RELATIVITY EFFECTS ON TRACKING DATA *

Jean-Paul Richard
University of Maryland
College Park, Maryland

February 13, 1970

* Work supported by NASA Grant NGL 21-002-214

TABLE OF CONTENTS

1. INTRODUCTION

Recently, new radar techniques have provided a powerful tool for the study of the dynamicsof the solar system. By bouncing off signals from the surface of inferior planets, Range measurements can be made with an accuracy of a few kilometers.[1] With emitters or transponders on artificial planets, orbiters or landers, Range and Range Rate measurements can be made with an accuracy of a few meters and of about 1 mm/sec respectively.[2] At this point, the general relativistic corrections to the transmission and to the propagation of electromagnetic signals and the corrections to the orbital motions of the Earth and of the emitter (or transponder) become important. The purpose of this study is to formulate within the frame of General Relativity the relations between the various quantities measured in tracking experiments: Range, Range Rate and time. These results will be applied to simple cases where the differences between the General Relativistic predictions and the classical predictions will be evaluated in an invariant way. The separability of the general relativistic corrections from possible alternate classical explanations will also be discussed.

Emphasis will be on interplanetary experiments where the departure from a spherically symmetric gravitational field are more amenable to a relativistic treatment. Near-earth satellites are subjected to perturbations of the spherically symmetric field which are relatively larger. The relativistic treatment of these perturbations has not been done yet. There still is little reason to do it since these perturbations could not be accounted for with a high enough accuracy to put relativistic corrections into evidence.[3,4]

2. PROCEDURE

In order to analyse experiments carried within the solar system it will be helpful to neglect at first the departures from a spherically symmetric field as described by the generalized metric (§4). Thus, classical perturbations such as the oblateness of the Sun and the field of other planets will not be included in the initial relativistic treatment. Within that frame however, it shall still be possible to discuss the separability of the relativistic corrections from possible alternate classical effects not included in the relativistic solution. Later on, relativistic treatment of the many-body problem including Lense-Thirring type effects will be considered.

After a few remarks (§3), relativistic expressions of Range and Doppler shift will be derived as functions of the (coordinate) time of travel of the tracking data (§5,6,7) and then, as functions of the positions and the velocities of the stations (§9 and 10). Then the equations of motion will be solved for elliptical motion of the stations in the spherically symmetric field (§11 - 15). Using these solutions, simple cases will be considered where the relativistic corrections in tracking experiments will be evaluated in an invariant way (independent of the choice of the coordinate system) and discussed (§16 - 19).

3. REMARKS

As usual, the four dimensional line element between two events is identified with the element of proper time elapsed along this line element. It is also assumed that atomic clocks indicate proper time along their trajectories as long as their instability and the dependence of their operation on various factors (temperature, pressure, etc.) are taken into account. Proper time shall refer to earth proper time. Proper time along other trajectories should be specified as such. Time shall indicate coordinate time.

A distinction has to be made between the Doppler shift which refers to the instantaneous value of a phenomenon and the Range Rate measurement which is the average value of the Doppler shift over a period of time. This period of time can be an appreciable fraction of an hour and in such a case the Range Rate has to be related to the value of the Doppler shift at a specific instant within the counting time. At this point, expressions of the Range and Doppler shift have been derived only since these are adequate tools for the analysis of special simple cases.
The relation between the Range Rate and the Doppler Shift shall be derived later.

PART I - SPHERICAL SYMMETRY

4. GENERALIZED METRIC:

At first, the gravitational field to be described within the frame of General Relativity will be a simplified model of the actual one. The gravitational field of the planets and of the asteroids will be neglected. Effects due to the rotation[5] and to a possible oblateness of the Sun[6] are also neglected. Under these assumptions, the interplanetary gravitational field has spherical symmetry.

Following Eddington[7], Robertson[8] and Schiff[9], the field will be described by the generalized metric which is, within the context of a curved spacetime, the most general expression of a spherically symmetric field. In isotropic coordinates, it reads:

$$ds^2 = \left[1 - \frac{2\alpha_s GM}{c^2 r} + 2\beta_s \left(\frac{GM}{c^2 r}\right)^2 --\right] c^2 dt^2 - \left(1 + \frac{2\gamma}{r}\right) d\sigma^2$$

with (1)

$$d\sigma^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 = dx^2 + dy^2 + dz^2 .$$

α_s, β_s and γ_s are dimensionless numbers, G is the gravitational constant, M, the mass of the Sun and c, the speed of light. r, θ, φ and x, y, z are the polar and rectangular coordinates. t is the time. The origin is at the center of gravity of the Sun and the orientation of the axis is fixed with respect to the stars.

The relativistic corrections are to be evaluated to first order in $(GM/c^2 r)$ only. It is well known to be adequate in such a case to retain the following terms of the metric:

$$ds^2 = c^2 d\tau^2 = \left[1 - \frac{2\alpha}{r} + \frac{2\beta}{r^2}\right] c^2 dt^2 - \left(1 + \frac{2\gamma}{r}\right) d\sigma^2 \qquad (2)$$

where $\alpha = \alpha_s (GM/c^2)$, $\beta = \beta_s (GM/c^2)^2$ and $\gamma = \gamma_s (GM/c^2)$. $d\tau$ is the element of proper time associated with $d\sigma$ and dt.

The coefficients α, β and γ will appear in the relativistic corrections calculated later so that it shall be possible to relate these corrections to the first or second order deviations of the metric from the flat space — time geometry. The General Relativistic corrections will be obtained by setting the coefficient $\alpha_s = \beta_s = \gamma_s = 1$. The corrections as predicted by the Brans-Dicke theory[10] will be obtained by setting $\gamma = 0.88$.

A) Range And Doppler In A Spherically Symmetric Field

5. TWO-WAY RANGE

The Range as measured from the Earth in a two-way experiment (Fig. 1) will be defined as:

$$R = c(\tau_3 - \tau_1) = c\,\Delta_{13}\tau \tag{3}$$

where $\Delta_{13}\tau$ is the time taken by the tracking signal to travel from the Earth to the transponder and back to Earth. The proper time τ is measured by a high accuracy time standard on Earth.

The Range can also be expressed in terms of wave length accumulated along the path of the tracking signal.

$$R_\lambda = \int_{\tau_1}^{\tau_3} \nu\, d\tau = \nu\,\Delta_{13}\tau \tag{4}$$

where ν is the frequency of the emitter on Earth also measured by the time standard on Earth. It is assumed that the frequency of the tracking signal is perfectly monitored by the time standard and is thus considered to be a constant.

From the definition (3) the Range can be related to the (coordinate) time:

$$R = c\left[\frac{1}{\Delta_{13}t}\int_{\tau_1}^{\tau_3} \left(\frac{d\tau}{dt}\right) dt\right] \Delta_{13}t \tag{5}$$

where $\Delta_{13}t = t_3 - t_1$ is the travel time of the tracking signal in units of coordinate time. The term between the brackets is simply the time average of the ratio (proper time/time) during the time of travel of the tracking signal.

6. ONE-WAY DOPPLER

The frequency of a sinusoidal tracking signal will be defined as the number of wave crests being observed per unit of proper time (use of amplitude zero crossings would be equivalent). On Figure 2, the propagation of N wave crests is shown. The frequency of the signal at emission is $\nu_1 = N/\delta\tau_1$ and the apparent frequency of the signal at reception is $\nu_{12} = N/\delta\tau_2$. $\delta\tau_1$ and $\delta\tau_2$ are the intervals of proper time for the emission and the reception of the N wave crests. If the intervals of (coordinate) time corresponding to $\delta\tau_1$ and $\delta\tau_2$ are $\delta_1 t$ and $\delta_2 t$, then:

$$\nu_{12}/\nu_1 = (N/\delta\tau_2)(\delta\tau_1/N) = \left(\frac{\delta\tau_1}{\delta_1 t}\right)\left(\frac{\delta_2 t}{\delta\tau_2}\right)\left(\frac{\delta_1 t}{\delta_2 t}\right) \tag{6}$$

If $\overset{f}{\Delta}_{12}t$ and $\overset{\ell}{\Delta}_{12}t$ are the travel time of the first and last wave crests, then the total time elapsed between the emission of the initial wave crest and the reception of the last wave crest is

$$\delta_1 t + \overset{\ell}{\Delta}_{12}t = \delta_2 t + \overset{f}{\Delta}_{12}t \tag{7}$$

In a metric which does not vary with time, the travel time of the tracking signal is a function of the position at emission ($\vec{r}_1$) and at reception ($\vec{r}_2$) only. Then the travel time of the last wave crest can be expressed as a development aroung the travel time of the first wave crest:

$$\overset{\ell}{\Delta}_{12}t = \overset{f}{\Delta}_{12}t + \frac{\partial(\Delta_{12}t)}{\partial\vec{r}_1}\cdot\dot{\vec{r}}_1\delta_1 t + \frac{\partial(\Delta_{12}t)}{\partial\vec{r}_2}\cdot\dot{\vec{r}}_2\cdot\delta_2 t \quad \ldots \tag{8}$$

where $(\partial/\partial\vec{r})\cdot\dot{\vec{r}}$ stands for $\sum_{i=1}^{3}(\partial/\partial x^i)\dot{x}^i$ with x^1=x, x^2=y, x^3=z. For $\delta_1 t \to 0$ and $\delta_2 t \to 0$ (instantaneous frequency measurements), Eqs. 6 and 8 where the δ are replaced by d lead to:

$$\frac{\nu_{12}}{\nu_1} = \left(\frac{d\tau_1}{dt}\right)\left(\frac{dt}{d\tau_2}\right)\frac{\left[1-\frac{\partial(\Delta_{12}t)}{\partial\vec{r}_2}\cdot\dot{\vec{r}}_2\right]}{\left[1+\frac{\partial(\Delta_{12}t)}{\partial\vec{r}_1}\cdot\dot{\vec{r}}_1\right]} \qquad (9)$$

$(d\tau_1/dt)$ and $(d\tau_2/dt)$ are the ratios proper time/time at emission and at reception. The $\cdot$ indicates a derivative with respect to (coordinate) time.

In actual one-way Doppler experiments, the signal received on Earth (ν_{12}) is compared with the signal of a local oscillator. If the (proper) frequency of this oscillator is ν_2, we then have:

$$\frac{\nu_{12}}{\nu_2} = \left(\frac{\nu_1}{\nu_2}\right)\left(\frac{d\tau_1}{dt}\right)\left(\frac{dt}{d\tau_2}\right)\frac{\left[1-\frac{\partial(\Delta_{12}t)}{\partial\vec{r}_2}\cdot\dot{\vec{r}}_2\right]}{\left[1+\frac{\partial(\Delta_{12}t)}{\partial\vec{r}_1}\cdot\dot{\vec{r}}_1\right]} \qquad (10)$$

If identical oscillators are used at emission and at reception (same frequency when brought together at a same location), ν_1 and ν_2 are numerically equal.

7. TWO-WAY DOPPLER

In a two way Doppler experiment (Figure 3), a signal with frequency ν_1 is emitted from the Earth. Its frequency as received at the transponder is ν_{12}. Assuming no frequency multiplication at the transponder, the frequency of the signal remitted is again ν_{12}. At reception on Earth, the apparent frequency ν_{23} of that signal is compared with the frequency ν_3 of the emitter at the moment of reception.

$$\frac{\nu_{23}}{\nu_3} = \left(\frac{\nu_1}{\nu_3}\right)\left(\frac{\nu_{12}}{\nu_1}\right)\left(\frac{\nu_{23}}{\nu_{12}}\right) \tag{11}$$

Of the two last factors, the first is given in Eq. 9, and the second is the same factor where the indices 1 and 2 are replaced by 2 and 3. Thus:

$$\frac{\nu_{23}}{\nu_3} = \left(\frac{\nu_1}{\nu_3}\right)\left(\frac{d\tau_1}{dt}\right)\left(\frac{dt}{d\tau_3}\right) \frac{\left[1 - \frac{\partial(\Delta_{12}t)}{\partial \vec{r}_2}\cdot\dot{\vec{r}}_2\right]}{\left[1 + \frac{\partial(\Delta_{12}t)}{\partial \vec{r}_1}\cdot\dot{\vec{r}}_1\right]} \frac{\left[1 - \frac{\partial(\Delta_{23}t)}{\partial \vec{r}_3}\cdot\dot{\vec{r}}_3\right]}{\left[1 + \frac{\partial(\Delta_{23}t)}{\partial \vec{r}_2}\cdot\dot{\vec{r}}_2\right]} \tag{12}$$

where $\Delta_{12}t$ is the travel time from the Earth to the transducer, and $\Delta_{23}t$, the travel time from the transducer to the Earth (not equal if the motion of the Earth during the time of travel of the signal is not neglected). In order to obtain explicit expressions for the Range and Doppler measurements in terms of the position and velocity of the emitter and receiver, expressions for the time of travel of the signal and for the ratio (proper time/time) shall be derived next in terms of these quantities.

8. TRAVEL TIME OF TRACKING SIGNAL

In the geometric limit, electromagnetic signals propagate along null geodesics of the space time. In the present metric, the condition

$$0 = c^2 dt^2 \left[1 - \frac{2\alpha}{r} + \frac{2\beta}{r^2}\right] - \left[1 + \frac{2\gamma}{r}\right] d\sigma^2 \tag{13}$$

is thus satisfied for all infinitesimal intervals along the path of the tracking data. The travel time is then given by integration of its infinitesimal value along the actual geodesic Γ (Figure 4):

$$t_2 - t_1 = \Delta_{12}t = \frac{1}{c}\int_\Gamma \left[1 + \frac{\alpha+\gamma}{r}\right] d\sigma + O\alpha^2 \; * \tag{14}$$

----

* The notation $O\alpha$, $O\alpha^2$ ....will be used to indicate that terms of order (GM/c^2r), $(GM/rc^2)^2$.......have been neglected.

It can be shown[11] that a Fermat's principle of least time holds in every static gravitational field (which is the case here) along the actual path Γ. The first derivative of the integral for transverse displacements of the path (along u in Figure 4) with end points fixed is thus zero and terms of second order only are neglected if Eq. 14 is evaluated along the straight line C joining the point of emission and the point of reception.* If the straight line C is chosen to lie within the $\theta = \pi/2$ plane, then:

$$c\Delta_{12}t = \int_C [1 + \frac{\alpha+\gamma}{r}]\, d\sigma \tag{15}$$

and, after integration:

$$c\Delta_{12}t = r_{12} + (\alpha+\gamma)\ \ln\left[\frac{r_2(r_{12}+r_2-r_1\ \cos\Delta\phi)}{r_1(r_{12}+r_2\ \cos\Delta\phi - r_1)}\right] \tag{16}$$

where r_{12} has the classical form:

$$r_{12} = [r_1^2 + r_2^2 - 2r_1r_2\ \cos\Delta\phi]^{1/2} \tag{17}$$

and $\Delta\phi = \phi_2 - \phi_1$. The indices 1 [2] refers to the point of emission [reception] at the time of emission [reception].**

The travel time in a static metric should be independent of the direction of propagation or symmetric with respect to $\vec{r}_1$ and $\vec{r}_2$. This symmetry is evident when formula (16) is rewritten in the following form:

$$c\Delta_{12}t = r_{12} + (\alpha+\gamma)\ \ln\left[\frac{(r_{12}+r_2-r_1\cos\Delta\phi)(r_{12}+r_1-r_2\cos\Delta\phi)}{r_1r_2\ \sin^2\Delta\phi}\right] \tag{18}$$

----

* These corrections of second order are neglected here since they are no larger than 30 cm across the Earth orbit.

** Except for being expressed in terms of different angular variables, this expression is equivalent to the one given by T. D. Moyer in Reference 12.

9. PROPER TIME/TIME RATIO

As indicated in the remarks (Section 3), the proper time is identified with the line element (within a factor c). We thus have

$$c^2 d\tau^2 = c^2 dt^2 [1 - \frac{2\alpha}{r} + \frac{2\beta}{r^2}] - [1 + \frac{2\gamma}{r}] \, d\sigma^2 \tag{19}$$

or

$$\frac{d\tau}{dt} = [1 - \frac{\alpha}{r} - \frac{\dot{\vec{r}}^2}{2c^2}] + O\alpha^2 \tag{20}$$

where $\dot{\vec{r}}^2$ is the classical expression of the velocity squared.*

10. EXPLICIT RANGE AND DOPPLER FORMULA

From the expressions obtained in the preceding section, it is now possible to write explicit formulas for Range and Doppler in terms of the coordinate position and velocity of the emitter, transducer and receiver.

From Eqs. (5), (16) [where $\cos\Delta\phi$ has been written as $\vec{r}_1 \cdot \vec{r}_2 / r_1 r_2$] and (20) the value of the two-way Range is:

$$R = c\overline{\left(\frac{d\tau}{dt}\right)}(r_{12}+r_{23}) + (\alpha+\gamma)\, \ln \left[\frac{r_3(r_{12}+r_2-\vec{r}_1\cdot\vec{r}_2/r_2)(r_{23}+r_3-\vec{r}_2\cdot\vec{r}_3/r_3)}{r_1(r_{12}+\vec{r}_1\cdot\vec{r}_2/r_1-r_1)(r_{23}+\vec{r}_2\cdot\vec{r}_3/r_2-r_2)}\right] \tag{21}$$

where

$$\overline{\left(\frac{d\tau}{dt}\right)} = 1 - \int_{t_1}^{t_3} \left(\frac{\alpha}{r} + \frac{\dot{\vec{r}}^2}{2c^2}\right) dt + O\alpha^2 \tag{22}$$

is the average of the ratio (proper time/time) over the time of travel of the tracking data.

The One-way Doppler is obtained by first deriving the expression for the travel time (Eq. 16) with respect to time and replacing in Eq. 10

---

* It is assumed that $\dot{\vec{r}}^2/c^2$ is of order α/r since this is the case for the orbits we are interested in (bound) orbits.

$\partial(\Delta_{12}t)/\partial\vec{r}$ by its calculated value. The result is:

$$\frac{\nu_{12}}{\nu_2} = \frac{\nu_1}{\nu_2} \frac{\left[1-\frac{\alpha}{r_1}-\frac{\dot{\vec{r}}_1^{\,2}}{2c^2}\right]\left[1-\frac{\vec{r}_{12}\cdot\dot{\vec{r}}_2}{cr_{12}}-(\frac{\alpha+\gamma}{c^3})G_{12}\right]}{\left[1-\frac{\alpha}{r_2}-\frac{\dot{\vec{r}}_2^{\,2}}{2c^2}\right]\left[1-\frac{\vec{r}_{12}\cdot\dot{\vec{r}}_1}{cr_{12}}\right]} \tag{23}$$

where

$$G_{12} = \frac{\dot{\vec{r}}_2\cdot[(r_2/r_{12})\vec{r}_{12}-\vec{r}_1+(2+r_{12}/r_2)\vec{r}_2]-\dot{\vec{r}}_1\cdot[\vec{r}_2+(r_2/r_{12})\vec{r}_{12}]}{[r_2r_{12}+r_2^{\,2}-\vec{r}_1\cdot\vec{r}_2]}$$

$$-\frac{\dot{\vec{r}}_1\cdot[\vec{r}_2-(r_1/r_{12})\vec{r}_{12}-(2-r_{12}/r_1)\vec{r}_1]+\dot{\vec{r}}_2\cdot[\vec{r}_1+(r_1/r_{12})\vec{r}_{12}]}{[r_1r_{12}-r_1^{\,2}+\vec{r}_1\cdot\vec{r}_2]} \tag{24}$$

Similarly, the two-way Doppler formula follows from its value as given in (Eq. 12):

$$\frac{\nu_{23}}{\nu_3} = \frac{\nu_1}{\nu_3} \frac{\left[1-\frac{\alpha}{r_1}-\frac{\dot{\vec{r}}_1^{\,2}}{2c^2}\right]\left[1-\frac{\vec{r}_{12}\cdot\dot{\vec{r}}_2}{cr_{12}}-\frac{(\alpha+\gamma)}{c^3}G_{12}\right]\left[1-\frac{\vec{r}_{23}\cdot\dot{\vec{r}}_3}{cr_{23}}-\frac{(\alpha+\gamma)}{c^3}G_{23}\right]}{\left[1-\frac{\alpha}{r_3}-\frac{\dot{\vec{r}}_3^{\,2}}{2c^2}\right]\left[1-\frac{\vec{r}_{12}\cdot\dot{\vec{r}}_1}{cr_{12}}\right]\left[1-\frac{\vec{r}_{23}\cdot\dot{\vec{r}}_2}{cr_{23}}\right]} \tag{25}$$

where G_{23} is given by Eq. 24 where the indices 1 and 2 are replaced by 2 and 3 respectively.*

The Range and Doppler formulas given above are to be used in conjunction with formulas for positions, velocities and proper time expressed in terms of the time, and derived within the context of the generalized metric. These

---

* This result is similar to the one given by T. D. Moyer p. 18, ref. 13. In our case, the field ϕ used by Moyer reduces to α/r. No attempt has been made to verify the identity of the quantities ε_{12} and G_{12} which are expressed in a different way following the expression of the travel time in terms of different variables (as noted at the bottom of page 10).

formulas will be derived next. From these formulas, it shall be possible to obtain, for simple cases, Range and Doppler formula in terms of orbital parameters and then in terms of measured quantities only (invariant formulation)

B) Free Orbital Motion In A Spherically Symmetric Field

11. FREE ORBITAL MOTION

In PART I, the effect of the gravitational fieldof the Sun is being taken into account only. Thus the Earth, the planets, artificial planets and satellites are considered to be test particles of negligible mass travelling along geodesics of the metric when in free fall (no solar radiation pressure, rocket thrust or others).

The easiest way to solve for the geodesics is by the Lagrangian method. The Euler's equations are:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial(dx^\mu/d\tau)}\right) - \frac{\partial L}{\partial x^\mu} = 0 \quad . \tag{26}$$

where $\mu = 0, 1, 2, 3$ and $x^0 = t$ and the Lagrangian is

$$L = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \quad . \tag{27}$$

These equations are equivalent to the geodesic equations in their standard form. From the definition of proper time (Eq. 2) we also have the condition

$$L = cte = 1 \quad . \tag{28}$$

It is convenient to use polar coordinates here. Within the metric $g_{\mu\nu}$ as defined in Eq. 2 , the equation for θ is:

$$[1+\frac{2\gamma}{r}]r^2\left[\frac{d^2\theta}{d\tau^2} - \sin\theta\cos\theta(\frac{d\phi}{d\tau})^2\right] + \frac{d\theta}{d\tau}\frac{d}{d\tau}[(1+\frac{2\gamma}{r})r^2] = 0 \tag{29}$$

This equation admits the solution θ = cte = $\pi/2$. This solution will be retained here without loss of generality since there is spherical symmetry. The Euler's equations for ϕ and t then are:

$$[1 - \frac{2\alpha}{r} + \frac{2\beta}{r^2}](\frac{dt}{d\tau}) + 0\alpha^3 = \text{cte} \tag{30}$$

$$[1 + \frac{2\gamma}{r}]r^2(\frac{d\phi}{d\tau}) - 0\alpha^3 = \text{cte} \quad . \tag{31}$$

The condition L = 1 is the fourth equation and reads:

$$[1- \frac{2\alpha}{r} + \frac{2\beta}{r^2}](\frac{dt}{d\tau})^2 - \frac{1}{c^2} [1+ \frac{2\gamma}{r}][(\frac{dr}{d\tau})^2 + r^2 (\frac{d\phi}{d\tau})^2] + 0\alpha^3 = 1 \; . \tag{32}$$

12. ORBITS

From Eqs. 30-32 the equation relating r and ϕ can be obtained. After integration and with a proper choice of constants of integration, the following solution is obtained:

$$r = \frac{p}{1+e \cos\left\{[1- \frac{\alpha}{p} (2+ \frac{2\gamma}{\alpha} - \frac{\beta}{\alpha^2})][\phi-\phi_\pi]\right\}} \tag{33}$$

where $p = a(1-e^2)$. e is the eccentricity and a, the semi-major axis of the orbit. ϕ_π is the angular position at the first passage at perihelion (which is not fixed). a, e and ϕ_π must be distinguished from their classical counterparts appearing in the solution of the Newtonian equations of motion. Their numerical values can be different.

Equation (33) is identical in form to the classical solution except for the factor of $(\phi-\phi_\pi)$ which contains a perihelion advance. It is easily verified that the point at which r = 0 and is a minimum occurs every $2\pi n$

$[1+ \frac{\alpha}{p} (2+ \frac{2\gamma}{\alpha} - \frac{\beta}{\alpha^2})]$ radians where n is an integer. This corresponds to a perihelion advance of $(2\pi \alpha /p)(2+ \frac{2\gamma}{\alpha} - \beta/\alpha^2)$ radian per revolution. The General Relativistic correction is obtained if the coefficients $\alpha, \sqrt{\beta}$ and γ are set equal to GM/c^2. The perihelion advance is then the familiar expression $(6\pi GM/c^2 p)$. This angular quantity is an invariant. With numerical values inserted, the displacement of the perihelion can be given as $27[a_E^{3/2}/a^{3/2}(1+e)]$ kms per year. a_E is the semi-major axis of the orbit of the Earth.

13. VELOCITY ALONG THE ORBITAL PATH

The angular motion as a function of time along the orbit defined by Eq. 33 follows from the equations of motion (30) and (31).

With the following change of variables:

$$f = [1- \frac{\alpha}{p} (2+ \frac{2\gamma}{\alpha} - \frac{\beta}{\alpha^2})][\phi-\phi_\pi] \tag{34}$$

and the usual definition of the eccentric anomally u:

$$Tg(\frac{u}{2}) = [\frac{(1-e)}{(1+e)}]^{1/2} Tg(\frac{f}{2}) \tag{35}$$

the integration of the equation of motion can be carried out. The result is:

$$u - [1- \frac{2(\alpha+\gamma)}{a}] e \sin u = \sqrt{\frac{c^2\alpha}{a^3}} [1- \frac{4\alpha+5\gamma}{2a}](t-t_\pi) \tag{36}$$

where t_π is the time at the first passage at perihelion $(\phi=\phi_\pi)$. The relation between $(\phi-\phi_\pi)$ and the time as given by Eqs. 34-36 exhibits two relativirtic corrections. A secular correction to Kepler's third law is contained in Eq. 34 and in the factor of $(t-t_\pi)$ of Eq. 35. The other is a short period correction appearing through the coefficient of $(\phi-\phi_\pi)$ (since ϕ is not directly proportional to time) and through the coefficient of e sin u. These corrections will be considered in more details later.

14. PROPER TIME/TIME RELATION ALONG THE ORBITAL PATH

The relation proper time/time is obtained through integration of Eq. 30 with proper evaluation of the corresponding constants of integration in terms of the adopted ones: p (or a) and e. The result is:

$$\tau - \tau_\pi = [1+ \frac{\alpha}{2a}] (t - t_\pi) -2 \sqrt{\frac{\alpha a}{c^2}} u + 0\alpha^2 \qquad (37)$$

where τ_π is the proper time at the first passage at perihelion. Since the coefficient of u is of the first order, u needs to be evaluated to 0th order only. According to Eq. 36, this is the classical relation

$$u - e \sin u = \sqrt{\frac{c^2\alpha}{a^3}} (t-t_\pi) + 0\alpha^2 \qquad . \qquad (38)$$

15. CIRCULAR ORBIT IN THE GENERALIZED METRIC

For future reference, the equations describing the motion along circular orbits are given here. They simply follow from the previous one (Eqs. 33, 34-36 and 37) when the eccentricity e is set to zero:

$$r = a = cte \qquad (39)$$

$$\phi-\phi_o = [1- \frac{\alpha}{2a} (\frac{\gamma}{\alpha} + \frac{2\beta}{\alpha^2})] \sqrt{\frac{c^2\alpha}{a^3}} (t-t_o) \qquad (40)$$

$$\tau-\tau_o = [1- \frac{3\alpha}{2a}] [t-t_o] \qquad . \qquad (41)$$

The indice o refers to an arbitrary point. Equation 40 contains the relativistic correction to Kepler's third law which, in General Relativity ($\alpha = \sqrt{\beta} = \gamma = GM/c^2$) becomes *: $\dot\phi = \sqrt{GM/a^3}[1-3GM/ 2c^2a]$.

----

* This result is in agreement with the one given in Reference 9.

C) Relativistic Corrections For Specific Experiments

Various experiments where Range and Doppler measurements are made by an observer on Earth will now be considered. Relativistic predictions will be expressed in terms of observables only*. Comparison with classical predictions will allow an invariant evaluation of the relativistic corrections. Relations derived in the previous sections shall be used so that the same assumptions hold: the field of the Sun is taken into account only and effects due to its rotation or to a possible oblateness are not taken into account in the evaluation of the relativistic corrections.

16. TIME DILATION EFFECTS

An experiment where time is indicated by a clock in orbit around the Sun is compared with the time indicated by an identical clock on Earth will now be considered. It is assumed that timing pulses are sent at regular proper time intervals of the orbiting clock. The time of arrival of these pulses on Earth, as indicated by the Earth based clock is measured. It is also assumed that Range measurements are made at the time where the pulses are emitted so that the travel time of these pulses can be evaluated. The orbit of the experimental station is elliptical. Its inclination with respect to the ecliptic is not specified. The eccentricity of the Earth is

---

* Observable refers to quantities which are measured directly (like Range and Doppler at definite value of Earth proper time) or to quantities which are determined by another set of measurements than the one considered (period of the Earth as determined by optical observations for example).

neglected.

Since the orbit of theEarth is assumed to be circular, the rate of a clock on Earth in terms of the coordinate time is given by Eq. 41. The corresponding relation for the clock on the orbiting station is Eq. 37 (where the eccentricity is not assumed to be zero). Developing Eq. 37 with respect to the eccentricity and keeping the term of first order in e only:

$$\tau = \tau_p + \frac{3}{2}\,\alpha(\tau-\tau_\pi)\left(\frac{1}{a_p} - \frac{1}{a_E}\right) + 2\,\frac{\sqrt{\alpha a_p}\,e_p}{c}\sin\left[\sqrt{\frac{c^2\alpha}{a_p^3}}\,(\tau-\tau_\pi)\right] + \frac{R}{c} + \text{cte.} \qquad (42)$$

where τ is the proper time of arrival on Earth of the timing pulse emitted at orbiting station proper time τ_p. $\frac{R}{c}$ is the proper time of travel of the timing pulse. The orbiting station is at perihelion at (Earth) proper time τ_π. The indice p in e_p and a_p refers to the orbiting station.

For $\alpha = GM/c^2$ (General Relativity value), the secular drift rate (coefficient of $\tau-\tau_\pi$) is of order 10^{-8}. Such an effect could be observed at the present time. However the accuracy in the measurement will be limited by the very long term components of the drift and a possible offset of the clock rate at launch. The observation of the short period terms would not have these sources of errors.

The amplitude of the short period term is $\sim 0.1\ e\ \sqrt{a_p/a_E}$ sec. For a highly eccentric orbit where e $\sim$ 0.5 and $(a_E/a_p) \sim 2$, the amplitude of this effect is $\sim$ 0.07 sec. and its period is 4 months. If the range R measured at the time of transmission of the timing pulses has an error of the order of 50 meters, the resulting error in the measurement of the effect is a negligible few parts in 10^6. For observations lasting over a few revolutions of the orbiting station, the error due to the instability of the clocks would be reduced. The error in the determination of the time of arrival of the pulses could also be reduced by repeated measurements. Using equivalent figures of 10^{-13} and two microsecond for these errors, the relativistic time dilatation could be measured to better than one part in 10^5. The latest laboratory measurements, to our knowledge, are accurate to about 1%. Such an experiment could provide a more accurate measurement of the time dilation effect than has now been obtained. It could not however test for second order effects which would be about $\sim 10^{-8}$ time the first order effect.

17. TWO-WAY RANGE MEASUREMENTS FOR CIRCULAR COPLANAR ORBIT

a) Synodic Period and Time of Inferior Conjunction Known in Units of Earth Proper Time.

The experiment to be discussed now is one where tracking data sent from Earth reaches a transducer in orbit around the Sun and is sent back to Earth (Figure 5). It is assumed that the orbit of the transducer is circular and contained in the ecliptic plane. It is also assumed that the synodic period and the time at inferior conjunction are known in units of observer (Earth) proper time. The eccentricity of the orbit of the Earth and the effects of the field and of the rotation of the Earth on the observer's position, velocity and proper time are neglected. The displacements of the Earth and of the transponder during the time of travel of the tracking data is also neglected for the purpose of simplification ($\vec{r}_1=\vec{r}_3$ in Eq. 21).

Since the travel time of the tracking data as given in Eq. 16 is symmetric with respect to the point of emission and the point of reception, the round trip time of travel is simply twice the one way time of travel. Both the synodic period and the ratio (Earth proper time rate/time rate) are constant with respect to time from Eqs. 40 and 41. The synodic period T_s can be chosen as the unit of proper time on Earth and the inferior conjunction as the origin of proper time. In such unit, the value of the Range is:

$$R = 2r + 2\overset{1}{R} + O\alpha^2 \quad . \tag{43}$$

$\overset{1}{R}$ is a first order term given by:

$$\overset{1}{R} = (\alpha+\gamma)\,\ln\left[\frac{\overset{0}{a}_E\,[\overset{0}{r}+\overset{0}{a}_E-\overset{0}{a}_T\cos(2\pi\tau)]}{\overset{0}{a}_T\,[\overset{0}{r}+\overset{0}{a}_E\cos(2\pi\tau)-\overset{0}{a}_T]}\right] \quad . \tag{44}$$

Also,

$$\begin{aligned} r &= [a_E{}^2+a_T{}^2-2a_E a_T\cos(2\pi\tau)]^{1/2} \\ \overset{0}{r} &= [\overset{0}{a}_E{}^2+\overset{0}{a}_T{}^2-2\overset{0}{a}_E\overset{0}{a}_T\cos(2\pi\tau)]^{1/2} \end{aligned} \tag{45}$$

a_E and a_T are the semi-major axis of the orbits of the Earth and of the transducer*. In $\overset{1}{R}$, the relativistic quantities r, a_E and a_T have been replaced by their Newtonian (classical) values $\overset{0}{r}$, $\overset{0}{a}_E$ and $\overset{0}{a}_T$ since terms of second order are neglected. In Eqs. 43 and 44, the Range is not expressed in terms of observables only since the radii a_E and a_T are not directly measured.

In order to relate the Range R to observables only, it will be assumed that a_E and a_T are to be determined from three measurements of the Range itself R_1, R_2 and R_3 performed respectively at proper time τ_1, τ_2 and τ_3.** a_E and a_T are then the solutions of:

$$\begin{aligned} 2r_2 + 2\overset{1}{R}_2 - 2r_1 - 2\overset{1}{R}_1 &= R_2-R_1 \\ 2r_3 + 2\overset{1}{R}_3 - 2r_1 - 2\overset{1}{R}_1 &= R_3-R_1 \end{aligned} \tag{46}$$

---

* This result is identical with the one given in Reference 15.

** This is only an approximation to the real data analysis where the elements of the orbits are determined by curve fitting using data distributed over the whole orbits.

where r_1 and $\overset{1}{R}_1$ are the functions of a_E and a_T given by Eqs. 44 and 45 when $\tau = \tau_1$. Since differences of range measurements are used in Eq. 46, errors resulting from constant bias in the Range measurements (transducer delay or other) are eliminated*. These equations can be used to evaluate the radii by successive approximation.

If $\overset{0}{a}$ and $\overset{1}{a}$ are zeroth and first order quantities in the following definitions:

$$a_E = \overset{0}{a}_E + \overset{1}{a}_E$$

$$a_T = \overset{0}{a}_T + \overset{1}{a}_T \qquad (47)$$

then, the zeroth order quantities $\overset{0}{a}_E$ and $\overset{0}{a}_T$ (Newtonian values of the radii) satisfy Eq. (46) where the relativistic terms $\overset{1}{R}_1$ are neglected. This is:

$$2\overset{0}{r}_2 - 2\overset{0}{r}_1 = R_2 - R_1$$

$$2\overset{0}{r}_3 - 2\overset{0}{r}_1 = R_3 - R_1 \qquad (48)$$

$\overset{0}{a}_E$ and $\overset{0}{a}_T$ can be used to evaluate the first order term $\overset{1}{R}_1$ in the Range. If a_E and a_T are written in terms of $\overset{0}{a}$ and $\overset{1}{a}$ in Eq. 46 and if r is developed around the $\overset{0}{a}$ up to the first order in the corrections $\overset{1}{a}$, then the quantities of zeroth order in a cancel with the quantities R_2-R_1 and R_3-R_1 according to Eq. (48) and the first order terms $\overset{1}{a}_E$ and $\overset{1}{a}_T$ obeys a simple set of equations which is immediately solved:

---

* Two range measurements would suffice to determine the radii. That case is considered in Reference 15 and leads to a simpler formulation of the relativistic corrections.

$$\overset{1}{a}_E = \frac{\left(\overset{1}{R}_3-\overset{1}{R}_1\right)\left[\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_2 - \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_1\right] - \left(\overset{1}{R}_2-\overset{1}{R}_1\right)\left[\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_3 - \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_1\right]}{\square}$$

$$\overset{1}{a}_T = -\frac{\left(\overset{1}{R}_3-\overset{1}{R}_1\right)\left[\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_2 - \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_1\right] - \left(\overset{1}{R}_2-\overset{1}{R}_1\right)\left[\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_3 - \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_1\right]}{\square}$$

$$\frac{\square}{2} = \left[\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_2 - \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_1\right]\left[\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_3 - \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_1\right] - \left[\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_2 - \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_1\right]\left[\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_3 - \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_1\right] \tag{49}$$

and where the derivatives are given by:

$$\left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_{E,T}}\right)_i = \left(\frac{1}{\overset{0}{r}_i}\right)\left[\overset{0}{a}_{E,T} - \overset{0}{a}_{T,E}\cos(2\pi\tau_i)\right] \tag{50}$$

The values of the radii as given by Eqs. 47-49 can now be inserted in the expression for the Range (Eqs. 43-45). The zeroth order part of the Range can be developed around the zeroth order value of the radii. In the first order term, $\overset{1}{R}$, a can be replaced by $\overset{0}{a}$ since terms of second order (α^2) are neglected. The Range at some proper time τ_ℓ thus becomes:

$$R_\ell = 2\left[\overset{0}{a}_E{}^2 + \overset{0}{a}_T{}^2 - 2\overset{0}{a}_E\overset{0}{a}_T\cos(2\pi\tau_\ell)\right]^{1/2} + 2\delta_\ell R \tag{51}$$

where the relativistic effect δR is:

$$\delta_\ell R = \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_E}\right)_\ell \overset{1}{a}_E + \left(\frac{\partial \overset{0}{r}}{\partial \overset{0}{a}_T}\right)_\ell \overset{1}{a}_T + (\alpha+\gamma)\,\ell n\left[\frac{\overset{0}{a}_E[\overset{0}{r}_\ell+\overset{0}{a}_E-\overset{0}{a}_T\cos(2\pi\tau_\ell)]}{\overset{0}{a}_T[\overset{0}{r}_\ell+\overset{0}{a}_E\cos(2\pi\tau_\ell)-\overset{0}{a}_T]}\right] \tag{52}$$

where the derivatives of $\overset{0}{r}$ are given by Eq. 50. This expression of the Range is implicitely in terms of the observables. R_1, R_2, R_3 and τ_ℓ: $\overset{0}{r}$ and $\overset{1}{a}$ (through $\overset{0}{r}$ and $\overset{1}{R}$) are related to the $\overset{0}{a}$, which are themselves related to the three measurements R_1, R_2 and R_3 by Eq. 48. For given values of R_1, R_2 and R_3 at proper time τ_1, τ_2 and τ_3, the computed value of R_ℓ at time τ_ℓ is independent of the choice of the coordinate system chosen (isotropic coordinates).

The numerical value of the General Relativistic correction $2\delta_\ell R$ (where $\alpha = \sqrt{\beta} = \gamma = GM/c^2$) is shown on Figure 6 for the case where the radius of the transducer orbit is 0.8 A.U.. The three measurements R_1, R_2 and R_3 are assumed to be made at $\tau = 0$, $\tau = T_s/8$ and $\tau = T_s/4$ ($\Delta\phi = 0$, 45° and 90°).* The effect which is more important near superior conjunction is essentially due to a modification of the optical length along the path of the tracking data.**

18. TWO-WAY RANGE MEASUREMENTS FOR CIRCULAR COPLANAR ORBIT

b) The Time of Inferior Conjunction and the Orbital Period of the Earth and of the Transducer Known in Units of Earth Proper Time.

This is essentially the same problem as the one treated in Section 17. It is assumed however that both the Earth and the transducer period have been determined in units of Earth proper time (by tracking data, optical observations or other means).*** These assumptions make it possible to look for the relativistic corrections to the third Kepler law given in Eq. 40.

The expression of the Range in terms of the new observables is easily obtained by replacing in Eq. 43 the quantities a_E and a_T by their value in terms of the orbital periods as given in Eq. 40. If this expression is divided by the value of the Range at inferior conjunction to be denoted

---

* This result is analog to the one given in Reference 14 based on a determination of the radii from two instead of three Range measurements.

** This effect was briefly discussed in Reference 17.

*** This problem is considered in References 15 and 16 where it is noted that one could use the value of the orbital periods as previously determined by optical observations.

by R_o, then:

$$R = R_o \left[\frac{[T_E^{4/3} + T_T^{4/3} - 2 T_E^{2/3} T_T^{2/3} \cos(2\pi\tau/T_s)]^{1/2}}{[T_E^{2/3} - T_T^{2/3}]} \right] + 2\,\delta R \qquad (53)$$

with

$$\delta R = [\alpha+\gamma]\,\ell n \left[\frac{\overset{0}{a}_E[\overset{0}{r}+\overset{0}{a}_E-\overset{0}{a}_T\cos(2\pi\tau/T_s)]}{\overset{0}{a}_T[\overset{0}{r}+\overset{0}{a}_E\cos(2\pi\tau/T_s)-a_T[}\right] - \left(\frac{\overset{0}{r}}{(\overset{0}{a}_E-\overset{0}{a}_T)}\right)\ell n \left(\frac{\overset{0}{a}_E}{\overset{0}{a}_T}\right)$$

$$- \frac{(\gamma+2\beta/\alpha)}{3}\left[\frac{\overset{0}{a}_E+\overset{0}{a}_T}{\overset{0}{r}}\right][1 - (\cos 2\pi\tau/T_s)] \qquad (54)$$

T_E and T_T are the orbital period of the Earth and of the transducer in units of proper time. $T_s^{-1} = T_T^{-1} - T_E^{-1}$ (for inferior transducer orbit). The quantities $\overset{0}{a}$ and $\overset{0}{r}$ in δR have not been replaced by their value in terms of the period for purpose of simplification. It is understood however that they are to be computed from the 0th order approximation of Eq. 40:

$$T = \frac{2\pi}{\overset{0}{\dot\phi}} = \frac{2\pi\ a^{3/2}}{\sqrt{c^2\alpha}} \qquad (55)$$

The Range as given in Eq. 53 is thus expressed in terms of observables only and is identical to the one obtained by I. I. Shapiro in Reference 15.

The first term in Eq. 53 is the classical expression of the Range in terms of the orbital periods. The first component of the relativistic correction (Eq. 54) is the analog of the corrections obtained in Section 17. It contains the important modification to the optical path due to the gravitational field of the Sun. The value of this effect predicted by General Relativity ($\alpha = \sqrt{\beta} = \gamma = GM/c^2$) if the radius of the transducer orbit is 0.8 A.U. is shown on Figure 6 (dotted curve). It is very similar to the correction predicted for the experiment discussed in the previous section. The second component of the relativistic correction (Eq. 54) contains a contribution in β resulting from the relativistic modification of Kepler's third law (Eq. 40). In General Relativity, its value is

$$-3.0 \left[\frac{\overset{0}{a}_E+\overset{0}{a}_T}{\overset{0}{r}}\right] [1 - \cos\left(\frac{2\pi\tau}{T_s}\right)] \text{ kms} . \tag{56}$$

66% of this effect is due to the term in β. Its value for $\overset{0}{a}_T = 0.8$ is shown on Figure 7. This effect does not exhibit the logarithmic dependence which makes the effect previously discussed (Figure 6) separable from classical effects. Within the specific case of circular coplanar orbits, it still remains to be considered if the effect given Eq. 56 could be distinguished from a classical effect due to a quadrupole moment of the Sun (related to its oblateness). That problem will now be considered.

19. CLASSICAL PREDICTIONS FOR TWO-WAY RANGE MEASUREMENTS

[The time of inferior conjunction and the orbital period of the Earth and of the transponder are assumed to be known. A solar quadrupole moment is taken into account.]

With the assumption that the Sun has a quadrupole moment, the potential energy per unit mass of a test particle in the field of the Sun is classically:

$$V = -\frac{GM}{r}\left[1 + \frac{J_{2,0}}{2r^2}(3\cos^2\theta - 1)\right] \tag{57}$$

where r, θ and ϕ are the polar coordinates. The ϕ plane is normal to the axis of the quadrupole moment and $J_{2,0}$ is the amplitude of the quadrupole moment. The equations of motion derived from the corresponding Lagrangian admit the solution $\theta = \pi/2$ = constant. For this value of θ, the differential equations left can be written as:

$$\frac{d^2(1/r)}{d\phi^2} + \frac{1}{r} - \frac{GM}{h^2} + \frac{3}{2}\,\frac{J_{2,0}GM}{h^2r^2} = 0$$

$$r^2\overset{0}{\phi} = h \tag{58}$$

For the special case where the orbits are circular, these equation are solved for:

$$\mathcal{T} = \frac{2\pi}{\overset{\circ}{\phi}} = \frac{1}{2\pi}\sqrt{\frac{\overset{\circ}{a}{}^3}{GM}}\left[1 + \frac{3J_{2,0}}{2\overset{\circ}{a}{}^2}\right]^* \tag{59}$$

where a is the radius of the orbit and $\mathcal{T}$, the orbital period (the ° have been reintroduced to denote classical values).

The classical value of the Range for circular and coplanar orbits (in the $\theta = \pi/2$ plane)**:

$$R = 2[\overset{\circ}{a}{}_E^2 + \overset{\circ}{a}{}_T^2 - 2\overset{\circ}{a}_E\overset{\circ}{a}_T\cos\left(\frac{2\pi\tau}{\mathcal{T}_s}\right)]^{1/2} \tag{60}$$

can now be expressed in terms of the orbital periods $\mathcal{T}_E$ and $\mathcal{T}_T$ (and $\mathcal{T}_s$). Using Eq. 59:

$$R = R_o\left[\frac{[\mathcal{T}_E^{4/3}+\mathcal{T}_T^{4/3}-2\mathcal{T}_E^{2/3}\mathcal{T}_T^{2/3}\cos(\frac{2\pi\tau}{\mathcal{T}_s})]^{1/2}}{[T_E^{2/3}-T_T^{2/3}]}\right] + 2\delta R \tag{61}$$

where

$$2\delta R = -2\,\frac{J_{2,0}}{\overset{\circ}{r}}\left[1 + \frac{\overset{\circ}{a}{}_E^2+\overset{\circ}{a}{}_T^2}{2\overset{\circ}{a}_E\overset{\circ}{a}_T}\right]\left[1 - \cos\left(\frac{2\pi\tau}{\mathcal{T}_s}\right)\right] \tag{62}$$

and R_o is the value of the Range measured at inferior conjunction.

The classical effect given in Eq. 62 is identical in form with the relativistic correction involving the factor β (second term in Eq. 54). This implies that within the experimental situation discussed in Sections

---

* $(J_{2,0}/\overset{\circ}{a}{}_E^2)$ is assumed to be small and terms of second order in $J_{2,0}$ are neglected.

** Here, the axis of rotation of the Sun is assumed to be parallel to the axis of the quadrupole moment. Its angular deviation (~3°) from the normal to the ecliptic is neglected.

18 and 19, two-way Range measurements could not establish the value of β or test for the relativistic correction to Kepler's third law (Eq. 40).

In order to separate the relativistic correction in β from an effect due to a possible quadrupole moment of the Sun, an additional observable has to be available. The residual variation in the inclination of the orbit of Mercury as determined by optical observations will be used as such. These variations set an upper limit to the amplitude of the quadrupole moment[19] such that

$$\frac{J_{2,0}}{a_E} \lesssim 50 \text{ meters} \qquad (63)$$

The classical correction to the Range in that case is shown (dotted curve) on Figure 7. It is significantly smaller than the relativistic effect shown on the same figure but should still be taken into account in high precision Range experiments.

REFERENCES

1. J. V. Evans et al, Astron. J. 70, 486 (1965).

2. F. O. Vonbun, N.A.S.A. technical note D-4507 p. 15, June 1968.

3. J.-P. Richard, Cahiers de Physique 20, nos 187-188, p. 85 (1966).

4. J.-P. Richard, Astronautica Acta, 13, 353 (1967).

5. J. Lense, H. Thirring, Phys. Zeits., 19, 156 (1918).

6. R. H. Dicke and M. Goldenberg, Phys. Rev. Lett., 18, 313 (1967).

7. A. S. Eddington, The Mathematical Theory of Relativity, (Cambridge University Press, New York, 1957) p. 105.

8. H. P. Robertson, Space Age Astronomy, A. J. Deutch and W. E. Klemperer editors (Academic Press, New York, 1962) p. 228.

9. L. I. Schiff, Proc. Nat'l. Acad. Sci., U. S. 46, 871 (1960).

10. C. Brans and R. H. Dicke, Phys. Rev., 124, 925 (1961).

11. C. Møller, The Theory of Relativity, (Oxford University Press, Oxford 1960) p. 309.

12. T. D. Moyer, Space Program Summary, 37-41, Vol. III, pp. 31-38.

13. T. D. Moyer, Space Program Summary, 37-42, Vol. III, p. 18.

14. I. I. Shapiro, Phys. Rev., 141, 1219 (1966).

15. I. I. Shapiro, Phys. Rev., 145, 1005 (1966).

16. D. K. Ross and L. I. Schiff, Phys. Rev., 141, 1215 (1966).

17. I. I. Shapiro, Phys. Rev. Lett., 13, 789 (1964).

18. D. Brouwer, Astron. J., 64, 378 (1959).

19. R. H. Dicke, Nature 202, 432 (1964).

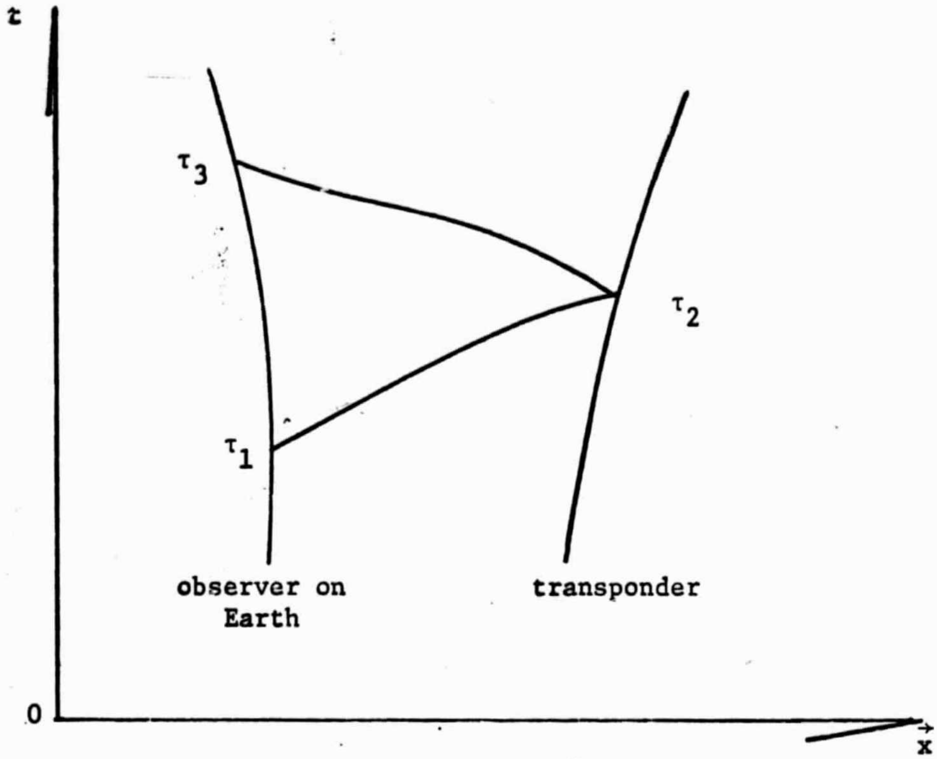


FIGURE 1

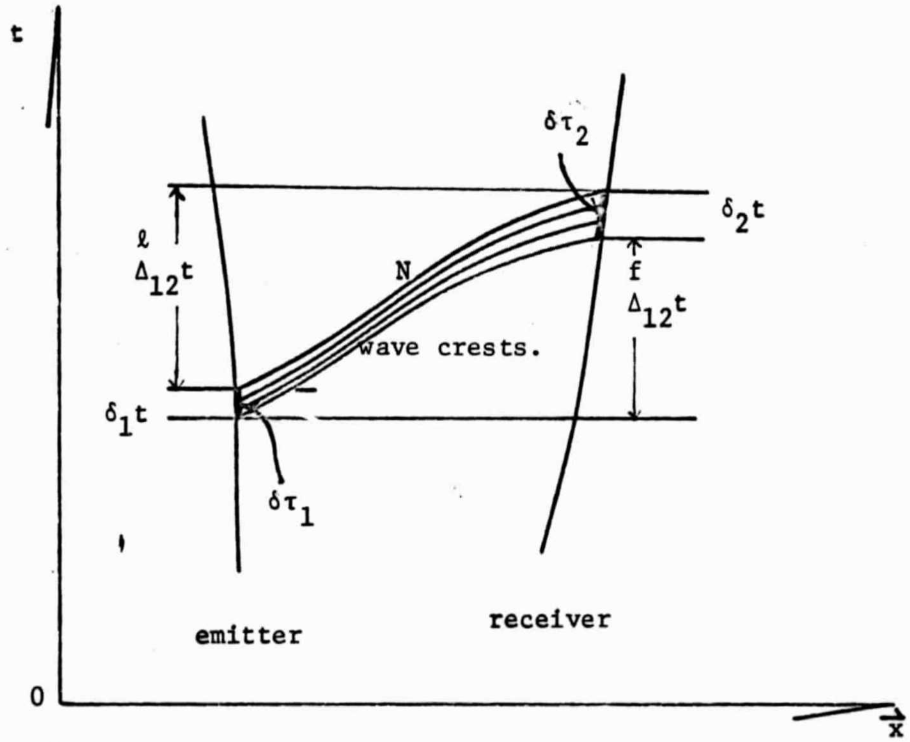


FIGURE 2

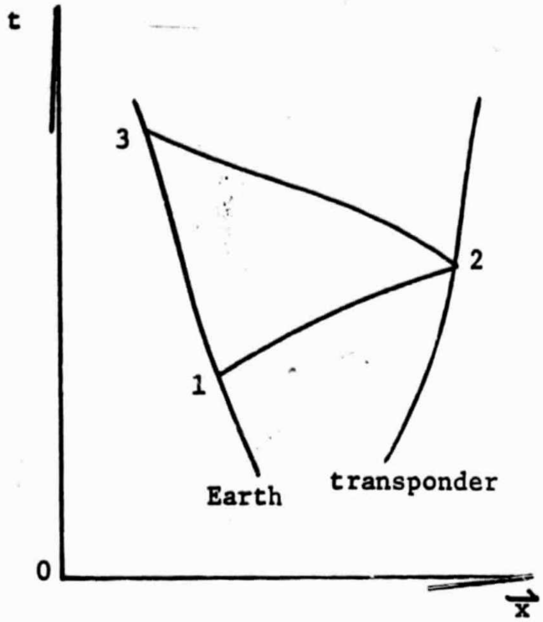


FIGURE 3

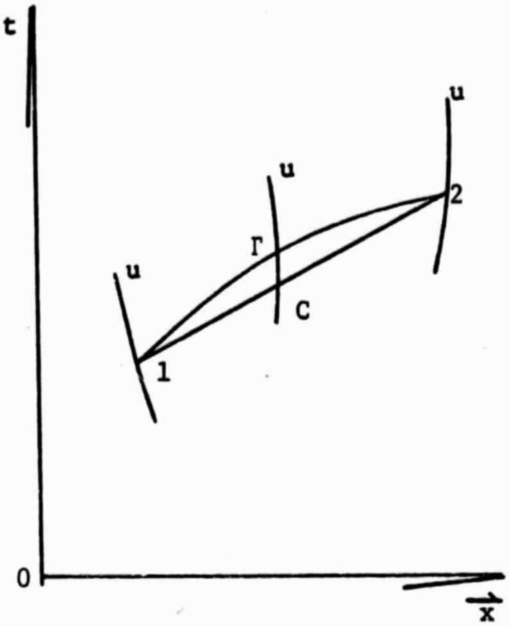


FIGURE 4

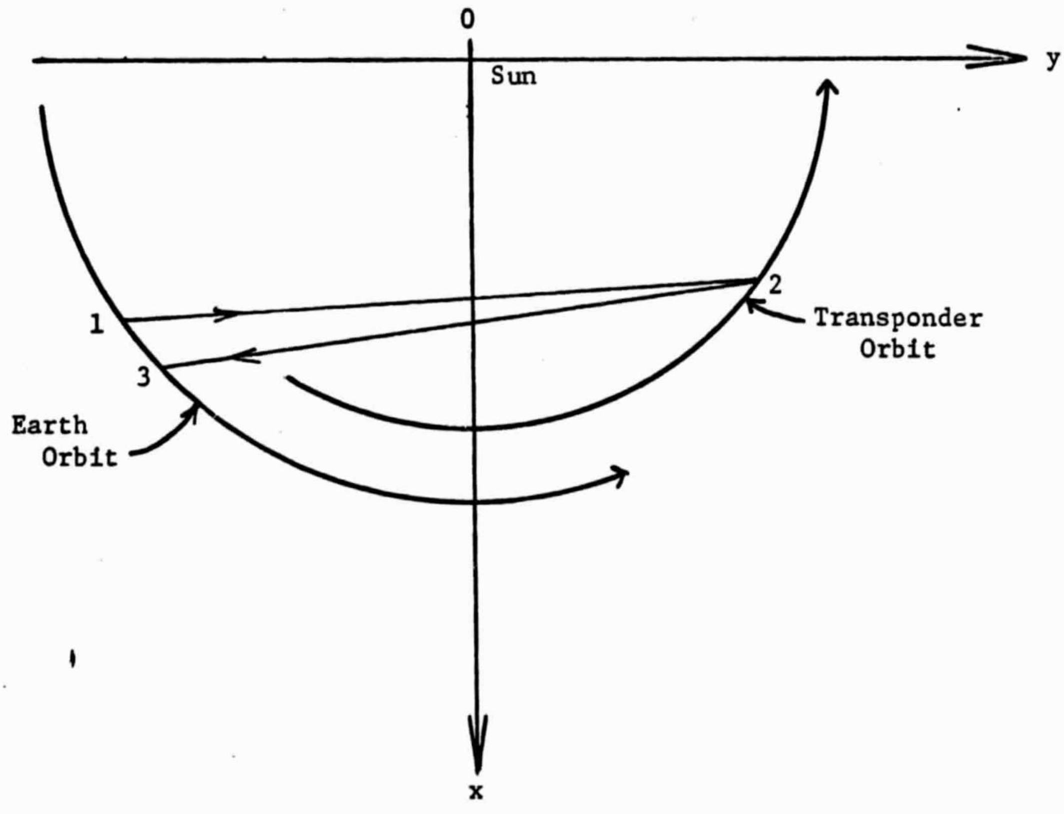


FIGURE 5

Relativistic Corrections in two-way range measurements ($2\delta_{\ell}R$ in eq. 52)
as a function of the angular distance ($\Delta\phi$) between the Earth and the transducer
(circular transducer orbit : a_T = 0.8 A.U.)

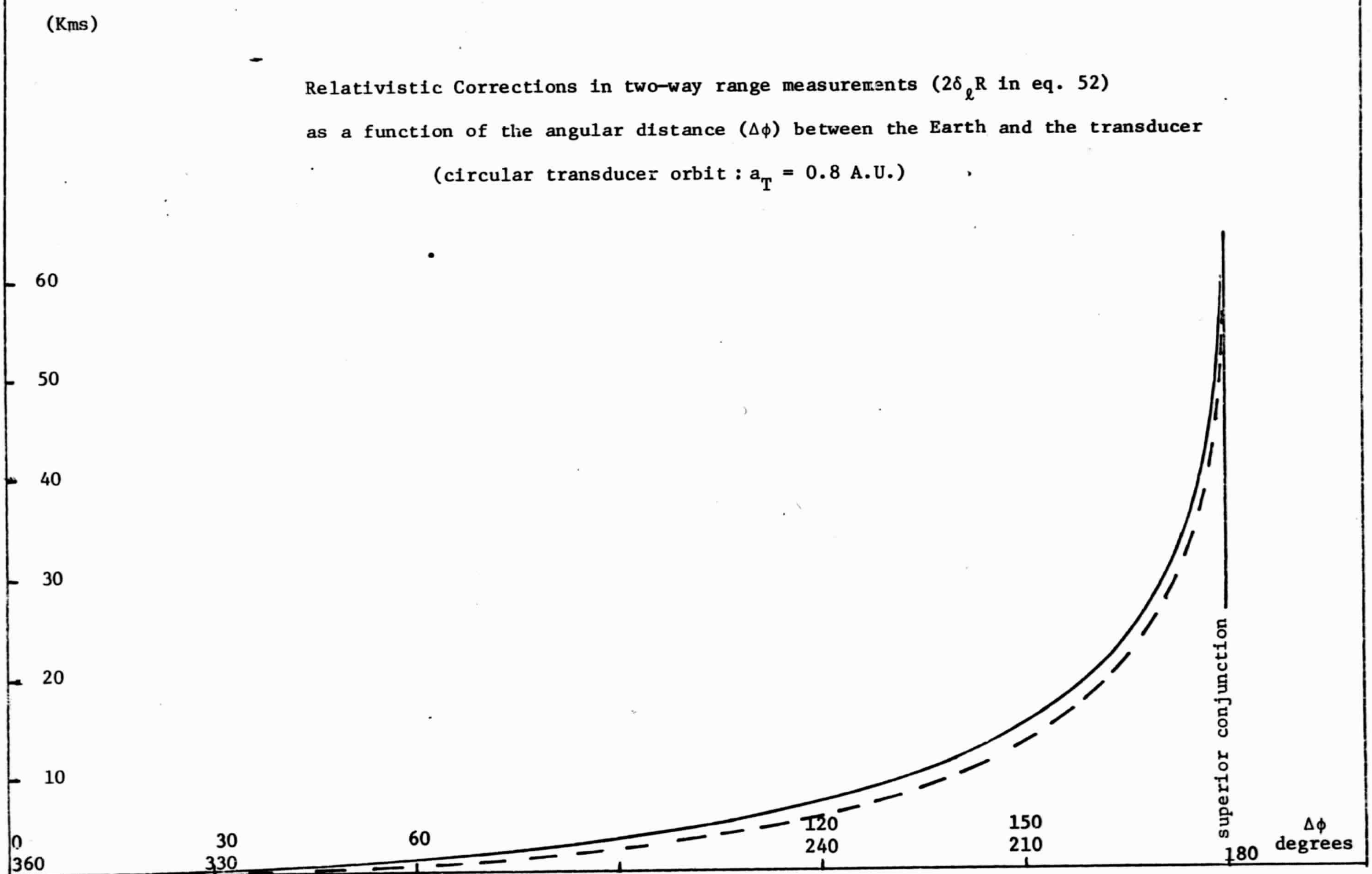
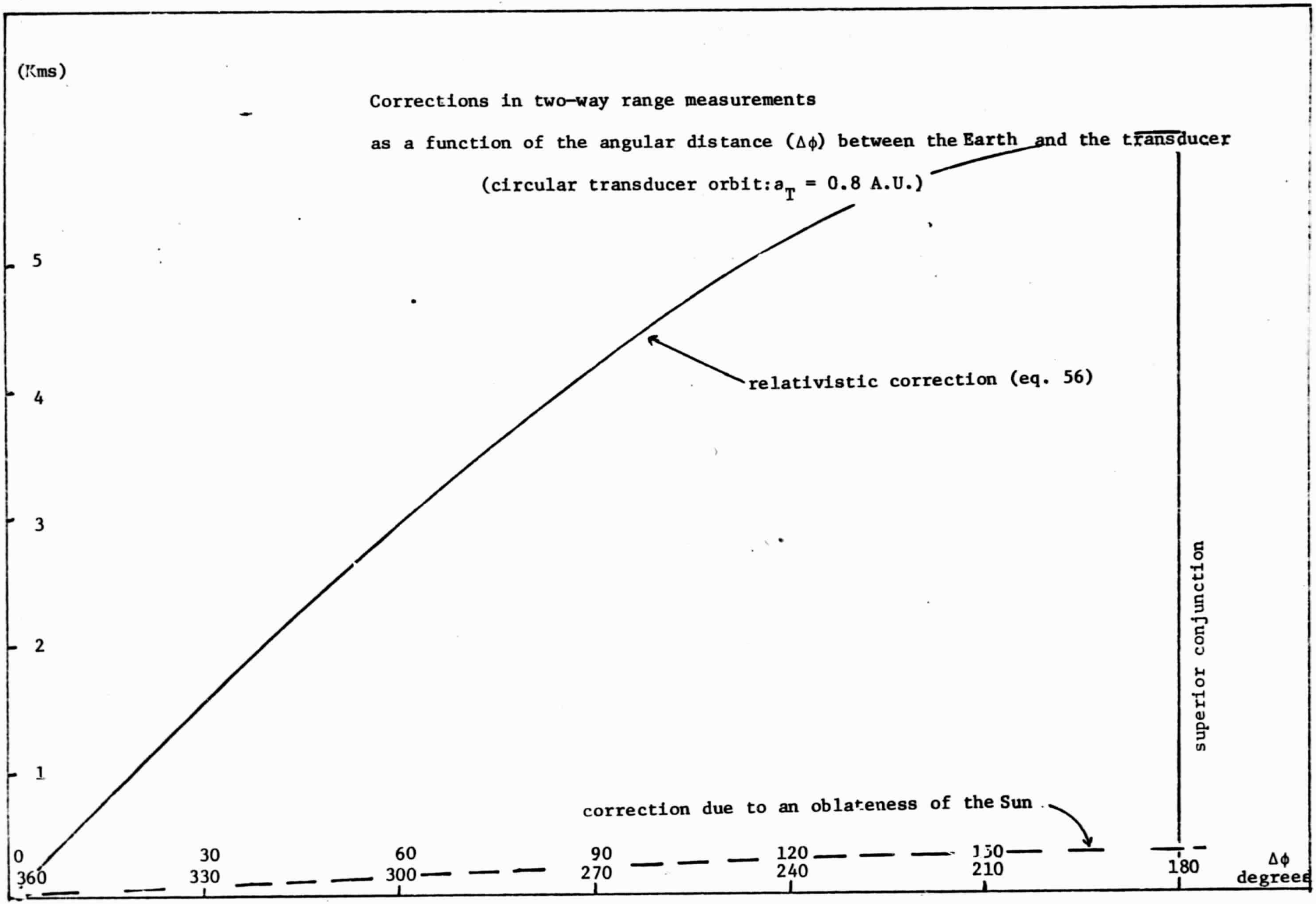


Fig. 6



(Kms)
Corrections in two-way range measurements
as a function of the angular distance (Δϕ) between the Earth and the transducer
(circular transducer orbit: a_T = 0.8 A.U.)
5
4
3
2
1
0
relativistic correction (eq. 56)
superior conjunction
correction due to an oblateness of the Sun
0 360
30 330
60 300
90 270
120 240
150 210
180
Δϕ degrees

Fig. 7