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TECHNICAL MEMORANDUM

STABILITY ANALYSIS OF LUMPED-DISTRIBUTED FEEDBACK SYSTEMS VIA OPEN-LOOP NYQUIST PLOTS

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ABSTRACT

In the study of linear lumped systems, system stability can often be determined by application of Nyquist's Criterion over a finite band of real frequencies. In the study of linear distributed systems, this may not be the case since such systems may have an infinity of singularities. The memorandum derives conditions under which the stability of distributed systems can be determined by Nyquist's Criterion over a finite band of frequencies, and applies these results to a simple system related to Saturn V POGO stability.

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INTRODUCTION

Open-loop Nyquist plots are widely used in the stability analysis of lumped-parameter (i.e., described by ordinary differential equations) feedback systems. Part of the reason for the acceptance of this technique is its practical convenience. It is easy to measure gain over a finite band of frequencies. It is desirable to have a similar technique when distributed (i.e., described by partial differential equations) elements are added.

The first difficulty encountered is that the frequency domain characterization of distributed systems results in an infinite number of poles or zeros. Hence it is no longer clear that any test over a finite frequency band will suffice to determine stability.

Another difficulty is that, although the characteristic equation of the system can be put into the form of a mixed polynomial in powers of the frequency variable and exponentials of the frequency variable, and although there exist techniques for determining the stability of such polynomials (to be described here), the usual case is that one is concerned with the ratio of such polynomials, for which no criteria exist. In the practical case, one would like to measure the open-loop gain and make conclusions concerning stability of the closed loop system. In this memorandum we derive conditions under which this can be done. We limit our examples to those systems which have as their only distributed element lossless transmission lines.

The memorandum consists of three parts. In the first part, the Pontryagin Criterion is presented and an attempt made to extend it to the open-loop type of analysis. Although certain interesting results are obtained, it is shown that open-loop results of the Pontryagin type are not feasible. In the second part, we prove Michailov's Criterion.

Although this has been proved previously by others, the proof required assumptions which are not met in our class of problems. This criterion is then extended to the open-loop type of analysis. Finally, we consider, in some detail, an example of a lumped-distributed system showing the applicability of these results. This example (in fact the entire memorandum) is strongly motivated by Bellcomm's POGO analysis to determine structural stability of the Saturn V [1].

Pontryagin's Criterion

Let $h(z,t)$ be a polynomial with complex coefficients in the two complex variables z and t. Pontryagin [2] has developed necessary and sufficient conditions that the function $H(z) = h(z,e^z)$ have zeros with only negative real parts. We will show later that such polynomials are related to the closed-loop characteristic equation of the class of systems under discussion. We now present one of Pontryagin's main results. Let r and s be the degrees of the polynomial $h(z,t)$ with respect to z and t. Then the principal term of $h(z,t)$ is the term containing the product $z^r t^s$. Pontryagin showed that if $h(z,t)$ does not contain the principal term then $H(z)$ has an infinity of zeros with arbitrarily large positive real parts.

Let $p(\cdot)$ and $q(\cdot)$ be real-valued functions of a real variable. We say that the zeros of these two functions alternate if: (1) They have no common zeros, (2) they have only simple zeros and (3) between every two zeros of one of these functions there exists at least one zero of the other. The result of [2] which will be used in the present study is:

Pontryagin's Theorem

Let $h(z,t)$ be a polynomial with the principal term and $H(iy) = F(y) + iG(y)$ where $F(y)$ and $G(y)$ take on real values whenever y is real. If all zeros of the function $H(z)$ have negative real parts then all zeros of $F(y)$ and $G(y)$ are real, alternate and

$$G'(y)F(y) - F'(y)G(y) > 0 \tag{1}$$

where superscript prime denotes the derivative. In order that all zeros of $H(z)$ have negative real parts, it is sufficient that all zeros of $F(y)$ and $G(y)$ are real and alternate and that $G'(y)F(y) - F'(y)G(y)$ be positive for some y.

To simplify the discussion we introduce the following notation. If the complex variable z has real part x and imaginary part y (i.e., z=x+iy), and p is a map in the complex plane, then we define the Pontryagin operator $\mathcal{P}$ by

$$\mathcal{P}(p(z)) = \mathcal{P}(p(iy)) = [\mathrm{Re}(p(iy)) \frac{d}{dy} \mathrm{Im}(p(iy))]$$

$$- [\mathrm{Im}(p(iy)) \frac{d}{dy} \mathrm{Re}(p(iy))] \qquad (2)$$

In view of (2), condition (1) becomes

$$\mathcal{P}(p(iy)) > 0 \qquad (3)$$

All the results stated in this section are easily proved, and the proofs are collected in Appendix A. To provide an appealing characterization of the Pontryagin criterion, we note that if $p(iy) = |p(iy)|e^{i\theta(y)}$, then

$$\mathcal{P}(p(iy)) = |p(iy)|^2 \frac{d}{dy} \theta(y) \qquad (4)$$

Thus, if p(z) has no purely imaginary zeros, the criterion simply states that a graph of p(iy) rotates ever counter-clockwise with increasing y.

Pontryagin's Criterion Applied to Open-Loop Analysis

In order to apply this criterion to open-loop analysis we will need to know the effect of the Pontryagin operator on the product of a scalar and a function, the product of two functions, the reciprocal of a function, etc. Some of the pertinent results are listed next. Here α is a complex number and p(z), q(z) denote functions.

$$P(\alpha p(z)) = |\alpha|^2 P(p(z)) \qquad (5)$$

$$P(\alpha) = 0 \qquad (6)$$

$$P(p(iy)q(iy)) = |q(iy)|^2 P(p(iy)) + |p(iy)|^2 P(q(iy)) \qquad (7)$$

$$P(1/q(z)) = \frac{-1}{|q(iy)|^4} P(q(z)) \qquad (8)$$

$$P(p(z)/q(z)) = \frac{1}{|q(iy)|^4} [|q(iy)|^2 P(p(z)) - |p(iy)|^2 P(q(z))] \qquad (9)$$

Interpretation of these results is instructive. Equation (5) says that if p(iy) has increasing angle so does any scalar multiple. (6) says that a scalar has constant angle. (7) shows that if both p(z) and q(z) have only left half plane zeros, so does their product. (8) demonstrates that if q(z) has increasing angle, its reciprocal has decreasing angle. (9) is, of course, the one of interest in open-loop analysis. In its present form it is not too illuminating. However, it is clear that, from (9), we can conclude that

$$\text{if } P(p(z)) > 0 \text{ and } P(p(z)/q(z)) \leq 0, \text{ or if } P(p(z)) \geq 0 \text{ and } P(p(z)/q(z)) < 0, \text{ then } P(q(z)) > 0\,. \qquad (10)$$

(10) explains the observation that stable, minimum-phase transfer functions have a Nyquist plot which tends to rotate clockwise about the origin. (An important special case, when p(z) is constant, is discussed in Example one.) In fact, (10) can be strengthened to

$$P(q(iy)) > 0 \text{ if and only if } P(p(iy)) > |q(iy)|^2 P(p(iy)/q(iy)) \qquad (11)$$

If we accept for the moment that stability and existence of roots only in the left half plane are synonomous, then (10) provides sufficient conditions for stability and (11) provides necessary and sufficient conditions for stability. However, they require knowledge of not only the open-loop gain (p/q) but also the numerator (p), and this information is often not available. In the same vein, one can obtain necessary and sufficient conditions for closed-loop stability as

$$P(q(z)+p(z)) > 0 \quad \text{if and only if}$$

$$P(1+p(iy)/q(iy)) > |q(iy)+p(iy)|^2 \; P(1/q(iy)) \qquad (12)$$

Here we require knowledge of q(z) which is usually not available. From (12) a sufficient condition for closed-loop stability can be derived for open-loop stable systems (i.e., $P(q(z)) > 0$).

$$P(q(z)) > 0 \text{ and } P(1+p(iy)/q(iy)) > 0 \text{ implies } P(q(z)+p(z)) > 0 \qquad (13)$$

The difficulty with this is that it is seldom true, in practice, that $P(1+p(iy)/q(iy)) > 0$. However, the following result shows it highly unlikely that criteria of the Pontryagin type will be found for open-loop systems. (Since it is important and easily derived, we include the derivation here, rather than in the Appendix.)

$$0 = P(1) = P\left(\frac{p(iy)}{q(iy)}\,\frac{q(iy)}{p(iy)}\right) = \left|\frac{q(iy)}{p(iy)}\right|^2 P\left(\frac{p(iy)}{q(iy)}\right) + \left|\frac{p(iy)}{q(iy)}\right|^2 P\left(\frac{q(iy)}{p(iy)}\right) \qquad (14)$$

Now consider a minimum-phase, stable open-loop system (i.e., both p(z) and q(z) have only left half plane zeros). The reciprocal is also a stable, minimum-phase gain. But (14) tells us that if one of these satisfies the Pontryagin Criterion, the other must fail to satisfy that criterion. Therefore, let us turn our attention to another criterion.

Michailov Criterion

As a starting point we consider an equation of the form

$$G(z) = \sum_{i=1}^{m} \sum_{j=0}^{n} \bar{a}_{ij} z^j e^{\omega_i z} = 0 \qquad (15)$$

where $\bar{a}_{ij}$ are complex and ω_i are real. If any of the ω_i were negative, we could multiply G(z) by $e^{|\omega_k| z}$, where ω_k is the most negative of the ω's. This would not change the zeros of G(z), so we assume $0 \leq \omega_1 < \omega_2 < \ldots < \omega_m$. Dividing by $e^{\omega_m z}$ and letting $\bar{a}_{m-i+1,j} = a_{ij}$, we transform (15) into

$$F(z) = \sum_{i=1}^{m} \sum_{j=0}^{n} a_{ij} z^j e^{-r_i z} \qquad (16)$$

where $r_i = \omega_m - \omega_{m-i+1} > 0$ for i=2,3,...,m and $r_1=0$. To relate this to the Pontryagin criterion, note that if the ω_i are rational (which can always be assumed in a practical situation) then a suitable scaling of the z variable will make G(z) of (15) into a polynomial like H(z) of the Pontryagin criterion.

Before continuing, it should be noted that a proof of the Michailov criterion for exponential polynomials has been presented in the literature [3]. However, this proof assumed that

$$|a_{1n}| > \sum_{i=2}^{m} |a_{in}| \qquad (17)$$

As will be seen by the example to be considered later, in the class of systems we are considering, (17) is almost never satisfied. Hence a proof is required which is free of this assumption.

However we do use the assumption that $a_{1n} \neq 0$. That this is no loss of generality can be seen by the following considerations. It is clear that $|r_m| \geq |r_i|$ $i = 2,\ldots,m$. Then multiplying (16) by $e^{r_m z}$ (which does not change the zeros) puts (16) into the Pontryagin form. If a_{1n} is zero, the principal term is missing and we are finished with the stability study. To aid in the subsequent developement, let us rewrite (16) as

$$F(z) = \sum_{j=0}^{n} z^j Q_j(e^{-z}) \tag{18}$$

where $Q_j(e^{-z}) = \sum_{i=1}^{m} a_{ij} e^{-r_i z}$, or as

$$F(z) = z^k F_k(z) + \sum_{j=0}^{k-1} z^j Q_j(e^{-z}) \tag{19}$$

where k is an integer between zero and n and

$$F_k(z) = \sum_{j=k}^{n} z^{j-k} Q_j(e^{-z}) \quad .$$

We now prove the following theorem:

<u>Theorem 1</u>

If there exists a non-negative integer $k \leq n$ such that $F_k(z)$ of (19) has at most a finite number of zeros on $\mathrm{Re}(z) \geq 0$, then $F(z)$ of (19) has at most a finite number of zeros on $\mathrm{Re}(z) > 0$.

Proof

Choose a real number $R' > 1$ such that, $\mathrm{Re}(z) > 0$ and $F_k(z) = 0$ implies $|z| < R'$. Define the set θ

$$\theta = \{z | \mathrm{Re}(z) > 0 \quad \text{and} \quad |z| > R'\}$$

Let

$$D_k = \inf_{z\varepsilon\theta} |F_k(z)| > 0$$

and

$$M_k = \sup_{j=0,\ldots,k-1} \sum_{i=1}^{m} |a_{ij}|$$

If $M_k = 0$, then $F(z) = F_k(z)$ and the theorem is trivially true. If $M_k > 0$, then for all $z\varepsilon\theta$ the following inequalities hold.

$$|F(z)| \geq |z^k F_k(z)| - \sum_{j=0}^{k-1} |z^j Q_j(e^{-z})|$$

$$\geq |z^k| D_k - \sum_{j=0}^{k-1} |z^j| \sum_{i=1}^{m} |a_{ij}| \; |e^{-r_i z}|$$

$$\geq D_k |z^k| - \sum_{j=0}^{k-1} |z^j| \sum_{i=1}^{m} |a_{ij}|$$

$$|F(z)| \geq D_k|z|^k - M_k \sum_{j=0}^{k-1} |z^j|$$

$$\geq D_k|z|^k - M_k \frac{|z|^k - 1}{|z| - 1}$$

$$\geq \frac{|z|^k[D_k(|z|-1) - M_k] + M_k}{|z| - 1}$$

$$\geq \frac{|z|^k}{|z| - 1} \{[D_k|z| - (D_k+M_k)] + M\}$$

which is positive for $|z| > 1 + M_k/D_k = R_k$. Hence the magnitude of all zeros of F(z) must be bounded by R, the larger of the numbers R' and R_k. The theorem follows at once by noting that F(z) is analytic and that an analytic function has at most a finite number of zeros in any finite region.

Corollary 1:

If there exists a non-negative integer $k \leq n$ such that $F_k(z)$ of (19) has no zeros on $Re(z) \geq 0$, then F(z) of (19) has at most a finite number of zeros on $Re(z) > 0$.

Corollary 2:

If $Q_n(e^{-z})$ of (18) has no zeros on $Re(z) \geq 0$ then F(z) of (18) has at most a finite number of zeros on $Re(z) > 0$.

Corollary 2 follows since $Q_n(e^{-z}) = F_n(z)$.

We now wish to derive an expression for the number of rhp zeros of F(z). We choose a contour Γ varying along the imaginary axis from -y to y (call this portion ω) where $y \geqq R_k$ of Theorem 1 and close it by a contour C outside the semi-circle of radius R of Theorem 1. Let

$$F(z) = [1 + \psi(z)]z^k F_k(z)$$

where

$$\psi(z) = \sum_{j=0}^{k-1} z^{(j-k)} \frac{Q_j(e^{-z})}{F_k(z)}$$

We choose contour C (and increase y, if necessary) so that $|\psi(z)| < 1$ along C. Let N be the number of zeros of F(z) inside Γ and let

$$\Delta_\Gamma (F(z))$$

be the net change in arg(F(z)) along Γ. Then N, the number of zeros of F(z) enclosed by Γ (assuming counter-clockwise travel) is given by

$$N = \frac{1}{2\pi} \Delta_\Gamma (F(z)) = \frac{1}{2\pi} \Delta_\omega (F(z)) + \frac{1}{2\pi} \Delta_C (F(z))$$

$$= \frac{1}{2\pi} [\Delta_\omega (F(z)) + \Delta_C (z^k) + \Delta_C (F_k(z)) + \Delta_C (1+\psi(z))]$$

if $F(z)$ has no zeros on Γ. Since $1 + \psi(z)$ does not wind around the origin (its real part being always positive), we have thus proven

Theorem 2:

The number of zeros of $F(z)$ with positive real part is

$$N = \frac{k}{2} - \frac{1}{2\pi} \Delta_{-\omega}(F(z)) + \frac{\Delta_C}{2\pi}(F_k(z))$$

$$+ \frac{1}{2\pi} [\arg(1+\psi(iy)) - \arg(1+\psi(-iy))]$$

assuming that $F(z)$ has no purely imaginary zeros and where $\Delta_{-\omega}(F(z))$ is the net change in $\arg F(z)$ along the imaginary axis from $-iy$ to $+iy$.

Theorem 2 is the desired statement of Michailov's criterion. To obtain tighter results, let us now assume that $Q_n(e^{-z})$ has no zeros on $\mathrm{Re}(z) \geq 0$ (i.e., Corollary 2). Further, let the r_i be rational and the a_{ij} be real. By virtue of rational r_i, $Q_k(e^{-z})$ is periodic in y, for fixed x. Let this period be P. By virtue of real a_{ij}, replacing z by its conjugate results in $Q_k(e^{-z})$ being replaced by its conjugate. Hence we need only consider the semi-infinite strip defined by $x>0$ and $P \geq y \geq 0$.

Michailov's criterion can be simplified if it can be shown that $Q_n(e^{-z})$ does not wind around the origin as z varies over a suitable C. We now consider this possibility. For $z = x + iy$

$$Q_n(e^{-z}) = \sum_{j=1}^{m} a_{jn} e^{-r_j x} (\cos r_j y - i \sin r_j y)$$

$$= a_{1n} + \sum_{j=2}^{m} a_{jn} e^{-r_j x} \cos r_j y - i \sum_{j=2}^{m} a_{jn} e^{-r_j x} \sin r_j y$$

$$= \mathrm{Re}[Q_n] - i\, I_m[Q_n] \quad .$$

If either $Re[Q_n]$ or $I_m[Q_n]$ does not vanish along C, then Q_n cannot wind around the origin. We now derive sufficient conditions for this. Let

$$r_k = \min_{j=2,\ldots,m} r_j$$

Then

$$\frac{e^{r_k x}}{a_{1n}} Re[Q_n] = e^{r_k x} + \sum_{j=2}^{m} \frac{a_{jn}}{a_{1n}} e^{-(r_j - r_k)x} \cos r_j y$$

$$\frac{e^{r_k x}}{|a_{1n}|} |Re[Q_n]| \geq e^{r_k x} - \sum_{j=2}^{m} \left|\frac{a_{jn}}{a_{1n}}\right|$$

Hence $|Re[Q_n]| > 0$ for $x > \frac{1}{r_k} \ln \alpha$ where

$$\alpha = \sum_{j=2}^{m} \left|\frac{a_{jn}}{a_{1n}}\right|$$

Thus we need only consider a rectangle defined by $0 < x \leq \frac{\ln\alpha}{r_k}$ $0 \leq y < P$. (Note that any contour C will work if $\alpha \leq 1$ which is the case if Assumption (17) is used.) $Q_n(e^{-z})$ does not wind around the origin for any contour with $x > \frac{\ln\alpha}{r_k}$, since $Re[Q_n]$ does not change sign. If $\sum_{j=1}^{m} a_{jn}{}^{-r_j x} \neq 0$ for $0 < x < \frac{\ln\alpha}{r_k}$ then $Re[Q_n]$ does not change sign for y any integer multiple of P, and $0 < x < \frac{\ln\alpha}{r_k}$. A suitable contour could then consist of a horizontal line at $y = KP$ from $x = 0$ to $x = \frac{\ln\alpha}{r_k}$, where K is an integer large enough so that $KP > R$ of Theorem 1. The rest of the contour in the first quadrant could be semi-circular. The contour is completed in the fourth quadrant by the mirror image of the first quadrant. Since $Re[Q_n] \neq 0$ along this contour, $Q_n[e^{-z}]$ does not wind around the origin. (Note that $a_{jn} > 0$, $j = 1,\ldots,m$ is sufficient to satisfy the conditions of this paragraph.) Furthermore, since $R[Q_n]$ is even in y, $\Delta_C(Q_n(e^{-z})) = 0$ along the contour chosen.

This result can be extended to include the case where $Re[Q_n]$ has simple zeros on $y = KP$. In this case, use semi-circular indentations around such points, in the direction to have $Im[Q_n] > 0$, <u>in both first and fourth quadrants</u>. (Hence the contour ceases to be symmetrical about the real axis.) Thus, along the deformed horizontal lines, the graph of $Q_n(e^{-z})$ remains in the upper half plane. Along the semi-circular portion it remains in either the right- or left-half plane. Hence no encirclements of the origin are possible and again $\Delta_C[Q_n(e^{-z})] = 0$.

We now assume that $\Delta_C[Q_n(e^{-z})] = 0$ and write the Michailov criterion as

$$N = \frac{n}{2} - \frac{1}{\pi}\Delta_{-w/2}(F(z)) + \frac{1}{\pi}\arg(1+\psi(iy)) \tag{20}$$

where $\Delta_{-w/2}$ is that part of the imaginary axis from 0 to iy. Thus $N = 0$ iff

$$\Delta_{-w/2}(F(z)) = \frac{n\pi}{2} + \arg(1+\psi(iy)) \tag{21}$$

We remark that $\arg(1+\psi(iy))$ can be made close to zero for y sufficiently large.

Michailov's Criterion Applied to Open-Loop Analysis

The Michailov Criterion, as well as its predecessor the Pontryagin Criterion, settle the problem of finding rhp zeros of polynomials in z and e^z. In many engineering applications, however, this polynomial is not directly available, but a related ratio of such polynomials can be found. In the study of feedback systems, for example, an open-loop gain can be measured and it is desired to find the poles of the closed-loop gain. These latter poles are the zeros of the polynomial which results from adding the two polynomials whose ratio is the open loop gain. Stability has been determined for non-distributed systems by counting encirclements of the open-loop gain along the imaginary axis. What we propose to do next is to provide a similar criterion for the distributed parameter problem. Thus let F(z) = D(z) + N(z). Then

$$\Delta_\Gamma (D(z) + N(z)) = \Delta_\Gamma [D(z)\ 1 + \frac{N(z)}{D(z)}\ ]$$

$$= \Delta_\Gamma \left(D(z)\right) + \Delta_\Gamma \left(1 + \frac{N(z)}{D(z)}\right) \tag{22}$$

Let the contour Γ be composed of a portion of the imaginary axis w and another (possibly semi-circular) contour C, such that Γ encloses all zeros of D(z) + N(z). Then

$$\Delta_\Gamma (D(z)+N(z)) = \Delta_\Gamma (D(z)) + \Delta_w \left(1 + \frac{N(z)}{D(z)}\right) + \Delta_C \left(1 + \frac{N(z)}{D(z)}\right) \tag{23}$$

It is the term $\Delta_w \left(1 + \frac{N(z)}{D(z)}\right)$ which is usually available for determining stability. We ask, "when does $\Delta_w \left(1 + \frac{N(z)}{D(z)}\right) = \Delta_\Gamma (D(z) + N(z))$?". The answer is that this happens exactly when

$$0 = \Delta_\Gamma (D(z)) + \Delta_C \left(1 + \frac{N(z)}{D(z)}\right) \quad . \tag{24}$$

To develop a more practical criterion let us rewrite this expression using n_F to be the highest power of z in F(z), and N_F to be the number of zeros of F(z) inside Γ. Then

$$0 = \Delta_\Gamma(D(z)) + \Delta_C(D(z) + N(z)) - \Delta_C(D(z)) \qquad (25)$$

$$0 = 2\pi N_D + n_{D+N}\pi - n_D\pi \qquad (26)$$

where we have neglected those terms which become small for large z. In most practical applications, the open loop gain is bounded at infinity, that is to say $n_N \leq n_D$. Hence $n_{D+N} \leq n_D$. Since $n_{D+N} < n_D$ requires $\lim_{w\to\infty} \frac{N(iw)}{D(iw)} = -1$, we conclude that counting encirclements of the open loop gain is a valid method for determining loop stable (i.e., $N_D = 0$) and whose open-loop gain is bounded but does not approach -1 for large frequencies (i.e., $n_{D+N} = n_D$). This includes the case, usually found in practice, that the open-loop gain approaches zero for large frequencies.

Example Based on the POGO Study

The system to be analyzed was chosen to be as simple as possible and yet retain most of the features of the POGO stability model [1]. Thus, for simplicity, we consider only one distributed parameter element. This is assumed to be a lossless, uniform transmission line. The partial differential equations of the line are linearized and Laplace Transformed in the usual manner [1]. The resulting equations can be expressed as

$$[A \quad -I] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad (27)$$

where X_i are 2-vectors whose first component P_i is transformed pressure and second component W_i is transformed flow. The subscripts 1 and 2 denote input and output, respectively. I is the 2 x 2 identity matrix and A is the matrix

$$A = \begin{bmatrix} \cosh ks & -Z_C \sinh ks \\ -\frac{1}{Z_C} \sinh ks & \cosh ks \end{bmatrix} \tag{28}$$

where k, Z_C are real positive numbers and s is the complex Laplace variable. In the derivation of these equations k is shown to be the ratio of the length of the transmission line to the wavespeed of sound in the line. Z_C is the characteristic impedance of the line.

The particular configuration to be considered here allows the input pressure P_1 to depend on output pressure P_2 through external feedback. (In the POGO study this feedback mechanism was due to "Pos Aos" forces exciting the structure causing acceleration of fluid in the fuel tanks which generated pressure variations at the input to the line.) Thus we have

$$P_1 = G(s)P_2 \tag{29}$$

The output flow W_2 is also assumed to depend on output pressure P_2. (In the POGO model this was due not only to the fluid load on the line (e.g., pump, discharge system, engine) but also due to the change in output flow caused by pump motion resulting from Pos Aos force.) Thus we write

$$W_2 = Y(s)P_2 \tag{30}$$

The system equations can be written in matrix form as

$$\left[\begin{array}{cc|cc} \cosh ks & -Z_C \sinh ks & -1 & 0 \\ -\frac{1}{Z_C}\sinh ks & \cosh ks & 0 & -1 \\ \hline -1 & 0 & G(s) & 0 \\ 0 & 0 & Y(s) & -1 \end{array}\right] \left[\begin{array}{c} P_1 \\ W_1 \\ \hline P_2 \\ W_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \tag{31}$$

or, using the indicated partitioning

$$\begin{bmatrix} A & -I \\ C & D \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{32}$$

where

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} G(s) & 0 \\ Y(s) & -1 \end{bmatrix}$$

For simplicity we will refer to this matrix as M(s), i.e.,

$$M(s) = \begin{bmatrix} A & -I \\ C & D \end{bmatrix} \tag{33}$$

Hence the form of the system equations to be considered is

$$M(s)X(s) = 0 \tag{34}$$

where

$$X(s) = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{34}$$

The following assumptions concerning G(s) and Y(s) are made:

A1. G(s) and Y(s) are each the ratio of two polynomials having real coefficients with no singularities on Re(s) > 0.

A2. $\lim_{s \to \infty} G(s) = k_1$ where k_1 is real and $|k_1| \leq 1$.

A3. $\lim_{s \to \infty} Y(s) = \lim_{s \to \infty} sC$ where C is non-negative real.

Assumption A1 requires G and Y to be stable transfer functions. Assumption A2 insists that the feedback gain at infinity be less than unity. Assumption A3 is physically appealing (and was satisfied in the POGO stability analysis).

General Eigenvalue Analysis

Let $\Delta(s)$ denote the determinant of M(s). From (33) and a well-known identity it follows that

$$\Delta(s) = \det(M(s)) = \det(D-CA^{-1}(-I))\det(A) \tag{35}$$

which is easily computed to be

$$\Delta(s) = \cosh ks - G(s) + Z_c Y(s) \sinh ks \tag{36}$$

The solutions of $\Delta(s) = 0$ are the system eigenvalues and are the natural modes of the system. Naturally it is desirable to determine these eigenvalues. However, this is a nonlinear eigenvalue problem involving a transcendental equation. Thus even in the simple model considered here, analytical solutions are usually not available and recourse to numerical methods is required. Before applying the Michailov criterion, let us "get a feel" for the system pecularities by means of some analysis.

As a preliminary, we simplify the notation as follows. Let

$s = \sigma + iw$ $(i = \sqrt{-1})$

$x = k\sigma$

$y = kw$

$Z_C \mathrm{Re}Y(s) = f_2(x,y)$

$Z_C \mathrm{Im}Y(s) = g_2(x,y)$

$\mathrm{Re}G(s) = f_1(x,y)$

$\mathrm{Im}G(s) = g_1(x,y)$

Since $\Delta(s) = 0$ exactly when $\mathrm{Re}\Delta(s)$ and $\mathrm{Im}\Delta(s)$ are both zero for some s, the following two conditions are of interest.

$\mathrm{Im}\Delta(s) = 0$ if and only if

$$\tanh x\,[\sin y + g_2(x,y)\cos y] + f_2(x,y)\sin y = \frac{g_1(x,y)}{\cosh x} \qquad (37)$$

$\mathrm{Re}\Delta(s) = 0$, if and only if

$$\cos y[1 + f_2(x,y)\tanh x] - g_2(x,y)\sin y = \frac{f_1(x,y)}{\cosh x} \qquad (38)$$

The problem is simplified somewhat by the following assertions:

Assertion 1: Complex solutions of $\Delta(s) = 0$ appear as conjugate pairs.

Proof

From A1, the inverse transforms of G and Y are real time functions. As is well-known, this requires

$$f_i(x,y) = f_i(x,-y)$$

and $$i = (1,2)$$

$$g_i(x,y) = -g_i(x,-y)$$

Under these conditions equations (37) and (38) are unchanged when y is replaced by -y. Thus, (x_1,y_1) satisfies (37) and (38) if and only if $(x_1,-y_1)$ does. The proof is complete since $\Delta(s) = 0$ if and only if (37) and (38) are satisfied.

Assertion 2: $\Delta(s) = 0$ has a purely real solution (x,0) if and only if

$$\cosh x + f_2(x,0) \sinh x = f_1(x,0) \tag{39}$$

Proof

A1 implies that $g_2(x,0) = 0 = g_1(x,0)$. Then, for y=0, (37) is trivial and (38) becomes the equation of the assertion.

Assertion 3: $\Delta(s) = 0$ has a purely imaginary solution (0,y) if and only if both of the following are satisfied

$$f_2(0,y)\sin y = g_1(0,y) \tag{40a}$$

$$\cos y = f_1(0,y) + g_2(0,y)\sin y \tag{40b}$$

Proof

Obvious.

During the POGO study, it was noted the G(s) had little effect on system stability. For this reason the following theorem is of interest.

Theorem 3

Let $f_1(x,y) = 0 = g_1(x,y)$. Let R denote the set of all positive, finite real numbers. If $\forall x \varepsilon R$, $\forall y \varepsilon R$, $f_2(x,y) > -\tanh x$ then $\Delta(s) = 0$ has no solution with $\sigma \varepsilon R$.

Proof

The case for $y<0$ need not be considered in view of Assertion 1. The case $y=0$ requires (by (39))

$$1 + f_2(x,0)\tanh x = 0$$

which is impossible by hypothesis. Equations (37) and (38), are now

$$\sin y[f_2(x,y) + \tanh x] + g_2(x,y)\cos y \tanh x = 0 \tag{41a}$$

$$\cos y[1 + f_2(x,y)\tanh x] - g_2(x,y)\sin y = 0 \tag{41b}$$

Since $x \varepsilon R$ implies $|f_2(x,y)| < \infty$ by A1 and since $|\tanh x| < 1$, both $(f_2(x,y) + \tanh x)$ and $(1 + f_2(x,y)\tanh x)$ are bounded on $x \varepsilon R$. Furthermore, they are positive on $x \varepsilon R$ by hypothesis. Solving (41a) and (41b) for sin y and cos y, respectively, yields

$$\sin y = \frac{-g_2(x,y)\cos y \tanh x}{f_2(x,y) + \tanh x} \tag{42a}$$

$$\cos y = \frac{g_2(x,y)\sin y}{1 + f_2(x,y)\tanh x} \tag{42b}$$

Simultaneous solution gives the condition

$$\cos y = \frac{-[g_2(x,y)]^2 \cos y \tanh x}{[1 + f_2(x,y)\tanh x][f_2(x,y) + \tanh x]} \qquad (43)$$

Since $\cos y = 0$ cannot be a solution to (42a) and (42b), (43) requires

$$[1 + f_2(x,y)\tanh x][f_2(x,y) + \tanh x] = -[g_2(x,y)]^2 \tanh x \qquad (44)$$

Since the left hand side of (44) is positive, there are no solutions on $x \varepsilon R$. The proof is completed by noting that $\sigma \varepsilon R$ exactly when $x \varepsilon R$.

Corollary

If $G(s) = 0$ and $Y(s)$ is the admittance of a passive, lumped, linear, time-invariant system, then $\Delta(s)$ has no zeros on $\sigma > 0$.

Proof

It is well known that such a $Y(s)$ is a positive real function. The restrictions of the theorem admit positive real functions.

A remark on the theorem and corollary is in order. If one accepts for the moment that $\Delta(s)$ having no right-half-plane roots is sufficient for system stability, then the corollary states the intuitively obvious proposition that a line, terminated in a passive load is stable. The theorem provides the further information that, loosely speaking, the real part of the load can be negative, as long as it is not too negative, and the system will still be stable.

The location of eigenvalues outside of a large circle is considered next. By the assumptions we see that

$$\lim_{s\to\infty} \Delta(s) = -k_1 + \lim_{s\to\infty}[\cosh ks + sCZ_C \sinh ks] \tag{45}$$

or, in our simpler notation, with $z = x + iy$

$$\lim_{z\to\infty} \Delta(z) = -k_1 + \lim_{z\to\infty}[\cosh z + C_1 z \sinh z] \tag{46}$$

where $C_1 = Z_C C/k$. The following theorem classifies the zeros of (46).

Theorem 4

Let C_1, k_1 be real numbers with $C_1 \geq 0$ and $|k_1| \leq 1$. Let z be a complex variable and consider

$$f(z) = \cosh z + C_1 z \sinh z - k_1 = 0 \quad . \tag{47}$$

Then f(z) has only imaginary zeros.

Proof

Express (47) in real and imaginary parts. For $z = x + iy$ this becomes

$$\cosh x\,(\cos y - C_1 y \sin y) + C_1\, x \sinh x \cos y - k_1 = 0 \tag{48}$$

$$\sinh x\,(\sin y + C_1 y \cos y) + C_1\, x \cosh x \sin y = 0 \tag{49}$$

Multiply (48) by sinh x cos y and (49) by cosh x sin y, add the results, and with some trigonometric manipulation the following identity is obtained:

$$\sinh 2x + C_1 x[\cosh 2x - \cos^2 y] = [k_1 \cos y] \sinh x \qquad (50)$$

The left hand terms have the same sign (that of x); thus,

$$|\sinh 2x| + |C_1 x||\cosh 2x - \cos^2 y| = |k_1||\cos y||\sinh x| \qquad (51)$$

and a necessary condition for (51) to be satisfied in

$$|\sinh 2x| \leqq |\sinh x| \qquad (52)$$

This can happen only when $x = 0$.

<u>Corollary</u>

If $C_1 > 0$ and $k_1 > 1$ then (47) has a single positive real solution.

<u>Proof</u>

$y = 0$ implies that (49) is trivially satisfied and that (48) becomes

$$\cosh x + C_1 x \sinh x - k_1 = 0 \qquad (53)$$

This is a concave function of x, is positive for large $|x|$ and is negative at $x = 0$. Hence (53) has a single positive real solution.

This theorem and corollary are interesting from a root locus point of view. It is intuitive that a lossless line, terminated in a lossless load, will have all its eigenvalues purely imaginary. It is somewhat surprising to note that this remains the case as feedback is applied, up to the point where the two smallest roots meet at the origin, and then break-away along the real axis. No matter how large the feedback becomes, no more real roots are generated.

Having thus discovered some of the unusual properties of this particular lumped-distributed system, let us now proceed to consider a stability analysis using Michailov's criterion.

First let us put (36) in polynomial form. Let

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad Z_C Y(s) = \frac{N_Y(s)}{D_Y(s)} \tag{54}$$

Then (36) can be written as

$$\Delta(s)[D_G(s)D_Y(s)] = [D_G(s)D_Y(s)]\cosh ks - N_G(s)D_Y(s) + D_G(s)N_Y(s)\sinh ks \tag{55}$$

By A1, $D_G(s)$ and $D_Y(s)$ have no right half plane zeros. Thus $\Delta(s)$ has right half plane zeros exactly when $\Delta(s)[D_G(s)D_Y(s)]$ has right half plane zeros. Thus (55) is the desired polynomial form.

Next let us consider suitable open- and closed-loop expressions. Assuming that G(s) "closes the loop" we solve (27) for $P_2(s)/P_1(s)$ using (30). The result is the "line transfer function".

$$\frac{P_2(s)}{P_1(s)} = \frac{1}{\cosh ks + Z_C Y(s)\sinh ks} \tag{56}$$

Hence the "loop gain", $G_0(s)$, is simply G(s) times (56), or

$$G_0(s) = \frac{G(s)}{\cosh ks + Z_C Y(s) \sinh ks} = \frac{D_Y(s) N_G(s)}{D_Y(s) D_G(s) \cosh ks + D_G(s) N_Y(s) \sinh ks} \tag{57}$$

Let us digress for a paragraph to relate these expressions to classical feedback theory of lumped systems. If we let $\Delta(s) = \Delta_0(s) - G(s)$ we have said that

$$G_0(s) = \frac{G(s)}{\Delta_0(s)} = \frac{\Delta_0(s) - \Delta(s)}{\Delta_0(s)} = 1 - \frac{\Delta(s)}{\Delta_0(s)} \tag{58}$$

where $\Delta_0(s) = \cosh ks + Z_C Y(s) \sinh ks$, or

$$\Delta(s) = \Delta_0(s)[1 - G_0(s)] \tag{59}$$

Note that equation (59) is identical to a formula from classical feedback theory of non-distributed systems, as originally derived by Bode [4]. In his terms, $G_0(s)$ is the return ratio and $1 - G_0(s)$ the return difference. Thus (59) suggests that the analysis of distributed feedback systems may have much in common with non-distributed systems. That this is true in general can be seen by noting that the Bode Theory is concerned with a matrix of functions of the complex variables. Most of the manipulations involve Cramer's rule, determinant identities, etc. and would also apply to (31). Hence such notions as null return differences, sensitivity and Blackman's equations will be meaningful in mixed, lumped and distributed parameter systems, in so far as the lumped portion is concerned.

Returning now to the present problem, let us transform these expressions so as to fit the form of Michailov's criterion. Thus, multiplying (55) by $2e^{-ks}$ gives

$$F(s) = 2e^{-ks}\Delta(s)[D_G(s)D_Y(s)] = D_G(s)[D_Y(s) + N_Y(s)]$$

$$-2N_G(s)D_Y(s)e^{-ks} + D_G(s)[D_Y(s) - N_Y(s)]e^{-2ks} \quad (60)$$

or

$$F(s) = \sum_{i=0}^{2} R_i(s)e^{-iks} \quad (61)$$

From Assumption A_2 we conclude that $\deg D_G(s) \geq \deg N_G(s)$. From A_3 we conclude that $\deg N_Y(s) > \deg D_Y(s)$. From this fact we see that the principal term is present and that the assumption (17) used in previous proofs of Michailov's criterion is not met. In fact one can readily convince oneself that this will be the case whenever lossless transmission lines are involved, since all exponential terms will involve hyperbolic functions.

Furthermore, even if the inequality of (17) is replaced by an equality, many practical problems will still not meet this criterion. For example, it can be shown that [4] in the case of lossless fluid flow in a cylindrical pipe, (17), even with equality, is violated if the fluid velocity is a significant fraction of the propagation speed of waves in the fluid.

Using the notation of (61), the open-loop gain (57) can be expressed as

$$G_0(s) = \frac{R_1(s)e^{-ks}}{R_0(s) + R_2(s)e^{-2ks}} \quad (62)$$

whose norm becomes small for large, right half plane values of s. Hence it is valid to count encirclements of the open-loop gain about the point +1 if the line and its termination is stable. As can be seen from (56) and (57), this latter condition guarantees stability of the open-loop gain.

Example 1

Assume that $D_G(s) = 1$, $N_G(s) = k_1$, $D_Y(s) = 1$, $N_Y(s) = sC_1 + g$. This corresponds to terminating the line in a capacitance C_1 and shunt conductance $g \neq 0$. The open-loop gain for $s = jw$ becomes

$$G_0(jw) = \frac{k_1}{\cos kw - wC_1 \sin kw + ig \sin kw} \tag{63}$$

which is real only when sin kw is zero. This implies that cos kw is ± 1. Thus if $|k_1| < 1$, $G_0(jw)$ cannot encircle the +1 point. This result is in agreement with Theorem 4, to which this problem corresponds if $g = 0$. It is intuitive that adding losses to a lossless sytem will enhance stability.

To show the necessity of the open-loop stable requirement, note that g can be either positive or negative. From (6) and (9) of the Pontryagin criterion, we see that (63) is then stable or unstable, respectively, and that for $|k_1| < 1$ the closed-loop system is stable or unstable, respectively. However, in either case there are no encirclements of the critical point by the open loop gain.

Example 2

Let

$$G(s) = \frac{\sum_{i=0}^{n} a_i s^i}{\sum_{j=0}^{n} b_j s^j} \qquad b_n \neq 0$$

$$Z_C Y(s) = \frac{\sum_{k=0}^{p} c_k s^k}{\sum_{\ell=0}^{p} d_\ell s^\ell} \qquad c_p \neq 0 \tag{64}$$

Using (54) and (64), (60) becomes

$$F(s) = s^{n+p}\{b_n(d_p + c_p) - 2a_n d_p e^{-ks} + b_n(d_p - c_p)e^{-2ks}\}$$

$$+ s^{n+p-1}\{[(d_{p-1} + c_{p-1})b_n + b_{n-1}(d_p + c_p)$$

$$- 2[a_{n-1}d_p + a_n d_{p-1}]e^{-ks}$$

$$+ [b_n(d_{p-1} - c_{p-1}) + b_{n-1}(d_p - c_p)]e^{-2ks}\}$$

$$+ \sum_{m=0}^{n+p-2} s^m\{ \sum_{j+i=m} b_j(d_i + c_i) - 2e^{-ks} \sum_{j+i=m} a_j d_i$$

$$+ e^{-2ks} \sum_{j+i=m} b_j(d_i - c_i)\} \qquad (65)$$

In this example we find conditions on (64) which are physically plausible, such that the corollaries of Theorem 1 apply. First we investigate the zeros of the coefficient of s^{n+p} in (65). (This coefficient corresponds to $Q_n(e^{-z})$ in Corollary 2.) For simplicity let $e^{ks} = z$. This maps the left-half s-plane into the unit circle in the z plane.

If $d_p = c_p$, then the coefficient of s^{n+p} has zeros whenever $b_n c_p = a_n c_p z^{-1}$. If a_n were zero, the coefficient in question would become constant, which satisfies the conditions of Corollary 2. If a_n is not zero then the condition under discussion simplifies to $z^{-1} = b_n/a_n$. All solutions of this will satisfy $|z| < 1$ if $|a_n| < |b_n|$. Hence all zeros of the coefficient of s^{n+p} in (65) will lie in the left half plane if $|a_n| < |b_n|$. This is intuitively appealing since this requires that G(s) have less than unity gain at large frequencies (as required by Assumption A2).

On the other hand, if $d_p \neq c_p$, then the zeros of interest are solutions of

$$(z^{-1})^2 - \frac{2a_n d_p}{b_n(d_p - c_p)}(z^{-1}) + \frac{d_p + c_p}{d_p - c_p} = 0 \tag{66}$$

It is well-known [6] that solutions of (66) (for z^{-1}) have magnitude less than unity if and only if the following three conditions are met.

$$\left|\frac{d_p + c_p}{d_p - c_p}\right| < 1 \tag{67a}$$

$$1 + \frac{d_p + c_p}{d_p - c_p} = \frac{2d_p}{d_p - c_p} > \frac{2a_n d_p}{b_n(d_p - c_p)} \tag{67b}$$

$$\frac{2d_p}{d_p - c_p} > - \frac{2a_n d_p}{b_n(d_p - c_p)} \tag{67c}$$

Thus conditions (67) are NAS for $|z| > 1$. Condition (67a) requires that d_p and c_p have opposite sign. Since this corresponds to terminating the line in a negative conductance at high frequencies (i.e., $\lim_{s\to\infty} Y(s) < 0$), we reject this case. If both d_p and c_p are non-zero, and have the same sign, (67a) is violated. If $d_p \neq 0$, (67b) and (67c) together require $|a_n| < |b_n|$ as before. The remaining possibility is that $d_p = 0$. This is a reasonable physical assumption; in fact, it is required by Assumption A3.

The coefficient in question now becomes $b_n c_p(1 - e^{-2ks})$ which has an infinity of purely imaginary zeros, and this example no longer satisfies Corollary 2.

To see if it satisfies Corollary 1, rewrite (65) as

$$F(s) = s^{n+p-1}\{(sb_n c_p + b_{n-1}c_p + b_n c_{p-1})(1 - e^{-2ks}) - 2a_n d_{p-1}e^{-ks}$$

$$+ b_n d_{p-1}(1 + e^{-2ks})\} \qquad (68)$$

$$+ \sum_{m=0}^{n+p-2} s^m\{\sum_{j+i=m} [b_j(d_i + c_i) - 2e^{-ks}a_j d_i + e^{-2ks}b_j(d_i - c_i)]\}$$

We complete this example by finding conditions under which the coefficient of s^{n+p-1} in (68) satisfies the conditions of Corollary 1. This coefficient can be written as

$$ks\frac{b_n c_p}{k} + b_{n-1}c_p + b_n c_{p-1} + b_n d_{p-1} - 2a_n d_{p-1}e^{-ks}$$

$$- e^{-2ks}(ks\frac{b_n c_p}{k} + b_{n-1}c_p + b_n c_{p-1} - b_n d_{p-1}) \qquad (69)$$

Let $w = ks$. Then (69) becomes

$$\frac{b_n c_p}{k}[w + \frac{kb_{n-1}}{b_n} + \frac{kc_{p-1}}{c_p} + \frac{kd_{p-1}}{c_p}] - 2a_n d_{p-1}e^{-w}$$

$$\frac{-b_n c_p}{k}[w + \frac{kb_{n-1}}{b_n} + \frac{kc_{p-1}}{c_p} - \frac{kd_{p-1}}{c_p}]e^{-2w} \qquad (70)$$

Zeros of (70) are given by the solutions of (71)

$$(w + \alpha) + \gamma e^{-w} - (w + \beta)e^{-2w} = 0 \tag{71}$$

where

$$\alpha = k\left(\frac{b_{n-1}}{b_n} + \frac{c_{p-1}}{c_p} + \frac{d_{p-1}}{c_p}\right)$$

$$\beta = k\left(\frac{b_{n-1}}{b_n} + \frac{c_{p-1}}{c_p} - \frac{d_{p-1}}{c_p}\right)$$

$$\gamma = -2\,k\,\frac{a_n}{b_n}\,\frac{d_{p-1}}{c_p}$$

We assume that $\alpha > \beta$ since $\alpha - \beta = \frac{2d_{p-1}}{c_p}k$ which is positive by Assumption A3.

It is also reasonable to assume $\alpha > 0$, since b_{n-1}/b_n must exceed zero for the denominator of G(s) to be strictly Hurwitz[1], and since c_{p-1}/c_p less than zero would imply zeros of Y(s) in the right half plane. These considerations also imply that $|\alpha| > |\beta|$. Using these assumptions (i.e., $\alpha > \beta$, $\alpha > 0$, $|\alpha| > |\beta|$) it follows that

$$|w + \alpha| > |w + \beta|$$

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[1]A well-known necessary condition for a polynomial to be Hurwitz is that all coefficients have the same sign (see, for example, [6], p. 281).

for Re(w) ≥ 0. Evaluating the magnitude of (71) on Re(w) ≥ 0 yields

$$|w+\alpha+\gamma e^{-w} - (w+\beta)e^{-2w}| \geq |w+\alpha| - |\gamma||e^{-w}| - |w+\beta||e^{-2w}|$$

$$> |w+\alpha| - |\alpha| - |w+\beta|$$

If $\lim_{s\to\infty} G(s) = 0$, then $a_n = 0$ and $\gamma = 0$. This means that (71), and hence (70) and (69) have no zeros in the right half plane and thus Corollary 1 is applicable to this problem.

Conclusions

It has been shown that the time-honored technique of determining the existence of unstable poles of a closed-loop gain by counting encirclements of the critical point of the open-loop gain along a finite segment of the imaginary axis remains valid for a large class of distributed parameter systems of practical importance in particular Bellcomm's POGO model, since the open-loop gain approached zero for large frequencies. The analysis reinforces the previously published opinion [3] that the Pontryagin criterion is "unsatisfactory except as a theoretical result", i.e., it is not directly applicable to existing practical problems. Existing limitations of the Michailov criterion have been removed so as to include physical systems of lossless transmission lines.

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1033-GCR-jf G. C. Reis

Attachments
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Appendix A

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APPENDIX A

In this appendix we prove the various results noted in that part of the memorandum devoted to the Pontryagin criterion. For convenience we use the same equation numbers here as in the main text

$$P(p(iy)) = |p(iy)|^2 \frac{d}{dy} \theta_p(y) \tag{4}$$

Proof

$$p(iy) = |p(iy)|e^{i\theta_p(y)} = |p(iy)|[\cos\theta_p(y)+i\sin\theta_p(y)]$$

$$P(p(iy)) = |p(iy)|\cos\theta_p(y) \frac{d}{dy} |p(iy)|\sin\theta_p(y)$$

$$- |p(iy)|\sin\theta_p(y) \frac{d}{dy} |p(iy)|\cos\theta_p(y)$$

$$= \{|p(iy)|^2[\cos\theta_p(y)]^2 + |p(iy)|^2[\sin\theta_p(y)]^2\} \frac{d}{dy} \theta_p(y)$$

$$+ |p(iy)|\cos\theta_p(y)\sin\theta_p(y) \frac{d}{dy} |p(iy)|$$

$$- |p(iy)|\sin\theta_p(y)\cos\theta_p(y) \frac{d}{dy} |p(iy)|$$

$$= |p(iy)|^2 \frac{d}{dy} \theta_p(y)$$

$$P(a) = 0 \tag{6}$$

<u>Proof</u>

Let $\alpha = |\alpha|e^{i\theta}$. By (4)

$$P(\alpha) = |\alpha|^2 \frac{d}{dy} \theta = 0$$

(7) $$P(p(iy)q(iy)) = |q(iy)|^2 P(p(iy)) + |p(iy)|^2 P(q(iy))$$

<u>Proof</u>

Let $p(iy) = p_r + ip_i$ and $q(iy) = q_r + iq_i$ and let superscript prime denote $\frac{d}{dw}$. Then

$$p(iy)q(iy) = p_rq_r - p_iq_i + i(p_rq_i+p_iq_r)$$

and

$$P(p(iy)q(iy)) = (p_rq_r-p_iq_i)(p_rq_i+p_iq_r)' - (p_rq_i+p_iq_r)(p_rq_r-p_iq_i)'$$

$$= (p_rq_r-p_iq_i)(p_r'q_i+p_rq_i'+p_i'q_r+p_iq_r')$$

$$- (p_rq_i+p_iq_r)(p_r'q_r+p_rq_r'-p_i'q_i-p_iq_i')$$

$$= p_r^2(q_rq_i'-q_iq_r') + p_i^2(q_rq_i'-q_iq_r')$$

$$+ q_r^2(p_rp_i'-p_ip_r') + q_i^2(p_rp_i'-p_ip_r')$$

$$+ p_rq_r(p_r'q_i+p_iq_r') - p_iq_i(p_rq_i'+p_i'q_r)$$

$$- p_r q_i (p_r' q_r - p_i q_i') - p_i q_r (p_r q_r' - p_i' q_i)$$

$$= (p_r^2 + p_i^2)(q_r q_i' - q_i q_r') + (q_r^2 + q_i^2)(p_r p_i' - p_i p_r')$$

$$= |p(iy)|^2 \; P(q(iy)) + |q(iy)|^2 \; P(p(iy))$$

(5) $P(\alpha p(z)) = |\alpha|^2 P(p(z))$

Proof

From (7)

$$P(\alpha p(z)) = |\alpha|^2 P(p(z)) + |p(iy)|^2 P(\alpha)$$

The desired result follows from (6).

(8) $P(1/q(z)) = -P(q(z))/|q(iy)|^4$

Proof

Using the notation $q(iy) = q_r + iq_i$ as before we have

$$P(1/q(z)) = P\left(\frac{q_r - iq_i}{|q|^2}\right)$$

$$= \left(\frac{q_r}{|q|^2}\right)\left(\frac{-q_i}{|q|^2}\right)' - \left(\frac{-q_i}{|q|^2}\right)\left(\frac{q_r}{|q|^2}\right)'$$

$$= \frac{q_r}{|q|^4}(-q_i)' - \frac{q_r q_i}{|q|^2}\left(\frac{1}{|q|^2}\right)'$$

$$+ \frac{q_i q_r}{|q|^2}\left(\frac{1}{|q|^2}\right)' + \frac{q_i}{|q|^4} q_r'$$

$$= -\frac{1}{|q|^4}(q_r q_i' - q_i q_r')$$

$$= -P(q(z))/|q|^4$$

(9) $P(p(z)/q(z)) = [|q(iy)|^2 P(p(z)) - |p(iy)|^2 P(q(z))]/|q(iy)|^4$

Proof:

By (7)

$$P(p(z)/q(z)) = \left|\frac{1}{q(iy)}\right|^2 P(p(z)) + |p(iy)|^2 P(1/q(z))$$

By (8)

$$P(p(z)/q(z)) = \frac{1}{|q(iy)|^2} P(p(z)) - \frac{|p(iy)|^2}{|q(iy)|^4} P(q(z))$$

which leads to (9).

(12) $P(q(z) + p(z)) > 0$ if and only if $P(1+p(iy)/q(iy)) >$

$$|q(iy) + p(iy)|^2 P(1/q(iy))$$

Proof:

$$P(q(z)+p(z)) = P[q(z)(1+p(z)/q(z)]$$

$$= \left|1 + \frac{p(iy)}{q(iy)}\right|^2 P(q(z)) + |q(iy)|^2 P(1+p(z)/q(z))$$

Therefore

$$\frac{1}{|q(iy)|^2} P(q(z)+p(z) = P(1+p(z)/q(z)) + \frac{|q(iy)+p(iy)|^2}{|q(iy)|^4} P(q(z))$$

$$= P(1+p(z)/q(z)) - |q(iy) + p(iy)|^2 P(1/q)$$

from (8). The desired result is an immediate consequence of this expression.