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THE UNIVERSITY OF TENNESSEE

DEPARTMENT OF MECHANICAL
AND AEROSPACE ENGINEERING

AN ANALYTICAL MODEL FOR THE PREDICTION OF VISCOSEAL PERFORMANCE IN THE REGIME FROM CONTINUUM TO FREE-MOLECULE FLOW

by

KERRY E. PATTERSON

Prepared Under
National Aeronautics and Space Administration
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FOREWORD

This document is submitted as an interim report on the rarefied-gas viscoseal investigation, which is part of the Fundamental Study in Low-Density Gas Dynamics currently being conducted in the Department of Mechanical and Aerospace Engineering at the University of Tennessee. Support for this work was provided by grant NGR-43-001-023 from the National Aeronautics and Space Administration Lewis Research Center.

This report was submitted to the University of Tennessee in partial fulfillment of the requirements for the degree of Master of Science with a major in Mechanical Engineering, and is presented here with minor changes in format.

Approved: M. H. Milligan Kerry E. Patterson

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Finally, the author expresses his appreciation to his wife for her patient typing of the preliminary manuscript.

ABSTRACT

This thesis contains a detailed investigation of the rarefied-gas viscoseal model proposed by Hodgson. Modifications to his basic model are made to make its application more general. The modified-Hodgson model is used to predict seal leakage rates. Since the model does not lead to an explicit determination of the pressure difference at zero flow, Newton's method is applied to solve for the pressure difference. An improved analytical model is developed to predict continuum and slip regime performance. This model leads to an explicit determination of the zero flow pressure difference and also lends itself to predicting optimum seal geometries. An optimization procedure is developed and applied to a specific application. Comparison of the two models with available continuum data shows that the agreement with the improved model is good, while the modified-Hodgson model shows only fair agreement. The improved model also shows good agreement with rarefied sealing coefficient and leakage data. The modified-Hodgson model agrees well with the more rarefied sealing coefficient data available, but its agreement with slip regime data is poor. The leakage prediction of the modified-Hodgson model is in poor agreement with all available leakage data. Based on the comparison with experimental data, the improved model appears to be the more promising of the two models.

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LIST OF SYMBOLS

SYMBOL	DESCRIPTION
English Alphabet	
a	Groove aspect ratio
a'	Hypothetical groove aspect ratio
A	Rotor induced flow coefficient
A_1	Slip constant
A_c	Annulus flow coefficient
A_L	Dimensionless annulus flow coefficient
A_{LC}	Continuum limit of A_L
A_p	Groove pressure flow coefficient
A_u	Rotor induced flow coefficient
b	Groove width
b'	Axial groove width, $b/\cos\alpha$
B	Groove flow coefficient
c	Radial seal clearance
C	Groove flow coefficient
C_1	Groove flow coefficient
C_2	Groove flow coefficient
C_3	Land flow coefficient
C_4	Land flow coefficient
C_5	Fourier expansion coefficient
C_{1np}, C_{2np}	Groove pressure flow integration constants

C_{1nu}, C_{2nu}	Rotor induced flow integration constants
d	Seal diameter
D	Land flow coefficient
E	Land flow coefficient
h	Groove depth
k	Boltzmann constant
K	Seal radius ratio, 1 - 2c/d
K_1	Groove flow coefficient
K_2	Land flow coefficient
K_3	Groove flow coefficient
K_c	Knudsen number based on clearance, λ/c
K_h	Knudsen number based on groove depth, λ/h
K'_h	Knudsen number based on groove depth, $\lambda/(h + c)$
K_α	Optimization parameter
ℓ	Flow passage length coordinate
ℓ_g	Groove length coordinate
ℓ_L	Land length coordinate
ℓ_t	Total groove length on seal
L	Axial seal length
m	Mass per molecule
n_s	Number of thread starts
n_t	Total turns of spiral on seal
n	Coordinate normal to flow boundary
$\dot{n}$	Molecular flow rate
$\dot{n}/\Delta P$	Specific molecular flow rate
N	Seal speed, rpm

P	Absolute pressure
P_1	Seal exit pressure
P_2	Seal inlet pressure
$\overline{P}$	Average seal pressure, $(P_1 + P_2)/2$
$\overline{P}_1$	Continuum regime value of $\overline{P}$
$\overline{P}_n$	Transition regime value of $\overline{P}$
ΔP	Pressure difference across seal, $P_2 - P_1$
$\Delta\overline{P}$	Incremental change in $\overline{P}$
Q	Volume flow rate at unit pressure, VP
Q_g	Volume flow rate at unit pressure in groove
Q_L	Volume flow rate at unit pressure over land
Q_N	Net volume flow rate at unit pressure
Q_R	Rotor induced volume flow rate at unit pressure
R	Modified groove pressure gradient
r_p	Seal pressure ratio, P_2/P_1
T	Absolute temperature
u	Dimensionless groove flow velocity
$\overline{u}$	Area-weighted average rotor induced velocity
U	Seal peripheral surface velocity
U_e	Effective rotor velocity
v	Groove flow velocity
v_R	Rotor induced flow velocity
v'_R	Rotor induced flow velocity in hypothetical groove
V	Volume flow rate
V_g	Total groove volume flow rate
V'_g	Volume flow rate in a single groove

V'_{gc}	Continuum limit of V'_g
V_L	Land leakage volume flow rate
V_T	Total volume flow rate in seal
w	Land width
w'	Axial land width, w/cos α
x	Dimensionless groove width coordinate
$\bar{x}$	Groove width coordinate
y	Dimensionless groove depth coordinate
y'	Dimensionless groove depth coordinate
$\bar{y}$	Groove depth coordinate
z	Groove length coordinate

Greek Alphabet

α	Seal helix angle, $\sin^{-1} [\frac{n_s(b + w)}{\pi d}]$
α_n	Eigenvalue
β	Tube flow coefficient
γ	Tube flow coefficient
θ_n	Modified eigenvalue
λ	Mean free path
Λ	Sealing coefficient
μ	Absolute viscosity
μHg	Pressure unit, micron of mercury
ν	Tube flow coefficient
ξ	Tube flow coefficient
Σ_c	Rotor induced flow correction
Σ_{cc}	Continuum limit of Σ_c

Σ_p	Groove pressure flow summation
Σ_{pc}	Continuum limit of Σp
Σ_u	Rotor induced flow summation
Σ_{uc}	Continuum limit of Σ_u
ϕ_n	Groove flow velocity function
Ω_n	Fourier coefficient

CHAPTER I

INTRODUCTION

In recent years much attention has been focused on the viscoseal as a highly efficient sealing mechanism. The viscoseal, a rotary shaft seal, consists either of a threaded shaft rotating in a smooth close fitting housing with a small, but finite clearance or a smooth shaft rotating in a threaded housing. In either case the sealing effect is produced by the balance between the flow induced by a pressure gradient and the flow induced by rotation.

A very important application of the viscoseal is found in shaft sealing devices operating in a space environment. The purpose of the sealing system would be to seal a shaft one end of which is exposed to a relatively high density working fluid while the other end is exposed to the vacuum of space. Since the working fluid undergoes a change from a continuum state at one end to a highly rarefied state before exiting to space, rarefied gas dynamics must be applied to the analysis of at least a portion of the seal.

To date much research, both analytical and experimental, has been done on viscoseals using liquids as the sealant, while relatively little research has dealt with gas-type viscoseals. Hodgson and Milligan (1)[1] conducted an analytical and experimental investigation of the performance

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[1] Numbers enclosed in parentheses refer to similarly numbered entries in the List of References.

of visco-type gas seals operating in the continuum flow regime. Baron (2) conducted an experimental investigation of viscoseal performance in the continuum regime with air and hydrogen as the sealants.

In the flow regime between continuum and the highly rarefied free-molecule flow, few theoretical and/or experimental investigations have been conducted. King (3) conducted a theoretical analysis in which he treats the flow in the viscoseal as continuum flow with slip boundary conditions. He presents experimental data in support of his theoretical analysis. Milligan and Wilkerson (4) have presented a viscoseal model which is a modification of the laminar continuum model of Boon and Tal (5) incorporating slip boundary conditions. Since the Boon and Tal model is based on the Reynolds lubrication equation, the model of Milligan and Wilkerson is known as the slip-modified Reynolds model. Wilkerson (6) has conducted an analytical and experimental investigation of rarefied viscoseal performance.

All of the analytical efforts mentioned above have dealt only with the slip flow regime. The only investigator to date who has attempted to analyze the entire flow spectrum including the free-molecule regime is Hodgson (7). He developed a model similar to the one proposed by King (3), but which is extended to the free-molecule regime. Hodgson formulates his flow models along the lines of the semi-empirical approach first taken by Knudsen (8) in his long tube work.

I. Statement of Problem

The investigation reported here concerns a complete examination of the rarefied viscoseal model proposed by Hodgson; modification of this model to make its application more general; and the development of a simpler improved model, the advantages of which will be pointed out. Previous work in the field of rarefied gas dynamics is adapted to the development of this improved model.

CHAPTER II

REVIEW AND MODIFICATION OF THE HODGSON MODEL

As was stated previously, the model developed by Hodgson (7) is by far the most ambitious attempt at solving the rarefied viscoseal problem. At this point it is desirable to review this particular model in detail and indicate where improvements can be made.

Hodgson chose to analyze the particular configuration where a smooth shaft rotates within a grooved housing. It will be shown later that the analysis for the grooved shaft and smooth housing is identical to this configuration. This being the case, Hodgson's model is not as restricted in this respect as it might appear.

Hodgson considers the flow in the viscoseal to be composed of three basic components: (1) the pressure induced flow along the groove, (2) the pressure induced flow over the lands, and (3) the rotor induced flow in the groove. This treatment of the flow is quite common and is exactly the way King (3) chose to break up the flow. Figure 1 shows a representation of the flow components. Q_L, Q_R, and Q_g represent the pressure induced land flow, the rotor induced groove flow and the pressure induced groove flow, respectively. Hodgson also restricts his considerations to a seal with a single thread start.

In his development of the two pressure induced flow components, he takes the semi-empirical approach of Knudsen (8) in describing the flow throughout the entire regime from continuum to free-molecule flow.

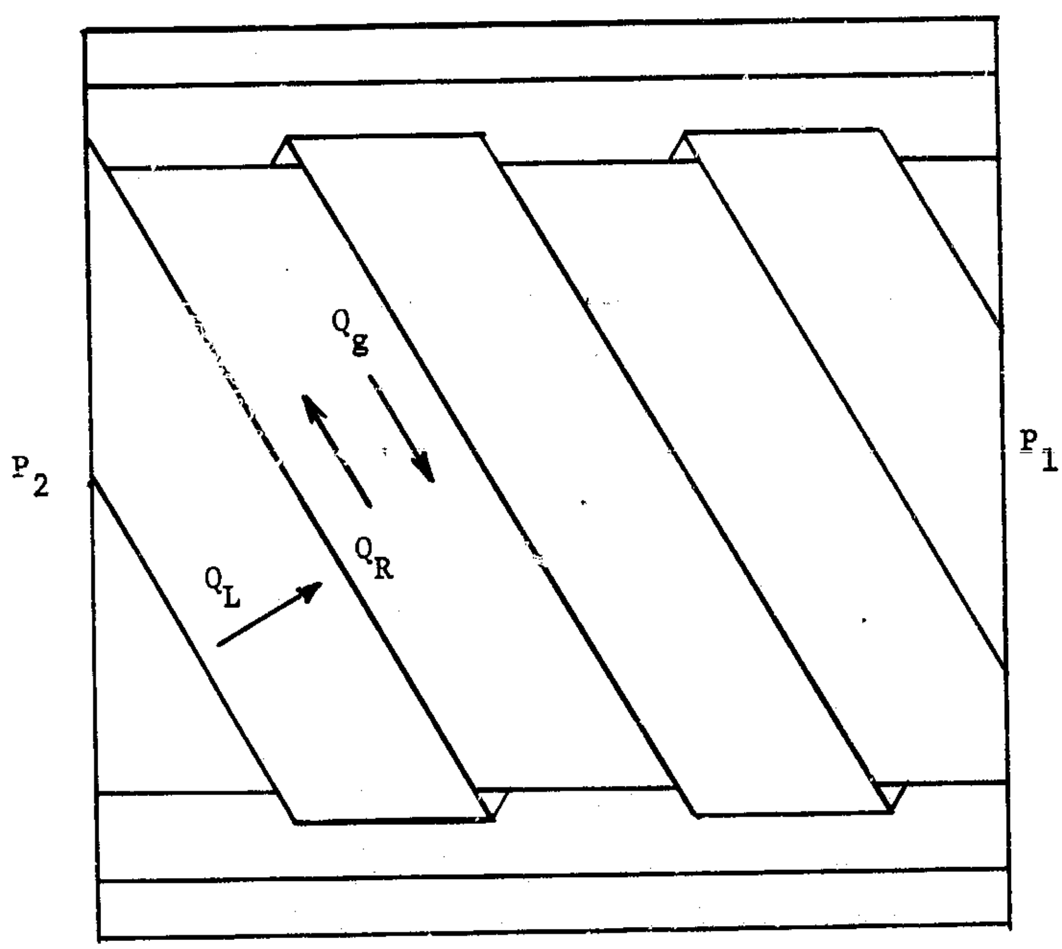

Figure 1. Viscoseal Flow Components.

In his long tube work Knudsen found that his experimental data could be described by an equation of the form

$$Q = [\gamma P + \beta\frac{1 + \nu P}{1 + \xi P}]\ \frac{dP}{d\ell} \qquad (1)$$

In the continuum limit Knudsen reduced Equation (1) to $Q = \gamma P\,\frac{dP}{d\ell}$ and by equating this to the known continuum solution, he was able to determine γ. In a similar manner as the free-molecule limit was approached he reduced Equation (1) to $Q = \beta\frac{dP}{d\ell}$ and by equating this to the known free-molecule solution he determined β. By considering the slip flow regime, Knudsen was able to determine the ratio ν/ξ by noting that Equation (1) becomes $Q = (\gamma P + \beta\frac{\nu}{\xi})\frac{dP}{d\ell}$ and equating this to the known continuum with slip solution. Next Knudsen determined the difference ξ-ν from a consideration of nearly free molecule flow and thus was able to determine both ξ and ν. He then applied experimentally determined corrections to ξ and ν so that the experimentally observed minimum in the $Q/\Delta P$ versus P curve would be correctly predicted. Knudsen applied the analysis above to a long circular tube. In his analysis Hodgson applies the identical procedure to his treatment of the pressure induced flow in the groove (a long rectangular duct) and the pressure induced flow over the lands which he takes to be a narrow slit.

I. Pressure Induced Flow in the Seal Groove

Hodgson assumes that the pressure varies continuously along the axis of the seal groove from a value of P_2 at the high pressure end to P_1 at the other end. In order to determine the pressure gradient along the groove axis, it is necessary to relate the length of the groove, ℓ_g, to the seal length, L. A development of a viscoseal is shown in Figure 2. Hodgson only considered seals with a single thread. The development that follows is generalized to any number of thread starts, n_s.

In Figure 2 line $\overline{AB}$ is drawn perpendicular to the grooves. The number of turns of spiral that $\overline{AB}$ crosses is equal to the total number, n_t, of complete turns on the seal which can be expressed as

$$n_t = \frac{L \cos \alpha}{w + b} .$$

The length of groove per turn of spiral is $\pi d/\cos \alpha$. The total groove length on the seal is

$$\ell_T = \frac{\pi d}{\cos \alpha} \cdot \frac{L \cos \alpha}{w + b} = \frac{L\pi d}{w + b} .$$

The length of each groove is then

$$\ell_g = \frac{\ell T}{n_s} = \frac{L\pi d}{n_s(w + b)} .$$

It follows that

$$\frac{dP}{d\ell_g} = \frac{dP}{d[\frac{L\pi d}{n_s(w + b)}]} = \frac{n_s(w + b)}{\pi d} \frac{dP}{dL} .$$

Hodgson indicates that the groove pressure gradient in a single threaded seal is

$$\frac{dP}{d\ell_g} = \frac{w + b}{\pi d} \frac{dP}{dL}$$

from which it follows that the flow in a groove of a multi-threaded seal is n_s times the flow in the groove of a single threaded seal of

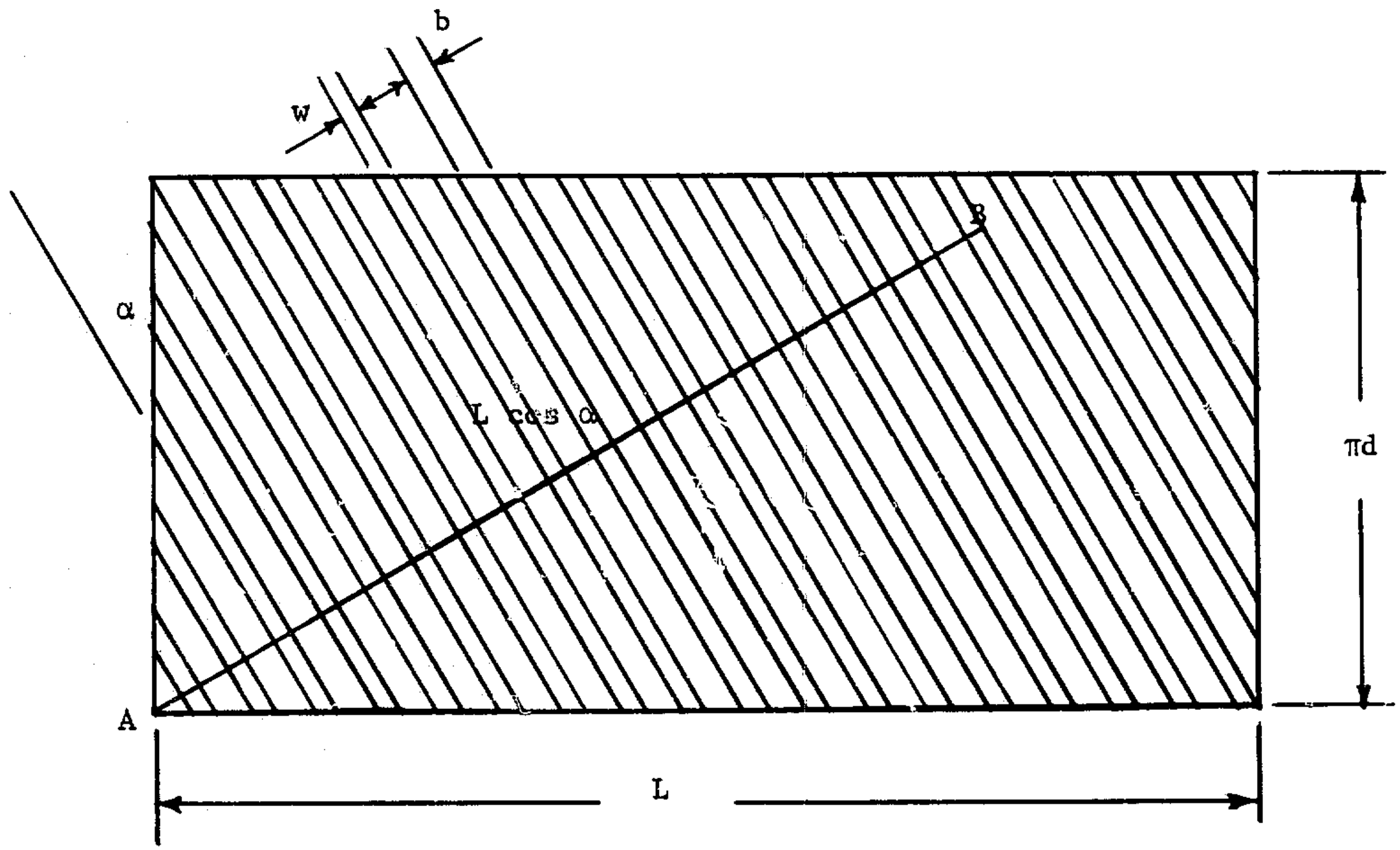

Figure 2. Viscoseal Development.

the same groove width, land width and diameter. Since there are n_s of these grooves, the total groove flow in the multi-threaded seal is n_s^2 time the flow in a comparable single threaded seal. The total groove flow is then

$$Q_g = n_s^2 \left[\left(BP + C \frac{1 + C_1 P}{1 + C_2 P}\right) \frac{dP}{dL} \right], \qquad (2)$$

where the bracketed term is the single threaded groove flow developed by Hodgson. The constants B, C, C_1, and C_2 depend on the geometry of the seal and the properties of the sealant and are given in Appendix A. Appendix A also contains a discussion of the flow models used by Hodgson to obtain these constants.

II. Pressure Induced Flow Over the Lands

Based on the assumption that the pressure varies continuously along the helical groove, Hodgson shows that the effective pressure gradient for the land flow is

$$\frac{dP}{d\ell_L} = \frac{w + b}{w} \cos \alpha \frac{dP}{dL} .$$

This pressure gradient applies equally well to both the single and multi-threaded seal. The land leakage flow for a seal of any number of threads is then

$$Q_L = \left[DP + E \frac{1 + C_3 P}{1 + C_4 P}\right] \frac{dP}{dL} \qquad (3)$$

where the constants D, E, C_3, and C_4 also depend on the seal geometry and the properties of the sealant and are given in Appendix A along with a discussion of their origin.

III. Rotor Induced Flow

Hodgson takes a simplified approach to the prediction of the rotor induced flow. In Figure 3 a groove cross section is shown with the rotor moving over the top of the groove. Hodgson develops the rotor induced flow on a molecular basis, but as he points out a continuum approach yields the same result. It is assumed that the rotor induced flow is the same as the flow obtained in a long rectangular duct of width b and height h in which the upper wall moves with velocity U cos α (the component of the circumferential velocity of the rotor along the axis of the groove). Rather than solve the describing differential equation for parallel flow, Hodgson chooses to compute the volume flow based on an area-weighted average velocity. This average velocity is

$$\bar{u} = \frac{(U \cos \alpha)b + (2h + b)(0)}{2(b + h)} = \frac{Ub}{2(b + h)} \cos \alpha.$$

The total rotor induced flow is thus

$$Q_R = n_s \left[\frac{Ub^2h}{2(b + h)} \cos \alpha\right]P = n_s AP \qquad (4)$$

where Q_R is generalized to a multi-threaded seal.

IV. Total Seal Flow

The flow rates given by Equations (2, 3, and 4) are superimposed to give the total flow in the seal which can be expressed as

$$Q_N = - n_s^2 \left[BP + C \frac{1 + C_1P}{1 + C_2P}\right] \frac{dP}{dL} - \left[DP + E \frac{1 + C_3P}{1 + C_4P}\right] \frac{dP}{dL} - n_s AP \qquad (5)$$

where flow in the direction of decreasing pressure is considered positive.

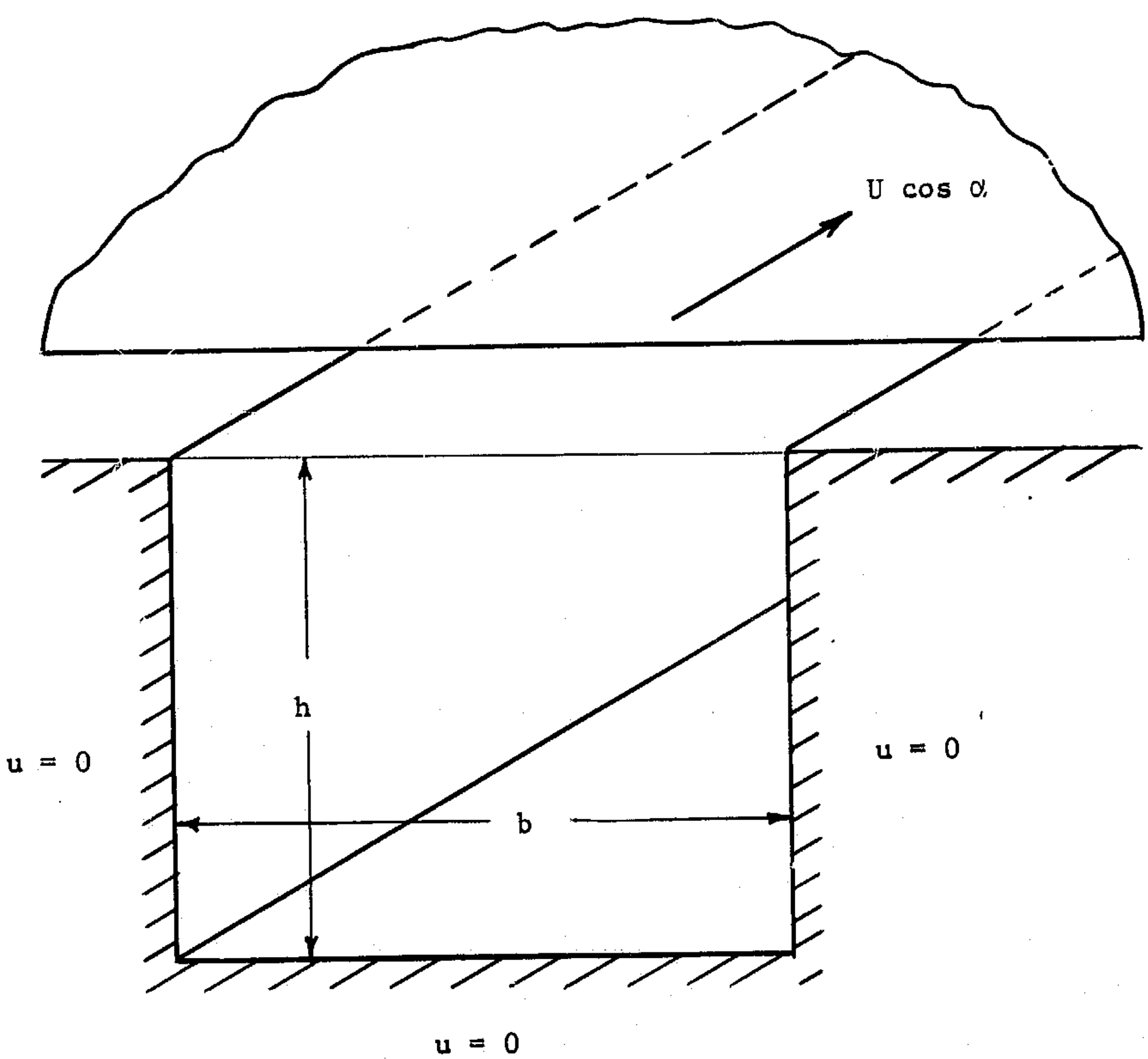

Figure 3. Groove Cross Section.

V. Solution of the Modified-Hodgson Equation

Unlike Hodgson's basic equation, Equation (5) is applicable to a seal with any number of threads. There are three solutions to Equation (5) which are of particular interest. These three cases are: (1) the flow rate through a non-rotating seal, (2) the flow rate through a rotating seal, and (3) the pressure difference across a rotating seal when the net flow is zero.

Flow Rate in a Static Seal

The volume flow rate at unit pressure, $Q = VP$, is related to the molecular flow rate, $\dot{n}$, by

$$\dot{n} = \frac{Q}{kT}.$$

For the case of a non-rotating seal ($Q_R = 0$), Equation (5) can be integrated to obtain the specific molecular flow rate

$$\frac{\dot{n}}{\Delta P} = \frac{1}{kTL} \{(n_s^2 B + D)\overline{P} + n_s^2 C \frac{C_1}{C_2} + E \frac{C_3}{C_4}$$

$$+ n_s^2 C \frac{C_2 - C_1}{2C_2^2 \overline{P}} (\frac{r_p + 1}{r_p - 1}) \ln [\frac{r_p(1 + 2C_2\overline{P}) + 1}{r_p + 2C_2\overline{P} + 1}]$$

$$+ E \frac{C_4 - C_3}{2C_4^2 \overline{P}} (\frac{r_p + 1}{r_p - 1}) \ln [\frac{r_p(1 + 2C_4\overline{P}) + 1}{r_p + 2C_4\overline{P} + 1}]\}. \qquad (6)$$

Flow Rate in a Rotating Seal

Assuming a constant pressure gradient along the groove, the total rotor induced flow on a molecule basis is

$$\frac{\dot{n}}{\Delta P} = -\frac{n_s A \overline{P}}{kT\Delta P} = -\frac{n_s A}{2kT}\left[\frac{r_p + 1}{r_p - 1}\right]. \tag{7}$$

The net flow through a rotating seal is found by combining the rotor induced flow of Equation (7) with the flow in the static seal, Equation (6). The net specific molecular leakage is

$$\frac{\dot{n}}{\Delta P} = \frac{1}{kTL}\{(n_s^2 B + D)\overline{P} + n_s^2 C\frac{C_1}{C_2} + E\frac{C_3}{C_4} + \frac{1}{2}\left[\frac{r_p + 1}{r_p - 1}\right][n_s^2 C\frac{C_2 - C_1}{C_2^2\,\overline{P}}\ln\left(\frac{r_p(1+2C_2\overline{P}) + 1}{r_p + 2C_2\overline{P} + 1}\right) + E\frac{C_4 - C_3}{C_4^2\,\overline{P}}\ln\left(\frac{r_p(1 + 2C_4\overline{P}) + 1}{r_p + 2C_4\overline{P} + 1}\right) - n_s AL]\}. \tag{8}$$

<u>Pressure Difference at Zero Net Leakage</u>

Since a rarefied viscoseal, in the ideal case of a true space environment, will normally operate with $P_1 = 0$, it would be impossible to maintain a zero net flow. However, the condition of zero net flow is of interest as far as experimentation and comparison to continuum performance are concerned, and in non-space applications where $P_1 \neq 0$.

The maximum pressure difference under which a seal can maintain a zero net flow is found by solving the modified-Hodgson model equation subject to the condition that $Q_N = 0$. Integrating Equation (5) subject to this condition one obtains

$$\left[\frac{2\overline{P} + \Delta P}{2\overline{P} - \Delta P}\right]^{n_s^2 C+E}\left[\frac{2 + C_2(2\overline{P} + \Delta P)}{2 + C_2(2\overline{P} - \Delta P)}\right]^{n_s^2 C(\frac{C_1}{C_2} - 1)}\left[\frac{2 + C_4(2\overline{P} + \Delta P)}{2 + C_4(2\overline{P} - \Delta P)}\right]^{E(\frac{C_3}{C_4} - 1)} = \exp\,[n_s AL - (n_s^2 B + D)\Delta P]. \tag{9}$$

In general Equation (9) cannot be solved explicitly for ΔP. For the special case of continuum flow $2\overline{P} >> \Delta P$, the continuum ΔP can be obtained from Equation (9) as

$$\Delta P = \frac{n_s AL}{n_s^2 B + D} . \qquad (10)$$

At the other extreme, free-molecule flow, Equation (5) can be reduced to its free-molecule limit,

$$Q_N = - (n_s^2 C + E)\frac{dP}{dL} - n_s AP. \qquad (10a)$$

Since in the free-molecule limit P is very small, the last term in Equation (10a) could presumably be very small; but since efficient seal performance requires that the rotor induced flow be of the same order of magnitude as the pressure induced flow, this term is retained. Solving Equation (10a) subject to the condition that $Q_N = 0$, the free-molecule pressure ratio becomes

$$r_p = \exp\,[n_s AL/(n_s^2 C + E)]. \qquad (11)$$

The ΔP across the seal can be expressed as

$$\Delta P = 2\overline{P}\;\frac{r_p - 1}{r_p + 1} . \qquad (12)$$

Combining Equations (11) and (12), the ΔP in the free-molecule regime is

$$\Delta P = 2\overline{P}\;\frac{\exp\,[n_s AL/(n_s^2 C + E)] - 1}{\exp\,[n_s AL/(n_s^2 C + E)] + 1} . \qquad (13)$$

Since no explicit solution to Equation (9) can in general be obtained, some approximate solution technique must be employed. Since

Newton's method of approximating roots is generally a rapidly converging iterative method, it is employed in solving Equation (9). Newton's method requires that the given relationship be differentiable. Equation (9) can certainly be differentiated. Another very important requirement is the ability to make a close initial approximation to the solution. This requirement is particularly important with a complicated relationship such as Equation (9).

In order to make a close approximation to the roots of Equation (9), one needs to know as much as possible before hand about the character of the solution. At this point two characteristics are known: (1) the continuum limit, Equation (10), and (2) the free-molecule limit, Equation (13). Figure 4 shows the general character of these limits for a given seal operating at a given speed. The solution of Equation (9) will simply define the behavior over the entire range of average pressures and approach the continuum and free-molecule limits at the two extremes. In order to make a close approximation to the value of ΔP which satisfies Equation (9) for a given $\overline{P}$, a high average pressure, $\overline{P}_1$, is chosen initially such that the continuum solution from Equation (10) is a good approximation to the root of Equation (9). Using this initial approximation for ΔP, Newton's method is employed to solve Equation (9) for the ΔP at $\overline{P}_1$. An incremental decrease, $\Delta\overline{P}$, in the average pressure is then taken and the ΔP obtained from the previous iteration at $\overline{P}_1$ is used as the initial estimate of the solution for ΔP at $\overline{P}_2$ and the iteration process is repeated to obtain ΔP. The entire process is repeated to obtain ΔP at P_3, P_4, etc.

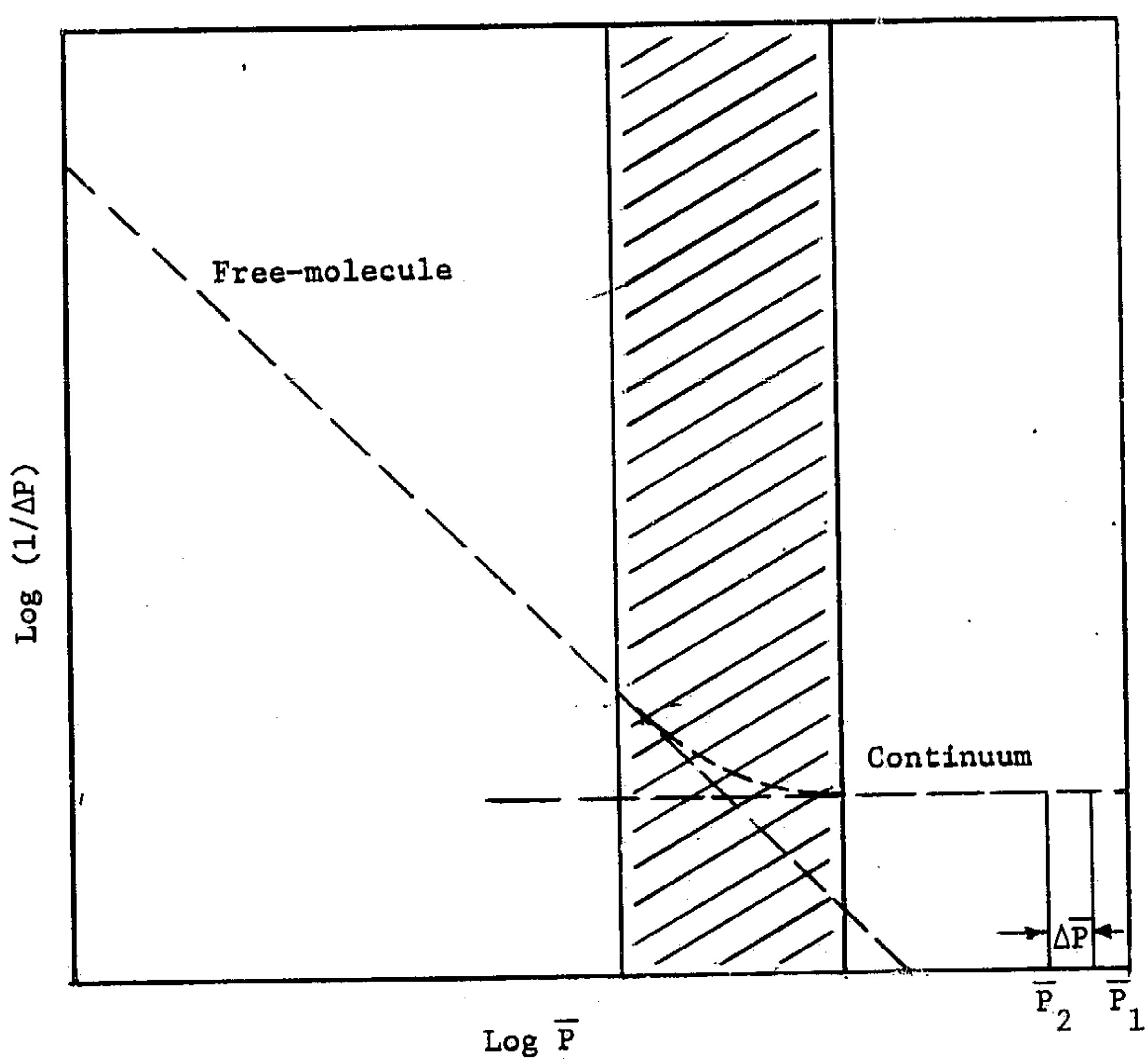

Figure 4. General Character of Modified-Hodgson Solution.

Essentially the above process could be continued until the entire spectrum of pressures had been traversed, but the process encounters difficulties in the shaded region of Figure 4 near the intersection of the continuum and free-molecule asymptotes. In the region to the right of the shaded area, the ΔP versus $\overline{P}$ solution of Equation (9) is fairly flat, thus making the solution at $\overline{P}_{n-1}$ a good approximation to the root at $\overline{P}_n$. In the shaded region, however, the rate of change is so large that the method used above for the initial estimation of the solution is not sufficiently accurate. Two simple modifications to the above method can help to ensure a close approximation of the root at $\overline{P}_n$. An obvious modification would be to reduce the step size, $\Delta\overline{P}$. An enlarged view of the curve in the shaded region is shown in Figure 5. Since all previous points $\overline{P}_1$ through $\overline{P}_{n-1}$ have been determined, these points can be used to extrapolate to an initial approximation at $\overline{P}_n$. Essentially an extrapolation of order n - 2 could be made, but experience has shown that a linear extrapolation combined with successive reductions in $\Delta\overline{P}$ is sufficient to ensure convergence at $\overline{P}_n$.

After passing through the critical region in the vicinity of the intersection of the asymptotes, the curve essentially assumes the straight line predicted by Equation (13). The solution is thus complete.

The Fortran programs for the solution of Equation (5) for the three cases outlined above are presented in Appendix D. The flow rate in a static seal is simply a special case ($Q_R = 0$) of the rotating solution. Consequently only one program is needed for the flow rate solutions.

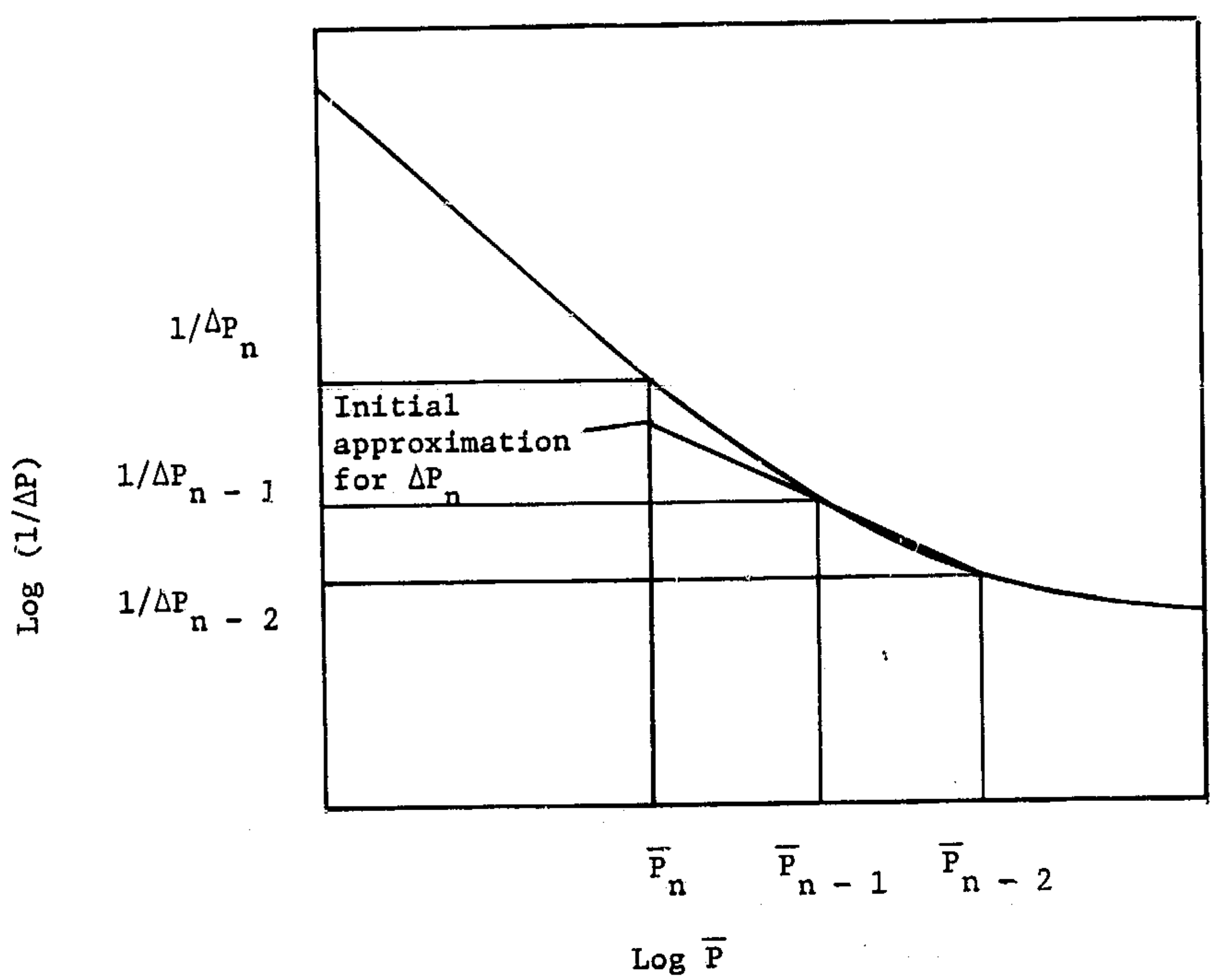

Figure 5. Linear Approximation in Critical Region.

The modified-Hodgson model provides a convenient means of predicting the performance of a given viscoseal. The limiting conditions of Equations (10) and (13) can conveniently be used to predict optimum seal geometries in the purely continuum and purely free-molecule regimes, respectively. In the transition regime, however, Equation (9) would be very difficult to use in an optimization study because of the time consuming solution method. In the next chapter a simpler transition regime model is developed in which the non-leaking pressure difference can be obtained explicitly.

CHAPTER III

DEVELOPMENT OF AN IMPROVED MODEL

In this chapter a simpler model than the modified-Hodgson model will be developed which will lend itself to a less arduous solution in the transition regime and will also correctly predict the continuum performance. The model also predicts the rotor induced flow in a more rigorous manner. Many of the concepts used in the development by Hodgson will be employed in this chapter.

I. Basic Model Development

The basic model will be essentially the same as the one used by Hodgson and which is basically the same as the simplified screw extruder theory presented by Carley, et al. (9). The assumptions inherent in these models are:

(1) the total flow in the seal can be treated as the superposition of the leakage flow in the grooves, the leakage flow over the lands and the rotor induced flow in the grooves,

(2) the pressure varies continuously along the groove and is constant over the cross section of a particular groove,

(3) the groove depth is small compared to the diameter of the seal, thus allowing curvature effects to be neglected in the groove flow development,

and

(4) the flow in a seal with a grooved housing is identical to the flow in a seal with a grooved shaft.

Assumption 1 obviously neglects the convective coupling of the flow components. The solution without this assumption is extremely complicated for even purely continuum flow (10). Its exclusion would certainly lead to an even more complex analysis when non-continuum boundary conditions are applied. Since the objective is the development of a simplified theory, the inclusion of assumption 1 is a necessity. Assumption 4 is discussed in greater detail in Appendix B. One important assumption that is usually made which is not made here is that the groove sidewall effects are negligible. This assumption is one of the prime distinctions between this analysis and the slip-modified Reynolds model in which this assumption is made.

The assumptions governing the development of the component flows are:

(1) the flow is steady, constant viscosity, fully developed, isothermal and Newtonian with negligible body forces,

(2) the Navier-Stokes equations with non-continuum boundary conditions are applicable,

and

(3) the non-continuum boundary conditions can be expressed as (11)

$$v\Big|_{wall} \sim \lambda \frac{\partial v}{\partial n}\Big|_{wall}.$$

In the development of the flow components, flow models which have been experimentally verified will be used and reference made to their verification.

II. Flow in the Groove

Although most previous investigators have initially treated the groove flow in two parts, this analysis will initially treat the groove flow as a single flow from which the two previously mentioned groove flows are eventually obtained.

Based on the assumptions stated above, the Navier-Stokes equations reduce to the single z-momentum equation

$$\frac{\partial^2 v}{\partial \bar{x}^2} + \frac{\partial^2 v}{\partial \bar{y}^2} = \frac{1}{\mu} \frac{dP}{dz} . \tag{14}$$

The applicable slip boundary conditions shown in Figure 6 are:

$$\frac{\partial v}{\partial \bar{x}} (0,\bar{y}) = 0 \tag{15a}$$

$$v(b/2,\bar{y}) = - A_1\lambda \frac{\partial v}{\partial \bar{x}} (b/2, \bar{y}) \tag{15b}$$

$$v(\bar{x}, 0) = A_1 \lambda \frac{\partial v}{\partial \bar{y}} (\bar{x}, 0) \tag{15c}$$

and

$$v(\bar{x}, h) = - U \cos \alpha - A_1\lambda \frac{\partial v}{\partial \bar{y}}(\bar{x}, h) \tag{15d}$$

where advantage has been taken of the symmetry about the $\bar{y}$-axis. The slip coefficient, A_1, is usually taken as unity as will be done here.

It is convenient at this point to non-dimensionalize the velocity and the coordinates and introduce an index of rarefication. The non-dimensionalized variables are taken as

$$x = \bar{x}/(b/2) \qquad y = \bar{y}/h \quad \text{and} \quad u = \frac{v}{U \cos \alpha} .$$

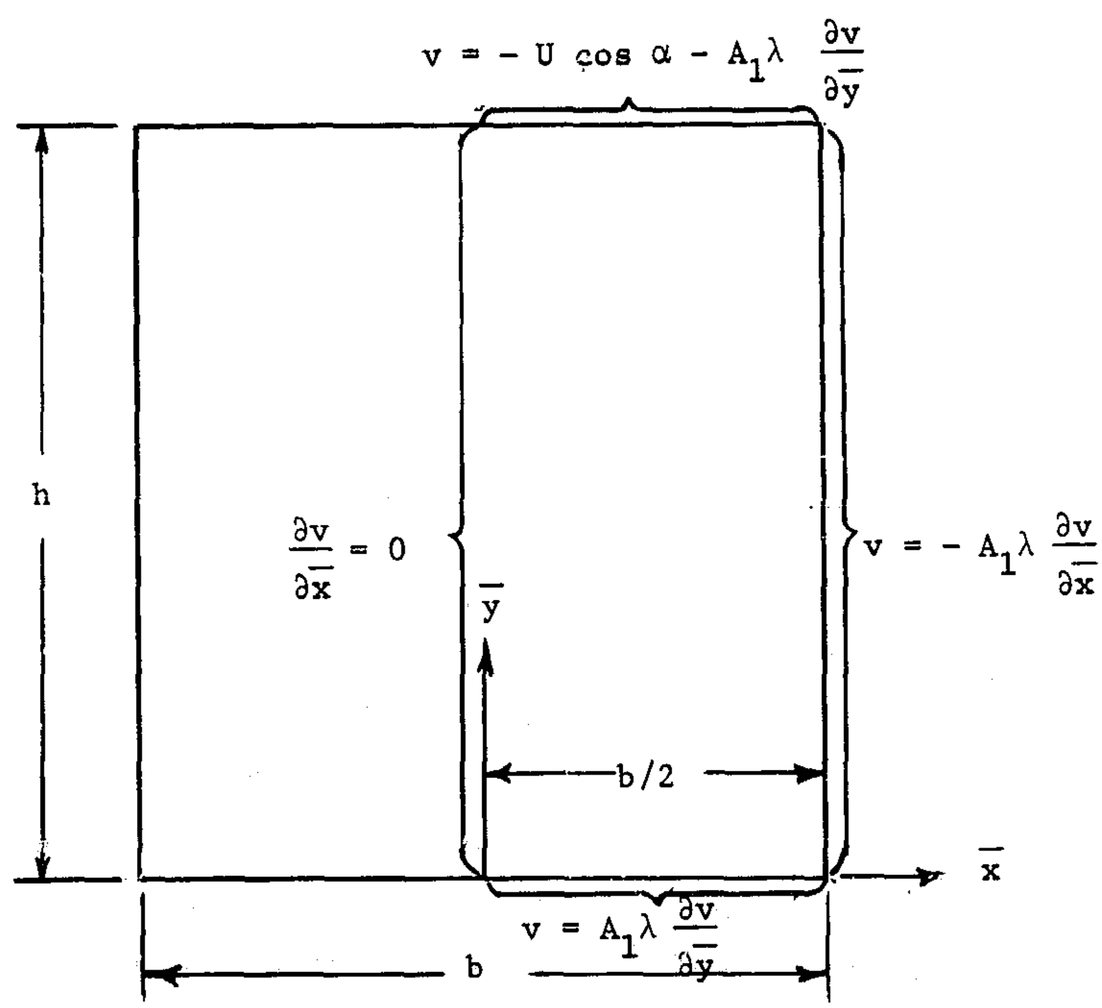

Figure 6. Groove Configuration.

A common index of rarefication, the Knudsen number is taken as

$$K_h = \lambda/h .$$

Introducing these new variables into Equations (14) and (15) one obtains

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{a}{2}\right)^2 \frac{\partial^2 u}{\partial y^2} = \frac{b^2}{4\mu U \cos\alpha} \frac{dP}{dz} = \frac{R}{U} \tag{16}$$

where

$$R = \frac{b^2}{4\mu\cos\alpha} \frac{dP}{dz}$$

and the boundary conditions are

$$\frac{\partial u}{\partial x}(0,y) = 0 \tag{17a}$$

$$u(1, y) = -\frac{2K_h}{a} \frac{\partial u}{\partial x}(1, y) \tag{17b}$$

$$u(x, 0) = K_h \frac{\partial u}{\partial y}(x, 0) \tag{17c}$$

and

$$u(x, 1) = -1 - K_h \frac{\partial u}{\partial y}(x, 1). \tag{17d}$$

Many techniques have been used to solve non-homogeneous problems of the type presented above. The least arduous is a modified form of the method of variation of parameters (12). This method has been employed successfully by Ebert and Sparrow (13) to solve Equation (14) with four homogeneous boundary conditions. In the case considered here Equation (17d) presents a non-homogeneous boundary condition, but the method used by Ebert and Sparrow still leads to a solution.

The method of variation of parameters is a generalization of the method of separation of variables, and as such, experience gained from employing the latter is helpful when applying the former method. In the method of separation of variables, the sign of the separation constant is taken so that the trigonometric solution is obtained for the homogeneous direction. In the method of variation of parameters the same reasoning is used in the assumption that the solution has the form

$$u(x, y) = \sum_{n=1}^{\infty} \phi_n(y)(\cos \alpha_n x + C_5 \sin \alpha_n x) \tag{18}$$

where the x-direction is the homogeneous direction determined by an inspection of Equations (17). The function $\phi_n(y)$ is a yet unknown function of y. One now proceeds to determine ϕ_n, α_n, and C_5.

Substituting Equation (18) into the boundary condition of Equation (17a) one obtains

$$\sum_{n=1}^{\infty} \phi_n(y) \, [C_5\alpha_n \cos (0) - \alpha_n \sin (0)] = 0$$

which implies that $C_5 = 0$ and that

$$u(x, y) = \sum_{n=1}^{\infty} \phi_n(y) \cos \alpha_n x \, . \tag{19}$$

Substituting Equation (19) into Equation (17b) gives

$$\sum_{n=1}^{\infty} \phi_n(y) \, [\, \cos \alpha_n - \frac{2K_h \alpha_n}{a} \sin \alpha_n] = 0$$

from which it follows that for a non-trivial solution

$$\cos \alpha_n - \frac{2K_h \alpha_n}{a} \sin \alpha_n = 0$$

or

$$\alpha_n \tan \alpha_n = \frac{a}{2K_h} \, . \tag{20}$$

The eigenvalues, α_n, of the eigenfunctions, $\cos \alpha_n x$, are the roots of the transcendental relationship of Equation (20).

The problem now reduces to determining $\phi_n(y)$. Substituting Equation (19) into Equation (16) yields

$$- \sum_{n=1}^{\infty} \alpha_n^2 \phi_n(y) \cos \alpha_n x + (\frac{a}{2})^2 \sum_{n=1}^{\infty} \phi''_n(y) \cos \alpha_n x$$

$$= \frac{R}{U} \sum_{n=1}^{\infty} \Omega_n \cos \alpha_n x \tag{21}$$

where Ω_n satisfies the Fourier series

$$1 = \sum_{n=1}^{\infty} \Omega_n \cos \alpha_n x. \tag{22}$$

The Fourier cosine series coefficient, Ω_n, must be

$$\Omega_n = \frac{2 \sin \alpha_n}{\alpha_n + \sin \alpha_n \cos \alpha_n} = \frac{2}{\alpha_n} [\frac{\sin \alpha_n}{1 + \frac{2K_h \sin^2 \alpha_n}{a}}] \tag{23}$$

where the last step comes from Equation (20).

In order for $\phi_n(y)$ to satisfy Equation (21), it must be that

$$\phi''_n(y) - (\frac{2\alpha_n}{a})^2 \phi_n(y) = \frac{4R\Omega_n}{Ua^2} . \tag{24}$$

The determination of ϕ_n thus reduces to the solution of a non-homogeneous second-order ordinary differential equation. The boundary conditions on $\phi_n(y)$ are obtained from Equations (17c) and (17d). From Equation (17c) one boundary condition is

$$\phi_n(0) = K_h \phi'_n(0). \tag{25a}$$

Substituting Equation (22) for the unity term in Equation (17d) leads to the second boundary condition,

$$\phi_n(1) = -\Omega_n - K_h\phi_n'(1). \qquad (25b)$$

The solution of Equation (24) subject to the boundary conditions in Equations (25) is

$$\phi_n(y) = (C_{1np} + C_{1nu})e^{-\theta_n y} + (C_{2np} + C_{2nu})e^{-\theta_n y} - \frac{R\Omega_n}{U\alpha_n^2} \qquad (26)$$

where

$$\theta_n = \frac{2\alpha_n}{a}$$

$$C_{1np} = \frac{R\Omega_n}{U\,\alpha_n^2}\left[\frac{(1 - K_h\theta_n)e^{-\theta_n} - (1 + K_h\theta_n)}{(1 - K_h\theta_n)^2 e^{-\theta_n} - (1 + K_h\theta_n)^2 e^{\theta_n}}\right]$$

$$C_{1nu} = \frac{\Omega_n(1 + K_h\theta_n)}{(1 - K_h\theta_n)^2 e^{-\theta_n} - (1 + K_h\theta_n)^2 e^{\theta_n}}$$

$$C_{2np} = \frac{R\Omega_n}{U\alpha_n^2}\left[\frac{(1 - K_h\theta_n) - (1 + K_h\theta_n)e^{\theta_n}}{(1 - K_h\theta_n)^2 e^{-\theta_n} - (1 + K_h\theta_n)^2 e^{\theta_n}}\right]$$

$$C_{2nu} = -\frac{\Omega_n(1 - K_h\theta_n)}{(1 - K_h\theta_n)^2 e^{-\theta_n} - (1 + K_h\theta_n)^2 e^{\theta_n}}\,.$$

By substituting Equation (26) back into Equation (19) the dimensionless velocity distribution becomes, after considerable simplification,

$$u(x, y) = \frac{b^2}{2\mu U \cos\alpha} \frac{dP}{dz} \sum_{n=1}^{\infty} \left[\frac{\sin\alpha_n}{1 + \frac{2K_h}{a}\sin^2\alpha_n}\right] \cdot \left[\frac{\cos\alpha_n x}{\alpha_n^3}\right]\left[\frac{\cosh\frac{2\alpha_n y}{a} - \tanh\frac{\alpha_n}{a}\sinh\frac{2\alpha_n y}{a}}{1 + 2\frac{K_h\alpha_n}{a}\tanh\frac{\alpha_n}{a}} - 1\right]$$

$$- \sum_{n=1}^{\infty} 2\left[\frac{\sin\alpha_n}{1 + \frac{2K_h}{a}\sin^2\alpha_n}\right]\left[\frac{\cos\alpha_n x}{\alpha_n}\right].$$

$$\left[\frac{\sinh\frac{2\alpha_n y}{a} + \frac{2K_h\alpha_n}{a}\cosh\frac{2\alpha_n y}{a}}{\left[1 + \left(\frac{2K_h\alpha_n}{a}\right)^2\right]\sinh\frac{2\alpha_n}{a} + \frac{4K_h\alpha_n}{a}\cosh\frac{2\alpha_n}{a}}\right]. \tag{27}$$

The volume flow rate in the groove is obtained by integrating the velocity distribution over the area of the groove. The volume flow rate thus obtained is

$$V_g' = \frac{b^3 h}{2\mu}\frac{dP}{dz}\sum_{n=1}^{\infty}\frac{a}{\alpha_n^5}\left[\frac{\sin^2\alpha_n}{1 + \frac{2K_h}{a}\sin^2\alpha_n}\right]\left[\frac{\tanh\alpha_n/a}{1 + \frac{2K_h\alpha_n}{a}\tanh\frac{\alpha_n}{a}} - \frac{\alpha_n}{a}\right]$$

$$- Ubh\cos\alpha\sum_{n=1}^{\infty}\frac{a}{\alpha_n^3}\left[\frac{\sin^2\alpha_n}{1 + 2\frac{K_h}{a}\sin^2\alpha_n}\right]\left[\frac{\tanh\alpha_n/a}{1 + \frac{2K_h\alpha_n}{a}\tanh\frac{\alpha_n}{a}}\right]. \tag{28}$$

The first summation term represents the pressure induced flow in the groove whereas the second term represents the rotor induced flow.

The linearity of Equation (14) has been demonstrated in the analysis above since the solution of Equation (28) can be shown to be the sum of two solutions. The first term is in agreement with the expression obtained by Ebert and Sparrow (13) for slip flow in a rectangular duct with stationary walls. Milligan and Patterson (14) have experimentally verified the solution of Ebert and Sparrow. The last term is the solution of $\nabla^2 v = 0$ with the boundary conditions given in Equation (15). It is thus demonstrated that it is permissible to obtain the two solutions mentioned above independently and then to add them together to arrive at the same result as Equation (28).

At this point it is interesting to determine V'_g when the Knudsen number approaches zero, the continuum flow regime. The solution for the eigenvalues becomes

$$\cos \alpha_n = 0$$

or

$$\alpha_n = \frac{n\pi}{2} \qquad n = 1, 3, 5, 7 \ldots$$

With these eigenvalues and with $K_h \to 0$, V'_g becomes

$$V'_{gc} = \frac{16b^3h}{\pi^5\mu} \frac{dP}{dz} \sum_{n=1}^{\infty} \frac{a}{n^5} \left[\tanh \frac{n\pi}{2a} - \frac{n\pi}{2a} \right]$$

$$- \frac{8Ubh \cos \alpha}{\pi^3} \sum_{n=1}^{\infty} \frac{a}{n^3} \tanh \frac{n\pi}{2a} \qquad (29)$$

$$n = 1, 3, 5, 7 \ldots$$

III. Correction to Rotor Induced Flow

In the analysis above it was assumed that the rotor velocity, $U \cos \alpha$, acted at the top of the groove ($\bar{y} = h$). No attempt was made to account for the fact that the rotor is not located at $\bar{y} = h$ but rather at $\bar{y} = h + c$. Since an exact analysis of the region above the groove, $h \leq \bar{y} \leq h + c$, has been shown to be very complex even for continuum flow (10), some sort of approximation must be made to obtain a simple solution. The following assumptions are made: (1) the flow induced by the rotor in the region $h \leq \overline{y} \leq h + c$ does not contribute to the seal discharge and (2) an effective inducing velocity less than $U \cos \alpha$ acts at $\overline{y} = h$ as a result of the clearance region $h \leq \overline{y} \leq h + c$. This effective inducing velocity, U_e, will be determined by calculating the average velocity in the plane $\overline{y} = h$ of a hypothetical groove of depth $h + c$ and width b with the upper boundary moving at $U \cos \alpha$ and with slip boundary conditions on all surfaces. The hypothetical groove and associated boundary conditions are shown in Figure 7. The brackets in Figure 7 indicate the portion of the boundary over which each boundary condition is applied.

From Equation (27) the velocity distribution induced by the rotor is

$$v_R = U \cos \alpha \sum_{n=1}^{\infty} \frac{2}{\alpha_n} \left[\frac{\sin \alpha_n \cos \alpha_n x}{1 + \frac{2K_h}{a} \sin^2 \alpha_n} \right] \cdot \left[\frac{\sinh \frac{2\alpha_n y}{a} + \frac{2K_h \alpha_n}{a} \cosh \frac{2\alpha_n y}{a}}{\left[1 + \left(\frac{2K_h \alpha_n}{a}\right)^2\right] \sinh \frac{2\alpha_n}{a} + \frac{4K_h \alpha_n}{a} \cosh \frac{2\alpha_n}{a}} \right] . \tag{30}$$

If Equation (30) is applied to a groove of depth $h + c$, the resulting distribution is

$$v_R' = U \cos \alpha \sum_{n=1}^{\infty} \frac{2}{\alpha_n} \left[\frac{\sin \alpha_n \cos \alpha_n x}{1 + \frac{2K_h'}{a'} \sin^2 \alpha_n} \right] \cdot \left[\frac{\sinh \frac{2\alpha_n y'}{a'} + \frac{2K_h' \alpha_n}{a'} \cosh \frac{2\alpha_n y'}{a'}}{\left[1 + \left(\frac{2K_h' \alpha_n}{a'}\right)^2\right] \sinh \frac{2\alpha_n}{a'} + \frac{4K_h' \alpha_n}{a'} \cosh \frac{2\alpha_n}{a'}} \right] ,$$

where

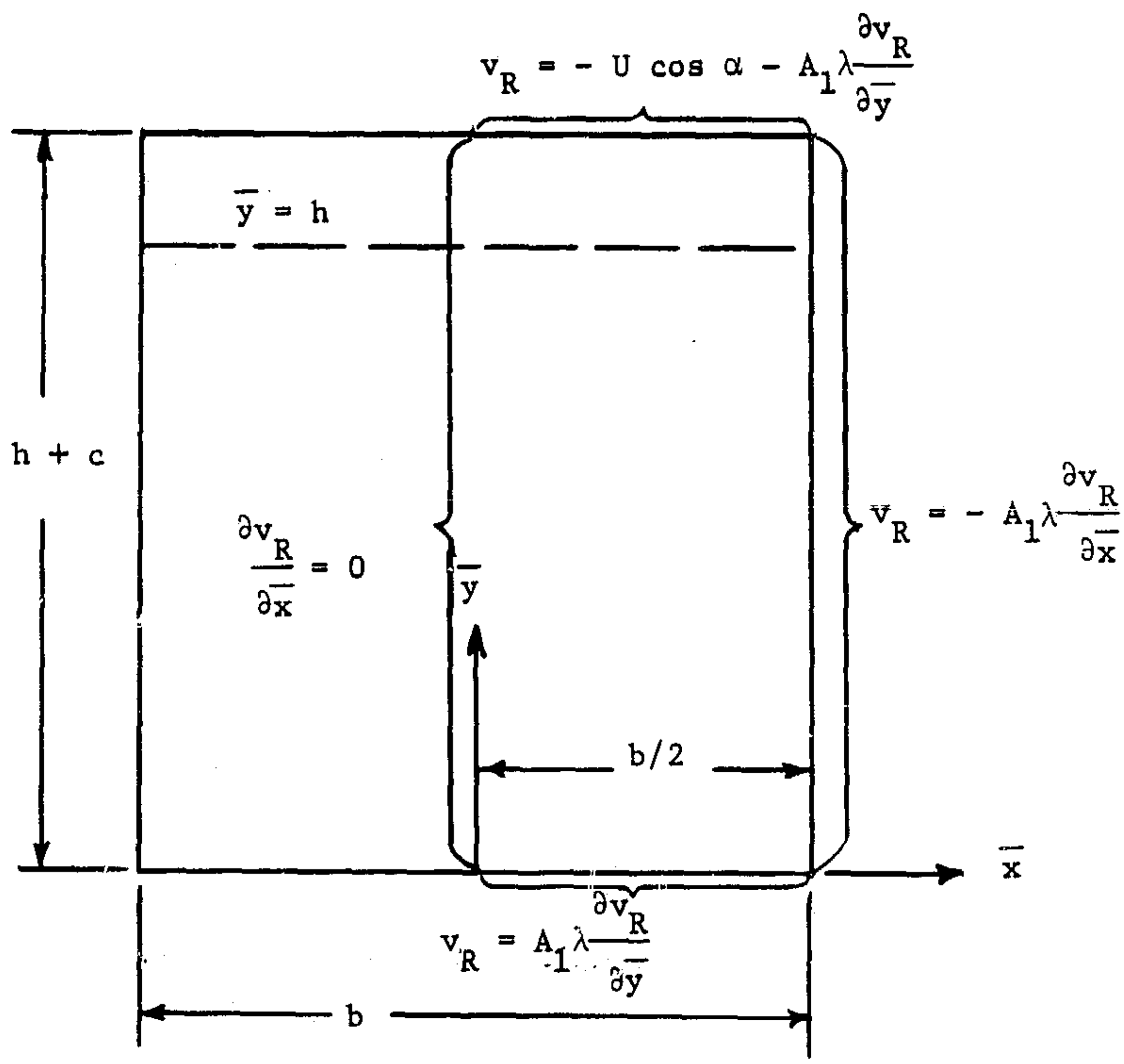

Figure 7. Hypothetical Groove.

$$a' = \frac{b}{h + c} = \frac{a}{1 + c/h}$$

$$K_h' = \frac{\lambda}{h + c} = \frac{K_h}{1 + c/h}$$

$$y' = \frac{\bar{y}}{h + c} .$$

The eigenvalues of Equation (20) are still the same because

$$\frac{a}{2K_h} = \frac{a'}{2K_h'} .$$

The average velocity at the plane $\bar{y} = h$ or $y' = \frac{h}{h + c}$ is

$$U_e = \int_0^1 v_R'(x, h/(h + c))dx.$$

Carrying through the integration, U_e becomes

$$U_e = 2U \cos \alpha \sum_{n=1}^{\infty} \frac{1}{\alpha_n^2} \left[\frac{\sin^2 \alpha_n}{1 + \frac{2K_h}{a} \sin^2 \alpha_n}\right]$$

$$\left[\frac{\sinh \frac{2\alpha_n}{a} + \frac{2K_h\alpha_n}{a} \cosh \frac{2\alpha_n}{a}}{\left[1 + \left(\frac{2K_h\alpha_n}{a}\right)^2\right] \sinh \frac{2\alpha_n}{a'} + \frac{4K_h\alpha_n}{a} \cosh \frac{2\alpha_n}{a'}}\right] \tag{31}$$

or

$$U_e = [U \cos \alpha] \Sigma_c \tag{32}$$

where the Σ_c is equal to twice the summation in Equation (31). The corrected groove flow is now obtained by replacing U cos α in Equation (28) with $[U \cos \alpha] \Sigma_c$. The total groove flow now becomes

$$V_g' = - \frac{b^3 h}{2\mu} \frac{dP}{dz} \Sigma_p - U\, bh \cos \alpha[\Sigma_c \Sigma_u] \tag{33}$$

where

$$\Sigma_p = \sum_{n=1}^{\infty} \frac{a}{\alpha_n^5} \left[\frac{\sin^2 \alpha_n}{1 + \frac{2K_h}{a} \sin^2 \alpha_n}\right]\left[\frac{\tanh \frac{\alpha_n}{a}}{1 + \frac{2K_h \alpha_n}{a} \tanh \frac{\alpha_n}{a}} - \frac{\alpha_n}{a}\right] \tag{34a}$$

$$\Sigma_u = \sum_{n=1}^{\infty} \frac{a}{\alpha_n^3} \left[\frac{\sin^2 \alpha_n}{1 + \frac{2K_h}{a} \sin^2 \alpha_n}\right]\left[\frac{\tanh \frac{\alpha_n}{a}}{1 + \frac{2K_h \alpha_n}{a} \tanh \frac{\alpha_n}{a}}\right] \tag{34b}$$

and

$$\Sigma_c = \sum_{n=1}^{\infty} \frac{2}{\alpha_n^2} \left[\frac{\sin^2 \alpha_n}{1 + \frac{2K_h}{a} \sin^2 \alpha_n}\right] \cdot$$

$$\left[\frac{\sinh \frac{2\alpha_n}{a} + \frac{2K_h \alpha_n}{a} \cosh \frac{2\alpha_n}{a}}{\left[1 + \left(\frac{2K_h \alpha_n}{a}\right)^2\right] \sinh \frac{2\alpha_n}{a'} + \frac{4K_h \alpha_n}{a} \cosh \frac{2\alpha_n}{a'}}\right]. \tag{34c}$$

In terms of the viscoseal geometry

$$\frac{dP}{dz} = \frac{dP}{d\ell_g} = \frac{n_s (b + w)}{\pi d} \frac{dP}{dL} = \sin \alpha \frac{dP}{dL} , \tag{35}$$

and since the pressure gradient in the viscoseal is assumed to be constant,

$$\frac{dP}{dz} = - \sin \alpha \frac{\Delta P}{L} . \tag{36}$$

The total groove flow is obtained by multiplying Equation (33) by n_s. Noting that $n_s = (\pi d) \sin \alpha/(b + w)$ the total groove flow becomes

$$\dot{V}_g = n_s V'_g = \frac{\pi d b^3 h \Delta P \sin^2 \alpha}{2\mu L \,(b + w)} \Sigma_p - \frac{U \pi d b h \cos \alpha \sin \alpha}{(b + w)} \Sigma_u \Sigma_c . \tag{37}$$

IV. Land Leakage Flow

The land leakage flow is taken as flow through a long annulus. Milligan et al. (15) have developed an expression for slip flow in an annulus and have obtained excellent experimental confirmation of the expression. The expression derived by Milligan is

$$V_L = - \frac{\pi d^4}{128\ \mu} \frac{dP}{d\ell_L} A_L \tag{38}$$

where

$$A_L = [(1 - K^4) + \frac{(1 - K^2)^2}{\ln K}] + [\frac{2K_c(1 - K^2)}{K \ln K - K_c(1 - K^2)}].$$

$$[2K(K^2 - 1) - 2K(\ln K)(K^2 - K + 1) - \frac{(1 - K^2)^2}{2 \ln K}$$

$$+ 2(1 - K^2)(1 - K)^2 K_c], \tag{39}$$

and $\frac{dP}{d\ell_L}$ is the effective land pressure gradient. For a single-threaded seal, Hodgson (7) showed based on assumption (2) at the beginning of this chapter that

$$\frac{dP}{d\ell_L} = \frac{w + b}{w} \cos \alpha \frac{dP}{dL}. \tag{40}$$

As was stated in Chapter II, Equation (40) remains valid for a multi-threaded seal. This point is verified in Appendix C.

Combining Equations (38) and (40) the land leakage flow becomes

$$V_L = - \frac{\pi d^4 (w + b) \cos \alpha}{128\mu\ w} A_L \frac{dP}{dL} = \frac{\pi d^4 (w + b) \cos \alpha}{128\mu\ wL} A_L\ \Delta P \tag{41}$$

V. Total Seal Flow and Sealing Coefficient

The total seal flow is obtained by combining Equations (37) and (41). The result is

$$V_T = \frac{\Delta P}{\mu L} [A_p \Sigma_p + A_c A_L] - \frac{6U}{c^2}[A_u \Sigma_c \Sigma_u] \tag{42}$$

where

$$A_p = \frac{\pi d b^3 h \sin^2 \alpha}{2(w + b)}$$

$$A_c = \frac{\pi d^4 (b + w) \cos \alpha}{128 w}$$

and

$$A_u = \frac{\pi d b h c^2 \cos \alpha \sin \alpha}{6(w + b)} .$$

The specific molecular flow rate is

$$\frac{\dot{n}}{\Delta P} = \frac{V_T}{\Delta P} \left(\frac{\bar{P}}{kT}\right)$$

or

$$\frac{\dot{n}}{\Delta P} = \frac{\bar{P}}{\mu L k T}[A_p \Sigma_p + A_c A_L] - \frac{3U}{c^2 kT}\left[\frac{r_p + 1}{r_p - 1}\right][A_u \Sigma_c \Sigma_u] \tag{43}$$

where Equation (12) is used to eliminate ΔP.

A common dimensionless viscoseal performance index (1, 4, 5, and 16) is called the sealing coefficient and is defined by

$$\Lambda \equiv \frac{6\mu UL}{c^2 \Delta P} \tag{44}$$

where ΔP is the pressure difference at zero flow. By equating V_T to zero in Equation (42), the sealing coefficient is obtained as

$$\Lambda = \frac{A_p \Sigma_p + A_c A_L}{A_u \Sigma_c \Sigma_u} . \tag{45}$$

Program three of Appendix D computes the specific molecular flow rate from Equation (43) and the sealing coefficient from Equation (45). In order to evaluate the summations, Σ_p, Σ_c, and Σ_u, it is necessary to determine the eigenvalues which satisfy Equation (20). Newton's method of approximating roots is employed to solve Equation (20) for these eigenvalues at each Knudsen number.

CHAPTER IV

OPTIMIZATION USING THE IMPROVED ANALYTICAL MODEL

Equation (45) presents an explicit determination of the performance in the transition regime, a vast improvement over the Hodgson model in which no explicit determination was possible. Even with this improvement the process of optimization in its broadest scope would require the determination of optimum relationships among six basic parameters: b, h, w, α, c, and d.

It seems appropriate to assume that for a given application the following will be known:

(1) the average operating pressure, $\overline{P}$;

(2) the shaft diameter, d;

and

(3) the minimum allowable clearance, c.

One important parameter that becomes fixed with the specification of $\overline{P}$ and c is the Knudsen number K_c based on the clearance. By definition

$$K_c \equiv \frac{\lambda}{c} \tag{46}$$

but for isothermal flow $\lambda = \lambda(\overline{P})$ and thus for constant $\overline{P}$ and c, K_c is constant. From this A_L also becomes a constant since specification of c and d fixes K.

The six basic geometric parameters have now been reduced to four: b, h, w, and α. Optimization requires minimizing the sealing coefficient. A necessary condition for the existence of a minimum is that

$$\frac{\partial \Lambda}{\partial \alpha} = \frac{\partial \Lambda}{\partial w} = \frac{\partial \Lambda}{\partial h} = \frac{\partial \Lambda}{\partial b} = 0 \tag{47}$$

The formation of the partial derivatives, $\frac{\partial \Lambda}{\partial h}$ and $\frac{\partial \Lambda}{\partial b}$, would be very difficult since the summations Σ_p, Σ_u and Σ_c are all functions of b and h and contain the involved transcendental relationship in Equation (20). But the other two partial derivatives can be determined.

Differentiating Equation (45) with respect to w and setting this equal to zero yields

$$\frac{1}{\Sigma_c \Sigma_u} \left[\frac{A_u \frac{\partial}{\partial w}(A_p \Sigma_p + A_c A_L) - (A_p \Sigma_p + A_c A_L) \frac{\partial A_u}{\partial w}}{A_u^2} \right] = 0 \tag{48}$$

from which it follows that

$$A_u \frac{\partial}{\partial w}(A_p \Sigma_p + A_c A_L) - (A_p \Sigma_p + A_c A_L) \frac{\partial A_u}{\partial w} = 0 \ . \tag{49}$$

Substituting the following:

$$\frac{\partial A_u}{\partial w} = - \frac{\pi dbhc^2 \cos \alpha \sin \alpha}{6(b + w)^2}$$

$$\frac{\partial A_p}{\partial w} = - \frac{\pi db^3 h \sin^2 \alpha}{2(b + w)^2} \tag{50}$$

$$\frac{\partial A_c}{\partial w} = - \frac{\pi d^4 b \cos \alpha}{128 \ w^2}$$

into Equation (49) it can be shown that

$$b = w \tag{51}$$

will satisfy the condition that $\frac{\partial \Lambda}{\partial w} = 0$.

Differentiating Equation (45) with respect to α and setting this equal to zero one obtains

$$\frac{1}{\Sigma_c \Sigma_u} \left[\frac{A_u \frac{\partial}{\partial \alpha} (A_p \Sigma_p + A_c A_L) - (A_p \Sigma_p + A_c A_L) \frac{\partial A_u}{\partial \alpha}}{A_u^2} \right] = 0$$

from which it follows that

$$A_u \frac{\partial}{\partial \alpha} (A_p \Sigma_p + A_c A_L) - (A_p \Sigma_p + A_c A_L) \frac{\partial A_u}{\partial \alpha} = 0. \qquad (52)$$

Performing the required differentiation and making use of Equation (51), Equation (52) becomes

$$K_\alpha \cos^3 \alpha + \cos^2 \alpha - 1 = 0 \qquad (53)$$

where

$$K_\alpha = \frac{A_L d^3}{16 b^2 h \Sigma_p}$$

Two optimization conditions have now been established. These conditions are that the land width and groove width must be equal and that Equation (53) must be satisfied. The process now reduces to finding the optimum value of b and h. Since the infinite series, Σ_p, Σ_c, and Σ_u, are involved the remainder of the process is carried out numerically.

The procedure is simply to set up acceptable limits for b and h. A value of b is chosen, w is determined from Equation (51), all values of h are combined with the chosen value of b, for each b - h combination the optimum helix angle is determined from Equation (53), and finally the sealing coefficient is computed from Equation (45). The sealing coefficients corresponding to all the b - h combinations are compared and the minimum value chosen.

For purposes of illustration, the procedure outlined above is carried out for a seal with the following characteristics:

$d = 2.00$ inch

$c = 0.004$ inch

$K_c = 0.2$

As stated above acceptable limits for the variables b and h must be established. The smallest practical value of b is chosen to be 0.010 inch. The smallest practical value of h was chosen to be 0.001 inch. The lower limits are in a sense arbitrary, but are subject to machining restrictions. The upper limit for the groove width is governed by the assumptions on which the basic model is derived. Since the groove flow is developed for a long groove, the groove width must be small compared to the length of the groove. The length of the groove is $L/\sin \alpha$. In general L and α are not known before the optimization study is conducted. Since the length of the groove cannot be established before the optimization study, a broad range of values for b is considered. In this instance b is allowed to take on values up to 0.5 inch. The upper limit for the groove depth, h, is fixed by the restriction that $h << d$. In the example considered here, if $\frac{d}{h} \geq 100$, h must be equal to or less than 0.020 inch.

With the limits on b and h established, the problem now reduces to finding the b - h combination within these limits that gives the minimum sealing coefficient. The Fortran program for computing and comparing these combinations is found in Appendix B.

The results of the optimization are shown in Table I. It should be kept in mind that the land width is equal to the groove width and that at each value of b there exists an optimum value of h from which the helix angle, α, is determined using Equation (53). The optimum sealing coefficient in Table I is computed using these values of h and α. The values in Table I are only approximate due to the fact that b and α must be chosen so that n_s will be an integer satisfying the relationship

$$n_s = \frac{\pi d \sin \alpha}{b + w} = \frac{\pi d}{2b} \sin \alpha \tag{54}$$

From Table I it is apparent that a wide shallow groove yields the best sealing coefficient and that any value of b above 0.2 inch will result in a good seal. It can also be seen that a groove depth of 13 or 14 mils will give the best results and that the optimum helix angle is about 22 degrees.

In order to satisfy the condition that the groove be long compared to its other dimensions, the smallest groove width that gives a good sealing coefficient should be chosen. In this case the sealing coefficient is still close to optimum for b = 0.2 inch. Substituting b = 0.2, α = 22.2°, and d = 2.0 into Equation (54), n_s = 5.94. But since n_s is the number of thread starts, it must be an integer. Taking n_s equal to 6 and α as 22.2 degrees, the groove width must be, from Equation (54), 0.198 inch. Once the seal geometry has been determined the sealing coefficient can be computed. With knowledge of the sealing coefficient, the clearance, the viscosity of the sealant, the operating

TABLE I

OPTIMUM SEALING COEFFICIENT FOR A
2.0 INCH DIAMETER SEAL WITH A CLEARANCE OF 4 MILS
OPERATING AT A KNUDSEN NUMBER OF 0.2

Groove Width (inches)	Groove Depth (inches)	Helix Angle (degrees)	Sealing Coefficient
0.010	0.007	47.2	68.6
0.020	0.009	37.6	35.6
0.030	0.010	33.5	28.5
0.040	0.011	30.1	25.6
0.050	0.012	27.3	24.0
0.060	0.012	27.1	23.1
0.070	0.012	26.9	22.4
0.080	0.013	24.6	22.0
0.090	0.013	24.5	21.7
0.100	0.013	24.4	21.4
0.200	0.014	22.2	20.5
0.300	0.014	22.1	20.3
0.400	0.014	22.1	20.2
0.500	0.014	22.1	20.2

speed, the seal diameter, and the pressure difference across the seal, the seal length can be computed from Equation (44).

The procedure outlined above can be followed to find the best seal configuration for any application where the diameter, clearance and average operating pressure can be established.

CHAPTER V

COMPARISON WITH EXPERIMENTAL DATA

Experimental data on gas-type viscoseal performance have been obtained under continuum and rarefied conditions. Wilkerson (6) has obtained both net leakage and sealing coefficient data. King (3) obtained sealing coefficient data but no leakage data, while Hodgson and Milligan (1) have obtained sealing coefficient data for purely continuum operating conditions. Comparison of these data with the predictions of Chapters II and III will be made.

I. Sealing Coefficient and Net Leakage for Modified-Hodgson Model

Combining Equations (10) and (44), the continuum sealing coefficient based on the modified Hodgson model is:

$$\Lambda = \frac{n_s(b + h)(b + w)h^2}{\pi dbc^2 \cos \alpha} [1 - 0.63 \frac{h}{b} \tanh \frac{\pi b}{2h}] + \frac{(b + h)(b + w)\pi dc}{n_s b^2 hw}. \quad (55)$$

From Equations (13) and (44), the free-molecule sealing coefficient is

$$\Lambda = \frac{3\mu UL}{\bar{P}c^2} [\frac{\exp [n_s AL/(n_s^2 C + E)] + 1}{\exp [n_s AL/(n_s^2 C + E)] - 1}]. \quad (56)$$

In the regime between continuum and free-molecule flow, the solution of the general pressure difference relationship, Equation (9), is combined with Equation (44) to obtain the sealing coefficient.

The net leakage for all flow regimes as predicted by the modified Hodgson model is given by Equation (8).

II. Continuum Sealing Coefficient for the Improved Analytical Model

The sealing coefficient expressed in Equation (45) is applicable to seals operating in the continuum and slip flow regimes. In the continuum limit, as $K_c \to 0$ and $K_h \to 0$, the sealing coefficient can be written as:

$$\Lambda = \frac{A_p \Sigma_{pc} + A_c A_{LC}}{A_u \Sigma_{cc} \Sigma_{uc}} ,$$

where Σ_{pc}, A_{LC}, Σ_{cc}, and Σ_{uc} are the continuum limits of Σ_p, A_L, Σ_c and Σ_u, respectively, and are expressed as:

$$\Sigma_{pc} = -\frac{32}{\pi^5} \sum_{n=1}^{\infty} \frac{a}{n^5} \left[\tanh \frac{n\pi}{2a} - \frac{n\pi}{2a}\right]$$

$$\Sigma_{cc} = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\sinh \frac{n\pi}{a}}{\sinh \frac{n\pi}{a'}}$$

$$A_{LC} = (1 - K^4) + \frac{(1 - K^2)^2}{\ln K}$$

and

$$\Sigma_{uc} = \frac{8a}{\pi^3} \sum_{n=1}^{\infty} \frac{\tanh \frac{n\pi}{2a}}{n^3} ,$$

where n = 1, 3, 5, 7 . . . for all the summations.

III. Comparison with Continuum Data

Table II shows a comparison of the sealing coefficient predicted by Equations (55) and (57) with data presented by various investigators.

TABLE II

COMPARISON WITH EXPERIMENTAL CONTINUUM DATA

Seal	Sealing Coefficient		
	Experimental	Modified-Hodgson	Improved Model
Hodgson Milligan No. 2	12.5	10.4	12.3
Hodgson Milligan No. 4	14	10.5	12.4
Hodgson Milligan No. 5	23.5	17.0	26.3
Wilkerson	25	14.6	24.6
King	45	40.5	47.4

A detailed description of the seal configurations is given in Table III. The agreement of Equation (57) with the data is good for all five seals. It should be pointed out that both the grooved housing and grooved shaft type seals are represented, the seal tested by Wilkerson being of the grooved shaft type, while the others are the grooved housing type. This lends support to the applicability of the models to either type seal. The modified-Hodgson model shows poor agreement for all the seals except the seal tested by King. The reason for the good agreement with this seal is due to the fact that the groove depth for this seal is much larger than the clearance, thus making the assumption valid that the rotor acts at the top of the groove. Also, the groove of this seal has an aspect ratio of 1.77. With this aspect ratio the area-weighted average velocity approach for the rotor induced flow is a good approximation to the exact parallel flow solution.

IV. Comparison with Rarefied Data

To the author's knowledge, few investigators have presented a significant amount of experimental data for rarefied viscoseals. The most thorough investigation is presented by Wilkerson (6). This data will be compared to the two analytical models. The seal tested by Wilkerson is the same seal as his continuum seal of Table III with one exception. Using a rubbing contact vacuum seal to seal the shaft where it penetrated the test section, Wilkerson encountered small changes in clearance due to thermal growth of the shaft. The clearance varied with speed and was different for leakage and sealing tests. All other dimensions are the same as those given in Table III.

0

TABLE III

EXPERIMENTAL SEAL CONFIGURATIONS

Seal	Hodgson and Milligan No. 2	Hodgson and Milligan No. 4	Hodgson and Milligan No. 5	Wilkerson	King
d(in.)	2.5037	2.5037	2.5037	2.0005	3.147
c(in.)	0.00175	0.00175	0.00435	0.00418	0.0031
L(in.)	2.30	2.30	2.30	4.53	1.70
n_s	4	4	4	16	1
b'(in.)	0.125	0.125	0.125	0.03111	0.197
w'(in.)	0.125	0.125	0.125	0.03235	0.110
h(in.)	0.010	0.0105	0.009	0.03065	0.111
α(deg)	7.27	7.27	7.27	9.30	1.47

Figure 8 shows sealing coefficient data obtained by Wilkerson at speeds of 5,000, 10,000, and 30,000 rpm. The theoretical curves for the sealing coefficient for the modified-Hodgson model and the improved model are also shown. The clearances for the 5,000 and 10,000 rpm data were essentially the same, whereas the clearance for the 30,000 rpm data differed by about 14 percent from the 5,000 and 10,000 clearances. Note the speed sensitivity of the modified-Hodgson model. The difference between the two curves for the improved model is due to the difference in clearances and in no way indicates a speed dependence of this model. The improved model gives fair agreement for all speeds. The modified-Hodgson model also gives good agreement for the more rarefied data at each speed, but gives poorer agreement with the less rarefied and continuum data.

Figure 9 shows net leakage data for the same seal for speeds of 0, 5,000, 10,000, and 30,000 rpm. The flow rates of Equations (8) and (43) are also shown. Equations (8) and (43) are pressure ratio dependent. The average experimental pressure ratios at each speed were used in these equations. The average pressure ratios given by Wilkerson are 177, 146, and 807 for 5,000, 10,000, and 30,000 rpm, respectively. The agreement of both models with the static leakage is good. The improved model shows good agreement for the less rarefied data at 5,000 and 10,000 rpm but is in poor agreement with the more rarefied data at these speeds. It is in good agreement with the data taken at 30,000 rpm. The modified-Hodgson model gives poor agreement for all speeds.

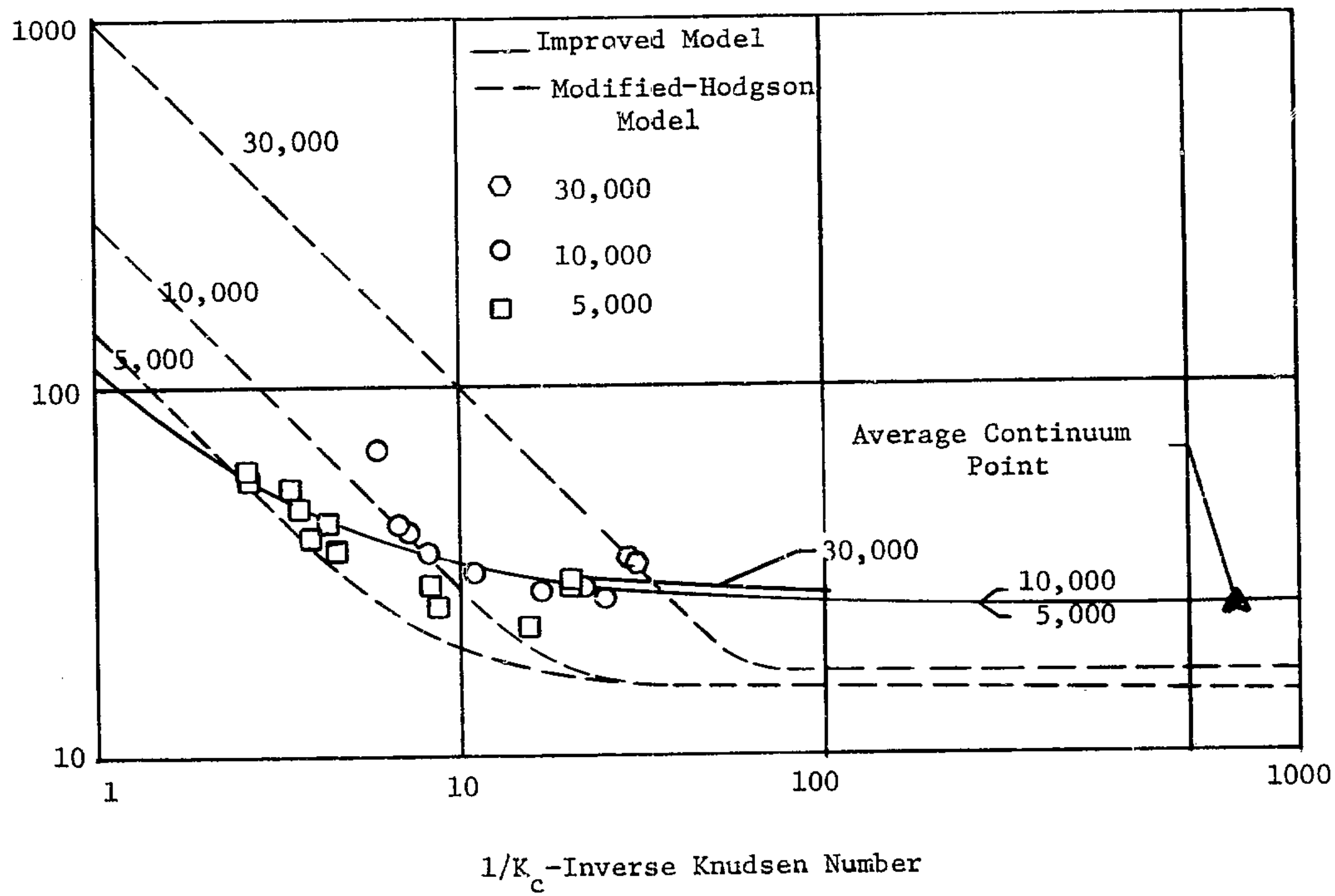

Figure 8. Sealing Coefficient Versus Inverse Knudsen Number.

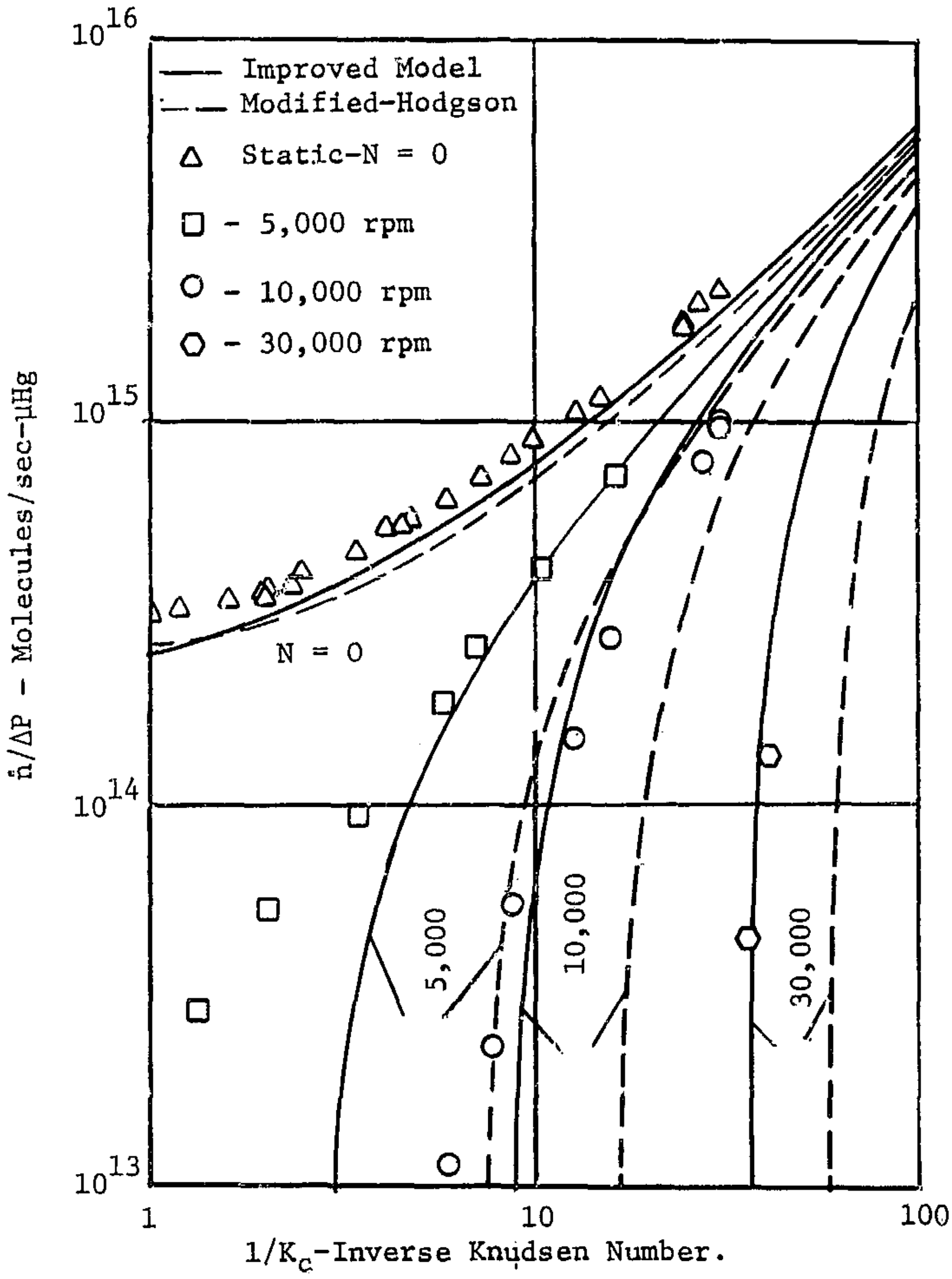

Figure 9. Seal Leakage Flow Versus Inverse Knudsen Number.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

I. Conclusions

This investigation has dealt with a thorough evaluation of the Hodgson model for a viscoseal. Modifications of the model have been made to make it applicable to a multi-threaded seal. From the basic flow equation, expressions have been obtained for seal leakage and sealing coefficient. A technique has been presented for solving the expressions for the pressure difference at zero leakage.

Due to the rather lengthy process of solving the pressure difference equation for the modified-Hodgson model in the slip regime, a simpler model which better lends itself to seal optimization studies has been developed. The improved model formulation results in an explicit determination of the pressure difference at zero leakage, a vast improvement over the Hodgson formulation in which only an implicit determination could be made. The optimization process is demonstrated for a specific application and the results presented.

In the Hodgson and modified-Hodgson models, no attempt was made to account for the clearance effect on the rotor induced flow. The improved model not only takes a more exact approach to the groove flow, but it also attempts to account for the clearance effect on the groove flow.

A comparison of the modified-Hodgson and the improved model with continuum data show that the agreement of the improved model is better. Since the data used represented a wide variation in seal geometries, it appears that the improved model can be used with confidence to predict continuum performance.

Although rarefied experimental viscoseal data are quite limited, a comparison was made with the data obtained by Wilkerson. The improved model in general shows better agreement with these data. Wilkerson did discover a speed dependence in the sealing coefficient, a characteristic which the Hodgson model predicts. The improved model predicts no such speed dependence. Comparison of the models with the leakage data shows that the improved model does a much better job of predicting the leakage for a rotating seal. Since the improved model does not account for the molecular diffusion which previous investigators (15 and 17) have shown to be the primary flow component in highly rarefied flow, it fails to predict the flow at high values of Knudsen number. This can be seen in Figure 9 on page 50 by the failure of the model at higher Knudsen numbers. It can also be seen that both models predict about the same static leakage.

In general, this investigation has shown that the Hodgson model can be modified to account for multiple thread starts, that the model does not lend itself to optimization studies, and that its prediction of leakage rates is poor. An improved model has been shown to correctly predict seal behavior in the continuum and slip regimes and also lends itself to optimization. The major difference between the two models was found to be in the rotor induced flow component.

II. Recommendations

The obvious weakness of the improved model developed in this investigation is its exclusion of molecular diffusion. A model to predict seal behavior in the lower slip and free-molecule regimes must include this effect. Wilkerson (6) presents a model for this flow component in a viscoseal. Combining this component with the improved model in a manner analogous to Weber's (17) long tube formulation and Milligan's (15) annulus treatment would result in a model applicable to all flow regimes.

Although the Hodgson model is applicable to all flow regimes, its rotor induced flow component is inadequate. A further improvement in the Hodgson model would be to replace its rotor induced flow component with the rotor induced flow in the improved model of this investigation.

Regardless of what analytical studies are undertaken in the future, considerable experimentation is needed to validate or invalidate the models now under consideration and those to be developed in the future.

LIST OF REFERENCES

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1. Hodgson, J. W., and M. W. Milligan, "Visco-Type Gas Sealing," ASLE Special Publication SP-2, "Recent Developments in Seal Technology,"pp. 83-87, 1969.

2. Baron, H. W., "Experimenten aan een hydrodynamisch werkende spiraalgroef - asafdichting," De Ingenieur, Vol. 74, No. 32, pp. 144-150, 1962.

3. King, A. E., "Screw Type Shaft Seals for Potassium Lubricated Generators," Supplement to IEEE Transactions on Aerospace, June 1965, pp. 471-479.

4. Milligan, M. W., and H. J. Wilkerson, "Fundamental Study in Low-Density Gas Dynamics," Mechanical and Aerospace Engineering Research Report AE-67-023-5,University of Tennessee, June 1969.

5. Boon, E. F., and S. E. Tal, "Hydrodynamische Dichtung fur rotierende Wellen," Chemie-Ing-Technik, Vol. 31, No. 3, January 31, 1959, pp. 202-212.

6. Wilkerson, H. J., "Rarefied-Gas Viscoseal," Ph.D. Dissertation, The University of Tennessee, Knoxville, Tennessee, 1970.

7. Hodgson, J. N., "SNAP - 8 Seals-to-Space Development Test Program - Volume II - Molecular Pump," NASA-CR-54234, May 1964.

8. Loeb, L. B., The Kinetic Theory of Gases, New York: Dover Publications, 3rd ed., pp. 296-300, 1961.

9. Carley, J. F., R. S. Mallouk, and J. M. McKelvey, "Simplified Flow Theory for Screw Extruders," Industrial and Engineering Chemistry, Vol. 45, No. 5, May 1953, pp. 974-978.

10. Vreeburg, J. P. B., "Investigation of Laminar Flow Patterns and Pressure Generation in a Viscoseal Geometry," Ph.D. Dissertation, The University of Tennessee, Knoxville, Tennessee, 1969.

11. Kennard, E. H., Kinetic Theory of Gases, New York: McGraw-Hill Book Company, Inc., 1938.

12. Berg, P. W., and J. L. McGregor, Elementary Partial Differential Equations, San Francisco: Halden-Day, Inc., 1966, pp. 102-110.

13. Ebert, W. A., and E. M. Sparrow, "Slip Flow in Rectangular and Annular Ducts," ASME Paper 65-FE-20, 1965.

14. Milligan, M. W., and K. E. Patterson, "Rarefied Gas Flow Through Long Square Tubes," Unpublished, Submitted for presentation at 1970 Winter Annual Meeting of American Society of Mechanical Engineers, June 1970.

15. Milligan, M. W., P. W. Cowling, and H. J. Wilkerson, "Fundamental Study in Low-Density Gas Dynamics," Mechanical and Aerospace Engineering Research Report AE-67-023-3, University of Tennessee, November 1967.

16. Stair, W. K., "Theoretical and Experimental Studies of Visco-Type Shaft Seals," Mechanical and Aerospace Engineering Research Report ME-65-587-4, University of Tennessee, October 1965.

17. Weber, S., "Uber Deu Zusammenhang Zwischen Der Laminaren Stromung Der Reineu Gase Durch Rohre Und Dem Selbstcliffusions Koeffizienten," Math-Fysiske Meddlelser, Band 28, No. 2, 1954.

18. Guthrie, A., and R. K. Wakerling, Vacuum Equipment and Techniques, New York: McGraw-Hill Book Company, Inc., 1949.

APPENDICES

APPENDIX A

CONSTANTS FOR HODGSON'S MODEL

The following expressions for the coefficients in Hodgson's model are taken from reference (7).

$$A = \frac{N\pi d b^2 h}{120(b + h)} \cos \alpha$$

$$B = \frac{b h^3 (b + w) K_3}{12\mu\pi d}$$

$$C = \frac{8K_1 (bh)^2 (b + w)}{3\pi d (b + h)} \left(\frac{kT}{2\pi m}\right)^{1/2}$$

$$D = \frac{\pi d c^3 (b + w)}{12\mu w}$$

$$E = \frac{8K_2 c^2 \pi d (b + w)}{3w} \left(\frac{kT}{2\pi m}\right)^{1/2}$$

$$C_1 = \frac{K_3^2 (b + h)^2 h}{8K_1 b\mu \left(\frac{kT}{2\pi m}\right)^{1/2} [2K_1 b - K_3 (b + h)]}$$

$$C_2 = \frac{2K_1 b}{K_3 (b + h)} C_1$$

$$C_3 = \frac{c}{8\mu K_2 (2K_2 - 1) \left(\frac{kT}{2\pi m}\right)^{1/2}}$$

$$C_4 = 2K_2 C_3$$

$$K_1 = \frac{3(1 + a)}{8a^2} \{a \ln (a + (1 + a^2)^{1/2}) + a^2 \ln \frac{1 + (1 + a^2)^{1/2}}{a}$$

$$+ \frac{1}{3} [1 + a^3 - (1 + a^2)^{3/2}]\}$$

$$K_3 = 1 - 0.63 \frac{h}{b} \tanh \frac{\pi b}{2h}$$

Values of K_2

w/c	0.1	0.2	0.4	0.8	1	2	3	4	5	10
K_2	0.036	0.068	0.13	0.22	0.26	0.40	0.52	0.60	0.67	0.94

When $w/c > 10$, $K_2 = \frac{3}{8} \ln \frac{w}{c}$.

The groove flow coefficients, B, C, C_1, and C_2 are obtained from a consideration of flow in a long rectangular tube. The continuum groove flow coefficient, B, is obtained from the continuum solution for Poiseuille flow in a long rectangular tube. From a consideration of purely molecular flow, the free-molecule flow coefficient, C, can be derived. The basic free-molecule flow equation used by Hodgson is given in Reference 18 and is written as

$$Q = \frac{16}{3} K_n \left(\frac{kT}{2\pi m}\right)^{1/2} \frac{O^2}{S} \frac{dP}{d\ell} . \qquad (58)$$

Equation (58) is attributed to Knudsen (18, p. 35), but the constant K_n, which is equal to K_2, is obtained by Clausing (18, p. 40). O and S are the groove cross sectional area and perimeter, respectively. The two remaining groove flow constants, C_1 and C_2, are obtained in the same way Knudsen determined ν and ξ in Equation (1). As was pointed out in Chapter II, the determination of the ratio ν/ξ requires knowledge of the continuum with slip solution. Hodgson does not take an exact approach at this point since he determines C_1/C_2 from the continuum with slip solution for flow between infinite parallel flat plates rather than flow in a long rectangular tube.

Hodgson obtains the land leakage flow coefficients, D, E, C_3, and C_4, from a consideration of flow in a thin slit-like tube. In this case the continuum coefficient, D, is obtained from the continuum solution for Poiseuille flow between parallel flat plates. The same basic model, Equation (58), for free-molecule flow is used to determine the free-molecule flow coefficient, E. K_n now becomes K_1, which is also attributed to Clausing. O and S now represent the cross sectional area of the land leakage passageway and its perimeter, respectively. The constants C_3 and C_4 are also determined using the same approach Knudsen used to determine ν and ξ in Equation (1). For the continuum with slip solution, Hodgson once again uses the continuum with slip solution for flow between infinite parallel flat plates which in this case is the proper solution to use.

APPENDIX B

EQUIVALENCE OF GROOVED HOUSING AND GROOVED SHAFT SOLUTION

The only flow component which requires separate consideration for the grooved housing and grooved shaft configurations is the rotor induced flow. At first it would appear that the same boundary conditions can not be applied in both cases. It would seem that the proper boundary conditions for the grooved shaft would be three moving walls and a stationary upper wall. Consideration of the movement of the fluid in the groove will show that these are improper boundary conditions.

The rotor induced flow component can be considered as the rate of progress of the fluid along the groove due to the rotor motion. The proper boundary conditions are thus obtained from a coordinate system moving with the groove. In this system the boundary conditions are those of a moving upper wall with three stationary walls. These are the same boundary conditions as those for the grooved housing configuration. The two configurations are thus equivalent.

APPENDIX C

THE LAND LEAKAGE PRESSURE GRADIENT

In Chapter III the land leakage pressure gradient is taken to be

$$\frac{dP}{d\ell_L} = \frac{b + w}{w} \cos \alpha \frac{dP}{dL} \tag{40}$$

This expression which is developed by Hodgson (7) can be derived in a much simpler manner than he used.

Figure 10 shows a land-groove pair. The component of the axial pressure gradient, $\frac{dP}{dL}$, perpendicular to the land-groove pair is $\cos \alpha \frac{dP}{dL}$. This represents the pressure gradient from point a to point c and can be represented as

$$\cos \alpha \frac{dP}{dL} = \left.\frac{dP}{d\ell}\right)_{a - c} = \frac{P_a - P_c}{b + w}$$

The land pressure graident, $\frac{dP}{d\ell_L}$, is

$$\frac{dP}{d\ell_L} = \left.\frac{dP}{d\ell}\right)_{b - c} = \frac{P_b - P_c}{w}$$

but from assumption 2 in Chapter III, $P_a = P_b$. By combining the above expressions, it can be shown that Equation (40) is valid.

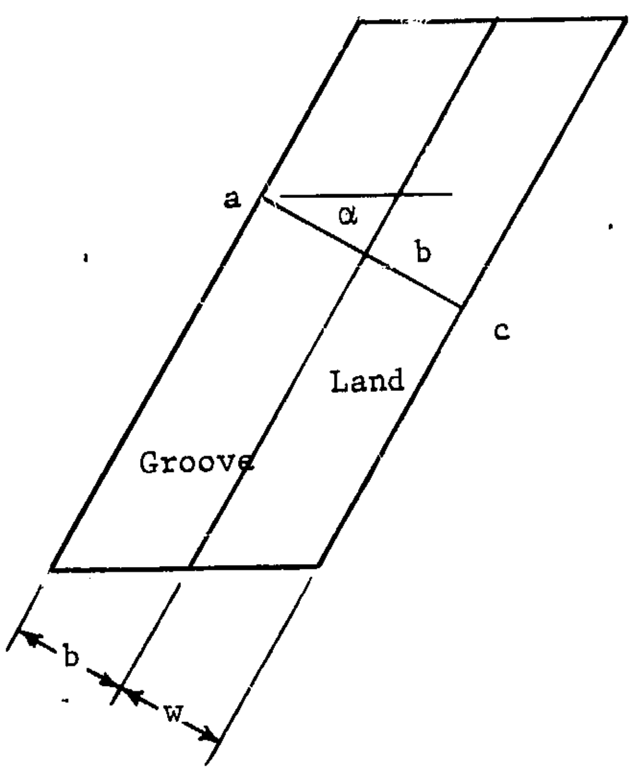

Figure 10. Land Leakage Pressure Gradient.

APPENDIX D
FORTRAN PROGRAMS

```
C
C   THE USER OF THESE PROGRAMS SHOULD TAKE CARE TO ASSIGN
C   THE PROPER VALUE TO ALL SEAL VARIABLES.
C
                         PROGRAM ONE
C                   MODIFIED-HODGSON LEAKAGE
C
C   THIS PROGRAM COMPUTES THE SEAL LEAKAGE BASED ON THE
C   MODIFIED-HODGSON MODEL
C   PL = SEAL LENGTH; ALPHA = HELIX ANGLE; RPM = SEAL SPEED
C   DEL = SEAL CLEARANCE; R2 = K2(APPENDIX 'A'); ANS = NO.
C   OF THREADS; DIA = SEAL DIAMETER; VIS = VISCOSITY OF
C   SEALANT; W = LAND WIDTH; HI = GROOVE DEPTH; BASE =
C   GROOVE WIDTH; TEMP = ABSOLUTE TEMPERATURE; AKO = BOLTZ-
C   MANN CONSTANT; CC = PSI/MICRON OF MERCURY; AMM = MASS PER
C   MOLECULE;
      PL = 4.53
      ALPHA = 9.24/57.3
      N = 1
   16 READ 17,RPM,DEL,R2
   17 FORMAT(F20.1,F20.6,F20.3)
      N = N + 1
      ANS = 16.0
      DIA = 2.00
      VIS = 1.26E-06
      W = 0.03235*COS(ALPHA)
      HI = 0.03065
      BASE = 0.03111*COS(ALPHA)
      TEMP = 535.0
      AKO = 5.655E-24
      CC = 1.935E-05
      AMM = 1.459E-25
      V = SQRT(AKO*TEMP*193.0*12.0/(AMM*3.1416))
      GM = HI/BASE
      FGM = SQRT(1.0+(GM**2))
C   R1 AND R3 IN THIS PROGRAM REFER TO K1 AND K3, RESPEC-
C   TIVELY, IN APPENDIX 'A'.
      R1 = (3.0*(1.0+GM)/(GM**2))*(GM*ALOG(GM+FGM)+(GM**2)*
     1 ALOG((1.0+FGM)/GM)+(1.0/3.0)*(1.0++M**3-FGM**3))/8.0
      R3 = 1.0-0.63*GM*TANH(3.14/(2.0*GM))
C   THE FLOW COEFFICIENTS OF EQUATION 8 ARE NOW COMPUTED
      C1 = (HI*193.0*((R3*(BASE+HI))**2))/(4.0*R1*BASE*VIS*
     1    V*(2.0*R1*BASE -R3*(BASE + HI)))
      C2 =(193.0*R3*HI*(HI+BASE))/(2.0*VIS*V*(2.0*R1*BASE-
     1 R3*(BASE+HI)))
      C3 = (193.0*DEL)/(4.0*VIS*R2*(2.0*R2-1.0)*V)
      C4 = (193.0*DEL)/(2.0*VIS*(2.0*R2-1.0)*V)
      A = ANS*RPM*3.1416*DIA*BASE**2*HI*COS(ALPHA)/
     1 (120.0*(BASE+HI))
```

```
      B = 386.0*BASE*HI**3*(BASE + W)*ANS**2*R3/
     1 (12.0*VIS*3.1416*DIA)
      C= 8.0*R1*(BASE*HI)**2*(BASE+W)*ANS**2*V/
     1 (3.0*3.1416*DIA*(HI+BASE))
      D = (3.14*DIA*(DEL**3)*(W+BASE))/(12.0*VIS*W)*386.0
      E = (8.0*R2*(DEL**2)*3.14*DIA*(W+BASE)*V)/(3.0*W)
      R4 =      (C*C1/C2)+(E*C3/C4)
      R5 = C*(C2-C1)/(C2**2)
      R6 = E*(C4-C3)/(C4**2)
      PRINT 79,RPM,DEL
   79 FORMAT(1H1,6HRPM = ,F10.1,12HCLEARANCE = ,F10.6)
      PRINT 101
  101 FORMAT(1H0,10X,5HPD/PE,12X,4H1/NK,14X,4HPBAR,13X,
     1   9H STATIC   ,14X,5HROTOR,14X,8HNET FLOW)
    1 EE = -2.0
    2 VV = 1.0
C   ANKINV IS THE INVERSE KNUDSEN NUMBER BASED ON CLEARANCE
    3 ANKINV = VV*10.0**EE
      PBAR = ANKINV*VIS*V*3.1416/(DEL*386.0)
C   RP = SEAL PRESSURE RATIO
      RP = 20.0
   13 CONTINUE
      Z2 = 1.0 + 2.0*C2*PBAR
      Z4 = 1.0 + 2.0*C4*PBAR
C   FLOW IS THE SPECIFIC MOLECULAR FLOW RATE IN A STATIC SEAL
      FLOW = (1.0/(12.0*AKO*TEMP*PL))*CC*((B+D)*PBAR + R4
     1    + (0.5/PBAR)*((RP+1.0)/(RP-1.0))*(R5*ALOG((RP*Z2+
     2   1.0)/(RP+Z2))+R6*ALOG((RP*Z4+1.0)/(RP+Z4))))
      PBARP = PBAR/CC
C   FLOWR IS THE ROTOR INDUCED SPECIFIC MOLECULAR FLOW RATE
      FLOWR = A*(RP + 1.0)*CC/(2.0*AKO*TEMP*(RP - 1.0)*12.0)
C   FLOWT IS THE NET SPECIFIC MOLECULAR FLOW RATE IN A
C   ROTATING SEAL
      FLOWT = FLOW - FLOWR
      PRINT 40,RP,ANKINV,PBARP,FLOW,FLOWR,FLOWT
   40 FORMAT(1H0,10X,F6.1,5X,E13.3,5X,E13.3,5X,E16.4,5X,
     1 E16.4,5X,E16.4)
      RP = RP + 20.0
      IF(RP.LT.220.0)GO TO 13
      VV = VV + 1.0
      IF(VV-9.0)3,3,4
    4 EE = EE + 1.0
      IF(EE- 3.0)2,5,5
    5 CONTINUE
      IF(N.LT.5) GO TO 16
    6 CONTINUE
      CALL EXIT
      END
$ENTRY
    0.0                  0.00418              0.83
    5000.0               0.00385              0.87
```

```
$IBFTC
C                               PROGRAM TWO
C
C    MODIFIED-HODGSON PRESSURE DIFFERENCE AT ZERO FLOW
C
C   THIS PROGRAM SOLVES EQUATION 9 FOR THE PRESSURE DIFFER-
C   ENCE AT ZERO FLOW FOR FLOW REGIMES FROM CONTINUUM TO
C   FREE MOLECULE FLOW AND OUTPUTS THE RESULTS IN THE FORM
C   OF A SEALING COEFFICIENT
C
C   THIS READ STATEMENT INPUTS THE VALUES OF K2 FROM APPENDIX
C   A INTO AN ARRAY SO THAT INTERPOLATION MAY BE USED TO
C   DETERMINE PROPER VALUE OF K2;X = W/DEL; Y = K2
      DIMENSION X(11),Y(11)
      READ 800,(X(J),J=1,11)
  800 FORMAT(11F7.3)
      READ 801,(Y(J),J=1,11)
  801 FORMAT(11F7.3)
      DELA = 0.0
C   PL = SEAL LENGTH; ANS = NO. OF THREADS; DIA = SEAL DIA-
C   METER; RPM = SEAL SPEED; VIS = VISCOSITY OF SEALANT;
C   GC= G SUB C; AKO = BOLTZMANN CONSTANT; AMM = MASS PER
C   MOLECULE; TEMP = ABSOLUTE TEMPERATURE; DEL = SEAL CLEAR-
C   ANCE; W = LAND WIDTH; HI = GROOVE DEPTH; BASE = GROOVE
C   WIDTH; AA = GROOVE ASPECT RATIO.
      PL = 4.53
      ANS = 4.0
      DIA = 2.00
      RPM = 1.0E04
      VIS = 1.26E-06
      GC = 386.0
      AKO = 6.80E-23
      AMM = 1.459E-25
      TEMP = 535.0
      DEL = 0.004
      W = 0.0125
      HI = 0.0125
      BASE = 0.0125
      V = SQRT(AKO*TEMP*GC/(2.0*3.1416*AMM))
      AA = BASE/HI
      DP1 = 0.0
      DP2 = 0.0
      DP3 = 0.0
      ANK1 = 0.0
      ANK2 = 0.0
      ANK3 = 0.0
C   STATEMENTS 8 THROUGH 200 INTERPOLATE TO FIND THE PROPER
C   VALUE OF K2 WHICH IS HERE GIVEN THE SYMBOL R2
    8 THETA = W/DEL
      IF(THETA.GT.10.0)GO TO 199
      DO 110 J=1,11
```

```
      IF(THETA-X(J)) 112,111,110
  110 CONTINUE
  111 R2 = Y(J)
  112 R2 = Y(J-1)+(Y(J)-Y(J-1))/(X(J)-X(J-1))*(THETA-X(J-1))
      GO TO 200
  199 R2 = 3.0*ALOG(THETA)/8.0
  200 CONTINUE
C  THE FLOW COEFFICIENTS OF EQUATION 9 ARE NOW COMPUTED
C  R1 AND R3 IN THIS PROGRAM REFER TO K1 AND K3, RESPEC-
C  TIVELY, IN APPENDIX 'A'.
      GM = HI/BASE
      FGM = SQRT(1.0+(GM**2))
      R1 = (3.0*(1.0+GM)/(GM**2))*(GM*ALOG(GM+FGM)+(GM**2)*
     1 ALOG((1.0+FGM)/GM)+(1.0/3.0)*(1.0+GM**3-FGM**3))/8.0
      R3 = 1.0-0.63*GM*TANH(3.14/(2.0*GM))
      ALPHA = ATAN(ANS*(BASE+W)/(3.1416*DIA))
      ALPHAP = ALPHA*57.3
      C1 = (HI*193.0*((R3*(BASE+HI))**2))/(4.0*R1*BASE*VIS*
     1   V*(2.0*R1*BASE -R3*(BASE + HI)))
      C2 =(193.0*R3*HI*(HI+BASE))/(2.0*VIS*V*(2.0*R1*BASE-
     1 R3*(BASE+HI)))
      C3 = (193.0*DEL)/(4.0*VIS*R2*(2.0*R2-1.0)*V)
      C4 = (193.0*DEL)/(2.0*VIS*(2.0*R2-1.0)*V)
      A = ANS*RPM*3.1416*DIA*BASE**2*HI*COS(ALPHA)/
     1 (120.0*(BASE+HI))
      B = 386.0*BASE*HI**3*(BASE + W)*ANS**2*R3/
     1 (12.0*VIS*3.1416*DIA)
      C= 8.0*R1*(BASE*HI)**2*(BASE+W)*ANS**2*V/
     1 (3.0*3.1416*DIA*(HI+BASE))
      D =((3.14*DIA*(DEL**3)*(W+BASE))/(12.0*VIS*W))*386.0
      E = (8.0*R2*(DEL**2)*3.14*DIA*(W+BASE)*V)/(3.0*W)
      F = C + E
      G = C*((C1/C2) - 1.0)
      H = E*((C3/C4) - 1.0)
      PRINT 100,BASE,HI,W,DIA,DEL,ALPHAP
 100  FORMAT(1H1,10X,6HBASE= ,F7.4,5X,4HHI= ,F7.4,5X,3HW= ,
     1 F7.4,5X,5HDIA= ,F7.4,5X,5HDEL= ,F7.4,5X,7ALPHA= ,
     2 F8.4/1H0,10X,15HINVERSE KNUDSEN,10X,
     3 19HSEALING COEFFICIENT)
C  DELP IS THE PRESSURE DIFFERENCE AT ZERO FLOW
C  THE CONTINUUM DELP IS COMPUTED
      DELP = A*PL/(B+D)
      EEE = 2.0
    2 VVV=10.0
C  ANKINV IS THE INVERSE KNUDSEN NUMBER BASED ON CLEARANCE
    3 ANKINV = VVV*10.0**EEE
C  THE AVERAGE SEAL PRESSURE IS COMPUTED FROM THE KNUDSEN
C  NUMBER
   77 PBAR = VIS*V*3.1416*ANKINV/(386.0*DEL)
C  IF THE INVERSE KNUDSEN NUMBER IS ABOVE 30.0 THE LINEAR
C  APPROXIMATION OF THE NEXT DELP IS OMITTED
```

```
      IF(ANKINV.GT.30.0) GO TO 30
C   IF THE INVERSE KNUDSEN NUMBER IS LESS THAN 30.0 THE NEXT
C   DELP IS APPROXIMATED BY A LINEAR EXTRAPOLATION
      DELP =EXP(ALOG(DP2)+ALOG(ANKINV/ANK2)*ALOG(DP3/DP2)/
     1 ALOG(ANK3/ANK2))
C   THE NEXT SECTION OF THE PROGRAM UP TO STATEMENT 40
C   EMPLOYS NEWTON'S METHOD TO SOLVE EQUATION 9 FOR DELP
   30 Z2 = 1.0 + C2*PBAR
      Z4 = 1.0 + C4*PBAR
      PE = PBAR - DELP/2.0
      Z=((2.0*PBAR+DELP)/(2.0*PBAR-DELP))**F*((2.0*Z2+C2*
     1 DELP)/(2.0*Z2-C2*DELP))**G*((2.0*Z4+C4*DELP)/(2.0*
     2 Z4-C4*DELP))**H*EXP(B*DELP)*EXP(D*DELP)
      U = Z - EXP(A*PL)
      DY=Z*(B+D+4.0*PBAR*F/(4.0*PBAR**2-DELP**2)+4.0*Z2*C2*G
     1 /(4.0*Z2**2-(C2*DELP)**2)+4.0*Z4*C4*H/(4.0*Z4**2-(C4*
     2 DELP)**2))
      DELP1 = DELP - U/DY
      PO = PE + DELP
      IF(ABS(DELP1-DELP).LT.1.0E-07) GO TO 40
      DELP = DELP1
      GO TO 30
C   WITH DELP  DETERMINED THE SEALING COEFFICIENT IS COMPUTED
   40  ALAMBD = VIS*3.1416*DIA*RPM*PL/(3860.0*DELP*(DEL**2))
      PRINT 50,ANKINV,ALAMBD
   50 FORMAT(1H0,13X,F10.4,16X,F10.3)
      DP1 = DP2
      DP2 = DP3
      DP3 = DELP
      ANK1 = ANK2
      ANK2 = ANK3
      ANK3 = ANKINV
      IF(ANKINV.GT.30.0)GO TO 76
C IF THE INVERSE KNUDSEN NUMBER IS BETWEEN 16 AND 30 THE
C  INCREMENTAL DECREASE IN THE INVERSE KNUDSEN NUMBER IS
C  REDUCED TO 1.0
      IF(ANKINV.LE.30.0.AND.ANKINV.GT.16.0)DELA = 1.0
C IF THE INVERSE KNUDSEN NUMBER IS LESS THAN 16 THE
C  INCREMENTAL DECREASE IN THE INVERSE KNUDSEN NUMBER IS
C  REDUCED TO 0.25
      IF(ANKINV.LE.16.0)DELA = 0.25
C   THE SOLUTION WILL USUALLY BECOME THE LINEAR FREE MOLECULE
C   SOLUTION BEFORE AN INVERSE KNUDSEN NUMBER OF 1.0 IS
C   REACHED, THE SOLUTION IS THEREFORE TERMINATED.
      IF(ANKINV.LT.1.0)GO TO 5
      ANKINV = ANKINV - DELA
      GO TO 77
   76 VVV = VVV-1.0
      IF(VVV-2.0)4,3,3
    4 EEE = EEE-1.0
      IF(EEE+4.0)5,5,2
```

```
    5 CONTINUE
      CALL EXIT
      END
C   DATA CARDS HAVE BEEN OMITTED FROM THIS LISTING
```

```
$IBFTC
C                          PROGRAM THREE
C
C   LEAKAGE RATE AND SEALING COEFFICIENT FOR IMPROVED MODEL
C
C   THIS PROGRAM COMPUTES THE SPECIFIC MOLECULAR FLOW RATE
C   AND SEALING COEFFICIENT BASED ON THE IMPROVED MODEL
C
C   EXECUTE IN DOUBLE PRECISION
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ALPHA(50,50),ANKIC(50),ANKIH(50),ANKC(50),
     1   ANKH(50),ALAMB(50)
C   BASE = GROOVE WIDTH; H = GROOVE DEPTH; ANS = NO. OF
C   THREADS; W = LAND WIDTH; DIA = SEAL DIAMETER; C=CLEARANCE
C   A = GROOVE ASPECT RATIO; AN = ASPECT RATIO OF HYPOTHET-
C   ICAL GROOVE; PL = SEAL LENGTH; AK = SEAL RADIUS RATIO;
C   AMM = MASS/MOLECULE; AKO = BOLTZMANN CONSTANT; GC=G SUB C
C   YU = LOCATION OF ACTUAL GROOVE TOP IN HYPOTHETICAL GROOVE
C   TEMP = ABSOLUTE TEMPERATURE; C1 = MICRONS OF MERCURY PER
C   PSI; ANU = VISCOSITY OF SEALANT
 888  READ 1,BASE,H,ANS,W,DIA,C
    1 FORMAT(6D12.4)
      K = 1
      A = BASE/H
      AN = BASE/(H + C)
      PL = 4.53
      AK = 1.0 - 2.0*C/DIA
      AMM = 1.459E-25
      AKO = 0.680E-22
      GC = 386.0
      YU = H/(H + C)
      TEMP = 537.0
      C1 = 51700.0
      ANU = 0.126D-05
      EE = -2.0
      VV = 1.0
      DO 4 N=1,45
C   ANKIC = INVERSE KNUDSEN NUMBER BASED ON CLEARANCE
      ANKIC(N) = VV*10.0**EE
      VV = VV+1.0
      IF(VV.LT.10.0)GO TO 4
      EE = EE + 1.0
      VV = VV - 9.0
    4 CONTINUE
C   THE DO LOOP ENDING WITH STATEMENT 10 COMPUTES THE
C   EIGENVALUES FROM EQUATION 20 USING NEWTONS METHOD OF
C   APPROXIMATING ROOTS
      DO 10 N=1,45
C   ANKIH = INVERSE KNUDSEN NUMBER BASED ON GROOVE DEPTH
      ANKIH(N) = H*ANKIC(N)/C
      CC = 0.5*A*ANKIH(N)
```

```
      ALPHA(N,1) = 1.5707
      DO 9 J = 1,15
    7 FA = ALPHA(N,J)*DTAN(ALPHA(N,J)) - CC
      FPA=DTAN(ALPHA(N,J))+ALPHA(N,J)/(DCOS(ALPHA(N,J))**2)
      ALPH = ALPHA(N,J) - FA/FPA
      IF(DABS(FA/FPA).LT.1.0E-05)GO TO 8
      ALPHA(N,J) = ALPH
      GO TO 7
    8 ALPHA(N,J+1) = ALPHA(N,J) + 3.14159
    9 CONTINUE
   10 CONTINUE
      DO 13 N = 1,45
      PRINT 11,ANKIC(N),ANKIH(N),A
   11 FORMAT(1H0,5X,D10.2,5X,D10.2,5X,D10.4)
      PRINT 12,(ALPHA(N,J),  J=1,15)
   12 FORMAT(15F8.4)
   13 CONTINUE
C  RPM = SEAL SPEED, IF ONLY SEALING COEFFICIENT IS DESIRED
C  ANY VALUE OF RPM MAY BE USED HERE
      RPM = 10000.0
  105 PRINT 106,RPM
  106 FORMAT(1H0,10X,6HRPM = ,D20.3)
C  SINA = SINE OF HELIX ANGLE; COSA = COSINE OF HELIX ANGLE
      SINA = ANS*(BASE+W)/(3.1416*DIA)
      COSA = DSQRT(1.0- SINA**2)
C  THE FLOW COEFFICIENTS FOR LEAKAGE AND SEALING ARE NOW
C  COMPUTED
      AU = 64.0*C**2*BASE*H*COSA*ANS
      AAC =(3.0*3.1416*DIA**4*(BASE + W)/W)*COSA
      AP = 192.0*ANS*BASE**3*H*SINA
      A1 = AAC/(384.0*ANU*PL)
      A2 = AP/(384.0*ANU*PL)
      A3 = AU*(3.1416*DIA*RPM     )/(C**2*7680.0)
      DO 23 N=1,45
      ANKC(N) = 1.0/ANKIC(N)
      AC = 1.0 - AK**4 + (1.0-AK**2)**2/DLOG(AK) - (2.0*
     1 ANKC(N)*(1.0-AK**2)/(AK*DLOG(AK)-ANKC(N)*(1.0-AK**2
     2 )))*(2.0*AK*(AK**2-1.0)-2.0*AK*DLOG(AK)*(AK**2-AK+
     3 1.0)-(1.0-AK**2)**2/(2.0*DLOG(AK))+2.0*(1.0-AK**2)
     4 *(1.0-AK)**2*ANKC(N))
      ANKH(N) = 1.0/ANKIH(N)
C  SUMP = SUMMATION ASSOCIATED WITH THE GROOVE PRESSURE
C  FLOW; SUM4 = SUMMATION ASSOCIATED WITH THE ROTOR VELOCITY
C  CORRECTION; SUMU = SUMMATION ASSOCIATED WITH THE ROTOR
C  INDUCED FLOW
      SUMP = 0.0
      SUMU = 0.0
      SUM4 = 0.0
      DO 22 J=1,10
      TERM1=DTANH(ALPHA(N,J)/A)/(1.0+2.0*ALPHA(N,J)*ANKH(N)*
     1 DTANH(ALPHA(N,J)/A)/A)
```

```
      TERM2 =DSIN(ALPHA(N,J))**2/(1.0+2.0*ANKH(N)*
     1  DSIN(ALPHA(N,J))**2)
      TERMP =  (A/(ALPHA(N,J)**5))*TERM2*(ALPHA(N,J)/A-TERM1)
      TERMU =  (A/(ALPHA(N,J)**3))*TERM2*TERM1
      SUMP = SUMP + TERMP
      SUMU = SUMU + TERMU
      TER1=DSIN(ALPHA(N,J))**2/((1.0+2.0*ANKH(N)*
     1  DSIN(ALPHA(N,J))**2/A)*ALPHA(N,J)**2)
      TER2 =DCOSH(2.0*ALPHA(N,J)*YU/AN)/DCOSH(2.0*ALPHA(N,J)
     1  /AN)
      TER3 =(2.0*ANKH(N)*ALPHA(N,J)/A  +DTANH(2.0*ALPHA(N,J)
     1  YU/AN))/((1.0+(2.0*ANKH(N)*ALPHA(N,J)/A)**2)*DTANH(
     2  2.0*ALPHA(N,J)/AN)+4.0*ANKH(N)*ALPHA(N,J)/A)
      TER4 = TER1*TER2*TER3
      SUM4 = SUM4 + TER4
   22 CONTINUE
C  RU IS THE ROTOR VELOCITY CORRECTION FACTOR
      RU = 2.0*SUM4
      SUMU = RU * SUMU
C  RP, THE SEAL PRESSURE RATIO VARIES FROM 20.0 TO 200.0
C  IN STEPS OF 20.0
      RP = 20.0
C  PBAR IS THE AVERAGE SEAL PRESSURE
      PBAR =ANU*DSQRT(2.0*3.1416*AKO*TEMP*GC/AMM)/
     1  (2.0*ANKC(N)*C)
C  ANDDP IS THE SPECIFIC MOLECULAR FLOW RATE
  100 ANDDP=(PBAR/(C1*AKO*TEMP))*(A1*AC+A2*SUMP-A3*(RP+1.0)
     1  *SUMU/(PBAR*(RP-1.0)))
      PRINT 99,ANKIC(N),RP,ANDDP,RU
   99 FORMAT(1H0,10X,D10.2,5X,D10.2,5X,D20.5,5X,D20.5)
      RP = RP + 20.0
      IF(RP.NE.220.0) GO TO 100
C  ALAMB IS THE SEALING COEFFICIENT
      ALAMB(N) = (AAC*AC + AP*SUMP)/(AU*SUMU)
  23  CONTINUE
      PRINT 14,DIA,C,BASE,W,H,ANS
  14  FORMAT(1H1,5HDIA =,D11.4,2X,11HCLEARANCE =,D11.4,2X,
     2  11HGROOVEWID =,    D11.4,2X,12HLAND WIDTH =,D11.4,2X,
     3 7HDEPTH =,          D11.4,2X,9HGROOVES =,D9.2/1H0,5X,
     4   15HINVERSE KNUDSEN,5X,6HLAMBDA)
      DO 25 N = 1,45
      PRINT 24,ANKIC(N),ALAMB(N)
  24  FORMAT(1H0,7X,D20.4,7X,D20.4)
   25 CONTINUE
  26  CONTINUE
      GO TO 888
      CALL EXIT
      END
C  DATA CARDS HAVE BEEN OMITTED FROM THIS LISTING
```

```
C
                            PROGRAM FOUR
C
C          OPTIMIZATION USING THE IMPROVED MODEL
C
C THIS PROGRAM CARRIES OUT THE OPTIMIZATION FOR A 2.0 INCH
C DIAMETER VISCOSEAL WHICH HAS A CLEARANCE OF 0.004 INCH
C AND OPERATES AT A KNUDSEN NO. BASED ON THE CLEARANCE OF .2
C
C  EXECUTE IN DOUBLE PRECISION
       IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ALAM(50),ANKIH(50),ANKH(50),H(50),ALPHA(50),
     1 A(50),AN(50),YU(50),SP(50),SU(50)
      L = 1
C  BASE = LOWER LIMIT ON GROOVE WIDTH;DIA = SEAL DIAMETER;
C  C = CLEARANCE;ANKIC = INVERSE KNUDSEN NUMBER
  888 READ 1,BASE,DIA,C,ANKIC
  1   FORMAT(4D15.4)
C  AK = THE SEAL RADIUS RATIO; AC = THE DIMENSIONLESS
C  ANNULUS FLOW COEFFICIENT
 721  AK = 1.0 - 2.0*C/DIA
      PRINT 722,BASE,DIA,C,ANKIC
 722  FORMAT(1H0,5HBASE=,F10.4,5X,4HDIA=,F8.2,5X,2HC=,F10.4,
     1  5X,5HI/NK=,F12.3)
C  THE NEXT SECTION THROUGH CARD NO. 9 COMPUTES THE
C  EIGENVALUES FOR THE SUMMATIONS
      CC = 0.5*BASE*ANKIC/C
      ALPHA(1) = 1.5707
      DO 9 J=1,4
  7   FA = ALPHA(J)*DTAN(ALPHA(J)) - CC
      FPA = DTAN(ALPHA(J)) + ALPHA(J)/(DCOS(ALPHA(J))**2)
      ALPH = ALPHA(J) - FA/FPA
      IF(DABS(FA/FPA).LT.1.0E-05)GO TO 8
      ALPHA(J) = ALPH
      GO TO 7
  8   ALPHA(J+1) = ALPHA(J) + 3.14159
  9   CONTINUE
      ANKC = 1.0/ANKIC
      AC = 1.0 - AK**4 + (1.0-AK**2)**2/DLOG(AK) - (2.0*ANKC
     1  *(1.0-AK**2)/(AK*DLOG(AK)-ANKC*(1.0-AK**2)))*(2.0*AK
     2  *(AK**2-1.0)-2.0*AK*DLOG(AK)*(AK**2-AK+1.0)-(1.0-
     3  AK**2)**2/(2.0*DLOG(AK))+2.0*(1.0-AK**2)*(1.0-AK)**2
     4  *ANKC)
C  H = GROOVE DEPTH;ANKH = GROOVE KNUDSEN NUMBER;A =  THE
C  GROOVE ASPECT RATIO; AN = ASPECT RATIO OF HYPOTHETICAL
C  GROOVE OF DEPTH H+C; YU = ACTUAL GROOVE DEPTH IN
C  HYPOTHETICAL GROOVE
      DO 10 J=1,20
      H(J) = DFLOAT(J)*0.001
      ANKH(J) = C/(ANKIC*H(J))
      A(J) = BASE/H(J)
```

```
      AN(J) =BASE/(H(J) + C)
      YU(J) = H(J)/(H(J) + C)
C  SUMP = SUMMATION ASSOCIATED WITH THE GROOVE PRESSURE
C  FLOW; SUM4 = SUMMATION ASSOCIATED WITH THE ROTOR VELOCITY
C  CORRECTION; SUMU = SUMMATION ASSOCIATED WITH THE ROTOR
C  INDUCED FLOW
      SUMP = 0.0
      SUM4 = 0.0
      SUMU = 0.0
      DO 12 K=1,4
      AL = ALPHA(K)
      TER1 =DSIN(AL)**2/((1.0+2.0*ANKH(J)*DSIN(AL)**2/
     1  A(J))*AL**2)
      TER2 = DCOSH(2.0*AL*YU(J)/AN(J))/DCOSH(2.0*AL/AN(J))
      TER3 = (2.0*ANKH(J)*AL/A (J) + DTANH(2.0*AL*YU(J)/
     1  AN(J)))/((1.0+(2.0*ANKH(J)*AL/A(J))**2)*DTANH(2.0
     2 *AL/AN(J))+ 4.0*ANKH(J)*AL/A(J))
      TER4 = TER1*TER2*TER3
      TERM1 = DTANH(AL/A(J))/(1.0 + 2.0*AL*ANKH(J)*
     1  DTANH(AL/A(J))/A(J))
      TERM2 = DSIN(AL)**2/(1.0 + 2.0*ANKH(J)*DSIN(AL)**2)
      TERMP = (A(J)/(AL**5))*TERM2*(AL/A(J) - TERM1)
      TERMU = (A(J)/(AL**3))*TERM2*TERM1
      SUMP = SUMP + TERMP
      SUMU = SUMU + TERMU
      SUM4 = SUM4 + TER4*2.0
   12 CONTINUE
      SP(J) = SUMP
      SU(J) = SUMU*SUM4
   10 CONTINUE
C  THE OPTIMUM HELIX ANGLE FOR EACH H IS COMPUTED
C  USING NEWTON'S METHOD OF APPROXIMATING ROOTS
      DO 27 J=1,20
      AKANG = DIA**3*AC/(16.0*BASE**2*H(J)*SP(J))
      ANG = 3.14159/8.0
   55 FANG = AKANG*DCOS(ANG)**3 + DCOS(ANG)**2 - 1.0
      FPANG = -3.0*AKANG*DCOS(ANG)**2*DSIN(ANG) -
     1  2.0*DCOS(ANG)*DSIN(ANG)
      ANG1 = ANG - FANG/FPANG
      IF(DABS(ANG-ANG1).LT.1.0E-05) GO TO 56
      ANG = ANG1
      GO TO 55
   56 CONTINUE
      AAC = 6.0*3.14159*DIA**4*DCOS(ANG)
      AU = 32.0*C**2*H(J)*3.14159*DIA*DCOS(ANG)*DSIN(ANG)
      AP = 96.0*3.14159*DIA*BASE**2*H(J)*DSIN(ANG)**2
C  ALAM = SEALING COEFFICIENT; ANG1 = HELIX ANGLE
      ALAM(J) = (AAC*AC + AP*SP(J))/(AU*SU(J))
      ANG1 = ANG*180.0/3.14159
      PRINT 29,ALAM(J),H(J),ANG1,AKANG,SP(J),AC,SU(J),SUM4
   29 FORMAT(1H0,8D13.5)
```

```
  27  CONTINUE
C  A NEW VALUE OF BASE IS INTRODUCED AND THE PROCESS
C  IS REPEATED UNTIL THE UPPER LIMIT FOR BASE IS REACHED
      BASE = BASE + 0.005
      IF(BASE.LT.0.50)GO TO 721
   69 CONTINUE
      L = L + 1
      IF(L.NE.4) GO TO 888
      CALL EXIT
      END
C  A REVIEW OF THE OUTPUT WILL RESULT IN THE DETERMINATION
C  OF THE BEST CONFIGURATION
$ENTRY
  0.010D 00     0.200D 01        0.400D-02        0.0500D 02
```