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Gradient Techniques for Aerospace Applications

FINAL REPORT(*)

by

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(A) Introduction

The object of this investigation is a contribution to the gradient techniques of interest in optimization theory with particular regard to aerospace applications. Typical mathematical areas are: (a) gradient methods for unconstrained variational problems, (b) gradient methods with restoration of constraints, (c) memory gradient methods, (d) supermemory gradient methods, (e) gradient methods with bounded control, (f) gradient methods with bounded state, and (g) gradient methods with free final time. The main analytical results, summarized in 6 reports and 3 articles (see references) are described below.

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(B) References: Reports

1. MIELE, A., Gradient Methods in Control Theory, Part 1, Ordinary Gradient Method, Rice University, Aero-Astronautics Report No. 60, 1969.
2. MIELE, A., and PRITCHARD, R. E., Gradient Methods in Control Theory, Part 2, Sequential Gradient-Restoration Algorithm, Rice University, Aero-Astronautics Report No. 62, 1969.
3. DAMOULAKIS, J. N., Gradient Methods in Control Theory, Part 3, Sequential Gradient-Restoration Algorithm: Numerical Examples, Rice University, Aero-Astronautics Report No. 65, 1969.
4. DAMOULAKIS, J. N., Gradient Methods in Control Theory, Part 4, Sequential Gradient-Restoration Algorithm: Further Numerical Examples, Rice University, Aero-Astronautics Report No. 67, 1970.
5. DAMOULAKIS, J. N., Gradient Methods in Control Theory, Part 5, Sequential Gradient-Restoration Algorithm: Additional Numerical Examples, Rice University, Aero-Astronautics Report No. 73, 1970.
6. MIELE, A., Gradient Methods in Control Theory, Part 6, Combined Gradient-Restoration Algorithm, Rice University, Aero-Astronautics Report No. 74, 1970.

(C) References: Articles

7. MIELE, A., HUANG, H. Y., and HEIDEMAN, J. C., Sequential Gradient-Restoration Algorithm for the Minimization of Constrained Functions: Ordinary and Conjugate Gradient Versions, Journal of Optimization Theory and Applications, Vol. 4, No. 4, 1969.

8. MIELE, A., PRITCHARD, R. E., and DAMOULAKIS, J. N., Sequential Gradient-Restoration Algorithm for Optimal Control Problems, Journal of Optimization Theory and Applications, Vol. 5, No. 4, 1970.

9. MIELE, A., Recent Advances on Gradient Methods in Control Theory, Paper Presented at the 22nd Annual Southwestern IEEE Conference and Exhibition, Dallas, Texas, 1970.

(D) Abstract of Reports

1. MIELE, A., Gradient Methods in Control Theory, Part 1, Ordinary Gradient Method, Rice University, Aero-Astronautics Report No. 60, 1969.

Abstract. An analytical approach to the gradient method is presented within the framework of the Bolza problem of the calculus of variations. The first variation is minimized subject to the linearized differential constraint and an isoperimetric constraint on the control variation. Since the resulting Euler equations are linear, the differential system describing the optimum corrections is linear. The properties of this system are studied, and the solutions are related to the stepsize α. Next, the optimization of α is performed by minimizing the sum of the first variation and the second variation; an analytical expression is derived for the optimum value of α. Thus, the present method is a hybrid, in that the shape of the system of variations is obtained from first-order considerations while the scale factor for the variations is obtained from second-order considerations. Two numerical examples illustrating the convergence properties of the algorithm are supplied.

2. MIELE, A., and PRITCHARD, R. E., Gradient Methods in Control Theory, Part 2, Sequential Gradient-Restoration Algorithm, Rice University, Aero-Astronautics Report No. 62, 1969.

Abstract. This paper considers the problem of minimizing a functional I which depends on the state $x(t)$, the control $u(t)$, and a parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state x is prescribed. At the final point, the state x and the parameter π are required to satisfy

q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations. A sequential algorithm composed of the alternate succession of gradient phases and restoration phases is presented.

In the gradient phase, nominal functions $x(t)$, $u(t)$, π satisfying all the differential equations and boundary conditions are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ leading to varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are determined so that the value of the functional is decreased. These variations are obtained by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter.

Since the constraints are satisfied only to first order during the gradient phase, the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ may violate the differential equations and/or the boundary conditions. This being the case, a restoration phase is needed prior to starting the next gradient phase. In this restoration phase, the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are assumed to be the nominal functions. Variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ leading to varied functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ consistent with all the differential equations and boundary conditions are determined. These variations are obtained by requiring the least-square change of the control and the parameter subject to the linearized differential equations and the linearized boundary conditions. Of course, the restoration phase must be performed iteratively until the cumulative error in the differential equations and boundary conditions becomes smaller than some preselected value.

If the gradient stepsize is α, an order of magnitude analysis shows that the gradient corrections are $\Delta x = O(\alpha)$, $\Delta u = O(\alpha)$, $\Delta\pi = O(\alpha)$, while the restoration corrections are $\Delta\tilde{x} = O(\alpha^2)$, $\Delta\tilde{u} = O(\alpha^2)$, $\Delta\tilde{\pi} = O(\alpha^2)$. Hence, for α sufficiently small, the restoration phase preserves the descent property of the gradient phase: the functional I decreases between any two successive restoration phases.

To obtain a reasonable convergence rate, the gradient stepsize α must be determined in an optimal fashion. In this connection, two methods are presented: one is based on information available at the end of the gradient phase and one is based on information available at the end of the restoration phase.

3. DAMOULAKIS, J. N., Gradient Methods in Control Theory, Part 3, Sequential Gradient-Restoration Algorithm: Numerical Examples, Rice University, Aero-Astronautics Report No. 65, 1969.

Abstract. In Ref. 1, Miele and Pritchard developed the sequential gradient-restoration algorithm for minimizing a functional subject to certain differential constraints and boundary conditions. In this report, nine examples are presented, six pertaining to the fixed-final-time case and three pertaining to the free-final-time case. The numerical results show the rapid convergence characteristics of the sequential gradient-restoration algorithm.

4. DAMOULAKIS, J. N., Gradient Methods in Control Theory, Part 4, Sequential Gradient-Restoration Algorithm: Further Numerical Examples, Rice University, Aero-Astronautics Report No. 67, 1970.

Abstract. In Ref. 1, Miele and Pritchard developed the sequential gradient-restoration algorithm for minimizing a functional subject to certain differential constraints and boundary conditions. In this report, four examples are presented, two pertaining to the fixed-final-time case and two pertaining to the free-final-time case. The numerical results show the rapid convergence characteristics of the sequential gradient-restoration algorithm.

5. DAMOULAKIS, J. N., Gradient Methods in Control Theory, Part 5, Sequential Gradient-Restoration Algorithm: Additional Numerical Examples, Rice University, Aero-Astronautics Report No. 73, 1970.

Abstract. In Ref. 1, Miele and Pritchard developed the sequential gradient-restoration algorithm for minimizing a functional subject to certain differential constraints and boundary conditions. In order to reach a more complete understanding of the properties of the sequential gradient-restoration algorithm, several modifications and extensions are studied. These modifications and extensions are concerned with (i) the scheme for updating the state, the control, and the parameter, (ii) the possibility of employing an incomplete restoration phase at the end of each gradient phase, and (iii) the search technique for the gradient stepsize. Several numerical examples are given.

6. MIELE, A., Gradient Methods in Control Theory, Part 6, Combined Gradient-Restoration Algorithm, Rice University, Aero-Astronautics Report No. 74, 1970.

Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state x is prescribed. At the final point, the state x and the parameter π are required to satisfy q scalar relations. Along the interval of integration, the state, and the control, and the parameter are required to satisfy n scalar differential equations. A combined gradient-restoration algorithm is presented: this is an iterative algorithm characterized by variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ leading toward the minimal condition while simultaneously leading toward constraint satisfaction. These variations are computed by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter. The resulting linear, two-point boundary-value problem

is solved via the method of particular solutions. The descent properties of the algorithm are studied, and schemes to determine the optimum stepsize are discussed.

(E) Abstract of Articles

7. MIELE, A., HUANG, H. Y., and HEIDEMAN, J. C., Sequential Gradient-Restoration Algorithm for the Minimization of Constrained Functions: Ordinary and Conjugate Gradient Versions, Journal of Optimization Theory and Applications, Vol. 4, No. 4, 1969.

Abstract. The problem of minimizing a function $f(x)$ subject to the constraint $\varphi(x) = 0$ is considered. Here, f is a scalar, x an n-vector, and φ a q-vector. A sequential algorithm is presented, made up of the alternate succession of gradient phases and restoration phases.

In the gradient phase, a nominal point x satisfying the constraint is assumed; a displacement Δx leading from point x to a varied point y is determined such that the value of the function is reduced. The determination of the displacement Δx incorporates information at only point x for the ordinary gradient version of the method (Part 1) and information at both points x and $\hat{x}$ for the conjugate gradient version of the method (Part 2). Here, $\hat{x}$ is the point preceding x.

In the restoration phase, a nominal point y not satisfying the constraint is assumed; a displacement Δy leading from point y to a varied point $\tilde{x}$ is determined such that the constraint is restored to a prescribed degree of accuracy. The restoration is done by requiring the least-square change of the coordinates.

If the stepsize α of the gradient phase is of $O(\epsilon)$, then $\Delta x = O(\epsilon)$ and $\Delta y = O(\epsilon^2)$. For ϵ sufficiently small, the restoration phase preserves the descent property of the gradient phase: the function f decreases between any two successive restoration phases.

The ordinary gradient version of the algorithm exhibits asymptotic convergence but not quadratic convergence. On the other hand, the conjugate gradient version

exhibits quadratic convergence in the neighborhood of the minimum point. In particular, for a quadratic function subject to a linear constraint, the minimum point is obtained in no more than n-q iterations.

8. MIELE, A., PRITCHARD, R. E., and DAMOULAKIS, J. N., Sequential Gradient-Restoration Algorithm for Optimal Control Problems, Journal of Optimization Theory and Applications, Vol. 5, No. 4, 1970.

Abstract. This paper considers the problem of minimizing a functional I which depends on the state x(t), the control u(t), and a parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state x and the parameter π are required to satisfy q scalar relations. Along the interval of integration, the state, the control, and the parameter are required to satisfy n scalar differential equations. A sequential algorithm composed of the alternate succession of gradient phases and restoration phases is presented.

In the gradient phase, nominal functions x(t), u(t), π satisfying all the differential equations and boundary conditions are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ leading to varied function $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are determined so that the value of the functional is decreased. These variations are obtained by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter.

Since the constraints are satisfied only to first order during the gradient phase, the functions $\tilde{x}(t)$ $\tilde{u}(t)$, $\tilde{\pi}$ may violate the differential equations and/or the boundary conditions. This being the case, a restoration phase is needed prior to starting the next gradient phase. In this restoration phase, the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are assumed

to be the nominal functions. Variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}(t)$, $\Delta\tilde{\pi}$ leading to varied functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ consistent with all the differential equations and boundary conditions are determined. These variations are obtained by requiring the least-square change of the control and the parameter subject to the linearized differential equations and the linearized boundary conditions. Of course, the restoration phase must be performed iteratively until the cumulative error in the differential equations and boundary conditions becomes smaller than some preselected value.

If the gradient stepsize is α, an order of magnitude analysis shows that the gradient corrections are $\Delta x = O(\alpha)$, $\Delta u = O(\alpha)$, $\Delta\pi = O(\alpha)$, while the restoration corrections are $\Delta\tilde{x} = O(\alpha^2)$, $\Delta\tilde{u} = O(\alpha^2)$, $\Delta\tilde{\pi} = O(\alpha^2)$. Hence, for α sufficiently small, the restoration phase preserves the descent property of the gradient phase: the functional I decreases between any two successive restoration phases.

Several examples are presented for both the fixed-final-time case and the free-final-time case. The numerical results show the rapid convergence characteristics of the sequential gradient-restoration algorithm.

9. MIELE, A., Recent Advances on Gradient Methods in Control Theory, Paper Presented at the 22nd Annual Southwestern IEEE Conference and Exhibition, Dallas, Texas, 1970.

Abstract. This lecture summarizes recent work performed at Rice University under the sponsorship of the NASA-Manned Spacecraft Center. The problem considered is that of minimizing a functional I which depends on the state $x(t)$, the control $u(t)$, and the parameter π. Here, I is a scalar, x an n-vector, u an m-vector, and π a p-vector. At the initial point, the state is prescribed. At the final point, the state x and the parameter π are required to satisfy q scalar relations. Along the interval of integration,

the state, the control, and the parameter are required to satisfy n scalar differential equations. Two algorithms have been developed: (a) the sequential gradient-restoration algorithm and (b) the combined gradient-restoration algorithm.

(a) Sequential gradient-restoration algorithm. This algorithm is composed of the alternate succession of gradient phases and restoration phases. This sequential algorithm is constructed in such a way that the differential equations and boundary conditions are satisfied at the end of each iteration, that is, at the end of a complete gradient-restoration phase; hence, the value of the functional at the end of one iteration is comparable with the value of the functional at the end of any other iteration.

In the gradient phase, nominal functions $x(t)$, $u(t)$, π satisfying all the differential equations and boundary conditions are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ leading to varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are determined so that the value of the functional is decreased. These variations are obtained by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control parameter.

Since the constraints are satisfied only to first order during the gradient phase, the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ may violate the differential equations and/or the boundary conditions. This being the case, a restoration phase is needed prior to starting the next gradient phase. In this restoration phase, the functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are assumed to be the nominal functions. Variations $\Delta\tilde{x}(t)$, $\Delta\tilde{u}$, $\Delta\tilde{\pi}$ leading to varied functions $\hat{x}(t)$, $\hat{u}(t)$, $\hat{\pi}$ consistent with all the differential equations and boundary conditions are determined. These variations are obtained by requiring the least-square change of the control and the parameter subject to the linearized differential equations and the linearized boundary conditions. Of course, the restoration phase must be performed iteratively until the cumulative error in the differential equations and boundary conditions becomes smaller than some preselected value.

If the gradient stepsize is α, an order to magnitude analysis shows taht the gradient corrections are $\Delta x = O(\alpha)$, $\Delta u = O(\alpha)$, $\Delta\pi = O(\alpha)$, while the restoration corrections are $\Delta\tilde{x} = O(\alpha^2)$, $\Delta\tilde{u} = O(\alpha^2)$, $\Delta\tilde{\pi} = O(\alpha^2)$. Hence, for α sufficiently small, the restoration phase preserves the descent property of the gradient phase: the functional I decreases between any two successive restoration phases.

Methods to determine the gradient stepsize in an optimal fashion are discussed. Examples are presented for both the fixed-final-time case and the free-final-time case. The numerical results show the rapid convergence characteristics of the sequential gradient-restoration algorithm.

(b) Combined gradient-restoration algorithm. In this algorithm, the gradient phase and the restoration phase are joined together in a single phase. Nominal functions $x(t)$, $u(t)$, π not satisfying all the differential equations and boundary conditions are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ leading to varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ are determined by minimizing the first-order change of the functional subject to the linearized differential equations, the linearized boundary conditions, and a quadratic constraint on the variations of the control and the parameter. The descent properties of the algorithm are studied, and schemes to determine the optimum stepsize are discussed. Examples are presented for both the fixed-final-time case and the free-final-time case.

(c) Integration technique. In the sequential gradient-restoration algorithm, both the gradient phase and the restoration phase require the solution of a linear, two-point boundary value problem. In the combined gradient-restoration algorithm, the joint gradient-restoration phase also requires the solution of a linear, two-point boundary value problem. The technique employed is the recently discovered method of particular solutions. It consists of combining linearly $q+1$ particular solutions of the nonhomogeneous differential system. This linear combination satisfies the differential equations and the initial conditions if the constants are such that their sum is one. This and the q final conditions yield a set of $q+1$ scalar equations determining the $q+1$ constants of the combination.