CONFIGURATION SPACE THREE-BODY SCATTERING THEORY

E. Gerjuoy

Chapters 4 - 5

4. TRANSITION AMPLITUDES

The previous chapter has discussed the asymptotic behavior of the outgoing Green's function $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$ at large $\underset{\sim}{r}$. This chapter examines the limit of $\Psi_i^{(+)}(\underset{\sim}{r})$ as $\underset{\sim}{r} \to \infty$, and interprets the reaction rates inferred therefrom, concentrating primarily on directions $\underset{\sim}{\nu}$ corresponding to three-body elastic scattering, i.e., on directions $\underset{\sim}{\nu}$ for which no $r_{\alpha\beta}$ remains finite as $r \to \infty$. In this connection suppose the scattered part $\Phi_i^{(+)}$ of $\Psi_i^{(+)}$ really were everywhere outgoing at infinity in the laboratory system. Then, according to arguments which have been given elsewhere[(2)], in the laboratory system the outward flow of probability current (associated with $\Phi_i^{(+)}$) across the sphere at infinity should be

$$\mathcal{J} = \frac{1}{i\hbar}\int d\underset{\sim}{S} \cdot \underset{\sim}{W}\left[\Phi_i^{(+)*}, \Phi_i^{(+)}\right] \tag{117a}$$

where $\underset{\sim}{W}$ is defined as in (45a); and where the surface element $d\underset{\sim}{S}$ is perpendicular to $\underset{\sim}{\nu}$ and has magnitude $dS = r^8 d\underset{\sim}{\nu}$ given by Eqs. (89). As was mentioned in Chapter 1, in the time-independent configuration space formulation of scattering theory, reaction coefficients are computed from the probability current at infinity. Thus, to be sure that computation of the three-body elastic scattering rate does not involve divergent expressions, it is necessary that along most directions $\underset{\sim}{\nu}$

$$\left|\underset{\sim}{W}\left[\Phi_i^{(+)*}, \Phi_i^{(+)}\right]\right| \sim \frac{1}{r^8} \tag{117b}$$

or even smaller. If $|\underset{\sim}{W}|$ decreases more slowly than $1/r^8$, the integrand in the surface integral (117a) at infinity is not bounded. Correspondingly, the integral for $\mathcal{J}$ diverges as $r \to \infty$, unless (at fixed large r) the angular integrations over $d\underset{\sim}{\nu}$ vanish; of course, the integrations over $d\underset{\sim}{\nu}$ in (117a) could not vanish if $\Phi_i^{(+)}$ really were everywhere outgoing at infinity, since then $(i\hbar)^{-1} d\underset{\sim}{S} \cdot \underset{\sim}{W}$ always would have the same sign.

Eq. (117b) would hold if $\Phi_i^{(+)}(\underset{\sim}{r};E)$ behaved asymptotically at large $\underset{\sim}{r}$ like $G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}';E)$ [recall Eqs. (90)], i.e., if $\Phi_i^{(+)}(\underset{\sim}{r};E)$ at large $\underset{\sim}{r}$ represented three particles moving freely (as if under no forces) outwards from the laboratory system origin and from each other. But Eq. (55b) shows the center of mass motion associated with $\Phi_i^{(+)}$ is that of a plane wave, not an outgoing spherical wave; correspondingly, $|\underset{\sim}{W}|$ actually decreases

no more rapidly than r^{-5}, and the integrand in (117a) does turn out to be divergent [see subsection 4.1.1]. For three-particle collisions involving two incident bodies only, as, e.g., reactions (17b) and (17c), this divergence of (117a) is not a cause for serious concern, however,

because[2]: (i) the divergence is interpretable physically, and (more importantly) (ii) manipulations with divergent quantities can be wholly avoided by computing the center of mass frame probability current flow

$$\bar{\mathcal{J}} = \frac{1}{i\hbar}\int d\bar{\underset{\sim}{S}} \cdot \bar{\underset{\sim}{W}}\left[\bar{\Phi}_i^{(+)*}, \bar{\Phi}_i^{(+)}\right] \tag{118a}$$

In particular, in the cited two-body reactions (17b) and (17c), when $\bar{r} \to \infty$ along directions $\bar{\nu}$ corresponding to breakup into three particles, $\bar{\Phi}_i^{(+)}(\bar{r};\bar{E})$ behaves[11] like $\bar{G}_F^{(+)}(\bar{r};\bar{r}';\bar{E})$, and

$$\left| \bar{W}\left[\bar{\Phi}_i^{(+)*}, \bar{\Phi}_i^{(+)} \right] \right| \sim \frac{1}{\bar{r}^5} \tag{118b}$$

which suffices to keep finite the total scattered current flow across the sphere at infinite $\bar{r}$, whose surface element $d\bar{S}$ is of order $\bar{r}^{-5}$.

On the other hand, for collisions induced by the incident wave (21a), wherein all three particles are initially free, $\bar{\Phi}_i^{(+)}(\bar{r};\bar{E})$ does not behave asymptotically like $\bar{G}_F^{(+)}(\bar{r};\bar{r}';\bar{E})$; correspondingly, Eq. (118b) does not hold and use of (118a) generally does not avoid infinite probability current flows. In fact, Eqs. (61), (68) and (72) make it obvious that $\bar{\Phi}_i^{(+)}$ generated by (20) contains contributions $\bar{\Phi}_{12}^{(+)}$, $\bar{\Phi}_{23}^{(+)}$, $\bar{\Phi}_{31}^{(+)}$ possessing plane wave factors. For such terms, (118b) fails [see subsection 4.1.2] because whereas lim $\bar{G}_F^{(+)}(\bar{r};\bar{r}';E)$ as $\bar{r} \to \infty$ along $\bar{\nu}$ is of order $\bar{r}^{-5/2}$ [recall Eqs. (90) and (92)], the corresponding limit of $\bar{\Phi}_{\alpha\beta}^{(+)}(\bar{r};\bar{E})$ is of order $\bar{r}_{\alpha\beta}^{-1} \approx \bar{r}^{-1}$ along directions $\bar{\nu}$ for which $r_{\alpha\beta}$ becomes infinite with $\bar{r}$. These considerations indicate that at the very least $\bar{\Phi}_{12}^{(+)}$, $\bar{\Phi}_{23}^{(+)}$ and $\bar{\Phi}_{31}^{(+)}$ must be subtracted from $\bar{\Phi}_i^{(+)}$ before there can be any hope of computing—via Eq. (118a), but now using $\bar{\Phi}_i^{s(+)}$ from Eq. (62) in place of $\bar{\Phi}_i^{(+)}$—non-diverging center of mass frame scattered current flows.

Unfortunately (as particularly subsection 4.1.3 will show), use of $\bar{\Phi}_i^{s(+)}$ instead of $\bar{\Phi}_i^{(+)}$ in (118a) still does not eliminate all sources of divergent $\bar{\mathfrak{J}}$. To put it differently, it will be shown in subsection 4.1.3 that—for short range forces and directions $\bar{\underset{\sim}{\nu}}$ corresponding to three-body elastic scattering—$\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}};\bar{E})$ still is not identical with that part of $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}};\bar{E})$ which as $\bar{\underset{\sim}{r}} \to \infty$ along $\bar{\underset{\sim}{\nu}}$ behaves like the corresponding limit of $\bar{G}_F^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}';\bar{E})$ holding $\bar{\underset{\sim}{r}}'$ constant; it is this [behaving like $\bar{G}_F^{(+)}$] part of $\bar{\Phi}_i^{(+)}$ which in Chapter 1 was termed its "truly three-body" part $\bar{\Phi}_i^{t(+)}$. Note that the foregoing definition of $\bar{\Phi}_i^{t(+)}$ is not uniquely prescriptive because it permits adding to [or subtracting from] $\bar{\Phi}_i^{t(+)}$ any part of $\bar{\Phi}_i^{t(+)}$ which at infinity is negligible compared to $\bar{\rho}^{-5/2}$; this indeterminateness in $\bar{\Phi}_i^{t(+)}$ is inconsequential, however, since terms negligible compared to $\bar{\rho}^{-5/2}$ make no contribution to (118a) [when $\bar{\Phi}_i^{t(+)}$ replaces $\bar{\Phi}_i^{(+)}$]. The definition does rule out of $\bar{\Phi}_i^{t(+)}$ any terms which at infinity in the center of mass frame decrease less rapidly than $\bar{\rho}^{-5/2}$, or which are not everywhere outgoing [i.e., which contain contributions proportional to $e^{-i\bar{\rho}\sqrt{\bar{E}}}$ instead of $e^{i\bar{\rho}\sqrt{\bar{E}}}$]. It is understood, of course, that for our present purposes—namely, determining elastic scattering coefficients—nothing need be said nor has been said about the permitted behavior of $\bar{\Phi}_i^{t(+)}$ along directions $\bar{\underset{\sim}{\nu}} = \bar{\underset{\sim}{\nu}}_{\alpha\beta}$ corresponding to keeping $r_{\alpha\beta}$ finite as $\bar{\underset{\sim}{r}} \to \infty$. As a matter of fact, recombination reactions, e.g., (17a), are "truly three-body"; moreover, when, e.g., particle 1 can be bound to 2,

$$\lim_{q_{12}\to\infty \parallel \bar{\nu}_{12}} \bar{\Phi}_i^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{q}_{12};\bar{E}) \sim \sum_j a_j(\bar{\underset{\sim}{\nu}}_{12}) u_j(\underset{\sim}{r}_{12}) \frac{e^{iK_{12j}q_{12}}}{q_{12}} \qquad (119)$$

where $a_j(\bar{\underset{\sim}{\nu}}_{12})$ is a number, and where K_{12j} is defined by (114b). At infinite q_{12}, the right side of (119) is proportional to $\bar{\rho}^{-1}$ [recall Eq. (116b)]. However, because (119) dominates $\bar{\rho}^{-5/2}$ only on that subspace of $d\underset{\sim}{\bar{S}}$ corresponding to finite r_{12}, the total contribution of (119) to $\bar{\mathfrak{J}}$ of (118a) remains finite [and can be taken[2] to represent the flow of probability current corresponding to reactions such as (17a)].

4.1 Divergences in Transition Amplitudes

This section will show that (in our configuration space formulation) the divergences encountered in transition amplitudes typically are associated with failure to recognize the implications of the above introduction to this chapter. More specifically, this section provides further illustrations of the principle that the δ-functions (even if physically interpretable) encountered in the configuration space formulation of scattering theory are associated with improper mathematical manipulations. The δ-functions considered in this section are those appearing in transition amplitudes; it will be seen that these δ-functions generally are a consequence of an invalid interchange of order of integration and limit $r \to \infty$ in integrals for $\Phi_i^{(+)}$, $\bar{\Phi}_i^{(+)}$, $\Phi_i^{s(+)}$, etc., of the sort discussed in section 3.1 in connection with integrals for $G^{(+)}$ [e.g., Eq. (99)]. Failure to recognize that such interchange of order of integration and limit $r \to \infty$ is invalid typically leads to incorrect assumptions about the asymptotic behavior of the relevant scattered parts [e.g., of $\Phi_i^{(+)}$], and thus to incorrect computations of the scattered current flow [e.g., of $\mathfrak{J}$ via (117a)]. In particular, subsection 4.1.3 will show that assuming $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}};\bar{E})$ behaves like $\bar{G}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}';\bar{E})$ leads to a divergent transition amplitude, from which follows the (independently verifiable, see section E.3) conclusion that $\bar{\Phi}_i^{s(+)}$ indeed cannot be identical with $\bar{\Phi}_i^{t(+)}$.

4.1.1 Divergences Associated with Momentum Conservation

With the introduction to this chapter in mind, consider the asymptotic behavior of the integral (52a), which is the simplest expression we have found for the scattered part of $\Psi_i^{(+)}$ when the incident wave is **(21a)**, representing three initially free particles. As has been discussed[in sections 2.2 and A. 4 - A.5], the integral (52a) is divergent when two-body or three-body bound states can occur, so that Eq. (52a) is not expected to be a generally useful starting point for determining the asymptotic behavior of $\Phi_i^{(+)}$. Suppose, nevertheless, Eq. (100a) [which is valid providing $\Psi_f^{(-)*}$ is given by Eqs. (106)] is employed in (52a) to infer

$$\lim_{r\to\infty \parallel \underset{\sim}{\nu}_f} \Phi_i^{(+)}(\underset{\sim}{r}) = -C_3(E)\frac{e^{i\rho\sqrt{E}}}{\rho^4}T(\underset{\sim}{k}_i\to\underset{\sim}{k}_f) \tag{120a}$$

where the laboratory system transition amplitude

$$T(\underset{\sim}{k}_i\to\underset{\sim}{k}_f) = \Psi_f^{(-)*}V_i\psi_i \equiv \int d\underset{\sim}{r}'\,\Psi_f^{(-)*}(\underset{\sim}{r}')\left[V_{12}(r'_{12})+V_{23}(r'_{23})+V_{31}(r'_{31})\right]\psi_i(\underset{\sim}{r}') \tag{120b}$$

and where the dependences of ψ_i and $\Psi_f^{(-)*}$ on $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_f$ are specified by Eqs. **(21a)** and (100c), together with (106). Then, as has been shown previously[2], use of Eq. (120a) in (117a), together with Eqs. (87) - (92), yields

$$\mathcal{I} = \int dk_{2f}\, dk_{3f}\, d\eta_{1f}\, d\eta_{2f}\, d\eta_{3f}\, k_{2f}^2 k_{3f}^2 k_{1f} \frac{1}{(2\pi)^8} \frac{m_1}{\hbar^3} \left| T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \right|^2 \equiv \int w(i \to f) \tag{121a}$$

wherein the (unphysical, see below) laboratory system three-body scattering coefficient

$$w(i \to f) \equiv w(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \frac{2\pi}{\hbar} \frac{1}{(2\pi)^9} \left| T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \right|^2 \delta(E_f - E_i)\, d\underset{\sim}{k}_{1f}\, d\underset{\sim}{k}_{2f}\, d\underset{\sim}{k}_{3f} \tag{121b}$$

The energy-conserving $\delta(E_f - E_i)$ factor is employed in (121b) merely as an artifice, to put (121b) into a simple form consistent with the results of time-dependent scattering and the "golden rule"; the directly derived integrand of (121a) contains no $\delta(E_f - E_i)$, and the specification of $\Psi_f^{(-)*}$ in (120b) automatically makes $E_f = E_i$. The laboratory frame quantity w in Eqs. (121a) and (121b) should be related to the observed scattering rate $\hat{w}$, defined beneath Eq. (2), by [see section 4.2]

$$\hat{w}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = N_1 N_2 N_3\, w(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \tag{121c}$$

The form of Eq. (120a) seems consistent with the result (117b) required for finite laboratory frame probability current flow $\mathfrak{J}$ from (117a). Actually $\mathfrak{J}$ is infinite, however (as expected from the introduction to this chapter), because of the customary total momentum-conserving δ-function factor occurring in laboratory system transition amplitudes. Specifically, employing (33a) and (102b), the integral (120b) reduces to

$$T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = (2\pi)^3 \delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i) \int d\underset{\sim}{r}' \, \bar{\Psi}_f^{(-)*}(\underset{\sim}{r}'; E_f) \left[V_{12}(\underset{\sim}{r}'_{12}) + V_{23}(\underset{\sim}{r}'_{23}) + V_{31}(\underset{\sim}{r}'_{31}) \right] \bar{\Psi}_i(\underset{\sim}{r}'; E_i) \qquad (122)$$

which, when inserted into (121a), causes $\mathfrak{J}$ to diverge by virtue of the $[\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)]^2$ factor under the integrand. Note that $\mathfrak{J}$ could remain finite if merely $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$ (rather than its square) appeared in the integrand of (121a); correspondingly, w from (121b) can be made physically meaningful only by somehow reinterpreting (and thus eliminating) one of the $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$ factors in $|T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)|^2$.

Of course, the fact that $T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ contains a momentum-conserving δ-function factor is gratifying on physical grounds. Nevertheless, from the standpoint of this work's configuration space formulation of scattering theory, this same fact must be regarded as a signal that the computation of the laboratory system transition amplitude has involved unjustified mathematical manipulations.

In particular, the assertion in (120a) that $\lim \Phi_i^{(+)}(\underset{\sim}{r})$ is $\sim e^{i\rho\sqrt{E}}/\rho^4$ is prima facie incorrect by virtue of (55b), as the introduction to this chapter has discussed. Moreover, to derive the pair of Eqs. (120) from Eq. (52a) it is necessary to assume [compare Eq. (99)]

$$\lim_{r\to\infty \| \underset{\sim}{\gamma}_f} \int d\underset{\sim}{r}' G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}') V(\underset{\sim}{r}') \Psi_i(\underset{\sim}{r}') = \int d\underset{\sim}{r}' \lim_{r\to\infty \| \underset{\sim}{\gamma}_f} G^{(+)}(\underset{\sim}{r},\underset{\sim}{r}') V(\underset{\sim}{r}') \psi_i(\underset{\sim}{r}') . \quad (123)$$

Thus the interchange of order of integration and limit $r \to \infty$ in (123) also must be incorrect, as can be directly verified by comparing [as in the case of (99)] the contributions to the left side of (123) from the regions $r' < r$ and $r' > r$ as $r \to \infty$ [see section C.4].

Similar remarks [see section C.4] pertain to the result for $T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ if—still for ψ_i of **(21a)**—Eqs. (90) together with interchange of order of integration and limit $r \to \infty$ are employed in Eq. (42); in this **fashion**, one again obtains (120a), but now with

$$T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \psi_f^* V_i \Psi_i^{(+)} \equiv \int d\underset{\sim}{r}' \psi_f^*(\underset{\sim}{r}') \left[V_{12}(\underset{\sim}{r}'_{12}) + V_{23}(\underset{\sim}{r}'_{23}) + V_{31}(\underset{\sim}{r}'_{31}) \right] \Psi_i^{(+)}(\underset{\sim}{r}') \quad (124a)$$

As in (122), Eq. (124a) reduces to

$$T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = (2\pi)^3 \delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i) \int d\bar{\underset{\sim}{r}}' \bar{\Psi}_f^*(\bar{\underset{\sim}{r}}'; \bar{E}_f) \left[V_{12}(\underset{\sim}{r}'_{12}) + V_{23}(\underset{\sim}{r}'_{23}) + V_{31}(\underset{\sim}{r}'_{31}) \right] \bar{\Psi}_i^{(+)}(\bar{\underset{\sim}{r}}'; \bar{E}_i) \quad (124b)$$

The integral (124a) has a $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$ factor even though the integral in (42) is convergent at real energies [recall section 2.2]. Similarly, the integral (120b) contains a $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$ factor whether or not (52a) diverges, i.e., whether or not two-body or three-body bound states exist. On the other hand, it is true (see subsection **4.1.4 below**) that bound states produce additional divergences in (120b), as well as in (124a).

4.1.2 Divergences Associated with Two-Body Scattering

The preceding subsection implies that if we wish to calculate the reaction coefficient by means which are mathematically valid and do not introduce divergent expressions, we must not make use of the expression (117a) for the laboratory system probability current flow. Let us examine, therefore, the possibility of **calculating** the probability current flow in the center of mass system, via Eq. (118a). In particular, consider the asymptotic behavior of the integral (52b), which is the center of mass analogue of (52a). Then, as in subsection 4.1.1, ignoring the bound state complications which make (52b) a dubious starting point, use in (52b) of the valid set of equations (102) and (106), together with

$$\lim_{\bar{r}\to\infty \| \underset{\sim}{\nu}_f} \int d\underset{\sim}{\bar{r}}' \bar{G}^{(+)}(\underset{\sim}{r};\underset{\sim}{r}') V(\underset{\sim}{r}') \bar{\Psi}_i(\underset{\sim}{r}') = \int d\underset{\sim}{\bar{r}}' \lim_{\bar{r}\to\infty \| \underset{\sim}{\bar{\nu}}_f} \bar{G}^{(+)}(\underset{\sim}{r};\underset{\sim}{r}') V(\underset{\sim}{r}') \bar{\Psi}_i(\underset{\sim}{r}') \tag{125}$$

yields

$$\lim_{\bar{r}\to\infty \| \underset{\sim}{\bar{\nu}}_f} \bar{\Phi}_i^{(+)}(\underset{\sim}{r}) = -C_2(\bar{E}) \frac{e^{i\bar{\rho}\sqrt{\bar{E}}}}{\bar{\rho}^{5/2}} \bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \tag{126a}$$

where

$$\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{\Psi}_f^{(-)*} V_i \bar{\Psi}_i$$

$$\equiv \int d\underset{\sim}{\bar{r}}' \bar{\Psi}_f^{(-)*}(\underset{\sim}{r}') \left[V_{12}(\underset{\sim}{r}'_{12}) + V_{23}(\underset{\sim}{r}'_{23}) + V_{31}(\underset{\sim}{r}'_{31}) \right] \bar{\Psi}_i(\underset{\sim}{r}') \tag{126b}$$

and where $C_2(\bar{E})$ and $\bar{\rho}$ are defined as in Eqs. (102). Correspondingly, using Eq. (126a) in (118a), the center of mass analogues of Eqs. (121a) and (121b) are found to be

$$\bar{\mathcal{J}} \equiv \int \bar{w}(i \to f) \tag{127a}$$

$$\bar{w}(i \to f) \equiv \bar{w}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \frac{2\pi}{\hbar} \frac{1}{(2\pi)^6} |\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)|^2 \delta(\bar{E}_f - \bar{E}_i) d\underset{\sim}{k}_{12f} d\underset{\sim}{K}_{12f} \tag{127b}$$

$$= \frac{2\pi}{\hbar} \frac{1}{(2\pi)^6} |\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)|^2 \delta(E_f - E_i) \delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i) d\underset{\sim}{k}_{1f} d\underset{\sim}{k}_{2f} d\underset{\sim}{k}_{3f} . \tag{127c}$$

where $\bar{w}$ is the reaction coefficient introduced in Eqs. (1) and (2).

In Eqs. (127b) and (127c), as in Eq. (121b), the δ-functions merely are convenient artifices for putting the final result into simple form; moreover the specification of $\bar{\Psi}_f^{(-)*}$ does not involve $\underset{\sim}{K}_f$, and automatically makes $\bar{E}_f = \bar{E}_i$. Therefore, if $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ contains no divergences, $\bar{\mathfrak{F}}$ and $\bar{w}$ given by (127) will be finite and well-defined. On the other hand, if $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ contains terms proportional to δ-functions whose arguments can vanish on the energy-momentum shell $E_f = E_i$ and $\underset{\sim}{K}_f = \underset{\sim}{K}_i$, then $\bar{\mathfrak{F}}$ will diverge because the integrand of (127a) will contain terms proportional to the squares of δ-functions; correspondingly, $\bar{w}$ from (127b) or (127c) will not be physically meaningful unless the singular terms in the integrand somehow can be reinterpreted so as to eliminate all powers of δ-functions higher than the first. Note that a factor $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$ in $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ actually would make the integrand of (127c) proportional to $[\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)]^3$. However, $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is independent of $\underset{\sim}{K}_f$ or $\underset{\sim}{K}_i$; in fact, Eqs. (120b) and (126b) immediately imply

$$T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = (2\pi)^3 \delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \tag{128}$$

consistent with the result (122) previously deduced.

As was mentioned in the introduction to this chapter, this subsection's procedure--namely calculating the probability current flow in the center of mass system--is mathematically valid for two-body reactions, but not for collisions induced by the incident wave (21a). To put it differently, $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ from (126b) generally is free from (on the energy-momentum shell) divergences for reactions produced by two-body collisions, even when these collisions cause breakup (e.g., ionization) for one of the incident bodies[28]; for the elastic scattering of three initially free particles, on the other hand, it is well known[4,8] that $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ contains δ-functions --associated with purely two-body single scattering--whose arguments can vanish on the energy-momentum shell. In particular, consider the contribution to (126b) from, e.g., the first two terms in the center of mass analogue of (106a), which validly specifies $\bar{\Psi}_f^{(-)*}$. In other words, recalling Eqs. (58a) and (72), replace $\bar{\Psi}_f^{(-)*}(\bar{\underset{\sim}{r}}')$ in (126b) by

$$\bar{\psi}_f^{*}(\bar{\underset{\sim}{r}}') + \bar{\Phi}^{(-)*}_{12f}(\bar{\underset{\sim}{r}}') = \bar{\Psi}^{(-)*}_{12f}(\bar{\underset{\sim}{r}}') = e^{-i\underset{\sim}{K}_{12f}\cdot\underset{\sim}{q}'_{12}}\left[e^{-i\underset{\sim}{k}_{12f}\cdot\underset{\sim}{r}'_{12}} + \varphi^{(-)*}_{12f}(\underset{\sim}{r}'_{12};\underset{\sim}{k}_{12f})\right]$$

$$\equiv e^{-i\underset{\sim}{K}_{12f}\cdot\underset{\sim}{q}'_{12}}\, u_c^{(-)*}(\underset{\sim}{r}'_{12};\underset{\sim}{k}_{12f}) \qquad (129a)$$

where $u_c^{(-)*}(\underset{\sim}{r}_{12})$ [which does not contain the $(2\pi)^{-3/2}$ normalization factor attached to $u(\underset{\sim}{r}_{12})$ of Eqs. (113)] obviously solves

$$\left[\frac{-\hbar^2}{2\mu_{12}}\nabla^2_{12} + V_{12}(\underset{\sim}{r}_{12}) - \frac{\hbar^2 k^2_{12f}}{2\mu_{12}}\right]u_c^{(-)*}(\underset{\sim}{r}_{12}) = 0 \qquad (129b)$$

and represents scattering of particles 1 and 2 in their own center of mass system when (in that center of mass system) the incident plane wave is $e^{-i\underset{\sim}{k}_{12f}\cdot\underset{\sim}{r}_{12}}$. Then one sees that there is a contribution $\bar{T}_{12}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ to $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$--from the V_{12} interaction in (126b)--of magnitude

$$\bar{\Psi}_{12f}^{(-)*} V_{12} \bar{\Psi}_i = \int d\underset{\sim}{r}'_{12} d\underset{\sim}{q}'_{12} e^{-i\underset{\sim}{K}_{12f}\cdot\underset{\sim}{q}'_{12}} u_c^{(-)*}(\underset{\sim}{r}'_{12};\underset{\sim}{k}_{12f}) V_{12}(r'_{12}) e^{i(\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}'_{12}+\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}'_{12})} \tag{130a}$$

$$= (2\pi)^3 \delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i}) \int d\underset{\sim}{r}_{12} u_c^{(-)*}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12f}) V_{12}(\underset{\sim}{r}_{12}) e^{i\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}_{12}} \tag{130b}$$

$$= (2\pi)^3 \delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i})\, t_{12}(\underset{\sim}{k}_{12i} \rightarrow \underset{\sim}{k}_{12f}) \tag{130c}$$

where $t_{12}(\underset{\sim}{k}_{12i} \rightarrow \underset{\sim}{k}_{12f})$ is the transition amplitude representing scattering of the completely isolated pair of particles 1 and 2 in their own center of mass system. Thus, for incident waves (21a), the quantities $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ and $T(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$, supposedly representing three-body transition amplitudes in the center of mass and laboratory frames respectively, actually contain a contribution (130c) representing purely two-body elastic scattering--of particles 1 and 2 without interaction with 3; according to the remarks following Eqs. (29), the

δ-function in (130c) guarantees that the laboratory velocity of the non-interacting particle 3 indeed remains unaltered. Similar [to (130c)] contributions to $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$, with similar interpretations, result of course from the other interactions in (126b).

Recalling the discussion in subsection 4.1.1, the divergent (on the energy-momentum shell) $\delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i})$ factor in (130c) is a signal that Eqs. (126) were derived using improper mathematical manipulations. Correspondingly, the interchange of order of integration and limit $\bar{r} \rightarrow \infty$ in (125) must be wrong, as can be directly verified [see section C.4]. Nevertheless, as has just been seen--and as in the case of the momentum-conserving $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$ factor discussed in the preceding subsection--the divergent term (130c) is readily interpretable physically. I point out that the above conclusion--namely that $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ contains contributions representing a single purely two-body scattering--was based solely on the form of the contribution (130c) to (126b). But (126b) has been derived from the admittedly not always valid formula (52b) for $\bar{\Phi}_i^{(+)}$; it would have been preferable to obtain $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ from an always legitimate formula for $\bar{\Phi}_i^{(+)}$, i.e., from (69) supplemented by the center of mass version of (61). However, starting in this latter fashion, it is immediately obvious that $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ defined as in (126a) must contain a contribution--stemming from the $\bar{\Phi}_{12}^{(+)}$ term in $\bar{\Phi}_i^{(+)}$--representing the single purely two-body scattering of 1 and 2. Furthermore, Eqs. (68) and (72) show explicitly that the assertion $\lim \bar{\Phi}_i^{(+)}(r) \cong e^{i\bar{\rho}\sqrt{\bar{E}}}/\bar{\rho}^{5/2}$ [in (126a)] is prima facie incorrect, and that a $\delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i})$ factor in the $\bar{\Phi}_{12}^{(+)}$ contribution to $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is to be expected. Alternatively, if—ignoring the accurate result (72) for $\bar{\Phi}_{12}^{(+)}$—one starts from the admittedly not always correct [cf. Eqs. (60) or (105d)] analogue of (52b)

$$\bar{\Phi}_{12}^{(+)}(\underset{\sim}{\bar{r}};\bar{E}) = -\int d\underset{\sim}{\bar{r}}'\,\bar{G}_{12}^{(+)}(\underset{\sim}{\bar{r}};\underset{\sim}{\bar{r}}')\,V_{12}(\underset{\sim}{r}_{12}')\,\bar{\Psi}_i(\underset{\sim}{\bar{r}}') \qquad (131a)$$

and then employs the analogue of Eq. (102a) [i.e., Eqs. (105), in effect] after performing the interchange analogous to (125), one finds

$$\lim_{\bar{r}\to\infty \| \underset{\sim}{\bar{r}}_f} \bar{\Phi}_{12}^{(+)}(\underset{\sim}{\bar{r}}) = -C_2(\bar{E})\frac{e^{i\bar{\rho}\sqrt{\bar{E}}}}{\bar{\rho}^{5/2}}\,\bar{\Psi}_{12f}^{(-)*}V_{12}\bar{\Psi}_i \qquad (131b)$$

Comparing with Eq. (130a), one sees that the contribution to $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ stemming from the $\bar{\Phi}_{12}^{(+)}$ term in $\bar{\Phi}_i^{(+)}$ is precisely the contribution (130c) previously obtained and interpreted. Moreover the contribution to the integral (131a) from $\bar{r}' > \bar{r}$ is not negligible compared to $\bar{\rho}^{-5/2} \cong \bar{r}^{-5/2}$ [see section C.4], so that the interchange of order of integration and limit $\underset{\sim}{\bar{r}} \to \infty$ in (131a) really is unjustified.

The foregoing discussion [especially in the last paragraph] is relevant also to the expression for $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ obtained from (unjustified) employment of (90) in the center of mass system version of (42); this procedure again yields (126a), but with

$$\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{\Psi}_f^* V_i \bar{\Psi}_i^{(+)} \equiv \int d\bar{\underset{\sim}{r}}' \bar{\Psi}_f^*(\underset{\sim}{r}')\left[V_{12}(r_{12}') + V_{23}(r_{23}') + V_{31}(r_{31}')\right]\bar{\Psi}_i^{(+)}(\bar{\underset{\sim}{r}}') \tag{131c}$$

Eqs. (124) and (131c) are consistent with Eq. (128). Replacing $\bar{\Psi}_i^{(+)}$ in (131c) by $\bar{\Psi}_{12}^{(+)} = \bar{\psi}_i + \bar{\Phi}_{12}^{(+)}$, and using the V_{12} interaction, once more yields the two-body contribution (130c) to $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$.

If the fact that $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ given by (131c) is divergent is overlooked, and if (52b) or the center of mass analogue of (51c) is employed in (131c) despite the fact that (52b) and (51c) fail when two-body bound states exist, then

$$\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{\psi}_f^* \underset{\sim}{\bar{T}}(\bar{E}_i)\bar{\psi}_i \equiv \langle f|\underset{\sim}{\bar{T}}(\bar{E}_i)|i\rangle \tag{131d}$$

where $\underset{\sim}{\bar{T}}(\bar{E})$ is the operator defined by the center of mass analogue of (5). Eq. (131d) (but with $\bar{E}_f$ replacing $\bar{E}_i$) also follows from substituting--with similar inattention to questions of mathematical validity--the center of mass analogue of (100b) in (126b). Although (131d) is quite commonly employed, the foregoing remarks and the entire contents of this section 4.1 make it apparent that--for three independently incident particles described by ψ_i of Eq. (21a)--the seeming connection (131d) between the matrix element $\langle f|\underset{\sim}{\bar{T}}(\bar{E})|i\rangle$ and the asymptotic behavior of $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}})$ at large $\bar{\underset{\sim}{r}}$ [recall Eq. (126a) and its difficulties] has no real mathematical or physical justification. On the other hand, for the transition amplitude $t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})$ of (130c) it is mathematically justifiable to write

$$t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f}) = \int d\underset{\sim}{r}_{12}\, d\underset{\sim}{r}'_{12}\, e^{-i\underset{\sim}{k}_{12f}\cdot\underset{\sim}{r}_{12}}\, \underset{\sim}{t}_{12}(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};E_{12})\, e^{i\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}'_{12}}$$
$$\equiv \langle f \,|\, \underset{\sim}{t}_{12}(E_{12}) \,|\, i \rangle \qquad (131e)$$

where $k_{12f}^{\;2} = k_{12i}^{\;2} = k_{12}^{\;2}$ satisfying(74b),

$$\underset{\sim}{t}_{12}(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};\lambda) = V_{12}(\underset{\sim}{r}_{12})\,\delta(\underset{\sim}{r}_{12}-\underset{\sim}{r}'_{12}) - V_{12}(\underset{\sim}{r}_{12})\, g_{12}(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};\lambda)\, V_{12}(\underset{\sim}{r}'_{12}) \qquad (131f)$$

and g_{12} is the two-particle Green's function, defined as in (75); moreover, it really is true that as $\underset{\sim}{r}_{12} \to \infty$ along the direction of $\underset{\sim}{k}_{12f} = k_{12}\, \underset{\sim}{n}_{12f}$,

$$\lim_{r_{12}\to\infty \,\|\, \underset{\sim}{n}_{12f}} \varphi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i}) = -\frac{1}{4\pi}\,\frac{2\mu_{12}}{\hbar^2}\,\frac{e^{ik_{12}r_{12}}}{r_{12}}\,\langle f \,|\, \underset{\sim}{t}_{12}(E_{12}) \,|\, i\rangle \qquad (131g)$$

where $\phi_{12}^{(+)}$ is defined by Eqs. (73)-(74). Similarly, it is justified to write

$$\varphi_{12}^{(+)}(\underset{\sim}{k}_{12i}) = -g_{12}^{(+)} V_{12}\, \psi_{12i} = -\left[g_F^{(+)} - g_F^{(+)} V_{12}\, g_{12}^{(+)}\right] V_{12}\, \psi_{12i}$$
$$= -g_F^{(+)}\left[\underset{\sim}{V}_{12} - V_{12}\, g_{12}^{(+)} V_{12}\right]\psi_{12i} = -g_F^{(+)}\, \underset{\sim}{t}_{12}(E_{12})\, \psi_{12i} \qquad (131h)$$

as well as

$$V_{12}\left[\psi_{12i} + \phi_{12}^{(+)}\right] = V_{12}\left[1 - g_{12}^{(+)}V_{12}\right]\psi_{12i} = \underset{\sim}{t}_{12}(E_{12})\psi_{12i} \tag{131i}$$

Eq. (131i) is the two-particle analogue of the not necessarily valid (51c). Note that the first term on the right side of (131f), which term has been denoted by $\underset{\sim}{V}_{12}$ in (131h), is not identical with $\underset{\sim}{V}_{12}(\underset{\sim}{r};\underset{\sim}{r}')$ of Eq. (77a) [recall Eq. (27e)]; $\underset{\sim}{V}_{12}$ of (77a) operates in the nine-dimensional three-particle configuration space, whereas $\underset{\sim}{V}_{12}$ of (131h) operates in only a three-dimensional space.

4.1.3 Divergences After Subtraction of Two-Body Terms

The results of the preceding subsection imply that--whether or not divergent, i.e., whether or not bound states occur--the integrals (52) are an unsuitable starting point for mathematically unobjectionable derivations of formal expressions for the three-body amplitudes $T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ or $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$. Similarly, Eq. (42) and its center of mass version--though generally convergent whether or not bound states exist--also have been found to be mathematically unsuitable starting points for deriving matrix elements of $\underset{\sim}{T}$ or $\underset{\sim}{\bar{T}}$. To have any hope of deriving non-divergent expressions for $< f|\underset{\sim}{\bar{T}}|i >$, the purely two-body single scattering parts of $\bar{\Phi}_i^{(+)}$ apparently must be subtracted away at the very outset, before taking the limit $\underset{\sim}{r} \to \infty$ (as foreshadowed in the introduction to this chapter). Therefore, I now shall examine the contributions to $T(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ and $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ obtained from the asymptotic behavior of $\Phi_i^{s(+)}$ and $\bar{\Phi}_i^{s(+)}$, specified by Eqs. (67c) and (69) respectively. In any event, the starting point (69)--taken together with Eq. (72) and the center of mass version of (61)--has the virtue that it provides a specification of $\bar{\Phi}_i^{(+)}(\underset{\sim}{\bar{r}})$ free from divergences or ambiguities.

In Eq. (69), assume that

$$\lim_{\bar{r}\to\infty\|\underset{\sim}{\bar{\nu}}_f}\int d\underset{\sim}{\bar{r}}'\,\bar{G}^{(+)}(\underset{\sim}{\bar{r}};\underset{\sim}{\bar{r}}')V_{23}(\underset{\sim}{r}_{23}')\bar{\Phi}_{12}^{(+)}(\underset{\sim}{\bar{r}}') = \int d\underset{\sim}{\bar{r}}'\lim_{\bar{r}\to\infty\|\underset{\sim}{\bar{\nu}}_f}\bar{G}^{(+)}(\underset{\sim}{\bar{r}};\underset{\sim}{\bar{r}}')V_{23}(\underset{\sim}{r}_{23}')\bar{\Phi}_{12}^{(+)}(\underset{\sim}{\bar{r}}') \tag{132}$$

Then in the by now familiar fashion, there results

$$\lim_{\bar{r}\to\infty\|\underset{\sim}{\bar{\nu}}_f}\bar{\Phi}_i^{s(+)}(\underset{\sim}{\bar{r}}) = -C_2(\bar{E})\frac{e^{i\bar{\rho}\sqrt{\bar{E}}}}{\bar{\rho}^{5/2}}\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \tag{133a}$$

where

$$\bar{T}^{s}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \int d\underset{\sim}{\bar{r}}' \bar{\Psi}_f^{(-)*}(\underset{\sim}{\bar{r}}') \Big[(V_{23}+V_{31}) \bar{\Phi}_{12}^{(+)}(\underset{\sim}{\bar{r}}') + (V_{31}+V_{12}) \bar{\Phi}_{23}^{(+)}(\underset{\sim}{\bar{r}}') + (V_{12}+V_{23}) \bar{\Phi}_{31}^{(+)}(\underset{\sim}{\bar{r}}') \Big] \tag{133b}$$

Similarly, interchanging order of integration and limit $\underset{\sim}{r} \to \infty$ in (67c) yields

$$\lim_{r \to \infty \| \underset{\sim}{v}_f} \Phi_i^{s(+)}(\underset{\sim}{r}) = -C_3(E) \frac{e^{i\rho\sqrt{E}}}{\rho^4} T^{s}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \tag{134a}$$

where T^s turns out to obey

$$T^{s}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = (2\pi)^3 \delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i) \bar{T}^{s}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \tag{134b}$$

consistent with Eq. (128).

In view of the preceding two subsections, the momentum-conserving $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$ in $T^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is to be expected from Eq. (68), and requires no further discussion. On the other hand, there are no immediately obvious reasons why the asymptotic behavior of $\bar{\Phi}_i^{s(+)}$ from (69) should be inconsistent with (133a). Nevertheless, the integral (133b) also is divergent. In fact [see section B.2], the right side of (133b) contains contributions proportional to

$$\delta\left(k_{12i} - \left|\underset{\sim}{K}_{23f} + \frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}\right|\right) \tag{135a}$$

and cyclic permutations thereof. Using Eqs. (29) and the relation $\underset{\sim}{K}_i = \underset{\sim}{K}_f$, the expression (135a) takes the form

$$\delta\left(k_{12i} - \left|\underset{\sim}{k}_{12i} + \underset{\sim}{k}_{1f} - \underset{\sim}{k}_{1i}\right|\right) \tag{135b}$$

wherein the argument of the δ-function obviously can be zero on the energy-momentum shell. Consequently the contribution to (133b) made by (135a)--when squared in Eqs. (127) after replacing $\bar{T}$ by $\bar{T}^s$-- again causes $\bar{w}$ and $\bar{\mathfrak{F}}$ to diverge, although the divergence is of lower order with the one-dimensional δ-function (135a) than with the three-dimensional δ-function contribution (130c) to $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$.

Judging by our earlier experience in this chapter, therefore, the assertion in (133a) that $\lim \bar{\Phi}_i^{(+)}$ is $\sim e^{i\bar{\rho}\sqrt{E}}/\bar{\rho}^{5/2}$ must be incorrect. In fact, it is shown in section E.3 that there are contributions to $\bar{\Phi}_i^{s(+)}(\underset{\sim}{\bar{r}})$ behaving like $\bar{\rho}^{-2}$ as $\bar{\rho} \to \infty$. Correspondingly [see section E.2] it can be demonstrated that the contribution to the integral on the left side of (132) from the region $\bar{r}' > \bar{r}$ is non-negligible compared to $\bar{\rho}^{-5/2}$. Thus (as the result of section

E.3 confirms), the subtraction--of terms from $\bar{\Phi}_i^{(+)}$--yielding $\bar{\Phi}_i^{s(+)}$ is not yet sufficient to permit interchange of order of integration and limit $\bar{r} \to \infty$ in (132), although the $\bar{r}' > \bar{r}$ contribution to the left side of (132) is smaller than the corresponding contribution to the left side of (125) [compare the results of sections E.2 - E.3 and C.4]. It is additionally noteworthy that these results [of sections E.2 and E.3] hold whether or not bound states exist.

Moreover, still consistent with our previous experience, the result (135a) is physically interpretable. The particular δ-function (135a) arises in the contribution to (133b) made by the term $\bar{\Psi}_f^{(-)*} V_{23} \bar{\Phi}_{12}^{(+)}$; more specifically [see section B.2], (135a) is obtained from the

$$\bar{\Psi}_f^{*} + \bar{\Phi}_{23f}^{(-)*} = \bar{\Psi}_{23f}^{(-)*} \tag{136}$$

part of $\bar{\Psi}_f^{(-)*}$ in the aforementioned term. But one sees--using Eqs. (51), (60), (77a) and (81), together with the Lippmann-Schwinger equation for $\bar{\Psi}_{23f}^{(-)*}$ analogous to Eqs. (107)--that

$$\bar{\Psi}_{23f}^{(-)*} V_{23} \bar{\Phi}_{12}^{(+)} = - \bar{\Psi}_{23f}^{(-)*} V_{23} \left\{ \lim_{\varepsilon \to 0} \bar{G}_{12}(\bar{E} + i\varepsilon) V_{12} \bar{\Psi}_i \right\} . \tag{137a}$$

$$= - \lim_{\varepsilon \to o} \bar{\Psi}_{23f}^{(-)*} V_{23} \bar{G}_{12}(\bar{E} + i\varepsilon) V_{12} \bar{\psi}_i$$

$$= - \lim_{\varepsilon \to o} \bar{\Psi}_{23f}(\bar{E} + i\varepsilon) V_{23} \bar{G}_F(\bar{E} + i\varepsilon) \bar{\underset{\sim}{T}}_{12}(\bar{E} + i\varepsilon) \bar{\psi}_i \qquad (137b)$$

$$= - \lim_{\varepsilon \to o} \bar{\psi}_f^* V_{23} \bar{G}_{23}(\bar{E} + i\varepsilon) \bar{\underset{\sim}{T}}_{12}(\bar{E} + i\varepsilon) \bar{\psi}_i$$

$$= - \lim_{\varepsilon \to o} \bar{\psi}_f^* \bar{\underset{\sim}{T}}_{23}(\bar{E} + i\varepsilon) \bar{G}_F(\bar{E} + i\varepsilon) \bar{T}_{12}(\bar{E} + i\varepsilon) \bar{\psi}_i$$

where, for our present purely interpretative purposes, interchange of order of integration and limit $\varepsilon \to 0$ in (137a) is permissible. The matrix element (137b) is explicitly discussed on p. 59 of Watson and Nuttall[4], and obviously is representable by a double scattering diagram [see also section 5.3 below]. To be precise, (137b) corresponds to a diagram wherein there is first a purely two-body scattering of particles 1 and 2 (the factor $\bar{\underset{\sim}{T}}_{12}$), followed by a period of free propagation (the factor $\bar{G}_F$) and then a second final purely two-body scattering of particles 2 and 3.

The preceding two paragraphs justify the conclusion that the δ-functions (135) arise from contributions to $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$ which--because they arise from two successive purely two-body elastic scatterings--cannot (and do not) behave like truly three-body scattered waves at large $\bar{\underset{\sim}{r}}$. This conclusion is reinforced by the fact that the vanishing of the argument of the δ-function (135b) really does guarantee the necessary relations between initial and final momenta following the two independent successive two-particle scattering events associated with the diagram representing (137b)--namely first particle 2 is scattered by 1 with 3 playing no role, after which particle 1 plays no further role as 2 is scattered by 3. Without postulating that the total initial momentum $\hbar \underset{\sim}{K} = 0$, let the momenta (in units of $\hbar$) of 1, 2 respectively after the first scattering be $\underset{\sim}{k}_1'$, $\underset{\sim}{k}_2'$ where

$$\underset{\sim}{k}_1' + \underset{\sim}{k}_2' = \underset{\sim}{k}_{1i} + \underset{\sim}{k}_{2i} \tag{138}$$

Since the first scattering is an elastic collision between 1 and 2, $k_{12}' = k_{12i}$, i.e.,

$$(m_1+m_2)k_{12i} = |m_2\underset{\sim}{k}_1' - m_1\underset{\sim}{k}_2'| = |(m_1+m_2)\underset{\sim}{k}_1' - m_1(\underset{\sim}{k}_{1i}+\underset{\sim}{k}_{2i})| \tag{139a}$$

using (138). With the definition (29d) of $\underset{\sim}{k}_{12}$, Eq. (139a) can be put in the form

$$k_{12i} = |\underset{\sim}{k}_{12i} + \underset{\sim}{k}_1' - \underset{\sim}{k}_{1i}| \tag{139b}$$

But since particle 1 is unaffected in the second scattering, $\underset{\sim}{k}_1' = \underset{\sim}{k}_{1f}$, making (139b) identical with the condition for which the argument of the δ-function (135b) vanishes. Other permutations of such two successive two-particle scatterings are associated of course with corresponding permutations of (135b), which in turn correspond to other [than (137a)] terms in (133b).

Similar results (to those already discussed) pertain also to derivations of $T^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ or $\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ from (134a) or (133a) respectively, starting from the expressions for $\Phi_i^{s(+)}$ or $\bar{\Phi}_i^{s(+)}$ given by (84c) or its center of mass analogue. In particular, one thus finds

$$\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \Big[\bar{\Phi}^{(-)*}_{12f}(V_{23}+V_{31}) + \bar{\Phi}^{(-)*}_{23f}(V_{23}+V_{31}) + \bar{\Phi}^{(-)*}_{31f}(V_{12}+V_{23})\Big]\bar{\Psi}_i^{(+)} \qquad (140)$$

while $T^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ is given by the laboratory system analogue of (140) and obeys (134b). Eq. (140) obviously is the time-reversed analogue of (133b), and equally obviously suffers from the same deficiencies--i.e., contains the same double-scattering divergences--as does (133b).

I now observe that the on-shell δ-functions discussed in this subsection, and in subsections 4.1.1 - 4.1.2, illustrate what appears to be a general relation between the asymptotic behavior of any part of $\Phi_i^{(+)}$ (or $\bar{\Phi}_i^{(+)}$), e.g., $\Phi_{12}^{(+)}$ (or $\bar{\Phi}_{12}^{(+)}$), and the dimensionality of the δ-function in the contribution this same part makes to the laboratory or center of mass scattering amplitude. Specifically, as $\underset{\sim}{r} \rightarrow \infty$ along directions $\underset{\sim}{\nu}_f \neq \underset{\sim}{\nu}_{\alpha\beta}$, i.e., along directions $\underset{\sim}{\nu}_f$ not corresponding to the possibility of propagation in bound states: (a) $\bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}})$, cf. Eq. (72), decreases like $\phi_{12}^{(+)}(\underset{\sim}{r}_{12})$, i.e., like $\bar{r}^{\,-1} \cong \bar{\rho}^{\,-1}$, and the associated contribution to $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ contains the three-dimensional δ-function $\delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i})$ [recall the discussion preceding and following Eqs. (131)]; (b) the laboratory frame $\Phi_{12}^{(+)}(\underset{\sim}{r}_{12})$ still decreases like $\phi_{12}^{(+)}(\underset{\sim}{r}_{12})$, i.e., like $r^{-1} \cong \rho^{-1}$, and the associated contribution to $T(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ contains a six-dimensional δ-function, namely $\delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i})$ multiplied by $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$; (c) according to section E.3, there are parts of $\bar{\Phi}_i^{s(+)}(\bar{r})$ decreasing like $\bar{\rho}^{\,-2}$, and these parts apparently give rise to the one-dimensional δ-functions (135) contained in $\bar{T}^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$.

Evidently in the laboratory frame the rule is: as $\underset{\sim}{r} \rightarrow \infty \,||\, \underset{\sim}{\nu}_f \neq \underset{\sim}{\nu}_{\alpha\beta}$, if the part of $\Phi_i^{(+)}(\underset{\sim}{r})$ under consideration decreases like $\rho^{x/2}\rho^{-4}$, where x is an integer ≥ 0, then the associated contribution to $T(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ contains a δ-function of dimensionality x. Similarly, in the center of mass frame, if as $\bar{\underset{\sim}{r}} \rightarrow \infty \parallel \bar{\underset{\sim}{\nu}}_f \neq \bar{\underset{\sim}{\nu}}_{\alpha\beta}$ the part of $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}})$ under consideration decreases like $\bar{\rho}^{y/2}\,\bar{\rho}^{-5/2}$, where y is an integer ≥ 0, then the associated contribution to $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ contains a δ-function of dimensionality y. Of course, because (55b) holds, the δ-function dimensionalities associated with corresponding values of x and y are related by $x = y + 3$. Moreover, these rules can be understood. Along directions $\underset{\sim}{\nu}_f \neq \underset{\sim}{\nu}_{\alpha\beta}$, the scattered part of $\Psi_i^{(+)}(\underset{\sim}{r})$ normally would be expected to diverge like an outgoing spherical wave in nine dimensions, i.e., like $G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$, which is of order ρ^{-4} at large r. The amplitude with which $\Phi_i^{(+)}(\underset{\sim}{r})$ diverges along $\underset{\sim}{\nu}_f$ is measured by $T(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ of Eqs. (120). Because of special symmetries in the interaction V, however, all or parts of $\Phi_i^{(+)}(\underset{\sim}{r})$ may not be able to diverge in a fully nine-dimensional fashion along all $\underset{\sim}{\nu}_f \neq \underset{\sim}{\nu}_{\alpha\beta}$. These inabilities mean $\Phi_i^{(+)}(\underset{\sim}{r})$ or parts thereof are being forced to diverge in a restricted space of less than nine dimensions, i.e., that $\Phi_i^{(+)}(\underset{\sim}{r})$ or parts thereof actually will decrease asymptotically like $\rho^{x/2}\rho^{-4}$, where x is an integer ≥ 0, and where $x > 0$ corresponds to restricted propagation in the sense just described. Correspondingly, for $\Phi_i^{(+)}(\underset{\sim}{r})$ or parts thereof with $x > 0$, postulating (120a) is wrong; the resultant δ-functions in $T(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ reflect the failure of (120a), as has been discussed, but also express the x independent aforementioned restrictions on the directions $\underset{\sim}{\nu}_f$ into which—for given $\underset{\sim}{\nu}_i$—$\Phi_i^{(+)}(\underset{\sim}{r})$ or parts thereof can propagate. For example, the fact that V is independent of $\underset{\sim}{R}$ means $\Phi_i^{(+)}(\underset{\sim}{r})$ has a factor $e^{i\underset{\sim}{K}_i \cdot \underset{\sim}{R}}$, so that no part of $\Phi_i^{(+)}(\underset{\sim}{r})$ can be diverging in a space of more than six dimensions (the space of $\bar{\underset{\sim}{r}} \equiv \underset{\sim}{r}_{12}, \underset{\sim}{q}_{12}$), i.e., even $\Phi_i^{t(+)}(\underset{\sim}{r})$—the "truly" three-body scatted part of $\Phi_i^{(+)}(\underset{\sim}{r})$-- decreases asymptotically no more rapidly than $\rho^{-5/2}$; correspondingly, even the truly three-body scattering amplitude $T^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ will have the three-dimensional $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$ factor required by (128), which factor also expresses the fact that $\Phi_i^{t(+)}(\underset{\sim}{r})$ actually is propagating to infinity only along directions $\underset{\sim}{\nu}_f$ consistent with the three independent requirements $\underset{\sim}{K}_f = \underset{\sim}{K}_i$. **The** center of mass frame rule cited above is similarly understood. The considerations of this paragraph make it quite clear that the complicated analysis in section E.3 is basically correct, i.e., it now is quite clear that the presence of the one-dimensional δ-functions (135) deduced in section B.2 must be associated with the existence of contributions to $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$ behaving asymptotically like $\bar{\rho}^{-2}$.

4.1.4 Divergences Associated with Bound States

In addition to the on-shell δ-functions which have been discussed, the amplitudes T and $\bar{T}$ given respectively by Eqs. (120) and (126) contain off-shell δ-functions when two-body bound states exist. These off-shell δ-functions in T and $\bar{T}$ have essentially the same form as those [e.g., Eq. (47)] occurring in Eqs. (52), and their presence in the integrals (120b) and (126b) is demonstrated via essentially the same argument as was employed [in sections A.4 - A.6] for Eqs. (52). For example, because $\Psi_f^{(-)*}(\underset{\sim}{r}')$ in (120b)--like $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$ in (52a)--can contain a term proportional to $e^{i\rho'\sqrt{E - \varepsilon_j}}\, u_j(\underset{\sim}{r}_{12}')/\rho'^{5/2}$, the V_{12} term in (120b) contains a contribution behaving like [see section A.4]

$$\delta\left(\sqrt{E_f - \varepsilon_j} - \sqrt{E_i - (\hbar^2 k_{12i}^2/2\mu_{12})}\right) \tag{141a}$$

when there is a bound state $u_j(\underset{\sim}{r}_{12})$ of energy ε_j into which particles 1 and 2 can combine during the collision. The corresponding contribution to (126b) is proportional to [see section A.6]

$$\delta\left(K_{12jf} - K_{12i}\right) \tag{141b}$$

where Eq. (114b) defines K_{12jf} in terms of $\bar{E}_f$; the δ-function (141b) is the result to which (141a) reduces [except for constant factors] when $\underset{\sim}{K}_f$ is set equal to $\underset{\sim}{K}_i$. I note that these δ-function contributions to T or $\bar{T}$ arise from the asymptotic behavior of $\Phi_i^{(+)}$ or $\bar{\Phi}_i^{(+)}$ at large distances, and therefore are associated only with those bound states $u_j(\underset{\sim}{r}_{\alpha\beta})$ which actually can be formed during the collision of three initially free particles; in Eq. (52a), on the other hand, δ-functions are associated with all possible bound states of the three-particle system, because all such bound states are present in the asymptotic limit of $G^{(+)}(\underset{\sim}{r};\underset{\sim}{r}')$ at large $\underset{\sim}{r}'$. For example, because energy-momentum conservation prevents three initially free particles from combining into a three-body $u_j(\underset{\sim}{r}_{12}, \underset{\sim}{r}_{23})$, the existence of three-body states does not cause (120b) to diverge, though such states do produce divergences in (52a) [see section C.5].

The presence of the divergences (141a) or (141b) has the usual significance, namely that Eqs. (123) or (125) respectively must be invalid. In particular [see section C.5], the δ-functions (141a) indicate that at large r the integral on the left side of (123) has non-negligible contributions--compared to r^{-4}--from bound state propagation in the region r' > r along $\underset{\sim}{\nu}_{12}'$ (where r_{12}' remains finite as $r' \to \infty$), much as in the analogous integral on the left side of (99) [where $r'' \to \infty$ along $\underset{\sim}{\nu}_{12}''$, recall section C.1]. Of course, these contributions to the left side of (123) from $r' \to \infty$ along $\underset{\sim}{\nu}_{12}'$ are in addition to--and in no way negate--the contributions dominating r^{-4} from $r' \to \infty$ along arbitrary directions $\underset{\sim}{\nu}'$, to which we ascribed the failures of (123) discussed in the preceding subsections. Moreover, as (by now) is to be expected, the δ-functions signaling the failures of (123) or (125) due to bound states are readily interpretable. For instance, the δ-function (141b) corresponds to conservation of the energy of particle 3 relative to an observer moving with the center of mass of the entire system, as is physically reasonable for a contribution to the V_{12} term in (126b) associated with formation of the bound state $u_j(\underset{\sim}{r}_{12})$.

Nevertheless, despite this possibility of interpretation, it is doubtful that the δ-functions (141) occurring in (120) and (126) have any physical significance whatsoever. I now am contrasting the δ-functions (141) with those discussed in subsections 4.1.1 - 4.1.3. Admittedly the δ-functions in subsections 4.1.1 - 4.1.3, like the δ-functions of this subsection, are encountered in the configuration space formulation of scattering theory under present consideration solely because invalid mathematical manipulations have been performed. Naturally, such invalid manipulations always should be avoided if possible, especially if they lead to expressions for presumably physically meaningful quantities--namely transition amplitudes--involving non-convergent integrals. There is no immediately urgent physical reason for introducing valid mathematical procedures so as to avoid the δ-functions of this subsection, however, since these δ-functions make no contributions to the scattering coefficients w or $\bar{\bar{w}}$ computed from Eqs. (121b) or (127c), by virtue of the fact that their arguments do not vanish on the energy-momentum shell [e.g., remembering $\epsilon_j < 0$, Eqs. (35) and (114b) show the δ-function (141b) cannot be infinite on the center of mass system energy shell $\bar{E}_f = \bar{E}_i$]; the δ-functions of subsections 4.1.1 - 4.1.3, being on-shell, make infinite contributions to Eqs. (121b) or (127c), and so must be avoided via mathematically acceptable procedures--or at the very least via some sort of reinterpretation [recall the remarks following Eq. (122), and see section 4.2]--if physically sensible reaction coefficients are to be computed.

Moreover, the steps which must be taken to avoid the δ-functions of subsections 4.1.1 - 4.1.3 are physically as well as mathematically significant. Section 4.1.1 implies that the probability current flow must be computed in the center of mass frame; section 4.1.2 means that two-body scattering terms must be subtracted from $\bar{\Phi}_i^{(+)}$ before the computation of the three-body scattered current flow is initiated; and section 4.1.3 shows that it will be necessary to initially subtract from $\bar{\Phi}_i^{(+)}$ certain double-scattering terms as well. The δ-functions of this subsection, on the other hand, are eliminated without any subtraction merely by starting from the iterated formula for $\bar{\Phi}_i^{(+)}$ implied by (61) and (69), instead of--as heretofore in this section--from the formula (52b). More precisely, start from (61) and (69), but use the formula

$$\bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}}) = -\int d\bar{\underset{\sim}{r}}'\,\bar{G}_{12}^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')\,V_{12}(\bar{r}_{12}')\,\bar{\Psi}_i(\bar{r}') \tag{142}$$

in place of the known closed form result for $\bar{\Phi}_{12}^{(+)}$ given by (72). Then, performing on all integrals in the formula for $\bar{\Phi}_i^{(+)}$ the usual invalid interchange of order of integration and limit $\bar{r} \to \infty$, one again obtains (126a), but now with

$$\begin{aligned}\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = {} & \bar{\Psi}_{12f}^{(-)*} V_{12} \bar{\Psi}_i + \bar{\Psi}_{23f}^{(-)*} V_{23} \bar{\Psi}_i \\ & + \bar{\Psi}_{31f}^{(-)*} V_{31} \bar{\Psi}_i + \bar{T}^{\Delta}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)\end{aligned} \tag{143}$$

where $\bar{T}^S(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is given by (133b). Section B.2 shows that (133b), though of course still containing the δ-functions discussed in subsection 4.1.3, has no δ-functions of the type (141b) associated with bound states. Similarly, the other terms on the right side of (143) contain no δ-functions associated with bound states; in fact, recalling Eqs. (105) one sees that, e.g., the quantity defined by (130a) and evaluated in (130c) [which obviously contains no δ-functions of type (141b)] is identical with the quantity $\bar{\Psi}_{12f}^{(-)*} V_{12} \bar{\psi}_i$ on the right side of (143). Section E.2 shows that bound state propagation does not invalidate Eq. (132), consistent with the absence of bound state δ-function divergences in (133b).

I stress that the preceding two paragraphs do not mean that the presence of these δ-functions (141) in (120b) and (126b) is wholly inconsequential. As subsections 4.1.1 and 4.1.2 taken together illustrate, it may be easier to take account of some on-shell δ-functions than of the off-shell δ-functions (141), which [if the on-shell divergences were not present] would cause the integrals (120b) and (126b) to be oscillatory. In particular, approximate estimates of $\bar{\Psi}_f^{(-)*}$ can be constructed which—when inserted into (126b) so as to obtain approximate estimates of $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$—enable essentially exact subtraction of the single scattering two-body contributions [known exactly from (130c)] on the right side of (143); however, even if there were not the double scattering complications discussed in subsection 4.1.3, such approximate $\bar{\Psi}_f^{(-)*}$ probably would give very poor estimates of $\bar{T}^S(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ because of now non-vanishing contributions from the δ-functions (141b) [compare the discussion of the significance of the δ-functions (47), in section 2.2 following Eq. (48)]. Of course, this particular difficulty associated with the δ-functions (141) is perforce avoided when $\bar{T}^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ is estimated starting from Eq. (140). I also point out that the preceding two paragraphs must not be taken to imply that the exact $\bar{T}^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$--or better yet the exact truly three-body amplitude $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ obtained from the asymptotic behavior of $\bar{\Phi}_i^{t(+)}$--do not have singularities (as functions of $\underset{\sim}{k}_f$) where the arguments of the δ-functions (141b) vanish. I merely am insisting that the presence of off-shell singularities cannot be inferred legitimately from oscillatory (i.e., mathematically undefined) on-shell integrals for $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$. It is necessary to start with a convergent integral [e.g., $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ or $\bar{T}^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ with the double-scattering contributions (135) subtracted out]. The analytic continuation of this originally convergent integral well might have singularities at $K_{12jf} = K_{12i}$; on the other hand there is no reason to think these now legitimately inferred singularities at $K_{12jf} = K_{12i}$, if actually found to exist, would be of the δ-function (141b) type.

It is worth noting that the off-shell δ-functions we have been discussing show up in the expression (124a) for $T(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ even though such δ-functions do not appear in the real energy Lippmann-Schwinger integral equation (42) from which (124a) is derived. Correspondingly, bound state propagation invalidates the interchange of order of integration and limit $\underset{\sim}{r} \rightarrow \infty \parallel \underset{\sim}{\nu}_f$ in the integral on the right side of Eq. (42), even though (42)--unlike (52a)--is convergent; specifically, at large r the integral (42) has contributions of order $r^{-4} \cong \rho^{-4}$ from bound state propagation in the integration region $r' > r$, as shown in section C.5. Similar comments pertain to the expression (131) for $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$.

I conclude this subsection with some remarks stemming from the relation (131d). Although the argument leading to (131d) is unsatisfactory [as has been explained], nevertheless the results of this entire section 4.1 probably are relevant to the physical significance of $< f|\bar{\underset{\sim}{T}}(\bar{\lambda})|i >$ and the δ-functions contained therein, when i, f each denote center of mass plane wave states, and when $\bar{\lambda}$ equals one or both of $\bar{E}_i$, $\bar{E}_f$. At the moment, however, I am not prepared to state precisely how the considerations of this particular subsection 4.1.4 relate to the bound state singularities found by Rubin et al.[(5)], who [in the special case of Yukawa interactions $V_{\alpha\beta}$] examine $< f|\bar{\underset{\sim}{T}}(\bar{\lambda})|i >$ as a function of $\bar{\lambda}$ for fixed assigned physical values of the vectors $\underset{\sim}{y}_{\alpha i} = (2m_\alpha\bar{\lambda})^{-1/2}\underset{\sim}{k}_{\alpha i}$ and $\underset{\sim}{y}_{\alpha f} = (2m_\alpha\bar{\lambda})^{-1/2}\underset{\sim}{k}_{\alpha f}$ associated with the i, f plane waves respectively.

4.2 Volume Dependence of Reaction Rates

Eqs. (2) and (121c) imply that w defined by Eq. (121a) and $\bar{w}$ defined by Eq. (127a) are related by

$$w(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \tau \bar{w}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) \tag{144}$$

where τ is the large volume within which the three-body scattering of present interest is taking place. That the ratio $w/\bar{w}$ must be a quantity having the dimensions of volume can be seen simply from comparison of the right sides of Eqs. (117a) and (118a). The quantity $\bar{\underset{\sim}{W}}$ is defined via Green's Theorem in the center of mass space in complete analogy[16] with Eqs. (45), and therefore has the same dimensions as $\underset{\sim}{W}$ (since the center of mass kinetic energy operator $\bar{T}$ has the same dimensions as T); however, the laboratory system surface element at infinity dS is eight-dimensional in the present three-particle problem, whereas $d\bar{S}$ is merely five-dimensional. The particular relation (144) is obtained from an argument given previously[2]. From Eq. (128),

$$\left|T(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)\right|^2 = (2\pi)^6 \left[\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\right]^2 \left|\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)\right|^2 \tag{145}$$

But, as pointed out beneath Eq. (122), to make w physically meaningful, one of the δ-functions in (145a) must be eliminated, presumably via some reinterpretation of $\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$. A natural reinterpretation is

$$\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i) = \frac{1}{(2\pi)^3}\int d\underset{\sim}{R}\, e^{i(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\cdot \underset{\sim}{R}} \tag{146a}$$

$$[\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)]^2 = \frac{1}{(2\pi)^6}\int d\underset{\sim}{R}\, e^{i(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\cdot R}\int d\underset{\sim}{R}'\, e^{i(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\cdot \underset{\sim}{R}'}$$

$$= \frac{1}{(2\pi)^3}\,\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\int d\underset{\sim}{R}'\, e^{i(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\cdot \underset{\sim}{R}'} \tag{146b}$$

$$= \frac{1}{(2\pi)^3}\,\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\int d\underset{\sim}{R}' \cong \frac{1}{(2\pi)^3}\,\delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\,\tau$$

Using (145) as reinterpreted by (146b) in (121b), and comparing with (127c), yields (144).

Actually, because of the on-shell divergences discussed in subsections 4.1.2 and 4.1.3, use of (146b) is insufficient to make physically meaningful the quantities w and $\bar{w}$ of Eqs. (121b) and (127c). However, the procedure of Eqs. (146) can be employed to eliminate all troublesome squares of δ-functions in (121b) and (127c), thus ultimately yielding finite (in any finite volume τ) probability current flows $\mathfrak{F}$ and $\bar{\mathfrak{F}}$. Thus, in, e.g., the contribution (130c) to $\bar{T}(\underset{\sim}{k}_{i} \to \underset{\sim}{k}_{f})$

$$\left|\bar{\Psi}^{(-)*}_{12f} V_{12} \bar{\Psi}_i\right|^2 = (2\pi)^6 \delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i})^2 \left|t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})\right|^2$$
$$\cong (2\pi)^3 \tau\, \delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i}) \left|t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})\right|^2 \quad (147)$$

Inserting (147) into (127c), one sees that $\bar{w}(i \to f)$ has a contribution I will call $\bar{w}_{12}^{(3)}(i \to f)$--corresponding to purely two-body elastic scattering of 1 and 2 in the three-particle system--given by

$$\bar{w}^{(3)}_{12}(i \to f) = \frac{2\pi}{\hbar} \cdot \frac{1}{(2\pi)^3} \tau \left|t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})\right|^2 \delta(E_f - E_i)\, \delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)$$
$$\times\, \delta(\underset{\sim}{K}_{12f} - \underset{\sim}{K}_{12i})\, d\underset{\sim}{k}_{1f}\, d\underset{\sim}{k}_{2f}\, d\underset{\sim}{k}_{3f} \quad (148a)$$

$$= \frac{2\pi}{\hbar} \cdot \frac{1}{(2\pi)^3} \tau \left|t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})\right|^2 \delta(E_f - E_i)\, \delta(\underset{\sim}{K}_f - K_i)$$
$$\times\, \delta(\underset{\sim}{k}_{3f} - \underset{\sim}{k}_{3i})\, d\underset{\sim}{k}_{1f}\, d\underset{\sim}{k}_{2f}\, d\underset{\sim}{k}_{3f} \quad (148b)$$

using the second equality in (29c) for $\underset{\sim}{K}_{12}$. Therefore, integrating (148b) over $d\underset{\sim}{k}_{3f}$

$$\bar{w}^{(3)}_{12}(i \to f) = \frac{2\pi}{\hbar} \frac{1}{(2\pi)^3} \tau \left|t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})\right|^2 \delta\left\{\frac{\hbar^2}{2}\left[\left(\frac{k_1^2}{m_1} + \frac{k_2^2}{m_2}\right)_f - \left(\frac{k_1^2}{m_1} + \frac{k_2^2}{m_2}\right)_i\right]\right\}$$
$$\times\, \delta(\underset{\sim}{k}_{1f} + \underset{\sim}{k}_{2f} - \underset{\sim}{k}_{1i} - \underset{\sim}{k}_{2i})\, d\underset{\sim}{k}_{1f}\, d\underset{\sim}{k}_{2f} \quad (148c)$$

Or

$$\bar{W}_{12}^{(3)}(i \to f) = \Upsilon\, \bar{W}_{12}^{(2)}(i \to f) \tag{149a}$$

where

$$\bar{W}_{12}^{(2)}(i \to f) = \frac{2\pi}{\hbar} \frac{1}{(2\pi)^3} \left| t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f}) \right|^2 \delta\left(\frac{\hbar^2 k_{12f}^2}{2\mu_{12}} - \frac{\hbar^2 k_{12i}^2}{2\mu_{12}} \right) d\underset{\sim}{k}_{12f} \tag{149b}$$

$$= \frac{2\pi}{\hbar} \frac{1}{(2\pi)^3} \left| t_{12}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f}) \right|^2 \delta(E_f - E_i)\, \delta(\underset{\sim}{K}_f - \underset{\sim}{K}_i)\, d\underset{\sim}{k}_{1f}\, d\underset{\sim}{k}_{2f} \tag{149c}$$

In Eqs, (149), $\bar{w}_{12}^{(2)}(i \to f)$ represents the conventional elastic scattering coefficient for particles 1 and 2 in their center of mass frame; the definitions of the two-particle total energies E and total momenta $\underset{\sim}{K}$ in (149c) are obvious. It is understood that particle 3 never appears in the computation of $\bar{w}_{12}^{(2)}$ or its laboratory frame analogue $w_{12}^{(2)}$; in particular, these quantities are computed using Schroedinger's equation for particles 1, 2 only, with incident waves--in the laboratory and center of mass frames respectively--

$$\psi_i = e^{i(\underset{\sim}{k}_1 \cdot \underset{\sim}{r}_1 + \underset{\sim}{k}_2 \cdot \underset{\sim}{r}_2)} \tag{150a}$$

$$\bar{\psi}_i = e^{i \underset{\sim}{k}_{12} \cdot \underset{\sim}{r}_{12}} \tag{150b}$$

Furthermore, Eqs. (149b) - (149c), which are the two-particle system analogues of Eqs. (127b) - (127c), can be derived without any improper mathematical manipulations, because with the incident wave (150b) the problem of computing $\bar{w}_{12}^{(2)}(i \to f)$ reduces to potential scattering; correspondingly, $\bar{w}_{12}^{(2)}$ is assuredly τ-independent. But, using (144), to the relation (149a) corresponds

$$w_{12}^{(3)}(i \to f) = \tau^2 \bar{w}_{12}^{(2)}(i \to f) \tag{151}$$

Therefore, as foreshadowed in Chapter 1, Eq. (121b)--taken together with (121c) or (2)--implies the quantity $\tau^{-1}\,\hat{w}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$, supposedly representing the actually observed scattering rate per unit volume into wave number ranges $d\underset{\sim}{k}_{1f}, d\underset{\sim}{k}_{2f}, d\underset{\sim}{k}_{3f}$, will not be independent of the volume in the limit $\tau \to \infty$ for all $\underset{\sim}{k}_{1f}$, $\underset{\sim}{k}_{2f}$, $\underset{\sim}{k}_{3f}$. Rather, at

$\underset{\sim}{k}_{1f}$, $\underset{\sim}{k}_{2f}$, $\underset{\sim}{k}_{3f}$ consistent with the restrictions imposed by the three δ-functions in (148a) or (148b), $\tau^{-1}\hat{w}$ seemingly will be proportional to the reaction volume τ.

The above result is just another way of seeing that w and $\bar{w}$ of Eqs. (121b) and (127c) are not the "true" three-body elastic scattering coefficients; these, as discussed in Chapter 1, still will be computed from (121b) and (127c), except that the true three-body amplitudes $T^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ and $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$--determined by the asymptotic forms of $\Phi_i^{t(+)}$ and $\bar{\Phi}_i^{t(+)}$--will replace T and $\bar{T}$ respectively. This last remark suggests that the result (151)--having been deduced by a somewhat questionable argument (147), starting from formally divergent expressions (for T or $\bar{T}$) derived via invalid mathematical manipulations--does not have any physical significance. This suggestion is incorrect, however, as the immediately following subsection shows. Instead, the volume dependence of (151)--like the δ-functions of (128) and (130c) which are its source--is physically interpretable and, in fact, to be expected.

4.2.1 Volume Dependence and Incident Wave Normalization

One subject which has been ignored thus far in this work is the genesis of the relations (2) or (121c). To be more explicit, there is the following question which should be answered: Because the normalization of the incident wave (21a)--namely unit amplitude--is a purely arbitrary choice, how do I know that Eqs. (2) or (121c) relate the actually observed scattering rate to the probability current flows computed from ψ_i of (21a)? Or, to put it differently, granted I somehow have managed to determine the asymptotic forms of the truly three-body $\Phi_i^{t(+)}$ or $\bar{\Phi}_i^{t(+)}$ corresponding to the unit amplitude incident wave (21a), how do I know that the corresponding (presumably divergence-free and therefore τ-independent) center of mass frame "true" three-body coefficient $\bar{w}(i \rightarrow f)$ yields the expected laboratory frame reaction rate after multiplication by precisely $N_1N_2N_3\tau$?

Before trying to answer these questions for three-body scattering, let me try to answer their analogues for conventional two-body scattering of species 1 and 2, in the complete absence of species 3. In this latter event, the analogue of (2) is

$$\hat{w}_{12}^{(2)}(\underset{\sim}{k}_{1i},\underset{\sim}{k}_{2i} \rightarrow \underset{\sim}{k}_{1f},\underset{\sim}{k}_{2f}) = N_1N_2\tau\,\bar{w}_{12}^{(2)}(\underset{\sim}{k}_{1i},\underset{\sim}{k}_{2i} \rightarrow \underset{\sim}{k}_{1f},\underset{\sim}{k}_{2f}) \tag{152a}$$

where $\bar{w}_{12}^{(2)}$ is given by (149b) or (149c) and where $\hat{w}_{12}^{(2)}$ represents the observed scattering rate of particles 1, 2 into $d\underset{\sim}{k}_{1f}$, $d\underset{\sim}{k}_{2f}$, in a large volume τ containing particle species 1 and 2 only. Then the customary (and quite satisfactory) way of understanding the volume dependence of (152a) is as follows. One first observes that Eqs. (130) and (149b) imply

$$\bar{w}^{(2)}_{12}(i \to f) = |\underset{\sim}{v}_1 - \underset{\sim}{v}_2|\,\bar{\sigma}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})\,d\underset{\sim}{n}_{12f} \qquad (152b)$$

where $\bar{\sigma}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})$ is the conventional center of mass frame differential cross section for elastic scattering into the direction $\underset{\sim}{n}_{12f}$ of $\underset{\sim}{k}_{12f}$, computed as if for potential scattering of a particle having mass μ_{12} and incident wave vector $\underset{\sim}{k}_{12}$; $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ are the classical particle velocities, of course, and it is understood that scattering occurs only into $\underset{\sim}{k}_{1f}, \underset{\sim}{k}_{2f}$ consistent with energy-momentum conservation. By definition of the cross section, however, if a beam of particles 1--containing N_1 particles/cc with velocity $\underset{\sim}{v}_1$--is incident on a single particle 2, the number of elastic scattering events per second into $d\underset{\sim}{n}_{12f}$ is

$$N_1 |\underset{\sim}{v}_1 - \underset{\sim}{v}_2|\,\bar{\sigma}(\underset{\sim}{k}_{12i} \to \underset{\sim}{k}_{12f})\,d\underset{\sim}{n}_{12f} \qquad (152c)$$

The scattering rate $\hat{w}$ with $\hat{N}_2 = N_2\tau$ scatterers will be $\hat{N}_2$ times (152c), which--using (152b)--is precisely the result (152a).

The foregoing interpretation of (152a) is not readily generalized to collisions involving three incident particles because for three-body collisions it is not readily possible to find a quantity playing the role of the cross section; there is no useful analogue of the cross section because the three-particle center of mass frame incident wave (33b) propagates in six rather than three dimensions, so that going to the center of mass frame does not reduce the three-particle collision to potential scattering. However, I now shall give an alternative interpretation of (152a) which--because it rests on considerations of the laboratory frame six-dimensional two-particle incident wave (150a)--is easily generalizable to collisions between three (or more) particles.

Obviously the average scattering rate from a volume τ containing randomly and uniformly distributed particles 1 and 2, in numbers $\hat{N}_1 = N_1\tau$ and $\hat{N}_2$, will be $\hat{N}_1\hat{N}_2$ times the average scattering rate from the same volume containing only a single particle 1 and a single particle 2, assuming these single particles each may be found anywhere in τ with uniform probability per unit volume. The incident plane wave function corresponding to (150a), but normalized to one particle 1 in τ and one particle 2 in τ is

$$\psi_i' = \frac{1}{\tau} e^{i(\underset{\sim}{k}_1 \cdot \underset{\sim}{r}_1 + \underset{\sim}{k}_2 \cdot \underset{\sim}{r}_2)} \tag{153a}$$

because, e.g., the probability of finding particle 1 in any $d\underset{\sim}{r}_1$ within τ is

$$d\underset{\sim}{r}_1 \int_\tau d\underset{\sim}{r}_2 \left|\psi_i'(\underset{\sim}{r}_1, \underset{\sim}{r}_2)\right|^2 = d\underset{\sim}{r}_1 \frac{1}{\tau^2} \int d\underset{\sim}{r}_2 = \frac{1}{\tau} d\underset{\sim}{r}_1 \tag{153b}$$

On the other hand, because Eq. (50a) shows $\Phi_i(E + i\epsilon)$--and therefore also its limit $\Phi_i^{(+)}(E)$--rigorously is multiplied by τ^{-1} when ψ_i is multiplied by τ^{-1}, it follows from Eqs. (45a) and (117a) that the outgoing probability current flow computed with ψ_i' of (153a) is precisely τ^{-2} times the corresponding flow computed with ψ_i of (150a). In other words, recognizing that the definition (121a) of w applies to two-particle as well as to three-particle systems, the scattered probability current flow computed with ψ_i' of (153a) yields precisely

$$\hat{w}'^{(2)}_{12}(i \rightarrow f) = \tau^{-2} w^{(2)}_{12}(i \rightarrow f) = \tau^{-1} \bar{w}^{(2)}_{12}(i \rightarrow f) \tag{154}$$

wherein the second equality holds because the conventional laboratory and center of mass frame two-particle scattering coefficients--$w_{12}^{(2)}$ and $\bar{w}_{12}^{(2)}$ respectively--also satisfy (144). Moreover, $\hat{w}'^{(2)}_{12}$ of (154), with $\bar{w}_{12}^{(2)}$ given by (149b), represents the scattering rate when a single particle 1 and a single particle 2 are to be found in τ. Multiplying (154) by $\hat{N}_1\hat{N}_2 = N_1N_2\tau^2$ again yields (152a).

The fact that (154) represents the scattering rate for a single pair of particles also can be understood on the following less exact but very physical basis. In a genuinely two-body collision involving short range forces, it can be assumed that scattering takes place only if the two particles 1 and 2 manage to get within a (possibly dependent on $|\underset{\sim}{v}_1 - \underset{\sim}{v}_2|$) distance b of each other, where the total elastic scattering cross section $\bar{\sigma} \cong \pi b^2$. In effect this relation defines the (dependent on relative velocity) quantity b; of course, often--but not necessarily--b turns out to equal approximately the range at which the interaction $V_{12}(\underset{\sim}{r}_{12})$ becomes negligibly different from zero. Now again consider a large volume τ containing precisely one particle 1 and one particle 2, each of which may be located anywhere in τ with equal probability per unit volume. Then at any given instant, in any given volume $\tau_o = b^3$, the probability of finding particle 1 in τ_o is τ_o/τ. Hence the probability that particles 1 and 2 are scattering within τ_o at any given instant $= (\tau_o/\tau)^2$, the probability of simultaneously finding 1 and 2 within τ_o. The number of such volumes τ_o in τ is τ/τ_o. At any given instant, therefore, with one particle 1 and one particle 2 in τ, the probable number of scatterings taking place is τ_o/τ. To convert this result to a scattering rate per particle pair, one must divide by a time t_c representing the average "duration" of a collision, i.e., the average time a pair of particles remains within scattering range; this division by t_c recognizes that even with a large number of particle pairs in τ, scattering continues at a steady average rate only because particles complete one scattering event and move into a new volume τ_o, where they again have a chance τ_o/τ of scattering against any other given particle in τ. Hence the scattering rate per particle pair in τ is $\cong \tau_o/\tau t_c$. Since $t_c \cong |\underset{\sim}{v}_1 - \underset{\sim}{v}_2|^{-1} b$, this scattering rate per particle pair has exactly the form (154), recalling (152b) as well as the definitions in this paragraph relating τ_o and $\bar{\sigma}$ to b.

Now, having managed to give simple laboratory system interpretations of (152a), I turn to its analogous three-body relation (2). First, let me proceed inexactly, though **qualitatively** correctly, as in the preceding paragraph. A true three-body collision between particles 1, 2, 3 occurs only if the three particles simultaneously find themselves within some volume τ_o (possibly, but not necessarily, of the same order b^3 as in individual two-particle collisions between individual pairs α, β). With a single particle of each species α (α = 1, 2, 3) in a large volume τ, the probability of simultaneously finding all three particles in a given τ_o is $(\tau_o/\tau)^3$. Letting t_c again denote the average collision duration (now not as readily related as previously to the relative particle velocities), the true three-body scattering rate per triplet 1, 2, 3 in τ is

$$\hat{w}'(i \to f) \cong \left(\frac{\tau_o}{\tau}\right)^3 \frac{\tau}{\tau_o} \cdot \frac{1}{t_c} = \frac{\tau_o^2}{\tau^2 t_c} \tag{155a}$$

Therefore, the laboratory frame scattering rate with $\hat{N}_\alpha = N_\alpha \tau$ particles in τ is

$$\hat{w}(i \to f) = \hat{N}_1 \hat{N}_2 \hat{N}_3 \hat{w}' \cong N_1 N_2 N_3 \tau \frac{\tau_o^2}{t_c} \tag{155b}$$

Eq. (155b) has the form (2); in particular, it asserts that the measured laboratory scattering rate should be proportional to τ, as well as to $N_1N_2N_3$. If Eq. (2) now is regarded **merely** as a definition of the proportionality factor $\bar{w}$ between the actually observed three-body scattering rate $\hat{w}$ and $N_1N_2N_3\tau$, then (155b) shows

$$\bar{w} \cong \frac{r_o^2}{t_c} \tag{155c}$$

Thus, if Eq. (2) really provides a prediction of the measured $\hat{w}$ in terms of the true three-body reaction coefficient $\bar{w}$ determined from $\bar{\Phi}_i^{t(+)}$ [as this paper has been asserting] then calculations of this $\bar{w}$ from $\bar{\Phi}_i^{t(+)}$ should be consistent with (155c). In other words, the computed true three-body scattering coefficient $\bar{w}$ should turn out to be τ-independent, and should be interpretable as the square of a reaction volume divided by the collision duration.

I also can argue as in the next to the last paragraph above, wherein no approximations were made and no ill-defined average quantities (e.g., t_c) were introduced. The incident wave function corresponding to (21a), but normalized to one particle of each species 1, 2 and 3 in τ is [compare Eq. (153a)]

$$\psi_i' = \frac{1}{\tau^{3/2}} e^{i(\underset{\sim}{k}_1\cdot\underset{\sim}{r}_1 + \underset{\sim}{k}_2\cdot\underset{\sim}{r}_2 + \underset{\sim}{k}_3\cdot\underset{\sim}{r}_3)} \tag{156a}$$

Thus the true three-particle collision rate with one particle of each species in τ is precisely

$$\hat{w}'(i \to f) = \tau^{-3} w(i \to f) = \tau^{-2} \bar{w}(i \to f) \quad (156b)$$

where w, $\bar{w}$ here are supposed to be the true reaction coefficients determined from $\Phi_i^{t(+)}$, $\bar{\Phi}_i^{t(+)}$ corresponding to the conventional incident wave (21a), i.e., determined from the truly three-body parts of the conventional $\Phi_i^{(+)}$, $\bar{\Phi}_i^{(+)}$ whose asymptotic forms were examined in section 4.1. Multiplying the precise scattering rate (156b) per triplet by the number of triplets $\hat{N}_1\hat{N}_2\hat{N}_3 = N_1N_2N_3\tau^3$ in τ yields precisely Eq. (2); in other words, the argument of this paragraph implies that the measured scattering rate $\hat{w}$, and the reaction coefficient $\bar{w}$ determined as described in the preceding sentence, indeed must be related as in Eq. (2). Note that this present argument does not imply $\hat{w}$ is proportional to τ; $\bar{w}$ in (156b) might be τ-dependent, for all this present argument knows. However, the fact that the true three-body reaction coefficient $\bar{w}$ is independent of τ will become apparent when $\bar{w}$ is calculated correctly, i.e., starting from $\bar{\Phi}_i^{t(+)}$ and not employing any improper mathematical manipulations. Alternatively, having now shown $\bar{w}$ in (2) indeed must be the true three-body reaction coefficient, I can appeal to the considerations of the preceding paragraph--in particular to Eq. (155c)--thus inferring [without actual calculation of $\bar{w}$ from $\bar{\Phi}_i^{t(+)}$]--that such calculation will yield a $\bar{w}$ independent of τ. In fact, once Eqs. (155a) and (156b) each have been deduced, the relation (155c) for the true three-body reaction coefficient $\bar{w}$ of (156b) follows immediately, without any necessity for referring to Eq. (2).

This result answers the questions raised in the first paragraph of this subsection. I turn therefore to the problem of understanding (151). The quantity $w_{12}^{(3)}$ of Eq. (151) represents the laboratory frame coefficient for two-body scattering of 1, 2 when computed from the solution $\Psi_i^{(+)}$ to the three-particle Lippmann-Schwinger equation corresponding to the three-particle incident wave (21a). Now what two-body rate $w_{12}^{\prime(3)}$ of 1, 2 scattering should be expected with the incident wave ψ_i' of (156a)? The answer to this question, clearly, is the same rate (154) as was computed using the two-particle ψ_i' of (153a), because both these incident waves correspond to one particle 1 and one particle 2 in τ. In other words, it must be true that

$$\hat{w}_{12}^{\prime(3)}(i \to f) = \hat{w}_{12}^{\prime(2)}(i \to f) = \tau^{-1}\bar{w}_{12}^{(2)}(i \to f) \qquad (157a)$$

But, as explained previously following Eq. (153b), the probability current flow computed with ψ_i' of (156a) is precisely τ^{-3} times the corresponding flow computed with ψ_i of (21a). Therefore I see that with the incident wave (21a) I must expect to find a laboratory frame two-body coefficient

$$w_{12}^{(3)}(i \to f) = \tau^3 \hat{w}_{12}'^{(3)}(i \to f) = \tau^2 \bar{w}_{12}^{(2)}(i \to f) \qquad (157b)$$

which is precisely the result (151) obtained earlier from the expressions for $w_{12}^{(2)}$ and $w_{12}^{(3)}$ in terms of the matrix elements $\hat{T}_{12}(i \to f)$.

The argument in the preceding paragraph makes it apparent that the τ^2 dependence in $w_{12}^{(3)}(i \to f)$ is necessary if the predicted observed two-body scattering rate $\hat{w}_{12}^{(3)}$ using the three-body incident wave (21a) is to agree with the conventional prediction $\hat{w}_{12}^{(2)}$ of (152a) obtained using the two-body incident wave (150a). Indeed, one can say flatly that if adding an irrelevant particle 3 to the pair 1, 2 had changed the physical predictions, this publication's whole approach to many-particle collisions would have become very questionable. The preceding paragraph and earlier discussion in this subsection also suggest a simple series of rules for making the connection between collision theory and experiment, for any collision process and whatever the number of particles involved: (i) compute the reaction coefficient using unit amplitude waves; (ii) if the mathematics has involved invalid manipulations, so that on-shell δ-functions appear in the transition amplitudes, reinterpret them along the lines of Eqs. (146) - (147), permitting only the first powers of δ-functions to remain in w or $\bar{w}$; (iii) renormalize so as to correspond to an incident wave with one particle of each species in a volume τ; (iv) multiply by the appropriate number of particle pairs, triplets, tetrads, etc. (e.g., by $\hat{N}_1\hat{N}_2 = N_1N_2\tau^2$ for two-particle processes, by $N_1N_2N_3\tau^3$ for three-particle processes, etc.), to obtain the laboratory system reaction rate $\hat{w}$ in τ.

Granted I haven't proved the legitimacy of the above rules, this subsection makes it unlikely that they are not quite generally applicable. On the other hand, I must point out that especially rule (ii) above is dubious; certainly I have not shown that the prescribed replacement of powers of on-shell δ-functions by powers of τ always will make good physical sense, although the likelihood that this will be the case now seems much greater than previously might have been supposed. In particular, the next subsection will demonstrate that the τ-dependence implied by the double-scattering δ-functions (135) can be understood physically. Nevertheless, it is apparent that the results of this subsection in no way negate the results of previous sections. The presence of δ-functions in transition amplitudes still signals improper mathematical manipulations, generally reflecting the fact that erroneous assumptions have been made concerning the asymptotic dependence of the scattered wave terms whose limit as $\underset{\sim}{r} \to \infty$ is being extracted; the corresponding anomalous τ-dependences of computed reaction coefficients indicate the same fact from a different point of view, i.e., they indicate that physical processes other than those desired have been included, e.g., two-body scattering in the supposed three-body reaction coefficient.

Section 4.3 below illustrates the fact that qualitative arguments like those leading to Eqs. (155) can lead to a predicted center of mass reaction coefficient proportional to a negative power of τ. I believe that in this event the corresponding collision process either really is unobservably small in any large volume (in comparison with related competing processes), or at most has a laboratory system rate $\mathcal{R}(i \to f)$ proportional to τ; it also is possible that a predicted $\bar{\bar{w}}$ proportional to τ^{-z}, $z > 0$, means simply that the process under examination is essentially meaningless within the theoretical formulation adopted. In either case, the above rules probably are not applicable. It also seems reasonable that reaction coefficients $\bar{w}$ which really are physically proportional to τ^{-z}, $z > 0$, correspond to processes which--in the particular theoretical formulation adopted--depend on parts of $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}})$ decreasing more rapidly at large $\bar{\underset{\sim}{r}}$ than does the relevant free-space Green's function $\bar{G}_F^{J(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$, i.e., more rapidly than $\bar{r}^{-[3(J-1)-1]/2}$; here J is the number of independent aggregates moving outward to infinity in the laboratory system [J = 2 in a three-particle collision resulting in formation of bound states $u_j(\underset{\sim}{r}_{12})$ as, e.g., in Eq. (17a)], and $\bar{G}_F^{J(+)}$ has the dimensionality of the center of mass frame free space Green's function for a system of J elementary particles. Needless to say, I have not proved the immediately preceding assertion concerning $\bar{w} \sim \tau^{-z}$, $z > 0$; we have seen, however, that δ-functions in transition amplitudes generally lead to $\bar{w}$ proportional to positive powers of τ, and seem to be associated with terms in $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}})$ decreasing less rapidly than the relevant free space Green's function $\bar{G}_F^{J(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$ [recall the rules cited at the end of subsection 4.1.3].

4.2.2 Double Scattering Contributions

In this subsection I shall discuss the volume dependence implied by the δ-functions (135). To begin with, the briefest consideration of the contributions made by these δ-functions to w and $\bar{w}$ of (121b) and (127b) makes it evident that there is little hope of being able to compute precisely the anomalous τ-dependences these δ-functions yield. It is easily seen that the δ-functions (135) make contributions to $\bar{w}(i \to f)$ proportional to $\tau^{1/3}$, but the precise magnitudes of these contributions are essentially incalculable.

To make these last assertions more explicit, suppose I write, as in Eqs. (146) -(147),

$$\delta(k_{12i} - |\underset{\sim}{k}_{12f} + \underset{\sim}{k}_{1f} - \underset{\sim}{k}_{1i}|) \equiv \delta(k_{12i} - Q) = \frac{1}{2\pi}\int dx\, e^{ix(k_{12i} - Q)} \tag{158a}$$

$$[\delta(k_{12i} - Q)]^2 = \frac{1}{(2\pi)^2}\int dx\, e^{ix(k_{12i} - Q)} \int dx'\, e^{ix'(k_{12i} - Q)}$$

$$= \frac{1}{2\pi}\delta(k_{12i} - |\underset{\sim}{k}_{12f} + \underset{\sim}{k}_{1f} - \underset{\sim}{k}_{1i}|)\int dx' \tag{158b}$$

Then the one-dimensional integral over dx' in (158b) is not as readily interpretable as the three-dimensional integral over $d\underset{\sim}{R}'$ in (146b). Certainly the integral over dx' in (158b) can be assumed proportional to some average dimension of τ, i.e., proportional to $\tau^{1/3}$. The proportionality factor is ill-defined, however, and probably will depend on the shape of the large volume τ. In other words, the best I seem able to do is to replace (158b) by

$$\left[\delta\left(k_{12i} - \left|\underset{\sim}{k}_{12f} + \underset{\sim}{k}_{1f} - \underset{\sim}{k}_{1i}\right|\right)\right]^2 \cong \frac{1}{2\pi} C\, \delta\left(k_{12i} - \left|\underset{\sim}{k}_{12f} + \underset{\sim}{k}_{1f} - \underset{\sim}{k}_{1i}\right|\right) \tau^{1/3} \quad \text{(158c)}$$

where C is an unknown factor, dependent on the shape of the scattering region τ, but not on the magnitude of its volume. Recalling that (135a) is a contribution to $\bar{T}^s$ of (133b), and comparing with Eqs. (147) - (148), one sees that insertion of $\bar{T}^s$ into (127b) will yield a $\bar{w}(i \to f)$ containing terms surely proportional to $\tau^{1/3}$, but with unknown coefficients dependent on the shape of τ. The corresponding double scattering contributions to $w(i \to f)$ will be proportional to $\tau^{4/3}$, using the still applicable (144). I add that the rules cited at the end of subsection 4.1.3 now make it evident that when a part of $\Phi_i^{(+)}(\underset{\sim}{r})$ decreases like $\rho^{x/2}\,\rho^{-4}$ along $\underset{\sim}{\nu}_f \neq \underset{\sim}{\nu}_{\alpha\beta}$, $x \geq 0$, the associated contribution to $w(i \to f)$ will be proportional to $\tau^{x/3}$; equivalently, when a part of $\bar{\Phi}_i^{(+)}(\bar{\underset{\sim}{r}})$ decreases like $\bar{\rho}^{\,y/2}\,\bar{\rho}^{\,-5/2}$ along

$\bar{\underset{\sim}{\nu}}_f \neq \bar{\underset{\sim}{\nu}}_{\alpha\beta}$, the associated contribution to $\bar{w}(i \to f)$ will be proportional to $\tau^{y/3}$, assuming $y \geq 0$.

I now show that this $\tau^{4/3}$ dependence of double-scattering contributions to $w(i \to f)$--like the τ^2 dependence of two-body scattering contributions to $w(i \to f)$ discussed in subsection 4.2.1--is physically understandable and, in fact, to be expected. As in the case of true three-body collisions, in a large volume τ containing $\hat{N}_\alpha$ particles α, $\alpha = 1, 2, 3$, the double-scattering rate corresponding to (135b)--namely [recall the discussion of Eqs. (136) - (139)] the average number of times per second that a two-body scattering event between 1 and 2 is followed by a two-body scattering between 2 and 3--will be precisely $\hat{N}_1\hat{N}_2\hat{N}_3$ times the corresponding rate when τ contains a single particle of each species. The desired double-scattering rate in this latter situation will be the integral--over all possible intermediate momenta $\underset{\sim}{k}_2'$ resulting from the first scattering--of the product between the rate at which 1, 2 scatterings produce $\underset{\sim}{k}_2'$ and the probability that particle 2 will scatter from 3 as it moves through the volume τ with momentum $\underset{\sim}{k}_2'$. This latter probability is $\cong L\bar{\sigma}_{23}/\tau$, where $\bar{\sigma}_{23}$ is the cross section for two-body scattering of 2 by 3, and L is some average dimension of τ, depending on the site of the first scattering, the direction of $\underset{\sim}{k}_2'$, the shape of τ, etc.; evidently $L\bar{\sigma}_{23}$ is an estimate of the volume wherein scattering of 2 by 3 can occur as 2 moves through τ. The rate of 1, 2 scatterings for a single pair 1, 2 in τ is given by (154). Therefore, after performing all the complicated averages, the desired double scattering rate $\hat{w}_d'(12, 23)$ with one particle of each species in τ will turn out to be

$$\hat{w}_d'(12,23) \cong \left\langle \left(\frac{\bar{w}_{12}^{(2)}}{\tau}\right)\left(\frac{L\bar{\sigma}_{23}}{\tau}\right)\right\rangle_{av} \cong C\,\tau^{-5/3} \tag{159a}$$

where C again is an effectively unknown factor, dependent on the shape of the scattering region τ, but not on the magnitude of its volume. Here $\hat{w}_d'(12, 23)$ represents the double-scattering contribution to the probability current flow when the incident wave is ψ_i' of (156a). Hence the corresponding contribution to $w(i \rightarrow f)$ of (121b) must be proportional to $\tau^3\tau^{-5/3} = \tau^{4/3}$, with a shape-dependent factor C, as found in the preceding paragraph beneath (158c). The corresponding observed double scattering rate $\hat{w}_d(12, 23)$ when τ contains $\hat{N}_\alpha$ particles of each species will be

$$\hat{w}_d(12,23) = \hat{N}_1\hat{N}_2\hat{N}_3\hat{w}_d'(12,23) \cong N_1N_2N_3C\,\tau^{4/3} \tag{159b}$$

I want to contrast this result (159b) for the δ-functions (135) with the corresponding result in subsection 4.2.1 for the δ-function (130c). In the case of the δ-functions (135) the computed $w(i \rightarrow f)$ using ψ_i of (21a) has terms proportional to $\tau^{4/3}$; to these terms will correspond observed laboratory frame scattering rates $\hat{w}(i \to f)$ proportional to $N_1N_2N_3$ and to $\tau^{4/3}$. For the δ-function (130c) on the other hand, though the computed $w(i \to f)$ using ψ_i of (21a) is proportional to τ^2, the corresponding observable laboratory frame $\hat{w}(i \to f)$ is proportional merely to τ, as well as merely to N_1N_2 (being independent of N_3). Note also that the experimentalist attempting to measure the true three-body elastic scattering coefficient by crossing three beams (let's ignore the present utter infeasibility of such an experiment) will have to avoid placing his coincidence counters at directions and distances corresponding to vanishing of the arguments of the δ function (135b) and its analogues, if he wishes to avoid measuring double scattering rather than true three-body scattering. Of course, he also must avoid counter locations corresponding to a single purely two-body scattering.

4.3 Truly Three-Body Scattering

I return now to our original objective of determining the physical three-body $\bar{w}(i \to f)$, i.e., to the problem of finding expressions for the true three-body matrix elements $< f|\bar{T}^t|i >$ of Eq. (4). It is argued in subsection 4.3.1 immediately below that the contributions to $\bar{\Phi}_i^{s(+)}$ of (69) from triple and higher rescattering processes (namely, from processes involving any number $n \geq 3$ of successive purely two-body collisions between pairs of the three particles 1, 2, 3) behave asymptotically like $\bar{G}_F^{(+)}(\bar{r};\bar{r}';\bar{E})$ as $\bar{r} \to \infty$ along essentially all $\bar{\nu}_f$ which keep no $r_{\alpha\beta}$ finite. In other words, as this chapter (especially in its introduction and in subsection 4.1.3) has made abundantly clear, such $n \geq 3$ rescattering processes legitimately can be termed "truly three-body", and are expected to contribute neither δ-functions to $\bar{T}^s(k_i \to k_f)$ of (133b) nor anomalous τ-dependences to $\bar{w}(i \to f)$. A direct way of attaining our desired objective, therefore, is to develope a procedure for subtracting the double-scattering contributions to $\bar{\Phi}_i^{s(+)}$ of (69), thereby hopefully obtaining $\bar{\Phi}_i^{t(+)}$. If this could be done, one should be able to compute lim $\bar{\Phi}_i^{t(+)}$ as $\bar{r} \to \infty$ || $\bar{\nu}_f$, therewith determining $\bar{T}^t(k_i \to k_f)$ in closed form; correspondingly, using the center of mass analogues of (105) - (106), one would have a closed form expression for the true (or physical) three-body elastic scattering transition operator $\bar{T}^t$ introduced in Chapter 1. I add that because the shape-dependent factor C in Eq. (158c) is essentially incalculable, there seems to be no practical way to obtain the theoretical physical three-body $\bar{w}(i \to f)$ by subtraction of $\tau^{4/3}$ contributions from the $\bar{w}(i \to f)$ computed using $\bar{T}^s(k_i \to k_f)$ of (133b). Thus, to obtain the physical $\bar{w}(i \to f)$, the necessary subtraction of non-three-body contributions must be performed before carrying out the probability current flow computations and δ-function reinterpretations discussed in sections 4.1 - 4.2. The experimentalist, on the other hand, actually might be able to perform this subtraction by varying the scattering volume while keeping its shape constant, thus in effect determining the shape dependent factor C empirically.

4.3.1 Subtraction of Double Scattering Terms

Although there is no obvious reason why it should be impossible to do so, I have not been able to perform the desired subtraction of double scattering contributions to $\bar{\Phi}_i^{s(+)}$ described in the introduction to this section. The difficulty lies in the need not to subtract too much; otherwise there would be no problem. The two-body scattering δ-function appearing as a multiplicative factor in (130c) has its origin in the plane wave factor $e^{i\underset{\sim}{K}_{12}\cdot\underset{\sim}{q}_{12}}$ in the $\bar{\Phi}_{12}^{(+)}$ [Eq. (72)] part of $\bar{\Phi}_i^{(+)}$. But the presence of this plane wave factor means the entire term $\bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}})$ fails to behave like $\bar{G}_F(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$ as $\bar{\underset{\sim}{r}} \to \infty$ along $\bar{\underset{\sim}{\nu}}_f$ corresponding to elastic scattering, so that in seeking $\bar{\Phi}_i^{t(+)}$ one assuredly can subtract the entire term $\bar{\Phi}_{12}^{(+)}$ from $\bar{\Phi}_i^{(+)}$ [recall the discussion preceding Eq. (119)]. However, the double-scattering δ-functions (135), which appear as additive components of $\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$, correspondingly arise from additive components of $\bar{\Phi}_i^{s(+)}$. To obtain $\bar{\Phi}_i^{t(+)}$ from $\bar{\Phi}_i^{s(+)}$, one must subtract from $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$ all terms behaving asymptotically like $\bar{\rho}^{-2}$ as $\bar{\underset{\sim}{r}} \to \infty$ along elastic scattering $\bar{\underset{\sim}{\nu}}_f$, but--as is clear from the rules and discussion at the end of subsection 4.13--one must retain in $\bar{\Phi}_i^{t(+)}(\bar{\underset{\sim}{r}})$ all (outgoing) terms in $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$ behaving asymptotically like $\bar{\rho}^{-5/2}$.

To make more explicit the difficulty of performing this delicate subtraction, let me indicate the results of one reasonable attempt to single out the double-scattering terms in $\bar{\Phi}_i^{s(+)}$. According to subsection E.3.2, the $\bar{\rho}^{-2}$ contribution in the $\bar{G}^{(+)}V_{23}\bar{\Phi}_{12}^{(+)}$ part of $\bar{\Phi}_i^{s(+)}$ [Eq. (69)] is contained entirely in

$$\bar{G}_{23}^{(+)} V_{23} \bar{\Phi}_{12}^{(+)} = -\bar{G}_{23}^{(+)} V_{23} \left\{ \lim_{\varepsilon \to 0} \bar{G}_{12}(\bar{E}+i\varepsilon) V_{12} \bar{\Psi}_i(\bar{E}) \right\}$$
$$= -\bar{G}_{23}^{(+)} V_{23} \left\{ e^{i\underset{\sim}{K}_{12i}\cdot\underset{\sim}{\rho}_{12}} \left[g_{12}^{(+)}\left(\frac{\hbar^2 k_{12i}^2}{2\mu_{12}}\right)\right] V_{12} e^{i\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}_{12}} \right\} \tag{160}$$

recalling the center of mass versions of Eqs. (60) and (72). Therefore I will perform an iteration on Eq. (69), as follows. In the V_{23}

terms of (67c) use the second equality in (63b), proceeding as in Eqs. (64) - (67c), and similarly for the V_{31} and V_{12} terms in (67c). Then, after taking the limit $\varepsilon \to 0$, one finds

$$\Phi_i^{s(+)} = \Phi_{23}^{s(+)} + \Phi_{31}^{s(+)} + \Phi_{12}^{s(+)} + \Phi_i^{d(+)} \qquad (161a)$$

where

$$\Phi_{23}^{s(+)} = -G_{23}^{(+)} V_{23} \left[\Phi_{12}^{(+)} + \Phi_{31}^{(+)} \right] \qquad (161b)$$

$$\Phi_{31}^{s(+)} = -G_{31}^{(+)} V_{31} \left[\Phi_{23}^{(+)} + \Phi_{12}^{(+)} \right] \qquad (161c)$$

$$\Phi_{12}^{s(+)} = -G_{12}^{(+)} V_{12} \left[\Phi_{31}^{(+)} + \Phi_{23}^{(+)} \right] \qquad (161d)$$

and where the result of double iteration and subtraction [on the original formula (52a) for $\Psi_i^{(+)}$] is

$$\bar{\Phi}_i^{d(+)} = -G^{(+)}\left[(V_{23}+V_{31})\bar{\Phi}_{12}^{s(+)} + (V_{31}+V_{12})\bar{\Phi}_{23}^{s(+)} + (V_{12}+V_{23})\bar{\Phi}_{31}^{s(+)}\right] \quad (162)$$

The center of mass version of (162) involves no divergent integrals, and is the desired iteration of (69); obviously, I could have obtained the same result by iterating directly on (69), using the center of mass versions of Eqs. (63).

Because $\bar{\Phi}_{\alpha\beta}^{s(+)}(\bar{\underset{\sim}{r}})$ decreases no less slowly than $\bar{\rho}^{-2}$ as $\bar{r} \to \infty$ along $\bar{\underset{\sim}{\nu}}_f \neq \bar{\underset{\sim}{\nu}}_{\alpha\beta}$, it can be seen from the arguments at the end of Section E.2 that interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty \,||\, \bar{\underset{\sim}{\nu}}_f$ is justified in the integrals (162) for $\bar{\Phi}_i^{d(+)}(\bar{\underset{\sim}{r}})$, except possibly along certain special $\bar{\underset{\sim}{\nu}}_f$. As section E.2 explains, it has not been shown that these special $\bar{\underset{\sim}{\nu}}_f$ really exist; we merely have not ruled out the possibility that such $\bar{\underset{\sim}{\nu}}_f$ occur. However, the discussion in subsection 4.3.2 below strongly indicates that such special $\bar{\underset{\sim}{\nu}}_f$ [even if they actually occur] are inconsequential for the purposes of this work. Therefore we infer that for our present purposes interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty \,||\, \bar{\underset{\sim}{\nu}}_f$ in (162) is justified; correspondingly, we may conclude that $\bar{\Phi}_i^{d(+)}$ is outgoing and decreases no less rapidly than $\bar{\rho}^{-5/2}$ as $\bar{\underset{\sim}{r}} \to \infty \,||\, \bar{\underset{\sim}{\nu}}_f$, except possibly for these same special inconsequential $\bar{\underset{\sim}{\nu}}_f$. In other words, it appears legitimate to conclude that the anomalously propagating double-scattering contributions to $\bar{\Phi}_i^{s(+)}$ all are contained in $\bar{\Phi}_{23}^{s(+)}$, $\bar{\Phi}_{31}^{s(+)}$ and $\bar{\Phi}_{12}^{s(+)}$; thus subtracting these terms from $\bar{\Phi}_i^{s(+)}$ should leave a $\bar{\Phi}_i^{d(+)}$ which represents true three-body scattering only.

On the other hand, I see no reason to think that $\bar{\Phi}_{\alpha\beta}{}^{s(+)}$ represent anomalous double scattering only, i.e., contain no parts which should be included in $\bar{\Phi}_i{}^{t(+)}$. For one thing, subsection E.3.2 can be seen to imply

$$-\bar{G}_F^{(+)} V_{23} \bar{G}_{23}^{(+)} V_{23} \bar{\Phi}_{12}^{(+)} = \left[\bar{G}_{23}^{(+)} - \bar{G}_F^{(+)}\right] V_{23} \bar{\Phi}_{12}^{(+)} \tag{163}$$

decreases no less rapidly than $\bar{\rho}^{\,-5/2}$. But this result immediately means that $\bar{\Phi}_{23}{}^{s(+)}$ on the right side of (161a) contains a part--namely the left side of (163)--belonging in $\bar{\Phi}_i{}^{t(+)}$ (because it is most unlikely that the $\bar{\rho}^{\,-5/2}$ contribution to the left side of (163) is everywhere incoming); in other words, $\bar{\Phi}_i{}^{d(+)}$ of (162) does not contain all parts of $\bar{\Phi}_i{}^{(+)}$ contributing to $\bar{\Phi}_i{}^{t(+)}$.

Alternatively, Eq. (163) means

$$\bar{G}_F^{(+)} V_{23} \bar{\Phi}_{12}^{(+)} = -\bar{G}_F^{(+)} V_{23}\left\{ e^{i \underset{\sim}{K}_{12i}\cdot \underset{\sim}{\varrho}_{12}} \left[g_{12}^{(+)}\left(\frac{\hbar^2 k_{12i}^2}{2\mu_{12}}\right)\right] V_{12} e^{i \underset{\sim}{k}_{12i}\cdot \underset{\sim}{r}_{12}} \right\} \tag{164a}$$

contains the entire $\bar{\rho}^{\,-2}$ contribution in (160), just as (160) contains the entire $\bar{\rho}^{\,-2}$ contribution in $\bar{G}^{(+)} V_{23} \bar{\Phi}_{12}{}^{(+)}$. However, replacing $g_{12}{}^{(+)}$ in (164a) by $g_F{}^{(+)}$ would not retain the entire $\bar{\rho}^{\,-2}$ contribution. Sections E.2 - E.3 show the $\bar{\rho}^{\,-2}$ behavior in (164a) stems from the fact that the integral (73b) behaves like $r_{12}{}^{-1}$ as $\underset{\sim}{r}_{12} \to \infty$; evidently

$$\left[g_{12}^{(+)} - g_F^{(+)}\right] V_{12} e^{i \underset{\sim}{k}_{12i}\cdot \underset{\sim}{r}_{12}} = -\int d\underset{\sim}{r}'_{12} d\underset{\sim}{r}''_{12}\, g_F^{(+)}(\underset{\sim}{r}_{12}; \underset{\sim}{r}'_{12}) V_{12}(\underset{\sim}{r}'_{12}) \tag{164b}$$

$$\times\, g^{(+)}(r'; r'') V(r'') e^{i \underset{\sim}{k}_{12i}\cdot \underset{\sim}{r}''_{12}}$$

also behaves like r_{12}^{-1} as $\underset{\sim}{r}_{12} \to \infty$, recognizing that V_{12} is short range. In any event, even if one could find some iteration of (164a) that retained all $\bar{\rho}^{-2}$ terms in an integral of convenient or transparent form, there remains the complication that the $\bar{\rho}^{-2}$ contribution to (160) [i.e., to (164a)] was obtained in section E.3 by application of the principle of stationary phase. It readily can be seen that this method of obtaining the $\bar{\rho}^{-2}$ contribution amounts to computing the leading term in an expansion in powers of $\bar{\rho}^{-1/2}$. Therefore, along with the $\bar{\rho}^{-2}$ contribution to (160) or (164b), or to any $\bar{\rho}^{-2}$-retaining iteration thereof, there generally will be $\bar{\rho}^{-5/2}$ contributions.

It follows (from the material presented thus far in this subsection) that it is very difficult to find any set of scattering terms--or, equivalently, any set of scattering diagrams--which represent the anomalous double scattering part of $\bar{\Phi}_i^{(+)}$ without any truly three-body scattering contributions, and which therefore could be subtracted from $\bar{\Phi}_i^{s(+)}$ to yield the entire $\bar{\Phi}_i^{t(+)}$. In this connection it is worth noting that $\bar{\Phi}_i^{d(+)}$ of (162)--which, according to the penultimate paragraph above lies wholly in $\bar{\Phi}_i^{t(+)}$--can be thought to result from scattering processes involving no less than three successive purely two-body collisions. It has been explained above that interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty \,||\, \bar{\underset{\sim}{\nu}}_f$ in (162) is justified for essentially all $\bar{\underset{\sim}{\nu}}_f \neq \bar{\underset{\sim}{\nu}}_{\alpha\beta}$; thus we obtain as, e.g., in Eqs. (133)

$$\lim_{\bar{\underset{\sim}{r}} \to \infty \| \bar{\underset{\sim}{\nu}}_f} \bar{\Phi}_i^{d(+)}(\bar{\underset{\sim}{r}}) = -C_2(\bar{E}) \frac{e^{i\bar{\rho}\sqrt{\bar{E}}}}{\bar{\rho}^{5/2}} \bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \qquad (165a)$$

$$\bar{T}^{d}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \int d\bar{\underset{\sim}{r}}' \, \bar{\Psi}_f^{(-)*}(\bar{\underset{\sim}{r}}') \Big[(V_{23}+V_{31}) \bar{\Phi}_{12}^{s(+)}(\bar{\underset{\sim}{r}}') + (V_{31}+V_{12}) \bar{\Phi}_{23}^{s(+)}(\bar{\underset{\sim}{r}}') + (V_{12}+V_{23}) \bar{\Phi}_{31}^{s(+)}(\bar{\underset{\sim}{r}}') \Big] \tag{165b}$$

Now consider, e.g., the first term on the right side of (165b); in particular, consider the contribution to $\bar{\Psi}_f^{(-)*} V_{23} \bar{\Phi}_{12}^{s(+)}$ made by the $\bar{\Psi}_{23f}^{(-)*}$ part of $\bar{\Psi}_f^{(-)*}$, where $\bar{\Psi}_{23f}^{(-)*}$ is given by Eq. (136). From Eq. (161d),

$$\bar{\Psi}_{23f}^{(-)*} V_{23} \bar{\Phi}_{12}^{s(+)} = -\bar{\Psi}_{23f}^{(-)*} V_{23} \bar{G}_{12}^{(+)} V_{12} \Big[\bar{\Phi}_{31}^{(+)} + \bar{\Phi}_{23}^{(+)} \Big] \tag{166a}$$

The first term on the right side of (166a) can be reexpressed, as in Eqs. (137), in the interpretable form

$$\begin{aligned}
&\bar{\Psi}_{23f}^{(-)*} V_{23} \bar{G}_{12} V_{12} \bar{\Phi}_{31}^{(+)} \\
&\quad = -\lim_{\varepsilon \to 0} \bar{\Psi}_{23f}(\bar{E}+i\varepsilon) V_{23} \bar{G}_{12}(\bar{E}+i\varepsilon) V_{12} \bar{G}_{31}(\bar{E}+i\varepsilon) V_{31} \bar{\Psi}_i \\
&\quad = -\lim_{\varepsilon \to 0} \bar{\Psi}_{23f}(\bar{E}+i\varepsilon) V_{23} \bar{G}_F(\bar{E}+i\varepsilon) \bar{\underset{\sim}{T}}_{12}(\bar{E}+i\varepsilon) \bar{G}_F(\bar{E}+i\varepsilon) \bar{\underset{\sim}{T}}_{31}(\bar{E}+i\varepsilon) \bar{\Psi}_i \\
&\quad = -\lim_{\varepsilon \to 0} \bar{\Psi}_f^{*} \bar{\underset{\sim}{T}}_{23} \bar{G}_F \bar{\underset{\sim}{T}}_{12} \bar{G}_F \bar{\underset{\sim}{T}}_{31} \bar{\Psi}_i
\end{aligned} \tag{166b}$$

where $\bar{\underset{\sim}{T}}_{\alpha\beta}$ and $\bar{G}_F$ are evaluated at the complex energy $\bar{E} + i\varepsilon$. The integral (166b) obviously corresponds to a diagram wherein there are three successive purely two-body scatterings: first of the pair 3, 1; next of the pair 1, 2; and finally of the pair 2, 3. Similar results hold for the other terms on the right sides of (165b) and (166a). Moreover, further iterations of $\bar{\Psi}_f^{(-)*}$ in (165b), or, e.g., replacing $\bar{\Psi}_f^{(-)*}$ by $\bar{\Phi}_{12f}^{s(-)*}$ in $\bar{\Psi}_f^{(-)*}V_{23}\bar{\Phi}_{12}^{s(+)}$, yields integrals corresponding to even higher order scattering diagrams.

4.3.2 Volume Dependence of Triple Scattering Contributions

The integrals (165b) are convergent, except possibly along special directions $\underset{\sim}{k}_f$ [for given $\underset{\sim}{k}_i$] where some of the integrals in (165b) may be logarithmically divergent [see section B.2]. However, there is no reason to think that the center of mass frame probability current flow $\bar{\mathcal{J}}$ of Eqs. (127)--when integrated over an infinitesimal range $d\underset{\sim}{k}_f$ in the vicinity of these special (here meaning isolated) $\underset{\sim}{k}_f$ where $\bar{T}^d(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ from (165b) is undefined--receives finite contributions from these comparatively weak divergences [see section B.2]. Thus the possible existence of these special $\underset{\sim}{k}_f$ seemingly does not require reinterpretations along the lines of section 4.2, i.e., seemingly does not introduce any anomalously τ-dependent contributions into the reaction coefficient $\bar{w}(i \rightarrow f)$. Moreover, there is no indication that the integrals (165b) contain any other $\underset{\sim}{k}_f$-dependent parts which--after squaring--will be non-integrable over $d\underset{\sim}{k}_f$ [recall the form of Eq. (127c)]. Therefore, it does seem to be true that the doubly-iterated $\bar{\Phi}_i^{d(+)}(\bar{\underset{\sim}{r}})$ of (162)--comprising contributions from numbers $n \geq 3$ of successive two-body collisions--in essence behaves asymptotically like $\bar{G}_F^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}';\bar{E})$ and entirely represents truly three-body scattering [as concluded in subsection 4.3.1]. I remark that, as in the case of the special $\bar{\underset{\sim}{\nu}}_f$ discussed in subsection 4.3.1, it has not been shown that the special $\underset{\sim}{k}_f$ of the present subsection actually exist; rather, because their effects apparently are inconsequential for the purposes of this work, it is not worth the very considerable effort which would be required to decide whether or not the logarithmic divergences of (165b) ever can occur at physically allowed real $\underset{\sim}{k}_f$. Nor is there any evidence that the possible existance of real or imaginary values of $\underset{\sim}{k}_f$ where (165b) is logarithmically divergent should be associated with actual singularities of $\bar{T}^d(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$, computed via analytic continuation [as a function of $\underset{\sim}{k}_f$ for fixed $\underset{\sim}{k}_i$] from values of $\bar{T}^d(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ for which the integrals (165b) surely are well-behaved; it is conceivable, for instance, that the logarithmic divergences of the integrals (165b) at special $\underset{\sim}{k}_f$ have no physical significance, but simply are manifestations of the fact that the interchange of order of integration and limit $\bar{\underset{\sim}{r}} \rightarrow \infty \parallel \bar{\underset{\sim}{\nu}}_f$ in (162) is not justified at special $\bar{\underset{\sim}{\nu}}_f$. It does seem worthwhile to stress--much as in the discussion following Eq. (48)--that the integrals (165b) are convergent except possibly on an inconsequential subset of the allowed real $\underset{\sim}{k}_f$, whereas the integrals (133b) always are divergent, although it is true that the divergences in (133b) arise from δ-functions (135) which can be considered non-contributory except when their arguments vanish.

To further confirm our conclusion that $\bar{\Phi}_i^{d(+)}$ represents truly three-body scattering, I now shall demonstrate--by arguments along the lines of subsections 4.2.1 - 4.2.2--that three or more successive binary collisions cannot make contributions to the three-body reaction coefficient $\bar{w}(i \rightarrow f)$ which increase as any positive power of τ. Consider, e.g., the sequence of three two-body scatterings: 1, 2 collide; 2, 3 collide; 3, 1 collide. Then, as in subsection 4.2.2, I first compute the reaction rate for the above sequence under the circumstances that the volume τ contains precisely one particle of each species α. After the collision between 2 and 3, whenever it may take place, the laboratory frame speed and direction with which 3 moves through τ are strictly correlated. It follows that in order to rescatter from particle 1--whose trajectory has been fixed by the first collision between 1 and 2--particle 3 must be scattered by 2 into a very narrow solid angle, of the order $\bar{\sigma}_{31}/L^2$ where $L \stackrel{\sim}{=} \tau^{-1/3}$. Hence, referring to (159a), the postulated sequence of three binary collisions will have the rate

$$\hat{w}_t'(12;23;31) \cong \left\langle \frac{\bar{w}_{12}^{(2)}}{\tau} L \frac{\bar{\sigma}_{23}}{\tau} \frac{\bar{\sigma}_{31}}{L^2} \right\rangle_{av} \cong \tau^{-7/3} \tag{167}$$

This result for $\hat{w}_t'$ corresponds to the incident wave ψ_i' of (156a), so that the laboratory frame $w(i \to f)$ is proportional to $\tau^3 \tau^{-7/3} = \tau^{2/3}$, implying this sequence of three binary collisions makes a contribution to $\bar{w}(i \to f)$ which is proportional to $\tau^{-1/3}$. Therefore, recalling the discussion at the end of subsection 4.2.1, it is reasonable to infer that **successions of $n \geq 3$ purely two-body scatterings, if observable at all in a large volume τ, will be indistinguishable from [and apparently should be included in] what I have termed truly three-body scattering.** Note that the diagram corresponding to (166b) represents successive two-body scatterings in which energy is not necessarily conserved in the intermediate states (e.g., between the first 3,1 scattering and the second 1,2 scattering); the physical purely two-body scatterings yielding the just estimated $t^{-1/3}$ contribution to $\bar{w}(i \to f)$ are energy-conserving, and therefore are only a subset of the whole class of three successive two-body scatterings represented by the diagram corresponding to (166b) [see subsection 5.3.3].

5. THE PHYSICAL THREE-BODY TRANSITION AMPLITUDE

This, our final chapter, is concerned with an attempt to actually determine a useful expression for the physical three-body transition amplitude $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$. As section 5.1 explains, attempting to find $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ using mathematically defensible procedures is impractical; to avoid extremely difficult and complicated calculations, employment of some not obviously justified mathematical short cuts seems necessary. One such plausible attempt to determine $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ actually is carried out in section 5.1. The formula for $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ obtained in this fashion is shown to be consistent with detailed balancing in section 5,2, while its interpretation is discussed in section 5.3. Section 5.3 also compares the configuration space expression for $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$--as well as for the entire $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$--with the corresponding expressions inferred via the more customary momentum space procedures.

5.1 Derivation by Subtraction of δ-Functions

Among the concerns of the preceding section has been the possibility of expressing $\bar{\Phi}_i^{s(+)}$ in the form

$$\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}}) = \bar{\Phi}_i^{a(+)}(\bar{\underset{\sim}{r}}) + \bar{\Phi}_i^{t(+)}(\bar{\underset{\sim}{r}}) \tag{168}$$

where $\bar{\Phi}_i^{a(+)}$ is that part of $\bar{\Phi}_i^{s(+)}$ which represents unwanted contributions such as anomalous double scattering, but which is wholly devoid of any truly three-body scattering contributions. In fact, subsection 4.3.1 has examined--and found to be impractical though not obviously impossible--one suggested means of constructing $\bar{\Phi}_i^{a(+)}$, namely by seeking a set of scattering diagrams which separate out the anomalous double scattering terms from the truly three-body contributions to $\bar{\Phi}_i^{s(+)}$. Alternatively, one could try to find a closed form analytic expression for $\bar{\Phi}_i^{a(+)}(\bar{\underset{\sim}{r}})$ by carrying through the calculation--of the asymptotic form of $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$--outlined in section E.3. A glance at subsection E.3.1, however--especially Eq. (E40b) and the discussion immediately thereafter--makes it evident that this suggested procedure for finding $\bar{\Phi}_i^{a(+)}$ also is not very practical, though again not obviously impossible. It is understood, of course, that merely completing the calculation initiated with Eq. (E40b) [already an arduous enough task] would be insufficient, because the asymptotic form resulting from this calculation will be singular at $\bar{\underset{\sim}{r}} = 0$, and will include contributions behaving as $\bar{\rho}^{\,-5/2}$ at large $\bar{\rho}$; to correctly yield $\bar{\Phi}_i^{t(+)}$, what must be subtracted from $\bar{\Phi}_i^{s(+)}$ is an expression $\bar{\Phi}_i^{a(+)}(\bar{\underset{\sim}{r}})$ which is finite at $\bar{\underset{\sim}{r}} = 0$ and has no $\bar{\rho}^{\,-5/2}$ components at large $\bar{\rho}$, but whose leading $\bar{\rho}^{\,-2}$ part at large $\bar{\rho}$ is identical with the leading asymptotic part of $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$.

For many purposes--e.g., the construction of variational principles for three-body elastic scattering--complete knowledge of the asymptotic behavior of $\bar{\Phi}_i^{s(+)}$ may be essential(38). On the other hand, it is conceivable that only partial knowledge of the asymptotic behavior of $\bar{\Phi}_i^{s(+)}$--in particular, only partial knowledge of the asymptotic behavior of $\bar{\Phi}_i^{t(+)}$ in (168)--can suffice to determine the physical three-body scattering amplitude $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$. Thus it may be possible to find $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ without having to carry through either of the difficult calculations discussed in the preceding paragraph. It seems clear, however, that any argument which leads to $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ while avoiding exact construction of $\bar{\Phi}_i^{t(+)}(\underset{\sim}{\bar{r}})$--or of its leading $\bar{\rho}^{\,-5/2}$ part at the very least--will have to involve some mathematically questionable steps, i.e., will lead to a possibly erroneous result for $\bar{T}^t$. Nevertheless, because determination of $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ has been a major objective of this work [recall our opening remarks in chapter 1], I now shall describe an attempt to deduce $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ via a plausible argument which indeed does avoid finding first the leading $\bar{\rho}^{\,-5/2}$ part of $\bar{\Phi}_i^{t(+)}(\underset{\sim}{\bar{r}})$.

We have seen that $\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ of (133b) contains δ-functions (135), ascribable to the fact that the interchange of order of integration and limit $\underset{\sim}{\bar{r}} \to \infty$ in (132) was unjustified; this interchange led to the erroneous assertion (133a), whereas actually $\bar{\Phi}_i^{s(+)}(\underset{\sim}{\bar{r}})$ contains contributions behaving like $\bar{\rho}^{\,-2}$ at large $\bar{\rho}$. Suppose, therefore I am able to express $\bar{T}^s$ of Eqs. (133) in the form

$$\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{T}^a(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) + \bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \qquad (169a)$$

where $\bar{T}^t$ is wholly composed of convergent integrals [except possibly for inconsequential logarithmic divergences at special $\underset{\sim}{k}_f$, recall subsection 4.3.2], whereas $\bar{T}^a$ is a sum of terms proportional to δ-functions and thus has no finite part. Then from the entire body of this work, especially the rules and discussion at the end of subsection 4.1.3, it seems reasonable to infer that $\bar{T}^a(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ in (169a) represents the contribution to $\bar{T}^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ from that part of $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$ which behaves like $\bar{\rho}^{-2}$ at large $\bar{\rho}$, but which has no $\bar{\rho}^{-5/2}$ components at large $\bar{\rho}$. In other words, it seems reasonable to conclude that $\bar{T}^a(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ is the contribution to $\bar{T}^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ made by $\bar{\Phi}_i^{a(+)}(\bar{\underset{\sim}{r}})$ of (168), implying $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ of (169a) will be the desired entire truly three-body scattering amplitude associated with $\bar{\Phi}_i^{t(+)}(\bar{\underset{\sim}{r}})$. I stress that this conclusion, though reasonable enough, depends on a number of unproved assumptions. For instance, I am assuming that any mathematically well-behaved $\bar{\rho}^{-5/2}$ component of $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$--that is to say, any component of $\bar{\Phi}_i^{s(+)}(\bar{\underset{\sim}{r}})$ which is finite at $\bar{\underset{\sim}{r}} = 0$ and propagates to infinity without restriction in the six-dimensional $\bar{\underset{\sim}{r}}$-space [recall the discussion at the end of subsection 4.1.3]--indeed is everywhere outgoing, i.e., behaves everywhere at infinity like the outgoing $\bar{G}_F^{(+)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$, not like the incoming free space Green's function $\bar{G}_F^{(-)}(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}')$. I also am assuming that the unjustified interchange of order of integration and limit $\underset{\sim}{r} \rightarrow \infty \parallel \underset{\sim}{\nu}_f$ in (132) is not so wrong that (169a) becomes a quite misleading indication of the actual form of $\bar{\underset{\sim}{T}}^t$. Without making these and similar assumptions, there is little basis for arguing that $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ obtained from (169a) and (133b) can be identified with the "truly" three-body transition amplitudes of Eqs. (3) - (4). Of course, explicit verification of these assumptions would involve finding closed form analytic expressions for $\bar{\Phi}_i^{a(+)}(\bar{\underset{\sim}{r}})$ and $\bar{\Phi}_i^{t(+)}(\bar{\underset{\sim}{r}})$, an impractical task [as explained at the beginning of this subsection] whose performance--if achieved--simultaneously would obviate the need for computing the physical $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ via the present dubious argument.

Granting the legitimacy of using (133b) and (169a), $\bar{T}^t(k_i \to k_f)$ is found as follows. In the terms on the right side of (133b) involving the product $\bar{\Psi}_f^{(-)*}V_{12}$, use

$$\bar{\Psi}_f^{(-)*} = \bar{\Psi}_{12f}^{(-)*} - \bar{\Psi}_f^{(-)*}(V_{23}+V_{31})\bar{G}_{12}^{(+)} \tag{169b}$$

which is the time-reversed analogue of (86), written in the notational style of (100b) or (105d), i.e., with the Green's function on the right, as here will be convenient; Eq. (169b) also can be inferred directly from the center of mass analogue of the second equality in (65b), via the methods of chapter 3. Furthermore, in the $\bar{\Psi}_f^{(-)*}V_{23}$ and $\bar{\Psi}_f^{(-)*}V_{31}$ terms of (133b), use respectively the 2,3 and 3,1 analogues of (169b). Then, employing Eqs. (161) as well, one obtains

$$\begin{aligned}\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \int d\bar{\underset{\sim}{r}}' \Big\{ &\bar{\Psi}_{23f}^{(-)*}(\bar{\underset{\sim}{r}}')V_{23}\left[\bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}}') + \bar{\Phi}_{31}^{(+)}(\bar{\underset{\sim}{r}}')\right] \\ &+ \bar{\Psi}_{31f}^{(-)*}(\bar{\underset{\sim}{r}}')V_{31}\left[\bar{\Phi}_{23}^{(+)}(\bar{\underset{\sim}{r}}') + \bar{\Phi}_{12}^{(+)}(\bar{\underset{\sim}{r}}')\right] \\ &+ \bar{\Psi}_{12f}^{(-)*}(\bar{\underset{\sim}{r}}')V_{12}\left[\bar{\Phi}_{31}^{(+)}(\bar{\underset{\sim}{r}}') + \bar{\Phi}_{23}^{(+)}(\bar{\underset{\sim}{r}}')\right]\Big\} \\ &+ \bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)\end{aligned} \tag{169c}$$

in place of (133b), where $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is given by (165b). The same Eq. (169c) is obtained from the interchange of order of integration and $\lim \underset{\sim}{r} \to \infty \,||\, \underset{\sim}{\nu}_f$ in the right sides of Eqs. (161) - (162); of course, this interchange--though justified in Eq. (162) [recall the discussion in subsection 4.3.1]--is unjustified in the $\Phi_{\alpha\beta}^{s(+)}$ terms of Eqs. (161) [recall the discussion of Eqs. (132) - (135)].

Returning now to the discussion of Eqs. (135) - (137) in subsection 4.1.3, one sees from comparison of Eqs. (169c) and (133b) that all the double-scattering δ-functions of type (135) contributing to $\bar{T}^{s}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ are entirely contained in the integral on the right side of (169c); in particular, (a term proportional to) the specific δ-function (135a) arises entirely from the $\bar{\Psi}_{23f}^{(-)*}V_{23}\bar{\Phi}_{12}^{(+)}$ term in (169c). Let us consider this term, therefore. Using Eqs. (72), (105b) and (136), one finds

$$\bar{\Psi}_{23f}^{(-)*}V_{23}\bar{\Phi}_{12}^{(+)} = \int d\underset{\sim}{q}_{23}\, d\underset{\sim}{r}_{23}\, e^{-i\underset{\sim}{K}_{23f}\cdot\underset{\sim}{q}_{23}}\, u_{23cf}^{(-)*}(\underset{\sim}{r}_{23};\underset{\sim}{k}_{23f}) \times V_{23}(\underset{\sim}{r}_{23})\, e^{i\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}_{12}}\, \varphi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i}) \tag{170a}$$

where u_{23cf} is defined as was $u_c^{(-)*}$ in Eq. (129a), i.e.,

$$u_{23cf}^{(-)*}(\underset{\sim}{r}_{23};\underset{\sim}{k}_{23f}) = e^{-i\underset{\sim}{k}_{23f}\cdot\underset{\sim}{r}_{23}} + \phi_{23f}^{(-)*}(\underset{\sim}{r}_{23};\underset{\sim}{k}_{23f}) \tag{170b}$$

with $\phi_{23f}^{(-)*}$ given by the 2,3 analogue of (105c). Replacing the integration variable $\underset{\sim}{q}_{23}$ by $\underset{\sim}{r}_{12}$, the six-dimensional integral (170a) factorizes [compare Eqs. (E25b) - (E26b)] into a product of two independent three-dimensional integrals, namely

$$\int d\underset{\sim}{r}_{12}\, e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{Q}_{1f}}\, \varphi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i}) \int d\underset{\sim}{r}_{23}\, u_{23cf}^{(-)*}(\underset{\sim}{r}_{23};\underset{\sim}{k}_{23f})\, V_{23}(\underset{\sim}{r}_{23})\, e^{-i\underset{\sim}{Q}_{2f}\cdot\underset{\sim}{r}_{23}} \tag{170c}$$

where in this chapter I henceforth shall employ the notation [compare (E26c)]

$$\underset{\sim}{Q}_{1f} \equiv \underset{\sim}{A} = \underset{\sim}{K}_{23f} + \frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i}$$

$$\underset{\sim}{Q}_{2f} \equiv \underset{\sim}{B} = \frac{m_3}{m_2+m_3}\underset{\sim}{K}_{23f} + \underset{\sim}{K}_{12i} \tag{171a}$$

along with

$$\underset{\sim}{C} = \underset{\sim}{K}_{23f} + \frac{m_1}{m_3+m_1}\underset{\sim}{K}_{31i}$$

$$\underset{\sim}{D} = \frac{m_2}{m_2+m_3}\underset{\sim}{K}_{23f} + \underset{\sim}{K}_{31i} \tag{171b}$$

The integral over $d\underset{\sim}{r}_{23}$ in (170c) is convergent, and in fact can be identified with a matrix element of $\underset{\sim}{t}_{23}$ [see section E.4]. The integral over $d\underset{\sim}{r}_{12}$ in (170c) fails to converge, and in fact contains a contribution proportional to the δ-function (135a), as was originally shown in section B.2; the exact magnitude of this δ-function (135a) contribution to (170c), as well as of a second related δ-function contribution, is computed in section E.4.

Specifically, section E.4 shows that

$$\int d\underset{\sim}{r}_{12}\, e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}}\, \varphi^{(+)}_{12}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i})$$

$$= \int d\underset{\sim}{r}_{12} \Big\{ e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}}\, \varphi^{(+)}_{12}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i}) + \frac{i\mu_{12}}{\hbar^2}\frac{e^{ik_{12i}r_{12}}}{r_{12}} \langle k_{12i}\underset{\sim}{\nu}_{12}|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle$$

$$\times \left[\delta(\underset{\sim}{\nu}_{12}-\underset{\sim}{\nu}_{A})\frac{e^{-iAr_{12}}}{Ar_{12}} - \delta(\underset{\sim}{\nu}_{12}+\underset{\sim}{\nu}_{A})\frac{e^{iAr_{12}}}{Ar_{12}} \right] \Big\} \tag{172a}$$

$$-\frac{i\mu_{12}}{\hbar^2 A}\langle k_{12i}\underset{\sim}{\nu}_{A}|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle \int_0^\infty dr_{12}\, e^{i(k_{12i}-A)r_{12}}$$

$$+\frac{i\mu_{12}}{\hbar^2 A}\langle -k_{12i}\underset{\sim}{\nu}_{A}|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle \int_0^\infty dr_{12}\, e^{i(k_{12i}+A)r_{12}}$$

where $\underset{\sim}{r}_{12} = r_{12}\underset{\sim}{\nu}_{12}$; $A = A\underset{\sim}{\nu}_{A}$; while the truly two-body transition operators (third particle completely irrelevant and absent) $\underset{\sim}{t}_{\alpha\beta i}$, and their matrix elements, are defined by Eqs. (131e) - (131f), together with the shorthand notation

$$\underset{\sim}{t}_{12i} \equiv \underset{\sim}{t}_{12}(E_{12i}) = \underset{\sim}{t}_{12}\left(\frac{\hbar^2 k^2_{12i}}{2\mu_{12}}\right), \text{ etc.} \tag{172b}$$

$$\underset{\sim}{t}_{23f} \equiv \underset{\sim}{t}_{23}(E_{23f}) = \underset{\sim}{t}_{23}\left(\frac{\hbar^2 k^2_{23f}}{2\mu_{23}}\right), \text{ etc.} \tag{172c}$$

The notation (172c), though not employed in (172a), will be made use of below.

In (172a), integrals of the individual terms, e.g. of the term $e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}}\phi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i})$ originally appearing in (170c), do not converge. However, the δ-function terms inside the braces in (172a) cancel the leading ($\sim r_{12}^{-2}$) terms in the asymptotic expansion of $e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}}\phi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i})$; in other words, provided it is treated as a single $\underset{\sim}{r}_{12}$-dependent function, the quantity within the braces in (172a) is of order $r_{12}^{-1}e^{i(k_{12i} \pm A)r_{12}}$ at large r_{12}. Consequently the integral involving the braces in (172a) fails to converge only at the special values of $\underset{\sim}{k}_f$ satisfying $A^2 = k_{12i}^2$ for given $\underset{\sim}{k}_i$, whereas the left side of (172a) [the first integral factor in (170c)] diverges at all $\underset{\sim}{A},\underset{\sim}{k}_{12i}$. The remaining pair of one-dimensional integrals on the right side of (172a)--multiplying the matrix elements $< k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i} >$ and $< -k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i} >$ respectively--are obviously divergent, i.e., strictly speaking are mathematically undefined. To accomplish our present objective of finding an expression for $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ of (169a), it is necessary to somehow reinterpret these last two divergent integrals on the right side of (172a). There is no doubt but that the convergent first integral on the right side of (172a) contributes wholly to $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$. The problem is to decide whether or not the last two integrals on the right side of (172a) also contribute to $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$; referring to the discussion following Eq. (169a), this problem amounts to deciding whether or not the integrals in question plausibly can be interpreted as a sum of terms proportional to δ-functions, with no residual finite parts.

5.1.1 Formula for $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$

It is argued in section E.4 that the relations

$$\int_0^\infty dx\, e^{ikx} = \frac{i}{k} \qquad k \neq 0 \tag{173a}$$

$$\int_0^\infty dx\, e^{ikx} = \pi\,\delta(k) \qquad k = 0 \tag{173b}$$

provide a plausible interpretation--as a function of k--of the divergent integral on the left sides of Eqs. (173). Eqs. (173) are consistent with the more customary formula[42]

$$\int_0^\infty dx\, e^{ikx} = \pi\,\delta(k) + i\,P\,\frac{1}{k} \tag{174a}$$

where P signifies the principal part when integrated over k, i.e., where it is asserted that for any reasonably well-behaved function f(k)

$$\int_{-\infty}^{\infty} dk\, f(k) \int_0^\infty dx\, e^{ikx} = \pi f(0) + i \lim_{\varepsilon \to 0}\left[\int_{-\infty}^{\varepsilon} dk\,\frac{f(k)}{k} + \int_{\varepsilon}^{\infty} dk\,\frac{f(k)}{k}\right] \tag{174b}$$

Use of Eqs. (173) in (172a) specifies the magnitudes of the δ-function contributions to $\bar{\Psi}_{23f}^{(-)*} V_{23}\bar{\Phi}_{12}^{(+)}$ of (170a), thereby making it obvious how to express this $\bar{\Psi}_{23f}^{(-)*} V_{23}\bar{\Phi}_{12}^{(+)}$ part of $\bar{T}^{s}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ in the form (169a). Thus [see section E.4] we conclude from Eqs. (169a) and (169c) that

$$\begin{aligned}\bar{\bar{T}}^{t}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f) &= \bar{\bar{T}}^{d}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f) + \bar{\bar{T}}^{t}_{2312}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f) + \bar{\bar{T}}^{t}_{2331}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f)\\ &+ \bar{\bar{T}}^{t}_{3123}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f) + \bar{\bar{T}}^{t}_{3112}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f)\\ &+ \bar{\bar{T}}^{t}_{1231}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f) + \bar{\bar{T}}^{t}_{1223}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f)\end{aligned} \tag{175a}$$

where for $A^2 \neq k_{12i}^{\;2}$

$$\begin{aligned}&\bar{\bar{T}}^{t}_{2312}(\underset{\sim}{k}_i\to\underset{\sim}{k}_f)\\ &= \langle \underset{\sim}{k}_{23f}|\underset{\sim}{t}_{23f}|-\underset{\sim}{B}\rangle \int d\underset{\sim}{r}_{12}\Bigg\{ e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}}\,\varphi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i})\\ &\qquad + \frac{i\mu_{12}}{\hbar^2}\frac{e^{ik_{12i}r_{12}}}{r_{12}}\langle k_{12i}\underset{\sim}{\nu}_{12}|\underset{-}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle\left[\delta(\underset{\sim}{\nu}_{12}-\underset{\sim}{\nu}_A)\frac{e^{-iAr_{12}}}{Ar_{12}} - \delta(\underset{\sim}{\nu}_{12}+\underset{\sim}{\nu}_A)\frac{e^{iAr_{12}}}{Ar_{12}}\right]\Bigg\}\\ &+\frac{\mu_{12}}{\hbar^2}\langle \underset{\sim}{k}_{23f}|\underset{\sim}{t}_{23f}|-\underset{\sim}{B}\rangle\left[\frac{\langle k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle}{A(\underset{\sim}{k}_{12i}-A)} - \frac{\langle -k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle}{A(k_{12i}+A)}\right]\end{aligned} \tag{175b}$$

while for $C^2 \neq k_{31i}{}^2$

$$\bar{T}^t_{2331}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \langle \underset{\sim}{k}_{23f}|\underset{\sim}{t}_{23f}|\underset{\sim}{D}\rangle \int d\underset{\sim}{r}_{31} \Big\{ e^{i\underset{\sim}{r}_{31}\cdot \underset{\sim}{C}} \phi^{(+)}_{31}(\underset{\sim}{r}_{31}; \underset{\sim}{k}_{31i}) + \frac{i\mu_{31}}{\hbar^2} \frac{e^{ik_{31}r_{31}}}{r_{31}} \langle k_{31i}\underset{\sim}{\nu}_{31}|\underset{\sim}{t}_{31i}|\underset{\sim}{k}_{31i}\rangle \Big[\delta(\underset{\sim}{\nu}_{31}+\underset{\sim}{\nu}_c)\frac{e^{-iCr_{31}}}{Cr_{31}} - \delta(\underset{\sim}{\nu}_{31}-\underset{\sim}{\nu}_c)\frac{e^{iCr_{31}}}{Cr_{31}}\Big]\Big\}$$
$$+ \frac{\mu_{31}}{\hbar^2}\langle \underset{\sim}{k}_{23f}|\underset{\sim}{t}_{23f}|\underset{\sim}{D}\rangle \Big[\frac{\langle -k_{31i}\underset{\sim}{\nu}_c|\underset{\sim}{t}_{31i}|\underset{\sim}{k}_{31i}\rangle}{C(k_{31i}-C)} - \frac{\langle k_{31i}\underset{\sim}{\nu}_c|\underset{\sim}{t}_{31i}|\underset{\sim}{k}_{31i}\rangle}{C(k_{31i}+C)}\Big] \tag{175c}$$

In Eqs. (175) we employ the notation (172c), along with $\underset{\sim}{r}_{31} = r_{31}\underset{\sim}{\nu}_{31}$, $\underset{\sim}{C} = C\underset{\sim}{\nu}_c$; the two-particle scattered waves $\phi_{\alpha\beta}{}^{(+)}$ are given by Eqs. (73), as always. The quantities $\bar{T}^t_{3123}$, $\bar{T}^t_{1231}$ of Eqs. (175) are cyclic permutations of $\bar{T}^t_{2312}$; the quantities $\bar{T}^t_{3112}$, $\bar{T}^t_{1223}$ are cyclic permutations of $\bar{T}^t_{2331}$. Evidently $\bar{T}^t_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is the contribution to $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ made by $\bar{\Psi}_{23f}{}^{(-)*}\bar{V}_{23}\bar{\Phi}_{12}{}^{(+)}$ in (169c); $\bar{T}^t_{2331}$ is the contribution to $\bar{T}^t$ made by $\bar{\Psi}_{23f}{}^{(-)*}V_{23}\bar{\Phi}_{31}{}^{(+)}$. In (175b), as in its generating expression (172a), integrals of the individual terms within the braces [e.g., of the term $e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}}\phi_{12}{}^{(+)}$] do not converge, but the entire integral in (175b) does converge provided the quantity within the braces is treated as a single $\underset{\sim}{r}_{12}$-dependent function. Eq. (173a) means that the divergent integrals in (172a) have residual finite parts in addition to their δ-function parts from (173b); these residual parts are the terms not under the integral sign in (175b). Eq. (175b) does not specify $\bar{T}^t_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ at $A = \pm k_{12i}$ (although of course only $A = k_{12i}$ can occur for real $\underset{\sim}{k}_i, \underset{\sim}{k}_f$, where $A > 0$

by definition); in fact, as was the case for the corresponding integral in (172a), at $A = \pm k_{12i}$ the integral in (175b) is logarithmically divergent, i.e., strictly speaking is mathematically undefined. That these deficiencies of (175b) at $A = \pm k_{12i}$ are inconsequential for the purpose of this publication has been explained in subsection 4.3.2; in any event, the experimenter attempting to measure truly three-body scattering would not be placing his counters at locations consistent with $A = k_{12i}$, because these locations also are those at which the argument of the δ-function (135a) vanishes [recall the remarks at the end of subsection 4.2.2]. The rather awkward form of the right side of (175b) actually reduces to a quite convenient and readily interpretable expression for $\bar{\bar{T}}^{t}_{2312}$ [see subsection 5.2.3 below]. Similar remarks [to those of this paragraph] pertain to Eq. (175c), which converges except when $C = \pm k_{31i}$.

The derivation of Eqs. (175) [section E.4] simultaneously shows that in (169a)

$$\begin{aligned}\bar{\bar{T}}^{a}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = {} & \bar{\bar{T}}^{a}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) + \bar{\bar{T}}^{a}_{2331}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \\ & + \bar{\bar{T}}^{a}_{3123}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) + \bar{\bar{T}}^{a}_{3112}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \\ & + \bar{\bar{T}}^{a}_{1231}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) + \bar{\bar{T}}^{a}_{1223}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)\end{aligned} \tag{176a}$$

where

$$\bar{T}^{a}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \frac{-\mu_{12}\pi i}{\hbar^2 k_{12i}} \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} \rangle \langle -k_{12i} \underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle \delta(k_{12i} + A)$$

$$- \frac{\mu_{12}\pi i}{\hbar^2 k_{12i}} \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} \rangle \langle k_{12i} \underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle \delta(k_{12i} - A) \qquad (176b)$$

$$\bar{T}^{a}_{2331}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \frac{-\mu_{31}\pi i}{\hbar^2 k_{31i}} \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | \underset{\sim}{D} \rangle \langle k_{31i} \underset{\sim}{\nu}_C | \underset{\sim}{t}_{31i} | \underset{\sim}{k}_{31i} \rangle \delta(k_{31i} + C)$$

$$\frac{-\mu_{31}\pi i}{\hbar^2 k_{31i}} \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | \underset{\sim}{D} \rangle \langle -k_{31i} \underset{\sim}{\nu}_C | \underset{\sim}{t}_{31i} | \underset{\sim}{k}_{31i} \rangle \delta(k_{31i} - C) \qquad (176c)$$

The quantities $\bar{T}^{a}_{3123}$, $\bar{T}^{a}_{1231}$, in (176a) are cyclic permututions of $\bar{T}^{a}_{2312}$; the quantities $\bar{T}^{a}_{3112}$, $\bar{T}^{a}_{1223}$ are cyclic permutations of $\bar{T}^{a}_{2331}$. Evidently $\bar{T}^{a}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is the contribution to $\bar{T}^{a}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ made by $\bar{\Psi}_{23f}{}^{(-)*} V_{23} \bar{\Phi}_{12}{}^{(+)}$ in (169c); $\bar{T}^{a}_{2331}$ is the contribution to $\bar{T}^{a}$ made by $\bar{\Psi}_{23f}{}^{(-)*} V_{23} \bar{\Phi}_{31}{}^{(+)}$.

In Eq. (176b), only the second δ-function on the right can have a vanishing argument at real $\underset{\sim}{k}_i, \underset{\sim}{k}_f$. Referring to Eq. (171a), one sees explicitly that this second term on the right side of (176b) is proportional to precisely the δ-function (135a) interpreted in 4.1.3. In particular, Eqs. (137) - (139) and the discussion thereof showed the presence of the δ-function (135a) could be interpreted as resulting from two independent successive two-particle scatterings--namely first particles 1, 2 are scattered by each other, after which particle 2 is scattered by 3. The precise form of the $\delta(k_{12i} - A)$ term in (176b) is consistent with this interpretation, in that this term is proportional to the product of the truly two-body matrix elements $< k_{12i}\underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} >$ and $< \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} >$. Indeed, at $k_{12i} = A$, the final relative momentum $k_{12i}\underset{\sim}{\nu}_A$ in the two-body matrix element $< k_{12i}\underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} >$ associated with the first 1, 2 scattering is identical with $\underset{\sim}{A}$ of (171a), which in turn is identical with the intermediate (after the first scattering) $\underset{\sim}{k}'_{12}$ of Eqs. (138) - (139), because--using (29) and (138) as well as $\underset{\sim}{K}_i = \underset{\sim}{K}_f$ and $\underset{\sim}{k}_{1f} = \underset{\sim}{k}'_1$--

$$\underset{\sim}{A} = \underset{\sim}{K}_{23f} + \frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12i} = \underset{\sim}{k}_{1f} - \frac{m_1}{M}\underset{\sim}{K}_f + \frac{m_1}{m_1+m_2}\left(\underset{\sim}{k}_{3i} - \frac{m_3}{M}\underset{\sim}{K}_i\right)$$

$$= \underset{\sim}{k}_{1f} - \frac{m_1}{m_1+m_2}\left(\underset{\sim}{k}_{1i} + \underset{\sim}{k}_{2i}\right) = \underset{\sim}{k}'_1 - \frac{m_1}{m_1+m_2}\left(\underset{\sim}{k}'_1 + \underset{\sim}{k}'_2\right) = \underset{\sim}{k}'_{12} \qquad (177a)$$

Similarly,

$$-\underset{\sim}{B} = \frac{-m_3}{m_2+m_3}\underset{\sim}{K}_{23f} - \underset{\sim}{K}_{12i} = \frac{-m_3}{m_2+m_3}\left(\underset{\sim}{k}_{1f} - \frac{m_1}{M}\underset{\sim}{K}_f\right) - \left(\underset{\sim}{k}_{3i} - \frac{m_3}{M}\underset{\sim}{K}_i\right)$$

$$= -\underset{\sim}{k}_{3i} + \frac{m_3}{m_2+m_3}\left(\underset{\sim}{k}_{2f} + \underset{\sim}{k}_{3f}\right) = -\underset{\sim}{k}'_3 + \frac{m_3}{m_2+m_3}\left(\underset{\sim}{k}'_2 + \underset{\sim}{k}'_3\right) = \underset{\sim}{k}'_{23} \qquad (177b)$$

Eq. (177b) shows that--in the $\delta(k_{12} - A)$ term of (176b)--the initial momentum $-\underset{\sim}{B}$ in the two-body matrix element $< \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} >$ associated with the second 2,3 scattering is identical with the intermediate $\underset{\sim}{k}'_{23}$. Moreover, at $k_{12i} = A$ this second scattering matrix element $< \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} >$--like the first scattering matrix element $< k_{12i}\underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} >$--is on the two-body energy shell [i.e., $B = k_{23f}$] because

$$A^2 = K_{23f}^2 + \left(\frac{m_1}{m_1+m_2}\right)^2 K_{12i}^2 + \frac{2m_1}{m_1+m_2} \underset{\sim}{K}_{23f} \cdot \underset{\sim}{K}_{12i} = k_{12i}^2 \tag{178a}$$

implies

$$B^2 = \left(\frac{m_3}{m_2+m_3}\right)^2 K_{23f}^2 + K_{12i}^2 + \frac{2m_3}{m_2+m_3} \underset{\sim}{K}_{23f} \cdot \underset{\sim}{K}_{12i}$$

$$= \frac{m_3}{m_1} \frac{m_1+m_2}{m_2+m_3} k_{12i}^2 + \frac{m_2 M}{(m_2+m_3)(m_1+m_2)} K_{12i}^2 - \frac{m_2 m_3 M}{m_1(m_2+m_3)^2} K_{23f}^2 \tag{178b}$$

But conservation of the total energy of the three particles, along with $\underset{\sim}{K}_i = \underset{\sim}{K}_f$, further implies [via Eq. (35)] that

$$\frac{k_{23f}^2}{\mu_{23}} + \frac{K_{23f}^2}{\mu_{1R}} = \frac{k_{12i}^2}{\mu_{12}} + \frac{K_{12i}^2}{\mu_{3R}} \tag{178c}$$

Use of Eq. (178c) to eliminate k_{12i}^2 in (178b) immediately yields $B^2 = k_{23f}^2$ [recalling Eqs. (29e) and (29f)].

Similar considerations to those of the preceding two paragraphs pertain to the cyclic permutations of (176b), as well as to (176c) and its cyclic permutations. In particular, the precise form of the $\delta(k_{31i} - C)$ term in (176c) is consistent with the interpretation that this term results from the purely two-particle scattering of 3 and 1, followed by a second purely two-particle scattering of 3 by 2. More specifically, at $k_{31i} \simeq C$ it can be seen that: (i) the vector $-k_{31i}\underset{\sim}{\nu}_c = -\underset{\sim}{C}$ of (171b) is identical with the expected intermediate (after the first 3, 1 scattering) $\underset{\sim}{k}'_{31}$; (ii) the vector $\underset{\sim}{D}$ now is identical with the expected intermediate $\underset{\sim}{k}'_{23}$; (iii) now $D = k_{23f}$, so that each two-body matrix element multiplying $\delta(k_{31i} - C)$ in (176c) lies on the two-body energy shell.

5.2 Detailed Balancing

From very general time reversal considerations[39] one expects that the matrix elements of the total three-body transition operator $\bar{T}$ [defined by Eq. (5)] satisfy

$$\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{T}(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) \tag{179a}$$

Similarly, one expects that the truly three body part $\bar{T}^t$ of $\bar{T}$ obeys

$$\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{T}^t(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) \tag{179b}$$

For purely two-body collisions--where the integrals $\bar{\Psi}_f^{(-)*}V\bar{\psi}_i$ and $\bar{\psi}_f^{*}V\bar{\Psi}_i^{(+)}$ of Eqs. (126b) and (131c) always converge, and where correspondingly the scattered parts $\bar{\Phi}_i^{(+)}$, $\bar{\Phi}_f^{(-)*}$ of $\bar{\Psi}_i^{(+)}$, $\bar{\Psi}_f^{(-)*}$ are everywhere outgoing--the result (179a) easily is demonstrated[2] directly from the formulas (126b) and (131c). In the three-body case of present interest this previous demonstration[2] of (179a) is not applicable, however, because now the integrals (126b) and (131c) need not converge, and because correspondingly $\bar{\Phi}_i^{(+)}$ and $\bar{\Phi}_f^{(-)*}$ now are not everywhere outgoing. The truly three-body amplitudes of Eq. (179b) are expressible in terms of convergent integrals [as the last section 5.1 has shown], but these expressions are so complicated that the previous two-body proof[2] of the reciprocity relation (179a) also is inapplicable to (179b). Thus it is needful to investigate here whether or not the expressions (175) for $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ really are consistent with the reciprocity relation (179b).

This investigation is particularly necessary because the formulas (175) for $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ were derived on the basis of some mathematically questionable (though plausible) manipulations, as discussed in section 5.1. One also could adopt the viewpoint that Eq. (179b) obviously holds because--as will be discussed in section 5.3--the formulas (165b) and (175) for the various component parts of $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ reduce to momentum space matrix elements, for which there presumably are general proofs[39] of time reversal invariance; this viewpoint doesn't really simplify the problem of proving (179b), however, since the proofs of detailed balance in subsections 5.2.1 - 5.2.3 below largely involve carrying out this reduction of our configuration space expressions for $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ to recognizable momentum space matrix elements.

Recalling (175a), to demonstrate (179b) it is sufficient to show

$$\bar{T}^d(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \bar{T}^d(-\underset{\sim}{k}_f \rightarrow -\underset{\sim}{k}_i) \tag{180a}$$

$$\bar{T}^t_{2312}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \bar{T}^t_{1223}(-\underset{\sim}{k}_f \rightarrow -\underset{\sim}{k}_i) \tag{180b}$$

because (180b) obviously implies the cyclic relations

$$\bar{T}^t_{3123}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \bar{T}^t_{2331}(-\underset{\sim}{k}_f \rightarrow -\underset{\sim}{k}_i) \tag{180c}$$

$$\overline{\overline{T}}^{t}_{1231}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \overline{\overline{T}}^{t}_{3112}(-\underset{\sim}{k}_f \rightarrow -\underset{\sim}{k}_i) \tag{180d}$$

as well as

$$\overline{\overline{T}}^{t}_{1223}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \overline{\overline{T}}^{t}_{2312}(-\underset{\sim}{k}_f \rightarrow -\underset{\sim}{k}_i) \tag{180e}$$

Eq. (180e) is obtained from (180b) by replacing $\underset{\sim}{k}_i$, $\underset{\sim}{k}_f$ with $-\underset{\sim}{k}_f$, $-\underset{\sim}{k}_i$ respectively.

5.2.1 Triple and Higher Scattering Terms

First I shall prove (180a) holds, i.e., I shall prove detailed balancing for those terms in $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ associated with $n \geq 3$ successive two-body scatterings [recall the discussion at the end of subsection 4.3.1]. Using Eq. (165b), one sees (180a) is equivalent to the assertion that

$$\bar{\Psi}_f^{(-)*}(\underset{\sim}{k}_f)\Big[(V_{23}+V_{31})\bar{\Phi}_{12i}^{s(+)}(\underset{\sim}{k}_i)+(V_{31}+V_{12})\bar{\Phi}_{23i}^{s(+)}(\underset{\sim}{k}_i) + (V_{12}+V_{23})\bar{\Phi}_{31i}^{s(+)}(\underset{\sim}{k}_i)\Big]$$
$$= \bar{\Psi}_f^{(-)*}(-\underset{\sim}{k}_i)\Big[(V_{23}+V_{31})\bar{\Phi}_{12i}^{s(+)}(-\underset{\sim}{k}_f)+(V_{31}+V_{12})\bar{\Phi}_{23i}^{s(+)}(-\underset{\sim}{k}_f) + (V_{12}+V_{23})\bar{\Phi}_{31i}^{s(+)}(-\underset{\sim}{k}_f)\Big] \quad (181)$$

where the notation indicates that $\bar{\Psi}_f^{(-)*}$ on the left side of (181) is the limit as $\varepsilon \to 0$ of the solution to the center of mass version of the Lippmann-Schwinger equation (107a) with ψ_f of (100c), whereas $\bar{\Psi}_f^{(-)*}$ on the right side of (181) is the limit as $\varepsilon \to 0$ of the solution to the center of mass version of (107a) using

$$\psi_f \equiv \psi_f(\underset{\sim}{r}, -\underset{\sim}{k}_i) = e^{-i\underset{\sim}{k}_i\cdot\underset{\sim}{r}} \quad (182a)$$

Similarly $\bar{\Phi}_{12i}^{s(+)}(\underset{\sim}{k}_i)$ in (181) is the quantity defined by (161d) with the understanding that in (161d) the functions $\Phi_{\alpha\beta}^{(+)} \equiv \Phi_{\alpha\beta i}^{(+)}$ are defined by Eq. (58a) for ψ_i from (21a), whereas $\bar{\Phi}_{12i}^{s(+)}(-\underset{\sim}{k}_f)$ is defined by Eqs. (161d) and (58a) using

$$\psi_i \equiv \psi_i(\underset{\sim}{r}, -\underset{\sim}{k}_f) = e^{-i\underset{\sim}{k}_f \cdot \underset{\sim}{r}} \tag{182b}$$

Eqs. (182), together with Eqs. (8) and (107a), immediately imply

$$\bar{\Psi}_f^{(-)*}(-\underset{\sim}{k}_i) = \bar{\Psi}_i^{(+)}(\underset{\sim}{k}_i) \tag{183a}$$

Similarly, Eqs. (72) - (73) and (105)--along with Eqs. (106) and (160)--imply

$$\bar{\Phi}_{12i}^{(+)}(-\underset{\sim}{k}_f) = \bar{\Phi}_{12f}^{(-)*}(\underset{\sim}{k}_f) \tag{183b}$$

$$\bar{\Phi}_{12i}^{S(+)}(-\underset{\sim}{k}_f) = -\bar{G}_{12}^{(+)} V_{12}\left[\bar{\Phi}_{31f}^{(-)*}(\underset{\sim}{k}_f) + \bar{\Phi}_{23f}^{(-)*}(\underset{\sim}{k}_f)\right] \equiv \bar{\Phi}_{12f}^{S(-)*}(\underset{\sim}{k}_f) \tag{183c}$$

Thus the right sides of Eqs. (180a) and (181) become

$$\bar{T}^d(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i)$$

$$= \left[\bar{\Phi}_{12f}^{s(-)*}(V_{23}+V_{31}) + \bar{\Phi}_{23f}^{s(-)*}(V_{31}+V_{12}) + \bar{\Phi}_{31f}^{s(-)*}(V_{12}+V_{23})\right]\bar{\Psi}_i^{(+)} \quad \text{(184a)}$$

where now--as always in the past--the subscript i on the right side of a matrix element is associated with the incident wave vector $\underset{\sim}{k}_i$, while the subscript f on the left side of a matrix element is associated with the final wave vector $\underset{\sim}{k}_f$.

Using the symmetry relation (95), valid for all Green's functions employed in this work, Eq. (183c) permits rewriting (184a) in the form

$$\begin{aligned} \bar{T}^d(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) = -\Big[& \left(\bar{\Phi}_{31f}^{(-)*} + \bar{\Phi}_{23f}^{(-)*}\right) V_{12}\bar{G}_{12}^{(+)}(V_{23}+V_{31}) \\ & + \left(\bar{\Phi}_{12f}^{(-)*} + \bar{\Phi}_{31f}^{(-)*}\right) V_{23}\bar{G}_{23}^{(+)}(V_{31}+V_{12}) \\ & + \left(\bar{\Phi}_{23f}^{(-)*} + \bar{\Phi}_{12f}^{(-)*}\right) V_{31}\bar{G}_{31}^{(+)}(V_{12}+V_{23})\Big]\bar{\Psi}_i^{(+)} \end{aligned} \quad \text{(184b)}$$

In (184b), $\bar{\Psi}_i^{(+)}$ can be replaced by $\lim \bar{\Psi}_i(\bar{E} + i\varepsilon)$, from (34a); correspondingly, recalling Eqs. (60), (72) and (105), $\bar{\Phi}_{12f}^{(-)*}$ can be replaced by

$$-\lim_{\varepsilon\to 0}\left\{\bar{G}_{12}(\bar{E}+i\varepsilon)V_{12}\bar{\Psi}_f^*\right\} = -\lim_{\varepsilon\to 0}\left\{\bar{\Psi}_f^* V_{12}\bar{G}_{12}(\bar{E}+i\varepsilon)\right\} \quad \text{(184c)}$$

Moreover, as written the right side of (184b) is wholly composed of convergent integrals. According to sections 2.2 and A.8, therefore, it is legitimate to replace (184b) by

$$\bar{T}^d(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) = \lim_{\varepsilon\to 0} \bar{\Psi}_f^* \Big[(V_{31}\bar{G}_{31} + V_{23}\bar{G}_{23}) V_{12}\bar{G}_{12}(V_{23}+V_{31})$$

$$+ (V_{12}\bar{G}_{12} + V_{31}\bar{G}_{31}) V_{23}\bar{G}_{23}(V_{31}+V_{12})$$

$$+ (V_{23}\bar{G}_{23} + V_{12}\bar{G}_{12}) V_{31}\bar{G}_{31}(V_{12}+V_{23})\Big](1-\bar{G}V)\bar{\Psi}_i \tag{185a}$$

where all Green's functions are evaluated at the same complex energy $\bar{\lambda} = \bar{E} + i\varepsilon$. Similarly, the left sides of (180a) and (181) are

$$\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \lim_{\varepsilon\to 0} \bar{\Psi}_f^* (1 - V\bar{G}) \Big[(V_{23}+V_{31})\bar{G}_{12}V_{12}(\bar{G}_{31}V_{31}+\bar{G}_{23}V_{23})$$

$$+ (V_{31}+V_{12})\bar{G}_{23}V_{23}(\bar{G}_{12}V_{12}+\bar{G}_{31}V_{31})$$

$$+ (V_{12}+V_{23})\bar{G}_{31}V_{31}(\bar{G}_{23}V_{23}+\bar{G}_{12}V_{12})\Big]\bar{\Psi}_i \tag{185b}$$

I stress that Eqs. (185) hold even though the integral (52b) need not converge, i.e., even though it is not legitimate to replace $\bar{\Psi}_i^{(+)}$ by $[1 - \bar{G}^{(+)}V]\bar{\psi}_i$.

I now show that the matrix elements on the right sides of (185a) and (185b) are identical at every $\bar{\lambda} = \bar{E} + i\varepsilon$ ($\varepsilon > 0$), which is sufficient to demonstrate (180a). Because all the Green's functions in Eqs. (185) are exponentially decreasing at infinity for $\varepsilon > 0$, the orders of integration in Eqs. (185) [and in subsequent expressions in this subsection] can be and will be rearranged essentially at will. Moreover, to ease the notational complexity, for the moment I shall drop the bars in Eqs. (185), which here introduces no error even though the right sides of Eqs. (184a) - (184b) are not convergent in the laboratory system.

In Eq. (185b) use Eqs. (63) to replace, e.g., $G(V_{23} + V_{31})G_{12}$ by $G_{12} - G$. Then [also dropping temporarily the irrelevant lim as $\varepsilon \to 0$] the matrix element on the right side of (185b) reduces to

$$\begin{aligned}\psi_f^*\Big\{&(V_{23}+V_{31})G_{12}V_{12}(G_{31}V_{31}+G_{23}V_{23})\\&+(V_{31}+V_{12})G_{23}V_{23}(G_{12}V_{12}+G_{31}V_{31})+(V_{12}+V_{23})G_{31}V_{31}(G_{23}V_{23}+G_{12}V_{12})\\&+V\big[(G-G_{12})V_{12}(G_{31}V_{31}+G_{23}V_{23})+(G-G_{23})V_{23}(G_{12}V_{12}+G_{31}V_{31})\\&+(G-G_{31})V_{31}(G_{23}V_{23}+G_{12}V_{12})\big]\Big\}\psi_i\end{aligned} \tag{186a}$$

$$= \psi_f^* \Big\{ VG\big[(V_{12}+V_{23})G_{31}V_{31} + (V_{23}+V_{31})G_{12}V_{12} + (V_{31}+V_{12})G_{23}V_{23} \big]$$
$$- V_{12}G_{12}V_{12}(G_{31}V_{31}+G_{23}V_{23}) - V_{23}G_{23}V_{23}(G_{12}V_{12}+G_{31}V_{31}) \qquad (186b)$$
$$- V_{31}G_{31}V_{31}(G_{23}V_{23}+G_{12}V_{12}) \Big\} \psi_i$$

where in going from (186a) to (186b) I have rearranged the terms in G, and have noted that $V_{23} + V_{31} = V - V_{12}$, etc. Now in Eq. (186b), use Eqs. (63) again in the terms involving G, and recall Eq. (77a) as well as the manipulations in Eqs. (137) and (166b). Then Eq. (186b) further reduces to

$$\psi_f^* \Big\{ V\big[(G_{31}-G)V_{31} + (G_{12}-G)V_{12} + (G_{23}-G)V_{23} \big]$$
$$+ (\underset{\sim}{T}_{12} - V_{12})(G_{31}V_{31}+G_{23}V_{23}) + (\underset{\sim}{T}_{23} - V_{23})(G_{12}V_{12}+G_{31}V_{31})$$
$$+ (\underset{\sim}{T}_{31} - V_{31})(G_{23}V_{23}+G_{12}V_{12}) \Big\} \psi_i \qquad (186c)$$

$$= \psi_f^* \Big\{ -VGV + V_{31}G_{31}V_{31} + V_{12}G_{12}V_{12} + V_{23}G_{23}V_{23}$$
$$+ \underset{\sim}{T}_{12}\left(G_{31}V_{31} + G_{23}V_{23}\right) + \underset{\sim}{T}_{23}\left(G_{12}V_{12} + G_{31}V_{31}\right) \tag{186d}$$
$$+ \underset{\sim}{T}_{31}\left(G_{23}V_{23} + G_{12}V_{12}\right) \Big\} \psi_i$$

which, recalling our starting point for (184d) was (185b), implies finally

$$\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \lim_{\varepsilon \to 0} \bar{\psi}_f^* \Big\{ \bar{\underset{\sim}{T}} - \bar{\underset{\sim}{T}}_{12} - \bar{\underset{\sim}{T}}_{23} - \bar{\underset{\sim}{T}}_{31}$$
$$+ \bar{\underset{\sim}{T}}_{12}\bar{G}_F\bar{\underset{\sim}{T}}_{31} + \bar{\underset{\sim}{T}}_{12}\bar{G}_F\bar{\underset{\sim}{T}}_{23}$$
$$+ \bar{\underset{\sim}{T}}_{23}\bar{G}_F\bar{\underset{\sim}{T}}_{12} + \bar{\underset{\sim}{T}}_{23}\bar{G}_F\bar{\underset{\sim}{T}}_{31} \tag{187a}$$
$$+ \bar{\underset{\sim}{T}}_{31}\bar{G}_F\bar{\underset{\sim}{T}}_{23} + \bar{\underset{\sim}{T}}_{31}\bar{G}_F\bar{\underset{\sim}{T}}_{12} \Big\} \bar{\psi}_i$$

A similar [to those employed in Eqs. (186a) - (187a)] sequence of manipulations reduces the right side of (185a) to the right side of (187a). Therefore, the equality (180a) has been demonstrated.

It is worth remarking that the changes of sign on the right side of (187a) are consistent with the expectation that $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$--being the contribution to $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ associated with $n \geq 3$ successive two-body scatterings--must be identifiable with the matrix element of $\underset{\sim}{T}$ minus all single and double scattering contributions; these double scattering contributions, contained in $\bar{T}^s(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ of (133b), have been evaluated in Eqs. (137). Actually, if convergence difficulties are ignored, a much simpler sequence of iterations than was employed in deriving (187a) from (165b) yields [see section 5.3 below]

$$\bar{\Psi}_f^* \underset{\sim}{\bar{T}} \bar{\Psi}_i = \bar{\Psi}_f^* \Big\{ \underset{\sim}{\bar{T}}_{12} + \underset{\sim}{\bar{T}}_{23} + \underset{\sim}{\bar{T}}_{31} - \underset{\sim}{\bar{T}}_{23} \bar{G}_F \underset{\sim}{\bar{T}}_{12} - \bar{T}_{23} \bar{G}_F \underset{\sim}{\bar{T}}_{31} - \underset{\sim}{\bar{T}}_{31} \bar{G}_F \underset{\sim}{\bar{T}}_{23} - \underset{\sim}{\bar{T}}_{31} \bar{G}_F \underset{\sim}{\bar{T}}_{12} - \underset{\sim}{\bar{T}}_{12} \bar{G}_F \underset{\sim}{\bar{T}}_{31} - \underset{\sim}{\bar{T}}_{12} \bar{G}_F \underset{\sim}{\bar{T}}_{23} \Big\} \bar{\Psi}_i + \bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \tag{187b}$$

where now $\bar{G}_F, \underset{\sim}{\bar{T}}_{\alpha\beta}$ are evaluated at real center of mass energy $\bar{E}_i = \bar{E}_f$. The form of (187b) evidently is consistent with (187a), as asserted. However, because both the right and left sides of (187b) are composed of divergent integrals [containing the trivial and non-trivial δ-function contributions which have been discussed in section 4.1], Eq. (187b) is not a useful formula for actually computing $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$. Instead, one must use (187a), or--if one wants to avoid taking the limit $\varepsilon \to 0$--the original formula (165b).

I further remark that Eq. (180a) also can be demonstrated by showing that the right side of (184b) is precisely the expression one would deduce for $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ starting from the integral equation (84c). Specifically, in (84c)

$$\left\{G_F^{(+)} V_{12} G_{12}^{(+)}\right\} V_{23} \Psi_i^{(+)} = \left\{G_F^{(+)} V_{12} G_{12}^{(+)}\right\} V_{23} \left[\Psi_{23}^{(+)} - G_{23}^{(+)}(V_{12}+V_{31})\Psi_i^{(+)}\right] \quad (188a)$$

using (86). Thus one infers

$$\Phi_i^{S(+)} = \lim_{\varepsilon \to 0} \left\{G_F(\lambda) V_{12} G_{12}(\lambda) \left[V_{23}\Psi_{23}^{(+)} + V_{31}\Psi_{31}^{(+)}\right] + \text{etc.}\right\}$$

$$- \left\{G_F^{+} V_{12} G_{12}^{(+)}\right\} \left[V_{23} G_{23}^{(+)}(V_{12}+V_{31}) + V_{31} G_{31}^{(+)}(V_{23}+V_{12})\right] \Psi_i^{(+)} + \text{etc.} \quad (188b)$$

where, as always, $\lambda = E + i\varepsilon$. But in (188b)

$$\lim_{\varepsilon \to 0} \left\{G_F V_{12} G_{12} \left[V_{23}\Psi_{23}^{(+)} + V_{31}\Psi_{31}^{(+)}\right]\right\}$$

$$= \lim_{\varepsilon \to 0} \left\{G_{12} V_{12} G_F \left[V_{23}\Psi_{23}^{(+)} + V_{31}\Psi_{31}^{(+)}\right]\right\} \quad (189a)$$

$$= -G_{12}^{(+)} V_{12} \left[\Phi_{23}^{(+)} + \Phi_{31}^{(+)}\right] = \Phi_{12}^{S(+)}$$

Therefore, comparing with Eqs. (161) - (162), $\Phi_i^{d(+)}$ is given by the terms involving $\Psi_i^{(+)}$ in (188b). Correspondingly, using the defining Eq. (165a) for $\bar{T}^d(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$,

$$\bar{T}^d(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \bar{\Psi}_f^* V_{12} \bar{G}_{12}^{(+)} \left[V_{23} \bar{G}_{23}^{(+)} (V_{12} + V_{31}) + V_{31} \bar{G}_{31}^{(+)} (V_{23} + V_{12}) \right] \bar{\Psi}_i^{(+)} + \text{etc.}$$

$$= -\bar{\Phi}_{12f}^{(-)*} \left[V_{23} \bar{G}_{23}^{(+)} (V_{12} + V_{31}) + V_{31} \bar{G}_{31}^{(+)} (V_{23} + V_{12}) \right] \bar{\Psi}_i^{(+)} + \text{etc.} \qquad (189b)$$

The right side of (189b) is seen to be identical with the [slightly rearranged] right side of (184b).

5.2.2 Double Scattering δ-Function Terms

In this subsection I show that the amplitudes $\bar{T}^a(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ of (176a) also obey detailed balancing, i.e., that

$$\bar{T}^a(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{T}^a(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) \tag{190a}$$

In particular, I shall prove

$$\bar{T}^a_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{T}^a_{1223}(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) \tag{190b}$$

which is sufficient to demonstrate (190a) [recall the analogous case of Eqs. (180)]. It will be presumed (in the remainder of this subsection) that for any given $\underset{\sim}{k}_i$ the final $\underset{\sim}{k}_f$ are chosen consistent with energy and momentum conservation. These restrictions on $\underset{\sim}{k}_f$ are convenient, as will be seen; $\underset{\sim}{k}_f$ can be so restricted because--for the purposes of this section 5.2--detailed balancing need not be investigated for values of $\underset{\sim}{k}_i, \underset{\sim}{k}_f$ which cannot occur in actual collisions.

Referring to (176b), the left side of (190b) can be written in the form

$$\bar{\bar{T}}^{a}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \frac{-\mu_{12}\pi i}{\hbar^2 k_{12i}} \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} \rangle \langle \underset{\sim}{A} | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle \left[\delta(k_{12i} + A) + \delta(k_{12i} - A) \right]$$

$$= \frac{-2\mu_{12}\pi i}{\hbar^2} \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} \rangle \langle \underset{\sim}{A} | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle \delta(k_{12i}^2 - A^2) \qquad (191a)$$

where we have used

$$\delta[f(x)] = \sum_r \frac{1}{f'(x_r)} \delta(x - x_r) \qquad (191b)$$

summed over all roots x_r satisfying $f(x_r) = 0$. Similarly, from Eqs. (171b) and (176c),

$$\bar{\bar{T}}^{a}_{1223}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \frac{-2\mu_{23}\pi i}{\hbar^2} \langle \underset{\sim}{k}_{12f} | \underset{\sim}{t}_{12f} | \hat{\underset{\sim}{D}} \rangle \langle -\hat{\underset{\sim}{C}} | \underset{\sim}{t}_{23i} | \underset{\sim}{k}_{23i} \rangle \delta(k_{23i}^2 - \hat{C}^2) \qquad (192a)$$

where

$$\hat{\underset{\sim}{C}} = \underset{\sim}{K}_{12f} + \frac{m_3}{m_2+m_3}\underset{\sim}{K}_{23i}$$

$$\hat{\underset{\sim}{D}} = \frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12f} + \underset{\sim}{K}_{23i} \qquad (192b)$$

When $\underset{\sim}{k}_i, \underset{\sim}{k}_f$ are replaced by $-\underset{\sim}{k}_f$, $-\underset{\sim}{k}_i$ respectively, Eq. (192b) implies that $\hat{\underset{\sim}{C}}$, $\hat{\underset{\sim}{D}}$ are replaced by $-\underset{\sim}{B}$, $-\underset{\sim}{A}$ respectively, where $\underset{\sim}{A}$, $\underset{\sim}{B}$ again are the vectors defined in (171a). Therefore

$$\bar{T}^a_{1223}(-\underset{\sim}{k}_f \rightarrow -\underset{\sim}{k}_i)$$

$$= \frac{-2\mu_{23}\pi i}{\hbar^2}\left\langle -\underset{\sim}{k}_{12i}\middle|\underset{\sim}{t}_{12i}\middle|-\underset{\sim}{A}\right\rangle\left\langle \underset{\sim}{B}\middle|\underset{\sim}{t}_{23f}\middle|-\underset{\sim}{k}_{23f}\right\rangle\delta\left(k^2_{23f} - B^2\right) \qquad (193)$$

Now, as explained at the very beginning of this section, we know that the matrix elements of the two-body operators $\underset{\sim}{t}_{12i}$, $\underset{\sim}{t}_{23f}$ do obey detailed balancing, i.e., for any two vectors $\underset{\sim}{X}$, $\underset{\sim}{Y}$ (whether on the energy-momentum shell or not)

$$\left\langle \underset{\sim}{X}\middle|\underset{\sim}{t}_{12i}\middle|\underset{\sim}{Y}\right\rangle = \left\langle -\underset{\sim}{Y}\middle|\underset{\sim}{t}_{12i}\middle|-\underset{\sim}{X}\right\rangle \qquad (194a)$$

and similarly for $\underset{\sim}{t}_{23f}$. Eq. (194a) can be proved, e.g., by noting that

Eq. (108) and the definition (131f) of $\underset{\sim}{t}_{12}$ imply $\underset{\sim}{t}_{12}$ is a symmetric operator in the coordinate representation,

$$\underset{\sim}{t}_{12}(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};\lambda)=\underset{\sim}{t}_{12}(\underset{\sim}{r}'_{12};\underset{\sim}{r}_{12};\lambda) \tag{194b}$$

whereat (194a) follows immediately, recalling the fundamental defining relation (131e) for the matrix elements of $\underset{\sim}{t}_{12}$.

Comparing Eq. (191a) with Eq. (193), and employing (194a), we see that Eq. (190b) will hold if

$$\mu_{12}\delta(k^2_{12i}-A^2)=\mu_{23}\delta(k^2_{23f}-B^2) \tag{195a}$$

i.e., if

$$\frac{k^2_{12i}}{\mu_{12}}-\frac{A^2}{\mu_{12}}=\frac{k^2_{23f}}{\mu_{23}}-\frac{B^2}{\mu_{23}} \tag{195b}$$

But using Eqs. (178a) - (178b), we see that (195b) reduces to the relation (178c) required by conservation of energy and momentum. Therefore (195b) does hold--and the detailed balancing relations (190) are satisfied--for $\underset{\sim}{k}_i,\underset{\sim}{k}_f$ on the energy-momentum shell, Q.E.D.

5.2.3 Residual Terms

I now return to $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ of (175a); in particular I now shall investigate the detailed balancing properties of the residual terms $\bar{T}^t_{2312}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$, etc., not examined in subsection 5.2.1. If we ignore convergence questions casting doubt on the legitimacy of interchange of order of integration and limit $\varepsilon \rightarrow 0$, then--according to Eqs. (137)--the integral (170a) is

$$\bar{\Psi}^{(-)*}_{23f}(\underset{\sim}{k}_f) V_{23} \bar{\Phi}^{(+)}_{12}(\underset{\sim}{k}_i)$$

$$= -\lim_{\varepsilon \rightarrow 0} \bar{\psi}^*_f(\underset{\sim}{k}_f) \bar{\underset{\sim}{T}}_{23}(\bar{E}+i\varepsilon) \bar{G}_F(\bar{E}+i\varepsilon) \bar{\underset{\sim}{T}}_{12}(\bar{E}+i\varepsilon) \bar{\psi}_i(\underset{\sim}{k}_i) \tag{196a}$$

where $\psi_i(\underset{\sim}{k}_i)$, $\psi_f(\underset{\sim}{k}_f)$ are respectively the initial and final plane wave states ψ_i, ψ_f we have been employing throughout, defined by Eqs. (21a) and (100c). The time-reversed matrix element in $\underset{\sim}{\bar{T}}^s(-\underset{\sim}{k}_f \rightarrow -\underset{\sim}{k}_i)$ corresponding to (196a) would be [referring to (169c) and using the notation of Eqs. (181) - (182)]

$$\bar{\Psi}^{(-)*}_{12f}(-\underset{\sim}{k}_i) V_{12} \bar{\Phi}^{(+)}_{23}(-\underset{\sim}{k}_f)$$

$$= -\lim_{\varepsilon \rightarrow 0} \bar{\psi}^*_f(-\underset{\sim}{k}_i) \bar{\underset{\sim}{T}}_{12}(\bar{E}+i\varepsilon) \bar{G}_F(\bar{E}+i\varepsilon) \bar{\underset{\sim}{T}}_{23}(\bar{E}+i\varepsilon) \psi_i(-\underset{\sim}{k}_f) \tag{196b}$$

via manipulations as in Eqs. (137). But, as in Eqs. (183),

$$\bar{\Psi}_f^{*}(-\underset{\sim}{k}_i) = \bar{\Psi}_i(\underset{\sim}{k}_i)$$
$$\bar{\Psi}_i(-\underset{\sim}{k}_f) = \bar{\Psi}_f^{*}(\underset{\sim}{k}_f) \qquad (197)$$

Moreover, the fundamental definition (77a) implies the three-body $\underset{\sim}{T}_{12}$--like the purely two-body $\underset{\sim}{t}_{12}$ in (194b)--is a symmetric operator in the coordinate representation. Consequently, granting the validity of Eqs. (196),

$$\bar{\Psi}_{23f}^{(-)*}(\underset{\sim}{k}_f)\,V_{23}\,\bar{\Phi}_{12}^{(+)}(\underset{\sim}{k}_i) = \bar{\Psi}_{12f}^{(-)*}(-\underset{\sim}{k}_i)\,V_{12}\,\bar{\Phi}_{23}^{(+)}(-\underset{\sim}{k}_f) \qquad (198)$$

Section E.4 in essence shows that

$$\bar{\Psi}_{23f}^{(-)*}(\underset{\sim}{k}_f)\,V_{23}\,\bar{\Phi}_{12}^{(+)}(\underset{\sim}{k}_i) = \bar{T}^{t}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) + \bar{T}^{a}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \qquad (199a)$$

where the quantities on the right side of (199a) are given by Eqs. (175b) and (176b). Similarly,

$$\bar{\Psi}_{12f}^{(-)*}(-\underset{\sim}{k}_i)\,V_{12}\,\bar{\Phi}_{23}^{(+)}(-\underset{\sim}{k}_f) = \bar{T}^{t}_{1223}(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) + \bar{T}^{a}_{1223}(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) \qquad (199b)$$

Comparison of Eqs. (199a) - (199b), together with Eqs. (190b) and (198), now implies the desired reciprocity relation (180b) which--along with the already proved (180a)--is sufficient to guarantee (179b), as explained at the beginning of section 5.2.

The foregoing demonstration that $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ obeys (179b) is merely suggestive rather than compelling, for the following two reasons. First, the interchange of order of integration and limit $\varepsilon \to 0$ leading to the symmetric expression (196a) for $\bar{\Psi}_{23f}^{(-)*} V_{23} \bar{\Phi}_{12}^{(+)}$ really is not justified, for reasons amply discussed in this and earlier chapters. Second, even if the validity of Eqs. (196) is granted, it is not clear that the specific formulas (175b) and (175c) are consistent with detailed balancing, because these formulas were derived via some mathematically questionable manipulations, e.g., the use of Eqs. (173) to reinterpret the divergent integrals in (172a). What is required, therefore, is a proof that Eqs. (175b) - (175c) as they stand satisfy (180b). This proof I now proceed to give.

Recalling Eqs. (192) - (194), it is seen that Eq. (175c) yields

$$\begin{aligned}
&\bar{T}^t_{1223}(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) \\
&= \langle \underset{\sim}{A} | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle \int d\underset{\sim}{r}_{23} \Big\{ e^{-i\underset{\sim}{r}_{23}\cdot\underset{\sim}{B}} \varphi^{(+)}_{23}(\underset{\sim}{r}_{23}; -\underset{\sim}{k}_{23f}) \\
&\qquad + \frac{i\mu_{23}}{4\pi\hbar^2} \frac{e^{ik_{23f}r_{23}}}{r_{23}} \Big[\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{k}_{23f} \underset{\sim}{\nu}_B \rangle \frac{e^{-iBr_{23}}}{Br_{23}} \\
&\qquad - \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | \underset{\sim}{k}_{23f} \underset{\sim}{\nu}_B \rangle \frac{e^{iBr_{23}}}{Br_{23}} \Big] \Big\} \\
&+ \frac{\mu_{23}}{\hbar^2} \langle \underset{\sim}{A} | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle \left[\frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{k}_{23f} \underset{\sim}{\nu}_B \rangle}{B(k_{23f} - B)} - \frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | \underset{\sim}{k}_{23f} \underset{\sim}{\nu}_B \rangle}{B(k_{23f} + B)} \right]
\end{aligned} \tag{200a}$$

In (200a) it has been convenient to rewrite (175c) and its analogues in a fashion that trivially eliminates the δ-functions under the integral; as in (175c), the integral in (200a) is convergent provided the quantity within the braces is treated as a single $\underset{\sim}{r}_{23}$-dependent function. It is further convenient to rewrite (200a) as

$$\bar{T}^{t}_{1223}(-\underset{\sim}{k}_f \rightarrow -\underset{\sim}{k}_i)$$

$$= \langle \underset{\sim}{A} | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle \Big\{ F_{23}(\underset{\sim}{k}_{23f}; \underset{\sim}{B}; \underset{\sim}{t}_{23f})$$

$$+ \frac{\mu_{23}}{\hbar^2} \left[\frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -k_{23f} \underset{\sim}{\nu}_B \rangle}{B(k_{23f} - B)} - \frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | k_{23f} \underset{\sim}{\nu}_B \rangle}{B(k_{23f} + B)} \right] \Big\} \quad \text{(200b)}$$

where

$$F_{23}(\underset{\sim}{k}_{23f}; \underset{\sim}{B}; \underset{\sim}{t}_{23f}) = \int d\underset{\sim}{r}_{23} \Big\{ e^{-i \underset{\sim}{r}_{23} \cdot \underset{\sim}{B}} \varphi^{(+)}_{23}(\underset{\sim}{r}_{23}; -\underset{\sim}{k}_{23f})$$

$$+ \frac{i\mu_{23}}{4\pi\hbar^2} \frac{e^{ik_{23f} r_{23}}}{B r_{23}^2} \left[\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -k_{23f} \underset{\sim}{\nu}_B \rangle e^{-iBr_{23}} - \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | k_{23f} \underset{\sim}{\nu}_B \rangle e^{iBr_{23}} \right] \Big\}$$

(200c)

Rewriting (175b) in the same way, we obtain

$$\bar{T}^{b}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$$

$$= \langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} \rangle \left\{ F_{12}(\underset{\sim}{k}_{12i}; \underset{\sim}{A}; \underset{\sim}{t}_{12i}) \right. \tag{201a}$$

$$\left. + \frac{\mu_{12}}{\hbar^2} \left[\frac{\langle k_{12i} \underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle}{A(k_{12i} - A)} - \frac{\langle -k_{12i} \underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle}{A(k_{12i} + A)} \right] \right\}$$

where

$$F_{12}(\underset{\sim}{k}_{12i}; \underset{\sim}{A}; \underset{\sim}{t}_{12i}) = \int d\underset{\sim}{r}_{12} \left\{ e^{-i \underset{\sim}{r}_{12} \cdot \underset{\sim}{A}} \varphi^{(+)}_{12}(\underset{\sim}{r}_{12}; \underset{\sim}{k}_{12i}) \right.$$

$$\left. + \frac{i\mu_{12}}{4\pi\hbar^2} \frac{e^{ik_{12i} r_{12}}}{A r_{12}^2} \left[\langle k_{12i} \underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle e^{-iAr_{12}} - \langle -k_{12i} \underset{\sim}{\nu}_A | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle e^{iAr_{12}} \right] \right\} \tag{201b}$$

With the aid of (131f) and the two-particle analogue of (63a), Eq. (73b) takes the form

$$\varphi_{12}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{k}_{12i}) = -\int d\underset{\sim}{r}'_{12}\, g_{12F}^{(+)}(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};E_{12i})\int d\underset{\sim}{r}''_{12}\, \underset{\sim}{t}_{12}(\underset{\sim}{r}'_{12};\underset{\sim}{r}''_{12};E_{12i})\, e^{i\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}''_{12}} \tag{202a}$$

$$= -\lim_{\varepsilon\to 0}\int d\underset{\sim}{r}'_{12}\, d\underset{\sim}{r}''_{12}\, g_{12F}(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};\lambda)\, \underset{\sim}{t}_{12}(\underset{\sim}{r}'_{12};\underset{\sim}{r}''_{12};\lambda)\, e^{i\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}''_{12}} \tag{202b}$$

$$\equiv -\lim_{\varepsilon\to 0}\, g_{12F}(\lambda)\,\underset{\sim}{t}_{12}(\lambda)\,\psi_{12i}(\underset{\sim}{k}_{12i}) \tag{202c}$$

where Ψ_{12i} is defined by (74a), and

$$\lambda = E_{12i} + i\varepsilon = \frac{\hbar^2 k_{12i}^2}{2\mu_{12}} + i\varepsilon \tag{202d}$$

Then from (201b) we see

$$F_{12}(\underset{\sim}{k}_{12i};\underset{\sim}{A};\underset{\sim}{t}_{12i})$$

$$= \int d\underset{\sim}{r}_{12} \lim_{\varepsilon\to 0} \Bigg\{ -e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}} g_{12F}(\lambda)\underset{\sim}{t}_{12}(\lambda)\psi_{12i}(\underset{\sim}{k}_{12i})$$

$$+\frac{i\mu_{12}}{4\pi\hbar^2}\frac{e^{i\left(\frac{2\mu_{12}\lambda}{\hbar^2}\right)^{1/2}r_{12}}}{A r_{12}^2}\Big[\langle k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle e^{-iAr_{12}}$$

$$-\langle -k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle e^{iAr_{12}}\Big]\Bigg\}$$

(203a)

But, by our usual rule (section A.8), interchange of order of integration and limit $\varepsilon \to 0$ is permissible in (203a), because the integral (201b) is convergent [except at $A = \pm k_{12i}$]. Hence

$$F_{12}(\underset{\sim}{k}_{12i};\underset{\sim}{A};\underset{\sim}{t}_{12i})$$

$$= \lim_{\varepsilon\to 0}\int d\underset{\sim}{r}_{12} \Bigg\{ -e^{-i\underset{\sim}{r}_{12}\cdot\underset{\sim}{A}} g_{12F}(\lambda)\underset{\sim}{t}_{12}(\lambda)\psi_{12i}(\underset{\sim}{k}_{12i})$$

$$+\frac{i\mu_{12}}{4\pi\hbar^2}\frac{e^{i\left(\frac{2\mu_{12}\lambda}{\hbar^2}\right)^{1/2}r_{12}}}{A r_{12}^2}\Big[\langle k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle e^{-iAr_{12}}$$

$$-\langle -k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle e^{iAr_{12}}\Big]\Bigg\} \qquad (203b)$$

where now each of the terms inside the braces in (203b) are individually convergent integrals.

Using the expansion

$$g_{12F}(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};\lambda) = \frac{1}{(2\pi)^3}\int d\hat{\underset{\sim}{k}}_{12}\,\frac{e^{i\hat{\underset{\sim}{k}}_{12}\cdot(\underset{\sim}{r}_{12}-\underset{\sim}{r}'_{12})}}{\frac{\hbar^2\hat{k}_{12}^2}{2\mu_{12}}-\lambda} \tag{204}$$

the integrals (203b) yield

$$F_{12}(\underset{\sim}{k}_{12i};\underset{\sim}{A};\underset{\sim}{t}_{12i}) = \lim_{\varepsilon\to 0}\frac{2\mu_{12}}{\hbar^2}\left\{-\frac{\langle\underset{\sim}{A}|\underset{\sim}{t}_{12}(\lambda)|\underset{\sim}{k}_{12i}\rangle}{A^2-\frac{2\mu_{12}\lambda}{\hbar^2}}\right.$$

$$\left.+\frac{1}{2}\frac{\langle k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle}{A\left[A-\left(\frac{2\mu_{12}}{\hbar^2}\lambda\right)^{1/2}\right]}+\frac{1}{2}\frac{\langle -k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle}{A\left[A+\left(\frac{2\mu_{12}\lambda}{\hbar^2}\right)^{1/2}\right]}\right\} \tag{205a}$$

Hence, evaluating the limit $\varepsilon \to 0$ in (204a), we obtain finally

$$F_{12}(\underset{\sim}{k}_{12i}; A\underset{\sim}{\nu}_A; \underset{\sim}{t}_{12i}) = \frac{2\mu_{12}}{\hbar^2}\left\{ -\frac{\langle A\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle}{A^2 - k_{12i}^2} \right.$$

$$\left. + \frac{1}{2}\frac{\langle k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle}{A(A - k_{12i})} + \frac{1}{2}\frac{\langle -k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle}{A(A + k_{12i})} \right\} \tag{205b}$$

The result (205b) is well-defined and finite at all values of $\underset{\sim}{A}$, $\underset{\sim}{k}_{12i}$ such that $A \neq \pm k_{12i}$. In fact, (205b) is finite even at $A = \pm k_{12i}$-- where the integral (201b) [from which we deduced (205b)] diverges-- provided it is understood that the values of F_{12} at $A = \pm k_{12i}$ are given by the limits of (205b) as $A \to \pm k_{12i}$, namely

$$F_{12}(\underset{\sim}{k}_{12i}; k_{12i}\underset{\sim}{\nu}_A; \underset{\sim}{t}_{12i}) \equiv \lim_{A\to k_{12i}} F_{12} = \frac{\mu_{12}}{2\hbar^2 k_{12i}^2}\langle -k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle \tag{205c}$$

$$F_{12}(\underset{\sim}{k}_{12i}; -k_{12i}\underset{\sim}{\nu}_A; \underset{\sim}{t}_{12i}) \equiv \lim_{A\to -k_{12i}} F_{12} = \frac{\mu_{12}}{2\hbar^2 k_{12i}^2}\langle k_{12i}\underset{\sim}{\nu}_A|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle \tag{205d}$$

Eqs. (205c) - (205d) are consistent with each other, in the sense that changing $\underset{\sim}{\nu}_A$ to $-\underset{\sim}{\nu}_A$ in (205c) gives (205d).

Combining Eqs. (205b) and (201) leads to

$$\bar{T}^t_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$$
$$= \frac{-2\mu_{12}}{\hbar^2} \frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -B \rangle \langle \underset{\sim}{A} | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle}{A^2 - k_{12i}^2} \tag{206}$$

valid at $A^2 \neq k_{12i}^2$. Eq. (206), which has been deduced from (175b), obviously is a generally more convenient and more readily interpretable formula for $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ than is (175b) itself. Similarly, Eq. (200c) leads to

$$F_{23}(\underset{\sim}{k}_{23f}; B\underset{\sim}{\nu}_B; \underset{\sim}{t}_{23f}) = \frac{2\mu_{23}}{\hbar^2}\left\{ -\frac{\langle B\underset{\sim}{\nu}_B | \underset{\sim}{t}_{23f} | -\underset{\sim}{k}_{23f} \rangle}{B^2 - k_{23f}^2} \right.$$
$$\left. + \frac{1}{2}\frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -k_{23f}\underset{\sim}{\nu}_B \rangle}{B(B - k_{23f})} + \frac{1}{2}\frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | k_{23f}\underset{\sim}{\nu}_B \rangle}{B(B + k_{23f})} \right\} \tag{207}$$

which--when combined with Eqs. (200)--yields

$$\bar{T}^t_{1223}(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i)$$
$$= -\frac{2\mu_{23}}{\hbar^2} \frac{\langle \underset{\sim}{A} | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle \langle \underset{\sim}{B} | \underset{\sim}{t}_{23f} | -\underset{\sim}{k}_{23f} \rangle}{B^2 - k_{23f}^2} \tag{208}$$

valid at $B^2 \neq k_{23f}^2$.

Using Eqs. (194a) and (195b), we now see that Eqs. (206) and (208) indeed are consistent with the detailed balancing relation (180b), Q.E.D. I further note that according to Eq. (173b) [or, better, Eq. (E49b) in section E.4], the term involving $(k_{12i} - A)^{-1}$ in (175b) should be dropped at $A = k_{12i}$; correspondingly, the term in (175b) involving $(k_{12i} + A)^{-1}$ should be dropped when $A = - k_{12i}$. If these strictures are included in Eq. (201a), and then combined with Eqs. (205c) - (205d), we see that Eq. (206) should be supplemented by

$$\bar{\bar{T}}^{t}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = 0 \tag{209a}$$

at $A^2 = k_{12i}{}^2$. Similarly Eq. (208) is supplemented by

$$\bar{\bar{T}}^{t}_{1223}(-\underset{\sim}{k}_f \to -\underset{\sim}{k}_i) = 0 \tag{209b}$$

at $B^2 = k_{23f}{}^2$, again consistent with detailed balancing. For completeness, I also note that Eq. (175c) reduces to

$$\bar{\bar{T}}^{t}_{2331}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$$

$$= -\frac{2\mu_{31}}{\hbar^2} \frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | \underset{\sim}{D} \rangle \langle -\underset{\sim}{C} | \underset{\sim}{t}_{31i} | \underset{\sim}{k}_{31i} \rangle}{C^2 - k^2_{31i}} \tag{210a}$$

at $C^2 \neq k_{31i}{}^2$, supplemented by

$$\overline{T}^{t}_{2331}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = 0 \tag{210b}$$

at $C^2 = k_{31i}{}^2$.

I close this subsection with two remarks. First--because our conclusion that the detailed balancing relation (180b) is satisfied rests so heavily on the result (205b)--in Appendix A.10 I deduce Eq. (205b) from Eq. (201b) by a method which avoids the [made to seem reasonable, but not really proved in section A.8] interchange (203a) - (203b) of order of integration and limit $\varepsilon \to 0$; this alternative derivation confirms the conclusions of the present subsection and provides further evidence that our claims and arguments in section A.8 really are correct. Second, it readily can be verified that Eqs. (175b) - (175c) would not be consistent with (180b) if the contributions (173a) to the singular integrals in (172a) had been omitted.

5.3 Momentum Space Procedures

The iterations which have been employed in this work on numerous occasions--to obtain, e.g., Eqs. (64) or (162)--clearly are independent of representation, i.e., equally well could have been performed in momentum space. Moreover, relations such as Eqs. (63), (81) and the three-particle analogue of (131h) ultimately make it possible to express all iterations of the scattered wave $\Phi_i^{(+)}$--or of various contributions to $\Phi_i^{(+)}$ such as $\Phi_{\alpha\beta}^{s(+)}$ of Eqs. (161)--in terms of expressions beginning with $G_F^{(+)}$ [that is to say, expressions whose leftmost factor is $G_F^{(+)}$] and whose rightmost factor is ψ_i. In addition, as Eq. (96) argues and Eqs. (90) make explicit, it is the case that the limit of $G_F^{(+)}(\underset{\sim}{r};\underset{\sim}{r}';E)$ as $\underset{\sim}{r} \to \infty$ is proportional to $\psi_f^*(\underset{\sim}{r}')$ defined by (100c). Therefore, granting that the variety of possible iterations must lead to self-consistent results provided questions concerning convergence and interchanging orders of integration can be ignored, it really is not suprising that the transition amplitude matrix elements obtained from our configuration space approach agree formally with the corresponding matrix elements in the more customary momentum space procedures.

Thus, for example, it is no surprise that Eqs. (187) take the form they do. In the discussion of Eqs. (165) - (166) we have argued that $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ defined by Eq. (165b) represents the contribution to $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ resulting from scattering processes involving three or more successive purely two-body collisions. Hence Eq. (187b) merely states that $< f|\underset{\sim}{\bar{T}}|i >$ consists of $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ plus the contributions $< f|\underset{\sim}{\bar{T}}_{\alpha\beta}|i >$ from individual purely two-body collisions, plus the contributions $- < f|\underset{\sim}{\bar{T}}_{\alpha\beta}\bar{G}_F\underset{\sim}{T}_{\gamma\delta}|i >$ from all possible pairs of successive purely two-body collisions; that the minus signs in (187b) preceding the double scattering matrix elements are consistent with this interpretation follows from

Eqs. (133b) and (137). In the text I have given a long-winded derivation of (187a) [which except for its explicit inclusion of the limit $\varepsilon \to 0$ is the same as (187b)] only because I have insisted: (i) on starting from an expression (165b) for $\bar{\bar{T}}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ composed solely of convergent integrals, and (ii) on employing no mathematically illegitimate manipulations in going from (165b) to (187a). If I am willing to employ mathematically questionable operations, Eq. (187b) can be derived more readily than was (187a). Specifically, start from Eq. (131d), which with the aid of Eqs. (63) can be rewritten in the form [once again simplifying the notation by dropping the bars]

$$\begin{aligned}\psi_f^* \underset{\sim}{T} \psi_i &= \psi_f^*(V - VGV)\psi_i \\ &= \psi_f^*\Big\{V - V\left[G_{12} - G(V_{23}+V_{31})G_{12}\right]V_{12} \\ &\quad - V\left[G_{23} - G(V_{31}+V_{12})G_{23}\right]V_{23} \\ &\quad - V\left[G_{31} - G(V_{12}+V_{23})G_{31}\right]V_{31}\Big\}\psi_i\end{aligned} \tag{211a}$$

$$\begin{aligned} &= \psi_f^*\Big\{V - V_{12}G_{12}V_{12} - (V_{23}+V_{31})G_{12}V_{12} \\ &\quad + VG(V_{23}+V_{31})G_{12}V_{12} - V_{23}G_{23}V_{23} \\ &\quad - (V_{31}+V_{12})G_{23}V_{23} + VG(V_{31}+V_{12})G_{23}V_{23} \\ &\quad - V_{31}G_{31}V_{31} - (V_{12}+V_{23})G_{31}V_{31} \\ &\quad + VG(V_{12}+V_{23})G_{31}V_{31}\Big\}\psi_i\end{aligned} \tag{211b}$$

Next, employ Eqs. (60) and (77a), which reduce (211b) to

$$\begin{aligned}\psi_f^* \underset{\sim}{T} \psi_i &= \psi_f^* \left(\underset{\sim}{T}_{12} + \underset{\sim}{T}_{23} + \underset{\sim}{T}_{31}\right)\psi_i \\ &- \psi_f^* (-1 + VG)(V_{23} + V_{31})\, \Phi_{12}^{(+)} \\ &- \psi_f^* (-1 + VG)(V_{31} + V_{12})\, \Phi_{23}^{(+)} \\ &- \psi_f^* (-1 + VG)(V_{12} + V_{23})\, \Phi_{31}^{(+)}\end{aligned} \tag{211c}$$

and then use (100b), which converts (211c) to

$$\begin{aligned}\psi_f^* \underset{\sim}{T} \psi_i &= \psi_f^* \left(\underset{\sim}{T}_{12} + \underset{\sim}{T}_{23} + \underset{\sim}{T}_{31}\right)\psi_i \\ &+ \Psi_f^{(-)*} \Big[(V_{23} + V_{31})\, \Phi_{12}^{(+)} + (V_{31} + V_{12})\, \Phi_{23}^{(+)} \\ &\qquad + (V_{12} + V_{23})\, \Phi_{31}^{(+)} \Big]\end{aligned} \tag{211d}$$

The $\Psi_f^{(-)*}$ terms in (211d) are precisely $T^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ of (133b); Eqs. (137) and (169c) have shown

$$\begin{aligned}&T^s(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) \\ &= T^d(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) - \psi_f^* \Big[\underset{\sim}{T}_{23} G_F \underset{\sim}{T}_{12} + \underset{\sim}{T}_{23} G_F \underset{\sim}{T}_{31} + \underset{\sim}{T}_{31} G_F \underset{\sim}{T}_{23} \\ &\qquad + \underset{\sim}{T}_{31} G_F \underset{\sim}{T}_{12} + \underset{\sim}{T}_{12} G_F \underset{\sim}{T}_{31} + \underset{\sim}{T}_{12} G_F \underset{\sim}{T}_{23} \Big] \psi_i\end{aligned} \tag{211e}$$

Eqs. (211d) - (211e) yield Eq. (187b).

5.3.1 Diagrammatic Techniques

The above derivation of (187b)—as well as the derivation of (187a) in subsection 5.2.1—can be paralleled step by step in the momentum representation. Alternatively, one can obtain a quite direct and simple demonstration of (187b) via diagrammatic techniques[4,5,8]. Although it really isn't necessary to do so--we already have two independent derivations of Eq. (187b)--for completeness sake I shall give this diagrammatic derivation. It is convenient first to introduce as propagators the negatives of the Green's functions we have been using; in this section these negatives will be denoted by the carat, i.e., $\hat{G} = - G$, $\hat{G}_F = - G_F$, $\hat{g}_{12} = - g_{12}$, etc. Then Eq. (131f) becomes

$$\underset{\sim}{t}_{12} = \underset{\sim}{V}_{12} + V_{12}\hat{g}_{12}V_{12} \tag{212a}$$

Correspondingly, the two-particle analogue of Eqs. (81) is

$$\hat{g}_{12} = \hat{g}_F + \hat{g}_F V_{12}\hat{g}_{12} \tag{212b}$$

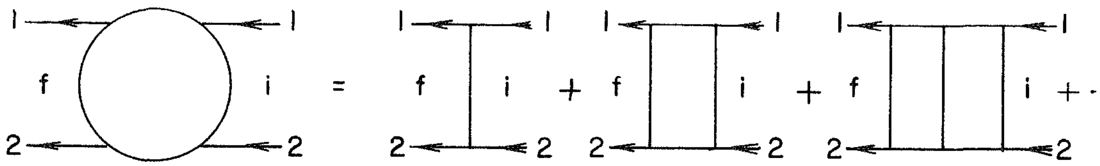
Using (212b) to iterate (212a) yields

$$\underset{\sim}{t}_{12} = \underset{\sim}{V}_{12} + V_{12}\hat{g}_F V_{12} + V_{12}\hat{g}_F V_{12}\hat{g}_F V_{12} + \cdots \tag{212c}$$

wherein all terms are of positive sign [the reason for introducing these carated propagators in place of our former g_F, g_{12}]. Taking matrix elements of (212c),

$$\bar{\Psi}^*_{12f}\underset{\sim}{t}_{12}\bar{\Psi}_{12i} = \bar{\Psi}^*_{12f}V_{12}\bar{\Psi}_{12i} + \bar{\Psi}^*_{12f}V_{12}\hat{g}_F V_{12}\bar{\Psi}_{12i} + \bar{\Psi}^*_{12f}V_{12}\hat{g}_F V_{12}\hat{g}_F\bar{\Psi}_{12i} + \cdots \tag{212d}$$

Eq. (212d) can be represented diagrammatically by



(213)

where the rules for constructing the matrix element counterpart to any individual diagram on the right side of (213) are obvious; it only is necessary to remember that between any pair of successive vertical lines connecting 1 and 2 the particles propagate freely, i.e., in each matrix element the free particle propagator $\hat{g}_F$ separates successive interactions V_{12}. Placing the initial state on the right and the final state on the left, as in the matrix elements (212d) themselves, minimizes possible confusion in interpreting the diagrams; in other words, we suppose the system evolves from right to left as indicated by the arrows. The bubble diagram on the left denotes the sum of the diagrams on the right; equivalently, the bubble diagram denotes the matrix element of $\underset{\sim}{t}_{12}(\underset{\sim}{r}_{12};\underset{\sim}{r}_{12}')$ itself.

Similarly, in the three-particle system [again dropping the bars] where

$$\underset{\sim}{T} = \underset{\sim}{V} + V\hat{G}V \tag{214a}$$

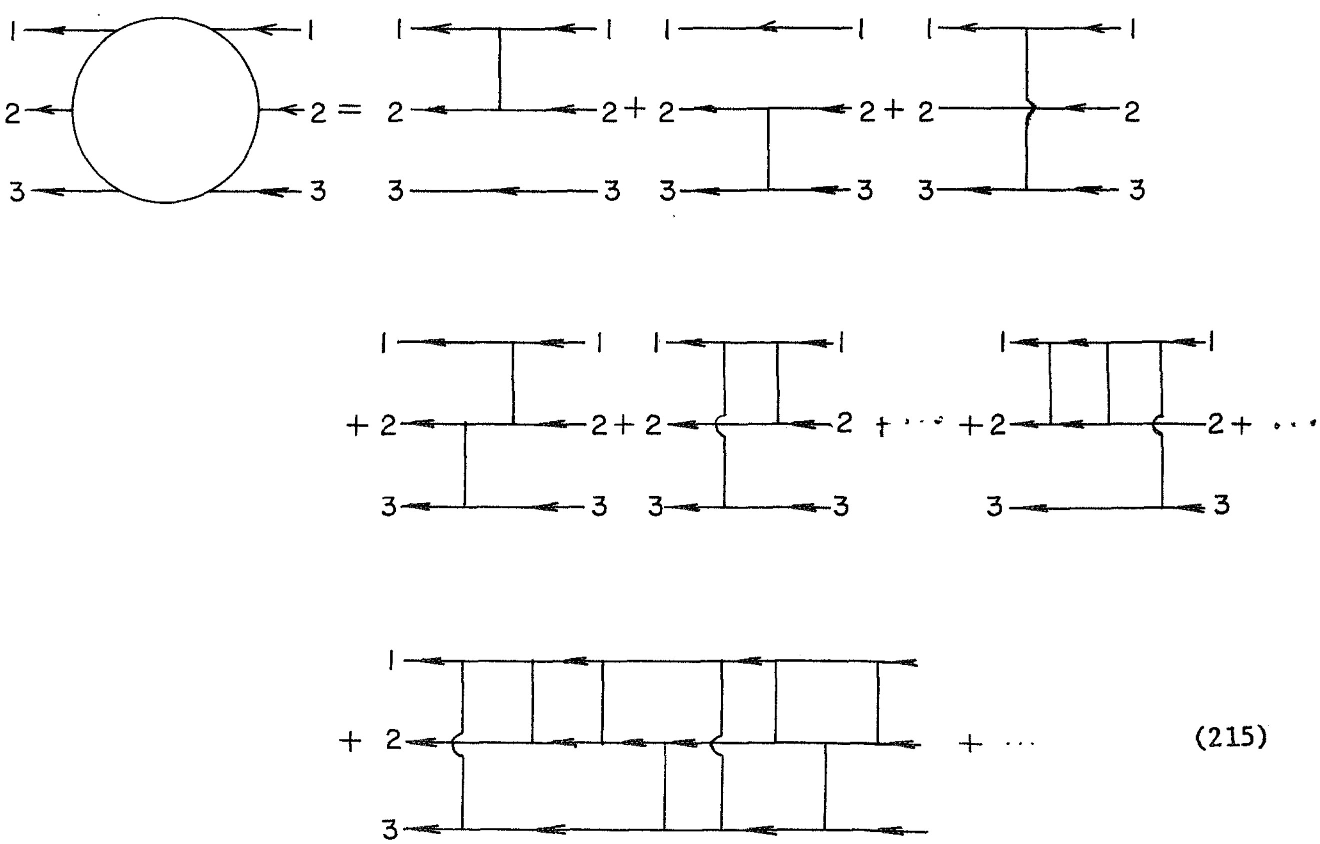
using

$$\hat{G} = \hat{G}_F + \hat{G}_F V \hat{G} \tag{214b}$$

obviously leads to

$$\underset{\sim}{T} = \underset{\sim}{V}_{12} + \underset{\sim}{V}_{23} + \underset{\sim}{V}_{31} + V_{23}\hat{G}_F V_{12} + V_{31}\hat{G}_F V_{12} + \cdots$$
$$+ V_{12}\hat{G}_F V_{12}\hat{G}_F V_{31} + \cdots \tag{214c}$$
$$+ V_{31}\hat{G}_F V_{12}\hat{G}_F V_{12}\hat{G}_F V_{23}\hat{G}_F V_{31}\hat{G}_F V_{12}\hat{G}_F V_{23}\hat{G}_F V_{12} + \cdots$$

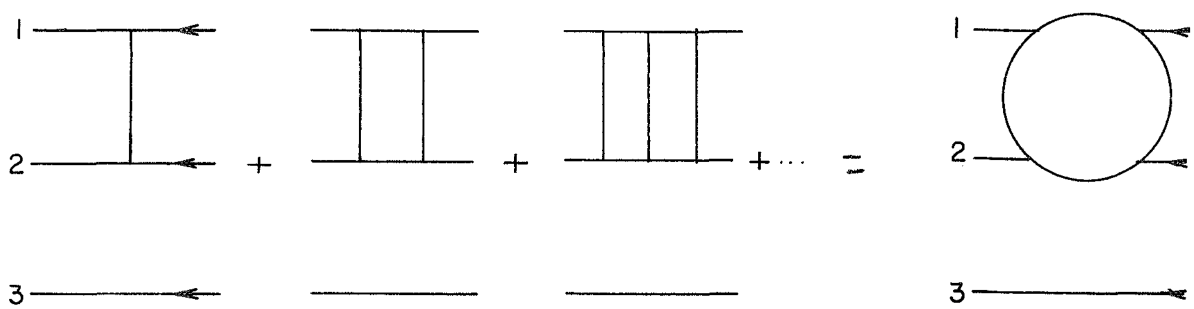
That is to say, every possible sequence (with repeats) of the three interactions V_{12}, V_{23}, V_{31} occurs in the iteration of $\underset{\sim}{T}$. Correspondingly we have the diagrammatic representation



(215)

where the super-bubble in (215) denotes $< f|\underset{\sim}{T}|i >$, and where on the right side of (215) there occurs every possible diagram constructed from successive vertical lines V_{12}, V_{23} or V_{31} connecting pairs of the horizontal lines 1, 2, 3. Actually (215) has been drawn so that the individual diagrams therein are the counterparts of the particular terms included in (214c).

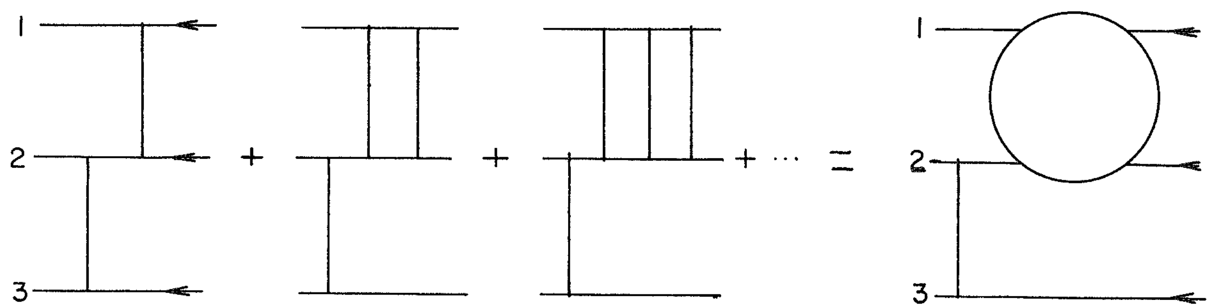
Now consider the first diagram on the right side of (215), representing the matrix element $< f|V_{12}|i >$. Evidently this diagram is the first in a whole sequence of diagrams on the right side of (215), each of which is composed solely of interactions V_{12}. In other words, on the right side of (215) I can single out the sum of diagrams



(216a)

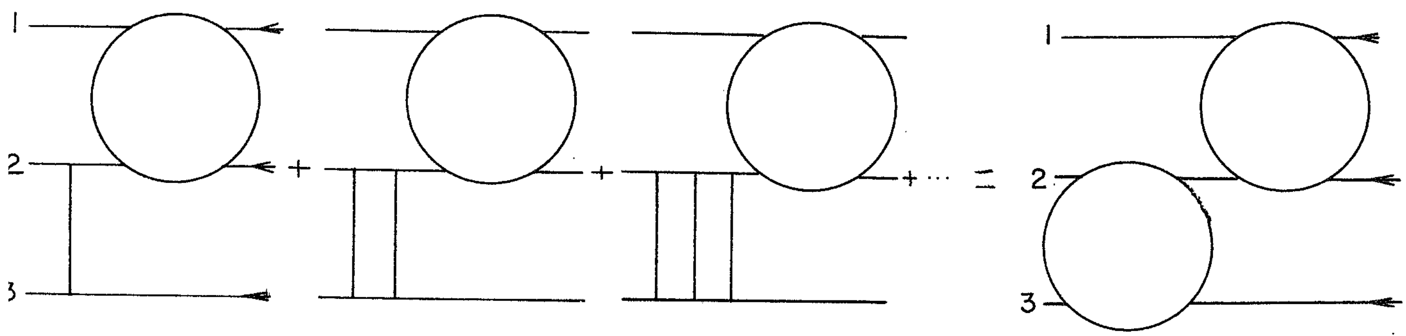
where the bubble on the right side of (216a) now denotes the matrix element $\psi_f^* \underset{\sim}{T}_{12}\psi_i$. This bubble must be identified with the matrix element of the three-particle $\underset{\sim}{T}_{12}$ of Eq. (77a)--rather than with the two-particle $\underset{\sim}{t}_{12}$ of Eqs. (131f) and the bubble in (213)--because in (214a) $\underset{\sim}{V}_{12}(\underset{\sim}{r};\underset{\sim}{r}')$ is proportional to the three-particle $\delta(\underset{\sim}{r} - \underset{\sim}{r}')$ of Eq. (27e), rather than merely to the two-particle $\delta(\underset{\sim}{r}_{12} - \underset{\sim}{r}_{12}')$ of Eq. (131f); correspondingly, the matrix element counterparts of the diagrams in (216a) involve the three-particle free space propagator $\hat{G}_F = -G_F$ we have used throughout, and are taken between the three-particle plane wave states ψ_f, ψ_i of Eqs. (100c), (21a) respectively.

Next consider, e.g., the fourth diagram on the right side of (215). Then on the right side of (215) I first can single out the sum of diagrams



(216b)

followed by the sum of sums



(216c)

The diagram on the right side of (216c) obviously denotes a $\psi_f^* \underset{\sim}{T}_{23} \hat{G}_F \underset{\sim}{T}_{12} \psi_i$ contribution to the overall sum on the right side of (216a). Similarly, starting with the last diagram (I'll call it D) on the right side of (215), I first sum those diagrams which repeat the rightmost interaction V_{12}, obtaining a sum represented by a diagram identical with D, except that $\underset{\sim}{T}_{12}$ replaces V_{12} on the right. Next I sum the sums in which $\underset{\sim}{T}_{12}$ has replaced V_{12} on the right side of D, but in which the interaction V_{23} immediately to the left of $\underset{\sim}{T}_{12}$ is repeated. Evidently, proceeding in this fashion, I single out in (215) a collection of diagrams associated with D whose sum represents the matrix element

$$\psi_f^* \underset{\sim}{T}_{31} \hat{G}_F \underset{\sim}{T}_{12} \hat{G}_F \underset{\sim}{T}_{12} \hat{G}_F \underset{\sim}{T}_{23} \hat{G}_F \underset{\sim}{T}_{31} \hat{G}_F \underset{\sim}{T}_{12} \hat{G}_F \underset{\sim}{T}_{23} \hat{G}_F \underset{\sim}{T}_{12} \psi_i$$

It is now apparent that (215) yields

$$\begin{aligned}\psi_f^* \underset{\sim}{T} \psi_i = \psi_f^* \Big\{ & \underset{\sim}{T}_{12} + \underset{\sim}{T}_{23} + \underset{\sim}{T}_{31} + \underset{\sim}{T}_{23} \hat{G}_F \underset{\sim}{T}_{12} \\ & + \underset{\sim}{T}_{23} \hat{G}_F \underset{\sim}{T}_{31} + \underset{\sim}{T}_{31} \hat{G}_F \underset{\sim}{T}_{23} + \underset{\sim}{T}_{31} \hat{G}_F \underset{\sim}{T}_{12} \\ & + \underset{\sim}{T}_{12} \hat{G}_F \underset{\sim}{T}_{31} + \underset{\sim}{T}_{12} \hat{G}_F \underset{\sim}{T}_{23} \Big\} \psi_i + T^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)\end{aligned} \tag{217a}$$

where $\underset{\sim}{T}^d$ can be thought to consist of all matrix elements corresponding to $n \geq 3$ successive two-body scatterings, i.e.,

$$\begin{aligned}T^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \psi_f^* \Big\{ & \underset{\sim}{T}_{23} \hat{G}_F \underset{\sim}{T}_{12} \hat{G}_F \underset{\sim}{T}_{31} \\ & + \underset{\sim}{T}_{23} \hat{G}_F \underset{\sim}{T}_{12} \hat{G}_F \underset{\sim}{T}_{23} + \cdots + \underset{\sim}{T}_{23} \hat{G}_F \underset{\sim}{T}_{12} \hat{G}_F \underset{\sim}{T}_{31} \hat{G}_F \underset{\sim}{T}_{23} + \cdots \Big\} \psi_i\end{aligned} \tag{217b}$$

Replacing $\hat{G}_F$ in Eqs. (217) by $-G_F$, and recalling Eqs. (165b) - (166b), we see that Eq. (217a), signs and all, is identical with (the laboratory frame version of) Eq (187b).

5.3.2 Single and Double Scattering Diagrams

Thus far this section 5.3 has made it clear that our configuration space expressions for $< f|\bar{T}|i > \equiv \bar{T}(k_i \to k_f)$ agree formally with expressions for $< f|T|i >$ derived via momentum space procedures, and that they should be expected to manifest such agreement. On the other hand, this assertion--important though it is--does not of itself imply that reaction rates computed using our configuration space expressions necessarily will agree with the results of reaction rate computations using momentum space expressions. In the first place, the whole possibility of demonstrating a correspondence between configuration space and momentum space formulations depends on being able to interchange order of integration and limit $\bar{r} \to \infty$ in various integral expressions for $\bar{\Phi}_i^{(+)}$ or parts of $\bar{\Phi}_i^{(+)}$, as discussed in chapter 4; without this interchange, the configuration space results for probability current flow cannot be expressed in terms of matrix elements [such as $\bar{\psi}_f^* V\bar{\Psi}_i^{(+)}$, $\bar{\Psi}_f^{(-)*} V_{23}\bar{\Phi}_{12}^{s(+)}$, etc.] ultimately identifiable with momentum space matrix elements composing all or part of $< f|T|i >$. Moreover, the aforementioned formal agreement between the configuration space and momentum space expressions for $< f|T|i >$ has been established without regard to the possible influences of manipulations such as: (i) interchange of orders of integration, (ii) interchange of order of integration and limit $\varepsilon \to 0$ in $< f|\bar{T}(E + i\varepsilon)|i >$, and (iii) Fourier transformation, i.e., transformation from the coordinate to momentum representations. Such manipulations, if not legitimate, can produce differences in the numerical values of matrix elements which are formally identical, and strict proofs of legitimacy are hard to come by; in fact, it already has been pointed out--in connection with Eqs. (51c), (51d) and (131d)--that the relations

$$\bar{\psi}_f^* V \bar{\Psi}_i^{(+)} = \bar{\psi}_f^* \bar{\underset{\sim}{T}}(\bar{E})\bar{\psi}_i = \lim_{\varepsilon \to 0} \bar{\psi}_f^* \bar{\underset{\sim}{T}}(E+i\varepsilon)\bar{\psi}_i \tag{218}$$

need not hold. However, as so often argued in this work, it is reasonable to assume that manipulations which at no step involve divergences indeed are justified.

Let me now assess the significance of Eq. (187b) in the light of the above remarks. The discussion of Eq. (165a) has explained that the interchange of order of integration and limit $\bar{\underset{\sim}{r}} \to \infty$ yielding (165b) is justified, except possibly along an inconsequential set of special $\bar{\underset{\sim}{\nu}}_f$. Correspondingly, $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is composed of convergent integrals, except possibly along an inconsequential set of special $\underset{\sim}{k}_f$; moreover, because the integrals in (165b) are convergent, manipulations such as those in Eq. (166b) are legitimate. It follows that the quantity $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$--here defined as the contribution to $\bar{T}^t(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ or $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ associated with $n \geq 3$ successive two-body scatterings--should yield the same values whether computed in momentum space or in configuration space [except possible along an inconsequential set of special $\underset{\sim}{k}_f$]. As it happens it is the carefully proved Eq. (187a)--rather than (187b)--which provides the mathematical statement of the immediately preceding assertion concerning $\bar{T}^d(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$, because in momentum space the scattering matrix elements comprised in $< f|\bar{\underset{\sim}{T}}(\bar{E})|i >$ customarily are computed from the limit $\varepsilon \to 0$ of the corresponding matrix elements in $< f|\bar{\underset{\sim}{T}}(\bar{E} + i\varepsilon)|i >$. To put it differently, in the momentum space formalism the contributions to the total scattering amplitude $\bar{T}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ made by e.g., the diagrams on the right sides of Eqs. (216a), (216c) are respectively

$$\bar{T}_{12}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \lim_{\varepsilon \rightarrow 0} \langle f | \bar{\underset{\sim}{T}}_{12}(\bar{E}+i\varepsilon) | i \rangle \tag{219a}$$

$$\bar{T}_{2312}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f) = \lim_{\varepsilon \rightarrow 0} -\langle f | \bar{\underset{\sim}{T}}_{23}(\bar{E}+i\varepsilon)\bar{G}_F(\bar{E}+i\varepsilon)\bar{\underset{\sim}{T}}_{12}(\bar{E}+i\varepsilon) | i \rangle \tag{219b}$$

where it is understood of course that for physically observable amplitudes $\bar{E} = \bar{E}_i = \bar{E}_f$. The relations (219) define the momentum space scattering amplitudes whether or not it is true that the corresponding relations

$$\lim_{\varepsilon \rightarrow 0} \langle f | \bar{\underset{\sim}{T}}_{12}(\bar{E}+i\varepsilon) | i \rangle = \langle f | \bar{\underset{\sim}{T}}_{12}(\bar{E}) | i \rangle \tag{220a}$$

and

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} -\langle f | \bar{\underset{\sim}{T}}_{23}(\bar{E}+i\varepsilon)\bar{G}_F(\bar{E}+i\varepsilon)\bar{\underset{\sim}{T}}_{12}(\bar{E}+i\varepsilon) | i \rangle \\ &= -\langle f | \bar{\underset{\sim}{T}}_{23}(\bar{E})\bar{G}_F^{(+)}(\bar{E})\bar{\underset{\sim}{T}}_{12}(\bar{E}) | i \rangle \end{aligned} \tag{220b}$$

[involving interchange of order of integration and limit $\varepsilon \rightarrow 0$] hold when these matrix elements are computed in the momentum representation.

In view of the foregoing, differences between configuration space and momentum space reaction rate predictions can stem only from the behavior of matrix elements representing single or double scattering, i.e., from matrix elements of the types written down in Eqs. (219) and (220); in the remainder of this section, therefore, we confine our attention to single and double scattering contributions to the scattering amplitude. For either of these types of scattering processes it was less obvious initially than in the case of $n \geq 3$ scattering processes that there would be a close agreement between configuration space and momentum space results, because for integral expressions representing those parts of $\Phi_i^{(+)}$ associated with $n = 1$ and $n = 2$ scattering processes interchange of order of integration and limit $\underset{\sim}{r} \to \infty$ is not justified [recall chapter 4]; correspondingly, when for these $n = 1$ and $n = 2$ processes this unjustified interchange of order of integration and limit $\underset{\sim}{r} \to \infty$ was performed, the configuration space matrix elements obtained were divergent, implying that the [seemingly required for agreement between configuration space and momentum space predictions] manipulations (i) - (iii) listed in the opening paragraph of this subsection would have dubious validity. Nevertheless the momentum space and configuration space results for single and double scattering processes gratifyingly turn out to be essentially identical, as is detailed below.

Consider first the typical single scattering process represented by the diagram (216a), whose contribution to the scattering amplitude is computed in momentum space via Eq. (219a). By definition [recalling also Eqs. (33b), (40b) and (77a)]

$$\langle f|\bar{T}_{12}(\bar{E}_i+i\epsilon)|i\rangle$$

$$=\int d\bar{r}\,d\bar{r}'\,e^{-i(k_{12f}\cdot r_{12}+K_{12f}\cdot q_{12})}\,\bar{T}_{12}(\bar{r};\bar{r}';\bar{E}_i+i\epsilon)\,e^{i(k_{12i}\cdot r_{12}'+K_{12i}\cdot q_{12}')}$$

$$=\int d\bar{r}\,d\bar{r}'\,\bar{\Psi}_f^*(\bar{r};k_f)\left[V_{12}(r_{12})\delta(r_{12}-r_{12}')\delta(q_{12}-q_{12}')-V_{12}(r_{12})\bar{G}_{12}(\bar{r};\bar{r}';\bar{E}_i+i\epsilon)V_{12}(r_{12}')\right]\bar{\Psi}_i(\bar{r}';k_i)$$

(221a)

But

$$\int dr_{12}\,dq_{12}\,dr_{12}'\,dq_{12}'\,e^{-i(k_{12f}\cdot r_{12}+K_{12f}\cdot q_{12})}V_{12}(r_{12})\delta(r_{12}-r_{12}')\delta(q_{12}-q_{12}')e^{i(k_{12i}\cdot r_{12}'+K_{12i}\cdot q_{12}')}$$

$$=(2\pi)^3\delta(K_{12f}-K_{12i})\int dr_{12}\,dr_{12}'\,e^{-ik_{12f}\cdot r_{12}}V_{12}(r_{12})\delta(r_{12}-r_{12}')e^{ik_{12i}\cdot r_{12}'}$$

(221b)

Also, using the center of mass analogue of Eq. (D3) in section D.1 below, the second integral involving $\bar{G}_{12}$ in (221a) becomes

$$-\frac{1}{(2\pi)^3}\int d\underset{\sim}{r}_{12}\, d\underset{\sim}{q}_{12}\, d\underset{\sim}{r}'_{12}\, d\underset{\sim}{q}'_{12}\, e^{-i(\underset{\sim}{k}_{12f}\cdot\underset{\sim}{r}_{12}+\underset{\sim}{K}_{12f}\cdot\underset{\sim}{q}_{12})}\, e^{i(\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}'_{12}+\underset{\sim}{K}_{12i}\cdot\underset{\sim}{q}'_{12})}$$

$$\times V_{12}(\underset{\sim}{r}_{12})V_{12}(\underset{\sim}{r}'_{12})\int d\hat{\underset{\sim}{K}}_{12}\, e^{i\hat{\underset{\sim}{K}}_{12}\cdot(\underset{\sim}{q}_{12}-\underset{\sim}{q}'_{12})}\, g_{12}\left(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};\bar{E}_i-\frac{\hbar^2\hat{K}_{12}^2}{2\mu_{3R}}+i\epsilon\right)$$

$$=-(2\pi)^3\delta(\underset{\sim}{K}_{12i}-\underset{\sim}{K}_{12f})\int d\underset{\sim}{r}_{12}\, d\underset{\sim}{r}'_{12}\, e^{-i\underset{\sim}{k}_{12f}\cdot\underset{\sim}{r}_{12}}\, V_{12}(\underset{\sim}{r}_{12})\, g_{12}\left(\underset{\sim}{r}_{12};\underset{\sim}{r}'_{12};\frac{\hbar^2 k_{12i}^2}{2\mu_{12}}+i\epsilon\right)V_{12}(\underset{\sim}{r}'_{12})\, e^{i\underset{\sim}{k}_{12i}\cdot\underset{\sim}{r}'_{12}} \tag{221c}$$

Thus,

$$\left\langle f\left|\bar{\underset{\sim}{T}}_{12}(\bar{\underset{\sim}{E}}_i+i\epsilon)\right|i\right\rangle=(2\pi)^3\delta(\underset{\sim}{K}_{12f}-\underset{\sim}{K}_{12i})\left\langle f\left|\underset{\sim}{t}_{12}(E_{12i}+i\epsilon)\right|i\right\rangle \tag{222a}$$

so that, from (219a), the momentum space contribution to $\bar{T}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ made by the diagram (216a) is

$$\bar{T}_{12}(\underset{\sim}{k}_i\rightarrow\underset{\sim}{k}_f)=(2\pi)^3\delta(\underset{\sim}{K}_{12f}-\underset{\sim}{K}_{12i})\left\langle f\left|\underset{\sim}{t}_{12}(E_{12i})\right|i\right\rangle \tag{222b}$$

where the matrix elements on the right sides of (222) are defined as in

Eq. (131e). Obviously Eq. (222b) is identical with Eq. (130c), recalling that the left side of Eq. (130a) is the matrix element for $\bar{T}_{12}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ in configuration space.

Next consider the typical double scattering process represented by the diagram (216c). Computed in momentum space the matrix element on the right side of (219b) is, by definition

$$\frac{1}{(2\pi)^{12}}\int d\underset{\sim}{k}_{12}\, d\underset{\sim}{K}_{12}\, d\underset{\sim}{k}'_{12}\, d\underset{\sim}{K}'_{12} \left\langle \underset{\sim}{k}_f \middle| \bar{\underset{\sim}{T}}_{23}(\bar{E}_i+i\varepsilon) \middle| \underset{\sim}{k} \right\rangle \left\langle \underset{\sim}{k} \middle| \bar{G}_F(\bar{E}_i+i\varepsilon) \middle| \underset{\sim}{k}' \right\rangle \times \left\langle \underset{\sim}{k}' \middle| \bar{\underset{\sim}{T}}_{12}(\bar{E}_i+i\varepsilon) \middle| \underset{\sim}{k}_i \right\rangle \tag{223a}$$

where the matrix elements of $\bar{\underset{\sim}{T}}_{\alpha\beta}$ are given by Eqs. (221) - (222), and where--again by definition

$$\left\langle \underset{\sim}{k} \middle| \bar{G}_F(\bar{E}_i+i\varepsilon) \middle| \underset{\sim}{k}' \right\rangle = \int d\bar{\underset{\sim}{r}}\, d\bar{\underset{\sim}{r}}'\, \bar{\psi}_f^*(\bar{\underset{\sim}{r}};\underset{\sim}{k})\, \bar{G}_F(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}';\bar{E}_i+i\varepsilon)\, \bar{\psi}_i(\bar{\underset{\sim}{r}}';\underset{\sim}{k}') \tag{223b}$$

Using the expansion

$$\bar{G}_F(\bar{\underset{\sim}{r}};\bar{\underset{\sim}{r}}';\bar{\underset{\sim}{E}}_i+i\varepsilon) = \frac{1}{\bar{T}-\bar{E}_i-i\varepsilon} = \frac{1}{(2\pi)^6}\int d\hat{\underset{\sim}{k}}_{12}\, d\hat{\underset{\sim}{K}}_{12} \frac{e^{i\left[\hat{\underset{\sim}{k}}_{12}\cdot(\underset{\sim}{r}_{12}-\underset{\sim}{r}'_{12})+\hat{\underset{\sim}{K}}_{12}\cdot(\underset{\sim}{q}_{12}-\underset{\sim}{q}'_{12})\right]}}{\frac{\hbar^2\hat{k}_{12}^2}{2\mu_{12}}+\frac{\hbar^2\hat{K}_{12}^2}{2\mu_{3R}}-\bar{E}_i-i\varepsilon} \tag{224a}$$

Eq. (223b) becomes

$$\langle \underset{\sim}{k} | \bar{G}_F(\bar{E}_i + i\varepsilon) | \underset{\sim}{k}' \rangle$$

$$= (2\pi)^6 \delta(\underset{\sim}{k}_{12} - \underset{\sim}{k}'_{12}) \delta(\underset{\sim}{K}_{12} - \underset{\sim}{K}'_{12}) \frac{1}{\frac{\hbar^2 k_{12}^2}{2\mu_{12}} + \frac{\hbar^2 K_{12}^2}{2\mu_{3R}} - \bar{E}_i - i\varepsilon} \tag{224b}$$

Inserting (224b) and (222a) into (223a) yields

$$\langle f | \bar{\underset{\sim}{T}}_{23}(E_i + i\varepsilon) \bar{G}_F(\bar{E}_i + i\varepsilon) \bar{\underset{\sim}{T}}_{12}(\bar{E}_i + i\varepsilon) | i \rangle$$

$$= \int d\underset{\sim}{k}_{12} \, d\underset{\sim}{K}_{12} \, \delta(\underset{\sim}{K}_{23f} - \underset{\sim}{K}_{23}) \delta(\underset{\sim}{K}_{12} - \underset{\sim}{K}_{12i})$$

$$\times \frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23}\left(\frac{\hbar^2 k_{23f}^2}{2\mu_{23}} + i\varepsilon\right) | \underset{\sim}{k}_{23} \rangle \langle \underset{\sim}{k}_{12} | \underset{\sim}{t}_{12i}\left(\frac{\hbar^2 k_{12i}^2}{2\mu_{12}} + i\varepsilon\right) | \underset{\sim}{k}_{12i} \rangle}{\bar{E}(\underset{\sim}{k}) - \bar{E}_i - i\varepsilon} \tag{225}$$

which can be seen to be equivalent to the result quoted on p. 59 of Watson and Nuttall[(4)]. In Eq. (225), $\bar{E}(\underset{\sim}{k})$ is given by either the 1, 2 or the 2, 3 analogues of (35), as one chooses. Hence, because of the $\delta(\underset{\sim}{K}_{12} - \underset{\sim}{K}_{12i})$ factor in the integrand, the denominator in (225) can be written as

$$\bar{E}(\underset{\sim}{k}) - \bar{E}_i - i\varepsilon = \frac{\hbar^2 k_{12}^2}{2\mu_{12}} - \frac{\hbar^2 k_{12i}^2}{2\mu_{12}} - i\varepsilon \tag{226a}$$

Also, from (29d)

$$\underset{\sim}{k}_{23} = -\left(\underset{\sim}{K}_{12} + \frac{m_3}{m_2+m_3}\underset{\sim}{K}_{23}\right) \tag{226b}$$

$$\underset{\sim}{k}_{12} = \underset{\sim}{K}_{23} + \frac{m_1}{m_1+m_2}\underset{\sim}{K}_{12} \tag{226c}$$

Eq. (226c) further implies that (for fixed $\underset{\sim}{K}_{12}$) $d\underset{\sim}{k}_{12} = d\underset{\sim}{K}_{23}$, i.e., in Eq. (225)

$$d\underset{\sim}{k}_{12}\, d\underset{\sim}{K}_{12} = d\underset{\sim}{K}_{23}\, d\underset{\sim}{K}_{12} \tag{226d}$$

Using (226d), the integrations in (225) are immediately performed, with the quantities $\underset{\sim}{k}_{23}$, $\underset{\sim}{k}_{12}$ being given by Eqs. (226b), (226c) respectively after making the replacements $\underset{\sim}{K}_{23} = \underset{\sim}{K}_{23f}$, $\underset{\sim}{K}_{12} = \underset{\sim}{K}_{12i}$. Recalling Eqs. (171a), we now see that Eqs. (219b) and (225) imply the momentum space amplitude

$$\bar{T}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$$

$$= -\lim_{\varepsilon \to 0} \frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23}(E_{23f}+i\varepsilon) | -\underset{\sim}{B} \rangle \langle \underset{\sim}{A} | \underset{\sim}{t}_{12}(E_{12i}+i\varepsilon) | \underset{\sim}{k}_{12i} \rangle}{\frac{\hbar^2 A^2}{2\mu_{12}} - \frac{\hbar^2 k_{12i}^2}{2\mu_{12}} - i\varepsilon} \qquad (227)$$

For $A^2 \neq k_{12i}{}^2$, the limit in (227) can be performed immediately, and obviously yields

$$\bar{T}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = -\frac{2\mu_{12}}{\hbar^2} \frac{\langle \underset{\sim}{k}_{23f} | \underset{\sim}{t}_{23f} | -\underset{\sim}{B} \rangle \langle \underset{\sim}{A} | \underset{\sim}{t}_{12i} | \underset{\sim}{k}_{12i} \rangle}{A^2 - k_{12i}^2} \qquad (228a)$$

For $A^2 = k_{12i}{}^2$, the limit $\varepsilon \to 0$ in (227) doesn't really exist, but in momentum space procedures it is customary[42] to make the interpretation

$$\lim_{\varepsilon \to 0} \frac{1}{\omega - i\varepsilon} = i\pi\delta(\omega) + P\frac{1}{\omega} \qquad (228b)$$

where P again signifies the principal part (after integration). According to (228b), at $A^2 = k_{12i}{}^2$, Eq. (227) should yield

$$\bar{T}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = -\frac{2\mu_{12}\pi i}{\hbar^2}\langle \underset{\sim}{k}_{23f}|\underset{\sim}{t}_{23f}|-\underset{\sim}{B}\rangle\langle \underset{\sim}{A}|\underset{\sim}{t}_{12i}|\underset{\sim}{k}_{12i}\rangle\delta(A^2-k_{12i}^2) \quad (228c)$$

The right sides of Eqs. (228a) and (206) are identical; the right sides of Eqs. (228c) and (191a) are identical. Therefore, recalling also Eq. (209b), we see that Eqs. (228a) and (228c) taken together show the momentum space $\bar{T}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ is identical with the configuration space

$$\bar{T}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) = \bar{T}^{t}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) + \bar{T}^{a}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f) \quad (229)$$

for all A^2, where the configuration space amplitudes on the right side of (229) are given by Eqs. (175b) and (176b).

5.3.3 Off-Shell Double Scattering

Section 4.2 makes it obvious that--whether arrived at via momentum space procedures or via the configuration space approach of section 4.1--the single scattering transition amplitude of (222a) or (130c) must not be included in $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$. In other words, there is complete agreement between the configuration space and momentum space single scattering contributions to the total transition operator $\underset{\sim}{T}$ and to its true three-body part $\underset{\sim}{T}^t$. For double scattering processes the preceding paragraph has shown that the momentum space and configuration space contributions to the total $\underset{\sim}{\bar{T}}$ are the same; however, the momentum space considerations, e.g., the diagrammatic derivation of (216c) in subsection 5.3.1, do not very convincingly indicate what part of $\bar{T}_{2312}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ should be included in $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$. Again, section 4.2 makes it obvious that--whether arrived at via momentum space procedures or via the configuration space approach of section 4.1--the quantity (228c) giving $\bar{T}_{2312}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ at $A^2 = k_{12i}{}^2$ must be excluded from $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$; otherwise the inferred three-body elastic scattering rate will have an anomalous $\tau^{4/3}$ dependence on the volume τ. But to decide whether (228a)--the value of $\bar{T}_{2312}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ at $A^2 \neq k_{12i}{}^2$--should or should not be excluded from $\bar{T}^t(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$, it seems necessary to fall back on our configuration space arguments.

Actually, once (175b) has been reduced to (206), the conclusion that it indeed represents a contribution to $\underset{\sim}{\bar{T}}^t$--i.e., the conclusion that (228a) should not be excluded from $\underset{\sim}{\bar{T}}^t$--apparently can be inferred merely from the rules at the end of subsection 4.1.3. Along most directions $\underset{\sim}{\nu}_f$ in the nine-dimensional configuration space, the scattered part $\Phi_i^{(+)}(\underset{\sim}{r})$ decreases asymptotically like $\rho^{-5/2}$; these are the directions corresponding to those $\underset{\sim}{k}_f$ for which the experimentalist expects to count truly three-body scattering events. Correspondingly, in general the allowed $\underset{\sim}{k}_f$ for given $\underset{\sim}{k}_i$ form a five-dimensional manifold [conservation of total momentum and total energy imposes four conditions on the otherwise arbitrary nine numbers specifying $\underset{\sim}{k}_{1f}$, $\underset{\sim}{k}_{2f}$, $\underset{\sim}{k}_{3f}$]. Now for specified $\underset{\sim}{k}_i$, $\underset{\sim}{k}_f$, the quantities $\underset{\sim}{A}$, $\underset{\sim}{B}$ are uniquely determined by Eqs. (171a), without imposition of any additional conditions; therefore, for physically allowed $\underset{\sim}{k}_f$ consistent with given $\underset{\sim}{k}_i$, the double scattering processes whose contributions are evaluated by (228a) are associated with the full five-dimensional manifold of final $\underset{\sim}{k}_f$. Consequently, in general (228a) represents a contribution to $\bar{T}_{2312}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ along directions $\underset{\sim}{k}_f$ corresponding to truly three-particle scattering; i.e., in general (228a) should be included in $\underset{\sim}{\bar{T}}^t$. Because of Eqs. (177), the result (228a) can be interpreted as resulting from a pair of successive purely two-body scatterings, each of which conserves momentum but not energy [though of course conservation of total energy in the overall transition from $\underset{\sim}{k}_i \to \underset{\sim}{k}_f$ is guaranteed, because $\underset{\sim}{k}_f$ is presumed to lie on the total energy shell]. The extra condition that energy shall be conserved in the individual two-body scattering events can be satisfied only on a four-dimensional manifold of final directions $\underset{\sim}{k}_f$, along which section E.3 shows $\Phi_i^{(+)}(\underset{\sim}{r})$ decreases asymptotically as ρ^{-2} [consistent with the fact that a $\delta(A^2 - k_{12i}^2)$ factor turns up in (228c)]; consequently the double scattering contributions (228a) along directions $A^2 = k_{12i}^2$ should be excluded from $\bar{T}^t$.

The factor $(A^2 - k_{12i}^2)^{-1}$ in the right side of (206) means that $\bar{w}(\underset{\sim}{k}_i \to \underset{\sim}{k}_f)$ of Eq. (3) will diverge when integrated over all final $\underset{\sim}{k}_f$ consistent with $A^2 \neq k_{12i}^2$. This result, for the elastic scattering processes here being discussed, can be interpreted along the lines of subsection 4.2.2. Although the diagram (216c) corresponds to a pair of purely two-body scatterings, nevertheless this diagram's off-shell contributions (206) or (228a) cannot occur unless all three particles somehow simultaneously interact; if particle 3 is infinitely far from the pair 1, 2, then the pair 1, 2 can only make a collision which conserves energy as well as momentum. To put it differently, after the first collision in (216c) the particles 1, 2 are in a state which lasts only a time Δt until particle 2 collides with 3. The magnitude of Δt is given by

$$\Delta t \sim \frac{X}{v_2'} \tag{230a}$$

where X is the distance traveled by particle 2 between its collisions with 1 and with 3, and $\underset{\sim}{v}_2{}' = \frac{\hbar \underset{\sim}{k}_2{}'}{m_2}$ is the velocity of particle 2 after its first collision. But the magnitude ΔE of the departure from energy conservation in the intermediate state is

$$\Delta E = \frac{\hbar^2}{2\mu_{12}}\left(A^2 - k_{12i}^2\right) \sim \frac{\hbar}{\Delta t} \tag{230b}$$

yielding

$$X \sim \frac{\hbar v_2'}{\Delta E} \tag{230c}$$

Now, as in subsection 4.2.2, suppose the volume τ contains precisely one particle of each species 1, 2, 3. The rate of double scatterings in which a collision between 1, 2 is followed by a collision between 2, 3 during the time for particle 2 to travel a distance X is [compare Eq. (159a)]

$$\sim \left\langle \left(\frac{\bar{w}_{12}^{(2)}}{\tau}\right)\left(\frac{X\,\bar{\sigma}_{23}}{\tau}\right) \right\rangle_{av} \tag{231a}$$

Therefore the rate of double scatterings in which the scattering between

2, 3 takes place after particle 2 has traveled a distance between X and X + dX is

$$\sim \left\langle \frac{\bar{w}_{12}^{(2)}}{\tau} \frac{\bar{\sigma}_{23}}{\tau} \right\rangle_{av} dX \tag{231b}$$

But, from (230c),

$$dX \sim \frac{\hbar v_2'}{(\Delta E)^2} d(\Delta E). \tag{231c}$$

so that, still with one particle of each species in τ, the rate of double scatterings in which energy conservation in the intermediate state fails by an amount between ΔE and $\Delta E + d(\Delta E)$ is [using (152b)]

$$\hat{w}'_{\Delta E}(1,2;2,3) \cong \left\langle \frac{\bar{w}_{12}^{(2)}}{\tau} \frac{\bar{\sigma}_{23}}{\tau} \hbar v_2' \right\rangle_{av} \frac{d(\Delta E)}{(\Delta E)^2} \tag{232a}$$

$$\cong C' \frac{\bar{\sigma}_{12}\bar{\sigma}_{23}}{\tau^2} \frac{d(\Delta E)}{(\Delta E)^2} \tag{232b}$$

where C' here is independent of the shape of the scattering region τ, and represents an average [over scattering directions and velocities] of the various primarily velocity-dependent factors in (232a) not explicitly included in (232b). The corresponding rate with $\hat{N}_\alpha$ particles of each species in τ is

$$\hat{w}_{\Delta E}(12;23) \doteq \hat{N}_1 \hat{N}_2 \hat{N}_3 \hat{w}'_{\Delta E}(12,23) \cong N_1 N_2 N_3 \tau \frac{C' \bar{\sigma}_{12} \bar{\sigma}_{23}}{(\Delta E)^2} d(\Delta E) \quad (232c)$$

Except for the factor C', the result (232c) has precisely the form obtained when the contribution $|\bar{T}^t{}_{2312}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)|^2$ from (206) is substituted into Eqs. (2)-(3), remembering that $\bar{\sigma}_{12}$ is proportional to $|< f|\underset{\sim}{t}_{12i}|i >|^2$, and that $d\underset{\sim}{k}_{1f} d\underset{\sim}{k}_{2f} d\underset{\sim}{k}_{3f}$ in (3) can be reexpressed in terms of $d(\Delta E)$ and other $\underset{\sim}{k}$-dependent differentials. The fact that (232c) is proportional to τ once again indicates that the expression (206) must be included in the physical three-body scattering amplitude.

I believe the above qualitative largely geometrical argument is basically consistent with the arguments of Iagolnitzer(45), who has examined the interpretability of a propagator pole in the scattering amplitude. He finds that the pole can be understood to represent a pair of successive real two-body collisions, but his analysis holds only in the limit that the distance X between the collisions is very large. It is to be noted that the geometrical argument in this subsection differs in one important aspect from those given in chapter 4; in chapter 4 it always was presumed that each individual collision under discussion [e.g., the individual two-body processes considered in the derivation of (159a)] was an actually occurring event, i.e., was consistent with energy and momentum conservation. Finally, I close this--the last section of the main text--with the remark that despite the consistency and interpretability of our result (206) for the contribution to $\underset{\sim}{\bar{T}}^t$ made by double scattering processes, it still would be desirable to confirm our conclusions via a configuration space calculation of $\bar{T}^t{}_{2312}(\underset{\sim}{k}_i \rightarrow \underset{\sim}{k}_f)$ which somehow avoids having to reinterpret singular integrals, as we were unable to avoid doing in deriving Eqs. (175).