THEORETICAL CHEMISTRY INSTITUTE

THE UNIVERSITY OF WISCONSIN

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WIS-TCI-400

17 August 1970

MADISON, WISCONSIN

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ABSTRACT

It is shown how the Green's function for any inhomogeneous Sturm-Liouville (S-L) problem can be constructed once two independent solutions of the homogeneous problem are known. The general technique is illustrated by considering two specific S-L problems. When the parameter appearing in the S-L equation is equal to an eigenvalue of the homogeneous S-L problem an added difficulty arises. The method of constructing a generalized Green's function which deals with this difficult case is also treated by example.

I. General Sturm-Liouville Theory

The inhomogeneous Sturm-Liouville (S-L) problem can be written in the following form[1]

$$\frac{d}{dx}\left(p\frac{d\psi}{dx}\right)-q\psi+\lambda\rho\psi=-f\,,\quad a\leq x\leq b \tag{1}$$

with the boundary conditions

$$\begin{aligned}\left(\alpha\frac{d\psi}{dx}+\beta\psi\right)_{x=a}&=\phi_a\\ \left(\alpha\frac{d\psi}{dx}+\beta\psi\right)_{x=b}&=\phi_b\quad.\end{aligned} \tag{2}$$

The known functions p(x), q(x), ρ(x), and f(x) are restricted only in that

(i) $p(x)\geq 0$, $\rho(x)\geq 0$ for $a\leq x\leq b$

(ii) p(x) is differentiable for $a\leq x\leq b$.

The constant λ is a parameter of the problem; α and β are constants which serve to make the boundary conditions as general as possible for the S-L problem.

The Green's function G(x,y) is introduced by seeking a solution of Eq. (1) having the specific form given below:

$$\psi(x)=\lim_{\eta\to 0+}\left\{\int_a^{x-\eta}G(x,y)f(y)\,dy+\int_{x+\eta}^{b}G(x,y)f(y)\,dy\right\}. \tag{3}$$

For convenience Eq. (3) is written in shorthand as follows

$$\Psi(x) = \int_a^b G(x,y)\, f(y)\, dy. \tag{4}$$

Substituting the expression for f(y) given in Eq. (1) into Eq. (4) results in

$$\psi(x) = -\int_a^b G(x,y)\left[\frac{d}{dy}\left(p\,\frac{d\psi}{dy}\right) - q\,\frac{d\psi}{dy} + \lambda\rho\psi\right] dy \tag{5}$$

After integration by parts Eq. (5) becomes

$$\psi(x) = -\int_a^b \left[\frac{d}{dy}\left(p\frac{dG}{dy}\right) - q\,G + \lambda\rho G\right]\psi(y)\,dy$$

$$-\lim_{\eta\to 0+}\left[p\left(\frac{d\psi}{dy}G - \frac{dG}{dy}\,\psi\right)\right]_a^{x-\eta}$$

$$-\lim_{\eta\to 0+}\left[p\;\left(\frac{d\psi}{dy}G - \frac{dG}{dy}\psi\right)\right]_{x+\eta}^{b} \tag{6}$$

Based on Eq. (6) we now present four conditions which the correct Green's function must satisfy for the right hand side of Eq. (3) to correctly yield the function $\Psi(x)$. We take as the first property of the Green's function that it satisfies the homogeneous S-L equation

$$\frac{d}{dy}\left(p\,\frac{dG}{dy}\right) - q\,G + \lambda\rho G = 0 \tag{7}$$

in the intervals $a \leq y < x \leq b$, $a \leq x < y \leq b$, i.e. everwhere except at the point $x = y$. With this the integral term in Eq. (6) vanishes identically. The second property is that G(x,y) satisfies the same boundary conditions (Eq. (2)) as Ψ at $y = a$ and $y = b$. This implies that two of the boundary terms appearing in Eq. (6) are equal to zero:

$$\left[p\left(\frac{d\psi}{dy}G - \frac{dG}{dy}\psi\right)\right]_a^b$$

$$= \frac{1}{\alpha}\left[p\left(\alpha\frac{d\psi}{dy}G - \alpha\frac{dG}{dy}\psi\right)\right]_a^b$$

$$= -\frac{1}{\alpha}\left[p\left(\beta\psi G - \beta G\psi\right)\right]_a^b = 0 . \tag{8}$$

Equation (6) is therefore reduced to the following simple form:

$$\psi(x) = -\lim_{\eta\to 0+}\left[p\left(\frac{d\psi}{dy}G - \frac{dG}{dy}\psi\right)\right]_{x+\eta}^{x-\eta}$$

$$= -\lim_{\eta\to 0+}\left\{p(x)\left[\frac{dG}{dy}(x,x+\eta) - \frac{dG}{dy}(x,x-\eta)\right]\psi(x)\right.$$

$$\left. + \; p(x)\left[G(x,x+\eta) - G(x,x-\eta)\right]\frac{d\psi}{dy}(x)\right\} . \tag{9}$$

Use has been made of the facts that p(x) and Ψ(x) are continuous functions in the range $a \leq x \leq b$.

The remaining two restrictions on G(x,y) are easily inferred from Eq. (9). Clearly if G(x,y) is a continuous function of y for <u>all</u> x and y in the interval [a,b], and if $\frac{dG(x,y)}{dy}$ is a continuous function of y <u>except</u> for a jump discontinuity of magnitude $-\frac{1}{p(x)}$ at the point y = x:

$$\lim_{\eta\to 0+}\left(\frac{dG}{dy}(x,x+\eta) - \frac{dG}{dy}(x,x-\eta)\right) = -1/p(x) , \tag{10}$$

then the right hand side of Eq. (9) will yield the desired function Ψ(x).

In review then, we see that the correct Green's function G(x,y), which when substituted into Eq. (3) gives the solution $\Psi(x)$ to the S-L problem, exhibits the following four properties:

(i) G(x,y) obeys the homogeneous S-L equation, Eq. (7)

(ii) G(x,y) obeys the same boundary condition as $\Psi(y)$ (i.e. Eq. (2))

(iii) G(x,y) is continuous for all x and y in [a,b]

(iv) $\frac{dG}{dy}$ (x,y) is continuous for all x and y in [a,b] <u>except</u> for a jump discontinuity of magnitude -1/p(x) at the point y = x, (Eq. (10)).

As we will soon demonstrate by example, these four properties are sufficient to determine G(x,y) unless the parameter λ is an eigenvalue of the homogeneous S-L equation. This more difficult case will be treated after we discuss two illustrative examples.

II. Two Simple Examples

To understand how the four properties (i) - (iv) can be used to determine the Green's function G(x,y), let us consider a specific S-L problem

$$\frac{d}{dx}\left(x\frac{d\psi}{dx}\right) - \frac{n^2}{x}\psi + k^2 x\psi = -f(x), \quad 0 \le x \le b$$

with boundary conditions

$$\begin{aligned} &\psi(b) = 0 \\ &\psi(0) \quad \text{finite}. \end{aligned} \tag{11}$$

In the interval $0 \le y < x \le b$ G(x,y) should satisfy

$$\frac{d}{dy}\left(y\frac{dG(x,y)}{dy}\right) - \frac{n^2}{y}G(x,y) + k^2 y\, G(x,y) = 0.$$

Therefore G(x,y) can be expanded in terms of the two independent solutions of the homogeneous differential equation:

$$G(x,y) = A(x)\,J_n(ky) + D(x)\,Y_n(ky), \quad 0 \le y \le x \le b. \tag{12}$$

The functions A(x) and D(x) are to be determined by using the four properties of G(x,y). $J_n(ky)$ and $Y_n(ky)$ are the two <u>independent</u> solutions of the Bessel equation

$$\frac{d^2}{dz^2} W(z) + \frac{1}{z}\frac{dW}{dz} - \frac{n^2}{z^2} W + W = 0, \tag{13}$$

for z = ky. Likewise for the interval $0 \le x < y \le b$, G(x,y) can be written as

$$G(x,y) = B(x)\,J_n(ky) + C(x)\,Y_n(ky), \quad 0 \le x < y \le b. \tag{14}$$

The boundary condition that G(x,o) be finite implies that D(x) = o because $Y_n(ky)$ is not finite at y = o. The second boundary condition G(x,b) = o takes the form

$$B(x)\,J_n(kb) + C(x)\,Y_n(kb) = 0. \tag{15}$$

Because G(x,y) is continuous everywhere in [a,b] it must be continuous at the point y = x for all x in [a,b]. This property is easily written as

$$A(x)\,J_n(kx) = B(x)\,J_n(kx) + C(x)\,Y_n(kx). \tag{16}$$

The condition that $\frac{dG}{dy}$ have a jump discontinuity at y = x leads to the equation

$$B(x)\, J_n'(kx) + C(x)\, Y_n'(kx) - A(x)\, J_n'(kx) \\ = -1/x \quad , \tag{17}$$

where

$$J_n'(kx) \equiv \frac{d\,J_n(kx)}{dx}$$

and

$$Y_n'(kx) \equiv \frac{d\,Y_n(kx)}{dx} \quad . \tag{18}$$

Equations (15) - (17) can be solved simultaneously for the three functions A(x), B(x), and C(x). The resulting solutions are given by:

$$A(x) = x^{-1}\left[J_n(kx)\,Y_n(kb) - J_n(kb)\,Y_n(kx)\right] \\ \left[J_n(kb)\,Y_n(kx)\,J_n'(kx) - J_n(kb)\,J_n(kx)\,Y_n'(kx)\right]^{-1} \tag{19}$$

$$B(x) = x^{-1}\,J_n(kx)\,Y_n(kb) \\ \left[J_n(kb)\,Y_n(kx)\,J_n'(kx) - J_n(kb)\,J_n(kx)\,Y_n'(kx)\right]^{-1} \tag{20}$$

$$C(x) = -x^{-1}J_n(kx)\left[Y_n(kx)\,J_n'(kx) - J_n(kx)\,Y_n'(kx)\right]^{-1} \tag{21}$$

By using various properties of the $J_n(kx)$ and $Y_n(kx)$ these expressions can be simplified considerably[1]. Once this is done the Green's function G(x,y) given in Eqs. (12) and (14) can be written as follows:

$$G(x,y) = -\frac{\pi\,J_n(ky)}{2\,J_n(kb)}\left[J_n(kx)\,Y_n(kb) - Y_n(kx)\,J_n(kb)\right]$$

$$\text{for} \quad 0 \leq y \leq x \leq b \quad \text{and} \tag{22}$$

$$G(x,y) = -\frac{\pi J_n(kx)}{2 J_n(kb)} \left[J_n(ky) Y_n(kb) - Y_n(ky) J_n(kb) \right] \tag{23}$$

$$\text{for} \quad 0 \leq x \leq y \leq b$$

provided that $J_n(kb) \neq 0$. That is, provided that k^2 is not an eigenvalue of the homogeneous Bessel equation.

Thus we see that the four properties of G(x,y) discussed above can be used to calculate the Green's function once we know two independent solutions of the homogeneous S-L differential equation. Notice that nothing is said about boundary conditions, until G(x,y) is considered, that is the two independent solutions of the homogeneous equation will not, in general, satisfy all the boundary conditions of the problem.

As a second example of the general method let us consider the following S-L problem[2]

$$\frac{d^2\psi}{dx^2} - \beta^2 x^2 \psi + \epsilon \psi = -f(x) \tag{24}$$

with boundary conditions

$$\psi(\pm\infty) = 0.$$

If we define the real parameter n by

$$n = \epsilon/2\beta - \frac{1}{2}$$

then two independent solutions of the homogeneous equation are given by[3] $D_n(x\sqrt{2\beta})$ and $D_{-n-1}(ix\sqrt{2\beta})$. These Weber functions can be expressed in terms of the confluent hypergeometric functions of the third kind (V_1 and V_2) as[3]:

$$D_m(z) = 2^{m/2} e^{-\frac{1}{4}z^2} e^{\frac{i\pi m}{2}} V_2\left(-\frac{m}{2} \middle| \frac{1}{2} \middle| \frac{z^2}{2}\right)$$

and

$$D_{-m-1}(iz) = 2^{-\frac{1}{2}(m+1)} e^{\frac{i\pi(m+1)}{2}} e^{\frac{1}{4}z^2} V_1\left(-\frac{m}{2} \middle| \frac{1}{2} \middle| -\frac{z^2}{2}\right) . \tag{25}$$

Because G(x,y) must also be a solution of the homogeneous equation for $y < x$ and for $x < y$, we can immediately write

$$G(x,y) = A(x)\, D_n(y\sqrt{2\beta}) + B(x)\, D_{-n-1}(iy\sqrt{2\beta}) , \quad y < x \tag{26}$$

and

$$G(x,y) = C(x)\, D_n(y\sqrt{2\beta}) + E(x)\, D_{-n-1}(iy\sqrt{2\beta}) , \quad x < y . \tag{27}$$

The conditions that $G(x, \pm\infty)$ be zero imply that $E(x) = 0$ and $A(x) = 0$ because $D_{-n-1}(i\,\infty\sqrt{2\beta})$ and $D_n(-\infty\sqrt{2\beta})$ are unbounded. Continuity of G(x,y) at the point $y = x$ (for all x in $(-\infty, \infty)$) requires

$$B(x)\, D_{-n-1}(ix\sqrt{2\beta}) = C(x)\, D_n(x\sqrt{2\beta}), \quad -\infty < x < \infty \tag{28}$$

The equation for the jump discontinuity of $\frac{dG}{dy}$ at $y = x$ reads as follows:

$$C(x)\, D_n'(x\sqrt{2\beta}) - B(x)\, D_{-n-1}'(ix\sqrt{2\beta}) = -1 , \quad -\infty < x < \infty. \tag{29}$$

By solving Eqs. (28) and (29) we arrive at the following expression for B(x) and C(x):

$$B(x) = D_n(x\sqrt{2\beta})\left[D_n(x\sqrt{2\beta})\,D'_{-n-1}(i x\sqrt{2\beta}) - D_{-n-1}(i x\sqrt{2\beta})\,D'_n(x\sqrt{2\beta})\right]^{-1} \quad (30)$$

$$C(x) = D_{-n-1}(i x\sqrt{2\beta})\left[D_n(x\sqrt{2\beta})\,D'_{-n-1}(i x\sqrt{2\beta}) - D_{-n-1}(i x\sqrt{2\beta})\,D'_n(x\sqrt{2\beta})\right]^{-1} \quad (31)$$

Defining $\Delta(x)$ as the denominator appearing in Eqs. (30) and (31), the final expression for $G(x,y)$ can be written as

$$G(x,y) = D_n(x\sqrt{2\beta})\,D_{-n-1}(i y\sqrt{2\beta})\,\Delta^{-1}(x)\,, \qquad y < x \quad (32)$$

$$G(x,y) = D_{-n-1}(i x\sqrt{2\beta})\,D_n(y\sqrt{2\beta})\,\Delta^{-1}(x)\,, \qquad x < y\,. \quad (33)$$

III. The Case When λ is an Eigenvalue

The treatment given above is only valid if the parameter ε is not an eigenvalue of the homogeneous differential equation with the same boundary conditions as the original S-L problem. If ε is equal to some eigenvalue, say ε_j, where j is a non-negative integer then a generalized Green's function[4], which can still be used in Eq. (2) to give a solution $\Psi(x)$ of the inhomogeneous equation, can be constructed if the inhomogeneity f(x) is orthogonal to the normalized solution $U_j(x)$ of the homogeneous S-L problem belonging to the eigenvalue ε_j, i.e.

$$\int_a^b U_j(x)\,f(x)\,dx = 0\,. \quad (34)$$

Assuming that Eq. (34) is true for the specific case of interest, we then proceed to construct G(x,y) by using the same four properties discussed above with one modification. The requirement that G(x,y) satisfy the homogeneous S-L differential equation for x < y and for y < x is replaced by the following property:

$$\frac{d^2}{dy^2} G(x,y) - \beta^2 y^2 G + \epsilon_j G = C_j e^{-\frac{\beta x^2}{2} - \frac{\beta y^2}{2}} H_j(x\sqrt{\beta}) C_j H_j(y\sqrt{\beta}) \quad (35)$$

where

$$C_j = \frac{1}{\sqrt{2^j j!}} \left(\frac{\beta}{\pi}\right)^{1/4} \quad (36)$$

is the normalization constant for the harmonic oscillator eigenfunction, and

$$H_j(z) = (-1)^j e^{z^2} \frac{d^j}{dz^j} e^{-z^2} \quad (37)$$

is the j^{th} Hermite polynomial. The function

$$C_j e^{-\frac{\beta x^2}{2}} H_j(x\sqrt{\beta})$$

is just the Weber function $D_j(x\sqrt{2\beta})$ normalized to unity.

To see that the quantity

$$\int_{-\infty}^{\infty} G(x,y) f(y)\, dy$$

is an acceptable solution $\Psi(x)$ of the S-L problem, we use the following identiy for f(y):

$$\frac{d^2\psi}{dy^2} - \beta^2 y^2 \psi + \epsilon_j \psi = - f(y) , \quad (38)$$

to write

$$\int_{-\infty}^{\infty} G(x,y)\, f(y)\, dy = -\int_{-\infty}^{\infty} G(x,y)\left[\frac{d^2\psi}{dy^2} - \beta^2 y^2 \psi + \epsilon_j \psi\right] dy\,. \tag{39}$$

Integrating by parts and using both Eq. (35) and the boundary conditions on G(x,y) and Ψ(y), we obtain

$$\int_{-\infty}^{\infty} G(x,y)\, f(y)\, dy = C_j\, e^{-\frac{\beta x^2}{2}} H_j(x\sqrt{\beta}) \int_{-\infty}^{\infty} C_j\, e^{-\frac{\beta y^2}{2}} H_j(y\sqrt{\beta})\, \psi(y)\, dy + \psi(x). \tag{40}$$

That Ψ(x) can, without loss of generality, be chosen to be orthogonal to $U_j(x)$ is easily shown:

Replace Ψ(x) by Ψ(x) − $cU_j(x)$; then

$$\frac{d^2}{dy^2}(\psi - c\,U_j) - \beta^2 y^2 (\psi - c\,U_j) + \epsilon_j (\psi - c\,U_j)$$

$$= -f(y) - c\left[\frac{d^2 U_j}{dy^2} - \beta^2 y^2 U_j + \epsilon_j U_j\right]$$

$$= -f(y) \tag{41}$$

Thus $\Psi - cU_j$ is an equally valid solution of the inhomogeneous equation. Hence we can choose the constant c such that

$$\int_{-\infty}^{\infty} (\psi - c\,U_j)(x)\, U_j(x)\, dx = 0, \tag{42}$$

i.e. we can choose to find a solution of the inhomogeneous S-L equation which is orthogonal to U_j.

With $\Psi(x)$ orthogonal to $U_j(x)$, Eq. (40) reads

$$\int_{-\infty}^{\infty} G(x,y)\, f(y)\, dy = \Psi(x) , \tag{43}$$

which is the desired identity. To assure that the function $\Psi(x)$ is orthogonal to $U_j(x)$ it is sufficient to constrain the Green's function $G(x,y)$ to be orthogonal to $U_j(x)$:

$$\int_{-\infty}^{\infty} G(x,y)\, U_j(y)\, dy = 0 . \tag{44}$$

This constraint will shortly be used along with the other four properties of $G(x,y)$ to construct the generalized Green's function $G(x,y)$.

Before we can write the form of $G(x,y)$ we must know the general solution of the differential equation given below:

$$\frac{d^2}{dy^2} F - \beta^2 y^2 F + \epsilon_j F = C_j\, e^{-\frac{\beta y^2}{2}} H_j(y\sqrt{\beta}) = U_j(y) . \tag{45}$$

The solution of Eq. (45) requires the knowledge of two independent solutions of the corresponding homogeneous equation

$$\frac{d^2}{dy^2} \phi - \beta^2 y^2 \phi + \epsilon_j \phi = 0. \tag{46}$$

Clearly one solution is given by

$$U_j(y) = C_j\, e^{-\frac{\beta y^2}{2}} H_j(y\sqrt{\beta}) \tag{47}$$

which is proportional to the Weber function $D_j(y\sqrt{2\beta})$.

A second solution $T_j(y\sqrt{2\beta})$ of Eq. (46) can be constructed as follows:

$$T_j(y\sqrt{2\beta}) = \lim_{n\to j}\left(\frac{D_n(y\sqrt{2\beta}) - (-1)^j D_n(-y\sqrt{2\beta})}{n-j}\right) . \tag{48}$$

Because both $D_n(y\sqrt{2\beta})$ and $D_n(-y\sqrt{2\beta})$ are solutions of the equation

$$\frac{d^2\phi}{dy^2} - \beta^2 y^2\phi + \epsilon\ \phi = 0,\ n = \frac{\epsilon}{2\beta} - \frac{1}{2} \tag{49}$$

the particular combination given by

$$\frac{D_n(y\sqrt{2\beta}) - (-1)^j D_n(-y\sqrt{2\beta})}{n-j}$$

must also be a solution of Eq. (49). Therefore as we allow n to approach the integer j, the solution $T_j(y\sqrt{2\beta})$ obeys the limiting value of Eq. (49), i.e. Eq. (46).

To simplify the expression for $T_j(y\sqrt{2\beta})$ given in Eq. (48) let us expand the "n-dependence" of the two Weber functions $D_n(y\sqrt{2\beta})$ and $D_n(-y\sqrt{2\beta})$ in a Taylor series about the point $n = j$:

$$D_n(z) = D_j(z) + (n-j)\left.\frac{dD_n(z)}{dn}\right|_{n=j} + \cdots \tag{50}$$

$$D_n(-z) = D_j(-z) + (n-j)\left.\frac{dD_n(-z)}{dn}\right|_{n=j} + \cdots \tag{51}$$

Using the fact that for n equal to a non-negative integer $D_n(-z) = (-1)^n D_n(z)$, we easily obtain from Eqs. (50) and (51) that

$$T_j(y\sqrt{2\beta}) = \left.\frac{dD_n(y\sqrt{2\beta})}{dn}\right|_{n=j} - (-1)^j\left.\frac{dD_n(-y\sqrt{2\beta})}{dn}\right|_{n=j} . \tag{52}$$

These derrivatives are assumed to be known functions.

Given the two solutions $D_j(y\sqrt{2\beta})$ and $T_j(y\sqrt{2\beta})$ of Eq. (46) we can immediately[5] write down the general solution of Eq. (45)

$$F(y) = \alpha_1 D_j(y\sqrt{2\beta}) + \alpha_2 T_j(y\sqrt{2\beta}) + \int dW\, \Delta^{-1}(w)$$

$$\left[D_j(w\sqrt{2\beta})\,T_j(y\sqrt{2\beta}) - T_j(w\sqrt{2\beta})\,D_j(y\sqrt{2\beta})\right] C_j\, e^{-\frac{\beta w^2}{2}} H_j(w\sqrt{\beta}) \tag{53}$$

where α_1 and α_2 are arbitrary constants and

$$\Delta(w) = D_j(w\sqrt{2\beta})\frac{d}{dW} T_j(w\sqrt{2\beta}) - T_j(w\sqrt{2\beta})\frac{d}{dW} D_j(w\sqrt{2\beta}) \tag{54}$$

The integral appearing in Eq. (53) is an <u>indefinite integral</u>. For brevity we will refer to this integral as $\bar{F}_j(y)$.

Knowing the general solution of Eq. (45), we can now write the form which G(x,y) must have if it is to be a solution of Eq. (35):

$$G(x,y) = A(x)\, D_j(y\sqrt{2\beta}) + B(x)\, T_j(y\sqrt{2\beta})$$

$$+ \mathcal{V}_j(x)\, F_j(y)\,, \quad \text{for} \quad y < x \tag{55}$$

and

$$G(x,y) = C(x)\, D_j(y\sqrt{2\beta}) + E(x)\, T_j(y\sqrt{2\beta})$$

$$+ \mathcal{V}_j(x)\, F_j(y)\,, \quad \text{for} \quad x < y\,. \tag{56}$$

The conditions that $G(x, \pm\infty)$ be zero imply that

$$B(x) = -\mathcal{V}_j(x) \lim_{y\to-\infty} \left(F_j(y) / T_j(y\sqrt{2\beta}) \right)$$

$$\equiv -b\, \mathcal{V}_j(x) \tag{57}$$

and

$$E(x) = -\mathcal{V}_j(x) \lim_{y\to\infty} \left(F_j(y) / T_j(y\sqrt{2\beta}) \right)$$

$$\equiv -e\,\mathcal{V}_j(x) \, . \tag{58}$$

Continuity of G(x,y) at the point y = x for all x in $(-\infty, \infty)$ requires

$$A(x)\, D_j(x\sqrt{2\beta}) - b\,\mathcal{V}_j(x)\, T_j(x\sqrt{2\beta}) = C(x)\, D_j(x\sqrt{2\beta}) - e\,\mathcal{V}_j(x)\, T_j(x\sqrt{2\beta}),$$

$$\text{for} \quad -\infty < x < \infty \tag{59}$$

The equation for the jump discontinuity of $\frac{dG}{dy}$ at y = x reads as follows:

$$C(x)\, D_j'(x\sqrt{2\beta}) - e\,\mathcal{V}_j(x)\, T_j'(x\sqrt{2\beta}) - A(x)\, D_j'(x\sqrt{2\beta})$$

$$+ b\,\mathcal{V}_j(x)\, T_j'(x\sqrt{2\beta}) = -1 \, , \quad -\infty < x < \infty \, . \tag{60}$$

Notice that Eqs. (59) and (60) can <u>not</u> be solved for both unknown functions A(x) and C(x); the difference A(x) - C(x) arises in both equations.

By using Eqs. (57) - (59) we can write the previous expression for G(x,y) as:

$$G(x,y) = C(x)\, D_j(y\sqrt{2\beta}) + (b-e)\left(\mathcal{V}_j(x)\, T_j(x\sqrt{2\beta}) / D_j(x\sqrt{2\beta})\right) D_j(y\sqrt{2\beta})$$

$$- b\,\mathcal{V}_j(x)\, T_j(y\sqrt{2\beta}) + \mathcal{V}_j(x)\, F_j(y) \, , \quad \text{for } y < x \tag{61}$$

$$G(x,y) = C(x)\, D_j(y\sqrt{2\beta}) - e\, U_j(x)\, T_j(y\sqrt{2\beta})$$

$$+ U_j(x)\, F_j(y), \quad \text{for} \quad x < y. \tag{62}$$

The remaining unknown C(x) is to be determined from Eq. (44):

$$\int_{-\infty}^{x} G(x,y)\, U_j(y)\, dy + \int_{x}^{\infty} G(x,y)\, U_j(y)\, dy = 0, \tag{44}$$

and the requirement[4] that G(x,y) be symmetric

$$G(x,y) = G(y,x). \tag{63}$$

From Eqs. (61) and (62) it is clear that the proper choice of C(x) is

$$C(x) = \alpha_j \left(F_j(x) - b\, T_j(x\sqrt{2\beta}) + \delta_j\, D_j(x\sqrt{2\beta}\right),$$

where the constant α_j arises from the fact that $U_j(x)$ is proportional to $D_j(x\sqrt{2\beta})$

$$U_j(x) = \left(\frac{\beta}{\pi}\right)^{1/4} \frac{1}{\sqrt{j!}}\, D_j(x\sqrt{2\beta}) \equiv \alpha_j\, D_j(x\sqrt{2\beta}), \tag{64}$$

and δ_j can be determined from Eq. (44).

The final expression for G(x,y) is then given by

$$G(x,y) = \alpha_j \Big[F_j(x)\, D_j(y\sqrt{2\beta}) + D_j(x\sqrt{2\beta})\, F_j(y) + \delta_j\, D_j(x\sqrt{2\beta})\, D_j(y\sqrt{2\beta})$$

$$- e\, T_j(x\sqrt{2\beta})\, D_j(y\sqrt{2\beta}) - b\, D_j(x\sqrt{2\beta})\, T_j(y\sqrt{2\beta}) \Big], \quad \text{for } y < x \tag{65}$$

$$G(x,y) = \alpha_j \left[F_j(x)\, D_j(y\sqrt{2\beta}) + D_j(x\sqrt{2\beta})\, F_j(y\sqrt{2\beta}) + \delta_j\, D_j(x\sqrt{2\beta})\, D_j(y\sqrt{2\beta}) \right.$$

$$\left. - b\, T_j(x\sqrt{2\beta})\, D_j(y\sqrt{2\beta}) - e\, D_j(x\sqrt{2\beta})\, T_j(y\sqrt{2\beta}) \right], \text{ for } x<y. \tag{66}$$

Hence the most general solution of Eq. (24) which vanishes at $x = \pm\infty$ $(\varepsilon = \varepsilon_j)$ can be written as follows:

$$\psi(x) = a_1\, D_j(x\sqrt{2\beta}) + \int_{-\infty}^{\infty} G(x,y)\, f(y)\, dy\,. \tag{67}$$

The constant a_1 is arbitrary.

ACKNOWLEDGMENTS

The author thanks Professors Saul T. Epstein and Joseph O. Hirschfelder for helpful discussions.

REFERENCES AND NOTES

1. The treatment presented here is taken largely from Dettman's book Mathematical Methods in Physics and Engineering, McGraw-Hill, 1962. This is not original research.
2. This equation arises in a quantum-mechanical treatment of the one-dimensional harmonic oscillator.
3. For a discussion of Weber functions and confluent hypergeometric functions of the third kind see Morse and Feshbach's book Methods of Theoretical Physics, McGraw-Hill, 1953.
4. See page 198 of Reference 1 for the logical development of the theory of generalized Green's functions.
5. See page 529 of Vol. I of Reference 3.