NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-439

The Probability Density Function of a Hardware Performance Parameter

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Preface

The work described in this report was performed by the Telecommunications Division of the Jet Propulsion Laboratory.

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Abstract

A study of the probability density function of a hardware performance parameter delivered by a contractor to a specification is presented. Because sufficient data were not available to estimate this probability density function experimentally, it was necessary to derive *a priori* probability density functions from assumed axioms determined from an investigation of contractor motivation and behavior. From these axioms, the problem was formulated and solved in the form of a calculus of variations problem with integral and inequality constraints.

Numerous examples are presented in which various strategies of contractor manipulation were used. The conclusions based on these examples are that incentive contracting is not an effective tool, and that close surveillance of the contractor is the most effective means for obtaining a hardware parameter to a specification.

The Probability Density Function of a Hardware Performance Parameter

I. Introduction

This report is a part of a continuing program for determining and specifying system-performance margin relating to telecommunication system methodology research. The first article written on this subject (Ref. 1) concerns a study of the criteria for acceptance of a system design. Essentially, that analysis furthered the answer to the question: Before investing more in the determination of the tolerances and their probability density functions, what would we do with them if we had them? To answer that question, dummy probability density functions were adequate for the demonstration. This analysis concerns itself with the question: What are the probability density functions? In particular, only the aspect of the probability density functions for hardware performance parameters delivered by a contractor to a specification along with methods of manipulating the contractor to obtain satisfactory probability density functions are considered.

One of the assumptions used in the referenced analysis was that the tolerances were independent. While dependency could be factored into the analysis with some difficulty, dependency was not thought to be prevalent enough to be of immediate concern. In the review process, attention was focused on the fact that the mode of obtaining hardware for the telecommunications system for the *Mariner VI* and *VII* spacecraft was different from that of previous spacecraft programs. Previously, the system was obtained by purchasing subsystems from numerous contractors; this method yielded essentially independent parameters. For *Mariners VI* and *VII*, however, the telecommunication system was obtained from one major contractor. This latter mode will probably be used in the future. One concern is that if the contractor does a poor job in one area, it is likely he will do a poor job in all areas, thus the dependency. Since the success of the mission depends largely on the contractor, it is quite important to understand how he will deliver to a specification under various influences and how to manipulate the contractor to match one's needs.

Probability density functions are either derived or estimated experimentally. In determining which method should be selected, one must realize that adequate data on parameters for the spacecraft part of the telecommunication system for the experimental determination of a parameter probability density function delivered to a specification is difficult to accomplish because of the small number of items delivered to any one design. It appears uncanny, nevertheless, how biased the delivery is toward the low performance side. Without adequate data (and if a probability density function is definitely wanted) the only recourse appears to be to derive *a priori* probability density functions. The objective of the remainder of this analysis, therefore, is to determine the *a priori* probability density function of a hardware parameter delivered by a contractor to a specification and to determine strategies for the manipulation of the contractor to obtain satisfactory probability density functions.

The analysis begins with the realization that the contractor acts to enhance his own self-interests and that these interests vary widely among contractors. A major part of the discussion is concerned with the determination of these self-interests and the behavior of the contractor. While all those that were evident are discussed individually, self-interests were found to fall into the following broad categories: (1) those that basically do not change but are only selectable among the contractors in the precontract stage and (2) those that can be changed or manipulated after the contract is entered. The amount of resources necessary to make a specified change under the second category is influenced by the contractor selection made under the first category.

Next, a short discussion as to the random state of nature, which is an "unintelligent adversary," is offered. Of course, without the random state of nature, there is no problem to be solved.

The assumption is made that the discussion describes a game between two players who also have an unintelligent adversary. From this assumption, an attempt is made to determine the rules under which they were playing. With this foundation, a formal statement of the problem is possible in a more or less axiomatic approach. With these rules, it is possible to infer other conclusions not obvious from the description of the observed play.

Once this formulation is accepted, the solution of the problem is undertaken. Solving the problem involves the determination of appropriate net payoff functions for the contractor and the use of the calculus of variations with integral and inequality constraints. To solve the problem, new techniques were developed; however, equivalent techniques developed by Koopman were found later (Ref. 2). This equivalence is demonstrated.

After the methodology is developed, numerous examples are given using both isolated and mixed strategies for manipulation of the contractor. These strategies are parameterized for various incentive-to-cost ratios. The general conclusion reached from the analysis and contrived examples is that there is a definite lack of sensitivity to the controllable parameters for effective influence over the resultant probability density functions. In an isolated strategy of incentive contracting, the prize offered must be overwhelming to have any pronounced effect for the stated objective. If mixed strategies are used with a fixed resource for incentives, none of the strategies is likely to yield any pronounced effect for the stated objective. Although the sensitivity of the controllable parameters is relatively the same, the isolated strategy of surveillance was found to be the most effective strategy to meet the stated objective. No effective substitutes were found by the analysis for the following rules for controlling the probability density functions of a hardware performance parameter delivered by a contractor to a specification:

(1) Know your contractors and know them well.

(2) Maintain very close surveillance over your contractors.

II. Discussion of Contractor Motivation

At the outset, a fundamental assumption made is that a contractor will act to enhance his own self-interests within some ethical framework. With this assumption, it is then necessary for the administrator to determine those self-interests or motivators if he is to get the most from a contractor or to manipulate the contractor. It is that purpose to which attention is now directed. (In this report, *administrator* is defined as the organization for whom work is performed, and *contractor* is the organization that performs the work.)

Since a contractor cannot exist long in the business world without profit, one would be inclined to believe that profit is the universal motivator. While this assumption is tolerable and is quite prevalent in system contracting as evidenced by the frequent use of incentive type contracts, this tactic is certainly not universally successful. This suggests that profit is not the sole motivator for good performance but is merely one of many factors influencing contractor behavior in executing his role as a system contractor.

In addition to profit, there are other common organizational objectives that are considered important to system contractors. The objectives include growth of sales and employment, organization survival, technical gain, reputation, freedom from harassment, advancement of science and technology, and the desire for public approbation. Institutions, however, do not truly have objectives because, in reality, their objectives are the aggregate of the individuals' objectives who make up an organization. Individuals may be manipulated, and the institution is only manipulated through them. The more closely motivation is related to the individuals, the more realistic this study of motivation will be.

To study motivation more closely, a survey was made of literature (Refs. 3 and 4), and numerous discussions were held with experienced people, both at JPL and in industry. The factors identified as influences of motivation are listed below (not necessarily in order of importance):

(1) Interests and personal background of the principals.

(2) Survival of the principals.

(3) Profit.

(4) Size of the contractor.

(5) Size of the contract.

(6) Relative size of the contract in relation to the contractor's business.

(7) Age of the contractor.

(8) Ancillary profits (such as developed technology).

(9) Condition of the business climate.

(10) Glamour of the work.

These factors are discussed in greater detail in succeeding paragraphs.

A. Interests and Personal Background of the Principals

A significant factor in the motivation of a contractor is the interests and personal background of its top management. Organizations frequently take on a character reflecting these interests and backgrounds. For example, a top manager who has attained his current position through significant experience in marketing is likely to be profit oriented but is conscious of a need to maintain long-term sales prospects. However, a top manager who has had significant experience through the financial control of the business would probably hold higher value in the monthly profit and loss statement and, therefore, pursue an immediate accounting profit. Although both managers are relatively profit oriented, each would see the needs for performance on a contract differently.

A research oriented manager might essentially "buy in" on a contract and sacrifice profit for an interesting research contract. In such a case, profit is only a means to an end. There are some instances where companies maintain entire divisions that are a dissipation for the interests of the principals even at no profit or some loss. On a smaller scale, some companies have found certain activities to be unprofitable but maintain them at a certain level as the price they must pay to maintain a staff capable of doing other profitable work.

Some motives can be of an altruistic nature. For instance, some executives have a strong feeling for public and social services and place high priority to this end. The recent job training program conducted through private enterprise in California aimed at helping minority groups and other underprivileged people in obtaining better job opportunities is but one of the examples. There are others who believe in creating useful work for people. Companies under management of these executives are inclined to emphasize growth and security in employment policy and are likely to actively and constantly participate in new product development. Yet there are others who have a strong desire to create new devices for labor-saving and personal enjoyment purposes. The preamble of this motive is to help improve technology, life, and as a consequence, living standards for all people. It is conceivable that many household and office labor-saving equipment and recreational gadgetry were brought about in this manner.

Although the above motives appear to be in direct conflict with the desire of achieving an immediate accounting profit, it is clear that those companies sponsoring programs of this nature will ultimately reap financial gains if the programs are conducted successfully. To exploit this motive for contractor performance, however, it is important to recognize that the actions of the contractor are restricted by the resources of the organization and the generosity of its principals.

B. Survival of the Principals

When a firm takes on the form of proprietorship or partnership, organization survival and the survival of the principals are essentially one and the same thing. For a corporation, however, these terms may assume an entirely different meaning. After a series of financial and technical blunders, while a corporation may have sufficient resources to survive, the top management team responsible for those mistakes is almost certain to be replaced. For this reason, the principals tend to be more conscious of their own survival with the corporation.

The performance of a contract and the subsequent profit or loss are impersonal things that generally cannot be attributed to any one individual. However, the decision process in conducting the project can be identified with specific people, and this puts the individual's security in jeopardy.

Survival within an organization is increasingly difficult for the members of top management. Not only is there less flexibility in the position and greater responsibility in crucial decision making, but also there is continual challenge from below to assume that position. It seems reasonable to believe that the desire of job survival is directly related to the ranking of the job position.

Of course, the principals can find other opportunities elsewhere if they are hard-pressed in maintaining their current position. However, this alternative can only be considered as a defensive weapon for those in the higher managerial ranks as it is not an easy task to achieve the desired status and remuneration on short notice. A good record of successful project management is mandatory if misunderstanding, embarrassment and harassment are to be avoided. The principals are highly responsive to being identified with any of these; therefore, close surveillance of the principals is a very effective tool for contract control. As illustrated in Fig. 1, surveillance can be effective at all organizational levels.

C. Profit

Profit is the most immediate reward in performing a job and probably is the least common denominator for the measure of success of a contractor. To a certain extent, profit permeates throughout a company as a motivator. However, it is unreasonable to assume that all employees at various levels (or for that matter, all organizations) take the same degree of interest in profits. For example, the sole proprietor of a firm might take more interest in profits than would any other individual

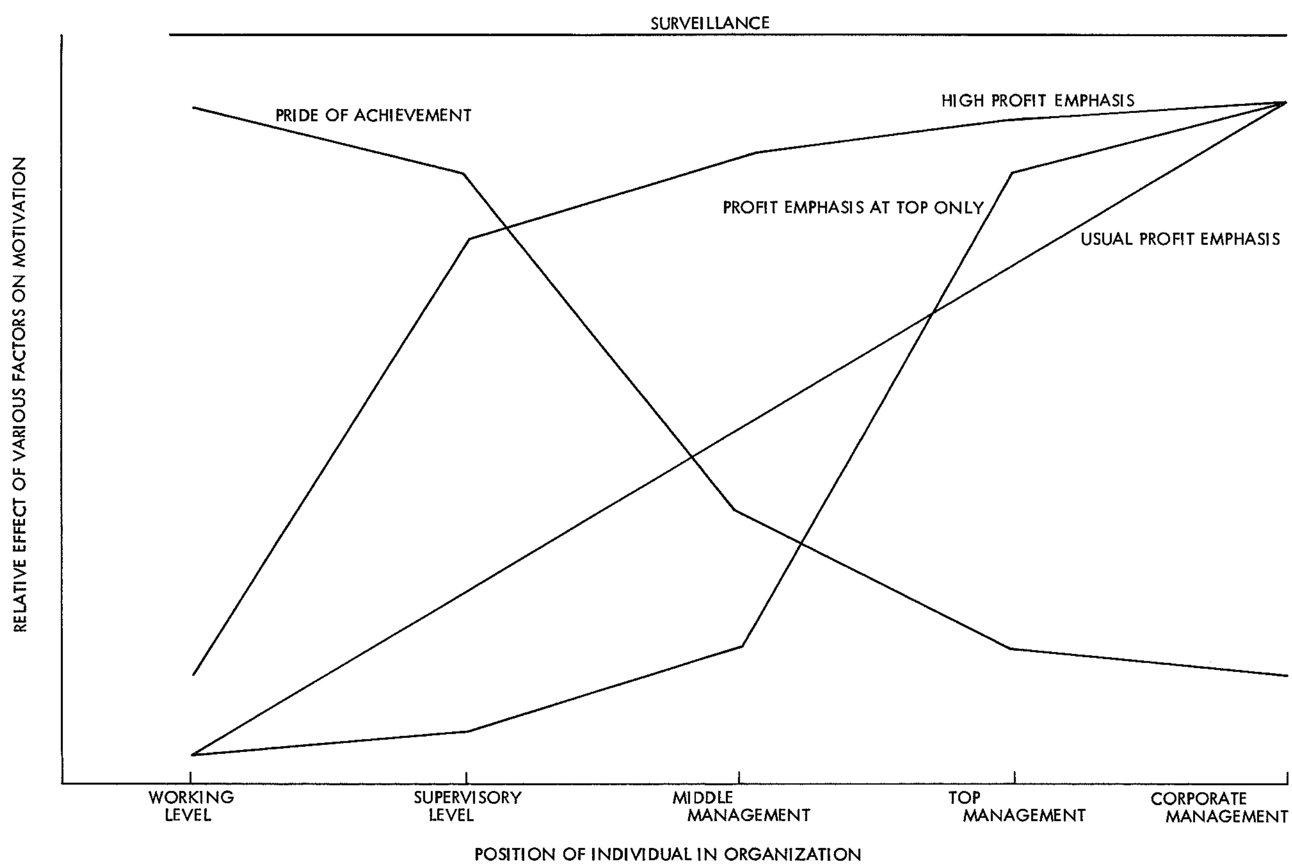

Fig. 1. Relative effect of various factors on motivation vs position of individual in organization

or group of individuals within the firm. When a company takes the more impersonal character of a corporation, the principals rarely list profits as their primary motivation or use it as the principal determinant of business action. Many corporation officials have expressed preference for other corporate objectives such as reputation and future growth over immediate profits.

The motivation of profit influences the individual less at the lower levels of an organization and the success of a motivation of profit strategy at each level depends distinctly upon the emphasis placed there by the top management (Fig. 1). Usually the emphasis on profits at the lower levels is nil as the company becomes amorphous with increasing size. Success of an incentive type of contract depends closely on this emphasis.

If the critical parameters of a contract depend heavily on the efforts at the working level, where profits are not a direct motivator, they may motivate in a negative sense by the emphasis placed on the performance by the management where profit is a higher motivator.

D. Size of the Contractor

Size of the contractor has a distinct influence on the performance of a contract with a parameter specification. Although distinctly limited in the tasks that can be performed, a small company emerging in the business world cannot survive far beyond their current job if it represents a sizable portion of their business, unless their response is satisfactory enough to create future business. In this circumstance, satisfaction of specification and delivery are paramount motives of the company. Because of this, the attention given by the principals of the company is direct and timely.

On the other hand, the pressure of corporate survival on a large firm with multi-program capability is less intense. The performance of a relatively small contract often receives insufficient managerial attention. This unconcerned attitude may change, however, if successful completion of this contract may lead to substantial production effort, or if the contract is performed in an unfavorable business climate.

E. Size of the Contract

The size of contract has an important impact on the behavior of contractors that is not directly related to profits. Assuming that one is considering a new system development contract, the larger the size of the contract, the larger is the size and scope of the system under consideration. In practice, large systems generally require numerous definitions and specifications of parameters in a time scale that is unreasonable to accomplish with an adequate degree of accuracy. Since the contractor realizes that this is the usual circumstance and is conditioned to this from previous experiences, many specifications are taken only as a guide or design target instead of a set of concrete design goals. If, during the performance of the contract, these prove to be easily attained, the adequacy of the specification is rarely questioned; on the other hand, if difficulty is encountered in attaining the specification, the contractor would likely bring this up for renegotiation on the ground that such targets were questionably determined. If a subsequent sensitivity study indicates that the specification is indeed too strenuous, the contractor is relieved of his obligation. The conditioning caused by the frequency of this situation creates an attitude among contractors that makes attainment of seriously submitted specifications difficult because of the lack of proper attention and planning on the part of the contractor.

F. Size of the Contract in Relation to the Contractor's Business

Size of the contract in relation to the size of the contractor can create thresholds in effectiveness of incentives. Although small contracts contribute heavily to the growth and survival of a company, small contracts in a relative sense probably will be handled in a routine manner. If a difficult specification exists in a contract that is in this category, an incentive award approach that tends to upgrade the critical parameters is usually ineffective. The reason for this is that if the contract is small, the proportionate fee probably will be so insignificant compared to the total sales volume of the company that it is not worth diverting management attention to pursue the incentive fee.

G. Age of the Contractor

The length of time that a company has been in existence can be both detrimental and beneficial. If time has not made the organization sedentary, age can provide an institutional pride for which there is motivation to maintain. This pride can be nurtured and taken advantage of during the execution of the contract.

H. Ancillary Profits

Many instances exist of companies that maintain divisions that are not chartered to show a direct monetary profit. Profits are expected to be reaped indirectly later from (1) developed technology, (2) acquisition of new procedures, and (3) training of personnel. Monetary incentives obviously will not influence the performance, and other incentives must be sought.

In the case of contracts with the government, ancillary profits may also be expected to be gained through low investment and risk associated with the use of government furnished equipment. Where this equipment is constructed near the contractor's facilities, subsequent contracts are almost automatic and give an advantage over competition.

I. Business Climate

The current business climate has a marked effect on the type of companies competing for contracts and the response given to the contracts. To elucidate, under a favorable business environment when large contracts are available, smaller contracts become a distraction to the large organizations; and they would rather not pursue them. In this circumstance, the small jobs will go to the small organizations unchallenged. This affects the response given to the contract because of less dependence on the contract to survive and because of the more restricted talents available. On the other hand, during a declining business cycle when contracts become difficult to obtain, the large organizations will suddenly show interest in any contract, irrespective of size, in an attempt to hold their development teams together. In many instances, they challenge and often successfully wrest jobs from the smaller companies. This is because the resources (money and talent) behind the larger firm can better prepare the proposals, which may be a sizable investment to a small firm, and can better cope with difficult situations, technical or otherwise.

J. Glamour of the Work

Glamour of the work definitely is a motivator that affects the performance of a contract. Space work, Hoover Dams, Panama Canals, ICBMs, atomic submarines, transcontinental railways, etc., all in their time were tremendous motivators. Even their mundane aspects could initially be motivated when the project was able to capture the imagination. Unfortunately, it is an uncontrollable force that cannot be created as an incentive for performance of a particular task by a contractor. However, its existence or nonexistence does provide a clue as to whether substitute motivators must be provided.

III. Discussion of Contractor Demotivators

Recognition of demotivators is equally as important as the recognition of the motivating factors to the manipulation of a contractor. The following are some of the more important demotivators:

(1) Heavy commitment by the administrator.

(2) Measure of performance and award fee.

A. Heavy Commitment by the Administrator

There are circumstances in which the system administrator is forced to acquiesce even though the contractor has performed unsatisfactorily. These circumstances exist frequently in space exploration programs where timing is critical and the available money is usually limited. After the selection among competitive contractors is finalized, the winning contractor essentially has the administrator heavily committed in schedule, money, and technique used. With this heavy commitment, the contractor fully realizes that his services will be accepted even though the performance is inadequate. This fact seriously undermines any substitute motivators the administrator may try to provide.

B. Measure of Performance and Award Fee

The use of incentive award approach is an attempt to influence contractor behavior by correlating performance and the size of the award received. Unfortunately, the measure of performance of a contractor is often based on factors that are replete with subjective judgments, and therefore seriously impedes the effectiveness of this approach in accomplishing its intended purpose. Because of this imperfection, the incentive award approach can serve to motivate, have a deadband of no motivation, or demotivate performance on the part of a contractor. As mentioned before, since performance cannot be adequately measured, there is no certainty or assurance in the mind of the contractor that tangible effort will change the result of the subjective judgment. Superior performance ratings do seem to incite continued superior performance. Average performance ratings seem to create no reaction one way or another, and thus a deadband of no motivation appears to develop. Poor reviews do not obtain immediate improvement but appear to cause resentment and defensiveness that may lead to further performance degradation. Such a large schism develops that ratings cannot be honestly applied any longer but are used as a bribe to bolster egos in hopes of gaining improved performance.

IV. Choices for Manipulation of the Contractor

Factors considered in the preceding section dichotomize into uncontrollable and controllable motives from the point of view of the administrator. From the list of considerations, the following are regarded as uncontrollable:

(1) Interests and personal background of the principals.

(2) Size of the contractor.

(3) Size of the contract.

(4) Relative size of the contract in relation to the contractor's business.

(5) Age of the contractor.

(6) Status of the business climate.

(7) Glamour of the work.

Since they are uncontrollable, these factors are not suitable for the purpose of contractor manipulation. Rather, they are used as guides to the selection of contractor best suited for a given project. For example, for a program of modest size with a substantial state-of-the-art technical advancement requirement, the ideal contractor appears to be a relatively small, young, and dynamic organization in a related technological expanding field whose management is highly technically oriented. As another example, if it is assumed technical competence is equal, the most suitable contractor for a development program with large follow-on production potential should be a larger size firm with adequate production capability whose principals strongly believe in long-term sales prospects, provided that the development cost is not a limiting factor. On the other hand, if development money is somewhat tight, the program may best be conducted by a company emphasizing budget and financial control and seeking an immediate accounting profit.

These factors are influential on the contract results, and, although uncontrollable, relate to the controllable factors by the amount of effort necessary through the controllable factors in making up for their deficiencies. The uncontrollable factors are only free parameters up to the point of contractor selection.

The preceding might be summarized as follows: (1) know your needs and requirements, (2) know your contractors and know them well, and (3) select the contractor best suited for your needs.

From the list of considerations, the remaining are regarded as controllable:

(1) Survival of the principals.

(2) Profit.

(3) Ancillary profits.

These may further be bifurcated into areas identified in this analysis as surveillance and incentive award under the general term "strategies," and will be considered for manipulation of the contractor.

Because of the consciousness of personal survival in the part of the contractor key personnel, surveillance appears to play an important role in shaping the course of successful contract performance. By surveillance, we shall mean the followup done as a consequence of deficiencies noted in the monitoring and supervision of the contractor. Surveillance is the incessant aiding and pressuring exerted on all management levels of the contractor to extract the desired performance; it is perhaps one of the more uniformly and personally felt incentives (or negative incentives) that an administrator can impose effectively.

A carefully planned surveillance procedure is essential if it is to successfully complement other tangible incentives or serve as a substitute when incentive alone appears to fail. For a system with parameter specifications, a proper surveillance function is the amount of followup that is accomplished as a function of the observed values of the hardware parameter detected during the execution of the contract. It should be recognized that, in actual practice, there are four difficulties in executing the surveillance function: (1) the difficulty in making certain that the surveillance function (the amount of pressure, etc.) corresponds to the followup actually accomplished, (2) the difficulty in recognizing needed action and in making the followup timely enough to be effectual in corrective action, (3) the difficulty of equating a subjective factor into money, and (4) the difficulty on the part of the administrator to ask for more than the intent of the original contract during the execution of the contract.

The incentive award takes advantage of the desire of the contractor to enhance accounting profit, and it is adaptable to system development contracts when parameter specifications are used as the yardstick for performance measurement. Thus, performance can be objectively evaluated and the incentive award can be formulated in direct relationship with the ultimate achievement of the parameter under consideration. In this case, the contractor knows where he stands in terms of the probable award, the necessary added expenditure, and effort needed. The hostility caused by subjective measures of performance can be avoided with this strategy.

In much of the previous discussion, the considerations have alluded to many of the motivations of the personnel in the contractor's organization. It seems appropriate to make one last reference to this before reducing the contractor to a "Black Box" to which we uniformly apply our strategies for convenience of analysis. Figure 1 depicts what is regarded as relative influences of various strategies on a relatively personal basis.

For a well-planned surveillance activity with appropriate pressure exerted on all levels of the contractor's organization, an ideal effect curve of constant magnitude over the entire range of organizational position would result. Under this condition, the surveillance strategy attains its highest effectiveness. The profit curves reflect the general tendency that the relative influence of incentive award strategy is directly related to ranking in the organizational structure. The characteristics of the profit curve, however, may take in many forms depending on such factors as individual attitude, company structure, and the degree of emphasis on profit as a corporate policy. It is conceivable that an unhealthy attitude taken by a certain key person in the middle management level could change the curve abruptly as shown in one of the example curves.

The pride of achievement has much to do with personal satisfaction of the working engineers to perfect the products they build. Scherer (Ref. 4) makes the following remark: "As one reaches further down into a development organization, the preoccupation with quality for its own sake appears to increase." As a probable result, unnecessary features may be built into the product at the expense of cost. Management is aware of this and usually tries to prevent this sort of activity.

V. Random Influences

With the current state of knowledge relating to random processes, one does not need to argue that all design and manufacturing processes have random factors associated with them. While at times these random factors appear to be uncontrollable, to assume such would be contrary to experience. Improvements are made continually and appear to be largely limited by the resources available to achieve them rather than fundamental limits.

In some cases, such random influences as variation of component parameters, aging, environment, inaccuracy in measurement, and human error may be known during the design and manufacturing process.

A parameter of a design is the result of its individual attributes. To control a parameter, the design (synthesis) process generally attempts to consider as many of these factors as possible within some resource frame and to eliminate or reduce the effect of each through decisions. Those factors that cannot be properly considered in the decision process are generally relegated to having their effects discovered in the testing of pilot models. These may be removed in whole or part by redesign, selection, or adjustment; or they may be ignored altogether. The manufacturing may only use selection and adjustment in altering the parameter.

Random influences in nature are obviously a part of the problem under consideration; to assume otherwise would result in a certainty and thus there would be no problem to solve. By convention of Decision Theory, the random state of nature will be referred to as an unintelligent adversary. By using decision, selection, and adjustment, the contractor has control of this unintelligent adversary and, thus, the probability density functions of a hardware parameter. Decision, selection, and adjustment have a definite cost associated with them, and their degree of success will be determined largely by the resources available for their removal.

VI. Formal Statement of the Problem

The preceding discussion of the factors surrounding this problem is a rather formidable and inconcise statement of the problem, which may be barely visible. Before a solution is attempted, it is expedient to extract from the above discussion a short, precise, and formal statement of the problem.

First of all, a reiteration of the objective should be made. The objective is to determine (1) the *a priori* probability density functions of a hardware performance parameter delivered by a contractor to specification and (2) strategies for manipulation of the contractor to obtain satisfactory probability density functions.

For the remainder of the formal statement of the problem it is convenient to think of the administrator and contractor roles as that of two players in a game with an unintelligent adversary included. The above discussion has served more or less as a description of the play; it is now necessary to determine from the play, the rules under which they were playing. This will be difficult since there never is a full meeting of the minds or formal statement of all the rules between the administrator and the contractor in an actual situation. Nevertheless, it is necessary to state these rules, no matter how weak the inference, that apparently control the parts of the game in which there is an interest.

From the previous sections, the following rules or axioms are hypothesized to suffice in describing the base from which the administrator/contractor relation proceeds:

(1) The administrator has the choice of strategies that are to be applied to the contractor.

(2) The contractor has knowledge of the strategies.

(3) The contractor is going to act in some rational fashion according to the strategies to enhance his own self-interests.

(4) The contractor has some unintelligent adversaries.

(5) The contractor has some choice in the control over the unintelligent adversaries.

By (1), is meant that options such as more money, incentives, surveillance, etc. may be applied as a choice by the administrator. This does not mean that these are applied without mutual consent. By (2), is meant that no strategy is to be applied in any clandestine manner. Any strategy is to be applied in full view of the contractor. It is assumed that it provides nothing to the performance of the contract if the contractor has to guess the rules under which he is to play. This factor carries over into (3) with the assumption that the contractor can only act rationally if he knows the rules under which he plays. In (3) it is also implied that irrational behavior is beyond the scope of this analysis. Axioms (4) and (5) have previously been discussed adequately.

It is believed that these axioms describe the game adequately; however, if one does not accept them, it is believed that the method of the solution to follow does not materially change with other formulations. The analysis now proceeds to results not found in the previous discussion of the play.

VII. Definition of Basic Concepts

The previous section provided a formulation of the problem as a set of axioms to either accept or reject. Once these axioms are accepted, and it appears reasonable that they be accepted, they must be converted into mathematical terms, and appropriate solutions then can be sought for quantitative results, which is the object of this section. This section also discusses some fundamental concepts on which the derivation of the sought-after probability density function is based.

A. Expected Payoff

The contractor's self-interests will be measured in monetary terms; although, they may in fact be quite subjective in nature. Subjective factors will essentially be converted to money with appropriate value constants. With this measure, the following accepted and practiced concept of net profit or payoff will be used: net profit = reward − costs, or symbolically,

$$N(x) = R(x) - C(x) \tag{1}$$

where

x = value of a hardware parameter delivered by a contractor

$N(x)$ = net profit or payoff for delivering a hardware parameter of x

$R(x)$ = reward for delivering a hardware parameter of x

$C(x)$ = cost to the contractor to deliver a hardware parameter of x

If the value x were purely deterministic, the maximization procedure would be simply to seek the value x_0 that maximized $N(x)$; and there would be no further problem. Having accepted the axiom that the contractor has unintelligent adversaries (nature), we have accepted the fact that the outcome is not altogether deterministic.

In the face of this adversity, instead of net profit, we can only measure the expected net profit.

$$E[N(x)] = E[R(x)] - E[C(x)] \tag{2}$$

where

$$E[f(x)] = \int f(x)p(x)dx$$

which is the expected value of $f(x)$.

Here, $f(x)$ is a dummy weighting function and $p(x)$ is the probability density function of delivering the value x. The factor that the contractor attempts to maximize in his own self-interests then is the expected payoff, rather than the payoff function itself. Now

$$E[N(x)] = \int_A^B [R(x) - C(x)]p(x)dx \qquad (3)$$

where A and B are, respectively, the lower and the upper allowable limits of x. They may represent an unbounded or infinite range. The only term not specified in Eq. (3) is the functional $p(x)$, the sought-after probability density function.

It is assumed that the functional $p(x)$ is selectable or controllable. Of course, if there is no control to be exercised by the contractor on the outcome of the parameter x, then the expected payoff is uncontrollable. The fact that the contractor has choice in the design, selection, and adjustment allows him to control $p(x)$ in exchange for cost. Consciously or unconsciously, the contractor will attempt to coordinate his efforts so that the resultant $p(x)$ maximizes his expected payoff.

B. Expected Reward

The reward function is straightforward and warrants only brief comment. This function will vary from situation to situation and is specified by the contract administrator. One of the primary reward functions that will be used in this analysis is a triangular reward function (Fig. 2). In this case $R(x) = k_r r(x)$ where $r(x)$ is a normalized function such that

$$\int_{-B}^{B} r(x)dx = 1$$

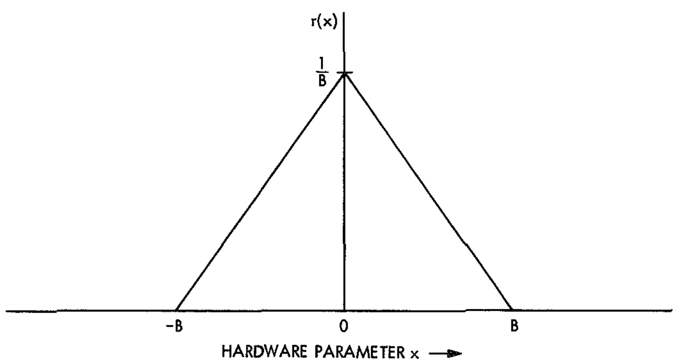

Fig. 2. Triangular reward function

and k_r is a constant for adjusting the appropriate unit of measure and magnitude.

This function was motivated by a situation where there was a technical conflict of interest. Specifically, a spacecraft transmitter design incentive was considered to increase the reward linearly beyond some wanted value of transmitter output power, and then to decrease beyond some higher value so as to discourage the unrestricted use of raw power for the transmitter.

C. Expected Costs

The cost function for the contractor is rather complicated, yet important to the analysis, and will require considerable discussion. This discussion will be broken into two aspects: (1) subjective costs as a result of surveillance, which will be equated in monetary terms, and (2) real costs.

Subjective costs will be lumped together into what will be called the surveillance function. (The term surveillance is discussed in Section IV.) As mentioned before, there are several difficulties in actual practice in formulating and executing a surveillance function realistically. It is also difficult to equate the subjective factors with a measurable quantity such as money.

In practice, management attention appears to increase rather rapidly as the hardware parameter deviates from the desired nominal value. This would indicate that the surveillance function selected should be such that it is zero at the nominal value and monotonically increasing as the value deviates away from the nominal value. The one selected for use here is $S(x) = k_s(x - x_0)^2$ where $S(x)$ is the surveillance function, k_s is a constant for adjusting the appropriate unit of measure and magnitude, and x_0 is the nominal value.

This function appears to adequately describe the observed attention management gives for values of the parameter other than nominal and will be the surveillance function used throughout this analysis. It should be noted that this is a cost to the contractor that is not reimbursable and that he would like to avoid or minimize.

Real cost functions afford special problems in their specification. Here again we will be talking about nonreimbursable costs to the contractor. The cost function reflects the relationship between the improvement in the probability density function $p(x)$ and the indirect economic burden imposed on the contractor for such an improvement.

A contractor has considerable choice during his design process. As each choice is pursued in an effort to improve the probability density function $p(x)$, facilities and personnel are involved in pursuing each of the various choices and delays are caused beyond that estimated. Each of these factors can dilute the contractor's profit on investment that he will try to avoid unless there is some other profit to be gained. How this cost is related to $p(x)$ is the subject of the remainder of this section.

Several methods were investigated for the purpose of establishing the real cost function; however, only one of these methods proved to be useful. The various unsuccessful methods used and the reasons for their rejection are presented in the following paragraphs.

An attempt at a microscopic point of view led to the consideration that the unperturbed, natural, or no-effort result is caused by many identifiable random factors. The contractor tries to modify parameters of the probability density function of each factor to obtain improvement of his product. When these factors are combined in a linear fashion, the natural probability density function $p_0(x)$ will have the form

$$p_0(x) = p_i(x) * p_2(x) * \cdots p_n(x) \tag{4}$$

where $p_i(x)$ is the probability density function of the ith contributor and * indicates convolution. When the first j contributors are removed by design, modification, and adjustment, the modified probability density function would then be

$$p(x) = F^{-1}\left\{\frac{F[p_1(x)]\,F[p_2(x)]\cdots F[p_n(x)]}{F[p_1(x)]\,F[p_2(x)]\cdots F[p_j(x)]}\right\}$$

$$= F^{-1}\left\{F[p_{j+1}(x)]\cdots F[p_n(x)]\right\} \tag{5}$$

where $F[p_i(x)]$ is the Fourier Transform of $p_i\,(x)$.

A cost function that might be envisioned would be a function of j such that cost $= f(j)$. Much further consideration was deemed fruitless because (1) the factors are not believed to be even vaguely linear, (2) mathematics other than for linear relations would be too cumbersome, and (3) the results would be too particular and difficult to obtain.

Another method was to consider the improvement of the parameter probability density function through design, selection, and adjustment as a Markov process. Then,

$$p(x) = p_0(x)\mathbf{P}^j \tag{6}$$

where $p_0(x)$ is again the natural probability density function and $\mathbf{P}$ is the Markov transition matrix. Again, a cost function that might be envisioned would be a function of j, the number of steps in the Markov process, and cost $= f(j)$. This process has the advantage of making it clear that each design, selection, and adjustment step, although it contributes to the improvement, does not yield a certainty in the result and, if j is large enough, $p(x)$ is not too sensitive to the assumed form of $p_0(x)$. However, it is difficult to specify a meaningful transition matrix $\mathbf{P}$. (This method was not considered seriously enough to argue the continuous case for x.)

The microscopic point of view was generally abandoned in favor of a more gross or macroscopic point of view. The next method involved the use of a statistical average as was used for the reward and surveillance functions. Assume that expected cost is equal to $f\{E[g(x)]\}$, where $g(x)$ is some function relating to cost.

One of the first forms of this type that was investigated assumed that the cost would be a function of the standard deviation σ of the probability density function such as, expected cost $\propto 1/\sigma$.

It appeared reasonable that the narrower, or the more peaked, the probability density function, or the smaller the standard deviation of the probability density function, the more the cost should be. These conditions are generally correlated. Although this reasoning is tenable, "pathological" examples can be illustrated to show that a function of the standard deviation σ is not a proper measure of cost for this analysis and it also appears untractable.

One would expect the probability density function shown in Fig. 3a to have a finite cost and that shown in Fig. 3b to have an infinite cost. These costs would appear to be consistent with the three factors listed above. If one were to decompose these probability density functions into those depicted in Fig. 4, one would again expect the probability density function shown in Fig. 4a to have a finite cost (equal to that of the previous example) and that shown in Fig. 4b to have an infinite cost. The fact that both have finite standard deviations in contrast to the previous examples would indicate that the standard

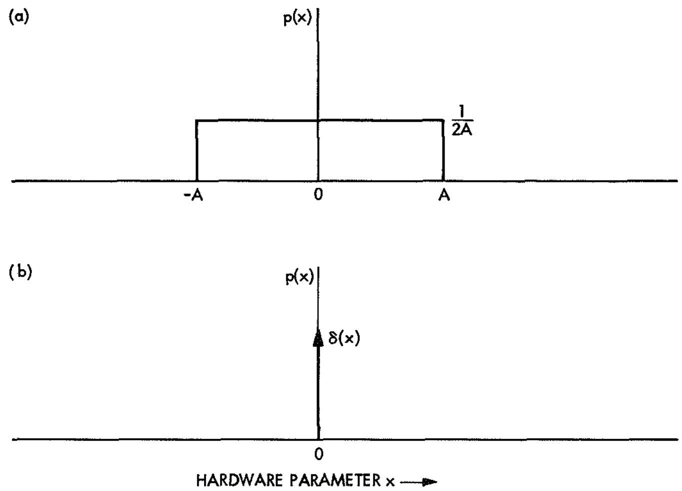

Fig. 3. Probability density functions, unimodal: (a) finite cost; (b) infinite cost

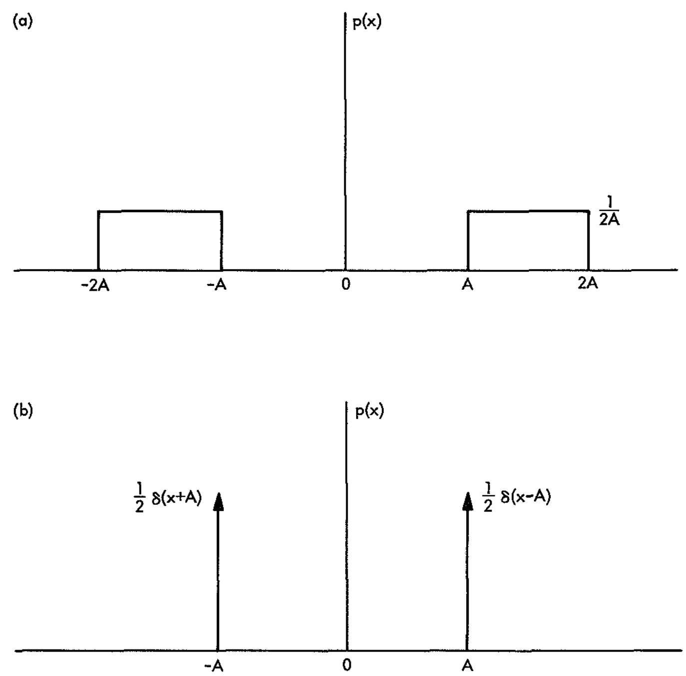

Fig. 4. Probability density functions, bi-modal: (a) finite cost; (b) infinite cost

deviation should not be used as a measure of the cost. It does, however, still appear to be related to the probability density function in its value and degree of spreading. Whether or not the values are clustered, or whether or not the probability density function is decomposed with translations, seems to be irrelevant to the cost. This leads to the belief that the cost function is a function of the probability density function itself and not the parameter x explicitly. The value of x has no bearing on the cost. Only the relative number that occurs at that value determines the cost. This same type of explanation was used for justification of the weighting function for information theory (Ref. 5). In measuring the information content, one can interpret the function $-\ln[p(x)]$ as being another measure of spread other than standard deviation.

Woodward (Ref. 5) explains the use of $-\ln[p(x)]$ as a measure of spread for attributes that are merely qualitative; unfortunately, it cannot be used as the cost function, since it yields results opposite to those outlined above. To obtain an alternate cost function, consider that the expected cost

$$C_p = \int g[p(x)]\, p(x) dx$$

contains a satisfactory function $g[p(x)]$ that is to be determined. The four characteristics or axioms that $g[p(x)]$ should satisfy are as follows:

(1) Continuity: The functional C_p as defined in the above integral is continuous in $p(x)$.

(2) Positive Bound: For any nonrestricted family of $p(x)$ (nonrestricted in the sense that the variable x is not bounded), $C_p = 0$ when

$$p(x) = \lim_{R \to \infty} \frac{1}{2R}, \qquad -R \leq x \leq R$$

For any restricted family of $p(x)$, C_p is a minimum when

$$p(x) = \frac{1}{B - A}, \qquad A \leq x \leq B$$

In both cases, $C_p \to \infty$ when $p(x) = \delta(x)$.

(3) Monotonicity: The function $g[p(x)]$ is a monotonic increasing function of $p(x)$; therefore, $g[p(x)] = 0$ when $p(x) = 0$, and $g[p(x)] \to \infty$ when $p(x) \to \infty$.

(4) Uniqueness: The functional C_p is unique with respect to $p(x)$.

There is an unlimited class of functions that satisfies these axioms, such as: $g[p(x)] = p^N(x)$, where N is a positive integer and $g[p(x)] = e^{p(x)} - 1$. To be more definitive in specifying a satisfactory function, define a differential cost in the neighborhood of x_0 as follows:

$$\Delta C_p = \int_{x_0 - \Delta x}^{x_0 + \Delta x} g[p(x)]\, p(x) dx \qquad (7)$$

and further specify that the rate of change of ΔC_p with respect to $p(x)$ be proportional to $p(x_0)$ or

$$\frac{d\Delta C_p}{dp(x)} = Kp(x_0) = \frac{d}{dp(x)} \int_{x_0-\Delta x}^{x_0+\Delta x} g[p(x)]\, p(x) dx \qquad (8)$$

With the relationship (Ref. 6)

$$f = \int_a^b F(x,y)dx$$

$$\frac{df}{dy} = \int_a^b \frac{\partial F(x,y)}{\partial y} dx - F(a,y,) \frac{\partial a}{\partial y} + F(b,y) \frac{\partial b}{\partial y}$$

then

$$Kp(x_0) = \int_{x_0-\Delta x}^{x_0+\Delta x} \left\{ g[p(x)] + p(x) \frac{\partial}{\partial p(x)} g[p(x)] \right\} dx$$

Now if

$$g[p(x)] = p(x)$$

then

$$Kp(x_0) = \int_{x_0-\Delta x}^{x_0+\Delta x} [p(x) + p(x)]dx$$

$$Kp(x_0) = p(x_0) \int_{x_0-\Delta x}^{x_0+\Delta x} 2dx = p(x_0)\, 4\Delta x$$

Therefore, $g[p(x)] = p(x)$ satisfies the above statement. On this basis, the required expected cost is

$$C_p = \int k_c p^2 (x) dx$$

In later sections of the report, this function will be shown to be tractable.

The above function can be interpreted as the average of the probability density function (first moment about the x axis) or clustering of the random variable x.

The arguments previously used are perfectly valid; however, they do not fully eliminate the strong intuitive feeling one has about the usefulness of the k_c/σ cost function. In reconsideration of this intuitive feeling, a study was made of a report by R. M. Stewart.* The cost function used in that report is of the form

$$C = \frac{k}{\sigma^\alpha}$$

where k and α are positive constants. This function was found to be quite useful in Stewart's report and can be interpreted very simply in that analysis.

The function $1/\sigma$ is very strongly entrenched as a measure of cost, yet E(cost) was the only form found tractable for the purpose of this analysis. Both functions have been said to be measures of spread of the probability density function and both have been shown to be useful for different purposes. This fact leads to the questions: (1) Are the two functions measures of spread? (2) Are they always or sometimes equivalent? (3) Is one more general than the other? (4) Is it possible that these two functions are not equivalent, yet are used to measure the same term for different purposes? To answer these questions, five specific examples of rather elementary probability density functions were examined. This examination expresses E(cost) in terms of $1/\sigma$.

As the first example, consider the Gaussian probability density function of

$$p(x) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

Evaluate then,

$$E_1(\text{cost}) = k_c \int_{-\infty}^{\infty} p^N(x)\, p(x) dx$$

with the Gaussian probability density function.

Now

$$E_1(\text{cost}) = 2k_c \left[\frac{1}{(2\pi)^{1/2}\sigma} \right]^{N+1} \int_0^\infty e^{-\frac{(N+1)x^2}{2\sigma^2}} \cdot d_x$$

But

$$\int_0^\infty e^{-r^2x^2} dx = \frac{(\pi)^{1/2}}{2r}$$

If

$$r = \frac{1}{\sigma} \left(\frac{N+1}{2} \right)^{1/2}$$

*"Component Cost and Accuracy Allocations in Guidance System Design," JPL Section Report 12-129, Nov. 16, 1953.

then

$$E_1(\text{cost}) = \frac{k_c}{(2\pi)^{N/2}(N+1)^{1/2}}\left(\frac{1}{\sigma^N}\right) = K_1(N)\left(\frac{1}{\sigma^N}\right) \tag{9}$$

which is the same form as used by Stewart.

For the special case of $N = 1$, then

$$E_1(\text{cost}) = \frac{k_c}{2(\pi)^{1/2}}\left(\frac{1}{\sigma}\right)$$

which (for this particular example) is equivalent to that used previously but found to be untractable in general.

As a second example, consider the rectangular probability density function of

$$p(x) = \frac{1}{2A}, \qquad -A \le x \le A$$

$$= 0, \qquad \text{elsewhere}$$

Now

$$E_2(\text{cost}) = k_c \int_{-A}^{A} \left(\frac{1}{2A}\right)^{N+1} dx$$

$$= k_c \left(\frac{1}{2A}\right)^N$$

But

$$\sigma^2 = \int_{-A}^{A} x^2 \left(\frac{1}{2A}\right) dx = \frac{A^3}{3}$$

or

$$A = (3)^{1/2}\sigma$$

Then

$$E_2(\text{cost}) = k_c\left[\frac{1}{2(3)^{1/2}}\right]^N \frac{1}{\sigma^N} = K_2(N)\frac{1}{\sigma^N} \tag{10}$$

As a third example, consider the exponential probability density function of

$$p(x) = \lambda e^{-\lambda x}, \qquad 0 \le x \le \infty$$

$$= 0, \qquad \text{elsewhere}$$

Now

$$E_3(\text{cost}) = k_c \int_0^{\infty} (\lambda e^{-\lambda x})^{N+1} dx$$

$$= k_c \lambda^{N+1} \int_0^{\infty} e^{-(N+1)\lambda x} dx$$

$$= \frac{k_c \lambda^N}{N+1}$$

But

$$\sigma^2 = E(x^2) - E^2(x)$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx - \left(\int_0^{\infty} x\lambda e^{-\lambda x} dx\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

or

$$\sigma = \frac{1}{\lambda}$$

Then

$$E_3(\text{cost}) = \frac{k_c}{(N+1)}\left(\frac{1}{\sigma^N}\right) = K_3(N)\frac{1}{\sigma^N} \tag{11}$$

As a fourth example, consider the triangular probability density function of

$$p(x) = \frac{2x}{B^2}, \qquad 0 \le x \le B$$

$$= 0, \qquad \text{elsewhere}$$

Evaluate then

$$E_4(\text{cost}) = k_c \int_0^{B} \left(\frac{2x}{B^2}\right)^{N+1} dx$$

$$E_4(\text{cost}) = k_c \left(\frac{2}{B^2}\right)^{N+1} \frac{B^{N+2}}{N+2}$$

$$E_4(\text{cost}) = k_c \frac{2^{N+1}}{N+2}\left(\frac{1}{B^N}\right)$$

But

$$\sigma^2 = E(x^2) - E^2(x)$$

$$= \int_0^B x^2 \frac{2x}{B^2}\,dx - \left(\int_0^B x \frac{2x}{B^2}\,dx\right)^2$$

$$= \frac{B^2}{2} - \left(\frac{2}{3}\right)^2 B^2$$

$$= \frac{B^2}{2(9)}$$

or

$$B = 3(2)^{1/2}\sigma$$

Then

$$E_4(\text{cost}) = k_c \frac{2^{N+1}}{(N+2)}\left[\frac{1}{3(2)^{1/2}}\right]^N \frac{1}{\sigma^N} = K_4(N)\frac{1}{\sigma^N} \tag{12}$$

As a fifth and final example, consider the triangular probability density function of

$$p(x) = \frac{1}{c^2}(x+c), \qquad -c \le x \le 0$$

$$= \frac{-1}{c^2}(x-c), \qquad 0 \le x \le c$$

$$= 0, \qquad \text{elsewhere}$$

Now

$E_5(\text{cost}) =$

$$k_c \left\{\int_{-c}^0 \left[\frac{1}{c^2}(x+c)\right]^{N+1} dx + \int_0^c \left[\frac{-1}{c^2}(x-c)\right]^{N+1} dx\right\}$$

Let

$$y = (x+c)$$

Then

$$dy = dx$$

and when

$$x = 0, \qquad y = c$$

$$x = -c, \qquad y = 0$$

Let

$$z = (x-c)$$

Then

$$dz = dx$$

and when

$$x = 0, \qquad z = -c$$

$$x = c, \qquad z = 0$$

Now

$$E_5(\text{cost}) = \frac{k_c}{c^{2N+2}}\left[\int_0^c y^{N+1}\,dy + (-1)^{N+1}\int_{-c}^0 z^{N+1}\,dz\right]$$

$$= \frac{k_c}{(N+2)c^{2N+2}}\left[y^{N+2}\big|_0^c + (-1)^{N+1} z^{N+2}\big|_{-c}^0\right]$$

$$= \frac{kc}{(N+2)c^{2N+2}}\left[c^{N+2} - (-1)^{N+1}(-c)^{N+2}\right]$$

$$= \frac{2k_c\, c^{N+2}}{(N+2)\,c^{2N+2}} = \frac{2k_c}{(N+2)c^N}$$

and σ^2 can be readily obtained by noting that the triangular probability density function of this example is the convolution of the probability density function of the second example with itself when $2A = C$. Since σ_2^2 was $A^2/3$ for the second example, then

$$\sigma_5^2 = 2\sigma_2^2 = \frac{2}{3}A^2$$

where

$$2A = C$$

or

$$\sigma_5^2 = \frac{c^2}{2(3)}$$

$$C = \sigma(2 \times 3)^{1/2}$$

Then

$$E_5(\text{cost}) = \frac{2k_c}{(N+2)[(2)(3)]^{N/2}\,\sigma^N} = K_5(N)\frac{1}{\sigma^N} \tag{13}$$

Table 1 summarizes the results obtained with the five types of probability density functions. It is obvious that all the results are equal within a multiplicative constant and that they are of the form intuitively thought to be reasonable. In the general form, the results are identical to those proposed by Stewart and, in the restricted case of $N = 1$, they are identical to those discussed previously but found to be untractable in general. This demonstration of equivalence for these five samples shows a compatibility between the two concepts that suppresses most of the criticism of the E(cost) function.

In retrospect, it seems that the equivalence in the above examples could have been predicted, because the probability density functions that were used as examples can all be expressed uniquely in terms of μ and σ as parameters. It is believed that any probability density function that does not change its form with σ, such as the examples above, will have an equivalence to E(cost). However, E(cost) is believed to have a broader meaning since it encompasses, as tractable, pathological cases not covered by $1/\sigma$. With these pathological cases, it has been shown that the cost functions are not always equivalent. With this fact in mind, there should be no difficulty in accepting E(cost) in the analysis by Stewart.

Table 1. Summary of results

Type of probability density function	E_i (cost, N)
Gaussian	$\frac{k_c}{(2\pi)^{N/2}(N+1)^{1/2}}\left(\frac{1}{\sigma^N}\right)$
Rectangular	$\frac{k_c}{[2(3)^{1/2}]^N}\left(\frac{1}{\sigma^N}\right)$
Exponential	$\frac{k_c}{(N+1)}\left(\frac{1}{\sigma^N}\right)$
Right triangle	$\frac{k_c(2^{N+1})}{(N+2)[3(2)^{1/2}]^N}\left(\frac{1}{\sigma^N}\right)$
Symmetric triangle	$\frac{2k_c}{(N+2)[(2)(3)]^{N/2}}\left(\frac{1}{\sigma^N}\right)$

Figure 5 shows E(cost) normalized by k_c/σ^N with N as a variable; the purpose of this is to determine the power N that makes all the cost factors the most uniform for all probability density functions. As can easily be seen, $N = 1$ is the best power for this purpose since the normalized E(cost) factors are within a factor of 2 for the cases analyzed.

VIII. Methods of Determining Probability Density Function

A. Calculus of Variations Technique

To obtain the maximizing probability density function $p(x)$ of Eq. (3), techniques of the calculus of variations may be applied. The calculus of variations is a mathematical method by which unknown functions that maximize or minimize certain integral expressions can be determined.

The manipulation of the calculus of variations is relatively simple. For a specific integral of the form

$$I = \int_{x_1}^{x_2} F(x,y,y')dx \tag{14}$$

where the integrand is twice-differentiable, a necessary condition for $y(x)$ to maximize (or minimize) the integral, if such a function exists, is that it satisfy Euler's equation (Ref. 7)

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \tag{15}$$

In the special case when the integrand is independent of y', the conditions

$$\frac{\partial F}{\partial y} = 0, \qquad \frac{\partial^2 F}{\partial y^2} \leq 0 \tag{16}$$

generally hold for maximizing (minimizing) the integral (Ref. 8).

When additional constraints such as

$$\int_{x_1}^{x_2} G_i(x,y,y')dx = K_i \tag{17}$$

are imposed, a modified integral may be formed such that

$$I^* = \int_{x_1}^{x_2} F^*(x,y,y')dx$$

$$= \int_{x_1}^{x_2}\left(F + \sum_{i=1}^{n}\lambda_i G_i\right)dx \tag{18}$$

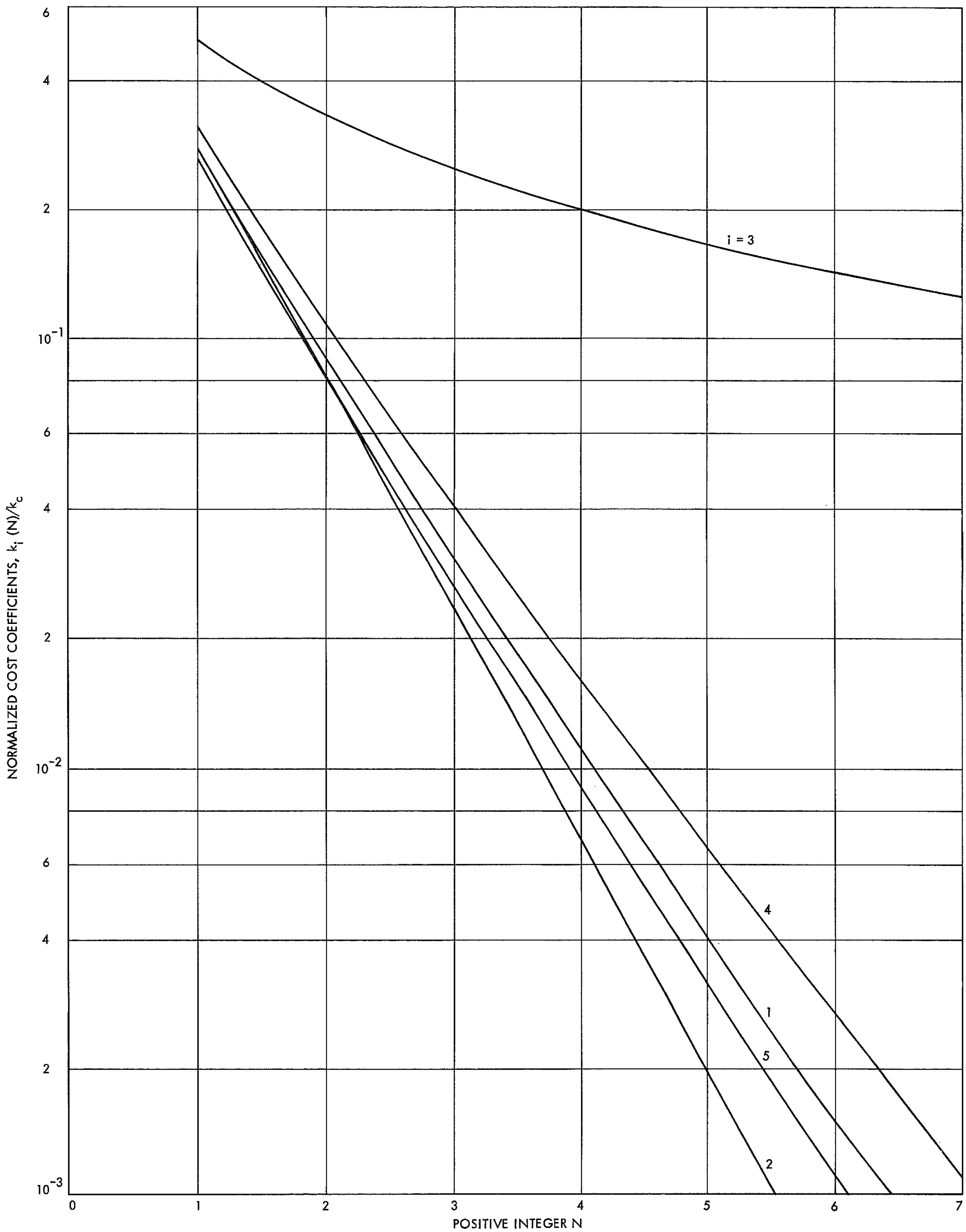

Fig. 5. Normalized cost coefficients vs positive integers

and this integral may be maximized with the prescribed method as though there were no constraints (Ref. 9). In Eq. (18), λ_i are constants known as Lagrange multipliers. These constants can be evaluated in such a way that the maximizing function satisfies all of the constraints as indicated by Eq. (17).

It should be mentioned that in the calculus of variations, the sufficient conditions for maxima (or minima) are difficult to derive. For many instances, however, numerical checks can be performed to show that the indicated results are indeed the desired optimizing function.

The integral equation in its general form, from which the maximizing probability density function is derived, can now be formulated. Thus,

$$I^* = \int_A^B [k_r r(x) - k_s s(x) - k_c p(x) + \lambda] p(x) dx \qquad (19)$$

subject to the conditions

$$\int_B^A p(x)dx = 1, \qquad p(x) \geq 0$$

This integral can be used to determine the Lagrange multiplier λ. These constraints are necessary to make the resultant $p(x)$ a proper probability density function.

The conventional techniques of the calculus of variations create a special problem when they are applied to probability problems because the derivation for the special problem does not include consideration of the inequality constraint. As a consequence, the resultant functional $p(x)$ comes from a class of both positive and negative functions and, therefore, may violate the inequality constraint $p(x) \geq 0$. To eliminate this difficulty, an operator $U(p)$ is introduced that has the following properties:

$$U(p) = 1, \qquad \text{if } p(x) > 0$$

$$U(p) = 0, \qquad \text{if } p(x) \leq 0$$

$$U(-p) = 1, \qquad \text{if } p(x) < 0$$

$$U(-p) = 0, \qquad \text{if } p(x) \geq 0$$

The functional $p(x)$ is then written as the sum of a positive functional $U(p)p(x)$ and a negative functional $U(-P)p(x)$ or

$$p(x) = U(p)p(x) + U(-p)p(x)$$

Hence, the desired probability density function is represented as the non-negative function $U(p)p(x)$ that satisfies the inequality constraint, rather than $p(x)$, which does not.

The use of $U(p)$ does not violate the basic considerations of the calculus of variations. It merely serves to sift out the positive functional from all functionals that satisfy the basic integral constraint. In the examples that follow, it will be demonstrated that $U(p)$ modifies the valid range or ranges of definition of the integrals in which the resultant $p(x)$ satisfies both constraints. In these ranges, techniques in elementary calculus of variations are applicable, as if $U(p)$ were not included; with this treatment, Eq. (19) becomes

$$I^* = \int_A^B [k_r r(x) - k_s s(x) - k_c p(x)U(p) + \lambda] p(x)U(p) dx \qquad (20)$$

subject to the single constraint

$$\int_A^B p(x)U(p)dx = 1$$

The associated Euler equation then becomes

$$\frac{\partial F^*}{\partial[U(p)p(x)]} - \frac{d}{dx}\frac{\partial F^*}{\partial[U(p)p(x)]'} = 0$$

which, for the case of concern to this problem, is

$$\frac{\partial F^*}{\partial[U(p)p(x)]} = 0$$

The treatment of the inequality constraint used herein is believed to be unique. It was developed because the only other technique known was the intuitive technique discussed in the following section. The intuitive technique shall be referred to as the area subtraction method. Another technique, developed by B. O. Koopman, was discovered later (Ref. 2). The equivalence of these three methods will be discussed in detail.

B. Area Subtraction Method (Graphical Solution)

Although the desired probability density function for any given strategy is readily obtainable by the prescribed analytical methods, the established procedures are not best suited for the generation of a computer program fitting all conceivable strategies. The difficulty is in the determination of the region in which the probability density function is valid (i.e., range for $U(p) = 1$). The ease of such determination varies depending upon the strategic function. The area subtraction method can overcome this difficulty and the step-by-step procedure for this method is as follows:

(1) An unrestricted probability density function $p(x)$ (unrestricted in the sense that $p(x)$ can assume negative values) is obtained by the conventional calculus of variations methods.

(2) This probability density function is checked over the entire permissible range R for negative $p(x)$.

(3) If $p(x)$ is non-negative in R, it becomes the desired probability density function.

(4) If $p(x)$ is negative in some region in R, the probability density function is to be modified by seeking a baseline P_b in such a manner that the total area in R bounded by $p(x)$ (upper bound) and P_b (lower bound) is equal to unity. This can be done by way of successive approximation.

(5) The desired probability density function in this case becomes

$$p^*(x) = p(x) - P_b, \qquad p(x) > P_b$$
$$= 0, \qquad \text{otherwise} \tag{21}$$

for all values of x in R.

This method was first conceived intuitively without rigorous mathematical proof. It is argued that the probability density function obtained directly from the conventional calculus of variations techniques has the form

$$p(x) = K\,[r(x) + \lambda] \tag{22}$$

in which the Langrange multiplier λ is a function of $\int_R r(x)dx$, where $r(x)$ is the general strategic or reward function. The constraint $p(x) \geq 0$ is violated when the minimum value of $r(x)$ is less than $(-\lambda)$. With the triangular incentive award case as an example, this situation is presented in Fig. 6.

Both Fig. 6 and Eq. (22) suggest that the modified probability density function $p^*(x)$ must assume the same form of $r(x)$ (Fig. 7) although the characteristic constants of the probability density function (in this particular example the limit value X) cannot be readily determined.

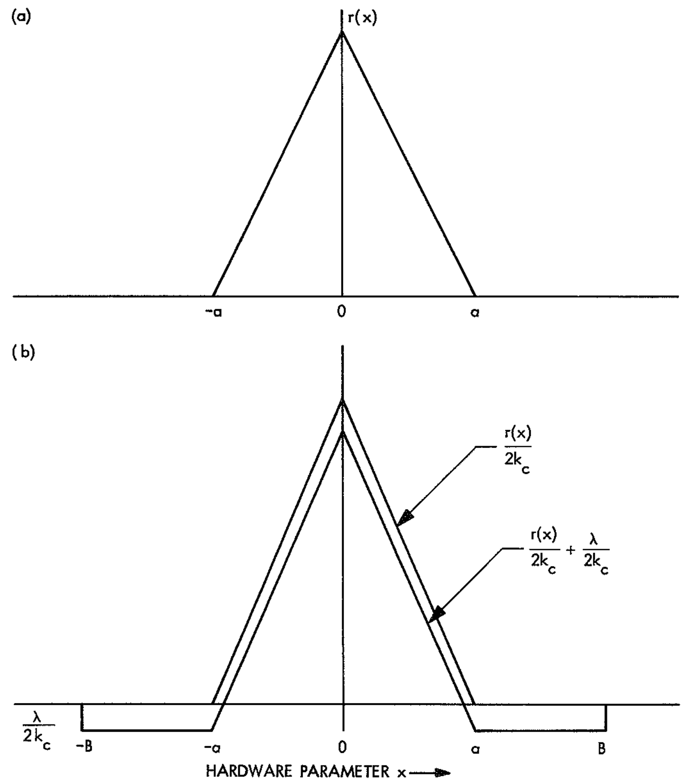

Fig. 6. Demonstrated example for violation of $p(x) \geq 0$: (a) strategic function; (b) $p(x)$ obtained through calculus of variations method

However, it can be realized that the slopes of the curve $p^*(x)$ must be the same as those of $p(x)$, and that the area under $p^*(x)$ must be equal to unity. The algorithm that satisfies both of these requirements is to subtract the area below the value $p(x) = 0$ from the area above $p(x) = 0$ by establishing the baseline P_b in the manner previously discussed. This method is graphically presented in Fig. 8.

From the geometry of the curve given in Fig. 8, the following relationship can be established.

$$\int_{-X}^{X} p(x)dx = 1 + 2Xp(X) \tag{23}$$

and the value X can be evaluated from this equation. This same approach is also applicable to other forms of $r(x)$ and is not restricted to linear symmetrical functions. The remaining question is whether the value of X so computed is indeed the desired characteristic constant that maximizes the expected payoff. The answer can be determined if the results of this method are compared to the analytical method described in the preceding

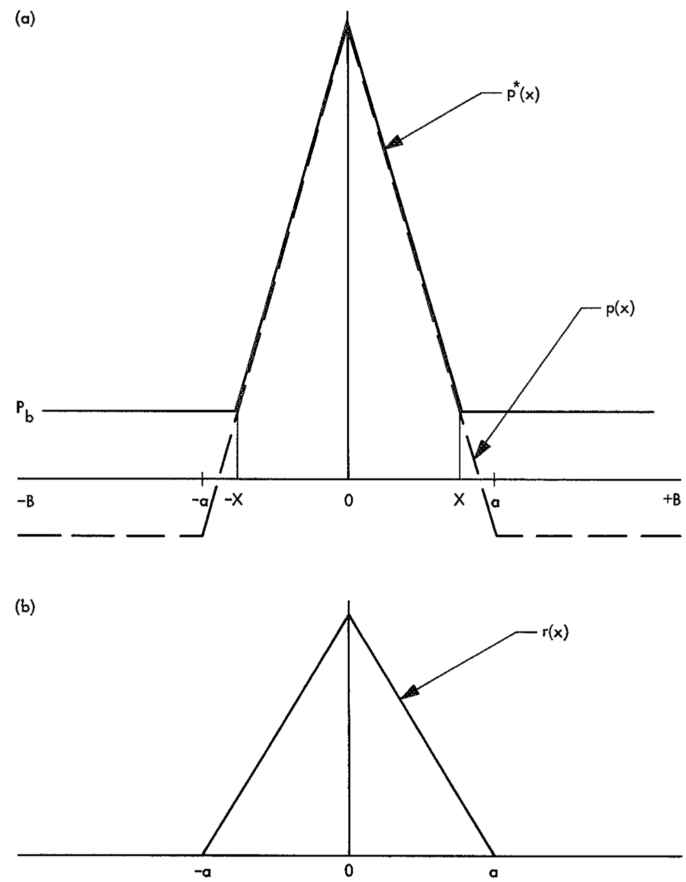

Fig. 7. Relationship between $p^*(x)$ and $r(x)$: (a) obtained and modified distributions; (b) strategic function

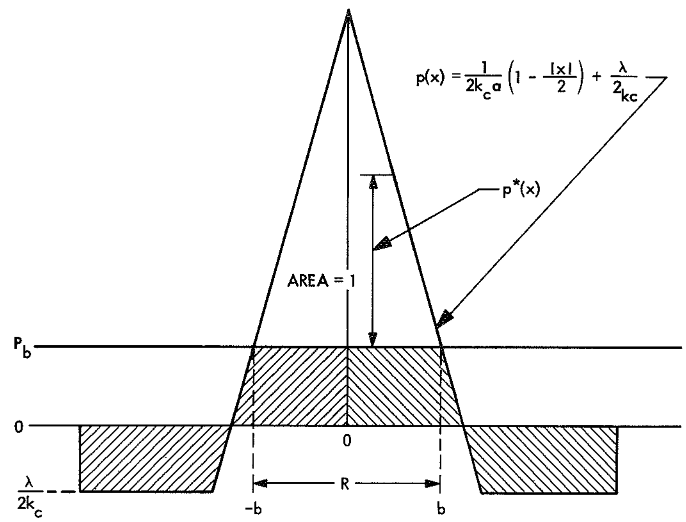

Fig. 8. Graphical presentation of area subtraction method

section, or if the integration of the expected payoff utilizing $p^*(x)$ is carried out and if the maximizing value of X is determined through the conventional calculus methods. Results of several example problems confirm the validity of the area subtraction method at least for those strategic functions considered.

C. Analogous Methods in the Search Problem

An article by B. O. Koopman (Ref. 2) discusses a problem in which an optimum distribution of search effort is sought. This problem is very similar to the one presented here, as far as mathematical application is concerned. In Ref. 2, a distribution of search effort $\varphi(x)$ that maximizes the overall probability of detecting a target is sought, given the probability density function of the location of the target. Since effort cannot be meaningfully measured in negative terms, a constraint of the problem is that

$$\int \varphi(x)dx = \Phi, \qquad \varphi(x) \geq 0$$

This constraint is comparable to the condition of the current problem (Eq. 19).

To solve this problem, Koopman devised a graphical method that is in essence the same as the area subtraction method. Mathematical treatment is also given in Ref. 2 to prove the validity of the graphical solution and to show the general solution of that problem.

It is encouraging to note that by applying the analytical method utilizing the operator $U(\varphi)$ to the search problem, the same resultant probability density function as realized by Koopman is obtained.

In the following, the analogy between the three methods is shown utilizing a general form of expected payoff function $E[N(x)]$.

*1. **Solution by the operator U(p) method.*** In the operator method, an operator $U(p)$ is used with the probability density function $p(x)$ to form a restricted distribution. The operator has the following properties:

$$U(p) = 1, \qquad \text{if } p(x) > 0$$

$$U(p) = 0, \qquad \text{if } p(x) \leq 0$$

Therefore, the following equations may be written

$$E[N(x)] = \int_R N[x, p(x)U(p)]dx \tag{24}$$

$$\int_{U(p)=1} p(x)U(p)dx = 1 \tag{25}$$

With this modification, the condition $p(x) \geq 0$ becomes an intrinsic part of the probability density function and the maximizing probability density function is limited to this class of functions.

Without loss of generality, the incentive award payoff function is used as an example to obtain a specific probability density function for comparison purposes. Thus,

$$E[N(x)] = \int_{-B}^{B} [k_r r(x) - k_c p(x)U(p)]p(x)U(p)dx$$

With the previously prescribed techniques, the following can be written:

$$F^* = k_r r(x)p(x)U(p) - k_c[p(x)U(p)]^2 + \lambda p(x)U(p)$$

and

$$\frac{\partial F^*}{\partial[p(x)U(p)]} = k_r r(x) - 2k_c p(x)U(p) + \lambda = 0$$

Rearranged,

$$p(x)U(p) = \frac{k_r r(x)}{2k_c} + \frac{\lambda}{2k_c}$$

The term $\lambda/2k_c$ may be evaluated according to the condition given in Eq. (25). Hence,

$$\frac{\lambda}{2k_c}\int_{U(p)=1} dx = 1 - \int_{U(p)=1} \frac{k_r r(x)}{2k_c} dx$$

Let us designate R as the area where $U(p) = 1$ within the permissible range $-B \leq x \geq B$, then

$$\frac{\lambda}{2k_c} = \frac{1}{R}\left[1 - \frac{k_r}{2k_c}\int_R r(x)dx\right]$$

Consequently,

$$p(x)U(p) = \frac{k_r r(x)}{2k_c} + \frac{1}{R}\left[1 - \frac{k_r}{2k_c}\int_R r(x)dx\right] \tag{26}$$

which is the desired probability density function.

The remaining problem is to determine the zero crossings of $p(x)$, if they exist, and to establish the valid region of R. This can be readily performed, given $r(x)$ and the relative magnitude of the associated constants. The complete solution and derivation of this probability density function are given in Section IX.

2. Solution by the area subtraction method. The area subtraction method is a numerical process for the solution of the desired probability density function. Concisely, the desired probability density function is a modified version of the probability density function directly obtained by the conventional calculus of variations techniques through the following relationship:

$$\begin{aligned} p^*(x) &= p(x) - P_b, \qquad p(x) > P_b \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{27}$$

for all values of x within the permissible range. Here, $p^*(x)$ is the desired probability density function, and P_b is a constant. Notice that the following conditions are true:

$$\int_R p^*(x)dx = \int_R [p(x) - P_b]dx = 1$$

$$\int p(x)dx = 1$$

The relationship among the three terms in Eq. (27) is shown pictorially in Fig. 8.

With the same incentive award function, the following can be written

$$\begin{aligned} p(x) &= \frac{k_r r(x)}{2k_c} + \frac{1}{2B}\left[1 - \frac{k_r}{2k_c}\int_{-B}^{B} r(x)dx\right] \\ &= \frac{k_r r(x)}{2k_c} + Q \end{aligned} \tag{28}$$

where Q is a constant.

Since, within the region R,

$$P_b = p(x) - p^*(x)$$

then

$$\int_R P_b dx = \frac{k_r}{2k_c}\int_R r(x)dx + \int_R Q dx - \int_R p^*(x)dx$$

or

$$P_b = \frac{k_r}{2k_c R}\int_R r(x)dx + Q - \frac{1}{R} \tag{29}$$

Now, when Eqs. (28) and (29) are substituted into Eq. (27), the result is

$$p(x) = \frac{k_r r(x)}{2k_c} + \frac{1}{R}\left[1 - \frac{k_r}{2k_c}\int_R r(x)dx\right] \tag{30}$$

which is identical to Eq. (26). Therefore, the proof of equivalence between the operator $U(p)$ method and the area subtraction method is complete.

3. ***Solution by the Koopman method.*** Koopman sought a distribution of search effort $\varphi(x)$ that maximized the probability of P_s of detecting a target that is given by

$$P_s = \int f_1(x)[1 - e^{-\varphi(x)}]dx$$

subject to

$$\int \varphi(x)dx = \Phi, \qquad \varphi(x) \geq 0$$

In this problem, $f_1(x)$ is the probability density function of the target location that is assumed to be known; and Φ is a constant representing the total allotted search effort.

In solving this problem, Koopman has formally introduced a mathematical treatment utilizing a "special variation" from which a condition analogous to the classical Euler equation can be derived. This treatment was later generalized by S. Zahl (Ref. 10) who reformulated the problem in the form:

Maximize

$$P_s = \int f[x,\varphi(x)]dx$$

subject to

$$\int g[x,\varphi(x)]dx = \Phi, \qquad \varphi(x) \geq 0$$

The results are presented in the following theorems:

(1) A necessary and sufficient condition that a function $\varphi^*(x)$ maximizes P_s is that there exist a constant λ such that for almost everywhere, $\varphi^*(x)$ maximizes

$$f[x,\varphi(x)] - \lambda g[x,\varphi(x)], \qquad \varphi(x) > 0$$

(2) Subject to certain conditions, there exists an element satisfying the condition in (1) (existence of a solution).

If (1) is applied to the specific problem of Koopman, and if, as before, R is designated as the area where $\varphi(x) > 0$, then the function to be considered in the maximization is

$$f_1(x)[1 - e^{-\varphi(x)}] - \lambda\varphi(x)$$

which results in

$$\frac{\partial}{\partial\varphi}\{f_1(x)[1 - e^{-\varphi(x)}] - \lambda\varphi(x)\} = 0$$

Accordingly,

$$f_1(x)\, e^{-\varphi(x)} = \lambda \tag{31}$$

Taking the logarithm of Eq. (31) gives

$$\varphi(x) = \log f_1(x) - \log\lambda \tag{32}$$

in which $\log\lambda$ can be determined by

$$\int_R \varphi(x)dx = \int_R [\log f_1(x) - \log\lambda]dx = \Phi$$

so that

$$\log\lambda = \frac{1}{R}\left[\int_R \log f_1(x)dx - \Phi\right]$$

By substitution, then

$$\varphi(x) = \log f_1(x) + \frac{1}{R}\left[\Phi - \int_R \log f_1(x)dx\right] \tag{33}$$

Equation (32) clearly indicates that $\varphi(x)$ can be obtained graphically in a manner similar to the area subtraction method. A comparison of Eq. (33) with Eq. (26) clearly indicates the analogy between the operator $U(p)$ method and the Koopman method. It should be noted that the logarithmic terms in Eq. (33) are the result of the form $f[x,\varphi(x)]$ chosen in the problem. If the latter method were applied to the problem presented here, an identical solution would have been obtained.

It is obvious that the three methods for the solution of the desired probability density function produce equivalent results. Although the Koopman method is the most elegant, the operator $U(p)$ method is the easiest to explain because it clearly defines the problem in a simple, direct way. For computational simplicity, however, the area subtraction method appears to be the most logical choice because it can be readily adapted to a computer for solution and presentation.

IX. Example Solutions

Now that the problem has been formulated and the method for solution has been developed, example problems should be contrived and solved. The theme of the examples will be: first, to devise isolated strategies so the sensitivity of these strategies may be understood; and second, to devise the more difficult mixed strategies that more closely resemble a true situation so that the sensitivity of these mixed strategies may be understood. These examples will be the basis for conclusions discussed in the succeeding section.

A. Isolated Strategies

1. Incentive award only. The first example to be described is the isolated strategy of offering an incentive award based on delivery of a measured hardware parameter. Here, as in all of the examples to follow, all direct costs are considered reimbursable; they do not enter into the analysis.

Consider that the incentive award formula is

$$r(x) = \frac{1}{a}\left(1 - \frac{|x|}{a}\right), \qquad 0 \leq |x| \leq a$$

that the triangular function is motivated as explained previously, and that the parameter x must be within the imposed specification limits $\pm B$ where B is greater than a. The coefficient $1/a$ in $r(x)$ adjusts the area under $r(x)$ equal to unity for convenience. With these conditions, the expected payoff is

$$E[N(x)] = \int_{-B}^{B} [k_r r(x) - k_c p(x)U(p)]p(x)U(p)dx \tag{34}$$

This formula includes the cost $k_c p(x)$ that is considered to be always with the contractor.

According to the prescribed techniques described in Section VIII,

$$F^* = k_r r(x)p(x)U(p) - k_c[p(x)U(p)]^2 + \lambda p(x)U(p) \tag{35}$$

and

$$\frac{\partial F^*}{\partial[U(p)p(x)]} = k_r r(x) - 2k_c p(x)U(p) + \lambda = 0 \tag{36}$$

The probability density function

$$p(x)U(p) = \frac{k_r r(x)}{2k_c} + \frac{\lambda}{2k_c} \tag{37}$$

results from Eq. (36).

Because

$$\int_{-B}^{B} p(x)U(p)dx = 1$$

it follows that

$$\int_{\substack{U(p)=1 \\ (x)\leq B}} \left[\frac{k_r r(x)}{2k_c} + \frac{\lambda}{2k_c}\right] dx = 1 \tag{38}$$

and

$$\frac{\lambda}{2k_c} = \frac{1}{\int_{U(p)=1} dx}\left[1 - \frac{k_r}{2k_c}\int_{U(p)=1} r(x)dx\right] \tag{39}$$

By substitution

$$p(x)U(p) = \frac{k_r r(x)}{2k_c} + \frac{1}{\int_{U(p)=1} dx}\left[1 - \frac{k_r}{2k_c}\int_{U(p)=1} r(x)dx\right] \tag{40}$$

If $p(x)$ is greater than or equal to zero over the entire range $|x| \leq B$, which is possible under certain conditions, then

$$\int_{U(P)=1} r(x)dx = \int_{-B}^{B} dx = 2B$$

and

$$\int_{U(P)=1} r(x)dx = \int_{-B}^{B} r(x)dx$$

The resultant probability density function for this case is given by

$$p(x)U(p) = \frac{k_r r(x)}{2k_c} + \frac{1}{2B}\left[1 - \frac{k_r}{2k_c}\int_{-B}^{B} r(x)dx\right] \tag{41}$$

Substitution of the appropriate $r(x)$ yields

$$p(x) = \frac{k_r}{2k_c a}\left(1 - \frac{|x|}{a}\right) + \frac{1}{2B}\left(1 - \frac{k_r}{2k_c}\right), \qquad 0 \leq |x| \leq a$$
$$= \frac{1}{2B}\left(1 - \frac{k_r}{2k_c}\right), \qquad a \leq |x| \leq B$$
$$= 0, \qquad \text{elsewhere} \tag{42}$$

The above probability density function holds only when the right-hand member of Eq. (41) is positive. The criterion is

$$1 - \frac{k_r}{2k_c}\int_{-B}^{B} r(x)dx \geq 0$$

or, in this particular example,

$$\frac{k_c}{k_r} \geq 0.5$$

Otherwise, the range in which $U(p) = 1$ must be established. In this example, when $r(x)$ is symmetrical about 0, let $\pm X$ be the boundaries of the range $U(p) = 1$ so that the resultant probability density function can be expressed as

$$p(x) = \frac{k_r r(x)}{2k_c} + \frac{1}{\int_{-X}^{X} dx}\left[1 - \frac{k_r}{2k_c}\int_{-X}^{X} r(x)dx\right]$$
$$= \frac{k_r r(x)}{2k_c} + \frac{1}{2X}\left[1 - \frac{k_r}{k_c a}\left(X - \frac{X}{2a}\right)\right], \qquad 0 \leq |x| \leq X$$

Since X is the zero crossing of the probability density function, its actual value can be obtained by setting $p(x)U(p) = 0$ with $x = X$. Thus,

$$\frac{k_r r(X)}{2k_c} + \frac{1}{2X}\left[1 - \frac{k_r}{k_c a}\left(X - \frac{X}{2a}\right)\right] = 0$$

which yields

$$X = \left(\frac{2k_c}{k_r}\right)^{1/2} a$$

By substitution and mathematical manipulation, it can be shown that

$$p(x) = \left(\frac{k_r}{2k_c}\right)^{1/2}\frac{1}{a}\left[1 - \left(\frac{k_r}{2k_c}\right)^{1/2}\frac{|x|}{a}\right]$$
$$= \frac{1}{X}\left(1 - \frac{|x|}{X}\right), \qquad 0 \leq |x| \leq X$$
$$= 0, \qquad \text{elsewhere}$$

With conventional calculus methods, it can be verified that the above value of X indeed maximizes the expected payoff of Eq. (34). Verification is accomplished by substitution of the resultant $p(x)U(p)$ into Eq. (34), by the assumption that

$$\frac{\partial E[N(x)]}{\partial X} = 0$$

and solving for X.

The above analytical procedures are equally applicable to the cases in which the incentive award formula is not symmetrical.

The result of this analysis indicates that if $k_c/k_r = 0$, the probability density function approaches a delta function. This condition essentially means that the contractor is able to obtain, without cost, any particular parameter value he wishes. It is expected that the contractor could deliver exactly the same parameter value for all units at a value that yields the highest net profit, if he indeed acts in a rational manner. Because $k_c = 0$ in this case, the cost function has served as a "kernel" for the solution regardless of its form but does not enter into the result itself. From the other extreme, if $k_c/k_r \to \infty$, the probability density function reduces to a rectangular form that is the most random of all distributions under a peak value constraint (Ref. 11). For this value of k_c/k_r condition, the contractor can no longer afford to pursue any particular parameter value but must settle for whatever value he may achieve with minimum effort, provided that this value is within the permissible range. In this case, one can reasonably expect that the parameter value of the units delivered would be uniformly distributed.

A family of curves is given in Fig. 9 to show the probability density function of this strategy for various k_c to k_r ratios. From these curves, it appears that unless k_r is relatively large compared to k_c, the contractor will not

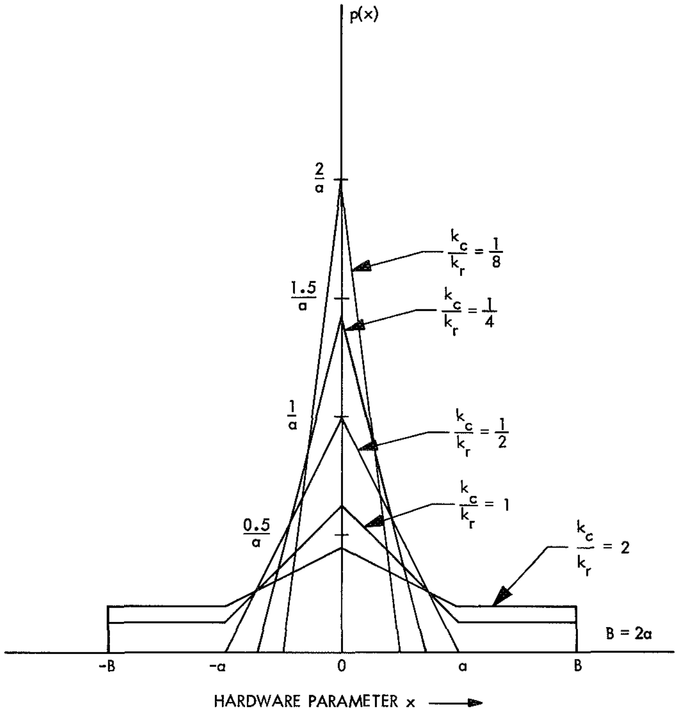

Fig. 9. Probability density function curves for various k_c/k_r ratios; incentive award strategy

make much effort to pursue the established incentive award. Furthermore, it takes a substantial increase in k_r to significantly improve the resultant probability density function. In other words, the resulting probability density function for this strategy is not very sensitive to the incentive award.

The word "sensitive" in this case is a relative term. When incentive contracts are structured, the payoffs offered are usually in the low percentages of the direct costs. A change of this payoff of say from 3% to 4% represents a substantial change in the opinions of the administrators. This, however, represents a change in the k_c/k_r ratios of 1/3, which can be seen in Fig. 9 not to change the shape of the resultant probability density function substantially. This is particularly true when k_c/k_r is already large.

Contract administrators seem to specify the incentive award for the purpose of modifying the probability density function without knowledge of its lack of sensitivity; yet, they have high hopes of good results. When the contractor cannot respond with performance matching those high hopes for reasons shown by the analysis, the administrator becomes unhappy with the contractor. With this result, the contractor would appear foolish to accept an incentive contract for the results specified in this problem. Many contractors do in fact refuse to accept incentive award contracts. In the face of this analysis, this appears to be rational behavior on the part of the contractor.

The reader should be cautioned in interpreting the results of this example as well as the following examples to note that they represent isolated strategies in most cases. This means that there are no external influences other than those specified in the example. In a real situation, this isolation would be difficult to approach.

2. ***More money.*** Another of the possible isolated strategies is to offer the contractor more money prior to contract commencement as long as he delivers within specification in the hope of obtaining higher quality of delivered products in return. In this case,

$$r(x) = Q, \qquad 0 \leq |x| \leq B$$

where

$$Q = \text{fixed amount of money offered}$$

Then, by the use of Eq. (40),

$$\begin{aligned} p(x) &= \frac{k_r Q}{2k_c} + \frac{1}{2B}\left(1 - \frac{k_r}{2k_c}\int_{-B}^{B} Q dx\right) \\ &= \frac{1}{2B}, \qquad 0 \leq |x| \leq B \\ &= 0, \qquad \text{elsewhere} \end{aligned} \tag{43}$$

which is independent of Q. This analysis indicates that, regardless of Q, the expected probability density function will remain rectangular, because the only concern of the contractor, under this strategy, is to keep his internal hidden cost down. This result is consistent with the intuitive reasoning that arbitrary offering of additional profit is an inferior strategy.

3. ***Surveillance.*** Surveillance, as defined in Section IV, is almost universally used and is perhaps the most isolatable of the isolated strategies.

The surveillance function $s(x)$ is assumed to be

$$s(x) = x^2, \qquad 0 \leq |x| \leq B$$

and the expected payoff for this example is

$$E[N(x)] = \int_{-B}^{B} [-k_s x^2 - k_c p(x) U(p)] p(x) U(p) dx \tag{44}$$

With this strategy, as with the incentive award strategy, the assumptions are: a specification limit of $\pm B$, and the contractor's cost of $k_c p(x)$.

Following the same procedures indicated for the case of incentive award strategy, it can be shown that

$$p(x) = -\frac{k_s x^2}{2k_c} + \frac{1}{2B}\left(1 + \frac{k_s B^3}{3k_c}\right), \qquad 0 \leq |x| \leq B$$
$$= 0, \qquad \text{elsewhere} \tag{45}$$

provided that the inequality

$$\frac{k_c}{k_s} \geq \frac{2}{3} B^3$$

can be assumed. When the two members of the above expression are equal, the distribution reduces to

$$p(x) = \frac{3}{4B^3}(B^2 - x^2), \qquad 0 \leq |x| \leq B$$
$$= 0, \qquad \text{elsewhere} \tag{46}$$

It can also be shown that when the k_c to k_s ratio is less than the critical value, the distribution will be of the same form as given in Eq. (46). Thus,

$$p(x) = \frac{3}{4b^3}(b^2 - x^2), \qquad 0 \leq |x| \leq b \tag{47}$$

in which

$$b = \left(\frac{3k_c}{2k_s}\right)^{1/3}$$

This criterion can be verified by maximizing the expected payoff (Eq. 44) through the conventional calculus methods.

Figure 10 shows the probability density functions for various values of the ratio k_c/k_s. As anticipated, the probability density function approximates the delta function as the said ratio becomes zero. Again, there is a general lack of sensitivity of the resulting probability density function. This lack of sensitivity is not, however, as restrictive as in the example with the incentive award. Surveillance can be changed drastically by the administrator and the contractor must respond to it. Because surveillance is a nonreimbursable cost, it is effective; however, it is an indirect and expensive cost to the administrator.

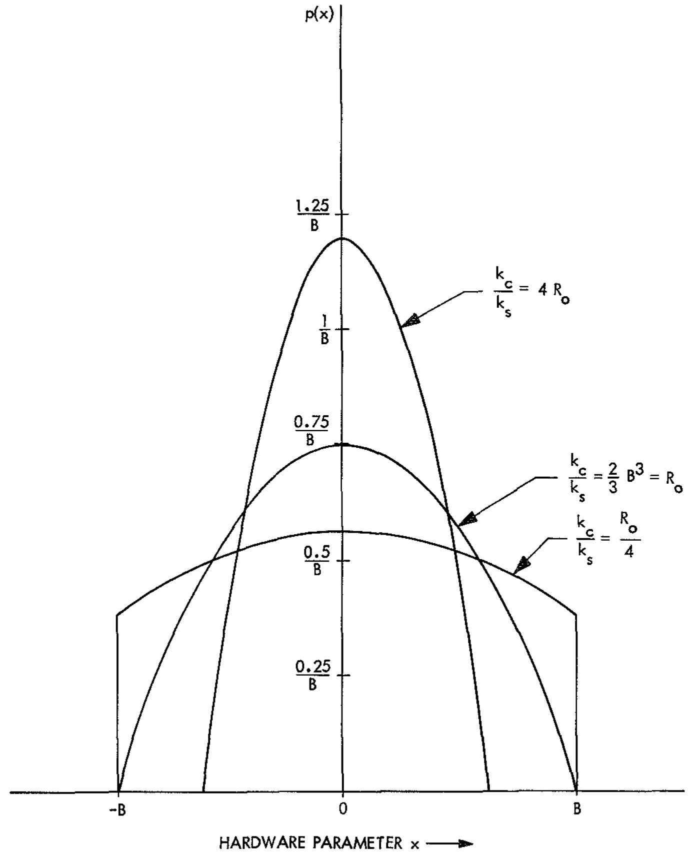

Fig. 10. Probability density function curves for various k_c/k_s ratios; surveillance strategy

4. State-of-the-art improvement. In the design of electronic devices, substantial development cost is often required so that certain parameters may be advanced beyond the currently available values. If the additional development cost for a particular parameter is linear with respect to the value achieved, the *a priori* probability density function (when the only restriction is the tolerance specifications) must be determined. The following example attempts to provide a method for determining the *a priori* probability density function.

Assume that the parameter x is specified to be between 0 and B, and that the state-of-the-art value of this parameter is $-A$. Then, the development cost is

$$C_d = k_d(x + A) \qquad (48)$$

and the expected net profit is

$$I = \int_0^B [-k_d(x + A) - k_c p(x)U(p)]p(x)U(p)dx \qquad (49)$$

which is a negative expected payoff function.

With the prescribed procedure, it can be shown that, for the condition $4k_c/k_d \geq B^2$,

$$p(x) = \frac{1}{B} + \frac{k_d}{2k_c}\left(\frac{B}{2} - x\right), \qquad 0 \leq x \leq B \qquad (50)$$

Otherwise,

$$p(x) = \left(\frac{k_d}{k_c}\right)^{1/2}\left[1 - \frac{1}{2}\left(\frac{k_d}{k_c}\right)^{1/2} x\right], \qquad 0 \leq x \leq 2\left(\frac{k_c}{k_d}\right)^{1/2} \qquad (51)$$

which is also valid for the case when no upper limit is specified (as B, in this case, becomes infinity). The same equation indicates that when $k_c = 0$, the probability density function approaches a delta function, $p(x) \to \delta(x)$. This fact indicates that the cost of improvement is the result of development alone, and that this cost is minimal when the parameter barely meets the requirement. When k_c is finite (nonzero), however, the probability density function tends to spread toward a higher x in a declining manner, depending upon the k_c to k_d ratio. The reason for this spread is that the idealized probability density function cannot be achieved at a finite cost, and the contractor tends to protect himself by expending additional effort to assure meeting the minimum requirements. These results tend to agree with observations made under actual conditions. (This example of isolated strategy perhaps clarifies the remark in the introduction to this report regarding the uncanny manner in which deliveries are biased toward the low performance side of a specification.)

Figure 11 shows the probability density function for various k_c to k_d ratios.

As with the previous examples, there is a lack of sensitivity to the parameters.

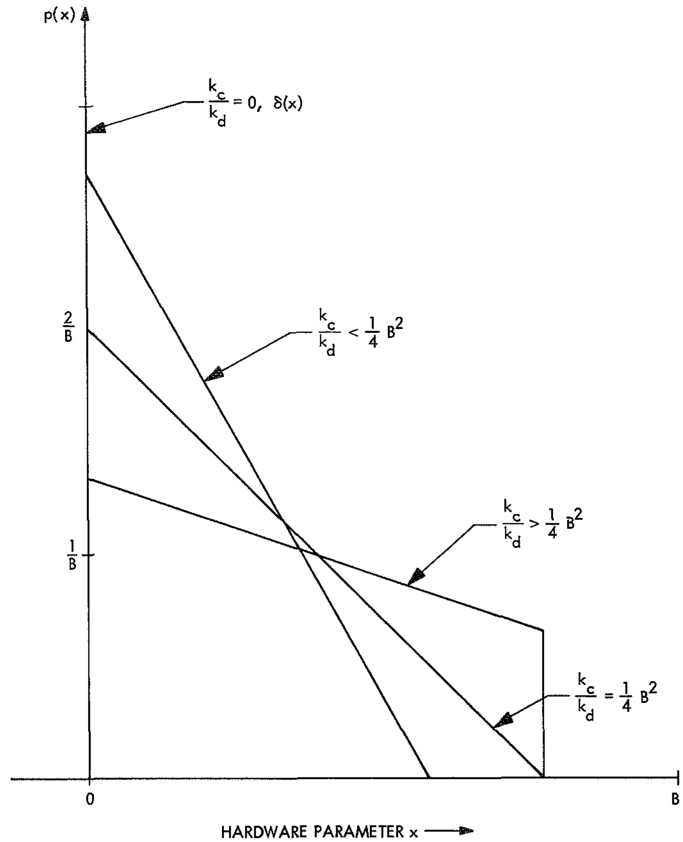

Fig. 11. Probability density function curves for various k_c/k_d ratios; state-of-the-art improvement

5. State-of-the-art improvement with linear incentive award. Consider the case of state-of-the-art improvement with a specified incentive award whose formula is

$$r(x) = k_r x, \qquad 0 \leq x \leq B$$

The relationship between the development cost function and the incentive award function is illustrated in Fig. 12.

Accordingly,

$$I = \int_0^B [k_r x - k_d(x + A) - k_c p(x)U(p)]p(x)U(p)dx \qquad (52)$$

and the basic probability density function equation is

$$p(x) = \frac{1}{B} + \left(\frac{k_r - k_d}{2k_c}\right)\left(x - \frac{B}{2}\right), \qquad 0 \leq x \leq B \qquad (53)$$

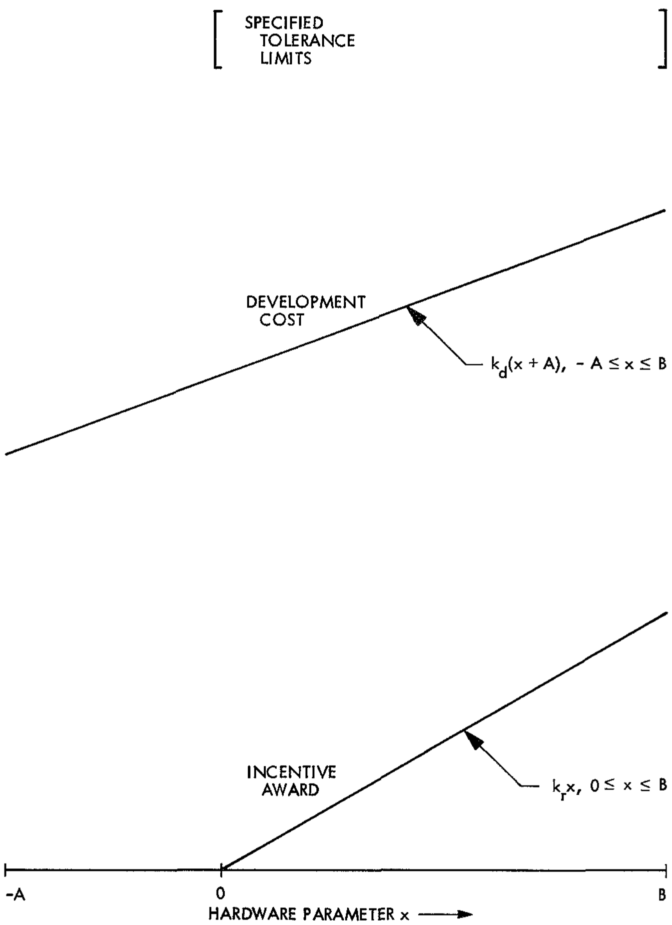

Fig. 12. Relationship between development cost and incentive award functions

There are three distinct cases to be considered in this example. The first case is $k_r = k_d = k$. In this case, the probability density function is rectangular $p(x) = 1/B$, $0 \le x \le B$ regardless of the values of k_c and k.

This result can be verified with the profit–loss diagram shown in Fig. 13. The maximum profit (more appropriately identified as the minimum loss in this case) occurs throughout the entire permissible range $0 \le x \le B$. As far as the contractor is concerned, there is no preference as to the value of x delivered, provided it is within the tolerance limits.

The second case for consideration is $k_r < k_d$, and from Eq. (53),

$$p(x) = \frac{1}{B} - \frac{\Delta k}{2k_c}\left(x - \frac{B}{2}\right), \qquad 0 \le x \le B \tag{54}$$

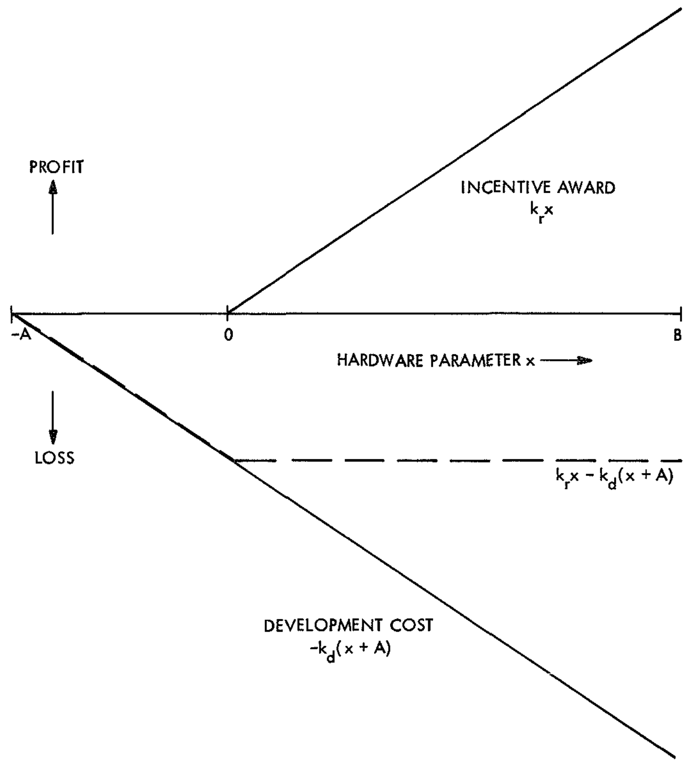

Fig. 13. Profit–loss diagram, $k_r = k_d$

subject to the condition that

$$\frac{4k_c}{\Delta k} \ge B^2$$

where

$$\Delta k = |k_r - k_d|$$

In the marginal case when $4k_c/\Delta k = B^2$, the probability density function reduces to

$$p(x) = \frac{2}{B}\left(1 - \frac{x}{B}\right), \qquad 0 \le x \le B \tag{55}$$

which is a declining right-angled triangle probability density function.

When $4k_c/\Delta k < B^2$, the probability density function will remain the same as the form given in Eq. (55).

$$p(x) = \frac{2}{b}\left(1 - \frac{x}{b}\right), \qquad 0 \le x \le b \le B \tag{56}$$

where b is a function of the k_c to Δk ratio and is found to be

$$b = 2\left(\frac{k_c}{\Delta k}\right)^{1/2}, \qquad \Delta k > 0$$

In the special case when $k_c = 0$, the probability density function approaches a delta function at $x = 0$. As can be seen in the profit–loss diagram of Fig. 14, the minimum loss for this case occurs at $x = 0$ with the specified range.

The last case to be considered is $k_r > k_d$ and from Eq. (53),

$$p(x) = \frac{1}{B} + \frac{\Delta k}{2k_c}\left(x - \frac{B}{2}\right), \qquad 0 \leq x \leq B \qquad (57)$$

subject to the same condition

$$\frac{4k_c}{\Delta k} \geq B^2$$

as in the second case of this example.

It is apparent that the probability density function for this case is the mirror image of the second case of this example with respect to $B/2$, provided that the upper limit of B is finite.

The profit–loss diagram for this case is shown in Fig. 15. The maximum profit occurs at $x = B$, which indicates that $p(x) \rightarrow \delta(x - B)$ when k_c is equal to zero.

The results of this example are summarized in three sets of probability density function curves. These curves are given in Figs. 16–18. As with previous examples, there is a lack of sensitivity to the parameters.

6. State-of-the-art improvement with linear incentive award (current parameter > 0). The case of state-of-the-art improvement with linear incentive award (current parameter > 0) is similar to the case described in Section IX-A-5; the only difference is that the current practical limit of the parameter exceeds the minimum tolerance specification; i.e., $x = 0$. Graphically, the development cost and the incentive award functions are related as shown in Fig. 19.

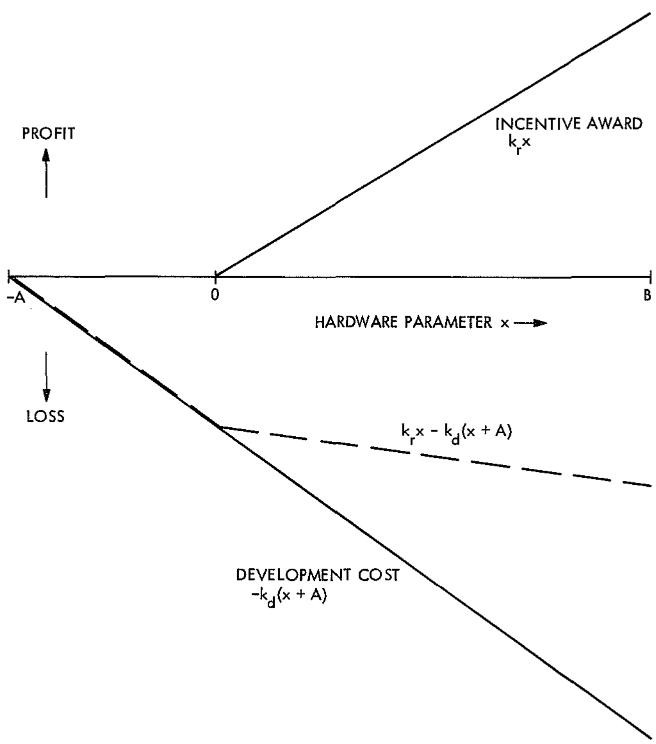

Fig. 14. Profit–loss diagram, $k_r < k_d$

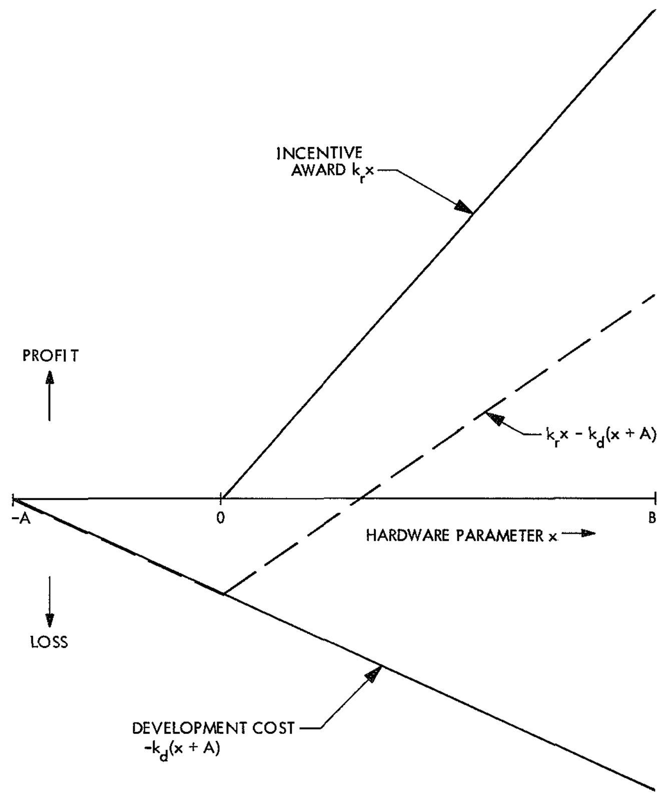

Fig. 15. Profit–loss diagram, $k_r > k_d$

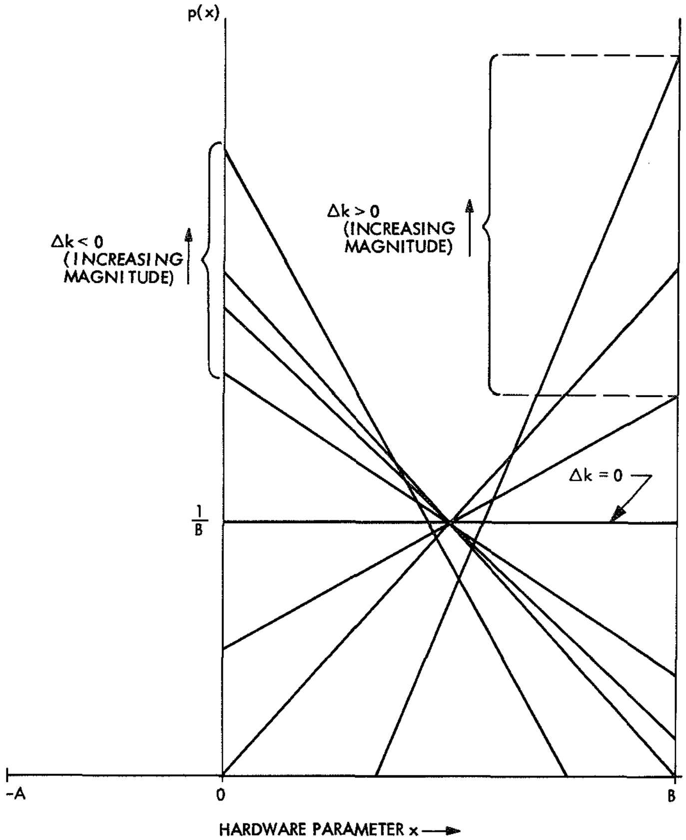

Fig. 16. Probability density function curves for constant k_c, $-\infty < \Delta k < \infty$; state-of-the-art improvement with incentive award strategy

In this case,

$$I = \int_0^A [k_r x - k_c p(x)U(p)]p(x)U(p)dx + \int_A^B [k_r x - k_d(x - A) - k_c p(x)U(p)]p(x)U(p)dx \quad (58)$$

It follows that

$$\begin{aligned} p(x) &= \frac{1}{B} + \frac{k_r}{2k_c}\left(x - \frac{B}{2}\right) + \frac{k_d}{4k_c}\frac{(B-A)^2}{B}, \qquad 0 \le x \le A \\ &= \frac{1}{B} + \frac{k_r - k_d}{2k_c}\left(x - \frac{B}{2}\right) + \frac{k_d}{4k_c}\frac{A^2}{B}, \qquad A \le x \le B \end{aligned} \quad (59)$$

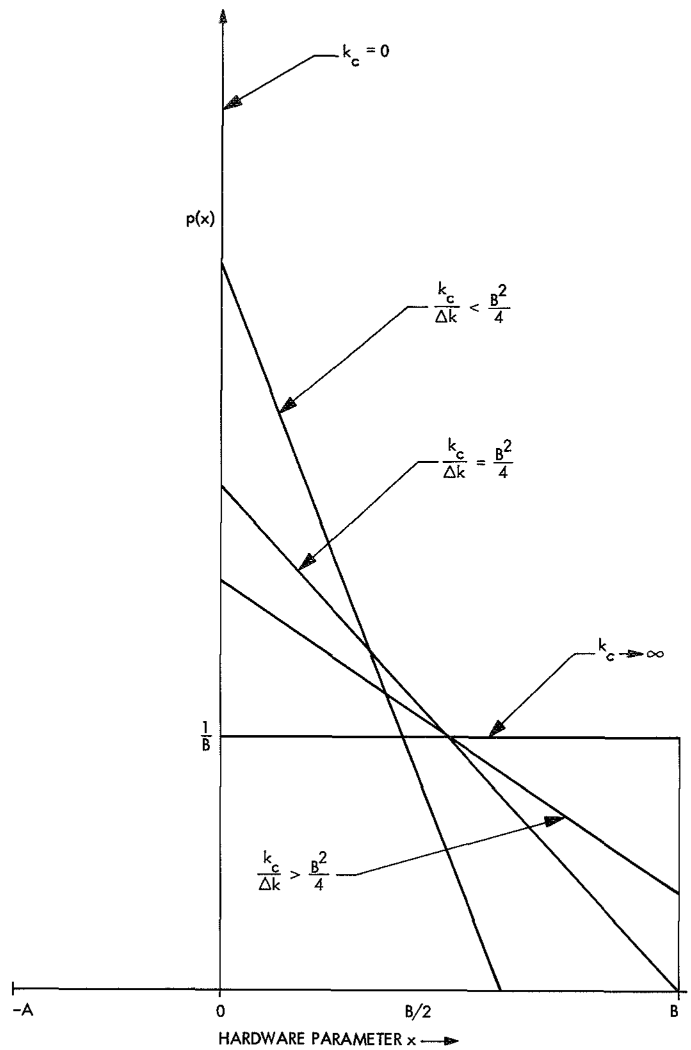

Fig. 17. Probability density function curves for various $k_c/\Delta k$ ratios, $k_r < k_d$; state-of-the-art improvement with incentive award strategy

There are three distinct cases to be considered in this example. The first case is $k_r = k_d = k$. The probability density function for this case is readily obtainable from Eq. (59).

$$\begin{aligned} p(x) &= \frac{1}{B} - \frac{kA}{2k_c}\left(1 - \frac{A}{2B}\right) + \frac{kx}{2k_c}, \qquad 0 \le x \le A \\ &= \frac{1}{B} + \frac{kA}{4k_c B}, \qquad A \le x \le B \end{aligned} \quad (60)$$

provided that the following inequality

$$\frac{k_c}{k} \ge \frac{A}{2}\left(B - \frac{A}{2}\right)$$

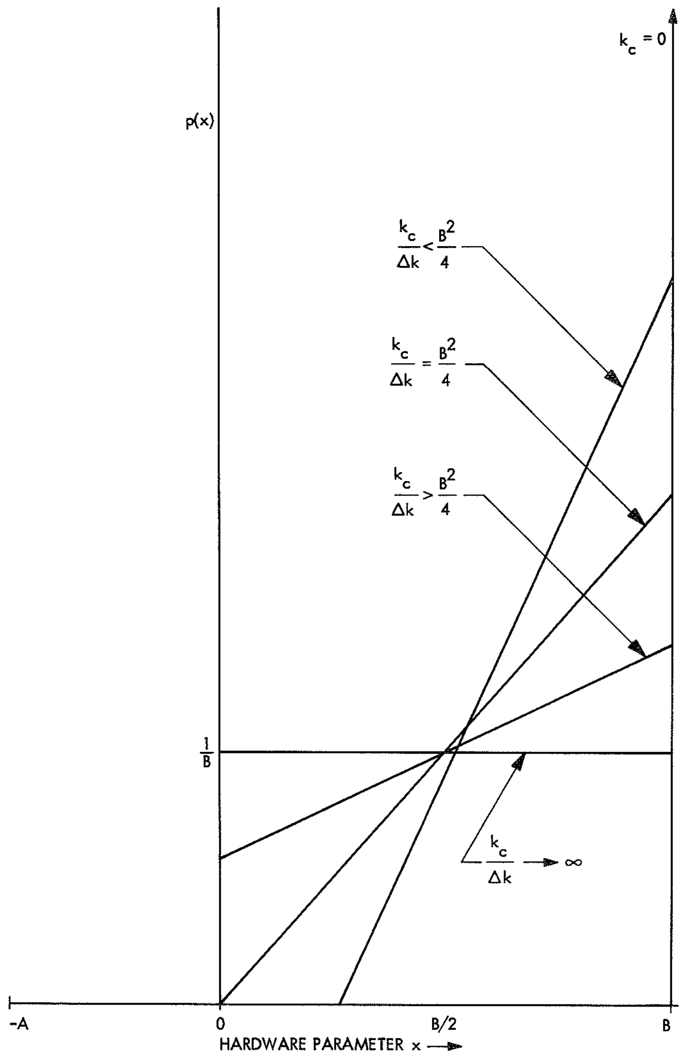

Fig. 18. Probability density function curves for various $k_c/\Delta k$ ratios, $k_r > k_d$; state-of-the-art improvement with incentive award strategy

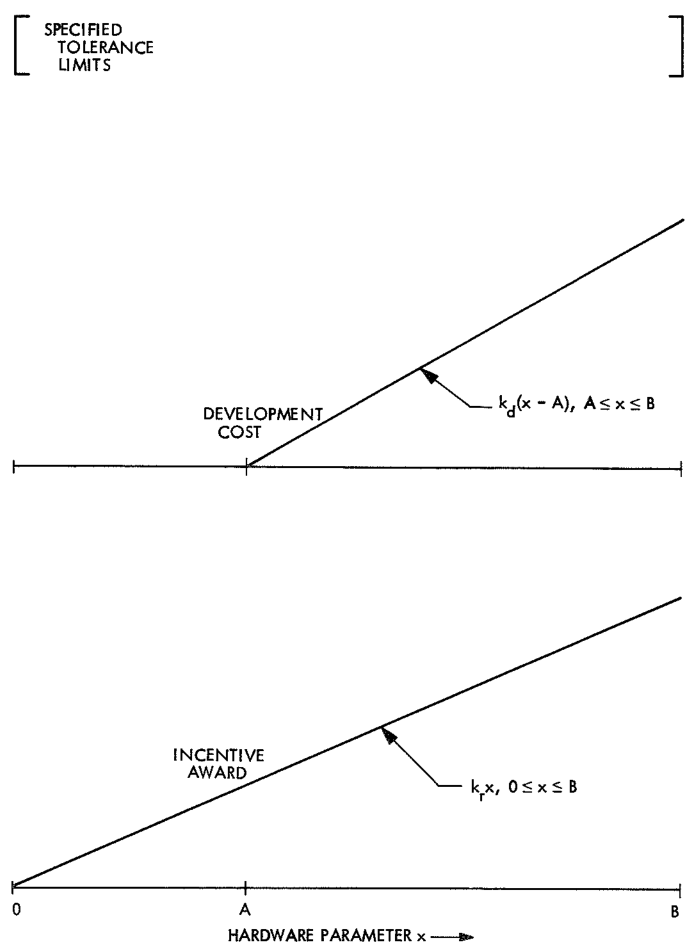

Fig. 19. Relationship between development cost and incentive award functions, $A > 0$

is satisfied. Otherwise, the slope of the curve in the first segment increases in the manner shown in Fig. 20 with the ultimate probability density function being rectangular between the limits A and B as k_c/k equals zero. This result can be verified with the corresponding profit–loss diagram given in Fig. 21. The maximum profit occurs for all x between A and B.

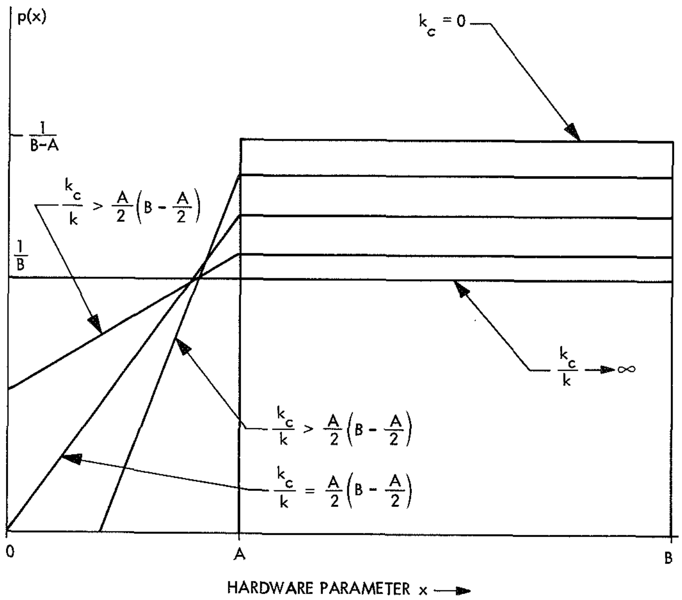

Fig. 20. Probability density function curves for various k_c/k ratios, $k_r = k_d = k$; state-of-the-art improvement with incentive award strategy, $A > 0$

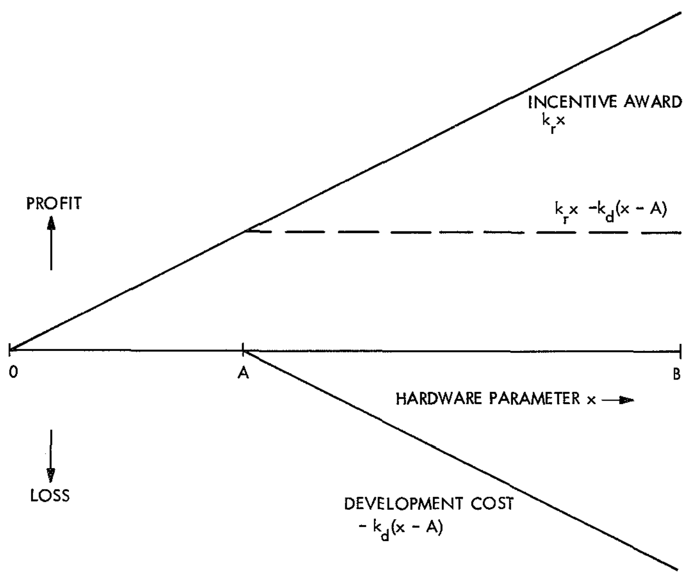

Fig. 21. Profit–loss diagram, $k_r = k_d$, $A > 0$

The second case for consideration is $k_r < k_d$. By rearranging Eq. (59), one obtains

$$p(x) = \begin{cases} \dfrac{1}{B} + \dfrac{\Delta k B}{4k_c} - \dfrac{k_d A}{2k_c} + \dfrac{k_d A^2}{4k_c B} + \dfrac{k_r x}{2k_c}, & 0 \le x \le A \\[2ex] \dfrac{1}{B} + \dfrac{\Delta k B}{4k_c} + \dfrac{k_d A^2}{4k_c B} - \dfrac{\Delta k x}{2k_c}, & A \le x \le B \end{cases} \qquad (61)$$

where $\Delta k = k_d - k_r$.

Evidently, both segments of the probability density function are linear functions of x.

It can be proved with elementary algebra that the following inequalities prevail:

$$p(A) \ge p(0), \qquad p(A) \ge p(B)$$

Therefore, the general shape of the probability density function is as shown in Fig. 22. In addition, depending upon the value of the k_c to Δk ratio, a family of curves can be constructed (Fig. 23). With $k_c = 0$, the probability density function becomes a delta function $\delta\,(x - A)$. The profit–loss diagram corresponding to this case is given in Fig. 24.

A special case of interest is when $k_r = 0$; i.e., no incentive award provision. In this case,

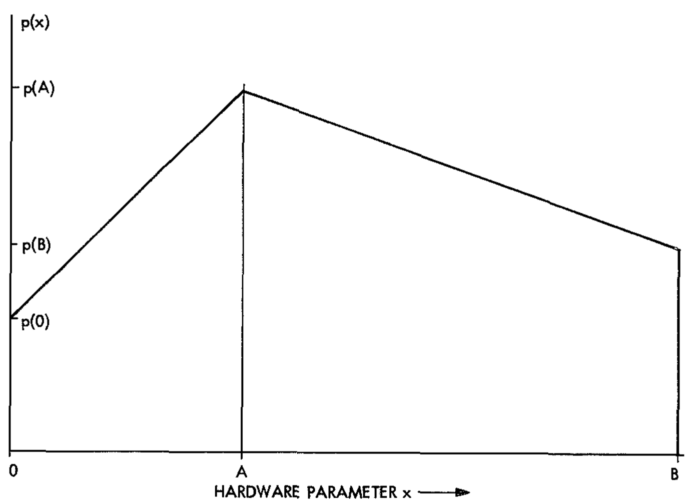

Fig. 22. General shape of probability density function curve; $k_r < k_d$, $A > 0$

$$p(x) = \frac{1}{B} + \frac{k_d}{4k_c}\frac{(B-A)^2}{B}, \qquad 0 \leq x \leq A$$

$$= \frac{1}{B} + \frac{k_d B}{4k_c} + \frac{k_d A^2}{4k_c B} - \frac{k_d x}{2k_c}, \qquad A \leq x \leq B \tag{62}$$

The probability density function curves and the associated profit–loss diagram are shown in Figs. 25 and 26, respectively.

The third case under consideration is $k_r > k_d$. The set of probability density function equations for this particular case is identical to the one given in Eq. (60). However, since the condition for the magnitude of k_r and k_d is reversed, the following inequalities become true:

$$p(A) \geq p(0), \qquad p(A) \leq p(B)$$

A sketch of the probability density function curves for various $k_c/\Delta k$ and the profit–loss diagram are presented in Figs. 27 and 28. And, as with previous examples, there is a lack of sensitivity to the parameters.

B. Mixed Strategies

1. Incentive award and surveillance. Consider a combined strategy of incentive award and surveillance such that

$$r(x) = \frac{1}{B}\left(1 - \frac{|x|}{B}\right), \qquad 0 \leq |x| \leq B$$

and

$$s(x) = x^2, \qquad 0 \leq |x| \leq B$$

with $x = \pm B$ being the imposed limits of the parameter. The expected payoff is

$$E[N(x)] = \int_{-B}^{B}\left[\frac{k_r}{B}\left(1 - \frac{|x|}{B}\right) - k_s x^2 - k_c p(x)U(p)\right]p(x)U(p)dx \tag{63}$$

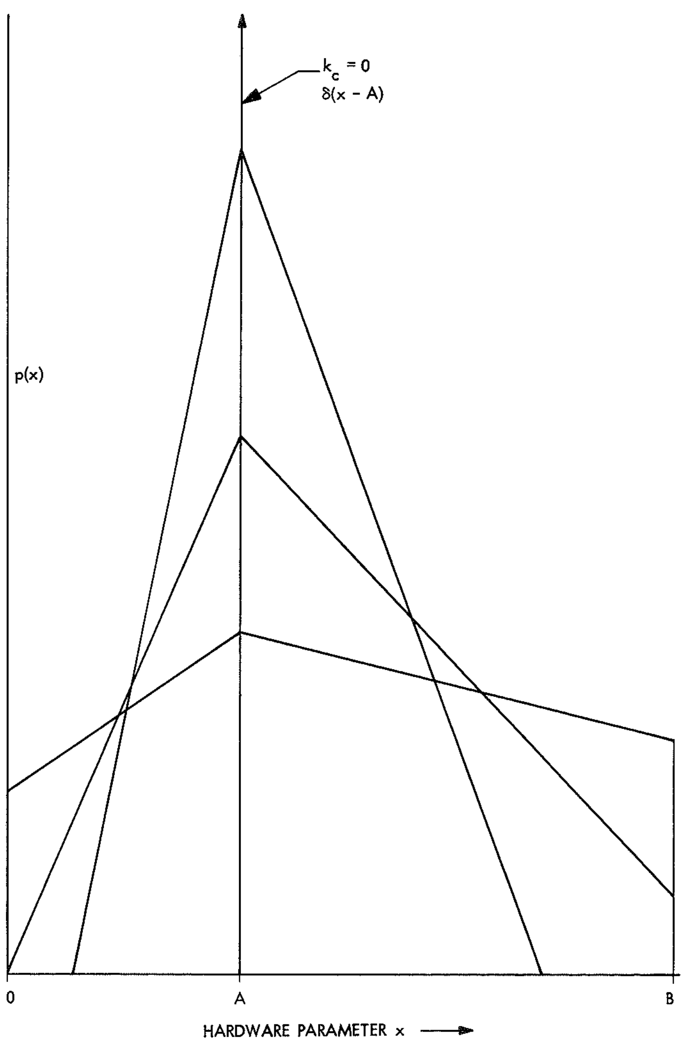

Fig. 23. Probability density function curves for various k_c; $k_r < k_d$, $A > 0$

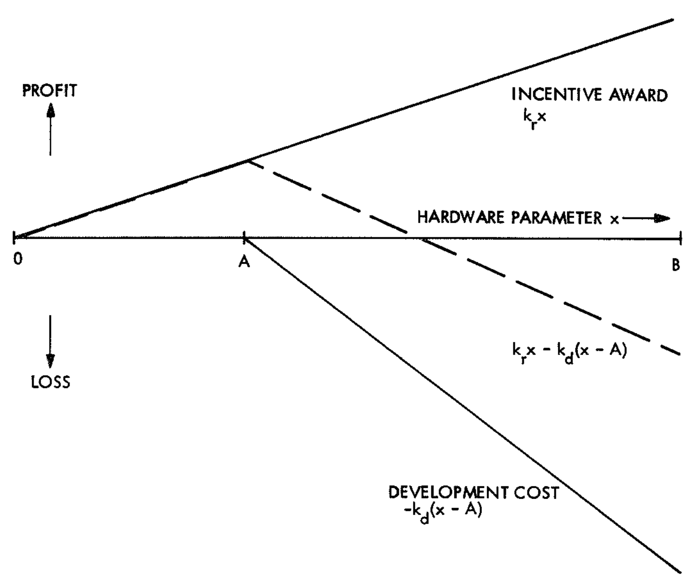

Fig. 24. Profit–loss diagram, $k_r < k_d$, $A > 0$

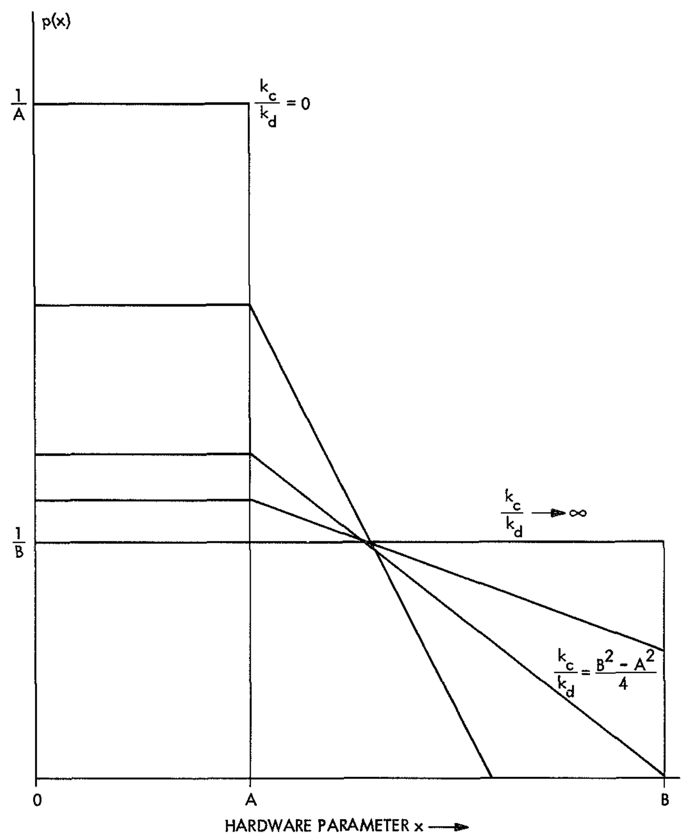

Fig. 25. Probability density function curves for various k_c/k_d ratios, $k_r = 0$, $A > 0$

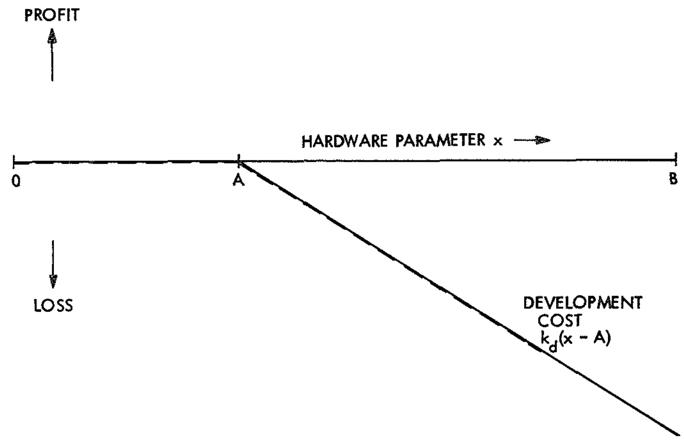

Fig. 26. Profit–loss diagram, $k_r = 0$, $A > 0$

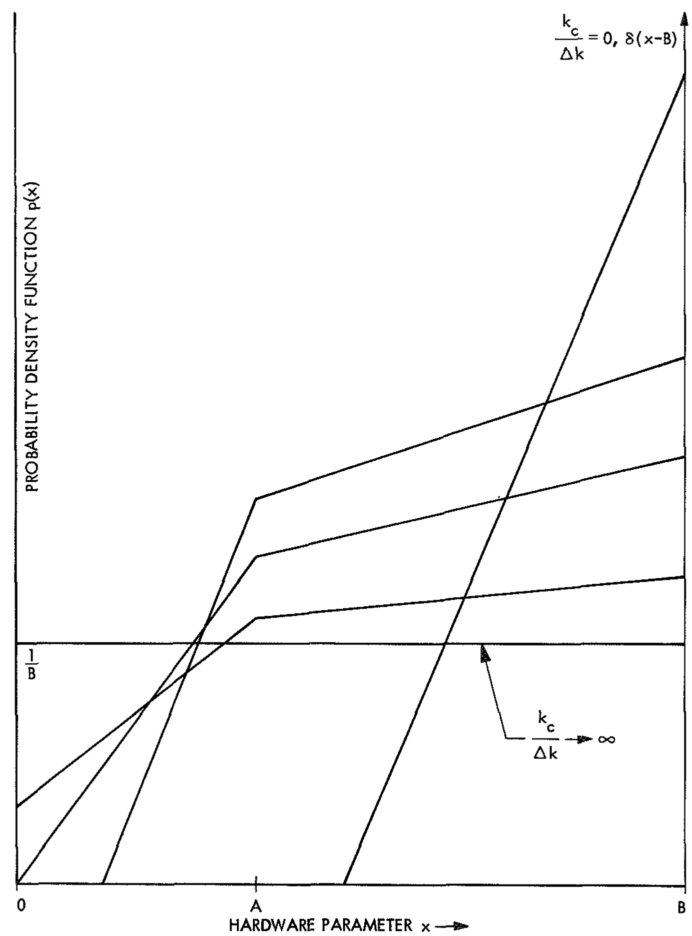

Fig. 27. Probability density function curves for various k_c; $k_r > k_d$, $A > 0$

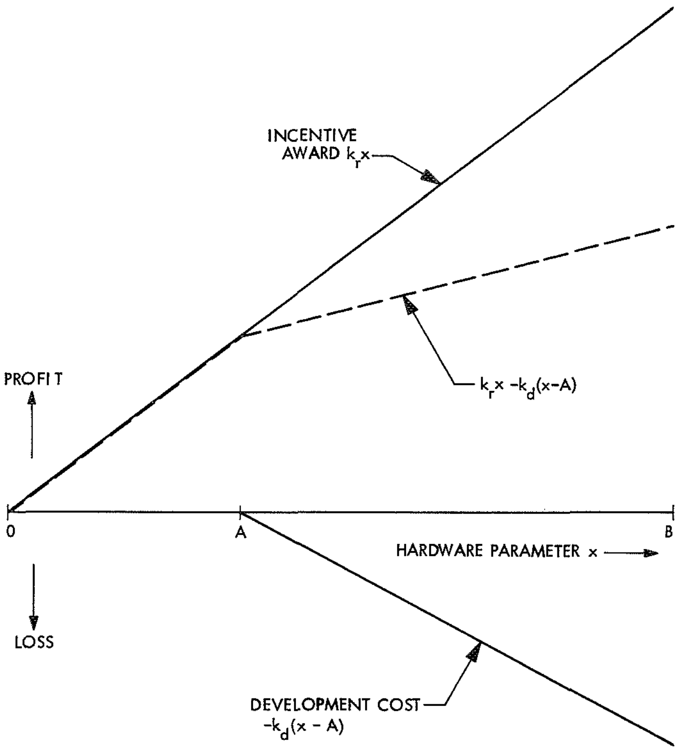

Fig. 28. Profit–loss diagram, $k_r > k_d$, $A > 0$

The combined function of $r(x)$ and $s(x)$ is depicted in Fig. 29. If this combined function is denoted by a single notation $v(x)$, then

$$v(x) = \frac{k_r}{B}\left(1 - \frac{|x|}{B}\right) - k_s x^2, \qquad 0 \leq |x| \leq B$$

and, according to Eq. (40),

$$p(x) = \frac{v(x)}{2k_c} + \frac{1}{2B}\left[1 - \frac{1}{k_c}\left(\frac{k_r}{2} - \frac{k_s B^3}{3}\right)\right], \qquad 0 \leq |x| \leq B \qquad (64)$$

where $v(x)$ represents the combined strategy of incentive award and surveillance. This probability density function is valid under the condition

$$k_c \geq \frac{4k_s B^3 + 3k_r}{6} = k_{c0}$$

The probability density function for the case when $k_c < k_{c0}$ can be obtained by the method described in the incentive award case. The necessary mathematical manipulation, however, is quite tedious.

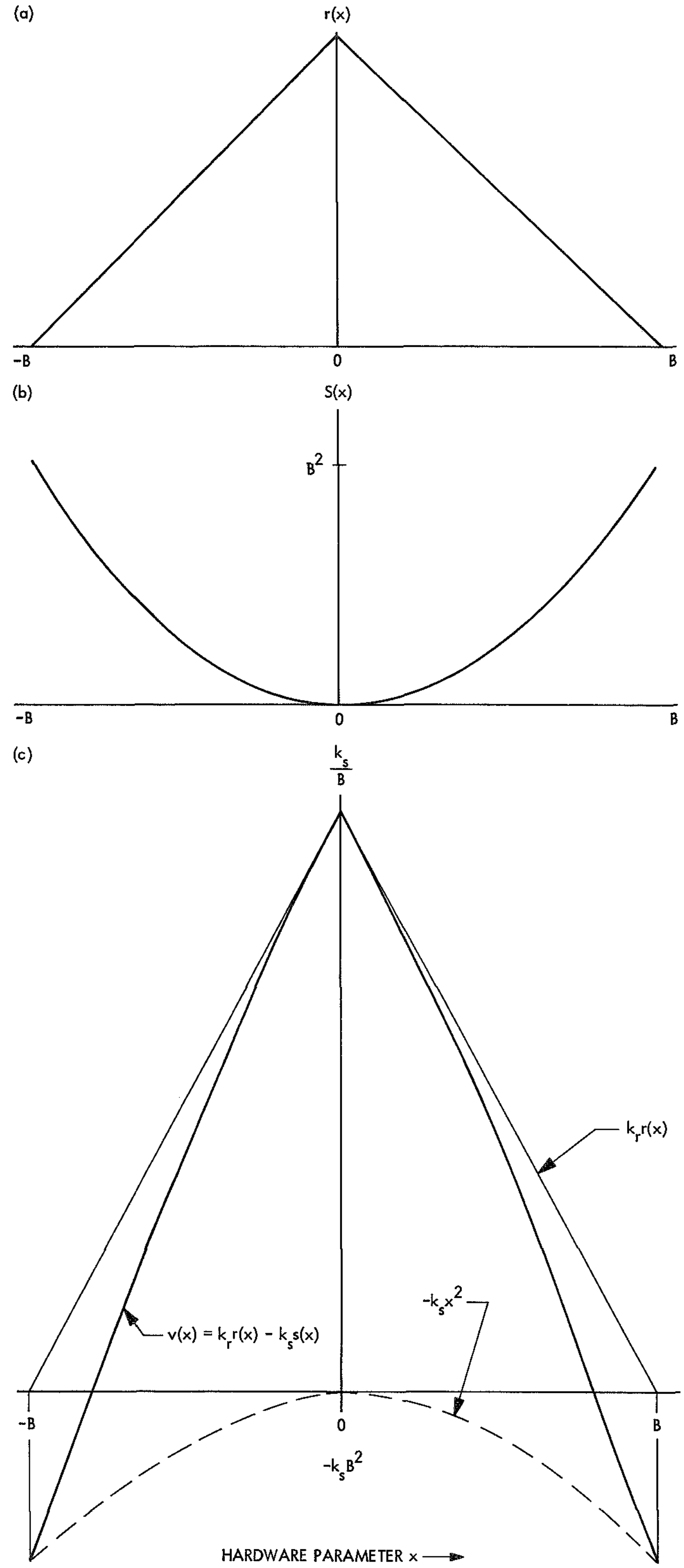

Fig. 29. Mixed strategy curves: (a) incentive award strategy; (b) surveillance strategy; (c) incentive award and surveillance strategies

The general form of the probability density function is

$$p(x) = C_1\left(1 - \frac{|x|}{b}\right) + C_2(b^2 - x^2), \qquad 0 \le |x| \le b$$
$$= 0, \qquad \text{elsewhere} \tag{65}$$

in which C_1, C_2, and b are constants that are best evaluated by the graphical method. Briefly, these constants can be obtained by the simultaneous solving of the following pair of partial derivatives:

$$\frac{\partial E}{\partial C_2} = 0, \qquad \frac{\partial E}{\partial b} = 0$$

where E is the expected payoff of Eq. (63) subject to the restrictions that

$$\int_0^b C_1\left(1 - \frac{x}{b}\right) dx + \int_0^b C_2(b^2 - x^2)dx = 0.5 \tag{66}$$

and that C_1, C_2, and b are non-negative. In addition, $b < B$.

A typical graphical solution for these constants is shown in Fig. 30. The intersection between the curves

$$\frac{\partial E}{\partial C_2} = 0, \qquad \frac{\partial E}{\partial b} = 0$$

within the shaded region bounded by the curves $C_2 = 0$, $b = 0$, and that of Eq. (66) with $C_1 = 0$ is the only correct solution. With the knowledge of C_2 and b, C_1 can be computed readily.

The same solution can be obtained readily with the aid of a digital computer through the area subtraction method described in Section VIII. Several families of curves for various ratios of k_s and k_r, with respect to the cost factor k_c, were obtained with the computer and are given in Figs. 31–38. Examination of these curves indicates a general lack of sensitivity of the resultant probability density function both for the variation of k_c with respect to k_s and k_r and for the ratio of k_s and k_r with constant k_c.

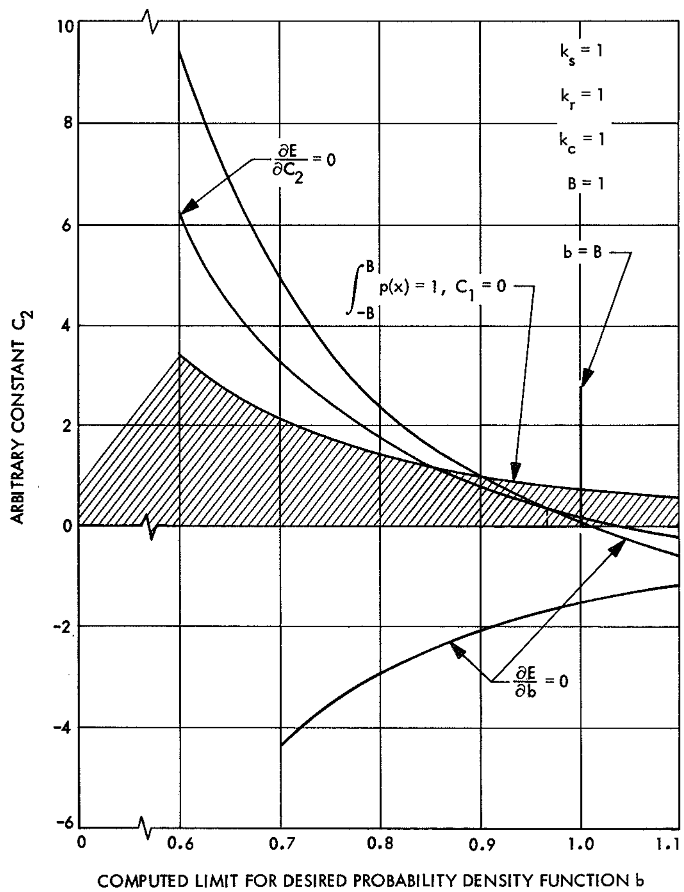

Fig. 30. Typical graphical solution for the mixed strategies of incentive award and surveillance

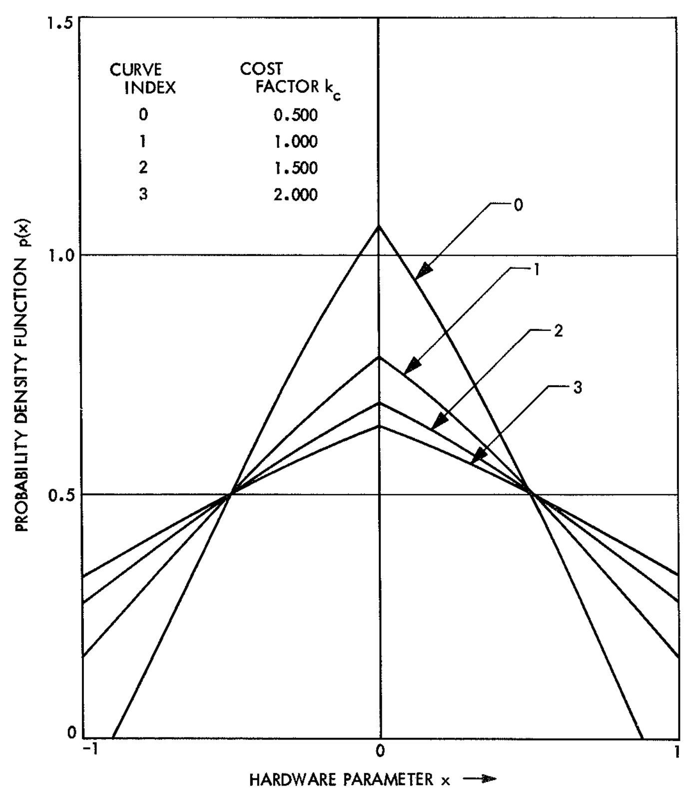

Fig. 31. Probability density function curves for incentive award and surveillance strategies; $k_r = 1$, $k_s = 0.25$

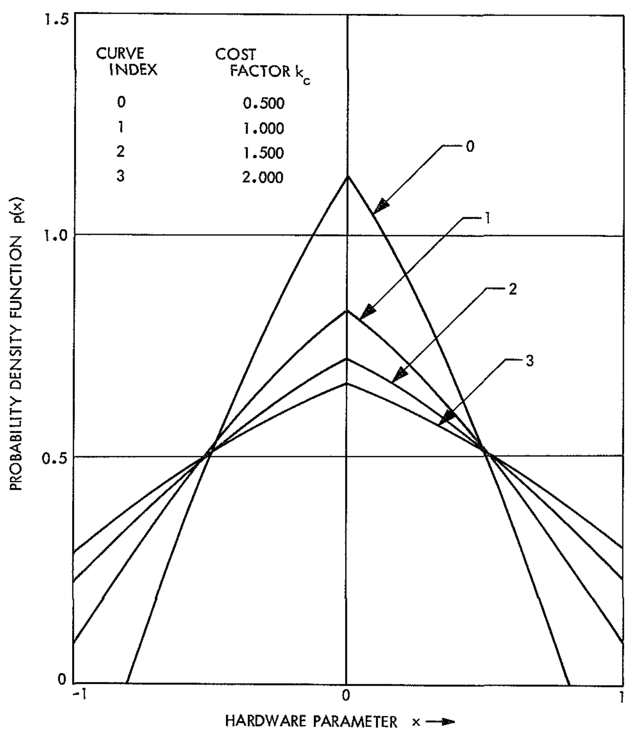

Fig. 32. Probability density function curves for incentive award and surveillance strategies; $k_r = 1$, $k_s = 0.5$

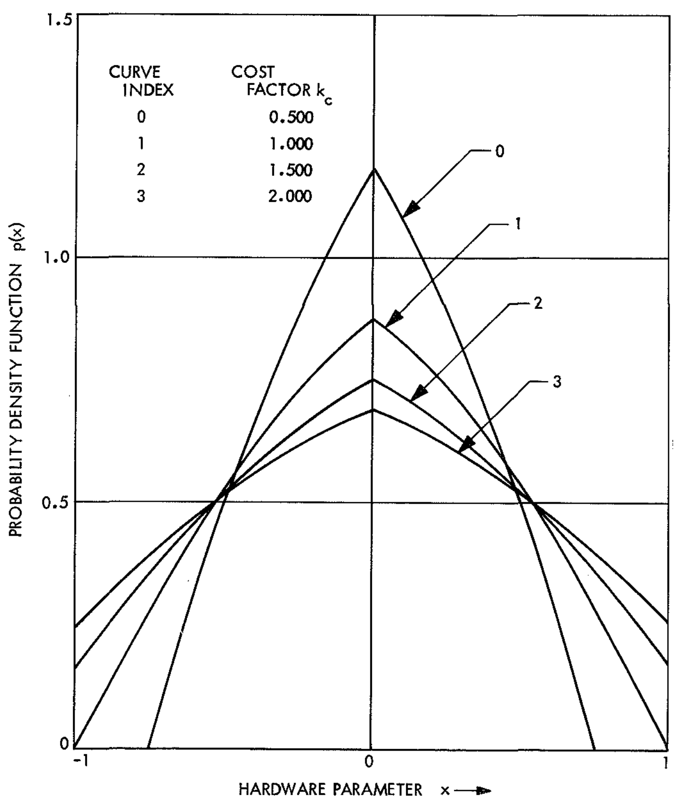

Fig. 33. Probability density function curves for incentive award and surveillance strategies; $k_r = 1$, $k_s = 0.75$

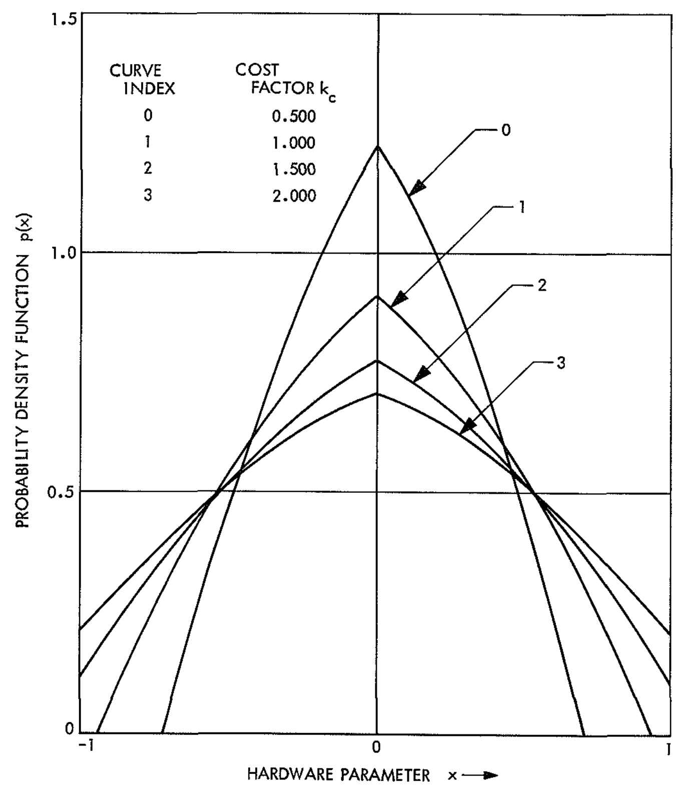

Fig. 34. Probability density function curves for incentive award and surveillance strategies; $k_r = 1$, $k_s = 1$

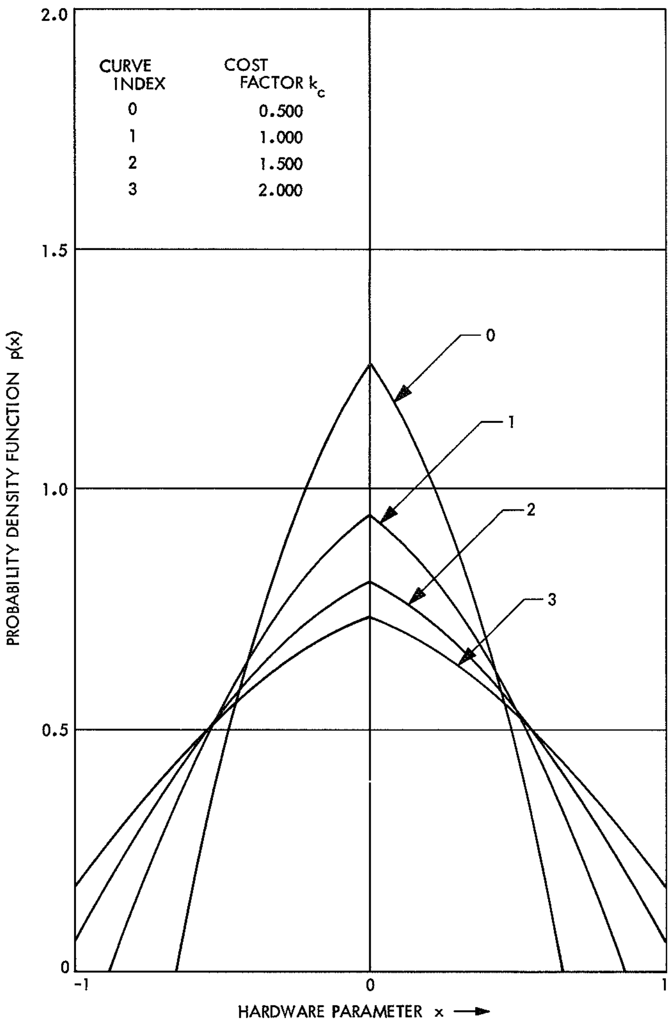

Fig. 35. Probability density function curves for incentive award and surveillance strategies; $k_r = 1$, $k_s = 1.25$

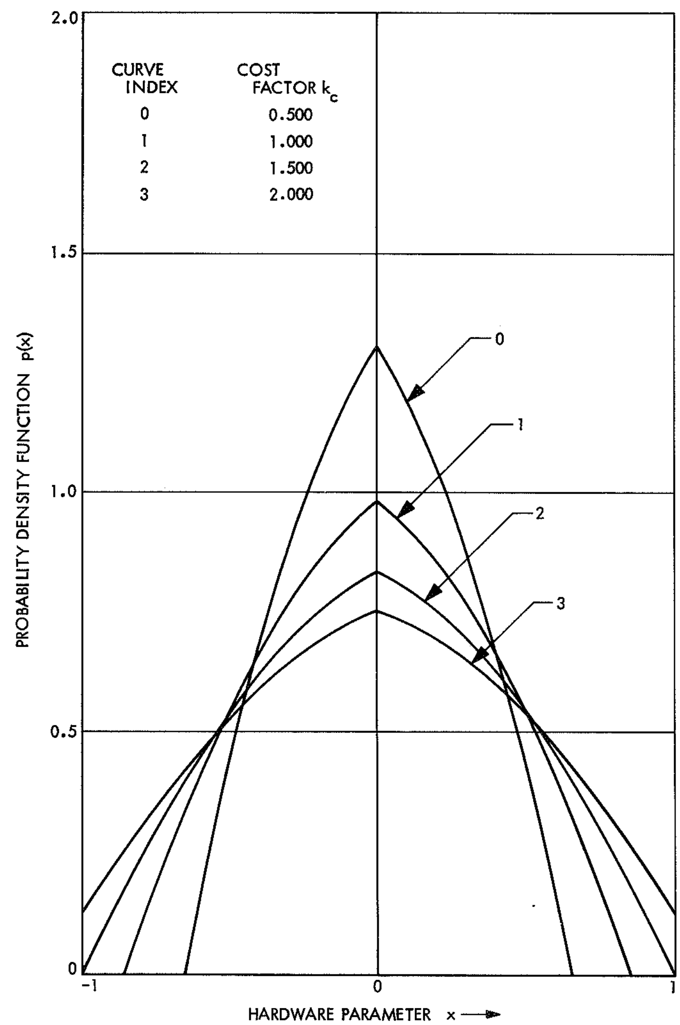

Fig. 36. Probability density function curves for incentive award and surveillance strategies; $k_r = 1$, $k_s = 1.5$

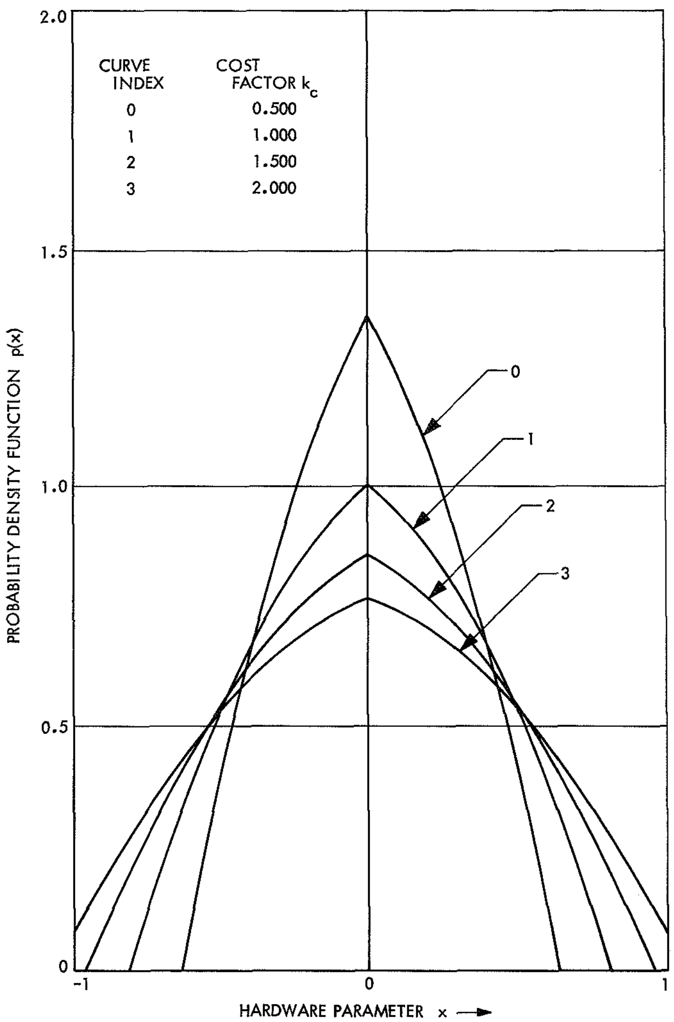

Fig. 37. Probability density function curves for incentive award and surveillance strategies; $k_r = 1$, $k_s = 1.75$

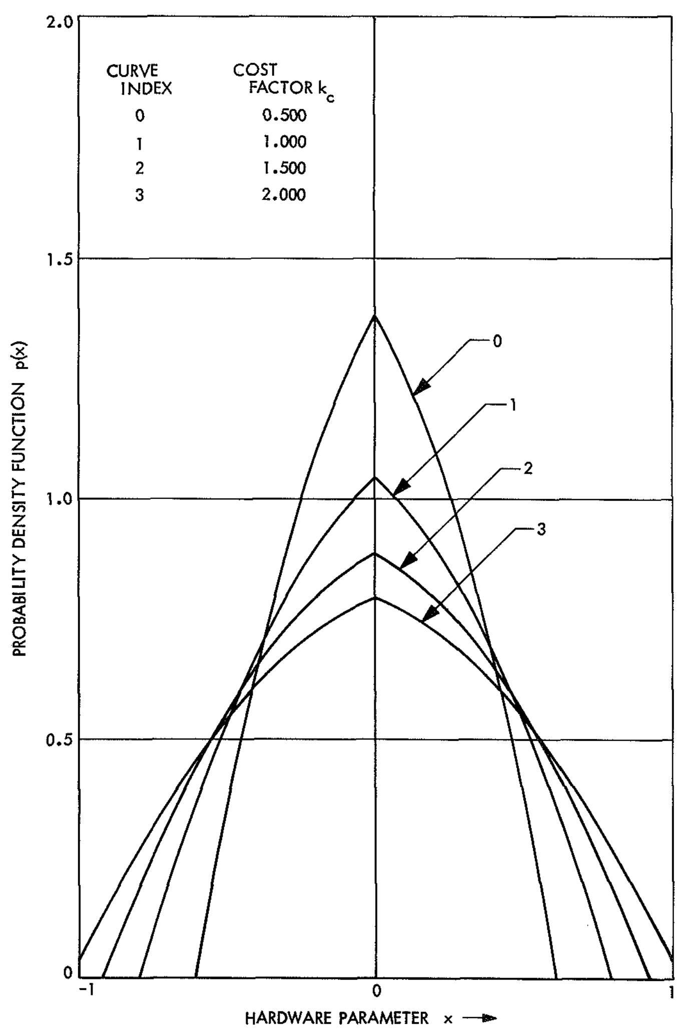

Fig. 38. Probability density function curves for incentive award and surveillance strategies; $k_r = 1$, $k_s = 2$

2. ***State-of-the-art improvement with incentive award and surveillance.*** Consider another mixed strategy example involving state-of-the-art improvement with both incentive award and surveillance. By use of the tolerance limits, development cost function, incentive award function, and a surveillance function of the form

$$s(x) = k_s(x-B)^2, \qquad \text{for } 0 \leq x \leq B$$

as shown in Fig. 39, we have

$$I = \int_0^B [k_r x - k_d (x + A) - k_s (x - B)^2 - k_c p(x)U(p)] \times p(x)U(p)dx$$

Although the resultant probability density function for this example can be obtained analytically, the process is tedious and time consuming because of the added complexity. Therefore, the detailed analytical solution for the probability density function is not presented in this report. Instead, computer solutions for various combinations of values of cost factors are plotted. These plots are given in Figs. 40–50. As in the previous cases, there is evidence of a general lack of sensitivity in the probability density function for this mixed strategy.

3. ***State-of-the-art improvement with surveillance.*** The example of state-of-the-art improvement with surveillance is a special case of the preceding example with $k_r = 0$ and is intended to show the effect on the resultant probability density functions of increased surveillance.

The strategic function used for this example is graphically presented in Fig. 51. The nominal value for the surveillance function in this case is 1. The resulting probability density function curves, with various values of k_s, are plotted in Fig. 52.

As shown in Fig. 52, significant improvement in the resultant probability density function can be realized if substantial surveillance effort is made by the administrator.

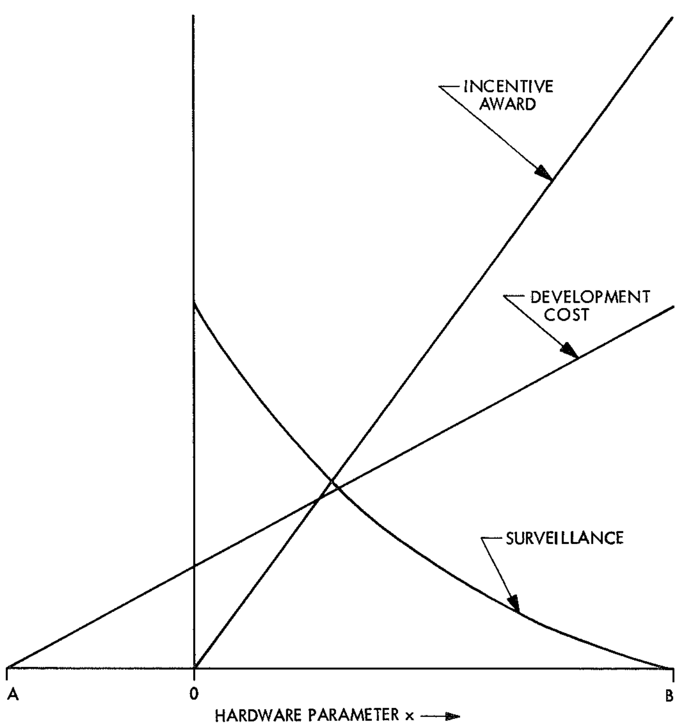

Fig. 39. Relationship between development cost, incentive award, and surveillance for a mixed strategy involving state-of-the-art improvement

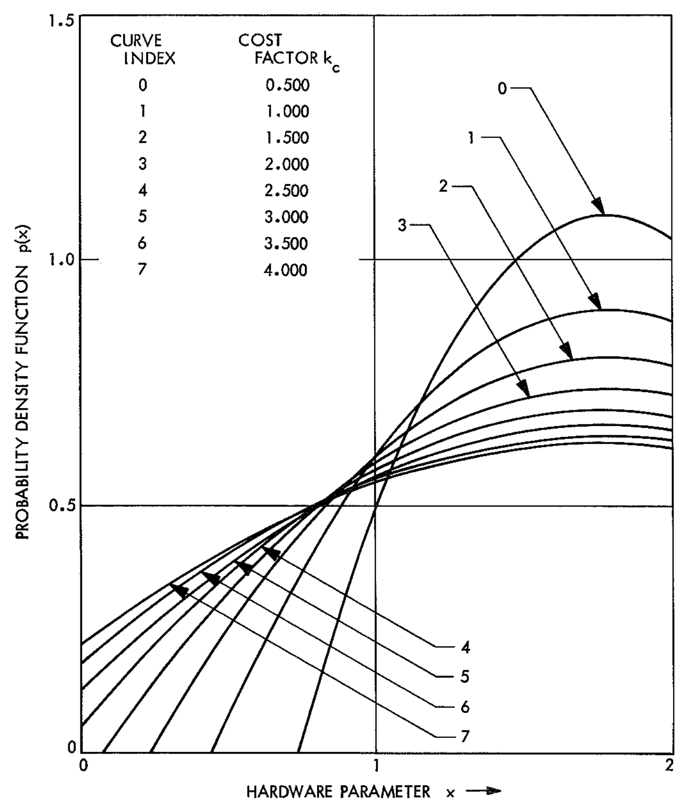

Fig. 40. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 0.1$, $k_s = 1$, $k_d = 1$

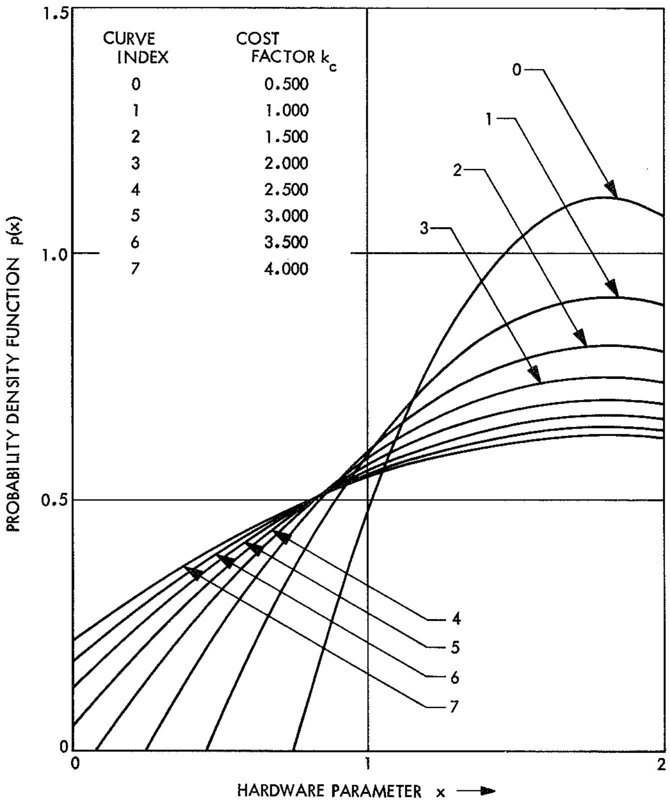

Fig. 41. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 0.2$, $k_s = 1$, $k_d = 1$

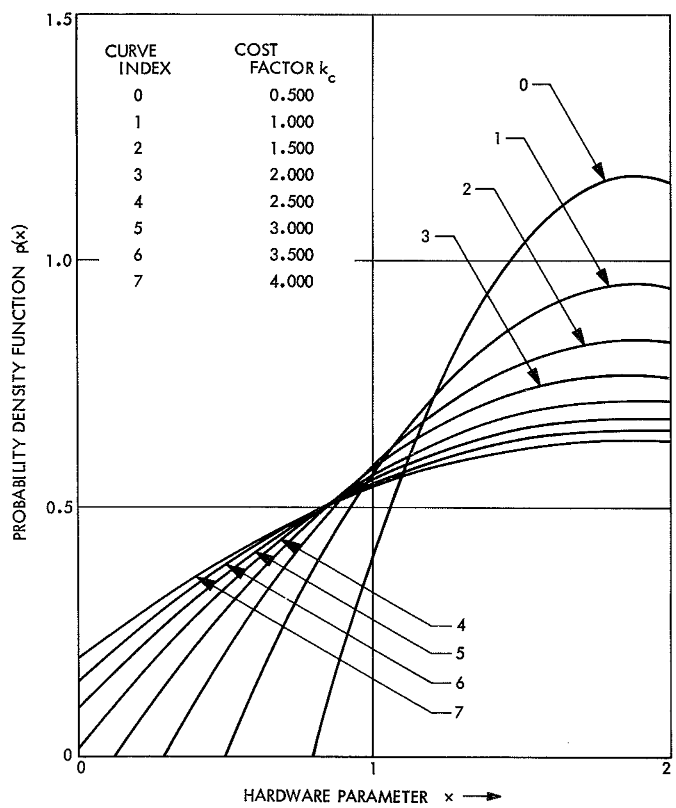

Fig. 42. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 0.5$, $k_s = 1$, $k_d = 1$

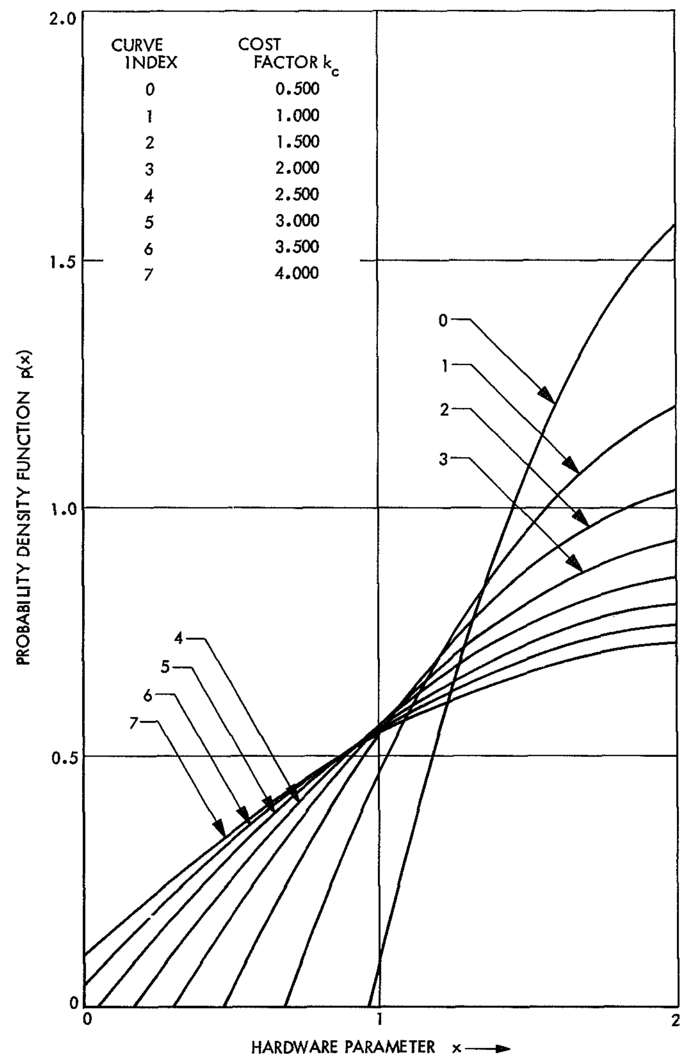

Fig. 44. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 2$, $k_s = 1$, $k_d = 1$

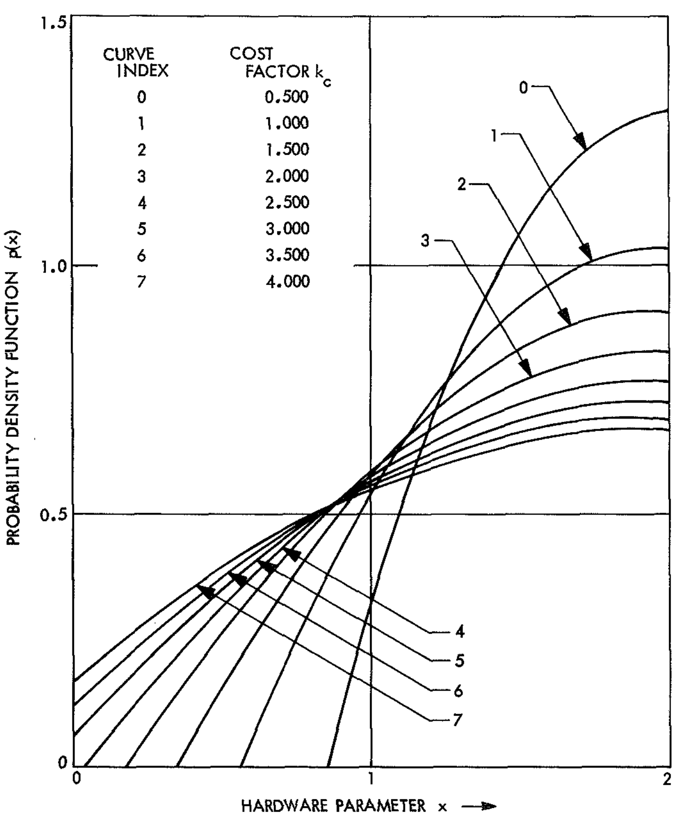

Fig. 43. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 1$, $k_s = 1$, $k_d = 1$

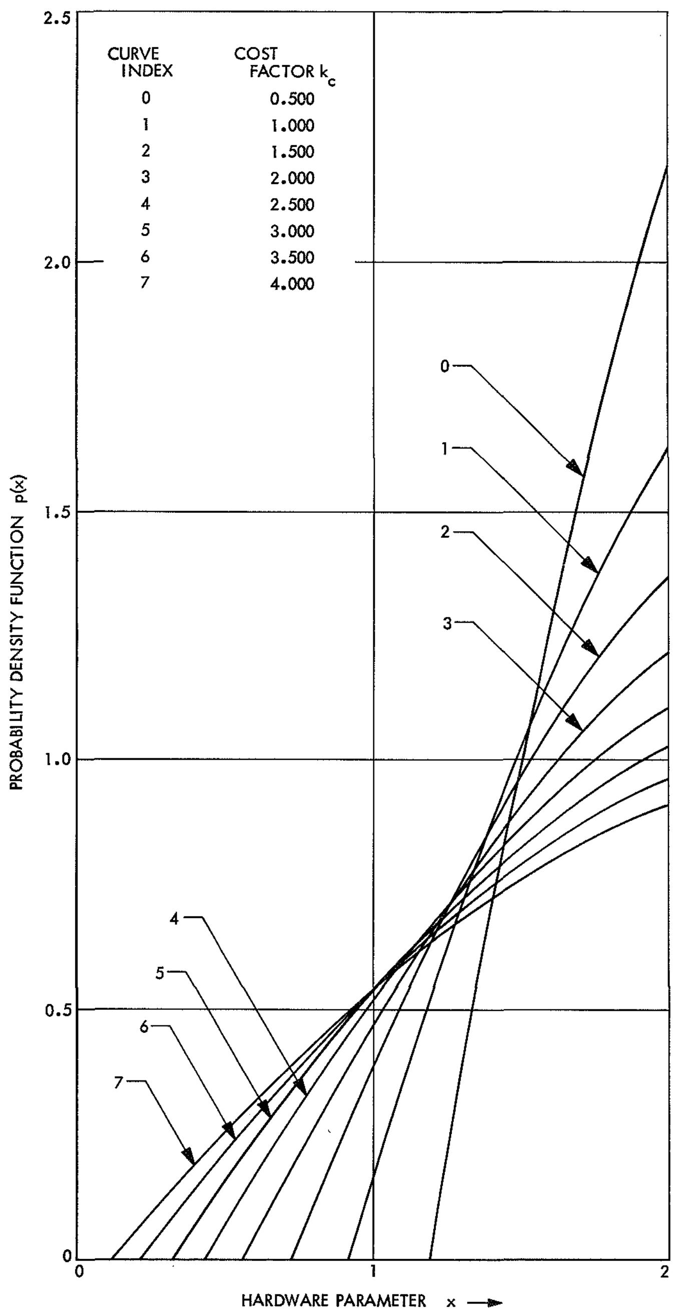

Fig. 45. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 5$, $k_s = 1$, $k_d = 1$

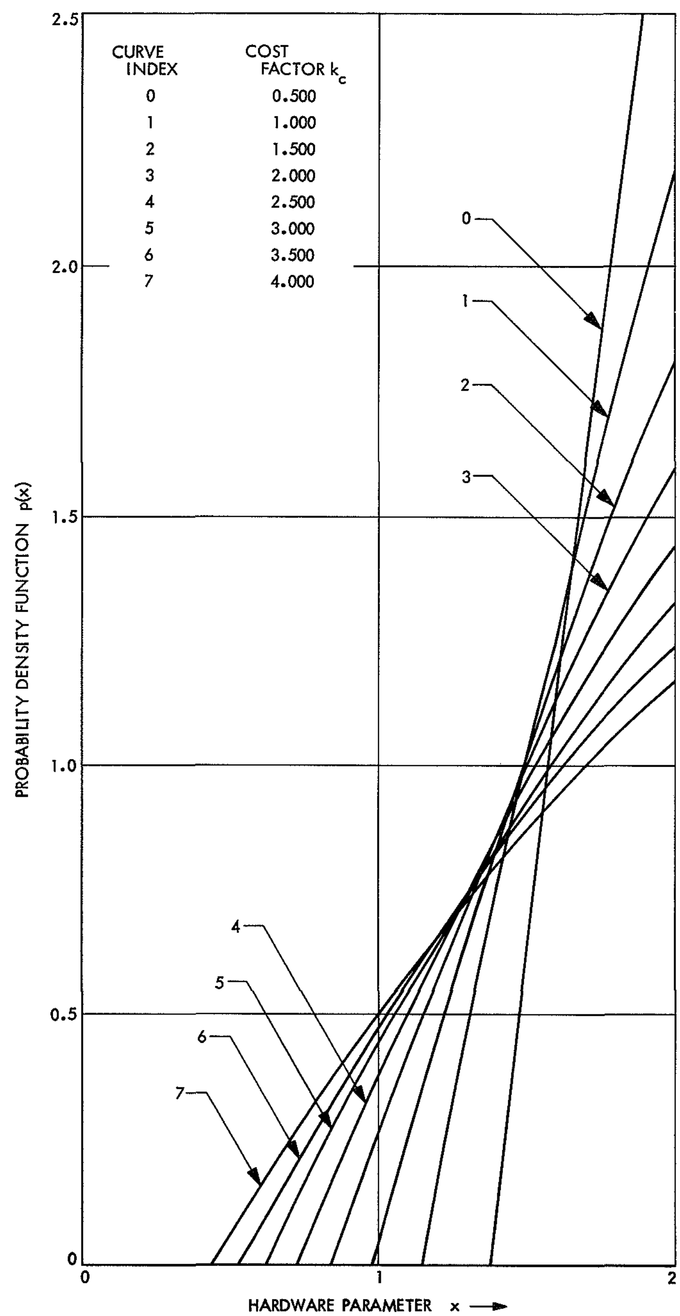

Fig. 46. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 10$, $k_s = 1$, $k_d = 1$

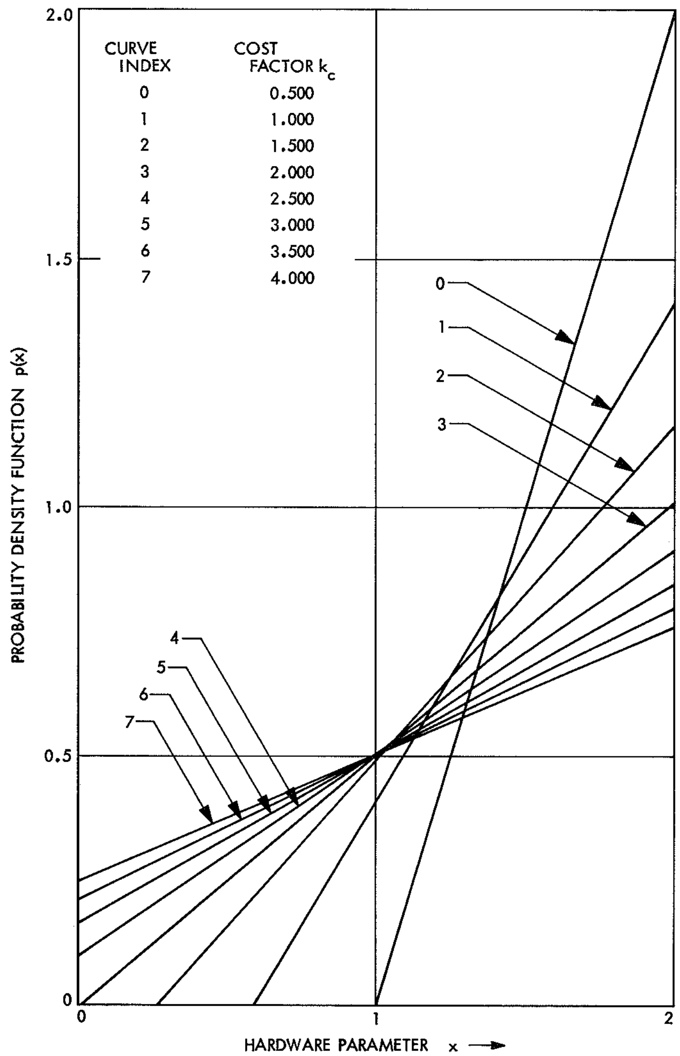

Fig. 47. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 5$, $k_s = 0$, $k_d = 1$

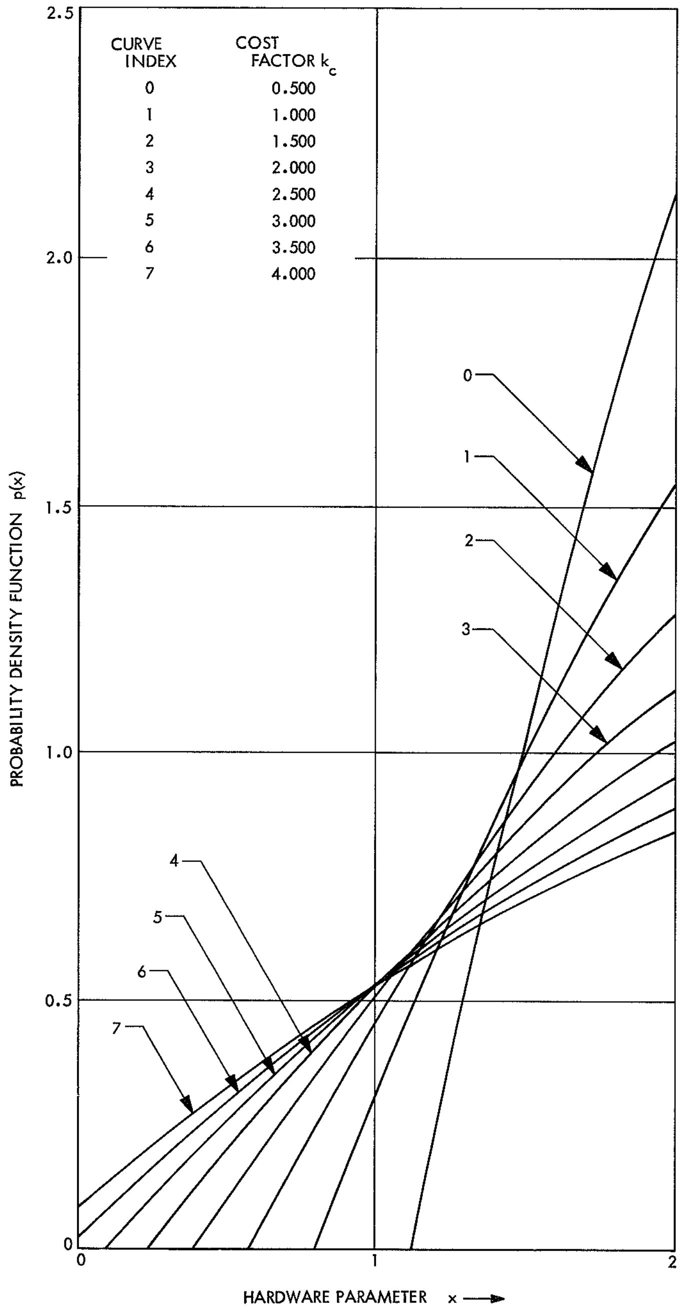

Fig. 48. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 5$, $k_s = 0.5$, $k_d = 1$

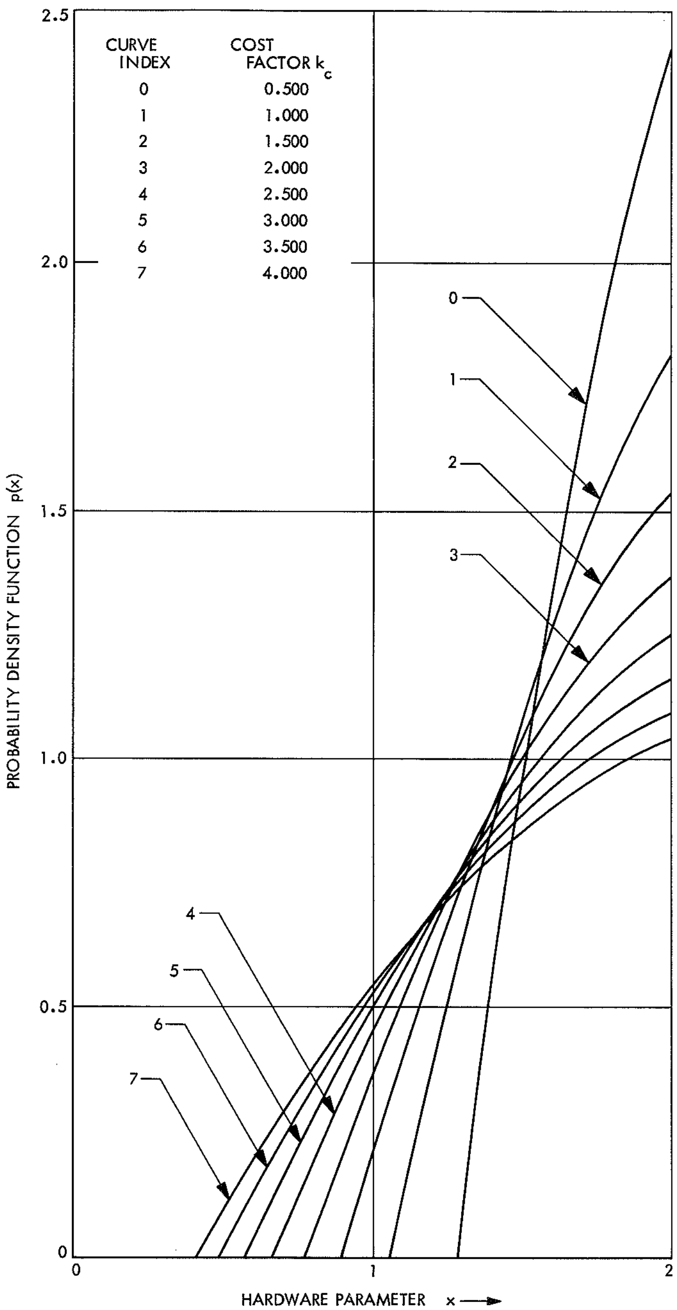

Fig. 49. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 5$, $k_s = 2$, $k_d = 1$

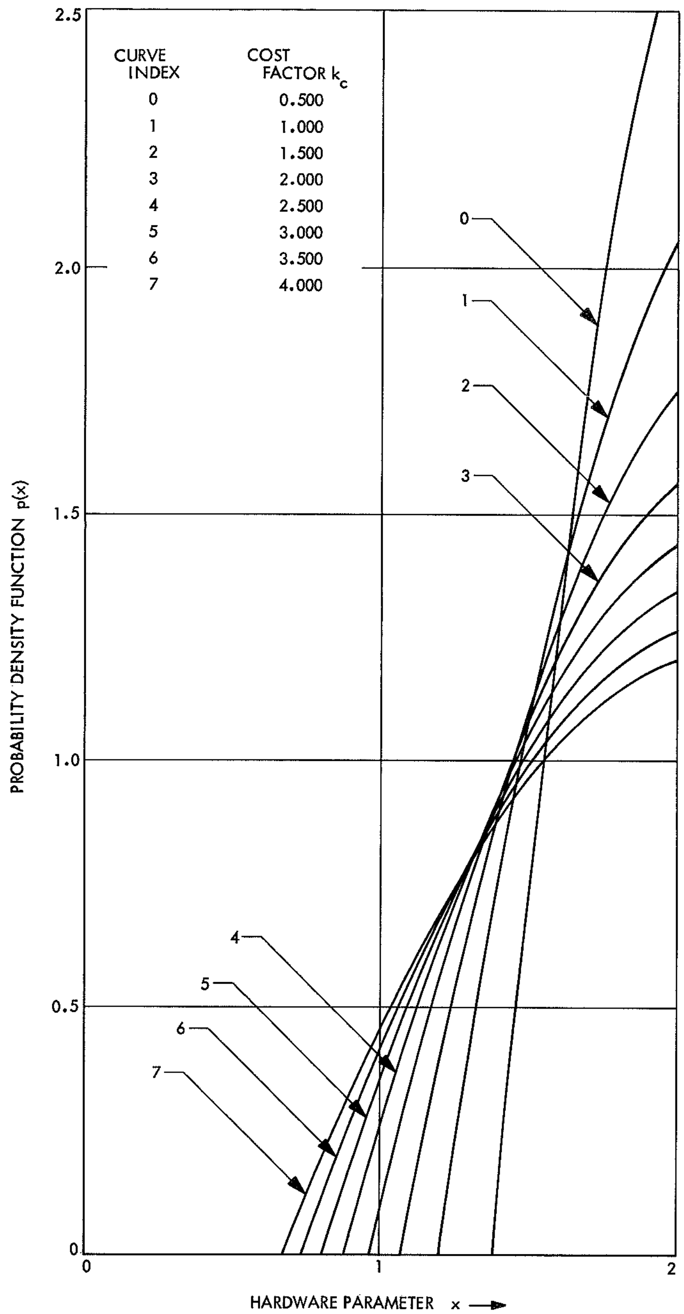

Fig. 50. Probability density function curves—state-of-the-art improvement with incentive award and surveillance strategies; $k_r = 5$, $k_s = 4$, $k_d = 1$

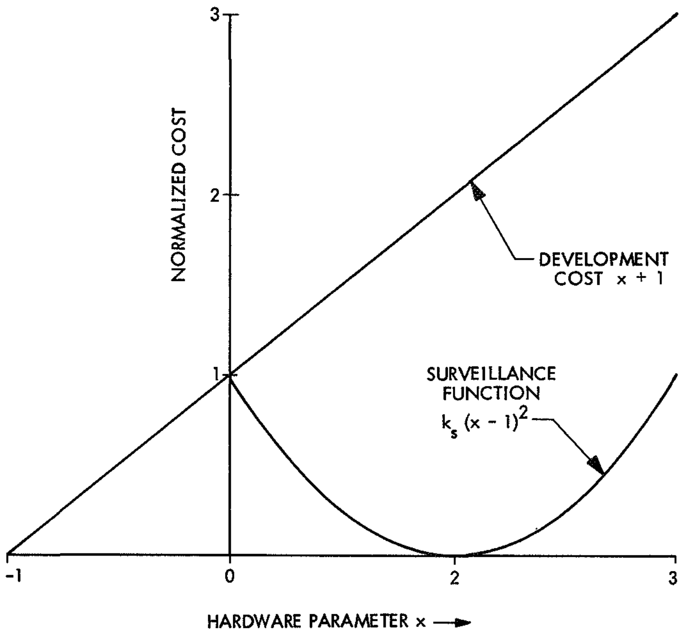

Fig. 51. Mixed strategy—state-of-the-art improvement with surveillance strategy

X. Summary and Conclusions

This effort was directed to establishing probability density functions of a hardware performance parameter delivered by a contractor to a specification. Because sufficient data would not be available to estimate this probability density function experimentally, it was necessary to derive *a priori* probability density functions from assumed axioms determined from an investigation of contractor motivation and behavior.

The analysis began with the realization that the contractor would act to enhance his own self-interests. One of the early conclusions of the investigation of contractor self-interests, motivation, and behavior was that profit is not a universal motivator, but that motivation is a many faceted problem. After examining many of these motivators, it was decided that some aid primarily in the selection of the contractor while others are concerned with what manipulations will motivate the contractor after the contract is placed.

While many of these motivations exist to varying degrees and are important to the success of a contract, it was found that they were not free parameters for the administrator to manipulate. It was concluded, however, that whether or not the free manipulable parameters exist is very important to the success of the contract and how it proceeds and that they affect the parameters of

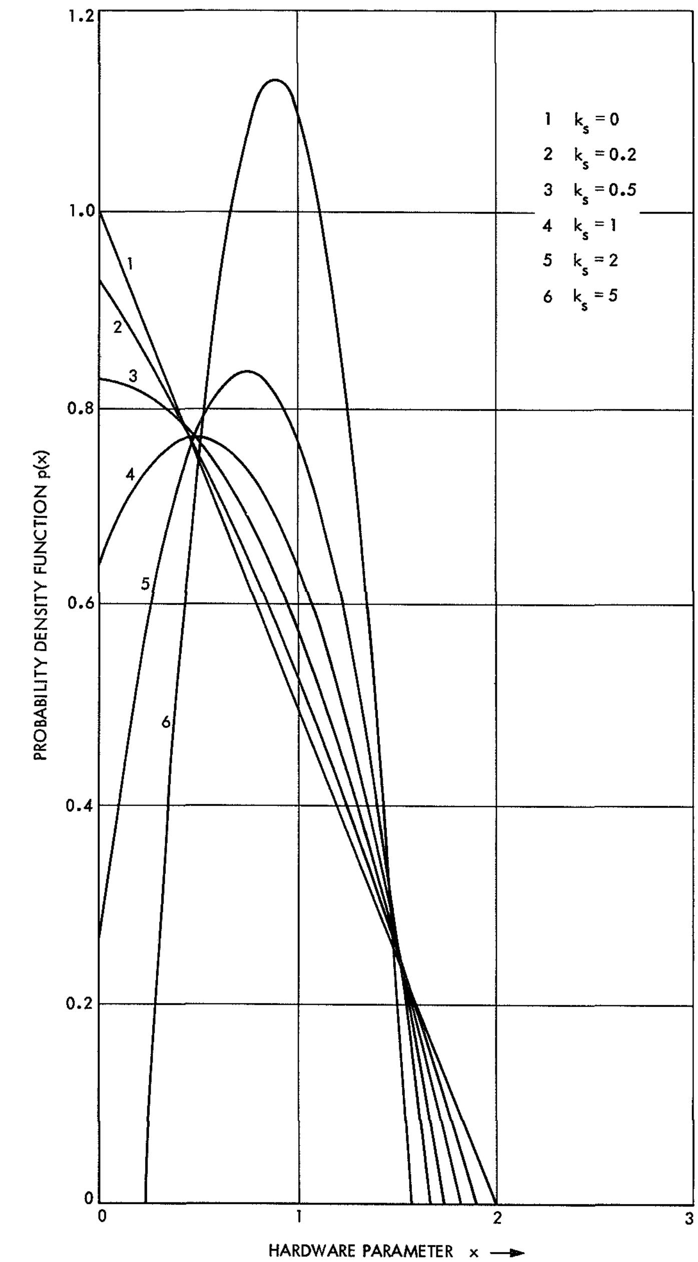

Fig. 52. Probability density function curves—state-of-the-art improvement with surveillance strategy; $k_c = 1$, $k_d = 1$

those motivators that can be manipulated. For instance, the glamour of the work is certainly a prime motivator to the contract success but is intrinsic to the work and cannot be created by the administrator. If it exists, however, the amount of substitute motivation required to accomplish the desired result is less.

On the other hand, survival of the principals is a factor that can be changed during the execution of the contract;

therefore, surveillance becomes a very effective strategy that is controllable by the administrator. Profit as a motivator enters in the problem, but its effectiveness as a strategy depends very heavily on how the contractor management emphasizes it. If its emphasis permeates the organizational structure, then it is an effective tool. If profit is only emphasized at the top levels, then it is not so effective. It was concluded that the assessment of the contractor's motivators during the precontract phase is a very important factor in determining the strategies to be applied to the contractor during the execution phase of the contract to obtain the desired results.

A short discussion concerning the random state of nature, the "unintelligent adversary," was presented. It was assumed that without the random state of nature there was no problem to be solved.

The assumption was made that the discussion of contractor self-interests, motivation, and behavior described a game between two players, who had an unintelligent adversary. From this fact, it was necessary to determine the rules under which they were playing. With this foundation, a formal statement of the problem was possible in a more or less axiomatic approach. With these rules, other conclusions not obvious from the description of the observed play were considered possible.

Once this formulation was accepted, the solution of the problem was undertaken. The solution involved determining appropriate, yet rather difficult, net payoff functions for the contractor and using the calculus of variations with integral and inequality constraints. New techniques for the solution were developed; although equivalent techniques developed by Koopman were found. This equivalence was shown.

After the methodology was developed, numerous contrived examples were given using both isolated and mixed strategies for manipulation of the contractor. These were parameterized for various incentive-to-cost ratios, etc. Both types of examples were investigated to determine the sensitivity of the resultant probability density functions to the parameters. A uniform finding in all of the examples was that there was a decisive lack of sensitivity to the parameters in affecting the resultant probability density functions. This conclusion was a complete surprise in that it was opposite to that expected when this research was begun. Preconditioning by the prevalence of incentive contracting had caused the supposition that this was an effective tool, and that the results sought would bear this out. For the particular factors investigated, this simply was not true.

The resources available for contract manipulation are simply not enough to cause substantial changes in the resultant probability density functions. Without realizing this, contract administrators use this strategy and expect good results. When good results are not forthcoming, the administrator unjustly blames the contractor. As a result, many contractors rightfully refuse to accept incentive award contracts. This analysis supports this reaction.

Surveillance was another major isolated strategy considered. This strategy also exhibited a lack of sensitivity to its parameters. This lack of sensitivity is, however, not as restrictive as was the example using the incentive award. Surveillance can be changed by large factors by the administrator more or less at his initiation and demand. The contractor must more or less endure and respond to this. Since this results in a nonreimbursable cost to the contractor, it becomes rather effective. The administrator must be willing to accept the cost of contractor surveillance as it influences his own operations.

For any strategy, the prize offered must be overwhelming to have any pronounced effect for the stated objective. If mixed strategies are used with a fixed resource for incentives, none of the strategies is likely to yield any pronounced effect for the stated objective. An implication, verified in the analysis, is that mixed strategies drawn from a fixed resource "water down" each factor below the threshold of effectiveness. Stated otherwise, too many separate incentives based on a fixed resource accomplishes nothing. If funds are limited, at most, a single incentive strategy should be attempted.

The conclusion, after all this investigation is that surveillance is the best strategy that the administrator can use. By surveillance, close and well-planned surveillance is meant, since the sensitivity for this strategy was as low as all others as a parameter in the analysis. However, it is the one most directly felt by contractor personnel, and it is a direct cost to the contractor in that it detracts from other possible profits. Since all incentives have limitations, one must recognize those of surveillance. Every Alumni Association realizes that it is important to select a good coach, but surveillance or harassment (i.e., burning him in effigy) only produces a level of results commensurate with what he is willing to pay in terms of health and personal discomfort; only an infinite reward-to-cost ratio will guarantee that perfect 10-0-0 season record.

To obtain satisfactory contract results as outlined, the following cardinal rules are offered for which no substitutes were found in this analysis:

(1) Know your contractors and know them well.

(2) Maintain very close surveillance over your contractors.

In the way of self-criticism, the above analysis was concerned largely with the reactions of the contractor to various manipulations to obtain "satisfactory" probability density functions. Consideration of this problem is a large investigation in itself, in that "satisfactory" probability density functions from a technical point of view may cost the administrator far beyond what he is willing to pay. (This perhaps should be the subject of another investigation.)

One of the prime considerations in deriving *a priori* probability density functions for contractor performance was that sufficient data would not be available to experimentally estimate a probability density function. After the analysis was done it was realized that some of the cost factors to the contractor are just as illusive to obtain as an experimentally estimated probability density function; thus, in some regards, one problem has been exchanged for another. However, knowledge of the trends and the sensitivities of various strategies, which is quite beneficial, has been obtained for directing the administrator.

A few words about the modeling of this problem are appropriate. Because of the nature of this problem, it is not a simple task to formulate an adequate model. As with most analyses, the modeling always can be questioned when examined closely. The fact that the applicability of the solution derived from the model depends entirely upon the accuracy and adequacy of the model is fully realized. In this particular investigation, although the modeling is perhaps inadequate, the trend and conclusions that were obtained are uniform, consistent, and generally in harmony with rational human behavior and common business practice. It is believed that the results of this investigation are in most aspects reasonable and, certainly so, within the restrictions of the assumptions made. Additionally, it is not felt that the conclusions will change substantially with other models.

As a final statement, it is not believed that incentive contracting is an effective tool in obtaining a hardware parameter to a specification; however, close surveillance is considered to be effective. This should not be construed to mean that either of these conclusions apply to contracts with a different purpose, although they are suspected to apply.

Appendix A

Computer Program for Plotting Desired Probability Density Function

Appendix A contains a condensed flow diagram (Fig. A-1) and a listing of the program, including plotting subroutines (Table A-1). The program is based on the procedure outlined in Section VIII-B.

This program is adaptable to a wide variety of strategic functions with linear incentive award, surveillance, and development cost. Up to seven segments can be accommodated to allow for those strategic functions with abrupt changes, or for better approximation should nonlinear incentive award functions be desired. The strategic function is read into the computer in the form of data to eliminate the need of modifying the main computer statements and recompiling every time a new strategic function is used.

The computer program is written so that linear incentive award, development cost, and surveillance are independently established through input data cards (Table A-2). Basically, the mixed strategic function $v(x)$ is programmed in the computer in the form given by

$$v(x) = Z_r \frac{Q_r}{X_{r2} - X_{r1}} (x - X_{r3}) - Z_d \frac{Q_d}{X_{d1} - X_{d2}} (x - X_{d3}) - Z_s(x - X_s)^2 \qquad \text{(A-1)}$$

where Z_r, Z_d, and Z_s are the binary coefficients (true or false) associated with incentive award, development cost, and surveillance portions of the function, respectively; and they assume a value of either zero or one. The various Qs and Xs in the same equation are characteristic constants that define the individual constituent functions. Thus, if the mixed strategic function does not involve an incentive award, the coefficient Z_r will be made zero; otherwise, it will be one, and so forth. The significance of the various constants is clearly illustrated in Fig. A-2.

The computer program also makes provision for strategies with a more irregular form. For example, the triangular incentive award function that has been used so frequently in this report consists of two segments of straight lines, each of which has a different slope value. To cope with the possibility of irregularly formed strategies, the internally stored strategic function is actually given by

$$V(x) = Z_2V_2(x) + Z_3V_3(x) + Z_4V_4(x) + Z_5V_5(x) + Z_6V_6(x) + Z_7V_7(x) + Z_8V_8(x) \qquad \text{(A-2)}$$

where $V_i(x)$ has the identical form of Eq. (A-1). In Eq. (A-2), Z_2 through Z_8 are again binary coefficients that are either zero or one, depending upon the number of segments defined in the strategic function. In the triangular incentive award case, (Fig. A-3), two segments exist so that only Z_2 and Z_3 are equal to one, and the remaining coefficients are zero. Here, Segment 1 and the last segment appear as a horizontal line with zero magnitude for better presentations. This procedure should be used for all cases.

All the binary coefficients and characteristic constants for all segments are read into the computer on input data cards. In addition, the boundary values of the relevant segments denoted by $B(1)$ through $B(N)$ $(N \leq 10)$ as shown in Fig. A-3 are also read into the computer as data.

There are additional data cards required for establishing grid lines of the plotted probability density function curves and for curve identification. The grid lines and the coordinate system for the plotted curve are shown in Fig. A-4.

To establish the graph format and the desired labeling of the grid lines, the following variables are used in the data entry of the plotting subroutine.

NGX = 1010 NLX

NGY = 1010 NLY

NLX = number of spaces for label on the x-axis

NLY = number of spaces for label on the y-axis

For the example shown in Fig. A-4, the following variables are used:

NGX = 4040	NLX = 4
NGY = 5050	NCY = 5

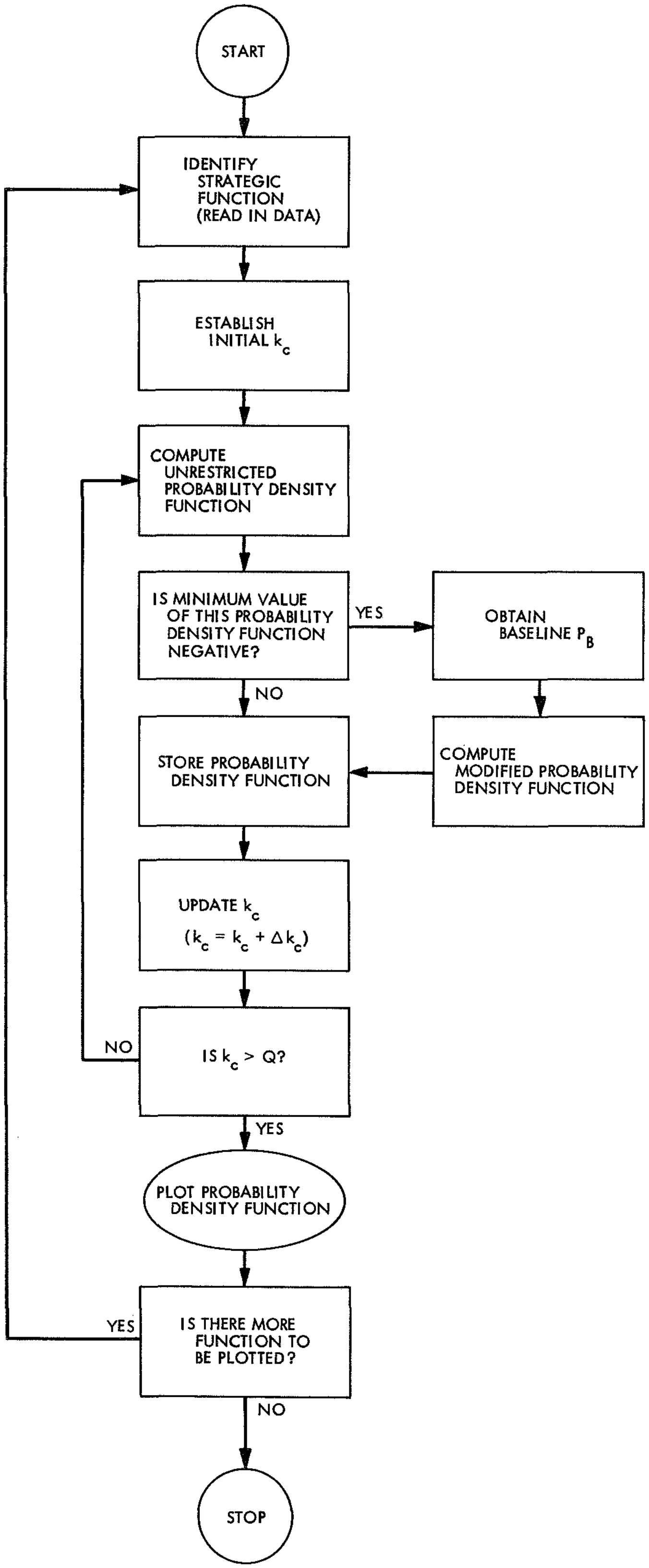

Fig. A-1. Condensed flow diagram of program

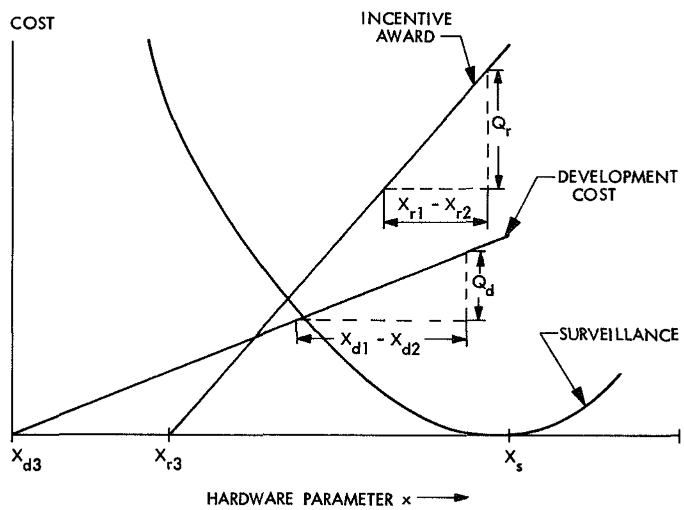

Fig. A-2. Relationship of various constants for the incentive award, development cost, and surveillance strategies

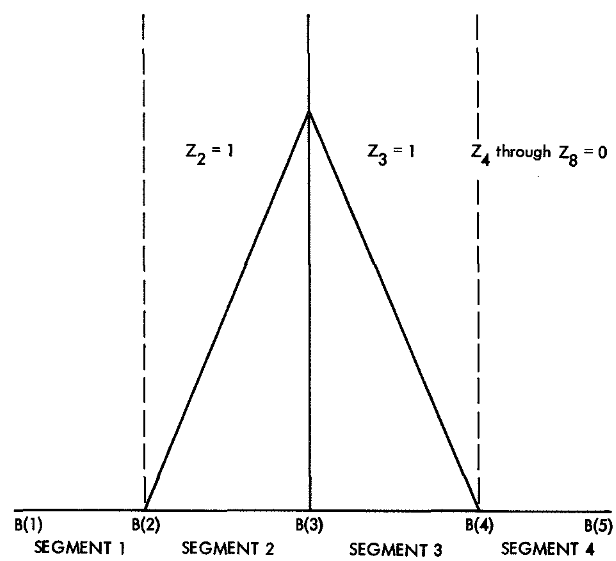

Fig. A-3. Triangular incentive award strategy

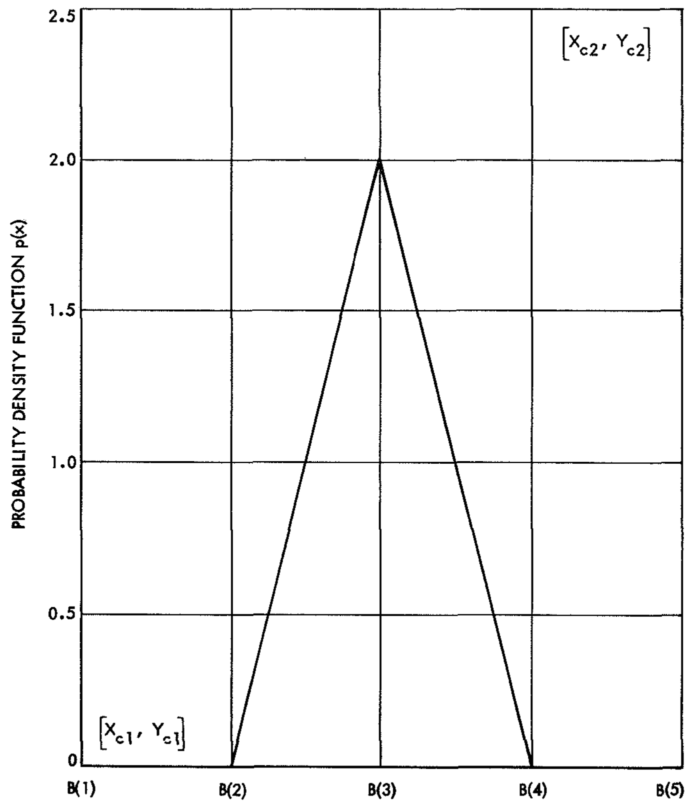

Fig. A-4. Probability density function curves and coordinate system

Table A-1. Program listing

```
$IBFTC D01701
C   SOURCE STATEMENTS - DISTRIBUTION PLOTTING PROGRAM
      DIMENSION B(10),ZD(8),ZH(8),ZS(8),QD(8),QS(8),AD(8,3),AS(8,3)
      DIMENSION TITLE(12)
      DIMENSION EX(501)
      DIMENSION DIST(501,8)
      COMMON /Z01701/ DCKC,M,EX,DIST,TITLE
      U(X)=Z2*(ZS2*CKS*QS2*(X-AS32)/(AS12-AS22)-ZD2*CKD*QD2*(X-AD32)/(AD
     112-AD22)-ZH2*CKH*(X-AH)*(X-AH))+Z3*(ZS3*CKS*QS3*(X-AS33)/(AS13-AS2
     13)-ZD3*CKD*QD3*(X-AD33)/(AD13-AD23)-ZH3*CKH*(X-AH)*(X-AH))+Z4*(ZS4
     1*CKS*QS4*(X-AS34)/(AS14-AS24)-ZD4*CKD*QD4*(X-AD34)/(AD14-AD24)-ZH4
     1*CKH*(X-AH)*(X-AH))+Z5*(ZS5*CKS*QS5*(X-AS35)/(AS15-AS25)-ZD5*CKD*Q
     1D5*(X-AD35)/(AD15-AD25)-ZH5*CKH*(X-AH)*(X-AH))+Z6*(ZS6*CKS*QS6*(X-
     1AD36)/(AS16-AS26)-ZD6*CKD*QD6*(X-AD36)/(AD16-AD26)-ZH6*CKH*(X-AH)*
     1(X-AH))+Z7*(ZS7*CKS*QS7*(X-AS37)/(AS17-AS27)-ZD7*CKD*QD7*(X-AD37)/
     1(AD17-AD27)-ZH7*CKH*(X-AH)*(X-AH))+Z8*(ZS8*CKS*QS8*(X-AS38)/(AS18-
     1AS28)-ZD8*CKD*QD8*(X-AD38)/(AD18-AD28)-ZH8*CKH*(X-AH)*(X-AH))
    1 READ(5,1000) N,CKD,CKH,CKS,DCKC,CKCMX,AH,UXMAX,UXMIN,(B(I),I=1,N)
      READ(5,1003) (ZD(L),ZH(L),ZS(L),QD(L),QS(L),AD(L,1),AD(L,2),AD(L,3
     1),AS(L,1),AS(L,2),AS(L,3),L=1,8)
 1000 FORMAT(I5/8F9.3/(F12.5))
 1003 FORMAT(11F6.2)
C
      READ(5,1200) TITLE
 1200 FORMAT(12A6)
      WRITE(6,1200) TITLE
C
      AS22=AS(2,2)
      AS23=AS(3,2)
      AS24=AS(4,2)
      AS25=AS(5,2)
      AS26=AS(6,2)
      AS27=AS(7,2)
      AS28=AS(8,2)
      AS32=AS(2,3)
      AS33=AS(3,3)
      AS34=AS(4,3)
      AS35=AS(5,3)
      AS36=AS(6,3)
      AS37=AS(7,3)
      AS38=AS(8,3)
      AD12=AD(2,1)
      AD13=AD(3,1)
      AD14=AD(4,1)
      AD15=AD(5,1)
      AD16=AD(6,1)
      AD17=AD(7,1)
      AD18=AD(8,1)
      AD22=AD(2,2)
      AD23=AD(3,2)
      AD24=AD(4,2)
      AD25=AD(5,2)
      AD26=AD(6,2)
      AD27=AD(7,2)
      AD28=AD(8,2)
      AD32=AD(2,3)
      AD33=AD(3,3)
      AD34=AD(4,3)
      AD35=AD(5,3)
```

Table A-1 (contd)

```
      AD36=AD(6,3)
      AD37=AD(7,3)
      AD38=AD(8,3)
      ZD2=ZD(2)
      ZD3=ZD(3)
      ZD4=ZD(4)
      ZD5=ZD(5)
      ZD6=ZD(6)
      ZD7=ZD(7)
      ZD8=ZD(8)
      ZH2=ZH(2)
      ZH3=ZH(3)
      ZH4=ZH(4)
      ZH5=ZH(5)
      ZH6=ZH(6)
      ZH7=ZH(7)
      ZH8=ZH(8)
      ZS2=ZS(2)
      ZS3=ZS(3)
      ZS4=ZS(4)
      ZS5=ZS(5)
      ZS6=ZS(6)
      ZS7=ZS(7)
      ZS8=ZS(8)
      QS2=QS(2)
      QS3=QS(3)
      QS4=QS(4)
      QS5=QS(5)
      QS6=QS(6)
      QS7=QS(7)
      QS8=QS(8)
      QD2=QD(2)
      QD3=QD(3)
      QD4=QD(4)
      QD5=QD(5)
      QD6=QD(6)
      QD7=QD(7)
      QD8=QD(8)
      AS12=AS(2,1)
      AS13=AS(3,1)
      AS14=AS(4,1)
      AS15=AS(5,1)
      AS16=AS(6,1)
      AS17=AS(7,1)
      AS18=AS(8,1)
      NN=N-1
      M=0
      CKC=0.
    5 CKC=CKC+DCKC
      IF(CKC.GT.CKCMX) GO TO 400
      M=M+1
      AREA=0.
      DO 100 I=2,NN
      BLOW=B(I)
      BHGH=B(I+1)
      Z2=0.
      Z3=0.
      Z4=0.
      Z5=0.
```

Table A-1 (contd)

```
      Z6=0.
      Z7=0.
      Z8=0.
      IF(I.EQ.2) GO TO 15
      IF(I.EQ.3) GO TO 20
      IF(I.EQ.4) GO TO 25
      IF(I.EQ.5) GO TO 30
      IF(I.EQ.6) GO TO 35
      IF(I.EQ.7) GO TO 40
      IF(I.EQ.8) GO TO 45
   15 Z2=1.
      GO TO 60
   20 Z3=1.
      GO TO 60
   25 Z4=1.
      GO TO 60
   30 Z5=1.
      GO TO 60
   35 Z6=1.
      GO TO 60
   40 Z7=1.
      GO TO 60
   45 Z8=1.
   60 DXS=(BHGH-BLOW)/2.
      FI1=U(BHGH)+U(BLOW)
      FI2=U(BLOW+DXS)
      FI3=0.
      FI=DXS*(FI1+4.*FI2)/3.
   62 FI3=FI2+FI3
      FI2=0.
      TDXS=DXS
      DXS=.5*DXS
      X=BLOW+DXS
   64 FI2=FI2+U(X)
      X=X+TDXS
      IF(X-BHGH)64,64,66
   66 FIP=DXS*(FI1+4.*FI2+2.*FI3)/3.
      IF(ABS(FIP-FI)-.001*ABS(FIP))80,80,68
   68 FI=FIP
      GO TO 62
   80 AREA=AREA+FIP
  100 CONTINUE
      VLM=(1.-.5*AREA/CKC)/(B(NN)-B(2))
  165 WRITE(6,170)M
  175 WRITE(6,180)VLM
  170 FORMAT(4H M =,I2)
  180 FORMAT(8H LAMDA =,F12.8)
C
      X=B(1)
      DX=.002*(B(N)-B(1))
      DO 200 K=1,501
      Z2=0.
      Z3=0.
      Z4=0.
      Z5=0.
      Z6=0.
      Z7=0.
      Z8=0.
      IF(X.LE.B(2)) GO TO 115
```

Table A-1 (contd)

```
      IF(X.LE.B(3)) GO TO 120
      IF(X.LE.B(4)) GO TO 125
      IF(X.LE.B(5)) GO TO 130
      IF(X.LE.B(6)) GO TO 135
      IF(X.LE.B(7)) GO TO 140
      IF(X.LE.B(8)) GO TO 145
      IF(X.LE.B(9)) GO TO 150
      IF(X.LE.B(10)) GO TO 155
C
  115 PDF=0.
      GO TO 160
  120 Z2=1.
      IF(NN.EQ.2) GO TO 155
      PDF=.5*U(X)/CKC+VLM
      GO TO 160
  125 Z3=1.
      IF(NN.EQ.3) GO TO 155
      PDF=.5*U(X)/CKC+VLM
      GO TO 160
  130 Z4=1.
      IF(NN.EQ.4) GO TO 155
      PDF=.5*U(X)/CKC+VLM
      GO TO 160
  135 Z5=1.
      IF(NN.EQ.5) GO TO 155
      PDF=.5*U(X)/CKC+VLM
      GO TO 160
  140 Z6=1.
      IF(NN.EQ.6) GO TO 155
      PDF=.5*U(X)/CKC+VLM
      GO TO 160
  145 Z7=1.
      IF(NN.EQ.7) GO TO 155
      PDF=.5*U(X)/CKC+VLM
      GO TO 160
  150 Z8=1.
      IF(NN.EQ.8) GO TO 155
      PDF=.5*U(X)/CKC+VLM
      GO TO 160
  155 PDF=0.
  160 DIST(K,M)=PDF
      EX(K)=X
      X=X+DX
  200 CONTINUE
  260 FORMAT(32H LISTING OF DIST (UNCOMPENSATED))
  240 FORMAT(1H ,5F12.8)
C
      SC=.5*UXMIN/CKC+VLM
      IF(SC.GE.0.) GO TO 5
      SUMM=0.
      BASLT=0.
      DBASL=.5*(.5*UXMAX/CKC+VLM)
  310 BASLT=BASLT+DBASL
      GO TO 325
  320 BASLT=BASLT-DBASL
  325 DO 350 K=1,501
      FXMOD=DIST(K,M)-BASLT
      IF(FXMOD.LE.0.) GO TO 350
      SUMM=SUMM+FXMOD
```

Table A-1 (contd)

```
  350 CONTINUE
C
      CHECK=SUMM*DX-1.
      IF(ABS(CHECK).LT..001) GO TO 375
      DBASL=DBASL/2.
      SUMM=0.
      IF(CHECK.GT.0.) GO TO 310
      GO TO 320
  375 BASLN=BASLT
  378 WRITE(6,379)BASLN
  379 FORMAT(8H BASLN =,F12.8)
C
      DO 380 K=1,501
      DIST(K,M)=DIST(K,M)-BASLN
      IF(DIST(K,M).GT.0.) GO TO 380
      DIST(K,M)=0.
  380 CONTINUE
  410 FORMAT(30H LISTING OF DIST (COMPENSATED))
      GO TO 5
  400 CONTINUE
C
      EXTERNAL S01702
      CALL S00202 (S01702)
C
      GO TO 1
      END
$IBFTC D01702
      SUBROUTINE S01702
C     PROGRAM NO. 017-02, REVISED 8/1/68
C
      DIMENSION EX(501),DIST(501,8)
      DIMENSION TITLE(12)
C
      COMMON /Z01701/ DCKC,M,EX,DIST,TITLE
      CALL S00206 (501,501,M,EX,DIST,0,240)
C
      CALL ADV (NT)
      CALL SETPL (0)
      WRITE(18,20) TITLE
   20 FORMAT(1H ,12A6)
      WRITE(18,30)
   30 FORMAT(1H ,3X,5HCURVE,10X,11HCOST FACTOR/1H ,3X,5HINDEX,14X,3HCKC)
C
      CKC=0.
      DO 100 I=1,M
      IC=I-1
      CKC=CKC+DCKC
      WRITE(18,40) IC,CKC
   40 FORMAT(1H ,I6,F20.3)
  100 CONTINUE
      CALL RESPL
      RETURN
      END
```

Table A-1 (contd)

```
      MAIN                    JPL,J264000,36010-0,50085,A   IC       IBJOB V13MI2
            D00202      -   EFN    SOURCE STATEMENT   -   IFN(S)   -

       SUBROUTINE S00202 (PLOT)
C      PROGRAM NO. 002-02, WRITTEN 2/16/66
C
       DIMENSION C(4),C1(4),BUF(1000),DATA(2,100)
       COMMON /A00202/ BUF
       EXTERNAL PLOT
C
       DATA J1 /0/
       DATA (C(I),I=1,4) /75.0,975.0,1023.0,0.0/
C
       READ (5,10) J2
    10 FORMAT(14I5)
       WRITE (6,11) J2
    11 FORMAT(1H1,14I5)
       IF(J2.EQ.0) RETURN
       CALL CAMERA (NT,J2)
C
       IF(J1.EQ.0) CALL SLABEL
C
    20 READ (5,10) J3,NDATA,NGX,NGY,NLX,NLY
       WRITE (6,21) J3,NDATA,NGX,NGY,NLX,NLY
    21 FORMAT(1H0,14I5)
       IF(J3.EQ.0) RETURN
C
    30 READ (5,40) (C1(I),I=1,4)
    40 FORMAT(7F10.8)
       WRITE (6,41) (C1(I),I=1,4)
    41 FORMAT (1H ,7F10.3)
C
       IF(J1.NE.0) CALL ADV (NT)
       J1=1
C
       CALL SDINIT (BUF,1000,C1,C,NT)
       CALL BGRID (BUF,NGX,NGY,1)
       CALL PLAB  (BUF,NLX,6H(F7.2),NLY,6H(F7.2))
C
       IF(NDATA.LE.0) GO TO 80
C
       READ (5,60) ((DATA(I,J),I=1,2),J=1,NDATA)
    60 FORMAT(2F10.8)
       WRITE (6,61) ((DATA(I,J),I=1,2),J=1,NDATA)
    61 FORMAT(1H ,2F10.3)
C
       DO 70 J=1,NDATA
    70 CALL PPLOT (DATA(1,J),DATA(2,J),BUF,1H*)
       CALL STERM (NT,BUF)
C
    80 CALL PLOT
C
       GO TO 20
       END
```

Table A-1 (contd)

```
      MAIN                              JPL,J264000,36010-0,50085,AF IC      IBJOB V13MI1
            D00203        -  EFN    SOURCE STATEMENT  -  IFN(S)  -

            SUBROUTINE S00203 (X,Y,N)
      C     PROGRAM NO. 002-03, WRITTEN 5/6/66, REVISED 2/16/67
      C
            DIMENSION BUF(1000)
            COMMON /A00202/ BUF
      C
            IF(N.LE.0) RETURN
            IF(N.GT.10) RETURN
      C
         20 GO TO (30,31,32,33,34,35,36,37,38,39),N
      C
         30 CALL PPLOT(X,Y,BUF,1H0)
            RETURN
         31 CALL PPLOT(X,Y,BUF,1H1)
            RETURN
         32 CALL PPLOT(X,Y,BUF,1H2)
            RETURN
         33 CALL PPLOT(X,Y,BUF,1H3)
            RETURN
         34 CALL PPLOT(X,Y,BUF,1H4)
            RETURN
         35 CALL PPLOT(X,Y,BUF,1H5)
            RETURN
         36 CALL PPLOT(X,Y,BUF,1H6)
            RETURN
         37 CALL PPLOT (X,Y,BUF,1H7)
            RETURN
         38 CALL PPLOT (X,Y,BUF,1H8)
            RETURN
         39 CALL PPLOT (X,Y,BUF,1H9)
            RETURN
            END
```

Table A-1 (contd)

```
      MAIN                            JPL,J264000,36010-0,50085,A   IC      IBJOB V13MI2
              D00206      -   EFN   SOURCE STATEMENT   -   IFN(S)   -

      SUBROUTINE S00206 (N,NX,NC,X,Y,INTRP,IA)
C     PROGRAM NO. 002-06, WRITTEN 2/16/67
C
      DIMENSION X(N),Y(N,1),BUF(1000)
      COMMON /A00202/ BUF
C
      DO 20 I=1,NC
      DO 10 J=1,NX
   10 CALL SDNPUT (X(J),Y(J,I),BUF,INTRP)
      CALL SINTRP (BUF)
   20 CALL S00203 (X(IA),Y(IA,I),I)
      CALL STERM (NT,BUF)
C
      RETURN
      END
```

Table A-2. Sequence of input data cards

Card sequence	Content	Field	Remarks
1	N	I5	Number of segments $= N - 1, (N \leq 10)$
2	$k_d,k_s,k_r,\Delta k_c,k_c$ max,X_s,$V(x)$max,$V(x)$min	8F9.3	
3	$B(1)$	F12.5	
4	$B(2)$	↓	
5	$B(3)$	↓	
•	•	↓	
•	•	↓	
•	•	↓	
•	•	↓	
•	•	↓	
$N+2$	$B(N)$	F12.5	
$N+3$	0.,0.,0.,0.,0.,0.,−1.,0.,0.,−1.,0.	11F6.2	Coefficients and constants for 1st segment
$N+4$	$Z_d,Z_s,Z_r,Q_d,Q_r,X_{d1},X_{d2},X_{d3},X_{r1},X_{r2},X_{r3}$	↓	Coefficients and constants for 2nd segment
$N+5$	↓	↓	Coefficients and constants for 3rd segment
$N+6$	↓	↓	Coefficients and constants for 4th segment
$N+7$	↓	↓	Coefficients and constants for 5th segment
$N+8$	↓	↓	Coefficients and constants for 6th segment
$N+9$	↓	↓	Coefficients and constants for 7th segment
$N+10$	$Z_d,Z_s,Z_r,Q_d,Q_r,X_{d1},X_{d2},X_{d3},X_{r1},X_{r2},X_{r3}$	11F6.2	Coefficients and constants for 8th segment
$N+11$	Title	12A6	
$N+12$	2	I5	
$N+13$	1,0,NGX,NGY,NLX,NLY	6I5	Grid lines and labeling
$N+14$	$X_{c1},Y_{c1},X_{c2},Y_{c2}$	4F10.8	Graph coordinate
$N+15$	0	I5	

Appendix B

Analysis of Available Spacecraft Transmitter Data

The purpose of Appendix B is to provide experimental verification of the analysis presented in this report by comparing (1) data of hardware parameters delivered to a specification and (2) data of the contract circumstances to data that would be predicted by the analysis. The hardware parameter chosen for this comparison was the power output of spacecraft transmitters. This parameter was chosen because it is critical in a space mission and it is generally difficult to achieve.

I. Data Analysis

Table B-1 is a listing of all readily available spacecraft transmitter power outputs. It is not an exhaustive listing of spacecraft missions and only reflects data that were easily available. Only minimum effort was made to qualify any of the data. All examples received are listed, although all have not proved to be useful examples for the purpose outlined above. The main cause for this lack of usefulness is that the number of samples is too small to permit strong inferences to be made. A second problem in the data analysis is that the true or complete strategies that were applied in the transmitter developments were not available during data collection.

The data have been plotted individually in cumulated histograms on which manual fits were estimated. These fitted curves were then differentiated to obtain estimated probability density functions. This procedure was found most useful for the small number of samples available where histograms display the characteristic of noise.

While some of the strategies were known, there appear to be implied or unstated strategies in the examples. In some instances, these implied strategies cause problems in identifying the individual contribution to the effect. Only the total strategy can be estimated because there are unlimited ways of breaking the total strategy into individual strategies. (In the problem described in the body of this report, the strategies were arbitrarily chosen.)

II. *Ranger–Mariner II*

Both the *Ranger* and *Mariner II* missions used essentially the same transmitter. This transmitter was one of the early space transmitter that was required to demonstrate many factors in a relatively new environment. Because much of the data on the power output of the transmitter has been lost in the intervening time since the missions, strong inferences cannot be made. However, the parameter transmitter power output was not a dominant consideration and was not regarded significant to the analysis.

The *Ranger–Mariner II* used an L-band cavity transmitter with a target power output of 3 W. A power output of 3 W at the L-band frequency was then well within the state-of-the-art. Dominant characteristics considered of greater importance were: bandwidth, lifetime, thermal control, and quality control as related to reliability.

The *Ranger–Mariner II* transmitter was developed at JPL. Although the analysis presented in this report is still applicable under these circumstances (if one considers suborganizations as the contractor), the situation potentially introduces a difficult complication. For example, it was impossible to determine whether the designer established the specification, or whether the design preceded the specification. For the production models that the data represent, a fixed price contract was placed for fabrication and cursory testing to JPL drawings and procedures. Pretested tubes were supplied to the vendor, and final tests and adjustments were made at JPL. The final established specification was 2.5 W.

The strategies applied to either the designer or the vendor cannot be determined. It was difficult to determine the personal interests of the designer or the vendor in regard to the transmitter power output because this parameter was not the dominant specification. Therefore, no strong inference can be drawn from inspection of the data in Fig. B-1.

III. *Mariner IV–Mariner V*

Surplus hardware from *Mariner IV* was used on *Mariner V*; therefore both *Mariners IV* and *V* are considered to have used the same hardware population.

By the time the *Mariner IV* mission was defined, procedures for telecommunication link performance had

Table B-1. Spacecraft transmitter RF power output

Ranger–Mariner II transponder		*Mariner IV, V* transponder (cavity)		*Mariner IV, V* TWT amplifier		*Mariner 69* TWT amplifier		*Apollo* (MSC) transponder		*Apollo* (MSFC) transponder		*Lunar Orbiter* TWT amplifier		*Pioneer* transponder	
Mission	Power output, W[a]	Serial number	Power output, W	Serial number	Power output, W[b]	Serial number	Power output, W[c]	Mission	Power output, W	Mission	Power output, W	Mission	Power output, W	Mission	Power output, W
R-1	—	C110	9.4	1	11.35	282	18.9	A501	11.48	A501	19.74	L01	12.2	P-6	9.3
R-2	—	C119	6.17	2	11.85	284	18.7	A502	11.75	A502	19.0	L02	11.9	P-7	8.3
R-3	—	C123	6.91	3	11.6	285	19.0	L205	13.2	A503	19.2	L03	11.9	P-8	9.3
R-4	—	C130	6.91	4	10.8	286	19.25			A504	20.42	L04	11.8		
R-5	—	C131	6.16	7	11.8	289	19.65					L05	12.2		
R-6	3.02			10	11.75	293	19.3								
R-7	—			12	12.65	296	19.7								
R-8	2.82			13	10.95	297	20.6								
R-9	2.76			15	12.2	312	20.0								
M-1	—			16	11.1	315	18.4								
M-2	—			18	11.35	316	18.42								
				20	11.4	317	19.92								
				21	11.6	320	19.45								
				22	11.6	323	20.75								
				24	10.9										
				25	11.1										
				27	11.35										
				28	13.3										
				29	12.6										
				30	11.8										

[a]Telemetered value measured at low gain antenna monitor.
[b]Nominal bench values measured by vendor.
[c]Worst-case over environment and voltage of bench values measured by vendor.

been firmly established. In the preliminary mission definition, two dominant characteristics of the transmitter were indicated:

(1) Power output greater than 10 W at S-band for greater than 6000 h.

(2) A raw power constraint requiring 30% efficiency.

A cavity amplifier was determined to be the best type of power amplifier to meet these requirements. Amplitrons, klystrons, and traveling-wave tube (TWT) amplifiers were also considered but were discarded for various reasons.

A cavity amplifier was developed at JPL for the *Mariner* program. Two major difficulties were identified in a late stage of the development. One problem was that a mechanical design choice in the final models made it difficult to obtain 10 W from the transmitters. A second problem was that life tests indicated a serious degradation of power with time. Major remedial action was undertaken to guarantee integrity of the *Mariner* program.

A reexamination of the availability of raw power showed that 30% efficiency was no longer required. This fact permitted consideration of a TWT amplifier that had demonstrated good lifetime characteristics. Subsequently, an alternate *Mariner IV* design was developed to include implementation of the TWT amplifier.

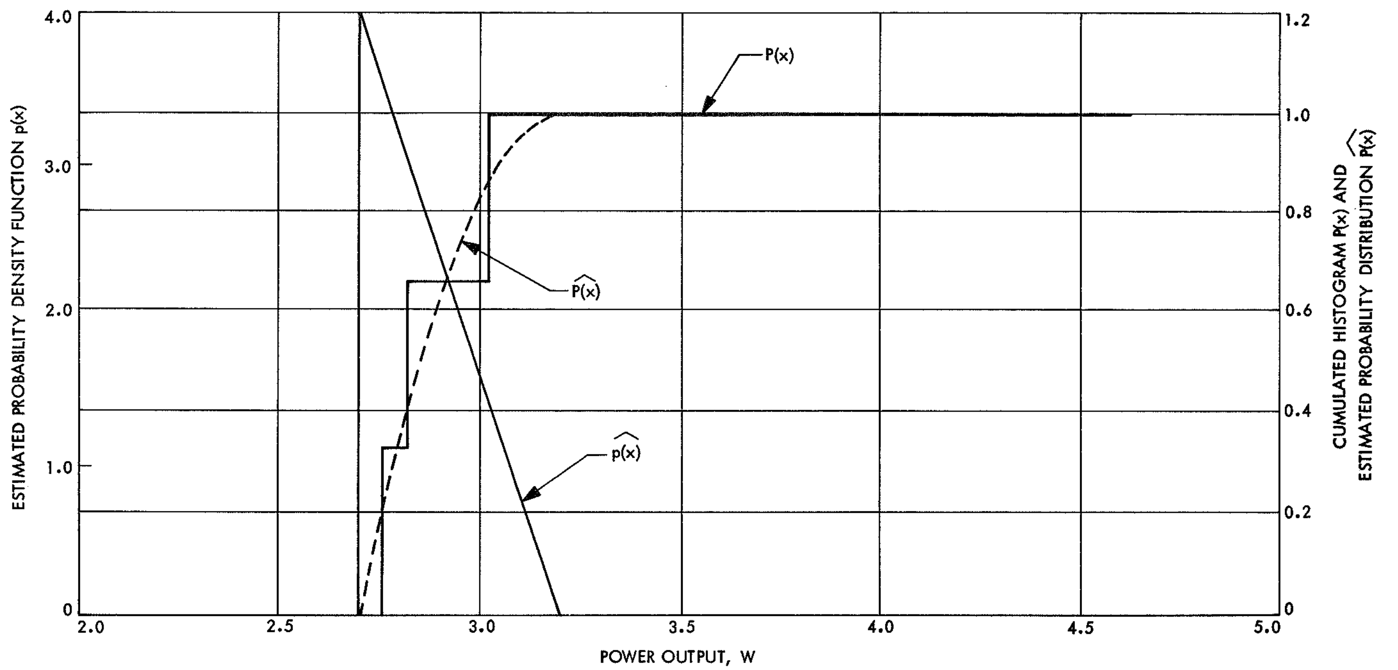

Fig. B-1. ***Ranger–Mariner II*** **transponder—power output, cumulated histogram, and estimated probability density function**

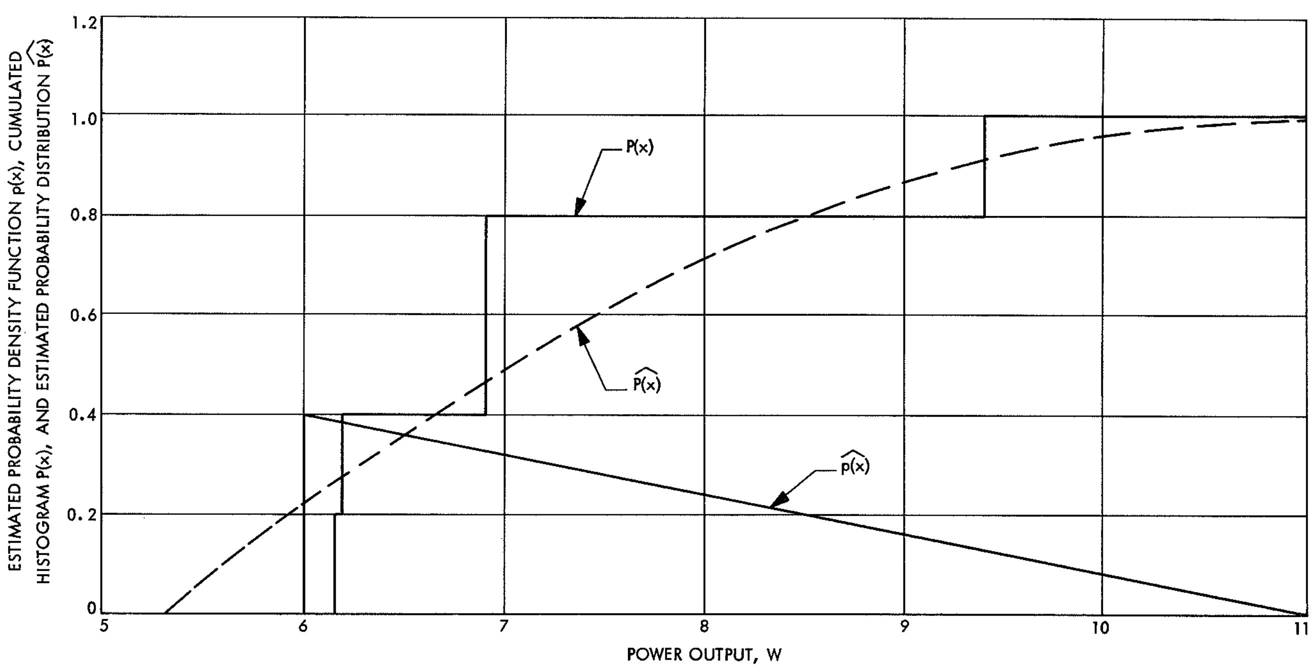

Fig. B-2. ***Mariner IV*** **transponder (cavity)—power output, cumulated histogram, and estimated probability density function**

Corrective mechanical measures eliminated the power output problem in the cavity amplifier, but power degradation continued to be a problem. This problem was partially rectified when the filament voltage was reduced, thereby lowering the power output. Final link analysis revealed that the resulting power output would accommodate encounter only if the cavity amplifier use was restricted to certain phases of the mission. The resulting design was a hybrid that included one cavity amplifier and one TWT amplifier.

The launch phase of the mission was accomplished with the cavity amplifier because it could survive the low pressure breakdown region. Later, the system was switched to the TWT amplifier so that the life of the cavity amplifier could be conserved for an emergency mode during encounter. (The preceding discussion indicates the strong surveillance given the *Mariner* transmitter problem.)

A. *Mariner* Cavity Amplifier Data

The *Mariner* cavity amplifier specification was subsequently reduced from the original 10-W minimum to a 6-W minimum. Figure B-2 shows the cumulated histogram of the delivered units. The fitted data follow closely that which would be expected from what was described previously as state-of-the-art improvement (Section IX-A-4). There is some hint of a mixed strategy of state-of-the-art improvement with surveillance centered at about 8 W (Section IX-B-3 and Fig. 52 for $k_s = 5$). However, with only five samples, strong inferences cannot be made. It appears that the surveillance influence should be stronger because so much attention was given to the *Mariner IV* cavity amplifier problem. However, it was difficult to recognize the need for action and to implement timely follow-up procedures that would have guaranteed the effectiveness of the corrective action. This explanation is satisfactory only for the cavity amplifier; however, if the complete mission is under consideration, it should be stated that the TWT amplifier did satisfy the specification in response to the surveillance given.

B. *Mariner* Traveling-Wave Tube Amplifier

The TWT amplifier selected for the *Mariner* mission had already been developed for the *Surveyor* program and had been subjected to some lifetime tests. By the time orders were placed, this amplifier was considered an off-the-shelf item.

The *Surveyor* specification was identical to the original *Mariner* specification (i.e., 10-W minimum); however, TWT amplifiers were not put in bonded stock unless they had demonstrated more than 11 W. For this reason, the target power output for the design had been 11.5 W.

There are two main selection and adjustment processes on the TWT amplifier assembly that can be made to change the basic power output of the tube by modest amounts. The focusing magnets can be restacked and the output matching stub can be changed. Both of these adjustments can be made inexpensively. Apparently, attention was only given to these adjustments if the power was below 11.8 W.

The cumulated histogram of the TWT amplifiers is presented in Fig. B-3. Strong inferences can be made from these data because a large number of samples were used in the analysis.

Figure B-3 also shows the fitted distribution function and the fitted probability density function. These are distinctive and unusual functions, and from the analysis presented in this report, it can be surmised that the probability density function must arise from a mixed or composite strategy as shown in Fig. B-4a. This strategy may be broken up in an unlimited number of ways; however, it seems that the strategy shown in Fig. B-4b is a reasonable choice. This anticipated strategy is obtained through an inverse process as given in Section VIII.

There is a distinct breaking point at 11.85 W that is suspected to be the point at which there is strong internal contractor management surveillance and procedural influence to leave the unit alone. However, there seems to be internal contractor management surveillance that increases linearly as the power output falls below 11.85 W. The difference between 11.85 W and the 11.5-W target value seems to be about the accuracy at which the power can be measured. In addition to the assumed surveillance function, there seems to be a modest cost that increases with output power and is referred to as development cost. Because the TWT amplifier was assumed to be an off-the-shelf item, "development cost" may be a misnomer.

IV. *Mariner*

The *Mariner 69* radio frequency subsystem (RFS), which includes the telecommunication radio and microwave components, was placed with a single system contractor. This contract included incentives on several parameters to assure performance of certain important functions of the subsystem.

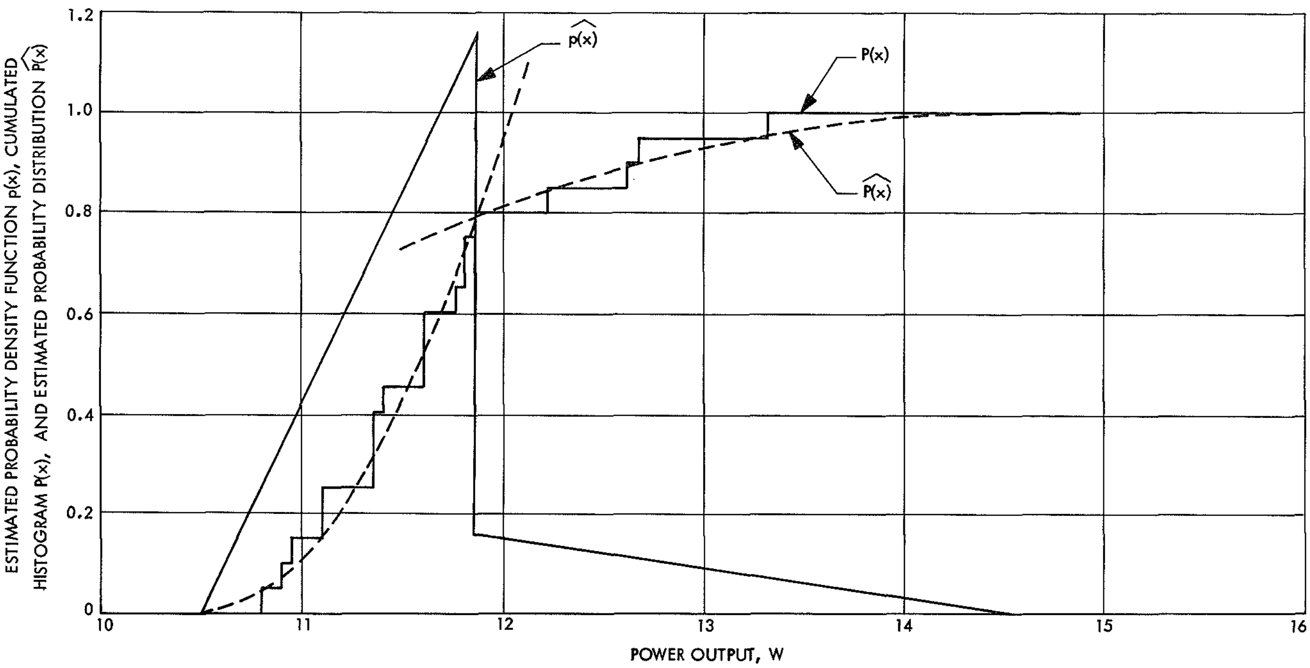

Fig. B-3. ***Mariner IV*** **TWT amplifier—power output, cumulated histogram, and estimated probability density function**

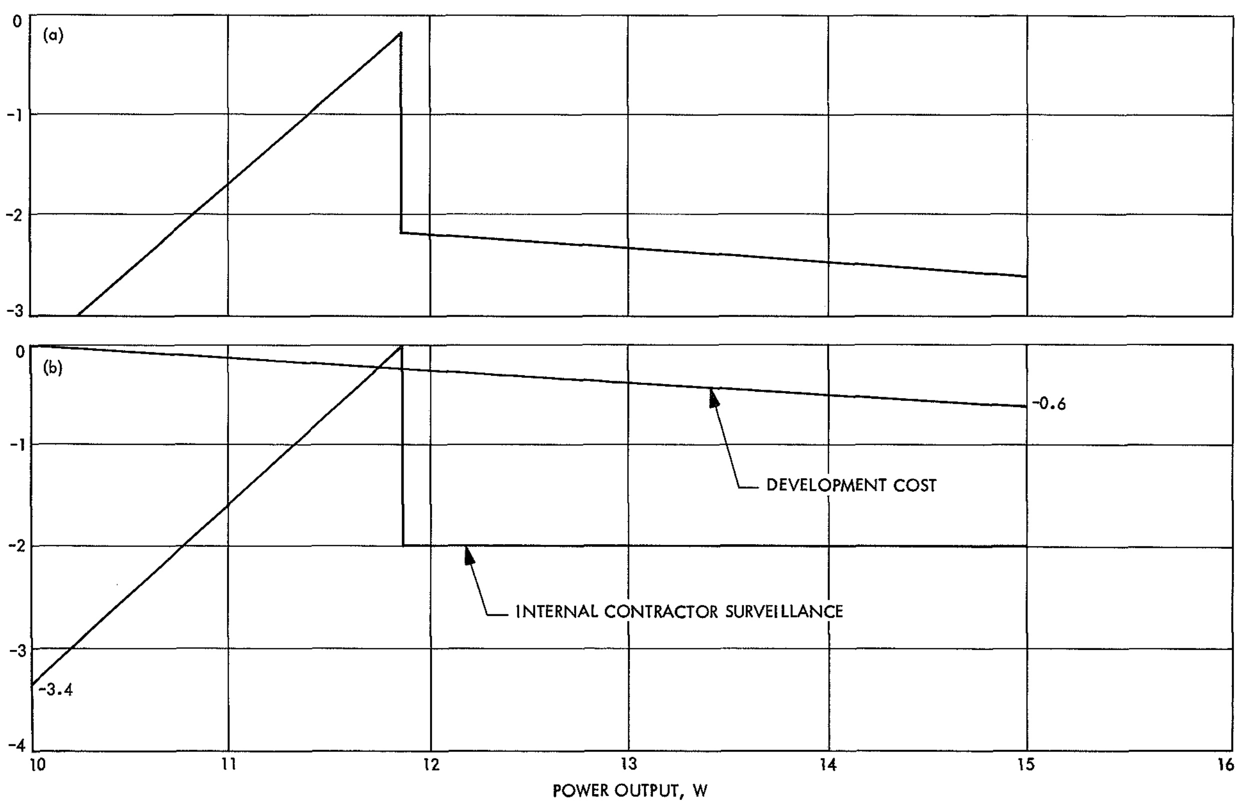

Fig. B-4. Anticipated strategy for the ***Mariner*** **TWT amplifier power output: (a) composite strategy; (b) anticipated components for the overall strategy**

Negotiations with the system contractor resulted in incentives on the total RF power delivered to the high gain antenna terminals and on the total circuit losses between the transmitter and the high gain antenna terminals. These incentives were as follows:

Power output, dBm	Award
> 41.95	$12,500
< 41.95	None
Circuit losses, dB	**Award**
< 0.95	$ 9,375
< 1.40	$ 1,875
> 1.40	None

Implementation of the subsystem was accomplished through a second group of contracts. Some of these contracts included incentives applied by the system contractor with strong insistence by JPL. In the particular parameter of interest here, the TWT amplifiers were subcontracted on an incentive schedule as follows:

Power output, W	Award
18.3 (minimum)	None
< 19.4	None
> 19.4	$1000/TWT amplifier
> 19.4 (less than 68 W raw power required)	Additional $1000/ TWT amplifier

These parameters were to be worst-case values under conditions of voltage variations and environmental conditions. To meet these conditions, an incentive was given and the price was increased by about 50%. This increase was requested so that the amplifiers could be given greater care and so a greater number of parts for adjustments could be purchased.

Table B-2 is a listing of the power outputs of the delivered TWT amplifiers. The power outputs are plotted in cumulated histograms with fitted distribution and density functions in Figs. B-5 to B-7.

Although the incentives were specified in the contract, the test procedures were negotiated during the execution

Table B-2. *Mariner* TWT amplifier—power output data

Serial number	Nominal power output, W	Worst-case power output, W	Conditional worst-case power output, W[a]
282	22.1	18.9	19.5
284	21.5	18.7	20.15
285	22.03	19.0	19.75
286	22.38	19.25	20.15
289	22.65	19.65	20.3
293	23.81	19.3	20.5
296	22.75	19.7	20.4
297	23.4	20.6	21.8
312	22.93	20.0	20.1
315	22.90	18.4	—[b]
316	22.10	18.42	19.3
317	23.50	19.92	21.30
320	22.24	19.45	20.15
323	24.00	20.75	22.8

[a]Conditional worst-case data are taken without a simulated power supply source resistance.

[b]No data available.

of the contract. These negotiations began about nine months before the tests began and continued through the early phase of the tests. The procedures to which the TWT amplifiers were being tested included provisions to compensate for the effects of flight on the internal impedance of the power supply for the TWT amplifier. (This test realistically represents the flight configuration.)

Plots of the nominal power output of the delivered TWT amplifiers are shown in Fig. B-5. This figure contains data that resulted from the methods that the subcontractor used in preparing to meet the worst-case tests; it is important to consider these methods because the TWT amplifiers were not modified to compensate for the worst-case test results. These curves closely resemble that of Section IX-A-1, which had a triangular incentive award. Of course, this is the result of subcontractor internal strategies and might be interpreted as either incentive or surveillance. It is apparent that the subcontractor is trying actively to pursue the award for high power output and low raw power requirements. Slight influence is also apparent for some linear development costs.

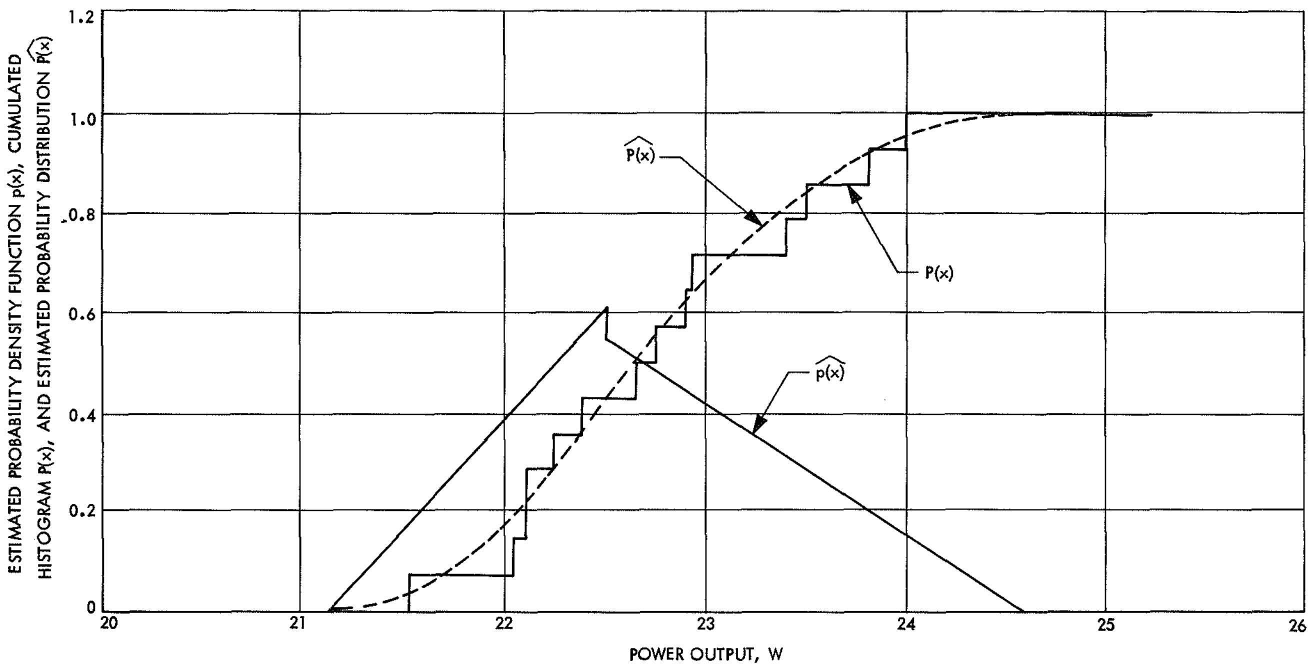

Fig. B-5. *Mariner VI* **and** *VII* **TWT amplifier—nominal power output, cumulated histogram, and estimated probability density function**

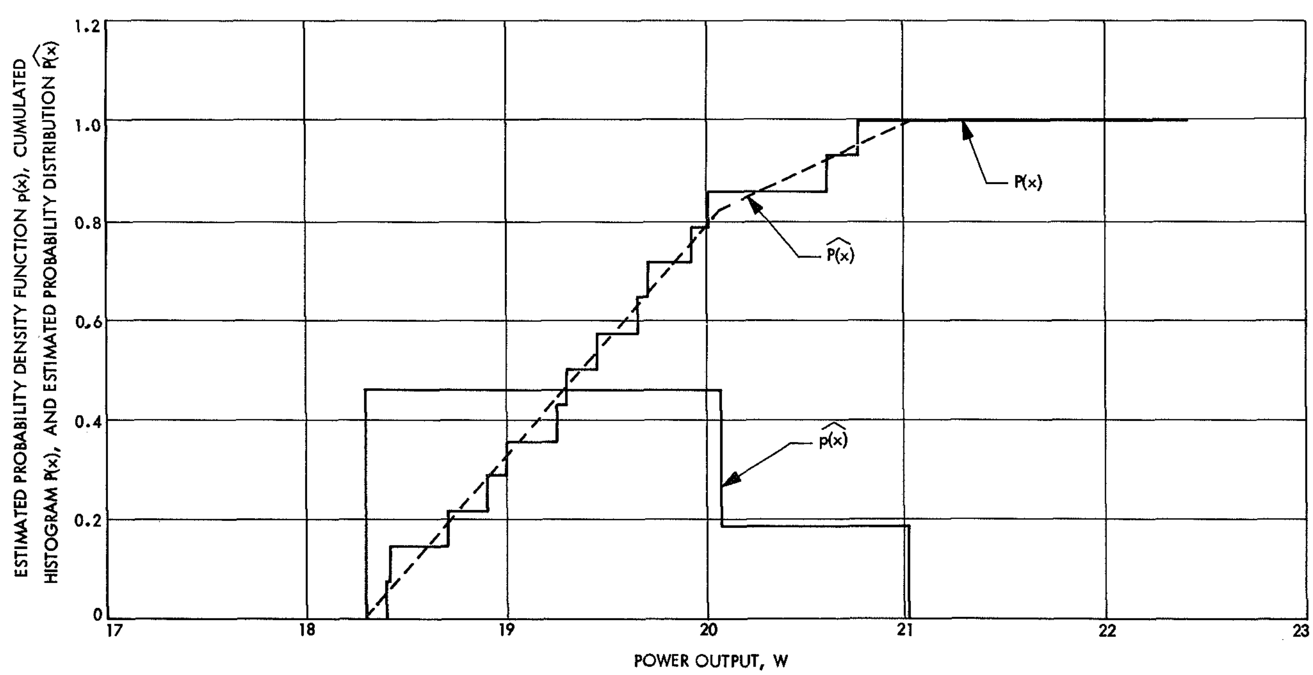

Fig. B-6. *Mariner VI* **and** *VII* **TWT amplifier—worst-case power output, cumulated histogram, and estimated probability density function**

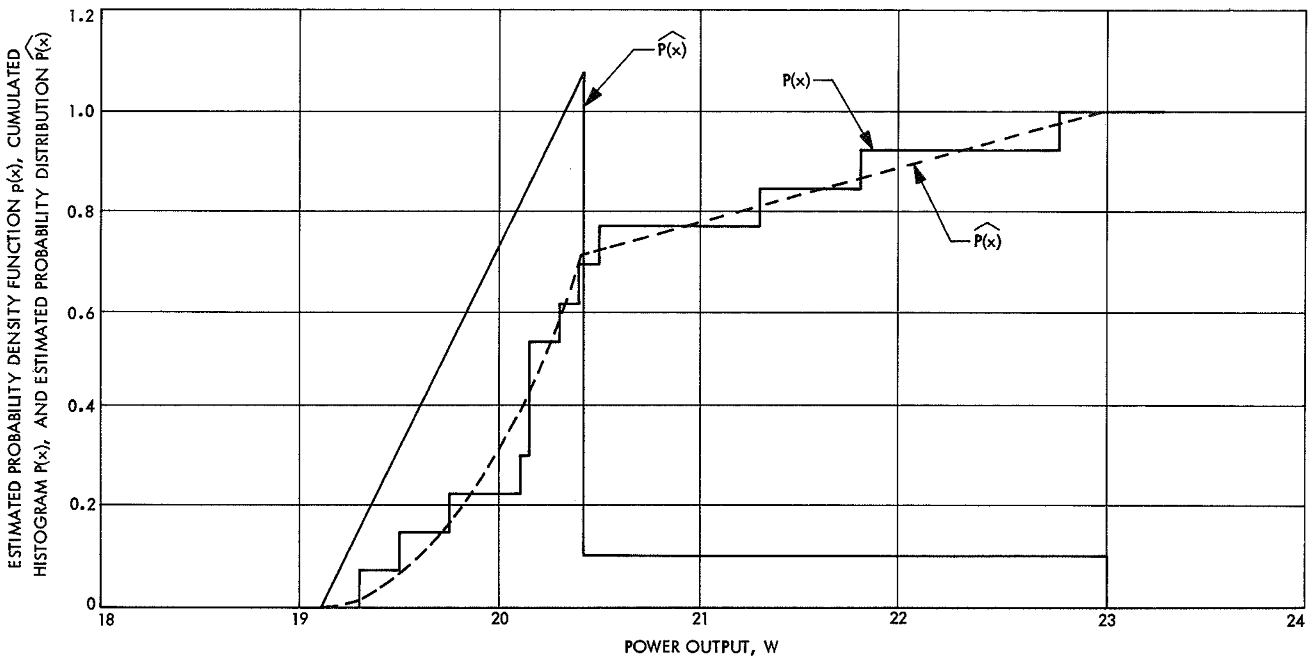

Fig. B-7. *Mariner VI* and *VII* TWT amplifier—power output, cumulated histogram, and estimated probability density function (with resistor modification)

Plots of the worst-case power output of the delivered TWT amplifiers are shown in Fig. B-6. The results indicate a complete indifference to the incentive award if the results are taken at face value. This indifference could be caused by the fact that there is no rework performed after the worst-case tests and that the worst-case conditions are the dominant factors in the results, completely nullifying some of the influence of the previous design effort.

All the units met the minimum specification (otherwise they would have been undeliverable), and only half the units met the incentive award criterion. Although the incentive award had no apparent effect (as indicated by the resulting probability density function of the hardware parameter), the incentive award did motivate the subcontractor to pursue the devious path of renegotiation. Without concurrence of JPL, the subcontractor was subsequently relieved (by the system contractor) of the requirement to compensate for the impedance of the TWT amplifier power supply. The result of this renegotiation completely negated the purpose of the incentive award and allowed the incentive to be paid on all but two of the delivered tubes.

Plots of the modified worst-case power output (relaxed test procedure) are shown in Fig. B-7. There is small difference in the shape of the curves in Fig. B-7 from that of the worst-case curves. They in fact look very similar to that of the *Mariner IV* TWT amplifiers, although the tests of the *Mariner IV* TWT amplifiers were not based on worst-case conditions.

V. *Apollo* Manned Spacecraft Center

A cumulated histogram of the *Apollo* Manned Spacecraft Center (MSC) transponder power output is shown in Fig. B-8. Because so few samples were available, the inference was weak, but it does appear to be a triangular probability density function sloping upward toward the low value of 11.0 W from 14.3 W. There was a minimum specification on this transponder of 11.2 W with no incentive offered. The assumed curve would indicate a linear development cost function.

The TWT amplifier used in this transponder was the same type as used on *Mariner IV*. Since exact details of the *Apollo* (MSC) measurements are not known and because there are so few samples, no attempt will be made to determine if they came from the same population.

VI. *Apollo* Marshall Space Flight Center

A cumulated histogram of the *Apollo* Marshall Space Flight Center (MSFC) transponder power output is

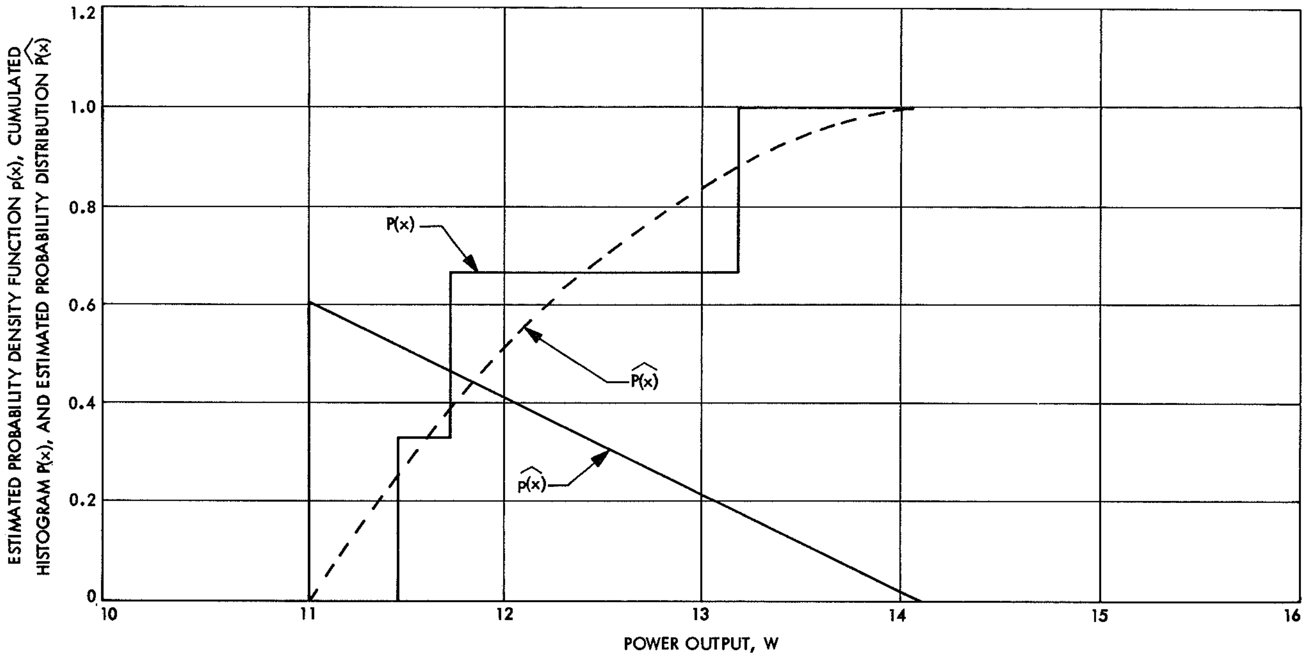

Fig. B-8. ***Apollo*** **(MSC) transponder—power output, cumulated histogram, and estimated probability density function**

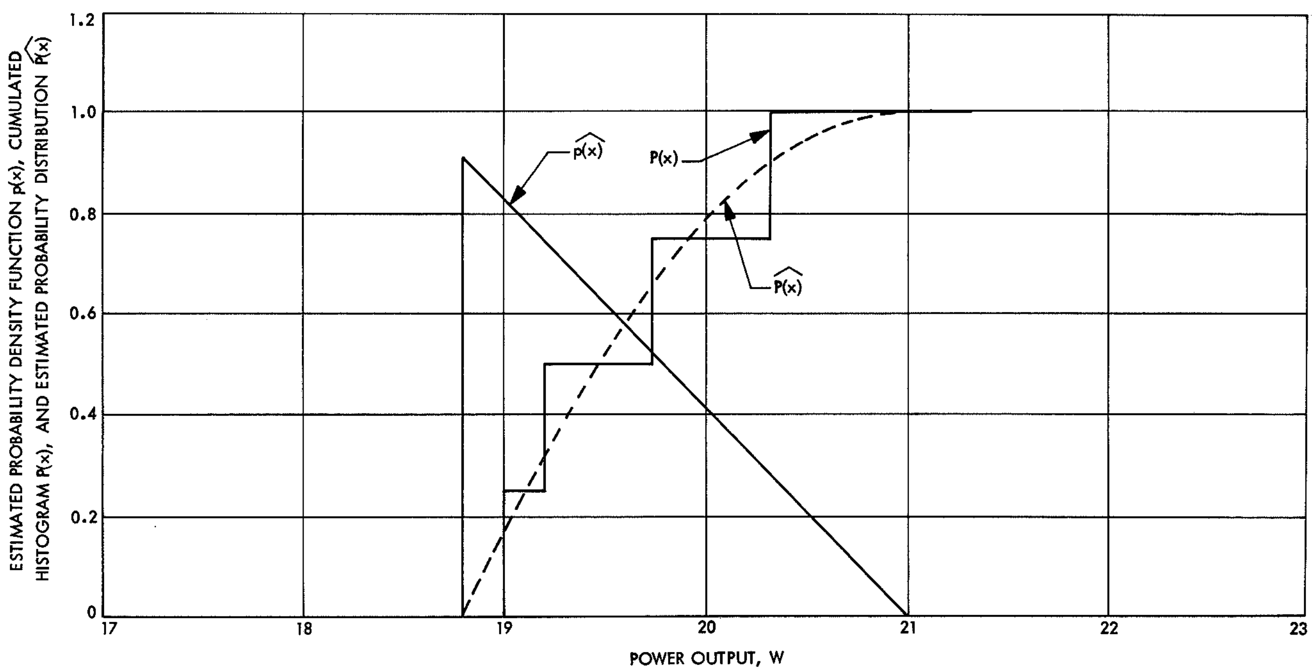

Fig. B-9. ***Apollo*** **(MSFC) transponder—power output, cumulated histogram, and estimated probability density function**

shown in Fig. B-9. The weakness of the inference is similar to that in the *Apollo* MSC data. A triangular probability density function sloping upward toward 18.8 W from 21.0 W might be reasonable. There was a minimum specification on this transponder of 15 W with no incentive offered. The assumed curve would indicate an inferred minimum specification of 18.8 W and a linear development cost function.

Although it is not clearly indicated in the cumulated histogram, it is quite possible that a considerable amount of surveillance in the same form as given in Section IX-B-3 may have actually been applied during the course of the development program of this transponder. This possibility can also be said for the *Apollo* MSC program.

VII. *Lunar Orbiter*

A cumulated histogram of the *Lunar Orbiter* TWT amplifiers is shown in Fig. B-10. The small number of samples makes the inference weak; however, fitted curves are drawn. There was a specification of 10, +3.0 −0.0 W for this function, but no incentive was included. However, load requirements were included. A triangular probability density function corresponding to a linear development cost seems to fit reasonably. The specification seems to have little influence on the shape of the curve.

Data were also available for the transponder power output that differs from the TWT amplifier power output because of interconnections and filters that are included. A cumulated histogram for this function is shown in Fig. B-11. This function had a 2.98-W minimum specification over a temperature range of 35–85°F. Load requirements were also included. The resulting fitted probability density function is a triangular one that is reversed in slope from that of the TWT amplifier. This seems to be a good example of the selective process the contractor has available to adjust the probability density function.

VIII. *Pioneer*

A cumulated histogram of the *Pioneer* 6, 7, and 8 transmitters is shown in Fig. B-12. A fit could not be attempted because the number of samples was too small.

IX. Conclusions

In all of the examples cited, the predominant influence on the resultant probability density functions appeared to be a linear development cost. In one transmitter example (for which incentives were applied to the contract), there was no perceptible effect on the results. There was only subtle hint of any effect of a surveillance function. Either the k_d to k_s ratio is extremely high or the function as defined is not the one used. Perhaps the surveillance function is more coincidental to the specification than was suspected. In one of the examples, there seems to be a clear-cut correspondence to that which was expected from the strategies. Nothing was found to contradict the theory presented in the body of this report.

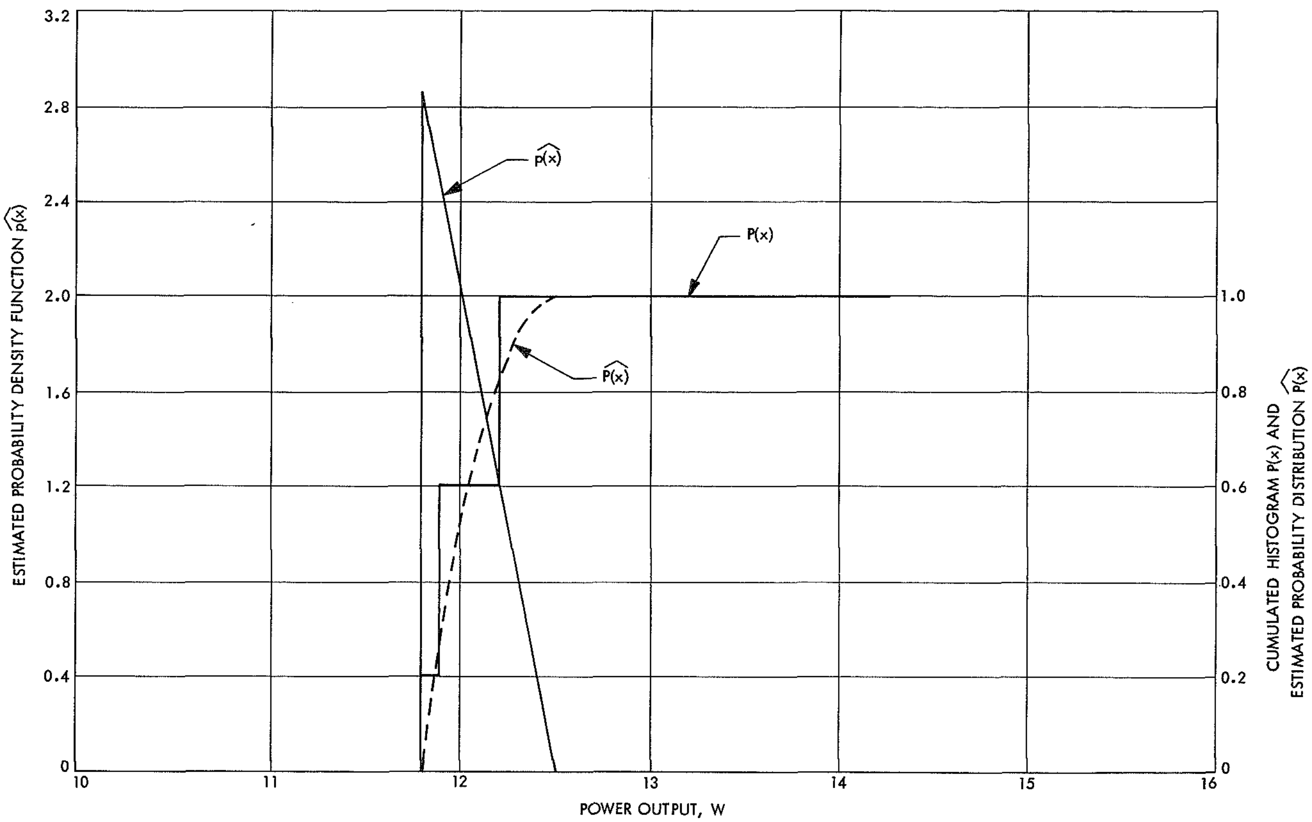

Fig. B-10. *Lunar Orbiter* TWT amplifier—power output, cumulated histogram, and estimated probability density function

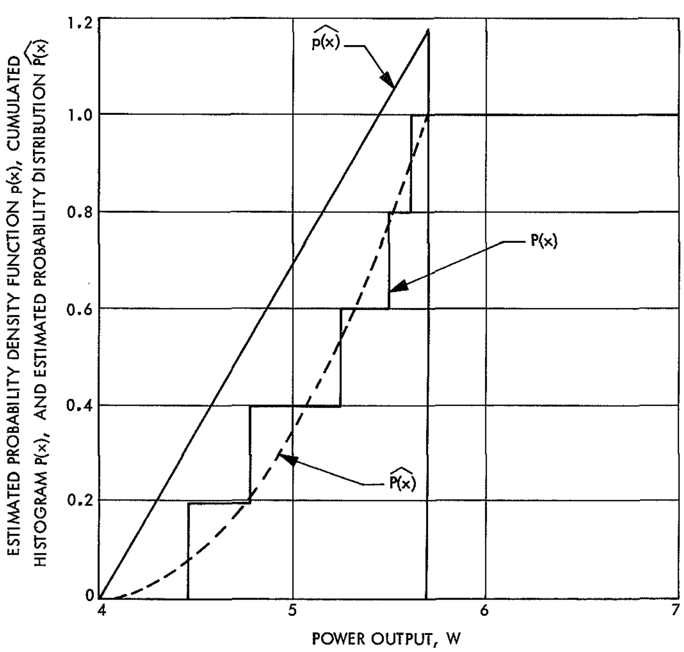

Fig. B-11. *Lunar Orbiter* transponder—power output, cumulated histogram, and estimated probability density function

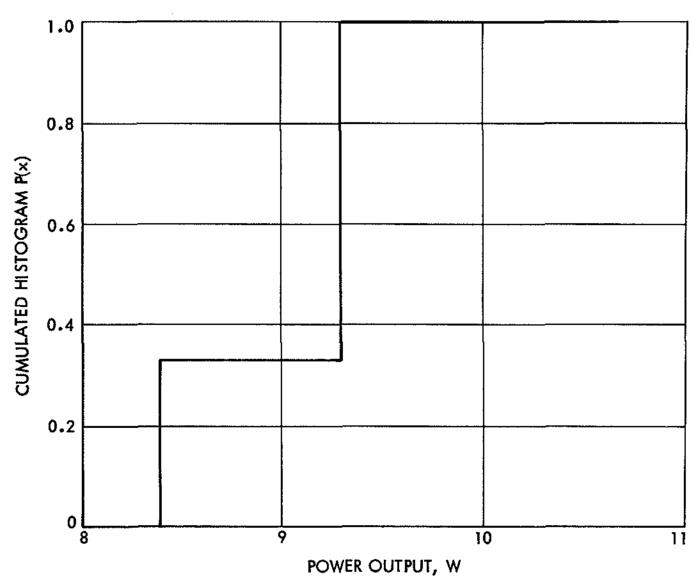

Fig. B-12. *Pioneer* transponder—power output and cumulated histogram

Nomenclature

A specification limit

a specification limit

B specification limit

b computed limit for the desired probability density function

C_1 arbitrary constant

C_2 arbitrary constant

C_p expected cost

$C(x)$ cost to contractor for delivering a hardware parameter of x

E expected net profit or payoff

$E[f(x)]$ expected value of function $f(x)$

$E[N(x)]$ expected net profit or payoff

F integrand of I

F^* integrand of I^*

$f(x)$ dummy weighting function

$f_1(x)$ probability density function of target location

G_i ith constraint function

$g[p(x)]$ functional of $p(x)$ related to cost

I integral (with no constraints) to be maximized (or minimized) with the techniques of the calculus of variations

I^* integral (with integral constraints) to be maximized (or minimized) with the techniques of the calculus of variations

i positive integer

j positive integer

K a constant

k cost factor used to represent k_r and k_d

k_c cost coefficient

k_{c0} critical value of cost coefficient

k_d state-of-the-art development cost coefficient

k_r incentive award coefficient

k_s surveillance coefficient

N positive integer

$N(x)$ net profit for delivering a hardware parameter of x

$\mathbf{P}$ Markov transition matrix

P_b baseline for modifying $p(x)$ when necessary

P_s the probability of detecting a target

$p(x)$ probability density function of delivering a hardware parameter of x, the desired probability density function

$p_0(x)$ natural probability density function

$p_i(x)$ probability density function of the ith item

$p^*(x)$ modified desired probability density function

Q arbitrary constant

R bounded region of an integral

$R(x)$ reward for delivering a hardware parameter of x

$r(x)$ normalized reward function

$S(x)$ surveillance function for an observed hardware parameter of x

$U(p)$ unit step function

$v(x)$ mixed strategic function

X some specific (or computed) value of x

x value of a hardware parameter

x_0 nominal value of x

x_1 some specific value of x

Δk $|k_r - k_d|$

λ Lagrange multiplier

λ_i Lagrange multiplier for the ith constraint

Φ total allocated resource of a search effort

$\varphi(x)$ distribution of search effort

$\varphi^*(x)$ optimum distribution of search effort

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