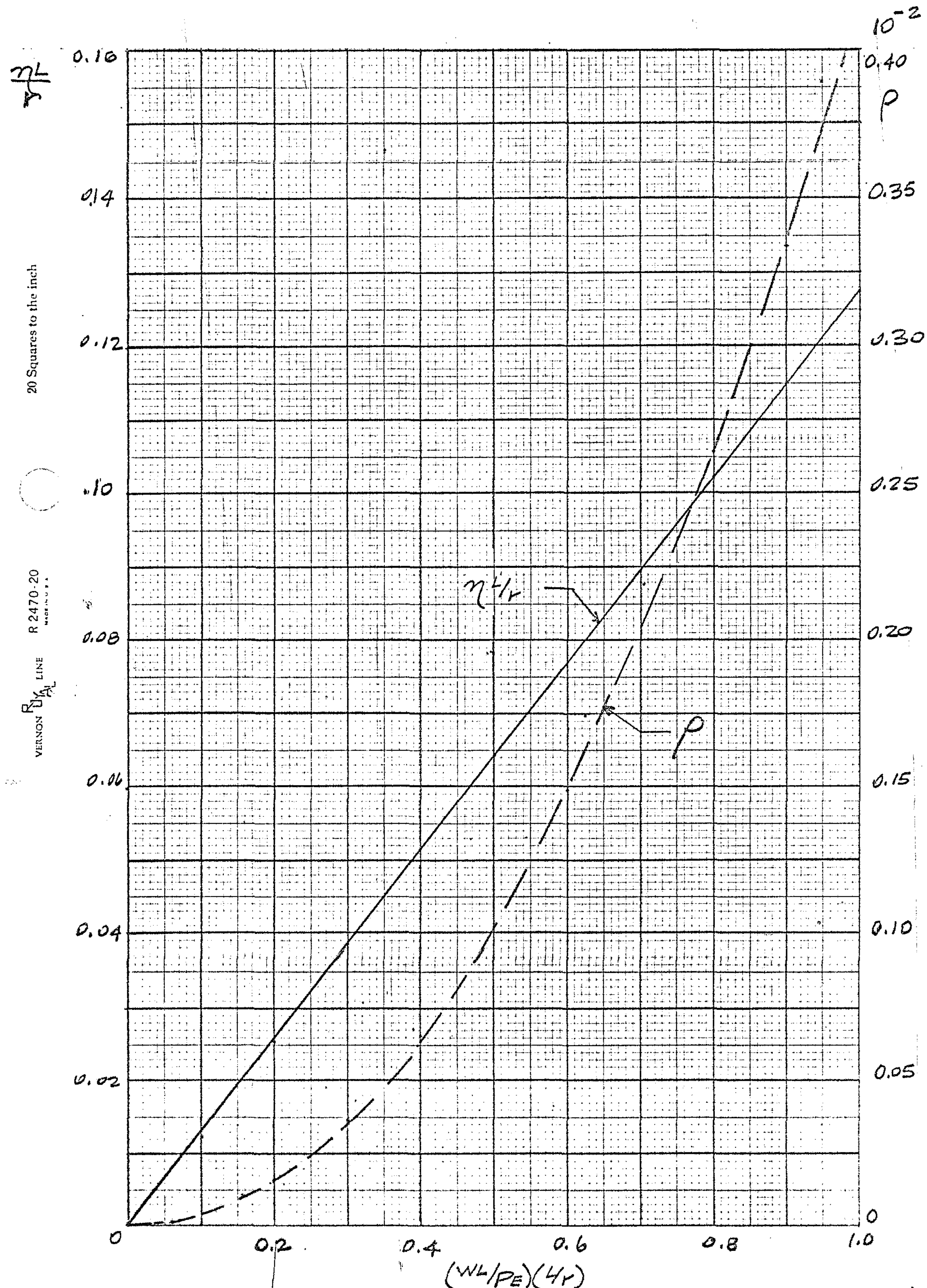

Fig. 5 Constraint Force and Mid-Span Deflection