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TECHNICAL NOTE R-165

EQUATIONS GOVERNING THE TWO-DIMENSIONAL DYNAMIC BEHAVIOR OF A LIQUID SURFACE IN A REDUCED GRAVITY FIELD

by E. H. Ingram

October 1965

RESEARCH LABORATORIES

BROWN ENGINEERING COMPANY, INC.

HUNTSVILLE, ALABAMA

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EQUATIONS GOVERNING THE TWO-DIMENSIONAL, DYNAMIC BEHAVIOR OF A LIQUID SURFACE IN A REDUCED GRAVITY FIELD

October 1965

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ABSTRACT

The equations for calculating the two-dimensional, unsteady, potential motion of a near-weightless liquid in a partially filled container are derived and discussed. The simplifications of the surface equations that must be made before practical solutions can be obtained by finite difference methods are discussed. It is not necessary that the surface equations be linearized if a central or forward time difference can be used.

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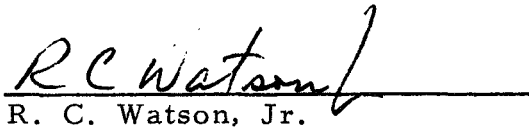

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Vice President
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LIST OF SYMBOLS

B	Bond number
F_r	Froude number
g	The effective acceleration of gravity, ft/sec^2
g_o	Standard acceleration of gravity, ft/sec^2
H	Height of static equilibrium surface, ft
h	Height of liquid surface, ft
P_o	Pressure of gas or vapor, lbf/ft^2
R	Radius of curvature, ft
T	Surface tension, lbf/ft
t	Time, sec
U	A characteristic velocity, ft/sec
U_s	Source velocity, ft/sec
u	Component of velocity in x-direction, ft/sec
V	Dimensionless group, $U/(g\tau)$
v	Component of velocity in y-direction, ft/sec
w	A characteristic length, ft
x, y	Orthogonal rectangular coordinates, ft
η	Difference between dynamic and static surface heights, ft
ρ	Density of fluid, slugs/ft^3
τ	A characteristic time, sec
ϕ	Potential function, ft^2/sec

Subscripts

g	Denotes gas
ℓ	Denotes liquid
s	Denotes solid

INTRODUCTION

A variety of problems have arisen concerning the static and dynamic behavior of liquids in a weightless or near-weightless condition. Many of these problems are concerned with the operation of liquid-propellant vehicles in a reduced gravity field. The main propellant tanks in these vehicles usually contain a large liquid-vapor interface that is subject to a number of disturbances, such as acceleration perturbations, jolts, disturbances in propellant feed lines, etc. When the confined liquid is almost weightless (relative to the space vehicle), one or more of these disturbances, if sufficiently severe, can cause violent reactions at the liquid surface. Furthermore, since most of the common fuels have small viscosities, the resulting motion may persist for a long period of time. Loss of fuel and/or oxidizer through tank vents, stability difficulties in the control of the space vehicle, and liquid-vapor mixing (which could affect the performance of the pumps upon engine restart) are some of the problems that could result from the sloshing of near-weightless liquids.

This is one of several reports being issued by the Research Laboratories, Brown Engineering Company, on the dynamic response of an almost-weightless liquid in a partially-filled tank to an oscillatory fluid disturbance. An oscillatory fluid disturbance occurs, for example, when the valves in the fuel or oxidizer feed lines are suddenly closed (or opened). The behavior of liquid hydrogen in the partially-filled tank of the Saturn SIV-B stage is of particular interest. This report is concerned with the derivation and discussion of the equations for calculating the two-dimensional (three-dimensional including time), dynamic behavior of a liquid-surface when the gravitational acceleration (relative to the tank) is small and the effects of surface tensions are important. The simplifications of the surface conditions that must be made in order to obtain practical numerical solutions by finite difference methods are discussed.

DISCUSSION OF PROBLEM

The calculation of the dynamic behavior of an almost-weightless liquid in a partially-filled container differs from the usual boundary-value problem in hydrodynamics in at least two important respects. First of all, the fluid is exposed to a new environment - that of a near-zero, effective-gravity, so that the effects of surface tensions are generally important. Secondly, a nonlinear boundary condition is to be satisfied along the liquid-gas interface; however, the shape and location of this surface is not known a priori and must therefore be determined as a part of the solution. Both of these characteristics greatly complicate the problem.

There are extensive discussions of the dynamics of free surfaces* for an unbounded surface (primarily concerned with the study of water waves, e.g., see Lamb[1] or Stoker[2]) and for a bounded surface when the effects of surface tensions can be ignored. However, very few studies of the dynamic behavior of bounded, liquid surfaces in a greatly reduced gravity field have been reported. These studies are mainly concerned with the analysis of the free oscillations of liquids in various containers. Methods for calculating limits of stability and natural frequencies (which are obtained by solving an eigenvalue problem based on a set of linearized surface conditions) dominate the literature. One of the most thorough analyses of free oscillations is given by Satterlee[3]. A good discussion of the fundamentals of hydrodynamics at low g is given by Reynolds[4]. Reynolds also gives an extensive list of references.

The study that is presently underway differs from those in the literature in two respects. First of all, the study is principally concerned

*A "free surface" is defined as a fluid surface along which no tangential stresses act and the pressure is constant. This term would seemingly not apply when surface tensions are considered since a pressure gradient along the liquid side of the liquid-gas interface exists; however, the term is commonly used for this case also.

with the forced oscillation of a near-weightless liquid in a partially-filled container. The calculation of natural frequencies is of secondary importance in this study. The second difference concerns the surface conditions. The surface equations are simplified for the case of small amplitude perturbations about the static equilibrium surface but the nonlinear terms are retained. These surface equations are more general than the usual linearized set and are discussed in this report.

The objective of the present study is the development of a numerical program for calculating velocities (throughout the flow field) and amplitudes versus time that are the direct result of a specified forced fluid disturbance. A specific problem which is being solved numerically is illustrated in Figure 1.

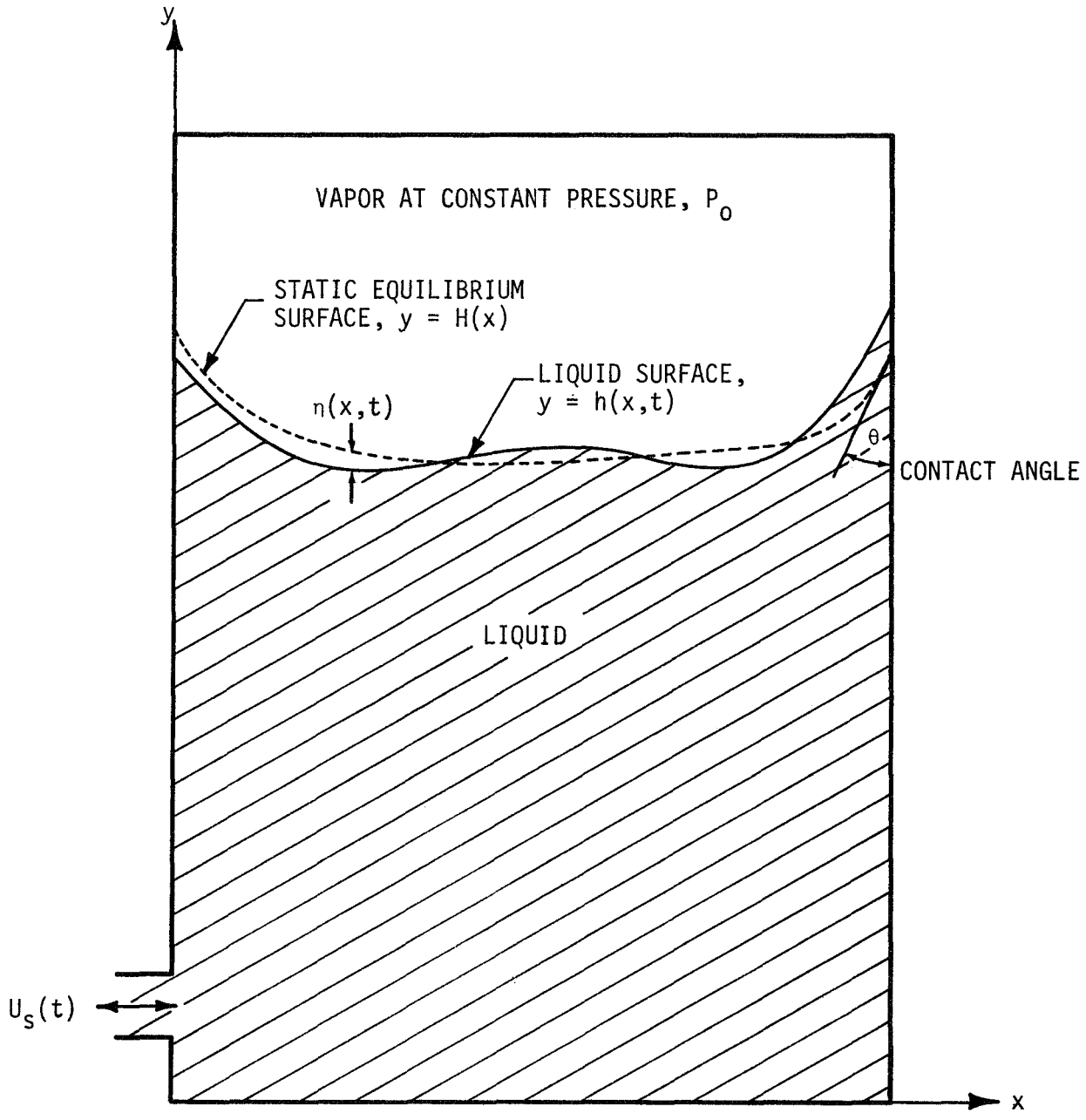

Figure 1. Geometry of Flow Model

SURFACE CONDITIONS

The dynamic and kinematic surface conditions are briefly discussed in this section for the case of a near-weightless liquid that is bounded by a gas at constant pressure above its surface and confined by rigid plane walls on the other sides, as illustrated in Figure 2. The effective acceleration of gravity (i.e., the acceleration of gravity relative to the container) is assumed to act in the negative y-direction. The unsteady (but not necessarily periodic), two-dimensional, inviscid motion of an incompressible, homogeneous fluid is assumed.

KINEMATICAL SURFACE CONDITION

Consider a plane, liquid surface, e.g., $y = h(x, t)$ in Figure 2, that is moving with the local fluid velocity at all points. The liquid and gas are assumed not to mix and the velocity components and their first derivatives are continuous. Under these conditions, the same fluid particles will always remain on the surface (e.g., see Lamb[1], pp. 6 - 8) so that,

$$\left.\frac{Dy}{Dt}\right|_{y=h(x,\,t)} = \left.\frac{Dx}{Dt}\right|_{y=h(x,\,t)} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \qquad (1)$$

or

$$v(x, h, t) = u(x, h, t)\,\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}$$

For irrotational flow, Equation 1 can be written in terms of the potential function $\phi(x, y, t)$, as

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=h(x,\,t)} = \left.\frac{\partial \phi}{\partial x}\right|_{y=h(x,\,t)} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \qquad (2)$$

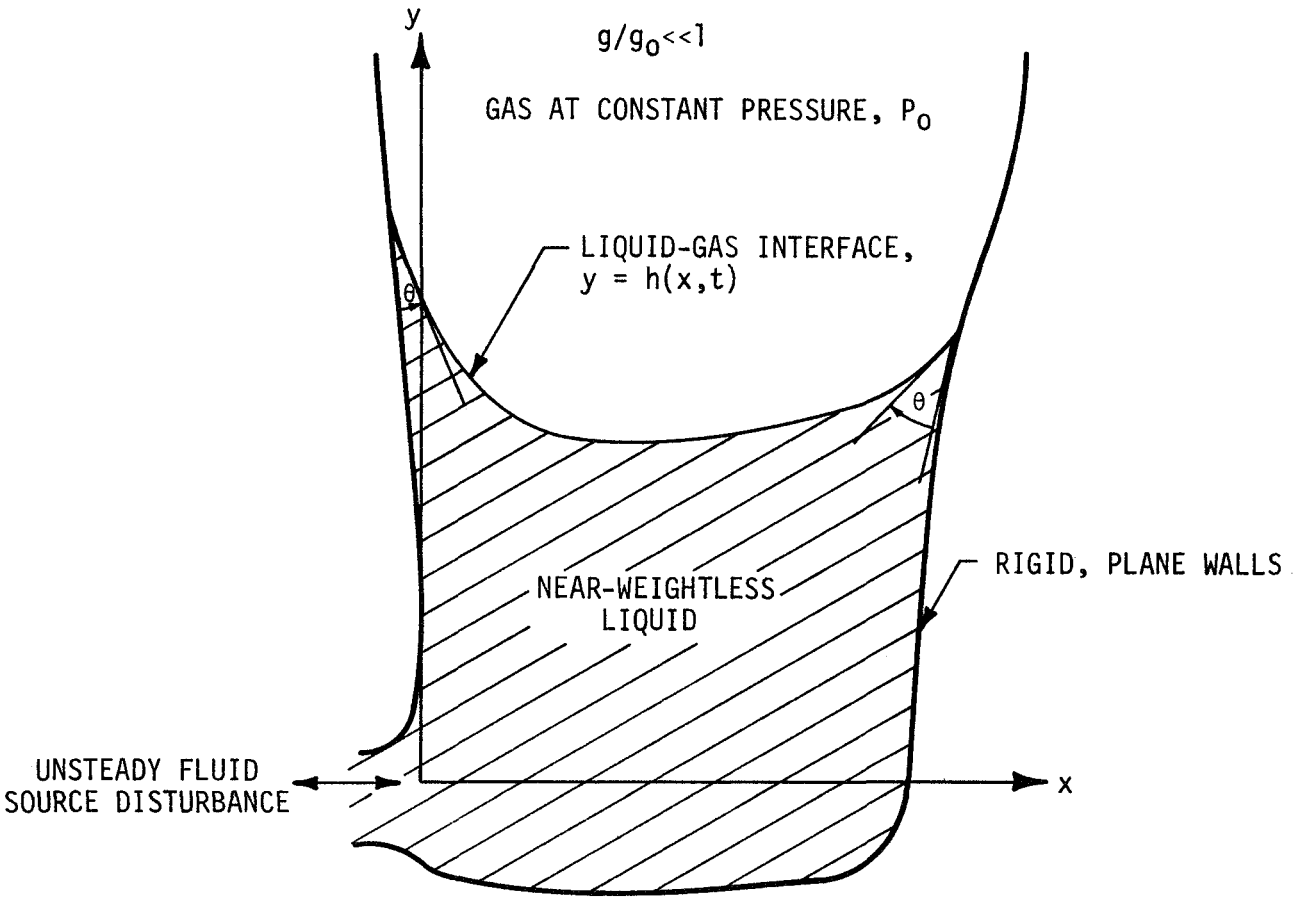

Figure 2. Illustration of General Problem

since

$$u \equiv \frac{\partial \phi(x, y, t)}{\partial x}$$

and

$$v \equiv \frac{\partial \phi(x, y, t)}{\partial y} \quad .$$

Equation 1 or Equation 2 is usually referred to as the kinematical or physical surface equation and provides a means for calculating the surface shape when the surface velocities are known.

It is interesting to note that Equation 1 applies to a viscous as well as an inviscid liquid. A question therefore arises concerning the behavior of the liquid surface near the walls. According to Equation 1, the surface particle at the wall will remain on the liquid surface; the no-slip boundary condition for a viscous fluid requires this particle to adhere to the wall. This would imply that the location of the liquid surface at the wall would never change; observations of the dynamic behavior of liquids in common containers confirm that this is not the case. There is no contradiction of boundary conditions for the inviscid case since motion tangential to the wall is allowed.

The types of surface waves that can be described by a theory that uses Equation 1 or Equation 2 is limited because of the no-mixing and continuity of derivatives assumptions. For example, the case of "breaking waves", i.e., when one part of the surface folds back on another portion of the surface, is one type of wave motion that is not considered. It is assumed that $y = h(x, t)$ is a continuous and single-valued function throughout this analysis.

THE PRESSURE-SURFACE TENSION CONDITION

At any point on a liquid surface the net force due to a discontinuity in pressure (normal stress for the viscous case) across the surface is

balanced by the surface tension forces* as illustrated in Figure 3 for a segment of a plane surface. A balance of these forces gives

$$P_s - P_o \simeq -T_{\ell g} \lim_{\Delta s \to 0} \frac{2\Delta\theta}{\Delta s} = -\frac{T_{\ell g}}{R} \tag{3}$$

where

$T_{\ell g}$ - the surface tension between the liquid and gas

R - the local radius of curvature

P_o - the constant pressure of the gas

P_s - the pressure on the liquid side of the surface.

The body forces need not be considered in this force balance since the surface forces are in local equilibrium. For the rectangular coordinate system of Figure 2, Equation 3 can be written as:

$$P(x,h,t) - P_o = -T_{\ell g} \frac{\frac{\partial^2 h}{\partial x^2}}{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{3/2}} = -T_{\ell g} \frac{\partial}{\partial x} \left\{ \frac{\frac{\partial h}{\partial x}}{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{1/2}} \right\} . \tag{4}$$

At a point where the liquid surface contacts a rigid wall, another condition (actually a boundary condition to Equation 3 or Equation 4) must be satisfied since the surface tensions between the liquid and wall, the vapor and wall must be considered in addition to the tension between the liquid and vapor. The equation which is widely used to relate these surface tensions is**

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*Surface tension is actually an energy/unit area although it behaves in every way like a force per unit length

**See Reynolds[4], pp 6-7, for a discussion of this equation.

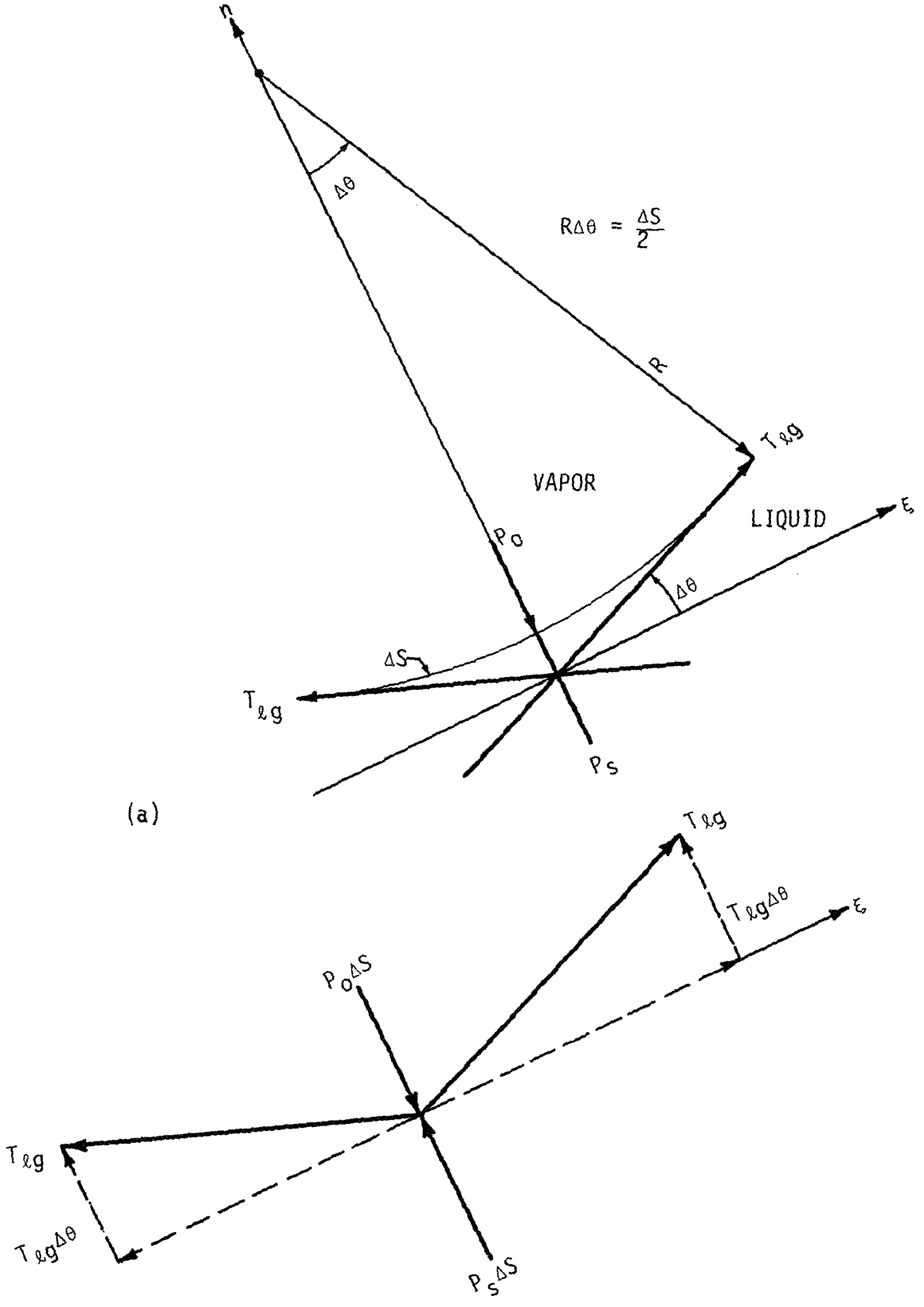

Figure 3. Force Diagram at Liquid-Vapor Interface

$$T_{gs} - T_{\ell s} = \cos \theta \, T_{\ell g} \tag{5}$$

where θ is the contact angle. Since the various surface tensions in Equation 5 depend only on the particular material, liquid, or gas and the temperature, the contact angle should theoretically be invariant with g or the dynamics of the flow field (provided that no significant temperature gradients exist). However, according to Satterlee[3], the constant contact angle condition at a solid-liquid-gas intersection is observed only when pure liquids and clean, carefully prepared solid surfaces are used. In most practical cases these conditions do not exist and a variation of the contact angle is observed. This variation is conventionally termed the hysteresis of the contact angle and must be determined experimentally. In the analyses of Satterlee[3], Reynolds[4], and others concerned with free oscillations, the cotangent of the contact angle is frequently assumed to be proportional to the difference between the dynamic and static equilibrium surface elevations at the walls.

SURFACE EQUATIONS FOR POTENTIAL MOTION AND PERTURBATIONS ABOUT H(x)

The pressures in the surface conditions for plane, potential motion can be eliminated by introducing the Bernoulli equation. Also, for reasons which will become apparent, the surface variables will be written in terms of perturbations about the static equilibrium surface.

The pressure on the liquid side of the surface is given by the Bernoulli equation,

$$\frac{P(x, h, t)}{\rho} = - \frac{\partial \Phi}{\partial t} - \frac{1}{2} q_s^2 - gh(x, t) + f(t) + A \tag{6}$$

for the dynamic case, and in the static case by

$$\frac{P[x, H(x)]}{\rho} = C - g H(x) \tag{7}$$

Any function of time (or constants) in Equation 6 can be absorbed in the $\partial\Phi/\partial t$ term without affecting any of the other equations since the Bernoulli equation is the only equation that contains a time derivative of the potential function. For convenience, a new potential function and the constant in Equation 6 are chosen so that

$$-\frac{\partial \Phi}{\partial t} + f(t) = \frac{\partial \phi}{\partial t}$$

and A = C. The Bernoulli equation for the dynamic case now becomes*,

$$\frac{P(x, h, t)}{\rho} = -\frac{\partial \phi}{\partial t}\bigg|_{y=h(x,t)} - \frac{1}{2} q_s^2 - gh(x, t) + C \quad . \tag{8}$$

From Equation 4, the pressure-surface tension condition for the static case is

$$\frac{P[x, H(x)]}{\rho} = -\frac{P_o}{\rho} - \frac{T_{\ell g}}{\rho} \frac{\dfrac{d^2 H}{dx^2}}{\left[1 + \left(\dfrac{dH}{dx}\right)^2\right]^{\frac{3}{2}}} \quad . \tag{9}$$

Equating the dynamic-static pressure difference obtained from Equations 7 and 8 to the same quantity obtained from Equations 4 and 9 yields

$$\frac{\partial \phi}{\partial y}\bigg|_{y=h(x,t)} + \frac{1}{2} q_s^2 + g\eta(x, t) - \frac{T_{\ell g}}{\rho} \left\{ \frac{\dfrac{d^2 \eta}{dx^2} + \dfrac{d^2 H}{dx^2}}{\left[1 + \left(\dfrac{\partial \eta}{\partial x} + \dfrac{dH}{dx}\right)^2\right]^{3/2}} - \frac{\dfrac{d^2 H}{dx^2}}{\left[1 + \left(\dfrac{dH}{dx}\right)^2\right]^{3/2}} \right\} = 0 \tag{10}$$

---

*This particular form of the Bernoulli equation satisfies the initial condition $P(x, h, o) = P[x, H(x)]$ when

$$\frac{\partial \phi}{\partial x}\bigg|_{y=h(x,0)} = \frac{\partial \phi}{\partial y}\bigg|_{y=h(x,0)} = \frac{\partial \phi}{\partial t}\bigg|_{y=h(x,0)} = 0 \quad .$$

where

$$\eta(x, t) \equiv h(x, t) - H(x) \quad . \tag{11}$$

Using Equation 11, the kinematic surface condition, Equation 2, can be written as

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=h(x,t)} = \left.\frac{\partial \phi}{\partial x}\right|_{y=h(x,t)} \left(\frac{\partial \eta}{\partial x} + \frac{dH}{dx}\right) + \frac{\partial \eta}{\partial t} \quad . \tag{12}$$

Equations 10 and 12 are the dynamic and kinematic surface conditions respectively in terms of differences between the dynamic and static equilibrium surface shapes. It is often more convenient to work with dimensionless forms of the surface equations. Let

$$X = \frac{x}{w} \; ; \qquad Y = \frac{y}{w} \; ; \qquad \bar{t} = \frac{tU}{w} \qquad \bar{\phi} = \frac{\phi}{Uw} \; ; \qquad \bar{\eta} = \frac{\eta}{w} \; ;$$

$$\bar{h} = \frac{h}{w} \; ; \qquad \bar{H} = \frac{H}{w}$$

where w and U are a characteristic length and velocity, respectively, e.g., the width of the tank and the maximum source velocity. Using these definitions, the following dimensionless forms of Equations 10 and 12 are obtained:

$$F_r \left.\frac{\partial \bar{\phi}}{\partial \bar{t}}\right|_{Y=\bar{h}} + \frac{1}{2} F_r \, Q_s^2 + \bar{\eta} - \frac{1}{B} \left\{ \frac{\dfrac{d^2\bar{\eta}}{dX^2} + \dfrac{d^2\bar{H}}{dX^2}}{\left[1 + \left(\dfrac{\partial \eta}{\partial X} + \dfrac{d\bar{H}}{dX}\right)^2\right]^{3/2}} - \frac{\dfrac{d^2\bar{H}}{dX^2}}{\left[1 + \left(\dfrac{d\bar{H}}{dX}\right)^2\right]^{3/2}} \right\} = 0 \tag{13}$$

and

$$\left.\frac{\partial \bar{\phi}}{\partial Y}\right|_{Y=\bar{h}} = \left.\frac{\partial \bar{\phi}}{\partial X}\right|_{Y=\bar{h}} \left(\frac{\partial \bar{\eta}}{\partial X} + \frac{d\bar{H}}{dX}\right) + \frac{\partial \bar{\eta}}{\partial \bar{t}} \tag{14}$$

where

$$Q_s^2 = \left[\left(\frac{\partial \bar{\phi}}{\partial X} \right)^2 + \left(\frac{\partial \bar{\phi}}{\partial Y} \right)^2 \right]_{Y=\bar{h}}$$

$$B = \frac{\rho g w^2}{T_{\ell g}}$$

$$F_r = \frac{U^2}{wg}$$

SIMPLIFICATIONS OF SURFACE CONDITIONS FOR NUMERICAL CALCULATIONS

Even for the ideal fluid behavior that was assumed to exist in the derivation of the surface conditions, i.e., the potential flow of an incompressible, homogeneous fluid, additional simplifications must be made before most forced oscillatory wave problems can be solved, even by numerical techniques on a high-speed digital computer. Analytical descriptions of forced wave-motion problems cannot, in general, be obtained and therefore are not discussed in this report. Perhaps what is most discouraging, however, is the fact that a particular flow may be unstable and therefore defies a mathematical treatment. Nevertheless, the calculation of stable wave motions and the ability to predict the conditions under which unstable motions will occur are very important.

The purpose of this chapter is to discuss the simplifications of the surface conditions that must be made before practical numerical solutions of forced oscillatory wave problems can be obtained. This can perhaps be best accomplished by examining a particular but rather general wave problem which is described below.

FORMULATION OF A GENERAL PROBLEM

Consider an almost-weightless, incompressible liquid that is bounded by a gas at constant pressure above its surface and confined by rigid, plane walls on the other sides as illustrated in Figure 2. The liquid which is initially motionless and in static equilibrium (e.g., t=0) is then disturbed by an oscillatory fluid source at some point (or points) below the surface. The flow field, shape and location of the liquid-gas interface versus time are to be determined. Two-dimensional, unsteady potential flow is assumed.

Mathematical Statement of the Problem

The problem can be stated mathematically as follows:

Laplace Equation

$$\nabla^2 \phi\,(x, y, t) = 0 \qquad \text{in } R(t) \tag{15}$$

Boundary Conditions at Rigid Walls

$$\frac{\partial \phi}{\partial N} = 0 \qquad \text{at the walls} \tag{16}$$

Surface Conditions, $y = H(x) + \eta(x, t) = h(x, t)$

(Kinematic)

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x}\left(\frac{\partial \eta}{\partial x} + \frac{dH}{dx}\right) + \frac{\partial \eta}{\partial t} \tag{17}$$

(Dynamic)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] + g\eta$$

$$- \frac{T_{\ell g}}{\rho}\left\{\frac{\dfrac{\partial^2 \eta}{\partial x^2} + \dfrac{d^2 H}{dx^2}}{\left[1 + \left(\dfrac{\partial \eta}{\partial x} + \dfrac{dH}{dx}\right)^2\right]^{3/2}} - \frac{\dfrac{d^2 H}{dx^2}}{\left[1 + \left(\dfrac{dH}{dx}\right)^2\right]^{3/2}}\right\} = 0 \tag{18}$$

Contact Angle Condition (No hysteresis)

$$\frac{\partial h}{\partial N} = \pm \cot \theta \quad \text{at the liquid-gas-wall intersections (+ at right wall)} \tag{19}$$

Source Disturbance

$$\frac{\partial \phi}{\partial x} = u_s \,, \quad \frac{\partial \phi}{\partial y} = v_s \quad \text{at the source} \tag{20}$$

Initial Conditions

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = \eta = 0 \quad \text{at} \quad t = 0 \tag{21}$$

R is the transient region occupied by the liquid and N is the coordinate that is normal to the wall.

Numerical Difficulties

There are four characteristics of this rather general problem that make a numerical solution of the equations in their present unsimplified forms impractical. These characteristics and the numerical difficulties are briefly discussed below.

(1) The shape of the liquid-gas interface and consequently the region, R, for which $\phi(x, y, t)$ must be calculated is not known a priori. The numerical solution of the equations in their present forms would therefore require the use of a time variable, finite-difference mesh. Furthermore, this mesh would have to be established by an iteration procedure.

(2) A number of nonlinear terms are present in the dynamic and kinematic surface equations. However, these terms are nonlinear in their space derivatives only. All terms involving time derivatives are linear. This is a very interesting point, for if either a central or forward time difference can be used, the nonlinear terms do not cause any significant difficulties and the surface variables can be obtained explicitly. However, if a backward time difference is necessary (in order to obtain a stable or accurate numerical solution) the nonlinear terms would cause tremendous difficulties and would have to be eliminated. It is not clear which of these finite difference methods should be used at the present time. The type of time difference scheme that is used seems especially important when it is observed that the dynamics of the problem appears through the surface conditions.

(3) The Laplace equation is one of the simplest (in mathematical form) and best known equations in fluid mechanics. However, a numerical solution of this equation that satisfies the specified boundary and surface conditions is very difficult. Because this equation contains second order space derivatives and no time derivatives, an implicit scheme must be used to solve a large number of finite-difference equations (as many as several thousand) for the same number of unknown potentials. A large number of iterations (sometimes, several hundred per time step) is usually necessary to solve the finite difference equations. A high-speed, large storage-capacity (preferably greater than 32K) digital computer is necessary for numerical solutions of this kind.

(4) The value of the normal derivative of the potential function at a rigid wall (Neumann type boundary condition) is specified instead of the value of the potential function itself (Dirichlet boundary condition). The Neumann boundary condition is more difficult to satisfy numerically especially for the case of curved walls.

Since neither the Laplace equation nor the Neumann-type boundary condition can be changed, nothing can apparently be done to eliminate the difficulties described in (3) and (4). The remaining numerical difficulties are directly associated with the surface conditions so that any further simplifications of the general problem must be the result of simpler surface equations.

SURFACE CONDITIONS FOR SMALL AMPLITUDE PERTURBATIONS ABOUT H(x)

The surface equations can be simplified (for numerical calculations) for the case of small amplitude perturbations about the static equilibrium surface. A surface variable, say $f(x, y, t)\Big|_{y=h(x,t)}$ can be expanded into a Taylor series as

$$f(x, y, t)\Big|_{y=h(x,t)} = f(x, y, t)\Big|_{y=H(x)} + \frac{\partial f(x, y, t)}{\partial y}\Big|_{y=H(x)} \eta(x, t) + \ldots \quad (22)$$

Using Equation 22 and assuming η is small enough to give a desired degree of accuracy when only the linear terms of the series are used, the surface Equations, 17 and 18, become

$$\frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi}{\partial y^2}\,\eta = \left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 \phi}{\partial x \partial y}\,\eta\right)\left(\frac{\partial \eta}{\partial x} + \frac{dH}{dx}\right) + \frac{\partial \eta}{\partial t} \tag{23}$$

and

$$\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial t \partial y}\,\eta + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] + \frac{1}{2}\,\frac{\partial}{\partial y}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]\eta$$

$$+ g\eta - \frac{T_{\ell g}}{\rho}\left\{\frac{\dfrac{d^2 H}{dx^2} + \dfrac{\partial^2 \eta}{\partial x^2}}{\left[1 + \left(\dfrac{\partial \eta}{\partial x} + \dfrac{dH}{dx}\right)^2\right]^{3/2}} - \frac{\dfrac{d^2 H}{dx^2}}{\left[1 + \left(\dfrac{dH}{dx}\right)^2\right]^{3/2}}\right\} = 0 \tag{24}$$

where now the surface variables are evaluated at $y = H(x)$. Therefore, for small amplitude perturbations about the static equilibrium surface, it is not necessary to use a time-variable finite-difference mesh. From a numerical point of view, this is a significant simplification.

In Equations 23 and 24, the nonlinear terms were retained. It should be emphasized that the assumption of small amplitude perturbations about the static equilibrium surface does not necessarily justify the neglect of all nonlinear terms. The critical magnitudes of η, i.e., the smallness of η, for which the nonlinear terms can be neglected (or the assumption of retaining only the linear part of the series) cannot be established before the solution itself is obtained. These assumptions must be justified by the solution itself and/or experimental data.

NUMERICAL SOLUTION FOR A PARTICULAR FLOW MODEL

Equations 15, 16, 19, 20, 21, and the surface equations for small amplitude perturbations have been solved numerically for the simple but useful flow model illustrated in Figure 1. A forward time difference was used so that the numerical difficulties associated with the nonlinear terms could be avoided. Central differences were used to evaluate the space derivatives. The solution of the finite difference equations by a relaxation technique is described by Young[5]. The static equilibrium surface shape was obtained from Geiger[6].

The dynamic response of near-weightless liquids to the "saw-toothed" source disturbance and pulse disturbance illustrated in Figure 4 have thus far received the most attention. Based on the results obtained for these disturbances, the use of a forward time difference (which permits the inclusion of the nonlinear terms) appears to be justified. However, results have only been obtained for a limited range of parameters (Bond, Froude numbers, etc.); therefore, final conclusions concerning the numerical procedures and behavior of the fluid cannot be reached until more results have been obtained and studied. The results of the study will be published in a future report.

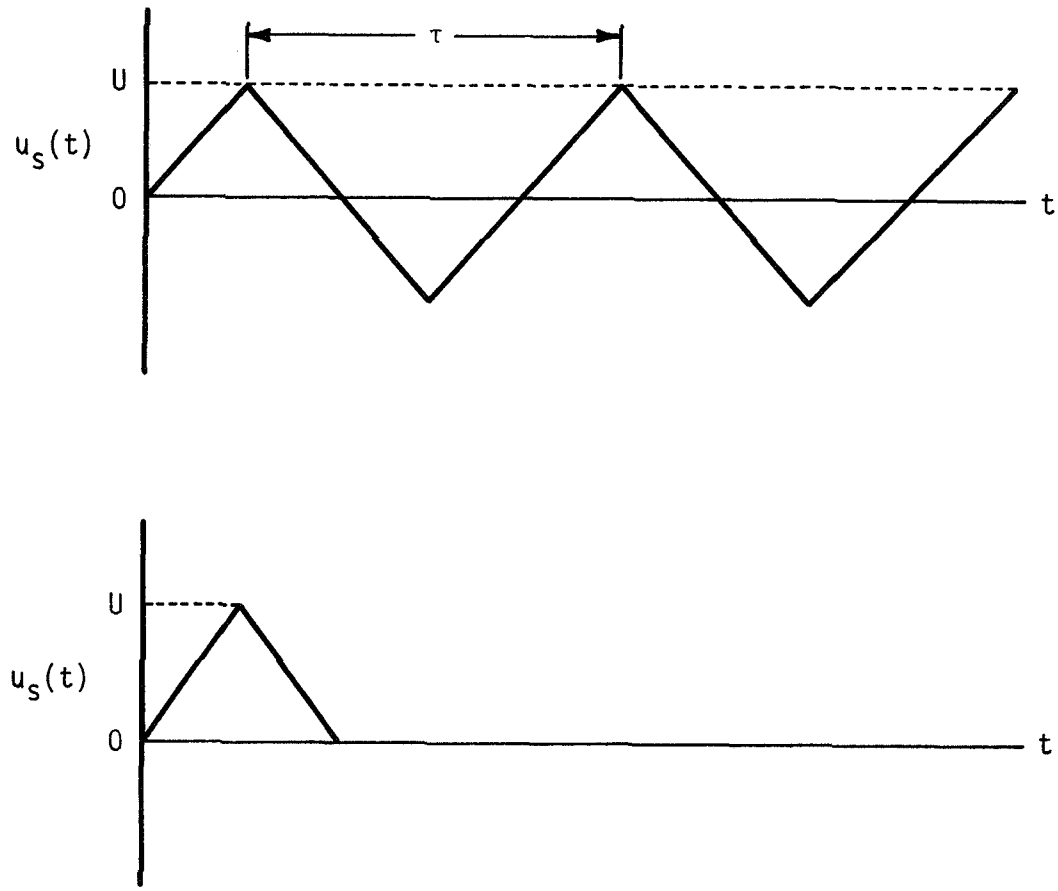

Figure 4. Typical Source Distributions Used in Analysis of the Simplified Flow Model (See Figure 1)

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13. ABSTRACT

The equations for calculating the two-dimensional, unsteady, potential motion of a near-weightless liquid in a partially filled container are derived and discussed. The simplifications of the surface equations that must be made before practical solutions can be obtained by finite difference methods are discussed. It is not necessary that the surface equations be linearized if a central or forward time difference can be used.

14. KEY WORDS

free surface
surface tension
forced oscillations
weightlessness
surface conditions
capillary