NUMERICAL METHODS OF INTEGRATION APPLIED IN THE NONLINEAR DYNAMIC ANALYSIS OF SHELLS OF REVOLUTION

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ABSTRACT

After evaluating numerous methods of numerically integrating the equations of motions formulated for nonlinear dynamic analyses of shells of revolution by the matrix displacement method, it is concluded that the Houbolt method is the most efficient and practical technique. A total of nine different integration techniques are formulated for use in nonlinear structural dynamic analyses by the finite element method. The results, conclusions, and observations of other investigators are discussed for each of the integration techniques. Each technique is initially evaluated in a nonlinear beam analysis (first vibratory mode only) to determine the most promising methods for use in shell analyses. Only these promising techniques are then applied in nonlinear shell analyses. Realistic test problems are used to evaluate three of the numerical integration procedures as applied in shell of revolution analyses. A comparison of the results obtained by each of the methods is made by analyzing the response data obtained for the lower modes of vibration. Since essentially the same response is obtained by each of the methods, a comparison of the efficiencies of the various techniques is made. This comparison reveals the superiority of the Houbolt method, so a critical test of this technique is conducted using a highly nonlinear test problem. Results of this test lend additional support

to the selection of the Houbolt method. For high frequency response other numerical techniques may be more advantageous than the Houbolt method.

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TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF FIGURES

NOMENCLATURE

Variable

A = crossectional area of beam

a_1, a_2, a_3 = coefficients in the expansion for shell slope representation

$C_1 = E_s t/(1 - \nu_{s\theta}\nu_{\theta s})$

$C_2 = E_\theta t/(1 - \nu_{s\theta}\nu_{\theta s})$

$D_1 = E_s t^3/[12(1 - \nu_{s\theta}\nu_{\theta s})]$

$D_2 = E_\theta t^3/[12(1 - \nu_{s\theta}\nu_{\theta s})]$

E = modulus of elasticity

$\hat{e}$ = linear strains and rotations of shell middle surface

F = generalized nodal force (including pseudo nonlinear force)

G = shear modulus

$G_1 = Gt$

$G_2 = Gt^3/12$

IA = number of Fourier cosine harmonics

K = structural stiffness matrix

L = length of beam element

M = moment resultant; mass matrix

N = stress resultant

O = order of the error in a finite difference expression

Q = generalized nodal force (excluding pseudo nonlinear force); shear resultant

q = generalized nodal displacement (cylindrical coordinates)

$\dot{q}$ = generalized nodal velocity

$\ddot{q}$ = generalized nodal acceleration

$\dddot{q}$ = third derivative of the generalized nodal displacements with respect to time

r = radial coordinate normal to the axis of revolution

s = meridional coordinate

T = kinetic energy

t = time; shell thickness

U = strain energy

u,v,w = meridional, tangential, and normal displacements, respectively

α = generalized coefficient of a displacement function

β = parameter of generalized acceleration

Δt = time increment

ε = midsurface nonlinear strain

Θ = circumferential angular coordinate

ν = Poisson's ratio

ρ = mass density

ϕ = angle between meridian and axis of revolution in the undeformed shell

ϕ' = $d\phi/ds$

χ = changes in curvature

Matrix

$[\ ]$ = square matrix

$\{\ \}$ = column matrix

Superscript

m = harmonic number

Subscript

i = degree of freedom

L = linear

n = time increment

NL = nonlinear

o = initial value

s = meridional direction

θ = circumferential direction

CHAPTER I

INTRODUCTION

The increased use of shells as structural elements in aircraft, spacecraft, missiles, and other structures requiring a minimum amount of weight and a maximum amount of structural integrity has promoted if not necessitated rapid advancement in the state of the art of shell structural analysis. Several papers have been presented which survey these advancements.[1-4]

Although considerable progress has been made in the development of closed-form solutions for shell analyses, these contributions are generally limited to specialized geometries which therefore restricts the usefulness of these advancements. As yet, closed form solutions for general shell equations have not been presented. Perhaps the most significant advancements in shell analysis in recent years utilize numerical techniques applied with the aid of high-speed digital computers. Computer codes which are capable of calculating the response of complex shell problems to realistic static and dynamic loading environments have been assembled. An assessment of current capabilities for computer analysis of shell structures has been presented by Hartung.[5] As reported in Ref. 5, "with the development of the computer, the solution to

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The citations on the following pages follow the style of the AIAA Journal.

complex shell problems involving such things as geometric and material nonlinearities, anisotropic and inhomogeneous material behavior, discrete stiffening and local reinforcement, arbitrary geometry and very general loading became possible."

While the most rapid advancements in shell structural analysis techniques have undoubtedly occurred in the development of computer codes[6-10] for static analyses, the development of codes[11-16] for dynamic analyses has been greatly accelerated in recent years. The development of these codes is directly related to increases in the storage space and the accompanying decreases in computation time allowed by the present generation of digital computers. Although dynamic analyses remain expensive, these analyses are well within the realm of practical applications providing care is taken to make the computer codes as efficient as possible.

Two basic methods of analysis can be utilized to effect a solution to the equations of motion derived for shell dynamic analyses. The mode superposition method[17] requires the solution of the eigenvalue problem associated with the free vibrations of the structure. A coordinate transformation is then made to uncouple the equations of motion. This transformation to principal coordinates is performed using the eigenvectors associated with the free vibration of the shell. This technique is not applied in many of the recent computer codes since computational problems are encountered in uncoupling the equations of motion when large systems of equations are to be solved. In addition, the mode superposition

method is based upon the assumption of linear structural behavior and therefore cannot be used for nonlinear analyses.

The second method of dynamic analysis uses numerical integration of the equations of motion without necessitating a transformation of coordinates. The numerical integration technique (also referred to as the step-by-step method) is characterized by the calculation of the response at some time, t, followed by the incrementing of the time by a finite (but usually relatively small) amount, Δt. The response at time, $(t + \Delta t)$, is then calculated. This process is repeated until the response is obtained for the desired period of time. Since the step-by-step method of solution is readily adaptable for use in nonlinear analyses, numerical integration procedures are employed in most of the dynamic response codes which have been recently developed. A large number of methods for numerically integrating the equations of motion are available.

The purpose of the present study is to investigate the efficiency, stability, and accuracy of a wide variety of numerical integration procedures in order to determine the optimum procedure for use in the computer code (DYNASOR) described in Refs. 14 and 15. This code has been developed at Texas A&M University to analyze the nonlinear dynamic response of shells of revolution resulting from a wide variety of mechanical and thermal loading conditions. By obtaining the most efficient numerical solution technique, the computer time and hence the cost required to obtain

the dynamic response of shell structures can be greatly reduced. By compiling, formulating, and discussing the variations of a relatively large number of numerical integration techniques it is hoped that this report will serve as a valuable reference for the many researchers who desire to employ numerical formulations for structural dynamic analyses.

The formulation of the equations of motion (Chapter II) for the DYNASOR code utilizes the matrix displacement method[18] of structural analysis. This method has been used extensively in linear structural analyses and has been employed effectively in a number of nonlinear analyses. Oden[19] and Martin[20] have compiled survey articles to describe the nonlinear contributions. In a number of studies geometric nonlinearities have been incorporated using a geometric stiffness matrix which must be incremented as the displacements vary. Stricklin, Haisler, MacDougall, and Stebbins[6] have presented a formulation which treats the nonlinear contributions as pseudo loads. Advantages accrued using this formulation are discussed in Ref. 14. Since the formulation of the equilibrium equations has been presented in detail in Ref. 14, an overview of the formulation is presented in Chapter II for completeness.

Several numerical integration procedures[21-25] are discussed in this study with each of the methods being formulated (Chapter III) for the solution of the equations of motion incorporated in the

DYNASOR code. In addition to presenting the various formulations, the results, observations, and conclusions obtained by other researchers[26-34] are discussed. The advantages and disadvantages of applying each numerical technique are discussed along with the relative amounts of storage space required by each procedure.

An evaluation of each of the methods is presented in Chapter IV. The accuracy, stability, and efficiency of each procedure are evaluated. A judicious choice of the methods tested in the nonlinear shell of revolution code is made. In this study the numerical procedures are initially evaluated in a relatively simple, but nevertheless nonlinear, analysis of a beam vibration problem. Only the most promising methods of solution are then evaluated in shell of revolution analyses. The choice of the optimum procedure is based upon the amount of computer time required to determine a satisfactory solution to the problems under consideration.

Many of the conclusions drawn in this study as well as the observations made concerning the integration procedures are not only applicable in the shell of revolution analyses conducted using the DYNASOR code but can also be generalized for application in a wide variety of important structural dynamic analyses, as well as in nonlinear analyses requiring the solution of systems of second-order differential equations.

CHAPTER II

FORMULATION FOR SHELL OF REVOLUTION ANALYSES

Equations of motion for the nonlinear dynamic analysis of shells of revolution by the matrix displacement method are formulated in this chapter. Documentation of the formulation used in the evaluation of numerical integration procedures is necessary since the stability as well as the speed of the numerical techniques is dependent upon this formulation.

Development of the DYNASOR code has been a logical step in extending the application of the finite element method of structural analysis. The curved element employed in this analysis was developed by Stricklin, Navaratna, and Pian.[35] Grafton and Strome[36] formulated the displacement function. Employing the nonlinear shell theory of Novozhilov,[37] static nonlinear analyses of shells of revolution can be performed using the formulation and computer code described in Ref. 6. The DYNASOR code[14] was developed using this static formulation and incorporating the inertia effects. A consistent mass matrix which includes the effects of rotary inertia is utilized in the equations of motion.

The nonlinearities considered in this analysis are the result of the geometry of the deformed shell and are not in any way the result of the material of which the shell is composed. In other words, elastic behavior of the material is assumed.

Structural Idealization

In this analysis the shell of revolution is idealized as a sequence of curved elements whose slope in the meridional direction (s) can be represented by a second order polynomial function

$$\phi = a_1 + a_2 s + a_3 s^2 \tag{1}$$

The coefficients of this polynomial are calculated by requiring the slopes of the idealized shell at the ends of each element to be identical to the slopes of the actual shell at those points. The elements of the idealized shell are assumed to be interconnected at a discrete number of nodal circles situated on the boundaries of the elements. The displacements of these nodes are the unknown quantities in this analysis. Figure 1 depicts the coordinization of the shell of revolution element employed in this analysis.

Equations of Motion

The matrix displacement method of structural analysis (an energy formulation) readily admits application of LaGrange's equation to derive the equilibrium equations. One equation corresponding to a particular degree of freedom, q_i, and a particular Fourier harmonic, the m^{th} one, is derived in the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i^m}\right) + \frac{\partial U}{\partial q_i^m} = Q_i^m \tag{2}$$

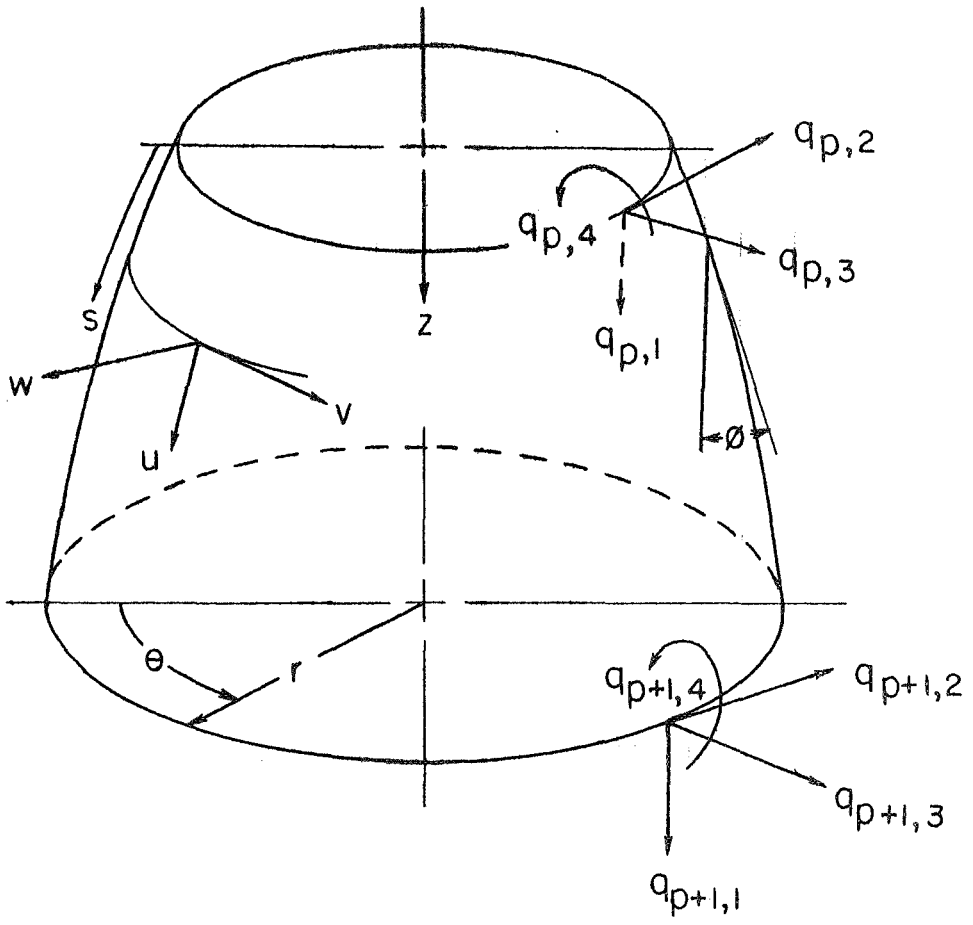

Fig. 1 GENERALIZED COORDINATES OF SHELL ELEMENT

where

q_i^m = generalized degree of freedom i in harmonic m.

The internal energy of any system is a scalar quantity and can therefore be determined (even in nonlinear analyses) by superimposing the contributions from various sources. In this analysis the internal energy is separated into two parts:

$$U = U_L + U_{NL} \tag{3}$$

where

U_L = strain energy based on linear strain-displacement relations

U_{NL} = strain energy resulting from nonlinearities

Equilibrium equations in matrix notation can be determined for a particular harmonic by substituting Eq. 3 into Eq. 2. These equations of motion for the m^{th} harmonic can be expressed as

$$[M^m]\{\ddot{q}^m\} + [K^m]\{q^m\} = \{Q^m\} - \frac{\partial U_{NL}}{\partial q^m} \tag{4}$$

It is noted that by using Eq. 4 the contribution of the geometric nonlinearities are treated as additional loads and are therefore termed pseudo nonlinear forces. The coupling between the various Fourier harmonics appears only in the expressions for these loads. The stiffness matrix which appears on the left-hand side of Eq. 4 can therefore be treated as a matrix of constants and does not have to be incremented as the time (and hence the displacement vector) varies.

The remaining sections of this chapter describe the calculation of the various terms which appear in Eq. 4.

Strain-Displacement Relations

The relationships between the strains and the displacements must be specified if the internal energy is to be formulated in terms of the displacements of the idealized shell. Geometric nonlinearities are incorporated into this analysis by assuming that the only important nonlinearities are the second-order contributions due to rotations about the shell coordinate axes. Utilizing this assumption the strain-displacement relations presented by Novozhilov[37] can be written for shells of revolution as

$$\begin{aligned} \varepsilon_s &= \hat{e}_s + \frac{1}{2}\hat{e}_{13}^2 \\ \varepsilon_\theta &= \hat{e}_\theta + \frac{1}{2}\hat{e}_{23}^2 \\ \varepsilon_{s\theta} &= \hat{e}_{s\theta} + \hat{e}_{13}\hat{e}_{23} \end{aligned} \tag{5}$$

where

$$\begin{aligned} \hat{e}_s &= (\partial u/\partial s) - \phi' w \\ \hat{e}_\theta &= (1/r)[(\partial v/\partial\theta) + u\sin\phi + w\cos\phi] \\ \hat{e}_{s\theta} &= (1/r)(\partial u/\partial\theta) - (v/r)\sin\phi + \partial v/\partial s \\ \hat{e}_{13} &= (\partial w/\partial s) + u\phi' \\ \hat{e}_{23} &= (1/r)(\partial w/\partial\theta) - (v\cos\phi)/r \end{aligned} \tag{6}$$

The expressions for the changes in curvature can be expressed

as

$$\chi_s = -\partial\hat{e}_{13}/\partial s$$

$$\chi_\theta = -(1/r)(\partial\hat{e}_{23}/\partial\theta) - (1/r)\sin\phi\ \hat{e}_{13} \tag{7}$$

$$\chi_{s\theta} = -(1/r)(\partial\hat{e}_{13}/\partial\theta) + (\sin\phi/r)\hat{e}_{23} - \partial\hat{e}_{23}/\partial s$$

Strain (Internal) Energy

An expression for the determination of the strain energy in an orthotropic shell as developed by Ambartsumian[38] is

$$U = \frac{1}{2}\int\int (C_1\varepsilon_s^2 + C_2\varepsilon_\theta^2 + 2\nu_{s\theta}C_1\varepsilon_s\varepsilon_\theta + G_1\varepsilon_{s\theta}^2 + D_1\chi_s^2 + D_2\chi_\theta^2 + 2\nu_{s\theta}D_1\chi_s\chi_\theta + G_2\chi_{s\theta}^2)rdsd\theta \tag{8}$$

The integrals around the circumference are evaluated in closed form for each of the harmonics while strip integration of the variables is employed over the length of each element.

The internal energy based upon linear theory, U_L, can be obtained by replacing the ε's in Eq. 8 by the corresponding $\hat{e}$'s. The terms of the element stiffness matrices are obtained from the internal energy of the elements and are transformed to the global (structural) coordinates before assembling the structural stiffness matrix.

Internal energy of an element resulting from geometric nonlinearities is given by the expression:

$$U_{NL} = \frac{1}{2}\iint [C_1\hat{e}_s\hat{e}_{13}^2 + C_2\hat{e}_\theta\hat{e}_{23}^2 + \nu_{s\theta}C_1(\hat{e}_s\hat{e}_{23}^2 + \hat{e}_\theta\hat{e}_{13}^2)$$
$$+ 2G_1\hat{e}_{s\theta}\hat{e}_{13}\hat{e}_{23} + \frac{1}{4}C_1\hat{e}_{13}^4 + \frac{1}{4}C_2\hat{e}_{23}^4 \qquad (9)$$
$$+ (\frac{1}{2}\nu_{s\theta}C_1 + G_1)\hat{e}_{13}^2\hat{e}_{23}^2]rdsd\theta$$

The retention of the fourth-order contributions has been shown[39] to be essential for analyses where the nonlinear contributions are significant. Conservative results are noted if the fourth-order terms are neglected.

The generalized forces due to the geometric nonlinearities can be calculated for each degree of freedom by taking the partial derivative of the nonlinear strain energy with respect to the generalized coordinates of the idealized shell. The generalized forces due to the nonlinearities are then combined at each point in time with the forces resulting from the external loads to generate the right-hand side of the equations of motion (Eq. 4). Calculation of the pseudo generalized forces is discussed in more detail in Ref. 14.

Displacement Functions

In order to apply the matrix displacement method of structural analysis, displacement functions must be chosen to uniquely define the state of displacement (and hence strain) within each element in terms of the displacements of the nodes of the element.

Certain requirements should be met by the assumed displacement functions.[18] Continuity of the $(j-1)^{th}$ derivative of the displacement in a particular direction is required between elements if the expression for the internal energy of an element is a function of the j^{th} derivative of the displacement in the same direction.

The linear strain energy depends upon the second derivative of the normal displacement, w, with respect to s. A third-order polynomial in s is therefore required in this expression to satisfy continuity between the elements. The nonlinear strain energy is a function of only the first derivative of w so a linear displacement function satisfies compatibility when substituted into Eq. 9. Different displacement functions can therefore be used, without violating compatibility, in the expressions for the linear and nonlinear strain energy.

The displacements of the elements are represented in this analysis by the following functions:

In U_{NL}

$$w = \sum_{i=0}^{IA} (\alpha_1^i + \alpha_2^i s + \alpha_3^i s^2 + \alpha_4^i s^3)\cos i\theta \tag{10}$$

In U_{NL}

$$w = \sum_{i=0}^{IA} (\alpha_5^i + \alpha_6^i s)\cos i\theta \tag{11}$$

In both U_L and U_{NL}

$$u = \sum_{i=0}^{IA} (\alpha_7^i + \alpha_8^i s)\cos i\theta \tag{12}$$

$$v = \sum_{i=0}^{IA} (\alpha_9^i + \alpha_{10}^i s)\sin i\theta \qquad (13)$$

It is interesting to note that polynomial functions are used to represent displacement variations in the meridional direction while a Fourier series expansion is utilized in the circumferential direction. Each node of the idealized structure is allowed four degrees of freedom: three translations (u, v, and w) and one rotation (ϕ).

Once the displacement functions have been chosen the element stiffness matrices and the ensuing structural stiffness matrix can be determined by considering the internal energy for each element.

Stress Resultants

The stress resultants for orthotropic shells may be expressed as functions of the strains and curvatures by the expression

$$\begin{Bmatrix} N_s \\ N_\theta \\ N_{s\theta} \\ M_S \\ M_\theta \\ M_{s\theta} \end{Bmatrix} = \begin{bmatrix} C_1 & \nu_{s\theta}C_1 & 0 & 0 & 0 & 0 \\ \nu_{\theta s}C_2 & C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_1 & \nu_{s\theta}D_1 & 0 \\ 0 & 0 & 0 & \nu_{\theta s}D_2 & D_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & G_2 \end{bmatrix} \begin{Bmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \varepsilon_{s\theta} \\ \chi_s \\ \chi_\theta \\ \chi_{s\theta} \end{Bmatrix} \qquad (14)$$

The approximate relations used to calculate the shear resultants are

$$Q_s = \frac{1}{r}\left[\frac{\partial}{\partial s}(rM_s) + \frac{\partial M_{s\theta}}{\partial \theta} - M_\theta \sin\phi\right]$$
$$Q_\theta = \frac{1}{r}\left[\frac{\partial}{\partial s}(rM_{s\theta}) + \frac{\partial M_\theta}{\partial \theta} + M_{s\theta}\sin\phi\right] \qquad (15)$$

The strains and curvature changes are calculated at the middle of each element using the assumed displacement functions. Equations 14 and 15 are then employed to determine the state of stress.

Mass Matrix

The element mass matrices are calculated from the expression for the kinetic energy of the element in a manner similar to the procedure employed to determine the element stiffness matrices from the strain energy. A consistent element mass matrix, first proposed by Archer,[40] is developed. Inclusion of the effects of rotary inertia makes the element mass matrix a function of the harmonic number. A different element mass matrix must therefore be calculated for each Fourier harmonic. The element mass matrices are assembled to form the structural mass matrix using transformation matrices identical to those used for the building of the structural stiffness matrix.

CHAPTER III

NUMERICAL INTEGRATION PROCEDURES

A variety of numerical integration procedures are discussed in this chapter. These procedures are formulated to solve the equations of motion for structural dynamic analyses. The nonlinear differential equations for each harmonic can be written in the form

$$[M]\{\ddot{q}\} + [K]\{q\} = \{F(t,q)\} \tag{16}$$

Equation 16 is equivalent to the expression for the equations of motion in the dynamic nonlinear analysis of shells of revolution providing the matrix of loads, $\{F(t,q)\}$, is defined as the right-hand side of Eq. 4.

The numerical procedures applied in this study employ difference equivalents to develop recurrence relations which may be used in a step-by-step calculation of the response. Utilizing these recurrence relations the response is obtained at a time t; the time is then incremented by an amount Δt and the response is obtained at the time $t + \Delta t$. This process is continued until the response has been calculated for the desired period of time. The procedures employed are systematic and are therefore easily adaptable for use on high-speed digital computers.

The initial conditions and the external loads must be known in order to solve the initial value problem set up in this analysis.

The initial velocity and displacement vectors are specified as

$$\dot{q}_o = \{\dot{q}_o\} \tag{17}$$

and

$$q_o = \{q_o\} \tag{18}$$

Although some numerical methods use the same formulas for calculating the response at each time step, most solution schemes require the development of special procedures for calculating the displacements at the end of the initial step. The procedure employed to start the solution is presented for each of the numerical solution techniques.

Each of the numerical integration procedures employed in this analysis is discussed along with the results and conclusions obtained by researchers who have utilized, investigated, or compared these solution techniques. The final section of this chapter is used to discuss extrapolation procedures employed to calculate the loads for the implicit methods of numerical integration employed in this analysis.

In this analysis the solution is considered unstable when the displacements become exceedingly large. The presence of such an instability can readily be detected by considering the program output. The critical time increment for stability, Δt_{crit}, is defined as the smallest time step for which the solution becomes unstable.

Fourth-Order Runge-Kutta Formulas

The equations of motion (Eq. 16) can be solved at any point in time to obtain the nodal accelerations. For the specialized set of equations derived for this shell analysis (neglecting damping effects), these accelerations are independent of the nodal velocities. The accelerations may be written in functional form as

$$\{\ddot{q}\} = G(t,\{q\}) \tag{19}$$

For this set of equations the general fourth-order forward integration Runge-Kutta formulas presented by Hildebrand[21] reduce to

$$\begin{aligned} \{q_{n+1}\} &= \{q_n\} + \Delta t\{\dot{q}_n\} + \frac{\Delta t}{6}\{m_o + m_1 + m_2\} + O(\Delta t^5) \\ \{\dot{q}_{n+1}\} &= \{\dot{q}_n\} + \frac{1}{6}\{m_o + 2m_1 + 2m_2 + m_3\} + O(\Delta t^5) \end{aligned} \tag{20}$$

where

$$\begin{aligned} \{m_o\} &= \Delta t\ G(t_n,\{q_n\}) \\ \{m\}_1 &= \Delta t\ G(t_n + \frac{\Delta t}{2},\{q_n + \frac{\Delta t}{2}\dot{q}_n\}) \\ \{m\}_2 &= \Delta t\ G(t_n + \frac{\Delta t}{2},\{q_n + \frac{\Delta t}{2}\dot{q}_n + \frac{\Delta t}{4}m_o\}) \\ \{m\}_3 &= \Delta t\ G(t_n + \Delta t,\{q_n + \Delta t\dot{q}_n + \frac{\Delta t}{2}m_1\}) \end{aligned} \tag{21}$$

Utilization of these one-step formulas provides a number of desirable advantages over other numerical schemes:

1. The formulas are the same for each time step and do not require any special starting procedures for the initial time increments.
2. Since the procedure does not require displacement vectors from previous time steps the time increment may be easily changed at any point in the calculations.
3. The explicit formulation of the Runge-Kutta technique readily permits application of these formulas in nonlinear analyses since iteration is not required.
4. The relatively high order of the truncation error [the truncation error in a fourth-order Runge-Kutta solution is of order $(\Delta t)^5$] permits the determination of very accurate results.
5. The formulas can possess at worst a weak instability[26] (i.e. the technique is unstable for $\Delta t > (\Delta t)_{crit}$ but is stable for smaller time steps).

These inherent advantages are offset in some cases by the disadvantages inherent in the procedure. The major disadvantage noted in applying these Runge-Kutta formulas is that it is necessary to calculate the acceleration vector four (4) times for each step of the advancing calculation. Since a set of simultaneous algebraic equations must be solved to determine the accelerations, a large amount of computer time may be required by this numerical solution technique. In order to be competitive with other numerical solution

routines (which require the solution of only one set of simultaneous equations per time step) the critical time increment for the Runge-Kutta method must be substantially larger (preferably by a factor of at least 4) than the step allowed by other methods.

Houbolt Method

The finite difference method of solution developed by Houbolt[22] for use in dynamic structural response studies of aircraft can be adapted for use in the present shell analysis. The acceleration vector in the equilibrium equations is replaced by an equivalent finite difference expression and the resulting matrix equation is solved for the displacements at the end of each time step.

The nodal accelerations are approximated by the following finite difference expression:

$$\{\ddot{q}_{n+1}\} = \frac{1}{(\Delta t)^2}\{2q_{n+1} - 5q_n + 4q_{n-1} - q_{n-2}\} \tag{22}$$

Substituting Eq. 22 into Eq. 16 and simplifying yields an expression of the form:

$$\begin{aligned}(2[M] + (\Delta t)^2[K])\{q_{n+1}\} &= (\Delta t)^2\{F(t,q)_{n+1}\} \\ &+ [M]\{5q_n - 4q_{n-1} + q_{n-2}\}\end{aligned} \tag{23}$$

This expression is used to determine the response at the end of each time increment except the first. A special procedure must be employed in order to calculate the displacements at the end of

the first time step (n = 0) since the values of the displacement vectors $\{q_{-1}\}$ and $\{q_{-2}\}$ are unknown.

In addition to the initial velocities and displacements which are specified, the initial accelerations may be calculated by considering Eq. 16 at time t = 0 or by employing Newton's second law of motion:

$$[M]\{\ddot{q}_o\} = \{F(0,q_o)\} - [K]\{q_o\} \tag{24}$$

Difference expressions can then be used to develop equations to relate the known initial conditions to the fictitious matrices $\{q_{-1}\}$ and $\{q_{-2}\}$. The difference expressions employed are

$$\{\ddot{q}_o\} = \frac{1}{(\Delta t)^2}\{q_1 - 2q_o + q_{-1}\} \tag{25}$$

and

$$\{\dot{q}_o\} = \frac{1}{6\Delta t}\{2q_1 + 3q_o - 6q_{-1} + q_{-2}\} \tag{26}$$

Equation 25 can be solved for $\{q_{-1}\}$ and written as

$$\{q_{-1}\} = (\Delta t)^2\{\ddot{q}_o\} + \{2q_o - q_1\} \tag{27}$$

Equations 26 and 27 can now be combined to yield

$$\{q_{-2}\} = 6(\Delta t)^2\{\ddot{q}_o\} + 6\Delta t\{\dot{q}_o\} + 9\{q_o\} - 8\{q_1\} \tag{28}$$

These last two expressions can be substituted into Eq. 23 to yield an expression to be solved to obtain the displacements at the end of the first time step (n = 0) providing the forces at the end of

the step are approximated by those at the start of the calculations:

$$(6[M] + (\Delta t)^2[K])\{q_1\} = (\Delta t)^2\{F(0,q_o)\} + [M]\{2(\Delta t)^2\ddot{q}_o + 6\Delta t\dot{q}_o + 6q_o\} \tag{29}$$

Having determined the displacements at the end of the first step a fictitious vector, $\{q_{-1}\}$, of displacements can be calculated using Eq. 27. Equation 23 can then be used to calculate the displacements at the end of the second and all succeeding time steps. It should be noted that utilizing this fictitious set of displacements does not violate in any way the prescribed initial conditions.

The stability of the Houbolt procedure has been investigated by researchers and has been proved in Ref. 27 to be unconditionally stable when utilized in the solution of linear structural dynamics problems. The procedure developed by von Neumann[41] was applied in this analysis in the manner previously employed by Leech, Hsu, and Mack[28] in a stability study of the second-order central difference approximation of the acceleration vector.

The research presented earlier by Levy and Kroll[29] predicts this inherent stability by noting the presence of a decaying exponential in the homogeneous solution. This exponential term tends to significantly damp the responses in the higher modes of vibration when large values of the time increment are used. In addition the procedure is shown to decrease the natural frequency of the system. Of the methods investigated by Levy and Kroll, "the Houbolt method is the only one which gives convergent results for large

time increments."

The Houbolt procedure has been evaluated in shell analyses by Johnson and Greif.[30] This technique was incorporated into a computer code to evaluate the linear, elastic response of thin cylindrical shells using a finite difference formulation for the spatial coordinates. The Houbolt (implicit) procedure was compared with the second-order central difference formulas used in many dynamic analyses. It was concluded that "the explicit method tends to be more efficient when the response varies rapidly, whereas the Houbolt method tends to be more efficient for the prediction of slower responses." The Houbolt procedure was found to be the more flexible of the two methods since unequal spacings (in the spatial directions) and large time increments could be readily utilized. The damping inherent in the Houbolt procedure was also noted in Ref. 30. In order for the output of a particular vibratory mode to appear undamped, the time increment, Δt, was required to be less than about 1/50 of the period of the mode being analyzed.

Stephens and Fulton[31] have successfully employed this method of numerical integration to obtain the axisymmetric dynamic response of spherical caps to centrally distributed pressure loadings. The nonlinear equations of motion were linearized using a Newton-Raphson procedure which necessitates iteration at each time step to determine a satisfactory solution.

Chan, Cox, and Benfield Procedure

Structural dynamics problems may be solved using the method of solution presented in Ref. 23 by Chan, Cox, and Benfield. This numerical method is derived directly from the equations of motion of the system. The resulting equations can be readily applied to a wide variety of multi-degree-of-freedom problems in structural dynamics.

As noted by other researchers,[23,32] this numerical technique is a specialized version of the more general technique developed by Newmark.[33] Unless this version of the Newmark formulation is used, damping (either positive or negative) is introduced into the response.

The numerical solution of the differential equations of motion for the dynamic system is accomplished utilizing the following finite difference relations:

$$\{\dot{q}_{n+1}\} = \{\dot{q}_n\} + \frac{\Delta t}{2}\{\ddot{q}_{n+1} + \ddot{q}_n\} \tag{30}$$

and

$$\{q_{n+1}\} = \{q_n\} + \Delta t\{\dot{q}_n\} + (\frac{1}{2} - \beta)(\Delta t)^2\{\ddot{q}_n\} + \beta(\Delta t)^2\{\ddot{q}_{n+1}\} \tag{31}$$

Eliminating the matrix of damping from the formulation in Ref. 23 the displacements for the $(n+1)^{th}$ time step can be calculated from the expression

$$[A]\{q_{n+1}\} = [B]\{q_n\} - [A]\{q_{n-1}\} + \beta(\Delta t)^2\{F_{n+1} + (\frac{1}{\beta} - 2)F_n + F_{n-1}\} \tag{32}$$

where

$$[A] = [M] + \beta(\Delta t)^2[K]$$
$$[B] = 2[M] - (1 - 2\beta)(\Delta t)^2[K] \tag{33}$$

The response at the end of each time step (except the first one) is calculated using Eq. 32.

A special procedure is necessary for calculating the displacements at the end of the first time step. This procedure is necessary since, for n = 0, the matrix of displacements $\{q_{-1}\}$ is unknown. The starting procedure presented in Ref. 23 simplifies to the following expression which is used to calculate the response for the first time step:

$$[A]\{q_1\} = [C]\{q_o\} + \Delta t[M]\{\dot{q}_o\} + \beta(\Delta t)^2\{F(\Delta t, q_1)\} + (\frac{1}{2} - \beta)(\Delta t)^2\{F(0, q_o)\} \tag{34}$$

where

$$[C] = [M] - (\frac{1}{2} - \beta)(\Delta t)^2[K] \tag{35}$$

An inherent disadvantage in applying the Chan, Cox, and Benfield procedure is evident if the form of Eq. 32 is considered. This formulation requires two matrix multiplications for each harmonic during each time step. If a large number of finite elements are utilized to approximate the structure, the matrix multiplications will require a relatively large amount of computer time making this method slower than the numerical procedures requiring only one matrix multiplication. The time increment which can be utilized

by this numerical procedure must (when compared with other methods) be large enough to offset the expected increase in computation time per step.

The Chan, Cox, and Benfield routine is an invaluable procedure when applied in a study of the effectiveness of various numerical procedures since at least five different numerical procedures can be investigated by varying the parameter of generalized acceleration, β. Each of these numerical procedures is consistent with a different assumption concerning the variation of the acceleration within the time increment. The acceleration parameter is used to describe the variation of the acceleration within the time increment. Five values of β have been employed in this study to investigate five different numerical solution procedures.

Constant Average Acceleration ($\beta = 1/4$)

Setting the value of β equal to 1/4 corresponds to using trapezoidal integration formulas to determine both the displacements and the velocities of the system. This trapezoidal integration procedure is consistent with the assumption that a constant acceleration exists within the interval. This constant acceleration has a value equal to the mean value of the initial and final accelerations of the increment.

This procedure ($\beta = 1/4$) has been shown to be unconditionally stable and to possess no artificial or inherent damping.[32] This

stability had been previously noted in Ref. 23.

Linear Acceleration ($\beta = 1/6$)

A second variation of the Chan, Cox, and Benfield formulas can be derived using Simpson's one-third rule to integrate the accelerations and trapezoidal integration of the velocities. These formulas correspond to using $\beta = 1/6$ and are consistent with the assumption of a linear variation of the acceleration within each time step. This numerical technique has also been presented by Wilson and Clough[24] and applied in dynamic analyses of structures subjected to earthquake loadings.

Step Function Acceleration Variation ($\beta = 1/8$)

The formulas which are derived utilizing $\beta = 1/8$ can also be shown to be consistent with the assumption that the acceleration within the time increment varies as a step function. The step function has a value equal to the initial value of the acceleration during the first half of the time increment and then uses the final value of the step function during the second half of the increment.

Fox and Goodwin Formulation ($\beta = 1/12$)

The set of formulas which evolve as a result of using $\beta = 1/12$ are identical to those presented by Fox and Goodwin[34] for the solu-

tion of ordinary differential equations. Newmark[33] has concluded that in most instances better results are obtained if larger values of β are used.

Second-Order Central Difference Formulation ($\beta = 0$)

One of the most commonly used formulations for dynamic structural analyses results from utilization of a zero value for the parameter of generalized acceleration. This variation is equivalent to making the assumption that the acceleration vector can be approximated by

$$\{\ddot{q}_n\} = \frac{1}{(\Delta t)^2} \{q_{n+1} - 2q_n + q_{n-1}\} \tag{36}$$

An explicit formulation for the displacements at the end of the $(n+1)^{th}$ time increment results from the substitution of Eq. 36 into Eq. 16.

The research conducted by Distefano[26] has established the fact that since an explicit formulation exists, the procedure can possess at worst a weak instability. In other words, the process will be stable for time increments smaller than a certain critical value ($\Delta t < \Delta t_{crit}$). For linear structural analyses the critical value of the time step as determined by Levy and Kroll[29] can be written as

$$\Delta t_{crit} = 2/w_{max} \tag{37}$$

where

w_{max} = largest natural frequency of the structure

In a more rigorous analysis, Leech, Hsu, and Mack[28] established the value of the critical time step using the procedure developed by von Newman.[41]

In addition to the stability studies which have been conducted, this explicit formulation has been utilized in a large number of structural dynamic analyses. Shell dynamic analyses using finite difference formulations for the spatial coordinates have been performed utilizing this formulation. Johnson and Greif[30] evaluated this method and compared the efficiency of this formulation with that of the Houbolt procedure in the linear elastic response of cylindrical shells. This study indicates that this explicit method tends to be more efficient when the response varies rapidly. Leech, Witmer, and Pian[16] and Wrenn, Sobel, and Silsby[12] employ this formulation in computer codes for the nonlinear analysis of general thin shells. Wrenn, Sobel, and Silsby[12] employ this formulation in computer codes for the nonlinear analysis of general thin shells. Wrenn, Sobel, and Silsby have found this method to be more efficient than either the fourth-order Runge-Kutta formulation or the Adams-Moulton predictor-corrector technique. Based upon the results obtained in numerous computer runs, they also report that "in most cases the critical time step is so small that the larger truncation error of the finite difference method does not create a serious accuracy problem."

Parabolic Acceleration Method

Another step-by-step integration procedure which may be employed in dynamic analysis allows the accelerations to vary parabolically within each increment of time. This procedure is presented in Ref. 24. Adapting these formulas for the solution of the equations of motion (Eq. 16), the dynamic response of a structure can be determined. The nodal accelerations at the end of each time step are calculated using

$$([M] + \frac{\Delta t^2}{12}[K])\{\ddot{q}_{n+1}\} = \{F(t,q)_{n+1}\} - [K]\{A\} \tag{38}$$

where

$$\{A\} = \{q_n\} + \Delta t\{\dot{q}_n\} + \frac{5}{12}\Delta t^2\{\ddot{q}_n\} + \frac{\Delta t^3}{12}\{\dddot{q}_n\} \tag{39}$$

The nodal displacements are then determined by applying the following equation:

$$\{q_{n+1}\} = \{A\} + \frac{\Delta t^2}{12}\{\ddot{q}_{n+1}\} \tag{40}$$

The values of the velocities and the third derivative of the displacements with respect to time are calculated for each step using

$$\{\dot{q}_{n+1}\} = \{\dot{q}_n\} + \frac{2\Delta t}{3}\{\ddot{q}_n\} + \frac{\Delta t^2}{6}\{\dddot{q}_n\} + \frac{\Delta t}{3}\{\ddot{q}_{n+1}\} \tag{41}$$

and

$$\{\dddot{q}_{n+1}\} = \frac{2}{\Delta t}\{\ddot{q}_{n+1}\} - \frac{2}{\Delta t}\{\ddot{q}_n\} - \{\dddot{q}_n\} \tag{42}$$

These vectors are used to determine the components of the vector {A} which is used to calculate the accelerations and displacements at the end of the next step. By successively applying Eqs. 38, 40, 41, and 42, the structural response can be calculated for each time step.

In order to start the calculations (first time increment) the initial accelerations and the initial values of the third derivatives must be calculated from the given initial velocity and displacement vectors. The initial accelerations are determined using the equations of motion at the initial time which is written as

$$[M]\{\ddot{q}_o\} = \{F(0,q_o)\} - [K]\{q_o\} \tag{43}$$

The initial values of the third derivatives are determined using

$$[M]\{\dddot{q}_o\} = \frac{1}{\Delta t}\{F(\Delta t,q_1) - F(0,q_o)\} - [K]\{\dot{q}_o\} \tag{44}$$

Having determined the values of these two derivatives, Eq. 38 can be used to calculate the displacements at the end of the first time step.

Utilization of this higher order method may result in an increase of the size of the critical time step. The increase in this step size must be substantial in order to justify employing this procedure since a relatively large number of calculations must be performed for each time step. This implicit formulation requires a knowledge of the force vector, {F(t,q)}, at the time increment at

which the displacements are to be calculated. The extrapolation procedures utilized to calculate these forces are presented in the final section of this chapter.

Third-Order Explicit Formulation

A third-order explicit formulation can be developed if the acceleration vector in Eq. 16 is approximated by the following expression[25]:

$$\{\ddot{q}_n\} = \frac{1}{12(\Delta t)^2}\{11q_{n+1} - 20q_n + 6q_{n-1} + 4q_{n-2} - q_{n-3}\} \quad (45)$$

The displacements at the end of the $(n+1)^{th}$ time step can then be determined if Eq. 45 is substituted into Eq. 16 to yield

$$[M]\{q_{n+1}\} = \frac{12(\Delta t)^2}{11}\{F(t,q)_n\} - \frac{12\Delta t^2}{11}[K]\{q_n\} + \frac{1}{11}[M](20q_n - 6q_{n-1} - 4q_{n-2} + q_{n-3}) \quad (46)$$

This method requires a knowledge of the response at the end of four previous time steps in order to calculate the response at the end of the fifth step. This requirement necessitates the use of a larger amount of storage space than is required by the other procedures. In addition to the requirement of additional storage space, this explicit procedure requires a relatively large amount of matrix operations since four displacement vectors must be summed and two matrix multiplications are required for each harmonic at each time step. In order to be competitive with other numerical solution

schemes this procedure must therefore be able to utilize a slightly larger time step since the computation time per step will undoubtedly be larger than for most of the other solution procedures.

A slightly different approach is taken to start the numerical solution using this formulation. Since relatively small errors in the values of the displacements of the initial time steps can cause large errors in the ensuing calculations, it is extremely important that the displacement vectors for the initial time step be accurately calculated. The fourth-order Runge-Kutta formulas presented in a previous section are utilized to calculate the response for the first three time steps. These formulas are applied for six time steps with one half the value of the desired time increment to insure accurate values for the vectors $\{q_1\}$, $\{q_2\}$, and $\{q_3\}$. Using these displacement vectors, the displacements at the end of the fourth (and all succeeding) steps can be calculated by employing Eq. 46.

Loads Matrix Approximation

In order to determine the displacements at the end of the $(n+1)^{th}$ time increment, the method of Chan, Cox, and Benfield, the Houbolt formulation, and the parabolic acceleration method require a knowledge of the loads, $\{F(t,q)\}$, at the end of the increment. As can easily be noted by considering Eq. 16, these loads are a function of the displacements which are to be calculated . These

loads cannot therefore be evaluated exactly.

The right-hand side of Eq. 16 will therefore be evaluated using a first-order Taylor's series expanded about the n^{th} time increment. This expansion may be written as:

$$\{F(t,q)_{n+1}\} = \{F(t,q)_n\} + \Delta t \frac{\partial}{\partial t} \{F(t,q)_n\} + O(\Delta t^2) \qquad (47)$$

Using a first-order backwards difference expression to approximate the partial derivative results in the following extrapolation formula, which is used to calculate the loads:

$$\{F(t,q)_{n+1}\} = 2\{F(t,q)_n\} - \{F(t,q)_{n-1}\} \qquad (48)$$

Employing Eq. 48 in the Houbolt and Chan, Cox, and Benfield formulations is consistent since the inherent error in these formulations is the same as the order of the truncation error (Δt^2) in Eq. 48. Applying Eq. 48 corresponds to using a linear extrapolation of the loads at the two previous time increments.

The loads may be approximated with an accuracy of order $(\Delta t)^3$ using the following second-order expression:

$$\{F(t,q)_{n+1}\} = 3\{F(t,q)_n\} - 3\{F(t,q)_{n-1}\} + \{F(t,q)_{n-2}\} \qquad (49)$$

This expression can be obtained by passing a parabola (second-order curve) through three points and extrapolating to obtain the fourth point on the load-time curve.

CHAPTER IV

EVALUATION OF NUMERICAL INTEGRATION PROCEDURES

In this chapter the results obtained using the various numerical integration techniques in selected test problems are presented, compared, and evaluated to determine the optimum procedure for use in the DYNASOR code. A one degree-of-freedom beam vibration problem is formulated to initially evaluate the integration techniques. After considering the results obtained in the beam analysis, two numerical integration techniques are eliminated from consideration. The remaining three techniques are then evaluated in shell of revolution analyses.

Test problems formulated for the evaluation of the numerical procedures in shell analyses are described since the relative stability of the integration procedures must be evaluated in light of these particular applications. A thorough evaluation of the Houbolt procedure is made after the technique is adjudged to be the optimum procedure. This critical test lends additional support to the conclusion that the Houbolt technique is the most advantageous procedure for use in the DYNASOR code.

Beam Response Study

The beam configuration depicted in Fig. 2 was selected as a test case for use in screening the numerical integration techniques.

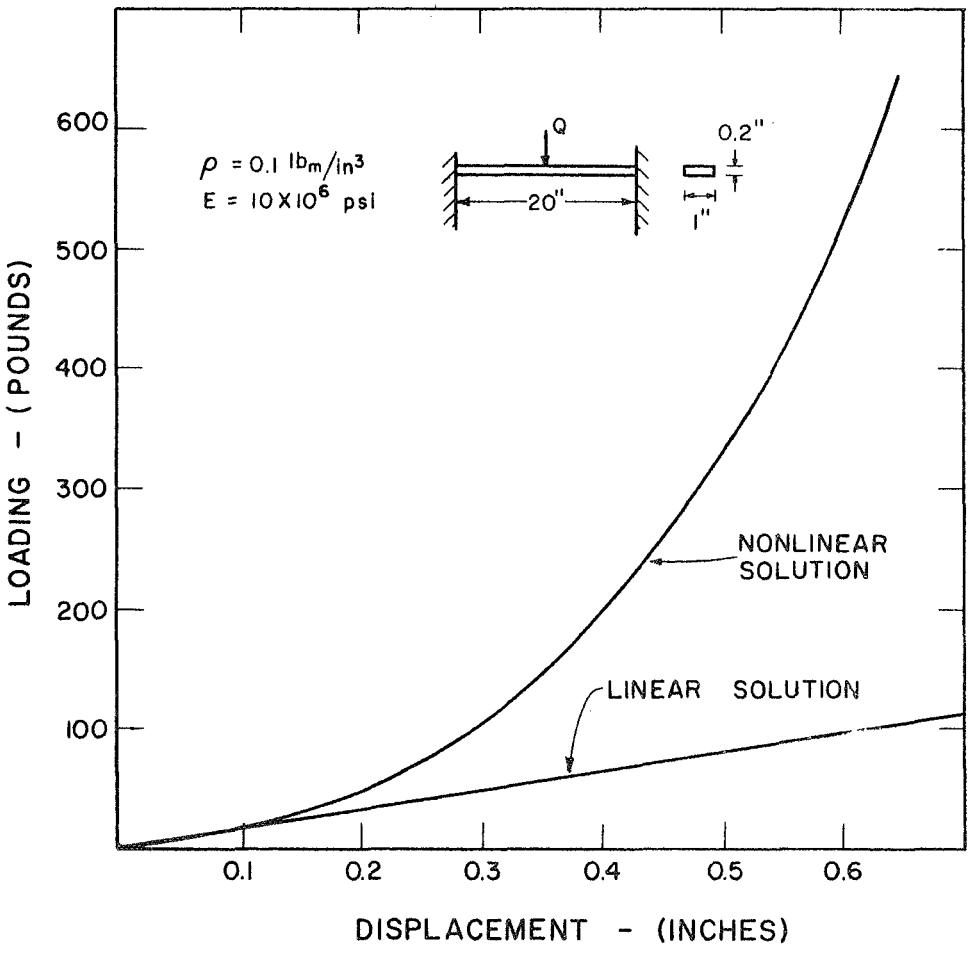

FIG. 2 STATIC RESPONSE OF FIXED BEAM

The concentrated loading was instantaneously applied at the beginning of the response calculations and does not vary with the time. Two elements were used to idealize the beam structure with the free nodal point being located at the center of the beam where the concentrated loading is applied. Although the two element idealization may appear somewhat crude, it should be kept in mind that this test problem was used primarily for testing the numerical integration techniques. Rigid investigation of nonlinear beam vibrations was not the purpose of this study. The one degree-of-freedom formulation is sufficient for the evaluation of the integration techniques providing only the response of the first vibratory mode is needed.

The equation of motion which describes the nonlinear dynamic response of the beam can be expressed as

$$\ddot{M}q + Kq = Q - \frac{\partial U_{NL}}{\partial q} \tag{50}$$

where

$$\begin{aligned} M &= \rho AL \\ K &= 24\ EI/L^3 \\ \frac{\partial U_{NL}}{\partial q} &= EA(q/L)^3 \end{aligned} \tag{51}$$

Equation 50 has been written in the same form as the equations of motion for shell of revolution analyses (Eq. 16). Similarities between the shell and beam analyses allow utilization of the beam

configuration for a preliminary analysis of the characteristics of the numerical integration techniques providing the material and geometric properties of the structure are judiciously selected.

Since the presence of nonlinearities affects the stability of the solution techniques, the nonlinear effects of the rotation of the beam upon the axial strain are included. The material properties selected are typical of those currently employed in the aerospace industry while the geometric parameters provide a test problem in which the nonlinearities are significant. The values of these parameters are as follows:

$$\begin{aligned} A &= 0.2 \text{ in}^2 \\ E &= 10.0 \times 10^6 \text{psi} \\ L &= 10 \text{ in} \\ \rho &= 0.1 \text{ lb}_m/\text{in}^3 \end{aligned} \tag{52}$$

The static response of this fixed beam configuration is presented in Fig. 2 to show the degree of nonlinearity present in the response. This response clearly indicates that the influence of the membrane effect becomes increasingly more significant as the loading is increased. The effect of the inclusion of the nonlinearities becomes increasingly significant as the loading is increased. Inclusion of the nonlinear terms causes a stiffening effect which results in smaller displacements for the same amount of load when compared to the linear solution.

In the dynamic analysis of this beam each of the numerical integration procedures was found to give essentially the same results when relatively small time steps were used. Results presented in Fig. 3 indicate the vertical response of the middle of the beam obtained by the various integration techniques with a time step of fifty microseconds. No significant discrepancies between the results obtained by the various procedures were noted for the small time increments.

As the time step was increased, two of the numerical integration procedures, the parabolic acceleration method and the third-order explicit formulation, were found to be unsatisfactory. A divergent oscillation was noted in the response obtained using the parabolic acceleration technique. A positive damping effect was noted in the response obtained by the third-order explicit formulation. The damped response converged to the static nonlinear solution. As the time increment was further increased, numerical instabilities were noted (i.e. the response became unrealistically large). Each of the other integration procedures provided satisfactory results for time increments at least as large as the step size at which the undesirable solutions were noted.

It should be noted that the mean point of the oscillations of this beam has a value less than the predicted static nonlinear deflection. This phenomena is expected since more energy per unit of deflection is required as the response increases.

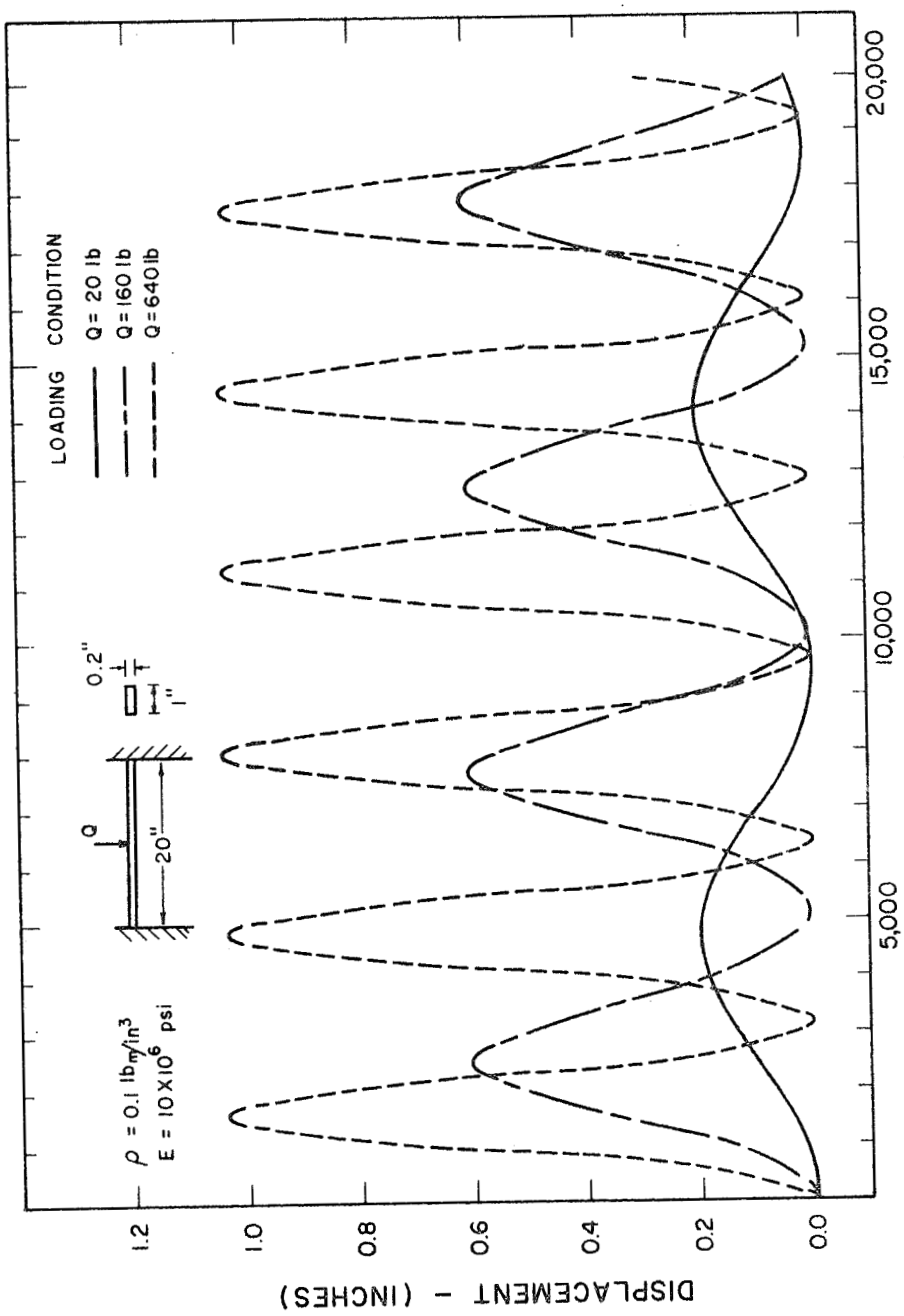

FIG. 3 LOADING CONDITION EFFECT UPON DISPLACEMENT RESPONSE OF FIXED BEAM

For exceedingly large values of the time increment the inherent damping of the Houbolt method was noted (Fig. 4). For small values of the load (hence only moderately nonlinear response) the dynamic response converged quickly to the static solution. A larger number of cycles was required for this convergence as the loading was increased. Only a small, almost negligible, amount of damping was apparent in solutions obtained using large values of both the load and the time increment; additional increases in the size of the time step resulted in unstable, not positively damped, solutions. The unconditional stability of the Houbolt technique as applied in linear dynamic analyses does not therefore exist for nonlinear dynamic analyses. As the degree of nonlinearity of the response increased the inherent damping of the Houbolt method became less significant since the time increment for stability became less than, or at least approached, the time increment at which the inherent damping became noticeable.

Satisfactory results were obtained using the method of Chan, Cox, and Benfield, the fourth-order Runge-Kutta formulas, and the Houbolt integration procedure. These solution techniques were therefore selected for application and evaluation in shell analyses.

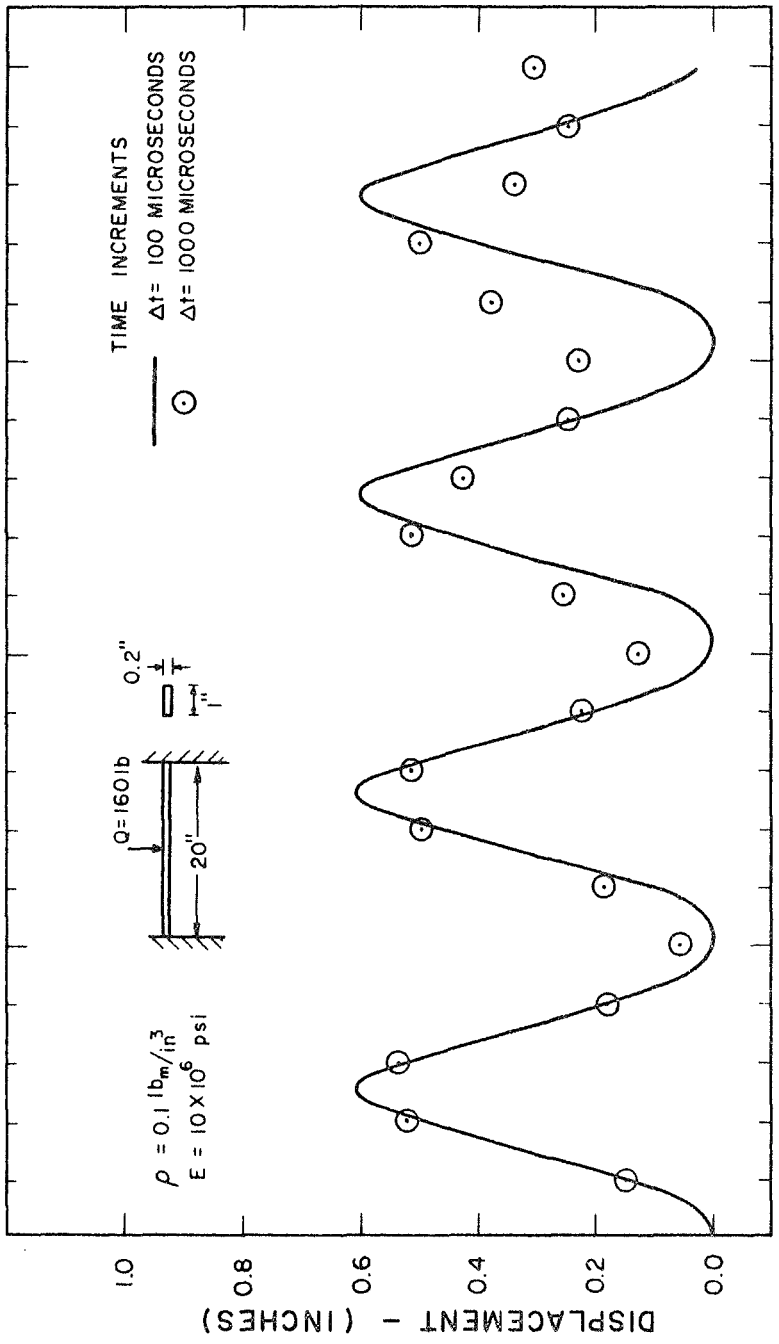

FIG. 4 EFFECT OF TIME INCREMENT VARIATION UPON BEAM DISPLACEMENT RESPONSE (HOUBOLT'S METHOD)

Shell of Revolution Response Studies

The most promising methods of numerical integration, as determined by the beam analyses, were efficiently incorporated into the DYNASOR code using the FORTRAN IV language. The calculations were then made using an IBM 360/65 computer. In addition to presenting an evaluation and comparison of the various numerical integration techniques, the results of a critical test of the Houbolt procedure are described in the remaining sections of this chapter.

Runge-Kutta Evaluation

A linear analysis of a shallow spherical cap (λ=6) with clamped edges was performed to evaluate the Runge-Kutta method of integration. A radius of 0.9 inches and a rise of 0.0859 inches were assumed for the cap. The slope of the shell at its base was selected as 10.9^{o}; a uniform thickness of 0.01576 inches was used in the calculations. The shell was subjected to an instantaneously applied (constant in time) internal pressure and the response of the zero harmonic was determined using a thirty element idealization of the shell.

After obtaining numerically unstable solutions for time increments of 2.0, 1.0, and 0.5 microseconds, a stable solution (linear) was obtained using a value of 0.005 microseconds. Utilizing this extremely small time step would necessitate prohibitive

amounts of computer time for dynamic analyses. Since the numerical instabilities consistently occurred after completing only a few time steps, a smaller time step was used to obtain the displacements at the end of the initial time steps. Employing this small initial step size increased the accuracy of the calculated displacements. Inaccurate calculations in an initial value problem greatly affect the numerical stability of ensuing calculations. Additional stability was afforded by this approach and the allowable time step size was increased to 0.05 microseconds. It was presumed that this time increment would be decreased if a nonlinear analysis was conducted. The allowable time increment was definitely too small to allow practical use of the Runge-Kutta formulas for nonlinear shell analyses.

The many advantages afforded by the simplicities of application and the high order of accuracy of the Runge-Kutta method are more than offset by the small critical time increment required for dynamic shell of revolution analyses conducted using this technique. The fourth-order Runge-Kutta formulation is therefore unsatisfactory for use in the DYNASOR code.

Chan, Cox, and Benfield Evaluations (β = 1/6, 1/8, 1/12, and 0)

Attempts were made in a variety of problems to use values of $\beta < 1/4$ in the Chan, Cox, and Benfield routine. A numerically stable response was never obtained using time increments as small

as 0.1 microseconds. The results of the Runge-Kutta evaluation indicated that a stable response could be obtained if further reductions were made in the size of the time step. A stable solution (zero harmonic only) was then obtained in one application for $\beta = 0$ using a time step of 0.01 microseconds. The same problem had been solved using another method (Houbolt) with a time increment of 0.25 microseconds. It is therefore impractical to employ the Chan, Cox, and Benfield procedure with values of β less than one-fourth for shell analyses using the DYNASOR code.

Comparison of Houbolt Method and Chan, Cox, and Benfield Routine ($\beta = 1/4$)

Selection of the most advantageous numerical solution technique can now be made by comparing the response obtained using the Chan, Cox, and Benfield method with $\beta = 1/4$ to the response determined using the Houbolt formulation. The shell selected for this comparision is the cap-torus-cylinder configuration depicted in Fig. 5. Fifty elements are used to idealize the shell with the distribution of the elements selected to yield a large number of elements near the cap-torus intersection and near the torus-cylinder intersection. The widely varying element sizes and the irregular shape of the shell combine to provide a problem which serves as a realistic test of the integration techniques.

The displacements and stresses for the zeroth harmonic were calculated using both methods of solution (single-precision

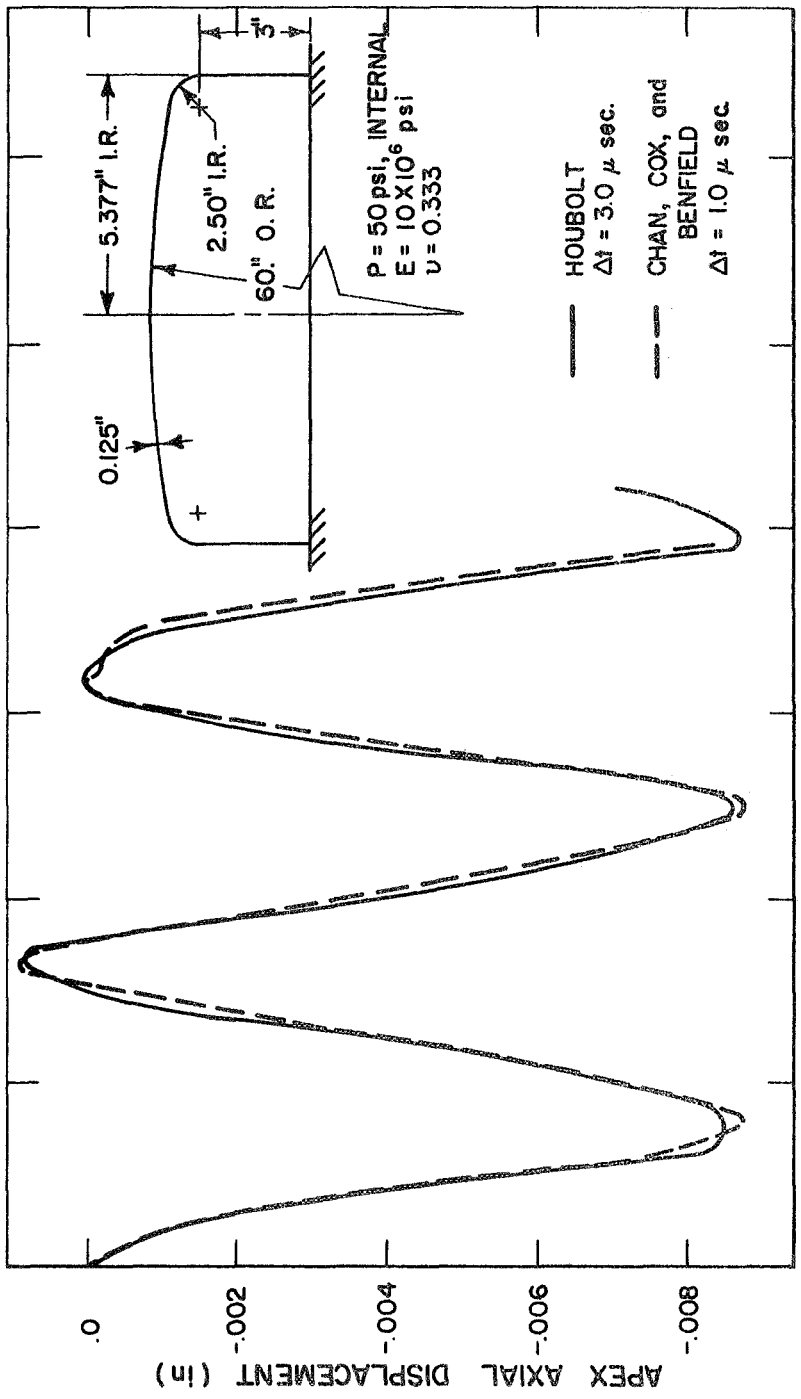

FIG. 5 DISPLACEMENT RESPONSE CALCULATED BY TWO NUMERICAL METHODS

arithmetic). The displacement response calculated by the two numerical procedures is almost identical (Fig. 5). Likewise excellent agreement is noted (Figs. 6 and 7) between the stresses calculated by the two integration techniques.

Selection of the technique for use in the DYNASOR code can be based upon economic considerations since the response curves are essentially the same. A time increment of 3.0 microseconds yielded stable solutions for the Houbolt method. The method of Chan, Cox, and Benfield could employ only as large as a 1.0 microsecond time step. In addition to allowing a larger time step the Houbolt method requires less computation time per step. A comparison of Eqs. 23 and 32 reveals that two matrix multiplications are required per time step for the Chan, Cox, and Benfield routine, but only one multiplication is necessary in the Houbolt formulation, thus explaining the difference in the amount of computer time required per step. As the number of finite elements is increased the savings in computation time per step becomes increasingly more significant.

In this test problem the combined effect of using a larger time step and decreasing the required computer time per step is to provide a solution to this problem almost four times faster using Houbolt's method. Comparable savings in computer time resulting from using the Houbolt procedure were noted in a number of other applications[14] of the two techniques.

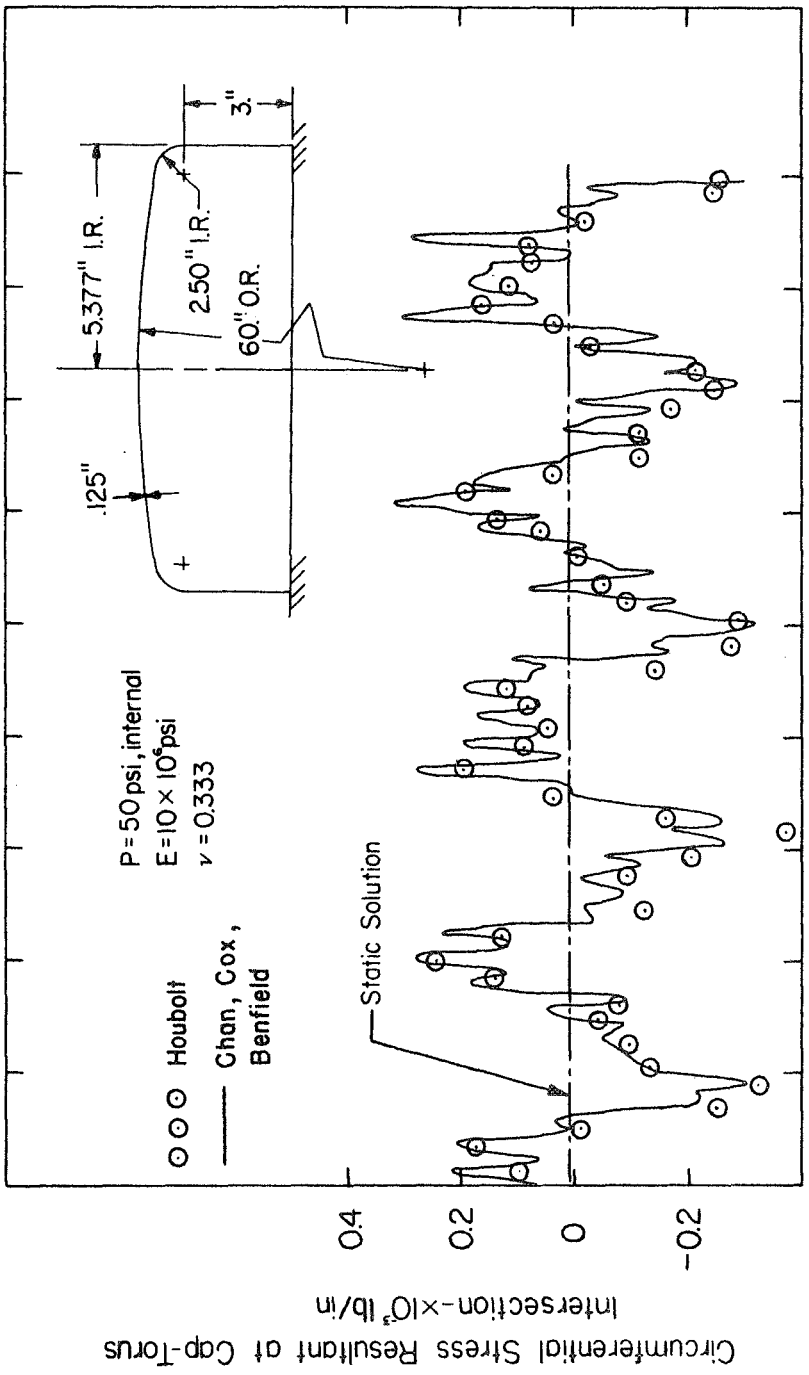

FIG. 6 CIRCUMFERENTIAL STRESS RESULTANT FOR SHELL UNDER STEP PRESSURE LOADING

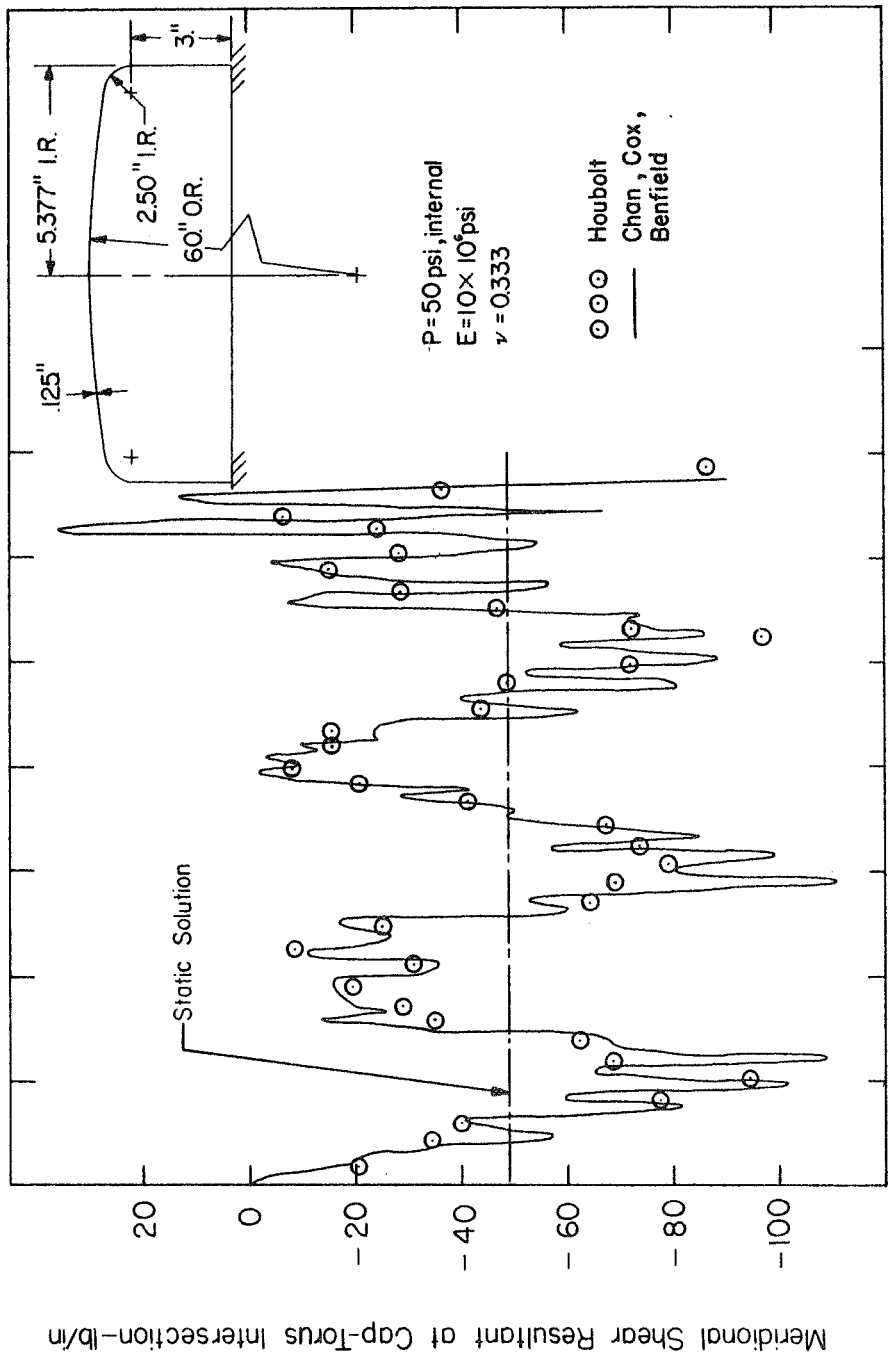

FIG. 7 MERIDIONAL SHEAR RESULTANT FOR SHELL UNDER STEP PRESSURE LOADING

Although the Houbolt scheme requires slightly more storage space than the Chan, Cox, and Benfield technique (three displacement vectors instead of two), the benefits and advantages accrued using the Houbolt scheme more than offset this slight disadvantage. The Houbolt method of numerical integration is therefore chosen as the most advantageous method for use in the shell analyses conducted using the DYNASOR code.

Comprehensive Evaluation of Houbolt's Method

A critical test of the Houbolt method was made to see if the procedure is capable of obtaining the dynamic response of shells whose behavior is highly nonlinear. A shallow spherical cap (λ=6) with clamped edges was used in the evaluative study. The geometric and material properties together with a descriptive drawing of the shell are presented in Fig. 8. A concentrated load applied at the apex of the shell was used to excite the response. This load was instantaneously applied at time t = 0 and remained constant for the duration of the calculations.

The problem was selected for the evaluation for two reasons. First, the response is highly nonlinear. The high degree of nonlinearity of the response can be established by considering the static load-deflection curve presented in Fig. 8. These results were obtained using the Newton-Raphson method of solution in a version of the SNASOR code similar to the version described in

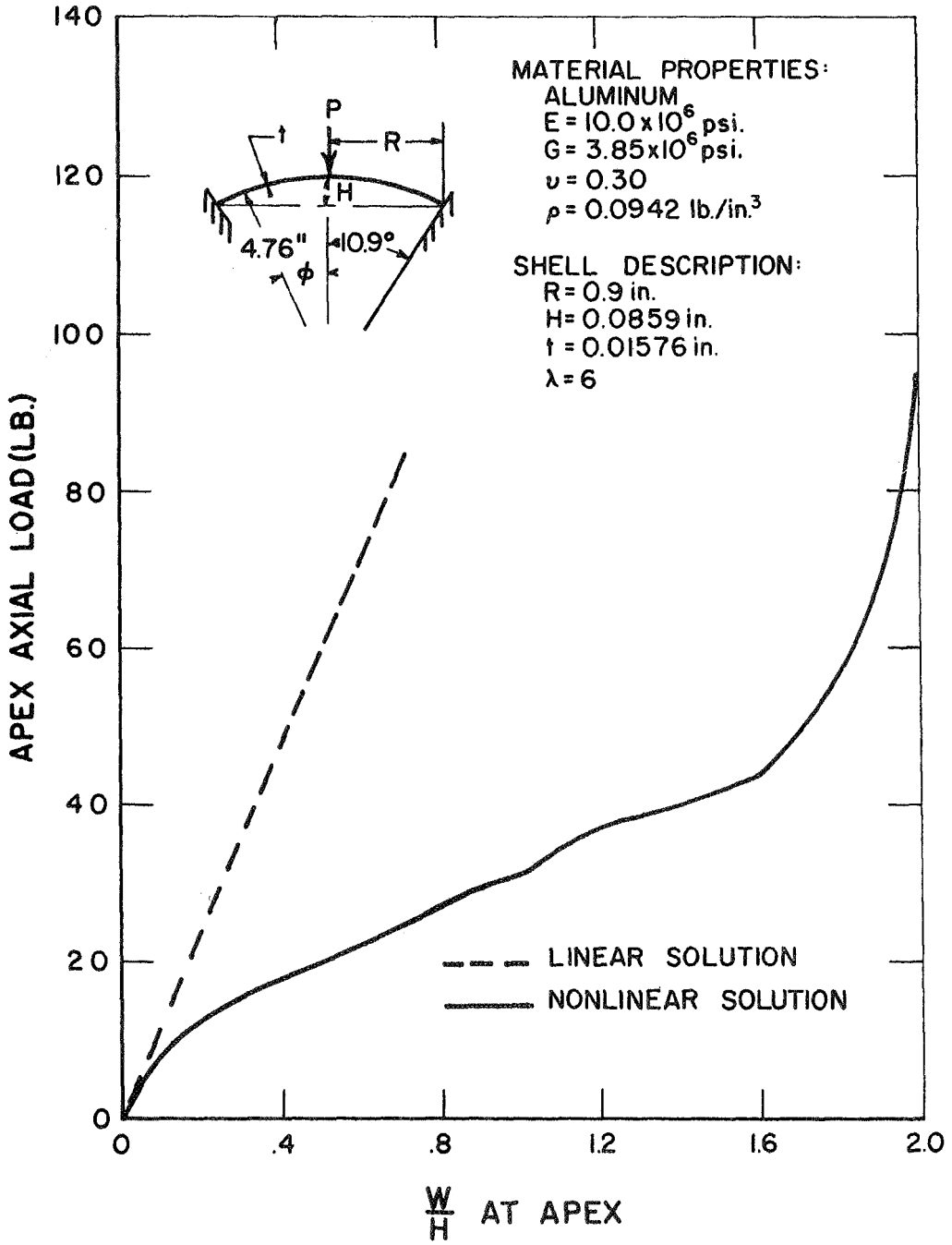

FIG. 8 STATIC RESPONSE OF SHALLOW SPHERICAL CAP

Ref. 6. These results agree well with the work of other researchers.[42,43] A forty pound load was selected for use in the dynamic evaluation studies. The nonlinearities are quite significant for this loading since the static nonlinear response is more than four times as large as the linear deflection at this loading.

A second reason for selecting this configuration centers around the existence of a singularity at the apex of the shell. Extremely large stiffness terms evolve as a result of this singularity; the corresponding terms in the mass matrices for the various harmonics are, however, rather small. The large stiffness to mass ratio provides an exceedingly large speed of sound in the medium. The criteria for selecting the size of the time step developed[12,30] for use in finite difference shell analyses are based upon the time required for a signal to travel from one mesh point to another. Applying these criteria to this problem results in the prediction of an unrealistically small time increment. It is hoped that by utilizing the Houbolt technique the time increment for stability will be much larger than the value predicted by the finite difference criteria.

Effect of Increased Numerical Accuracy

The effect of using a greater degree of numerical accuracy in the solution of this test problem was investigated. Using a thirty element idealization and a time increment of 0.125 micro-

seconds, response data was obtained with both single-precision (seven significant figures) and double-precision (sixteen significant figures) numerical accuracy. A comparison of the response curves can be made by considering Fig. 9. The decrease in the round-off error of the double-precision response results in an increase in the period and the amplitude of the motion in this highly nonlinear application. Single precision results for moderately nonlinear problems have been shown[14] to be in excellent agreement with the results obtained by other investigators. Double-precision arithmetic seems therefore to be necessary only when the behavior of highly nonlinear shells is to be analyzed using the DYNASOR code on computers whose inherent accuracy is equal to or less than that of an IBM 360/65 computer. Double-precision arithmetic is not thought to be necessary, even for highly nonlinear behavior, when the DYNASOR code is used in computers (such as the CDC 6600) which have a significantly longer word length than the IBM 360/65 system.

Effect of Load Extrapolation Procedure

Two extrapolation procedures (Ch. III, pp. 33 - 34) are utilized to calculate the loads at the end of the $(n+1)^{th}$ time step from the loads at the previous steps for the implicit Houbolt procedure. A comparison of the results obtained using both extrapolation procedures is presented in Fig. 10. These results

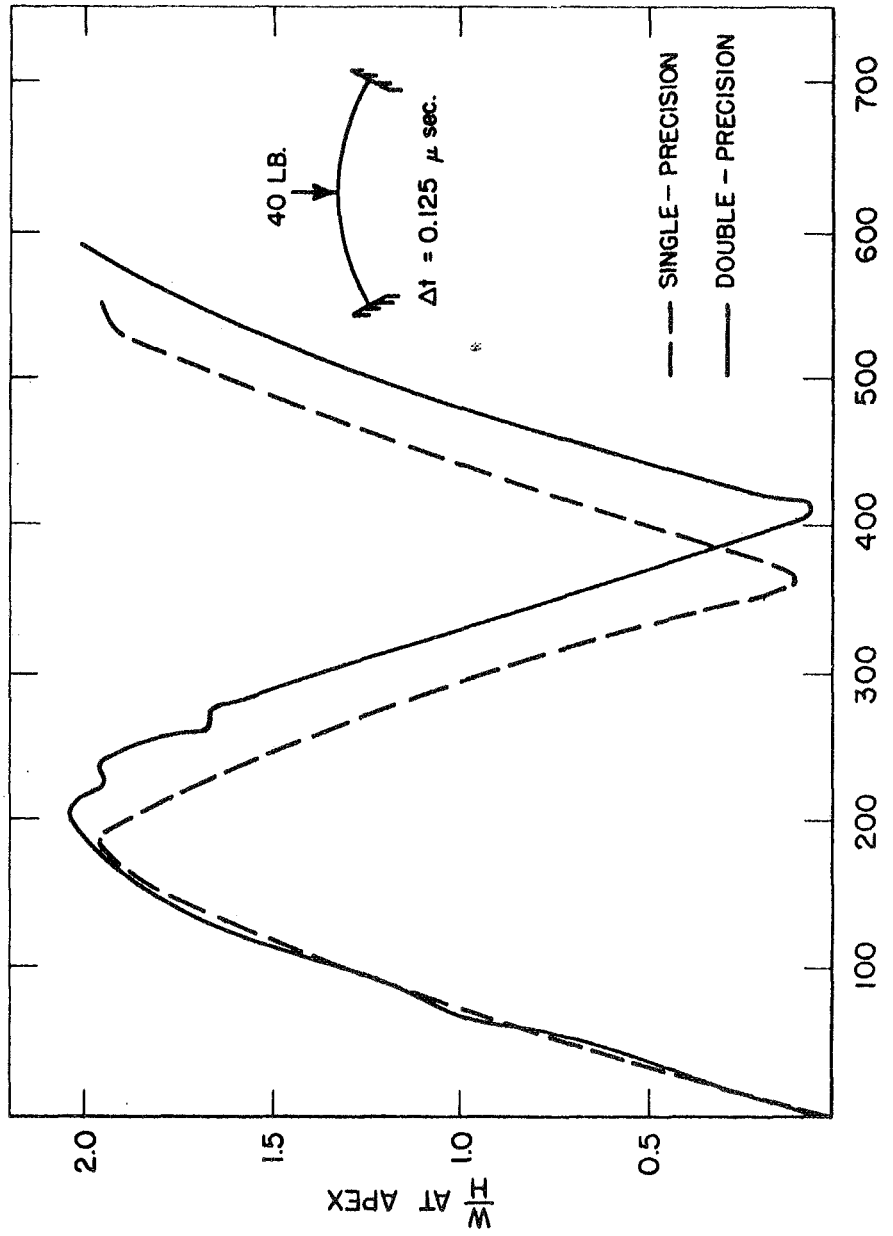

FIG. 9 EFFECT OF NUMERICAL ACCURACY UPON SHELL RESPONSE AT APEX

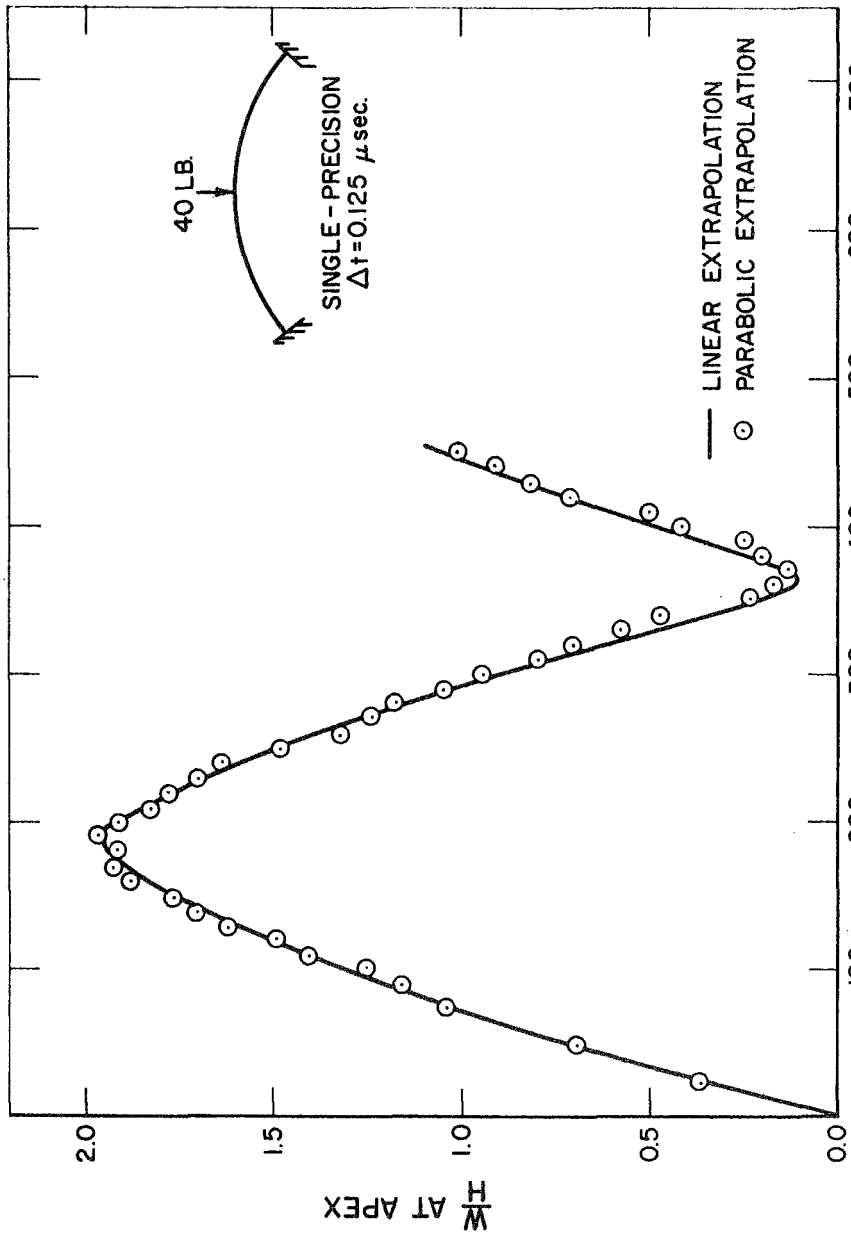

FIG. 10 EFFECT OF LOAD EXTRAPOLATION PROCEDURE UPON APEX AXIAL DISPLACEMENT

were obtained for the zeroth harmonic using a time increment of 0.125 microseconds and thirty elements to idealize the shell. No significant differences in the results obtained by the two extrapolation procedures are noticed in Fig. 10.

For a time step of 0.25 microseconds the solution obtained using the parabolic extrapolation procedure was found to be numerically unstable. A stable solution was, however, obtained using the linear extrapolation procedure to obtain the loads. Because of the additional stability obtained using the linear approximation, it is concluded that a linear extrapolation is more effective than a parabolic procedure.

In addition to the increased stability, the amount of storage space required by the code is lessened, as is the amount of computer time required per time step, since the force vectors at only two, instead of three, previous time steps must be retained and combined. Utilization of the linear extrapolation procedure is therefore an efficient and effective means of approximating the loads at the end of the $(n+1)^{th}$ time increment.

Solution Convergence with Improved Idealization

Since this analysis is based upon a finite element formulation , it is necessary to show that the idealization used for this test problem is fine enough to yield an accurate solution. As the number of elements is increased, the responses obtained by any

acceptable numerical method should converge. Solution convergence using the Houbolt method is demonstrated in Fig. 11. Responses were obtained using 15, 30, and 50 element idealizations of the shallow cap; these elements were concentrated near the apex and near the supports of the shell where the displacements and stresses vary rapidly.

For the formulation employed in this analysis it was expected that in highly nonlinear problems, such as this test problem, the convergence with improved idealization would be rather slow. Utilizing linear displacement functions in the nonlinear strain energy expression and employing strip integration across the elements was expected to result in rather slow convergence. However, as indicated in Fig. 11, the response, even in this highly nonlinear test problem, has completely converged for as few as 30 elements. Although the period of the motion is slightly damped, the response obtained using only 15 elements is accurate enough for many engineering purposes.

Obtaining accurate solutions using a relatively small number of elements results in considerable savings of computer time. Each element used to idealize the shell increases the size of the matrices which must be manipulated and increases, by four, the number of simultaneous equations which must be solved at each time step. Hence, the amount of computer time required per time step increases rapidly as the idealization of the shell is improved. In

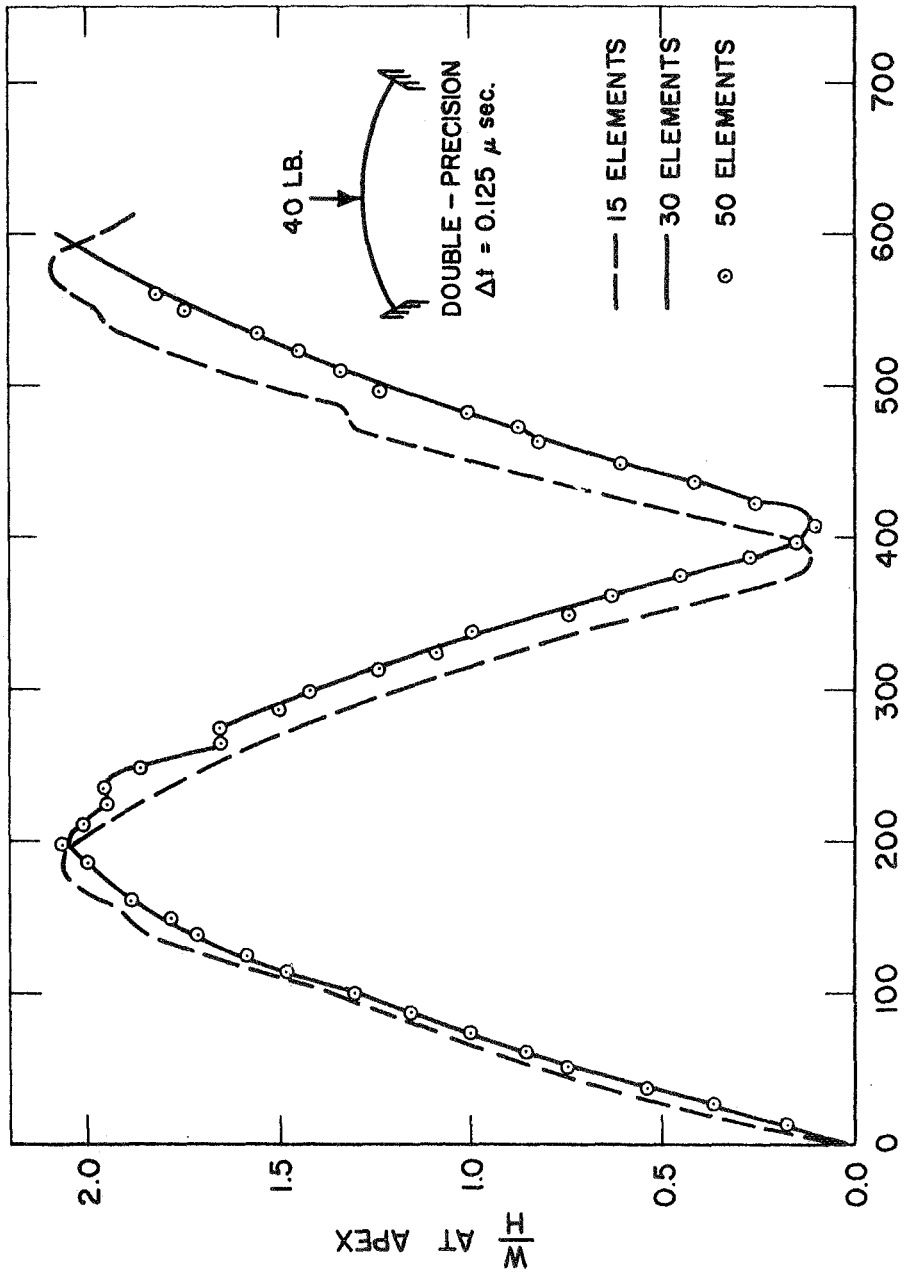

FIG. 11 SOLUTION CONVERGENCE WITH FINITE ELEMENT IDEALIZATION

this problem the computation time per step was 0.0067 minutes for the 50 element solution but only 0.0021 minutes for the 15 element case (only the response of the zeroth harmonic was determined). The Houbolt method as applied in the DYNASOR code provides convergent solutions using relatively few elements for idealizing the shell.

Effect of Time Increment Variation

The presence of inherent damping in the Houbolt method of integration necessitates demonstration that the time step size has been chosen small enough to ensure that the artificial viscosity of the method does not affect the calculated response. To determine the effect of varying the time increment in this problem, the test problem was run using time steps of 0.125 and 0.25 microseconds. Again, double-precision numerical accuracy was used for calculating the response of the thirty elements used to idealize the cap. As presented in Fig. 12, no discernable errors are introduced using the larger value of the time step. For a time step of 0.50 microseconds the solution became numerically unstable.

Selection of the Houbolt method as the optimum procedure for use in the DYNASOR code is further supported by results obtained for this test case. Using the method of Chan, Cox, and Benfield ($\beta = 1/4$) the responses obtained were numerically unstable for time steps as small as 0.125 microseconds. Using $\beta = 0$, a stable

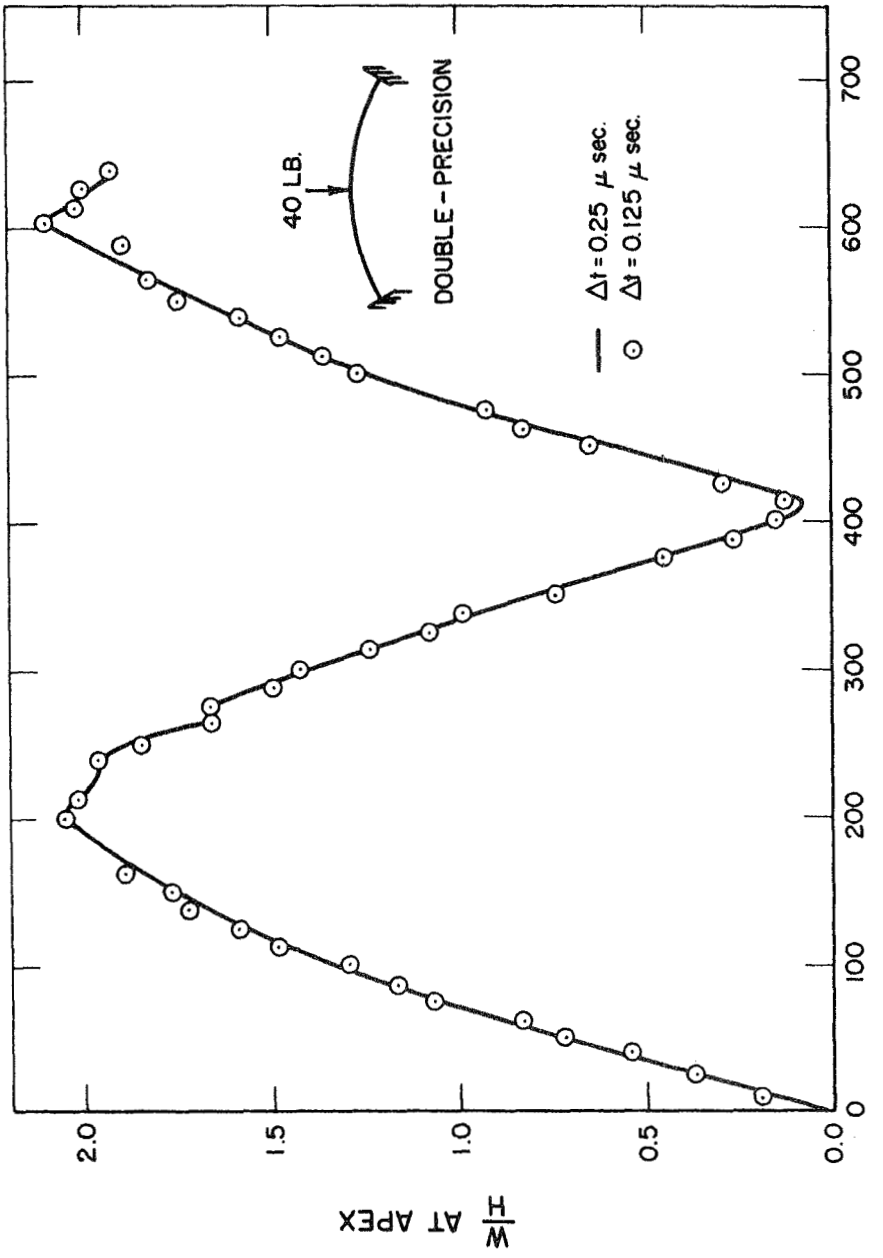

FIG. 12 EFFECT OF TIME INCREMENT UPON APEX AXIAL DISPLACEMENT

solution was finally obtained using a 0.01 microsecond time step. Again it is noted that the Houbolt procedure is numerically stable for time steps much larger than can be used with other methods.

If the Houbolt method is used to calculate high frequency response, a small time increment (in relation to the period of the motion) must be used. The step size must be chosen small enough (approximately 1/50 of the period) to keep the response from being damped. In these applications other numerical procedures may be deemed more efficient than the Houbolt procedure.

Time Increment Selection

Since the cost of a dynamic analysis is directly related to the size of the time step used in the calculations, judicious selection of the size of the time step is almost mandatory. Consideration must therefore be given to the factors which affect the size of the time step which may be used in nonlinear dynamic analyses of shells.

Techniques of estimating the size of the critical time step developed for finite difference analyses are based upon the minimum amount of time required for a signal to travel from one mesh point to another. Although applicable in a number of important analyses, these techniques are not applicable in many instances, such as for the test problem in Fig. 8. For the shallow

cap an unrealistically small time increment is estimated by these procedures. Results indicate much larger time steps may be used.

The size of the time step is greatly affected by the degree of nonlinearity present in the response. Hence, both the geometry of the shell and the loading applied to the shell affect the choice of the time step. In nonlinear analyses the influence of the loading on the size of the time step can be quite significant. As previously stated, a time increment of 0.25 microseconds can be effectively used for the shallow cap (Fig. 8) subjected to a forty pound apex loading. The response of the same shell subjected to a uniform external pressure was satisfactorily determined using a time step of 1.0 microsecond. This increase in the allowable time step is permitted since the response to the pressure loading does not exhibit the high degree of nonlinearity present for the concentrated loading.

Selection of the time increment for use in nonlinear dynamic analyses must therefore rely upon the judgement of the investigator. This judgement should be based upon the shell geometry and both the degree of nonlinearity and the frequency expected in the response. The choice should be aided by the estimated values obtained using finite difference techniques. Through experience an investigator can rapidly gain an insight into the selection of an efficient step size. Only when an acceptable level of pro-

ficiency has been attained can dynamic analyses be efficiently and effectively performed. Further study to develop techniques for estimating the size of the critical time step in nonlinear dynamic analyses is needed to correct the admittedly inefficient procedure presently being used.

CHAPTER V

CONCLUSIONS

The results obtained in this study strongly support the conclusion that, of the numerical integration procedures tested, the Houbolt method is the most efficient and practical method of solution for use in the DYNASOR code. For the test problems analysed in this study, as well as in the low frequency studies discussed in Ref. 14, the Houbolt method was the only technique which obtained numerically stable solutions using realistic values for the time increment.

When the high frequency response of higher modes of vibration is to be determined, the Houbolt procedure requires a small time step in order to keep the damping inherent in the technique from becoming significant. Other numerical techniques may, in these instances, be more efficient than the Houbolt procedure. Although applying another numerical technique would be more efficient for some of these cases, it is believed that the Houbolt technique is the best method for use in the vast majority of practical analyses to be conducted using the DYNASOR code.

In the present implicit formulation the loads applied at the end of the $(n+1)^{th}$ time step are obtained using a first-order Taylor's series expansion about the n^{th} step. Although utilizing the approximation undoubtedly contributes to making the method

unstable for large time increments, the Houbolt method is substantially more stable than the formulations which require no extrapolation of the loads.

Even for highly nonlinear analyses convergent solutions can be obtained using Houbolt's method. For a highly nonlinear test problem the response was shown to converge as the idealization of the shell was improved and as the size of the time step was decreased.

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