SN-168

# TRANSONIC FLOW IN A CONVERGING-DIVERGING NOZZLE

FINAL REPORT - CONTRACT NAS7-756



SEPTEMBER 1970

PREPARED BY:

JAY N. LEVINE

DOUGLAS E. COATS

DYNAMIC SCIENCE

A Division of Marshall Industries

2400 Michelson Drive, Irvine, California 92664 (714) 833-2670

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

SN-168

# TRANSONIC FLOW IN A CONVERGING-DIVERGING NOZZLE

Final Report - Contract NAS7-756

Prepared by: Jay N. Levine
Douglas E. Coats

Prepared for: National Aeronautics and Space Administration

September 1970

DYNAMIC SCIENCE
A Division of Marshall Industries
2400 Michelson Drive
Irvine, California 92664

## FOREWORD

This report describes the work performed at Dynamic Science, a Division of Marshall Industries, under NASA Contract No. NAS7-756, "Study of Transonic Flow in a Converging Diverging Nozzle."

The technical manager is Dr. Robert Cuffel of Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California.

The authors would like to acknowledge the contributions of Dr. James. R. Kliegel, who suggested this study, and Dr. Thomas J. Tyson. Their many helpful suggestions during the course of this study were appreciated.

## ABSTRACT

The transonic equations of motion for a converging diverging nozzle, including the effect of variable gamma, have been solved in toroidal coordinates using a combination of an asymptotic small parameter expansion and a double coordinate expansion. The analysis was kept general so that high order solutions could be recursively calculated. It was found that the use of toroidal coordinates and different expansion parameters did not significantly extend the range of normalized throat wall radii of curvature for which expansion solutions could be accurately calculated. An explanation of why expansion methods fail for small R is given. Calculations made, including the effect of variable gamma (for a homogeneous unstriated flow), indicate that its effect is negligible in the transonic region. A new technique for solving the subsonic portion of the nozzle flow is also described.

TABLE OF CONTENTS

## NOMENCLATURE

| | | |
|---|---|---|
| $a$ | — | sound speed |
| $a_n$ | — | coefficients in $\eta_w$ expansion, eq. (37) |
| $a_{N,m,n}$ | — | coefficients in the series expansion of $U_N$, eq. (51) |
| $A_i$ | — | known coefficients in subsonic difference equations, eq. (91) |
| $A_{N,m}$ | — | coefficients in the $\eta_w^N$ expansion, eq. (45) |
| $b_n$ | — | coefficients of the $1/\eta_w$ series, eq. (46) |
| $B_i$ | — | known coefficients in the subsonic difference equations, eq. (93) |
| $B_p$ | — | used in trig. and hyperbolic function expansions, eq. (41) |
| $\overline{B}_{p,m}$ | — | eq. (62) |
| $C_p$ | — | used in velocity derivative expansions, eq. (42) |
| $\overline{C}_{p,m,n}$ | — | eq. (60) |
| $D_k$ | — | groups of terms in the transonic momentum equation, eq. (43) |
| $\overline{D}_{T,u,v}$ | — | eq. (59) |
| $\mathscr{D}$ | — | parameter introduced to account for variable $\gamma$ effects |
| $e_n$ | — | coefficients in coth n series, eq. (47) |
| $E_Q$ | — | eq. (44) |
| $\overline{E}_{Q,K,L}$ | — | eq. (57) |
| $F_{N,M,Q,P}$ | — | used to represent products of velocity series multiplications, eq. (61) |
| $G_{i,j}$ | — | matrix of coefficients in the transonic equations |
| $h_1, h_2, h_3$ | — | metrics |
| $h$ | — | step size in x direction, $\Delta x$ |
| $H_j$ | — | column vector of unknowns is transonic equations |
| $K$ | — | step size in y direction, $\Delta y$ |
| $L_i$ | — | homogeneous terms in transonic equations |

| | | |
|---|---|---|
| $M$ | — | molecular weight |
| $P$ | — | pressure |
| $\underline{P}$ | — | eq. (26) |
| $q$ | — | velocity |
| $r$ | — | radial coordinate |
| $r_c$ | — | throat radius of curvature |
| $R$ | — | nondimensional throat radius of curvature, $r_c/r^*$ |
| $R_1, R_2$ | — | homogeneous terms in subsonic difference equations |
| $R$ | — | Universal Gas Constant |
| $S$ | — | entropy |
| $S_R$ | — | eq. (44) |
| $\overline{S}_{R,s,m}$, $\overline{S}_{R,q}$ | — | eq. (58) |
| $T$ | — | temperature |
| $u$ | — | longitudinal velocity |
| $v$ | — | transverse velocity |
| $u'$, $v'$ | — | velocity perturbations |
| $\overline{u}, \overline{v}$ | — | nondimensional velocities |
| $u_N, v_N$ | — | Nth order velocities |
| $u_{tr}$ | — | velocity on the subsonic "start line" |
| $x, y$ | — | cartesian coordinates, also transformed coordinates in subsonic analysis |
| $z$ | — | longitudinal coordinate |
| $\alpha$ | — | equals $(1 + \mathcal{D})/\Gamma_2$ |
| $\Gamma_1, \Gamma_2, \Gamma_3$ | — | constants, eq. (48) |
| $\gamma$ | — | ratio of specific heats |
| $\delta$ | — | special function, eq. (49) |
| $\epsilon$ | — | expansion parameter |
| $\eta$ | — | toroidal coordinate |
| $\bar{\eta}$ | — | normalized coordinate |
| $\theta$ | — | angle used in derivation of coordinate transformation |
| $\nu$ | — | special function, eq. (50) |

| | | |
|---|---|---|
| $\bar{\xi}$ | — | normalized coordinate |
| $\rho$ | — | density, complex variable in eq. (A-1) |
| $\sigma$ | — | direction cosine |

<u>Superscript</u>

| | | |
|---|---|---|
| $*$ | — | at the sonic condition (except $r^*$ is the throat radius) |

<u>Subscripts</u>

| | | |
|---|---|---|
| w | — | at the nozzle wall |
| 0 | — | at the throat axis point |

# I. INTRODUCTION

The transonic flow region in convergent divergent rocket nozzles has been widely studied, however, completely satisfactory solutions have yet to be achieved. The problem has been attacked by various expansion techniques (Refs. 1-6) ranging from double power series expansions to small parameter asymptotic expansions about the sonic condition. All of these expansion methods are similar, in the sense that they calculate perturbations about the one-dimensional flow solution. The deviation from one-dimensional flow is determined by the normalized throat wall radius of curvature, R, i.e., the ratio of the throat wall radius of curvature, $r_c$, to the throat radius, $r^*$. Although these techniques have been successfully applied to a variety of transonic flow problems, they have a common shortcoming; their inability to handle nozzles having small normalized throat radii of curvature, $R < 1$.

Another class of transonic solutions (actually combined subsonic-transonic solutions) consists of numerical solutions of the exact partial differential equations of motion, (e.g., Refs. 7-9). While such solutions are not subject to the limitation on R, they are subject to varying degrees of numerical instability and must go through lengthy iterations to satisfy the throat choked flow singularity. As a result, the numerical methods achieve solutions only at the cost of large amounts of computer time and money, and currently cannot be considered to be economically feasible engineering design aides.

In Reference 10, it was conjectured that the limitation of the expansion methods to $R \geq 1$ was due to the coordinate system employed (cylindrical) rather than a fundamental limitation of the method itself. In cylindrical coordinates, the nozzle wall boundary condition requires the flow angle to be equal to the local wall slope. The wall boundary is not a coordinate line in cylindrical coordinates and the boundary condition cannot be exactly satisfied. Also, the radial velocity, v, is proportional to the boundary slope, which can become large for $R < 1$. It was suggested that one could reasonably expect the accuracy of the solution to be improved by seeking a solution in toroidal coordinates, wherein both the wall and axis are coordinate lines and the boundary condition is reduced to its simplest form and can be exactly satisfied.

In addition, the normal coordinate lines would be approximately streamlines and the radial velocity would everywhere be small. It was also hypothesized that changing the expansion parameters from $1/R$ to $1/(1+R)^{+}$ would result in series expansions which were better behaved at small R, because the latter parameter does not become greater than unity for R less than one. Hall's solution was recast as series in $1/(1+R)$ in Reference 10 and the results, which were claimed to be the toroidal coordinate solution transformed to cylindrical coordinates, were very encouraging when compared to the data of Reference 11 for $R = .625$. These results provided the impetus for the current study to try and extend the region of applicability of the expansion technique by obtaining the transonic solutions directly in toroidal coordinates. It was also decided to formulate the equations so that the general nth order solution could be recursively generated.

The main thrust of the current effort was then directed towards obtaining nth order transonic expansion solutions in toroidal coordinates using a combination of an asymptotic parameter expansion and a double coordinate power series expansion. The development of the transonic equations and their solution is presented in Sections II and III and the computer program developed to perform the calculations is described in Appendix E.

In Section IV, a novel way of finding the flow field in the subsonic regime is developed, based on the assumption that a local transonic expansion solution can be used to generate a subsonic "start line" thereby eliminating the need to iterate to satisfy the mass flow singularity at the throat.

---

$^{+}(1/R)^{\frac{1}{2}}$ and $(1/R+1)^{\frac{1}{2}}$ is the terminology used herein.

## II. TRANSONIC EQUATIONS

### Transformation to Toroidal Coordinates

In order to write the equations of motion in toroidal coordinates, the transformations from cartesian and cylindrical coordinates to toroidal coordinates are required. The relationship between r, z and $\xi$, $\eta$ (the toroidal coordinates) is given in the following; together with the transformations for converting velocities in toroidal coordinates back to cylindrical coordinates. The derivations of these transformations and the metrics of the coordinate system, which are also needed, are outlined in Appendix A.

A circular arc throat forms a coordinate line in toroidal coordinates ($\eta$ = const.), hence, for a throat of height, $r^*$, and normalized radius of curvature, $R(= r_c/r^*)$, the transformation from cylindrical to toroidal coordinates becomes

$$\frac{r}{r^*} = \frac{(1 + 2R)^{\frac{1}{2}} \sinh \eta}{\cos \xi + \cosh \eta}$$

$$\frac{z}{r^*} = \frac{(1 + 2R)^{\frac{1}{2}} \sin \xi}{\cos \xi + \cosh \eta} \tag{1}$$

and the location of the throat wall is given by

$$\eta_w = \frac{1}{2} \ln \left[ \frac{1 + \frac{(1 + 2R)^{\frac{1}{2}}}{1 + R}}{1 - \frac{(1 + 2R)^{\frac{1}{2}}}{1 + R}} \right] \tag{2}$$

If $v_r$ and $v_z$ are used to denote the velocities in the r and z directions, respectively, and u and v are the velocities in the $\xi$ and $\eta$ directions; then

$$v_r = u \frac{\sinh \eta \sin \xi}{(\cos \xi + \cosh \eta)} + v\left[\cosh \eta - \frac{\sinh^2 \eta}{(\cos \xi + \cosh \eta)}\right]$$

$$v_z = u\left[\cos \xi + \frac{\sin^2 \xi}{(\cos \xi + \cosh \eta)}\right] - v \frac{\sin \xi \sinh \eta}{(\cos \xi + \cosh \eta)} \tag{3}$$

## Sound Speed Expansion

In order to account for the effects of variable gamma, the sound speed is expanded around the sonic condition as a function of pressure. However, the equations of motion will be written in terms of velocities, so Bernoulli's Equation is used to find pressure as a function of velocity. It is assumed that the equation of state is

$$P = \frac{\rho \mathcal{R} T}{M} \tag{4}$$

where $\mathcal{R}$ is the Universal Gas Constant and M is the molecular weight of the gas. The sound speed is given by

$$a^2 = \left.\frac{\partial P}{\partial \rho}\right|_s = \frac{\gamma P}{\rho} \tag{5}$$

where $\gamma$ is the ratio of specific heats, and is a function of the thermodynamic state of the gas. Since the nozzle flow is assumed to be irrotational and homentropic, $\gamma$ is a function of only one state variable.

$$\text{Let } \underline{P} = \int_{P^*}^{P} \frac{dP}{\rho}$$

$$\frac{\partial \underline{P}}{\partial P} = \frac{1}{\rho} \qquad \frac{\partial P}{\partial \underline{P}} = \rho \qquad \frac{\partial^2 P}{\partial \underline{P}^2} = \frac{\rho}{a^2} \tag{6}$$

Expand P as a function of $\underline{P}$ in a power series

$$\begin{aligned} P &= P^* + \left.\frac{\partial P}{\partial \underline{P}}\right|_* \underline{P} + \left.\frac{\partial^2 P}{\partial \underline{P}^2}\right|_* \underline{P}^2 + \ldots \\ &= P^* + \rho^* \underline{P} + \frac{\rho^*}{a^{*2}} \underline{P}^2 + \ldots \end{aligned} \tag{7}$$

Bernoulli's equation is

$$\frac{q^2}{2} - \frac{a^{*2}}{2} + \int_{P^*}^{P} \frac{dP}{\rho} = 0 \tag{8}$$

or

$$\underline{P} = \frac{a*^2 - q^2}{2}$$

Therefore, pressure as a function of velocity is

$$P - P* = \frac{\rho*}{2}(a*^2 - q^2) + \frac{\rho*}{4a*^2}(a*^2 - q^2)^2 + \ldots \tag{9}$$

Expanding the square of the sound speed, $a^2$, as a function of P gives

$$a^2 = a*^2 + \frac{\partial a^2}{\partial P}\Big|_*(P - P*) + \frac{\partial^2 a^2}{\partial P^2}\Big|_*(P - P*)^2 + \ldots \tag{10}$$

where

$$\begin{aligned} \frac{\partial a^2}{\partial P} &= \frac{\gamma - 1}{\rho} + \frac{a^2}{\gamma}\frac{\partial \gamma}{\partial P} \\ \frac{\partial^2 a^2}{\partial P^2} &= -\frac{(\gamma - 1)}{\rho^2 a^2} + \frac{(2\gamma - 1)}{\gamma\rho}\frac{\partial \gamma}{\partial P} + \frac{a^2}{\gamma}\frac{\partial^2 \gamma}{\partial P^2} \end{aligned} \tag{11}$$

Substituting (9) into (10) yields the desired sound speed expansion

$$\begin{aligned} a^2 = a*^2 + \frac{(\gamma* - 1)}{2}(a*^2 - q^2) &+ \left[\frac{a*^2\rho*}{\gamma*}\left(\frac{a*^2 - q^2}{2}\right)\right. \\ &\left. + 2\rho*\left(\frac{a*^2 - q^2}{2}\right)^2\right]\frac{d\gamma}{dP}\Big|_* + \frac{a*^2\rho*^2}{\gamma*}\left(\frac{a*^2 - q^2}{2}\right)^2\frac{d^2\gamma}{dP^2}\Big|_* \end{aligned} \tag{12}$$

Equations of Motion in Toroidal Coordinates

For inviscid, homentropic (and, therefore, irrotational) flows, the general equations of motion can be simply written in vector form as

$$\nabla \times \vec{q} = 0 \tag{13}$$

$$\vec{q} \cdot \frac{\nabla q^2}{2} - a^2\, \nabla \cdot \vec{q} = 0 \tag{14}$$

Equations for the curl, divergence and gradient in a general orthogonal coordinate system are given in Appendix A. Applying equations (A-17 - A-19) to the velocity vector $\vec{q} = u\,\vec{\alpha}_\xi + v\,\vec{\alpha}_\eta$, with the metrics and their derivatives given by equations (A-14 - A-16); and noting that $\partial/\partial\psi = 0$, yields the equations of motion in toroidal coordinates.

$$\frac{\sin\xi}{\cos\xi + \cosh\eta}\, v + v_\xi + \frac{u \sinh\eta}{\cos\xi + \cosh\eta} - u_\eta = 0 \tag{15}$$

$$(u^2 - a^2)u_\xi + (v^2 - a^2)v_\eta + uv(u_\eta + v_\xi) - a^2 \frac{2\sin\xi}{\cos\xi + \cosh\eta}\, u$$
$$- a^2 v\left[\coth\eta - \frac{2\sinh\eta}{\cos\xi + \cosh\eta}\right] = 0 \tag{16}$$

where $a^2$ is computed from (12), keeping only the first correction term

$$a^2 = a*^2 + \frac{(\gamma* - 1)}{2}(a*^2 - q^2) + \left[\frac{a*^2\,\rho*}{\gamma*}\left(\frac{a*^2 - q^2}{2}\right)\right]\frac{d\gamma}{dP}\Big|_* \tag{17}$$

It is better to work with nondimensional variables when series solutions are being sought, therefore, equations (15-17) have been nondimensionalized by assuming the following forms:

$$\overline{u} = \frac{u}{a*} \qquad \overline{v} = \frac{v}{a*} \qquad \overline{P} = \frac{P}{\rho*\, a*^2}$$

and a parameter, $\mathcal{B}$, which incorporates the effect of variable, $\gamma$, has been defined

$$\mathcal{B} = \frac{1}{\gamma*(\gamma*+1)}\,\frac{d\gamma}{d\overline{P}}\Big|_* \tag{18}$$

Inserting these relations into (15) - (17) results in the following nondimensional equations:

$$\frac{\sin \xi}{\cos \xi + \cosh \eta} \, \bar{v} + \bar{v}_{\xi} + \frac{\sinh \eta}{\cos \xi + \cosh \eta} \, \bar{u} - \bar{u}_{\eta} = 0 \tag{19}$$

$$\begin{aligned} &\left[(1+\mathcal{D}) - (1+\mathcal{D})\,\bar{u}^2 - \left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)\bar{v}^2\right]\bar{u}_{\xi} + \left[(1+\mathcal{D}) - (1+\mathcal{D})\,\bar{v}^2 - \left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)\bar{u}^2\right]\bar{v}_{\eta} \\ &- \frac{2}{\gamma^*+1}\,\bar{u}\,\bar{v}\,(\bar{u}_{\eta} + \bar{v}_{\xi}) + \left[(1+\mathcal{D}) - \left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)(\bar{u}^2 + \bar{v}^2)\right]\frac{2 \sin \xi}{\cos \xi + \cosh \eta}\,\bar{u} \\ &+ \left[(1+\mathcal{D}) - \left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)(\bar{u}^2 + \bar{v}^2)\right]\left(\coth \eta - \frac{2 \sinh \eta}{\cos \xi + \cosh \eta}\right)\bar{v} = 0 \end{aligned} \tag{20}$$

The equations are then specialized to the transonic regime by assuming that the velocity components can be written as

$$\begin{aligned} \bar{u} &= 1 + u' \\ \bar{v} &= v' \end{aligned} \tag{21}$$

Substitution of (21) into (19) and (20) gives the transonic equations of motion

$$\frac{\sin \xi}{\cos \xi + \cosh \eta} \, v' + v'_{\xi} + \frac{\sinh \eta}{\cos \xi + \cosh \eta} \, (1 + u') - u'_{\eta} = 0 \tag{22}$$

$$\begin{aligned} &\left[-(1+\mathcal{D})(2u' + u'^2) - \left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)v'^2\right]u'_{\xi} \\ &+ \left[\frac{2}{\gamma^*+1} - (1+\mathcal{D})\,v'^2 - 2\left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)u' - \left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)u'^2\right]v'_{\eta} - \frac{2}{\gamma^*+1}(v' + u'v')(u'_{\eta} + v'_{\xi}) \\ &+ \left[\frac{2}{\gamma^*+1} - 2\left(\frac{\gamma^*-2}{\gamma^*+1} + \mathcal{D}\right)u' - 3\left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)u'^2 - \left(\frac{\gamma^*-1}{\gamma^*+1} + \mathcal{D}\right)v'^2\right]\frac{2 \sin \xi}{\cos \xi + \cosh \eta} \end{aligned}$$

$$+\left[-\left(\frac{\gamma^*-1}{\gamma^*+1}+\mathcal{D}\right)u'^2-\left(\frac{\gamma^*-1}{\gamma^*+1}+\mathcal{D}\right)v'^2\right]\frac{2\sin\xi}{\cos\xi+\cosh\eta}\;u' \tag{23}$$

$$+\left[\frac{2}{\gamma^*+1}-2\left(\frac{\gamma^*-1}{\gamma^*+1}+\mathcal{D}\right)u'-\left(\frac{\gamma^*-1}{\gamma^*+1}+\mathcal{D}\right)u'^2-\left(\frac{\gamma^*-1}{\gamma^*+1}+\mathcal{D}\right)v'^2\right]\left[\coth\eta-\frac{2\sinh\eta}{\cos\xi+\cosh\eta}\right]v'=0$$

Parameter Expansion

The solution of equations (22) and (23) will be sought by a combination of an asymptotic parameter expansion and a double coordinate power series expansion. In carrying out these expansions, it is desirable to normalize the coordinates so that the scaled coordinates are of the same order. The proper forms for the velocity series must also be found.

Following Hall (Ref. 1 ) an expansion parameter involving the non-dimensional throat radius of curvature, R, will be used. The form of the expansion parameter, $\epsilon$, is dictated by the boundary condition at the throat wall.

$$v = 0 \text{ at } \eta = \eta_w \tag{24}$$

Without choosing $\epsilon$ it is shown in Appendix B that, in general, in order to obtain a nontrivial solution, the coordinates must be scaled as

$$\bar{\xi} = \frac{\xi}{\epsilon^2} \qquad \bar{\eta} = \frac{\eta}{\epsilon} \tag{25}$$

and the velocity expansions are of the form

$$\begin{aligned} u' &= \epsilon^2\, u_1(\bar{\xi}, \bar{\eta}) + \epsilon^4\, u_2(\bar{\xi}, \bar{\eta}) + \dots \\ v' &= \epsilon^3\, v_1(\bar{\xi}, \bar{\eta}) + \epsilon^5\, v_2(\bar{\xi}, \bar{\eta}) + \dots \end{aligned} \tag{26}$$

For large R, equations (25) and (26) should reduce to Hall's (Ref. 1 ) results if transformed back to cylindrical coordinates.

From equation (2) it can be shown that

$$\eta_w \doteq \frac{1}{R^{\frac{1}{2}}} \tag{27}$$

for large R, hence,

$$\lim_{R \to \infty} \epsilon \doteq \frac{1}{R^{\frac{1}{2}}} \tag{28}$$

and

$$\eta = 0\left(\frac{1}{R^{\frac{1}{2}}}\right) \qquad \xi = 0\left(\frac{1}{R}\right) \tag{29}$$

The order of $r/r^*$ and $z/r^*$ may then be found using equation (1)

$$\frac{r}{r^*} = 0(1) \quad \text{and} \quad \frac{z}{r^*} = 0\left(\frac{1}{R^{\frac{1}{2}}}\right) \tag{30}$$

and the velocity expansions become

$$\begin{aligned} u' &= \frac{u_1}{R} + \frac{u_2}{R^2} + \ldots \\ v' &= \frac{1}{R^{\frac{1}{2}}}\left[\frac{v_1}{R} + \frac{v_2}{R^2} + \ldots\right] \end{aligned} \tag{31}$$

As expected, Hall's results are reproduced in the limit of large R.

Hall treated only the case where $\epsilon = 1/R^{\frac{1}{2}}$, in cylindrical coordinates and his analysis was not applicable to nozzles with $R < 1$. Based on the favorable results presented in Ref. 10, it was felt that the use of toroidal coordinates, together with an expansion parameter which was well behaved for small R, would allow Hall type solutions to be extended to small values of R. Equation (28) represents the only restriction on allowable forms for $\epsilon$ and the solution to be outlined below does not add any additional constraints. While the method of solution will be valid for any $\epsilon$ (satisfying (28)) particular attention will be paid to the following two forms,

$$\epsilon = \frac{(1 + 2R)^{\frac{1}{2}}}{1 + R} \tag{32a}$$

and

$$\epsilon = \frac{1}{(1+R)^{\frac{1}{2}}} \tag{32b}$$

The first form is immediately suggested by the equation for $\eta_w$ (equation (2)), while the latter seemed to be a good choice based on the results of Ref. 10. Using equations (2) and (32a), the wall expansion becomes

$$\eta_w = \frac{1}{2} \ell n \frac{1+\epsilon}{1-\epsilon} = \epsilon + \frac{\epsilon^3}{3} + \frac{\epsilon^5}{5} + \ldots \tag{33}$$

while equations (2) and (32b) give

$$\eta_w = \epsilon \left[ 2^{\frac{1}{2}} + \sum_{n=1}^{\infty} a_n \epsilon^{2n} \right] \tag{34}$$

where

$$a_n = \frac{2^{(n+\frac{1}{2})}}{2n+1} + \sum_{\alpha=0}^{n-1} \frac{2^{(\alpha+\frac{1}{2})}}{2\alpha+1} \left[ \frac{(-\frac{1}{2})^{n-\alpha}}{(n-\alpha)!} \right] \prod_{j=0}^{(n-\alpha-1)} \left[ \alpha - \left( \frac{2j-1}{2} \right) \right] \tag{35}$$

Equations (33) and (34) both show that $\eta_w = 0(\epsilon)$, in agreement with equation (25). Since $\eta_w$ and $\epsilon$ are the same order, it was decided to first seek solutions with the coordinates normalized as follows:

$$\bar{\xi} = \frac{\xi}{\epsilon \eta_w} \qquad \bar{\eta} = \frac{\eta}{\eta_w} \tag{36}$$

so that the wall is always at $\bar{\eta} = 1$ and the resulting solution is universal for all R.

The nth order differential equations are obtained by expanding all of the terms in equations (22) and (23) and equating like powers of $\epsilon$. The velocities are expanded as in equation (26), the wall boundary, equation (2), is expanded in the form

$$\eta_w = \epsilon \sum_{n=0}^{\infty} a_n \epsilon^{2n} \tag{37}$$

where the $a_n$'s depend upon the choice of $\varepsilon$ (equations (33) - (35) give the $a_n$'s for two possible choices for $\varepsilon$). In addition to the above expansions, trigonometric and hyperbolic function expansions, products, and powers of series occur, and general formulae are required for expressing the resultant expansions in a form where all of the terms containing like powers of $\varepsilon$ are accumulated.

Given two series of the form

$$\sum_{n=c_2}^{\infty} b_n \varepsilon^{(c_3' + c_4 n)} \quad \text{and} \quad \sum_{m=c_1}^{\infty} a_m \varepsilon^{(\bar{c}_3 + c_5 n)}$$

A general formula has been derived for computing their product, wherein all terms of equal powers of $\varepsilon$ are explicitly determined. The formula is

$$\sum_{P=\alpha}^{\infty} \varepsilon^{c_4 P - \beta_1} \sum_{m=c_1}^{P-\beta_2} b_{\left[\left(\frac{c_4 P - \beta_1 - c_3}{c_4}\right) - m\right]} a_m \tag{38}$$

where

$$c_3 = c_3' + \bar{c}_3$$

$$\alpha = c_3 + c_4 (c_1 + c_2)$$

$$\beta_1 = \alpha (c_4 - 1)$$

$$\beta_2 = \alpha - c_1$$

and $\quad c_4 = c_5$

If $c_4 \neq c_5$, as occurs in the expansion of $\sin \xi$ and $\cos \xi$, a special function, $\delta$, can be defined which allows those expansions to be written in the same form as equation (38).

The diligent application of equation (38) to all of the terms in equations (22) and (23) yields the general, nth order, toroidal coordinate, differential equations. In order to make the analysis more practicable, several intermediate variables have been defined (A's, B's, C's, etc.). Appendix C has been included so that these variables may be more readily related to the terms appearing in equations (22) and (23).

The irrotational equation (22) contains only odd powers of $\epsilon$, say $\epsilon^{\alpha}$, $\alpha = 1, 3, 5 \ldots \infty$. If $P \equiv (\alpha + 1)/2$, then $P = 1, 2, 3 \ldots \infty$ is the order of the equation. The Pth order irrotational equation (containing all terms with $\epsilon^{2P-1}$) is

$$\nu(P-3) \sum_{R=1}^{P-2} B_{(P-1-R)_3} v_R + \sum_{R=1}^{P} \left(B_{(P-R)_4} + B_{(P-R)_2}\right)\left(C_{R_1} - C_{R_4}\right)$$
$$+ B_{P_1} + \nu(P-2) \sum_{R=1}^{P-1} B_{(P-R)_1} u_R = 0 \tag{39}$$

The P th order momentum equation, $P = 1, 2, 3 \ldots \infty$, contains all terms of order $\epsilon^{2P}$, and is given by

$$\sum_{K=1}^{P} E_{(P-K)_1} D_{K_1} + \Gamma_2 E_{P_2} + \nu(P-2) \sum_{K=1}^{P-1} E_{(P-K)_2} D_{K_2} - \nu(P-2)\Gamma_2 \sum_{K=1}^{P-1} E_{(P-K)_3} D_{K_3}$$
$$+ 2\Gamma_2 B_{P_3} + 2\nu(P-2) \sum_{K=1}^{P-1} B_{(P-K)_3} D_{K_4} + 2\nu(P-3) \sum_{K=1}^{P-2} E_{(P-K)_4} D_{K_5} \tag{40}$$
$$+ \Gamma_2 E_{P_5} + \nu(P-2) \sum_{K=1}^{P-1} E_{(P-K)_5} D_{K_6} = 0$$

where

$$B_{P_1} = \sum_{m=1}^{\infty} \frac{A_{(-1+2m),(P-m)}\,\bar{\eta}^{(-1+2m)}}{(-1+2m)!}$$

$$B_{P_2} = \sum_{m=0}^{P} \frac{\bar{\eta}^{2m}\,A_{2m,(P-m)}}{2m!}$$

$$B_{P_3} = \sum_{n=0}^{P-1} \frac{\delta\,(P-1-n)\,(-1)^{(P-1-n)/2}\,\bar{\xi}^{(P-n)}\,A_{(P-n),n}}{(P-n)!} \tag{41}$$

$$B_{P_4} = \sum_{n=0}^{P} \frac{\delta\,(P-n)\,(-1)^{(P-n)/2}\,\bar{\xi}^{(P-n)}\,A_{(P-n),n}}{(P-n)!}$$

$$B_{P_5} = \sum_{m=0}^{P} e_m\,\bar{\eta}^{2m-1}\,A_{(2m-1),(P-m)}$$

$$C_{P_1} = \sum_{n=1}^{P} b_{(P-n)}\,v_{n_{\bar{\xi}}} \qquad C_{P_3} = \sum_{n=1}^{P+1} b_{(P+1-n)}\,u_{n_{\bar{\xi}}}$$

$$C_{P_2} = \sum_{n=1}^{P} b_{(P-n)}\,v_{n_{\bar{\eta}}} \qquad C_{P_4} = \sum_{n=1}^{P} b_{(P-n)}\,u_{n_{\bar{\eta}}} \tag{42}$$

$$D_{K_1} = -(1+\mathcal{B})\left[2\,u_K + \nu\,(K-2)\sum_{n=1}^{K-1} u_{(K-n)}\,u_n\right] - \nu\,(K-3)\,\Gamma_1 \sum_{n=1}^{K-2} v_{(K-1-n)}\,v_n$$

$$D_{K_2} = -(1+\mathcal{B})\,\nu\,(K-3)\sum_{n=1}^{K-2} v_{(K-1-n)}v_n - 2\Gamma_1\,u_K - \Gamma_1\,\nu\,(K-2)\sum_{n=1}^{K-1} u_{(K-n)}\,u_n \tag{43}$$

$$D_{K_3} = v_K + \nu\,(K-2)\sum_{n=1}^{K-1} u_{(K-n)}\,v_n$$

$$D_{K_4} = -\Gamma_3 u_K - \nu(K-2)3\Gamma_1 \sum_{n=1}^{K-1} u_{(K-n)} u_n - \nu(K-3)\Gamma_1 \sum_{n=1}^{K-2} v_{(K-1-n)} v_n$$

$$D_{K_5} = -\nu(K-2)\Gamma_1 \sum_{n=1}^{K-1} u_{(K-n)} u_n - \nu(K-3)\Gamma_1 \sum_{n=1}^{K-2} v_{(K-1-n)} v_n \qquad (43)\ \text{Cont.}$$

$$D_{K_6} = -2\Gamma_1 u_K - \nu(K-2)\Gamma_1 \sum_{n=1}^{K-1} u_{(K-n)} u_n - \nu(K-3)\Gamma_1 \sum_{n=1}^{K-2} v_{(K-1-n)} v_n$$

$$E_{Q_1} = \sum_{R=0}^{Q} C_{(Q-R)_3}\left(B_{R_4} + B_{R_2}\right)$$

$$E_{Q_2} = \sum_{R=0}^{Q-1} C_{(Q-R)_2}\left(B_{R_4} + B_{R_2}\right)$$

$$E_{Q_3} = \sum_{R=0}^{Q-1} \left(C_{(Q-R)_4} + C_{(Q-R)_1}\right)\left(B_{R_4} + B_{R_2}\right) \qquad (44)$$

$$E_{Q_4} = \sum_{R=1}^{Q-1} u_{(Q-R)} B_{R_3}$$

$$E_{Q_5} = \sum_{R=0}^{Q-1} v_{(Q-R)} S_R - 2\nu(Q-2) \sum_{R=1}^{Q-1} v_{(Q-R)} B_{R_1}$$

$$S_R = \sum_{j=0}^{R} B_{(R-j)_5}\left(B_{j_4} + B_{j_2}\right)$$

$$A_{o,n} = 1$$

$$A_{1,n} = a_n$$

$$A_{N,n} = \sum_{i=0}^{n} a_{(n-i)} A_{N-1,i} \qquad N = 2 \ldots \infty \qquad (45)$$

$$A_{-1,n} = b_n$$

$$b_o = \frac{1}{a_o}$$

$$b_n = -\frac{1}{a_o} \sum_{m=0}^{n-1} a_{(n-m)} b_m \qquad n = 1, 2, \ldots \infty \tag{46}$$

$$e_o = 1$$

$$e_n = \frac{1}{(2n)!} - \sum_{m=0}^{n-1} \frac{e_m}{[2(n-m)+1]!} \qquad n = 1, 2 \ldots \infty \tag{47}$$

$$\Gamma_1 = \left(\frac{\gamma^* - 1}{\gamma^* + 1} + \mathscr{D}\right) \qquad \Gamma_2 = \frac{2}{\gamma^* + 1}$$

$$\Gamma_3 = 2\left(\frac{\gamma^* - 2}{\gamma^* + 1} + \mathscr{D}\right) \tag{48}$$

$$\delta(s) = -\tfrac{1}{2}\left[(-1)^{s+1} - 1\right] \tag{49}$$

i.e., for s even $\quad \delta(s) = 1$

for s odd $\quad \delta(s) = 0$

and

$$\nu(s) = 0 \text{ for } s < 0$$
$$= 1 \text{ for } s \geq 0 \tag{50}$$

Coordinate Expansion

To solve the differential equations for a given order, each of the velocity coefficients (i.e., $u_1$, $u_2$ ...; $v_1$, $v_2$...) is expanded in a double power series in $\overline{\xi}$ and $\overline{\eta}$. From the symmetry of the problem, the u coefficients must be even functions of $\overline{\eta}$ and the v coefficients odd functions. Thus,

$$u_N = \sum_{m=0}^{N} \sum_{n=0}^{N-m} a_{N,m,n} \overline{\xi}^{m} \overline{\eta}^{2n}$$

$$v_N = \sum_{m=0}^{N} \sum_{n=0}^{N-m} b_{N,m,n} \overline{\xi}^{m} \overline{\eta}^{(2n+1)} \tag{51}$$

In the course of expanding equations (39) and (40), many series multiplications must be carried out and the resultant product series expressed in a form wherein all of the terms involving like powers of $\overline{\xi}$ and $\overline{\eta}$ are explicitly collected. The use of the general formula (38), properly defined $\nu$ functions, and the following general formulae for interchanging the order of summations make this long and difficult task a bit more practicable.

Given a double summation of the form

$$\sum_{m=A}^{B} \sum_{n=C}^{m-\alpha}$$

it is equivalent to

$$\sum_{n=C}^{B-\alpha} \sum_{m=n+\alpha}^{B} \nu(m-A) \tag{52}$$

since $A - \alpha \geq C$

Also, for $A \geq K$

$$\sum_{m=0}^{K} \sum_{n=0}^{A-m} = \sum_{n=0}^{A} \sum_{m=0}^{A-n} \nu(K-m) \tag{53}$$

Using the formulae given by (38), (52), and (53), all of the series multiplications required in expanding equations (39) and (40) were carried out. Again, several new variables have been introduced to avoid writing long strings of summations. Appendix D has been included to indicate the origin of these variables. In solving the Pth order differential equations, it can be seen from equation (51) that there are a total of (P + 1) (P + 2) unknown coefficients in the velocity expansions ($a_{P,m,n's}$ and $b_{P,m,n's}$). Upon collecting terms having like powers of $\bar{\xi}$ and $\bar{\eta}$, it is found that the Irrotational Equation (39) generates (P/2)(P + 1) equations, the Momentum Equation (40) yields .5(P+1)(P+2) equations and the remaining (P+1) equations come from the boundary condition (24).

The (P + 1)(P + 2) equations that result from expanding the Pth order differential equations are presented below in a format in which all of the unknowns appear on the left hand sides of the equations.

<u>Irrotational Equations</u>

$$\sum_{K=0}^{P-1} \sum_{L=0}^{P-1-K} \left[ \frac{2(K+1)}{a_o} b_{P,K+1,L} - \frac{4(L+1)}{a_o} a_{P,K,L+1} \right] =$$

$$\sum_{K=0}^{P-1} \sum_{L=0}^{P-1-K} \Big\{ -\nu(K-1)\,\nu(P-3) \sum_{M=0}^{P-2-L} \nu(K-1-M) \sum_{R=M+L}^{P-2} \nu(R-1)\,\nu(P-1-R+M-K)$$

$$\bar{B}_{(P-1-R),(P-1-R+M-K)_3}\, b_{R,M,L} - \sum_{M=0}^{P-1-L} \nu(K-M)\,\nu(P-2-M-L) \tag{54}$$

$$\sum_{R=M+L+1}^{P-1} \nu(P-R+M-K)\,\bar{B}_{(P-R),(P-R+M-K)_4} \sum_{i=M+L+1}^{R} A_{-1,(R-i)}(M+1)\, b_{i,M+1,L}$$

$$-\,2\,\nu(P-2-K-L) \sum_{i=K+L+1}^{P-1} A_{-1,(P-1)}(K+1)\, b_{i,(K+1),L} - \sum_{M=0}^{L} \nu(P-2-K) \quad .$$

$$\sum_{R=K+1}^{P-1} \nu(R-1-K-M)\ \nu(P-R-L+M)\ \overline{B}_{(P-R),(L-M)_2} \sum_{i=K+M+1}^{R} A_{-1,(R-i)}\, (K+1)\, b_{i,K+1,M}$$

$$+ \sum_{m=0}^{P-1-L} \nu(K-M)\ \nu(P-2-M-L) \sum_{R=M+L+1}^{P-1} \nu(P-R+M-K)\overline{B}_{(P-R),(P-R+M-K)_4}$$

$$\sum_{i=M+L+1}^{R} A_{-1,(R-i)}\ 2(L+1)\, a_{i,M,L+1} + 2\ \nu(P-2-K-L) \quad .$$

$$\sum_{i=K+L+1}^{P-1} A_{-1,(P-1)}\, 2(L+1)\ a_{i,K,L+1} + \sum_{M=1}^{L+1} \nu(P-2-K) \qquad \text{(54) Cont.}$$

$$\sum_{R=K+1}^{P-1} \nu(R-K-M)\ \nu(P-R-L-1+M)\ \overline{B}_{(P-R),(L+1-M)_2}$$

$$\sum_{i=K+M}^{R} A_{-1,(R-i)}\, (2M)\ a_{i,K,M} - \nu(-K)\ \overline{B}_{P,(L+1)_1}$$

$$- \sum_{M=0}^{L} \sum_{R=K}^{P-1} \nu(R-1)\ \nu(R-K-M)\ \nu(P-R-L-1+M)\ \overline{B}_{(P-R),(L+1-M)_1} a_{R,K,M} \Big\}$$

<u>Momentum Equations</u>

$$\sum_{G=0}^{P} \sum_{H=0}^{P-G} \left\{ \frac{4\Gamma_2(H+1)}{a_o}\ b_{P,G,H} - 2(1+\vartheta)\left[ \frac{2a_{1,1,0}}{a_o}\ a_{P,G,H} + \right.\right.$$

$$\left.\left. \sum_{K=0}^{G} \nu(P-1-K) \sum_{L=0}^{H} \nu(P-1-K-L)\ \nu(1+K+L-G-H)\ a_{1,G-K,H-L}\left(\frac{2(K+1)}{a_o}\right) a_{P,K+1,L} \right]\right\} = \qquad (55)$$

$$\sum_{G=0}^{P} \sum_{H=0}^{P-G} \Big\{ -\nu(P-3) \sum_{T=2}^{P-1} \sum_{K=0}^{G} \nu(P-T-K) \sum_{L=0}^{H} \nu(P-T-L)\ \nu(T-G+K-H+L) \quad .$$

$$\overline{E}_{(P-T),K,L_1}\,\overline{D}_{T,(G-K),(H-L)_1} + \frac{2a_{1,1,0}}{a_o}\left[\nu(P-2)(1+\vartheta)\left(\sum_{j=1}^{P-1} F_{(P-j),j,G,H_1}\right) +\right.$$

$$\nu(P-3)\;\nu(H-1)\;\nu(P-1-G)\Gamma_1 \sum_{j=1}^{P-2} F_{(P-1-j),j,G,H_2}\Big] + 2(1+\vartheta) \sum_{k=0}^{G} \nu(P-1-K) \quad \cdot$$

$$\sum_{L=0}^{H} \nu(P-1-L)\;\nu(1+K+L-G-H)\,a_{1,(G-K),(H-L)}\,\overline{E}'_{(P-1),K,L_1} -$$

$$\Gamma_2\,\overline{E}'_{P,G,H_2} - 2\Gamma_2\;\nu(P-1-G-H) \sum_{i=G+H}^{P-1} \nu(i-1)\,b_{(P-i)}(2H+1)\,b_{i,G,H} \quad -$$

$$\sum_{T=1}^{P-1} \sum_{K=0}^{G} \nu(P-T-K) \sum_{L=0}^{H} \nu(P-T-L)\;\nu(T-G+K-H+L)\Big[\overline{E}_{(P-T),K,L_2} \quad \cdot$$

(55) Cont.

$$\overline{D}_{T,(G-K),(H-L)_2} + \overline{E}_{(P-T),K,L_5}\,\overline{D}_{T,(G-K),(H-L)_6}\Big] + \nu(P-1-G)\,\nu(H-1)\Gamma_2 \quad \cdot$$

$$\sum_{T=1}^{P-1} \sum_{K=0}^{G} \nu(P-T-1-K) \sum_{L=0}^{H-1} \nu(P-T-1-L)\;\nu(T-G+K-H+1+L)\,\overline{E}_{(P-T),K,L_3} \quad \cdot$$

$$\overline{D}_{T,(G-K),(H-1-L)_3} - \nu(G-1)\;\nu(-H)\;2\Gamma_2\,\overline{B}_{P,(P-G)_3} - \nu(G-1)2 \quad \cdot$$

$$\sum_{K=0}^{P-1-H} \nu(G-K)\;\nu(P-1-H-K) \sum_{T=H+K}^{P-1} \nu(T-1)\;\nu(K+P-T-G)\,\overline{D}_{T,K,H_4} \quad \cdot$$

$$\left.\overline{B}_{(P-T),(K+P-T-G)_3}\right\}$$

Boundary Condition

The boundary condition $V_N(\xi, 1) = 0$ becomes

$$\sum_{m=0}^{P} \left[ \sum_{n=0}^{P-m} b_{P,m,n} \right] = 0 \tag{56}$$

where

$$\bar{E}_{Q,K,L_1} = \sum_{M=0}^{Q-L} \nu(K-M) \sum_{R=0}^{Q-M-L} \nu(Q-R-M)\ \nu(M+R-K)\ \bar{C}_{(Q-R),M,L_3} \bar{B}_{R,(M+R-K)_4}$$

$$+ \nu(Q-K-L) \sum_{R=0}^{Q-K} \sum_{M=0}^{L} \nu(Q-R-K-M)\ \nu(R-L+M)\ \bar{C}_{(Q-R),K,M_3} \bar{B}_{R,(L-M)_2}$$

$$\bar{E}_{Q,K,L_2} = \sum_{M=0}^{Q-L} \nu(K-M) \sum_{R=0}^{Q-M-L} \nu(Q-1-R)\nu(Q-R-M-L)\nu(M+R-K)\ \bar{C}_{(Q-R),M,L_2}$$

$$\bar{B}_{R,(M+R-K)_4} \tag{57}$$

$$+ \nu(Q-K-L) \sum_{R=0}^{Q-K} \nu(Q-1-R) \sum_{M=0}^{L} \nu(Q-R-K-M)\nu(R-L+M)\ \bar{C}_{(Q-R),K,M_2} \bar{B}_{R,(L-M)_2}$$

$$\bar{E}_{Q,K,L_3} = \sum_{M=0}^{Q-1-L} \nu(K-M) \sum_{R=0}^{Q-1-M-L} \nu(Q-R-1-M)\ \nu(M+R-K)$$

$$\bar{B}_{R,(M+R-K)_4} \left[ \bar{C}_{(Q-R),M,L+1_4} + \bar{C}_{(Q-R),M,L_1} \right]$$

$$+ \nu(Q-1-K-L) \sum_{R=0}^{Q-1-K} \sum_{M=0}^{L} \nu(Q-1-R-K-M)\ \nu(R-L+M) \quad .$$

$$\bar{B}_{R,(L-M)_2} \left[ \bar{C}_{(Q-R),K,M+1_4} + \bar{C}_{(Q-R),K,M_1} \right]$$

$$\mathrm{E}_{Q,K,L_4} = \sum_{M=0}^{Q-L-1} \nu(K-M) \sum_{R=0}^{Q-M-L-1} \nu(Q-2-R)\ \nu(Q-R-M-1)\ \nu(M+R-K)$$

$$a_{(Q-R-1),M,L}\ \overline{B}_{(R+1),(M+R-K)_3}$$

$$\overline{E}_{Q,K,L_5} = \nu(Q-K-L)\left[\sum_{R=0}^{Q-1}\ \sum_{C=0}^{K}\ \sum_{d=0}^{L} \nu(Q-R-C-d)\,\nu(R-K+C-L+d)\ \overline{S}_{R,(K-C),(L-d)}\right.$$

$$b_{(Q-R),C,d}$$

$$+ \sum_{R=0}^{Q-K} \nu(Q-1-R) \sum_{d=0}^{L} \nu(Q-R-K-d)\ \nu(R-L+d)\ \overline{S}_{R,(L-d)}\ b_{(Q-R),K,d}$$

$$-\ 2\nu(L-1) \sum_{R=0}^{Q-1-K} \nu(Q-2-R) \sum_{d=0}^{L-1} \nu(Q-1-R-K-d)\ \nu(R+1-L+d)\quad .$$

$$\left.\overline{B}_{(R+1),(L-d)_1}\ b_{(Q-1-R),K,d}\right] \qquad (57)\ \text{Cont.}$$

$$\overline{E}'_{Q,K,L_1} = \sum_{M=0}^{Q-L} \nu(K-M) \sum_{R=1}^{Q-M-L} \nu(Q-R-M-L)\ \nu(M+R-K)\ \overline{C}_{Q-R,M,L_3}$$

$$\overline{B}_{R,(M+R-K)_4} + \nu(Q-K-L) \sum_{R=1}^{Q-K}\ \sum_{M=0}^{L} \nu(Q-R-K-M)\ \nu(R-L+M)\quad .$$

$$\overline{C}_{Q-R,K,M_3}\ \overline{B}_{R,(L-M)_2} + \nu(Q-K-L)\,2 \sum_{i=K+L+1}^{Q} b_{(Q+1-i)}^{\ (K+1)}\ a_{i,K+1,L}$$

$\overline{E}'_{Q,K,L_2} = \overline{E}_{Q,K,L_2}$ with sums on R beginning at 1 instead of 0.

$\overline{E}'_{Q,K,L_5} = \overline{E}_{Q,K,L_5}$ with sums on R beginning at 1 instead of 0, in the first two sums only. (57) Cont.

$$\overline{S}_{R,s,m} = \sum_{n=0}^{R-s-m} \overline{B}_{(R-n-s),m_5} \overline{B}_{(n+s),n_4}$$

(58)

$$\overline{S}_{R,q} = \sum_{n=0}^{q} \sum_{j=0}^{R} \nu(j-n)\,\nu(R-j-q+n)\ \overline{B}_{(R-j),(q-n)_5}\ \overline{B}_{j,n_2}$$

$$\overline{D}_{T,u,v_1} = -(1+\mathcal{B})\left[2\,a_{T,u,v} + \nu(T-2)\sum_{j=1}^{T-1} F_{(T-j),j,u,v_1}\right]$$

$$- \nu(T-3)\ \nu(v-1)\ \nu(T-1-u)\,\Gamma_1 \sum_{j=1}^{T-2} F_{(T-1-j),j,u,v_2}$$

$$\overline{D}_{T,u,v_2} = -\nu(T-3)\,\nu(v-1)\,\nu(T-1-u)\,(1+\mathcal{B}) \sum_{j=1}^{T-2} F_{(T-1-j),j,u,v_2} - 2\Gamma_1\, a_{T,u,v}$$

$$- \nu(T-2)\Gamma_1 \sum_{j=1}^{T-1} F_{(T-j),j,u,v_1}$$

(59)

$$\overline{D}_{T,u,v_3} = b_{T,u,v} + \nu(T-2) \sum_{j=1}^{T-1} F_{(T-j),j,u,v_3}$$

$$\overline{D}_{T,u,v_4} = -\Gamma_3\, a_{T,u,v} - \nu(T-2)\ 3\,\Gamma_1 \sum_{j=1}^{T-1} F_{(T-j),j,u,v_1}$$

$$- \nu(T-3)\ \nu(v-1)\ \nu(T-1-u)\ \Gamma_1 \sum_{j=1}^{T-2} F_{(T-1-j),j,u,v_2}$$

$$\overline{D}_{T,u,v_5} = -\nu(T-2)\,\Gamma_1 \sum_{j=1}^{T-1} F_{(T-j),j,u,v_1}$$

$$- \nu(T-3)\ \nu(v-1)\ \nu(T-1-u)\,\Gamma_1 \sum_{j=1}^{T-2} F_{(T-1-j),j,u,v_2}$$

$$\overline{D}_{T,u,v_6} = -2\Gamma_1 a_{T,u,v} - \nu(T-2)\Gamma_1 \sum_{j=1}^{T-1} F_{(T-j),j,u,v_1} \qquad (59)\ \text{Cont.}$$

$$- \nu(T-3)\ \nu(v-1)\ \nu(T-1-u)\ \Gamma_1 \sum_{j=1}^{T-2} F_{(T-1-j),j,u,v_2}$$

$$\overline{C}_{P,m,n_1} = \sum_{i=m+n+1}^{P} b_{(P-i)}\ b_{i,m+1,n}\ (m+1)$$

$$\overline{C}_{P,m,n_2} = \sum_{i=m+n}^{P} \nu_{(i-1)}\ b_{(P-i)}\ (2n+1)\ b_{i,m,n}$$

$$\overline{C}_{P,m,n_3} = \sum_{i=m+n+1}^{P+1} b_{(P+1-i)}\ (m+1)\ a_{i,m+1,n} \qquad (60)$$

$$\overline{C}_{P,m,n_4} = \sum_{i=m+n}^{P} b_{(P-i)}\ (2n)\ a_{i,m,n}$$

$$F_{N,M,Q,P_1} = \sum_{m=0}^{Q} \sum_{n=0}^{P} \nu(N-m-n)\ \nu(M-Q+m-P+n)\ a_{N,m,n}\ a_{M,(Q-m),(P-n)}$$

$$F_{N,M,Q,P_2} = \sum_{m=0}^{Q} \sum_{n=0}^{P-1} \nu(N-m-n)\ \nu(M-Q+m-P+1+n)\ b_{N,m,n}\ b_{M,(Q-m),(P-1-n)} \qquad (61)$$

$$F_{N,M,Q,P_3} = \sum_{m=0}^{Q} \sum_{n=0}^{P} \nu(N-m-n)\ \nu(M-Q+m-P+n)\ a_{N,m,n}\ b_{M,(Q-m),(P-n)}$$

$$\overline{B}_{P,m_1} = \frac{A_{(-1+2m),(P-m)}}{(-1+2m)!}$$

$$\overline{B}_{P,m_2} = \frac{A_{2m,(P-m)}}{(2m)!} \qquad (62)$$

$$\bar{B}_{P,m_3} \quad \frac{\delta(P-1-m)(-1)^{(P-1-m)/2} \; A_{(P-m),m}}{(P-m)!}$$

$$\bar{B}_{P,m_4} = \frac{\delta(P-m)(-1)^{(P-m)/2} \; A_{(P-m),m}}{(P-m)!} \qquad (62) \text{ Cont.}$$

$$\bar{B}_{P,m_5} = e_m \, A_{(2m-1),(P-m)}$$

and the A's, a's, b's, e's, $\Gamma$'s, $\delta$ and $\nu$ have all been defined earlier.

## III. TRANSONIC SOLUTIONS

### First Order Solutions

The first order solution must be known before the higher order solutions can be recursively solved; since, in general, the Nth order solution depends on the velocity coefficients up to the (N-1)st order.

The first order differential equations can either be obtained directly from the original differential equations (19 and 20) through the use of the general Nth order differential equations (39 and 40) with P=1, or from equations (54) and (55). The equations depend upon the choice of the expansion parameter, $\varepsilon$. For $\varepsilon$, given by equation (32a) they are

$$v_{1_{\bar{\xi}}} - u_{1_{\bar{\eta}}} + \frac{\bar{\eta}}{2} = 0 \tag{63}$$

$$-2\alpha u_1 u_{1_{\bar{\xi}}} + v_{1_{\bar{\eta}}} + \bar{\xi} + \frac{v_1}{\bar{\eta}} = 0 \tag{64}$$

where $\alpha = (1+\mathcal{B})/\Gamma_2$. While for $\varepsilon$ given by equation (32b) the first order equations are

$$v_{1_{\bar{\xi}}} - u_{1_{\bar{\eta}}} + \bar{\eta} = 0 \tag{65}$$

$$-2\alpha u_1 u_{1_{\bar{\xi}}} + v_{1_{\bar{\eta}}} + 2\bar{\xi} + \frac{v_1}{\bar{\eta}} = 0 \tag{66}$$

The boundary condition is:

$$v_1(\bar{\xi}, 1) = 0 \tag{67}$$

The first order equations are solved using the same method that will be used to solve the higher order equations, i.e., expanding the velocities in double power series in $\bar{\xi}$ and $\bar{\eta}$ and equating the resultant terms in the differential equations and boundary condition which contain like powers of

$\bar{\xi}$ and $\bar{\eta}$. For illustrative purposes, the solution of equations (63 and 64) is worked out below.

The solution proceeds as follows:

Let

$$u_1 = a_{00} + a_{01}\,\bar{\eta}^2 + a_{10}\,\bar{\xi}$$
$$v_1 = b_{00}\,\bar{\eta} + b_{01}\,\bar{\eta}^3 + b_{10}\,\bar{\xi}\,\bar{\eta} \tag{68}$$

Then inserting (68) into (63) and (64) and equating like powers of $\bar{\xi}$ and $\bar{\eta}$ yields

$$b_{10} = 2a_{01} - 1/2$$

and

$$-\alpha\, a_{00}\, a_{10} + b_{00} = 0$$
$$-\alpha\,(a_{01}\, a_{10}) + 2\, b_{01} = 0 \tag{69}$$
$$-\alpha\, a_{10}^{\,2} + b_{10} + 1/2 = 0$$

The boundary condition (67) supplies the remaining two equations

$$b_{00} + b_{01} = 0$$
$$b_{10} = 0 \tag{70}$$

Equations (69) and (70) are easily solved and give

$$u_1 = -\frac{1}{8} + \frac{1}{4}\,\bar{\eta}^2 + \frac{1}{(2\alpha)^{\frac{1}{2}}}\,\bar{\xi}$$
$$v_1 = -\frac{1}{8}\left(\frac{\alpha}{2}\right)^{\frac{1}{2}}\bar{\eta} + \frac{1}{8}\left(\frac{\alpha}{2}\right)^{\frac{1}{2}}\bar{\eta}^3 \tag{71}$$

Equations (65 and 66) can be solved in a similar manner to give

$$u_1 = -\frac{1}{4} + \frac{1}{2}\bar{\eta}^2 + \frac{1}{(\alpha)^{\frac{1}{2}}}\bar{\xi}$$
$$v_1 = -\frac{(\alpha)^{\frac{1}{2}}}{4}\bar{\eta} + \frac{(\alpha)^{\frac{1}{2}}}{4}\bar{\eta}^3 \tag{72}$$

For comparative purposes, the transonic solution has also been worked out assuming the coordinates are normalized as

$$\bar{\xi} = \frac{\xi}{\epsilon^2} \quad \text{and} \quad \bar{\eta} = \frac{\eta}{\epsilon} \tag{73}$$

instead of as in equation (36). When the coordinates are normalized in this manner, the $\eta_w$ expansion enters the solution only through the boundary condition

$$v' = 0 \quad \text{at} \quad \eta = \eta_w \tag{74}$$

hence, it does not directly enter into the differential equations and the differential equations remain the same for all $\epsilon$. To first order, the resulting differential equations are the same as (63) and (64), however, the boundary condition is changed to

$$v_1(\bar{\xi}, 1) = 0 \quad \text{at} \quad \bar{\eta} = \bar{\eta}_w = \frac{\eta_w}{\epsilon} \tag{75}$$

and the solution becomes

$$u_1 = -\frac{1}{8}\bar{\eta}_w^2 + \frac{1}{4}\bar{\eta}^2 + \frac{1}{\sqrt{2\alpha}}\bar{\xi}$$
$$v_1 = -\frac{1}{8}\left(\frac{\alpha}{2}\right)^{\frac{1}{2}}\bar{\eta}_w^2\,\bar{\eta} + \frac{1}{8}\left(\frac{\alpha}{2}\right)^{\frac{1}{2}}\bar{\eta}^3 \tag{76}$$

The effect of using different $\epsilon$'s is reflected in the value of $\bar{\eta}_w$.

### Higher Order Transonic Solutions

With the first order solutions given previously, higher order solutions can be found recursively, using equations (54) - (56). In general, the Pth order solution depends only upon the previous solutions up to (P-1)st order and contains (P + 1)(P + 2) unknown velocity coefficients. For second, and higher orders, the equations are all linear and may be conveniently written in matrix form as

$$G_{i,j} H_j = L_i \tag{77}$$

where G is a $(P+1)(P+2) \times (P+1)(P+2)$ matrix consisting of the coefficients of the unknown $a_{P,i,j}$ and $b_{P,i,j}$. $H_j$ is the column vector of unknowns and $L_i$ contains the homogeneous terms which depend upon the lower order solutions. These equations are easily solved, and in principle, there is no limit on the maximum order of solution which can be obtained. In practice, however, one is limited by the core size and machine time required to invert a $(P+1)(P+2) \times (P+1)(P+2)$ matrix, for large P. Also, the parameter expansion is asymptotic, hence, one would not expect the velocity series (equation 26) to be infinitely convergent. It can be expected that after an initial convergence trend, higher order solutions will begin to diverge. The number of terms that can be calculated before divergence occurs should be a function of the expansion parameter, $\epsilon$.

It is not computationally feasible to obtain high order solutions of equations (54)-(56) by hand, so a computer program, described in Appendix E, was written to solve the equations. Since the equations are so lengthy, and so difficult to both derive and program without error, every effort was made to continually check intermediate results. As part of this effort, the second order equations were derived and solved, by hand, two different ways (from the original differential equations and using equations (54)-(56)) to serve as a standard for checking out the computer program.

For $\epsilon = 1/(1+R)^{\frac{1}{2}}$, the second order solution was found to be

$$u_2 = \left[\frac{10\gamma+57}{288} + \frac{5}{144}(\gamma+1)\mathcal{D}\right] - \left[\frac{4\gamma+11}{24} + \frac{1}{6}(\gamma+1)\mathcal{D}\right]\bar{\eta}^2 + \left[\frac{2\gamma+5}{24} + \frac{1}{12}(\gamma+1)\mathcal{D}\right]\bar{\eta}^4$$

$$- \frac{11}{24}\frac{1}{\alpha^{\frac{1}{2}}}\bar{\xi} + \frac{\bar{\xi}\bar{\eta}^2}{2\alpha^{\frac{1}{2}}} - \frac{1}{\alpha}\left[\frac{2\gamma-3}{6} + \frac{(\gamma+1)\mathcal{D}}{3}\right]\bar{\xi}^2 \tag{78}$$

$$v_2 = \alpha^{\frac{1}{2}}\left[\frac{28\gamma+81}{288} + \frac{7}{72}(\gamma+1)\mathcal{D}\right]\bar{\eta} - \alpha^{\frac{1}{2}}\left[\frac{20\gamma+47}{96} + \frac{5}{24}(\gamma+1)\mathcal{D}\right]\bar{\eta}^3$$

$$+ \alpha^{\frac{1}{2}}\left[\frac{8\gamma+15}{72} + \frac{(\gamma+1)\mathcal{D}}{9}\right]\bar{\eta}^5 - \left[\frac{2\gamma+3}{6} + \frac{(\gamma+1)\mathcal{D}}{3}\right]\bar{\xi}\,\bar{\eta} + \left[\frac{2\gamma+3}{6} + \frac{(\gamma+1)\mathcal{D}}{3}\right]\bar{\xi}\,\bar{\eta}^3$$

These second order results are also of interest because the numerical results require $\gamma$ to be specified and do not show its effect explicitly.

The results obtained using the computer program are presented in Table 1. These results were computed with $\gamma = 1.4$ and have been rounded off to five significant figures. The velocity coefficients for four separate solutions are given to fifth order in the table. The first solution was obtained with $\epsilon = (1+2R)^{\frac{1}{2}}/(1+R)$ and $\mathcal{D} = 0$, the second and third with $\epsilon = 1/(1+R)^{\frac{1}{2}}$ and $\mathcal{D} = 0$ and $\mathcal{D} = 0.05$, respectively. The last set of results is the solution of the transonic equations in cylindrical coordinates and represents an extension of Hall's (Ref. 1) results to higher order$^+$. The second order solution,

$^+$In order to extend Hall's results, the analysis and computer program were modified as follows. The different equations of motion in cylindrical coordinates can be recovered from equations (15) and (16) by letting $\xi = z$, $\eta = r$, $\sin\xi = \cos\xi = \sinh\eta = 0$, $\cosh\eta = 1$ and $\coth\eta = 1/\eta$. These modifications were incorporated into the computer program by setting all of the $\bar{B}$'s = 0 except for $\bar{B}_{o,o_2} = 1$, and $e_o = 1$ and the other $e_n$'s = 0. In addition, all of the terms in the coefficient matrix $G_{i,j}$ (equation (77)) were halved, and $a_o$ was set equal to one and the other $a_n$'s to zero in order to eliminate the $\eta_w$ normalization of the coordinates. The boundary condition also had to be modified to reflect the change to cylindrical coordinates. In cylindrical coordinates, the boundary condition contains nonhomogeneous terms. Results were calculated to fifth order, and the calculations to third order checked identically with Hall's results (as corrected in Ref. 10 ). In comparing the two solutions, the slight differences in the definition of the axial coordinate and transverse velocity were accounted for.

TABLE 1

VELOCITY EXPANSION COEFFICIENTS – $\gamma = 1.4$

| | $\bar{\eta} = \eta/\eta_w$ | $\bar{\xi} = \bar{\xi}/\epsilon\,\eta_w$ | | Cylindrical Coordinates |
|---|---|---|---|---|
| | $\epsilon=(1+2R)^{\frac{1}{2}}/(1+R)$ | $\epsilon = 1/(1+R)^{\frac{1}{2}}$ | | $\epsilon=1/R$ |
| | $\vartheta = 0$ | $\vartheta = 0$ | $\vartheta = 0.05$ | $\vartheta = 0$ |
| $a_{1,0,0}$ | -.125 | -.250 | -.250 | -.250 |
| $a_{1,0,1}$ | .250 | .500 | .500 | .500 |
| $a_{1,1,0}$ | .64550 | .91287 | .89087 | .91287 |
| $a_{2,0,0}$ | .092882 | .24653 | .25069 | .24653 |
| $a_{2,0,1}$ | -.23542 | -.69167 | -.71167 | -.85833 |
| $a_{2,0,2}$ | .08125 | .32500 | .33500 | .49167 |
| $a_{2,1,0}$ | -.22861 | -.41840 | -.40832 | -.57054 |
| $a_{2,1,1}$ | .16137 | .45644 | .44544 | .91287 |
| $a_{2,2,0}$ | .013889 | .027778 | -.005291 | .027778 |
| $a_{3,0,0}$ | -.071174 | -.42861 | -.45340 | -.36764 |
| $a_{3,0,1}$ | .25622 | 1.6227 | 1.7442 | 1.7368 |
| $a_{3,0,2}$ | -.20716 | -1.3807 | -1.4934 | -1.8519 |
| $a_{3,0,3}$ | .048034 | .38427 | .41796 | .63566 |
| $a_{3,1,0}$ | .15904 | .88048 | .90869 | .84894 |
| $a_{3,1,1}$ | -.51039 | -2.4685 | -2.5721 | -2.3103 |
| $a_{3,1,2}$ | .19345 | 1.0943 | 1.1495 | 1.1894 |
| $a_{3,2,0}$ | -.039352 | -.14352 | -.12147 | -.15278 |
| $a_{3,2,1}$ | -.065972 | .26389 | .24735 | -.70833 |
| $a_{3,3,0}$ | .15809 | .44714 | .42070 | -.23752 |
| $a_{4,0,0}$ | .11883 | 1.3003 | 1.4521 | .85344 |
| $a_{4,0,1}$ | -.53961 | -6.2954 | -7.0860 | -4.7325 |
| $a_{4,0,2}$ | .58098 | 7.3272 | 8.2722 | 6.7616 |
| $a_{4,0,3}$ | -.26795 | -3.7012 | -4.2025 | -4.3085 |

TABLE 1 (Continued)

| | $\bar{\eta} = \eta/\eta_w$ | $\bar{\xi} = \bar{\xi}/\epsilon\,\eta_w$ | | Cylindrical Coordinates |
|---|---|---|---|---|
| | $\epsilon = (1+2R)^{\frac{1}{2}}/(1+R)$ | $\epsilon = 1/(1+R)^{\frac{1}{2}}$ | | $\epsilon = 1/R$ |
| | $\mathcal{B} = 0$ | $\mathcal{B} = 0$ | $\mathcal{B} = 0.05$ | $\mathcal{B} = 0$ |
| $a_{4,0,4}$ | .045212 | .72379 | .82768 | 1.0528 |
| $a_{4,1,0}$ | -.42932 | -3.5030 | -3.8162 | -2.2306 |
| $a_{4,1,1}$ | 1.3353 | 12.450 | 13.700 | 8.5123 |
| $a_{4,1,2}$ | -1.0032 | -10.178 | -11.256 | -8.0230 |
| $a_{4,1,3}$ | .23597 | 2.6697 | 2.9656 | 2.4097 |
| $a_{4,2,0}$ | .30257 | 2.2688 | 2.3883 | 1.1321 |
| $a_{4,2,1}$ | -.86077 | -6.5688 | -6.9569 | -.68020 |
| $a_{4,2,2}$ | .39619 | 3.1695 | 3.3707 | -1.1245 |
| $a_{4,3,0}$ | .032929 | .065190 | .077326 | .15542 |
| $a_{4,3,1}$ | .079865 | .45179 | .43307 | -1.3879 |
| $a_{4,4,0}$ | .018836 | .075345 | .042857 | .0059002 |
| $a_{5,0,0}$ | -.26076 | -6.2108 | -7.3586 | -2.9262 |
| $a_{5,0,1}$ | 1.3451 | 32.267 | 38.367 | 17.218 |
| $a_{5,0,2}$ | -1.7472 | -42.729 | -50.775 | -28.052 |
| $a_{5,0,3}$ | 1.1086 | 28.617 | 34.068 | 23.990 |
| $a_{5,0,4}$ | -.36103 | -10.178 | -12.172 | -10.903 |
| $a_{5,0,5}$ | .047950 | 1.5344 | 1.8463 | 2.0676 |
| $a_{5,1,0}$ | 1.0135 | 18.301 | 21.197 | 8.4791 |
| $a_{5,1,1}$ | -4.1995 | -76.419 | -88.797 | -38.727 |
| $a_{5,1,2}$ | 4.3112 | 82.177 | 95.594 | 49.529 |
| $a_{5,1,3}$ | -1.8928 | -38.604 | -45.030 | -28.108 |
| $a_{5,1,4}$ | .31500 | 7.1279 | 8.3556 | 6.1628 |
| $a_{5,2,0}$ | -1.1129 | -15.103 | -16.970 | -6.2714 |
| $a_{5,2,1}$ | 3.8406 | 53.753 | 60.698 | 18.055 |
| $a_{5,2,2}$ | -2.9542 | -43.584 | -49.334 | -10.544 |

TABLE 1 (Continued)

| | $\bar{\eta} = \eta/\eta_w$ | $\bar{\xi} = \bar{\xi}/\epsilon\eta_w$ | | Cylindrical Coordinates |
|---|---|---|---|---|
| | $\epsilon = (1+2R)^{\frac{1}{2}}/(1+R)$ | $\epsilon = 1/(1+R)^{\frac{1}{2}}$ | | $\epsilon = 1/R$ |
| | $\vartheta = 0$ | $\vartheta = 0$ | $\vartheta = 0.05$ | $\vartheta = 0$ |
| $a_{5,2,3}$ | .68911 | 11.026 | 12.527 | .71944 |
| $a_{5,3,0}$ | .46961 | 4.7614 | 5.0979 | 1.1050 |
| $a_{5,3,1}$ | -1.2087 | -13.201 | -14.270 | 1.3930 |
| $a_{5,3,2}$ | .57090 | 6.4590 | 7.0012 | -3.1781 |
| $a_{5,4,0}$ | -.014991 | -.086253 | -.087224 | -.017270 |
| $a_{5,4,1}$ | .015994 | .12795 | .10344 | .60503 |
| $a_{5,5,0}$ | .066452 | .37591 | .33671 | .085568 |
| | | | | |
| $b_{1,0,0}$ | -.96825 | -.27386 | -.28062 | -.27386 |
| $b_{1,0,1}$ | .96825 | .27386 | .28062 | .27386 |
| $b_{1,1,0}$ | 0.0 | 0.0 | 0.0 | 1.0 |
| | | | | |
| $b_{2,0,0}$ | .11713 | .45720 | .48158 | .50284 |
| $b_{2,0,1}$ | -.18760 | -.85582 | -.90501 | -1.0384 |
| $b_{2,0,2}$ | .070467 | .39862 | .42343 | .53555 |
| $b_{2,1,0}$ | -.24167 | -.96667 | -1.0067 | -1.7167 |
| $b_{2,1,1}$ | .24167 | .96667 | 1.0067 | 1.9667 |
| $b_{2,2,0}$ | 0. | 0. | 0. | .91287 |
| | | | | |
| $b_{3,0,0}$ | -.12536 | -1.0168 | -1.1143 | -.99582 |
| $b_{3,0,1}$ | .31826 | 2.6729 | 2.9409 | 2.9889 |
| $b_{3,0,2}$ | -.25080 | -2.3112 | -2.5575 | -3.0004 |
| $b_{3,0,3}$ | .057906 | .65513 | .73091 | 1.0073 |
| $b_{3,1,0}$ | .51138 | 3.0407 | 3.2794 | 3.4737 |
| $b_{3,1,1}$ | -.76521 | -5.0713 | -5.5021 | -7.4078 |
| $b_{3,1,2}$ | .25383 | 2.0306 | 2.2228 | 3.8140 |

TABLE 1 (Continued)

| | $\bar{\eta} = \eta/\eta_w$ | $\bar{\xi} = \bar{\xi}/\epsilon\,\eta_w$ | | Cylindrical Coordinates |
|---|---|---|---|---|
| | $\epsilon = (1+2R)^{\frac{1}{2}}/(1+R)$ | $\epsilon = 1/(1+R)^{\frac{1}{2}}$ | | $\epsilon = 1/R$ |
| | $\vartheta = 0$ | $\vartheta = 0$ | $\vartheta = 0.05$ | $\vartheta = 0$ |
| $b_{3,2,0}$ | -.33579 | -1.8995 | -2.0101 | -2.3103 |
| $b_{3,2,1}$ | .33579 | 1.8995 | 2.0101 | 2.3788 |
| $b_{3,3,0}$ | 0. | 0. | 0. | -.47222 |
| $b_{4,0,0}$ | .22280 | 3.3718 | 3.8736 | 2.5174 |
| $b_{4,0,1}$ | -.64770 | -10.375 | -11.953 | -9.1796 |
| $b_{4,0,2}$ | .69790 | 12.058 | 13.955 | 12.907 |
| $b_{4,0,3}$ | -.33012 | -6.3474 | -7.3910 | -8.2785 |
| $b_{4,0,4}$ | .057121 | 1.2926 | 1.5156 | 2.0340 |
| $b_{4,1,0}$ | -.99216 | -11.999 | -13.555 | -9.4649 |
| $b_{4,1,1}$ | 2.1474 | 27.377 | 31.028 | 27.046 |
| $b_{4,1,2}$ | -1.4958 | -20.825 | -23.718 | -25.851 |
| $b_{4,1,3}$ | .34051 | 5.4470 | 6.2460 | 8.4222 |
| $b_{4,2,0}$ | 1.2101 | 11.570 | 12.799 | 8.5123 |
| $b_{4,2,1}$ | -1.8601 | -18.924 | -21.001 | -16.046 |
| $b_{4,2,2}$ | .64999 | 7.3538 | 8.2027 | 7.2291 |
| $b_{4,3,0}$ | -.48064 | -3.8451 | -4.1038 | -.45347 |
| $b_{4,3,1}$ | .48064 | 3.8451 | 4.1038 | -1.4993 |
| $b_{4,4,0}$ | 0. | 0. | 0. | -.69395 |
| $b_{5,0,0}$ | -.49971 | -15.871 | -19.269 | -8.4867 |
| $b_{5,0,1}$ | 1.6273 | 52.142 | 63.264 | 33.285 |
| $b_{5,0,2}$ | -2.1107 | -69.990 | -84.992 | -55.514 |
| $b_{5,0,3}$ | 1.3841 | 49.046 | 59.742 | 49.215 |
| $b_{5,0,4}$ | -.46366 | -18.163 | -22.239 | -22.913 |
| $b_{5,0,5}$ | .062654 | 2.8358 | 3.4954 | 4.4144 |
| $b_{5,1,0}$ | 2.6444 | 63.083 | 75.123 | 34.436 |

TABLE 1 (Continued)

| | $\bar{\eta} = \eta/\eta_w$ | $\bar{\xi} = \bar{\xi}/\epsilon\,\eta_w$ | | Cylindrical Coordinates |
|---|---|---|---|---|
| | $\epsilon = (1+2R)^{\frac{1}{2}}/(1+R)$ | $\epsilon = 1/(1+R)^{\frac{1}{2}}$ | | $\epsilon = 1/R$ |
| | $\mathcal{B} = 0$ | $\mathcal{B} = 0$ | $\mathcal{B} = 0.05$ | $\mathcal{B} = 0$ |
| $b_{5,1,1}$ | -6.6742 | -164.06 | -195.40 | -112.21 |
| $b_{5,1,2}$ | 6.3278 | 164.17 | 195.90 | 143.94 |
| $b_{5,1,3}$ | -2.7566 | -77.875 | -93.327 | -87.221 |
| $b_{5,1,4}$ | .45862 | 14.676 | 17.697 | 20.676 |
| $b_{5,2,0}$ | -4.0157 | -73.881 | -86.063 | -38.727 |
| $b_{5,2,1}$ | 8.1822 | 156.58 | 182.65 | 99.057 |
| $b_{5,2,2}$ | -5.3603 | -109.71 | -128.34 | -84.323 |
| $b_{5,2,3}$ | 1.1938 | 27.011 | 31.746 | 24.651 |
| $b_{5,3,0}$ | 2.3864 | 33.824 | 38.338 | 12.036 |
| $b_{5,3,1}$ | -3.6644 | -54.272 | -61.661 | -14.058 |
| $b_{5,3,2}$ | 1.2780 | 20.448 | 23.323 | 1.4389 |
| $b_{5,4,0}$ | -.50714 | -5.7376 | -6.2511 | .69652 |
| $b_{5,4,1}$ | .50714 | 5.7376 | 6.2511 | -3.1781 |
| $b_{5,5,0}$ | 0. | 0. | 0. | .24201 |

$$u_N = \sum_{m=0}^{N} \sum_{n=0}^{N-m} a_{N,m,n}\, \bar{\xi}^m\, \bar{\eta}^{2n}$$

$$v_N = \sum_{m=0}^{N} \sum_{n=0}^{N-m} b_{N,m,n}\, \bar{\xi}^m\, \bar{\eta}^{(2n+1)}$$

equation (78) proved to be a valuable aid in checking out the computer program and is exactly reproduced by the corresponding results in Table 1. The numerical results have also satisfied every other cross check that has been carried out, including the correct reproduction of the previously known first three orders of Hall's solution, and are believed to be accurate. The reason for limiting the results presented in Table 1 to fifth order is discussed below.

Since it is difficult, if not impossible, to assess the relative merits of these solutions from the tabular results, the velocities at two selected points, the throat axis ($u_o$) and wall ($u_w$) points, have been computed from the tables and are presented in Figures 1-3.

It can be seen from these figures that, as indicated previously, the velocity series give results characteristic of asymptotic expansions. If a series which alternates in sign (like the current velocity series) is convergent, the results for each succeeding order should lie between the values obtained for the two previous orders. The present results show convergence initially for small values of the expansion parameter, however, after a certain value of $\epsilon$ is reached, which depends upon the order of the solution and the form of $\epsilon$, the series begin to diverge.

For $\epsilon = 1/(1+R)^{\frac{1}{2}}$, the third order solution begins to diverge for R small than about 0.66. For $\epsilon = (1+2R)^{\frac{1}{2}}/(1+R)$, the third order solution does not diverge, while the cylindrical coordinate (Hall's) third order solution diverges for R less than about 1.5. For all of the higher order solutions, divergence occurs for R less than about 2. For higher than fifth order the velocity coefficients get very large and the divergence rates become extreme. While the degree to which the various solutions diverge varies, none of them appear to be capable of yielding valid, accurate solutions for small R.

The toroidal coordinate solutions presented in Table 1 were obtained using coordinates normalized with $\eta_w$. As shown in the section on first order solutions, solutions can also be obtained with the toroidal coordinates scaled by $\epsilon$ instead of $\eta_w$. Higher order solutions were found using the alternate scaling by setting all of the $a_n$ in equation (37), except $a_o$, equal to zero; $a_o$ was set equal to one and the boundary condition was changed to agree with equation (74). These solutions turned out to be much worse than the original ones, and hence, were not presented. The reason these

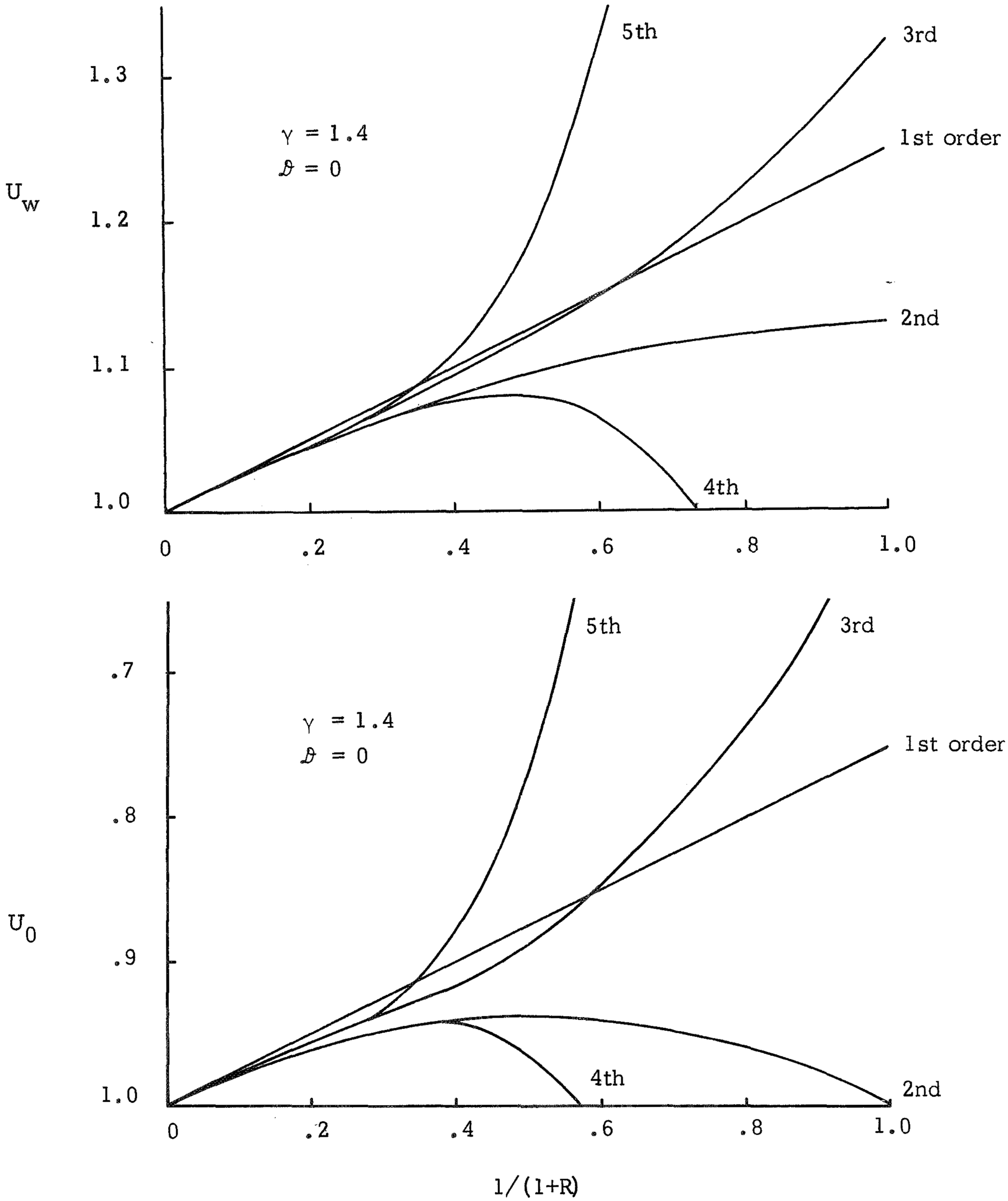



Figure 1. Throat Wall and Axis Velocities
$\varepsilon = 1/(1+R)^{\frac{1}{2}}$, Toroidal Coordinates

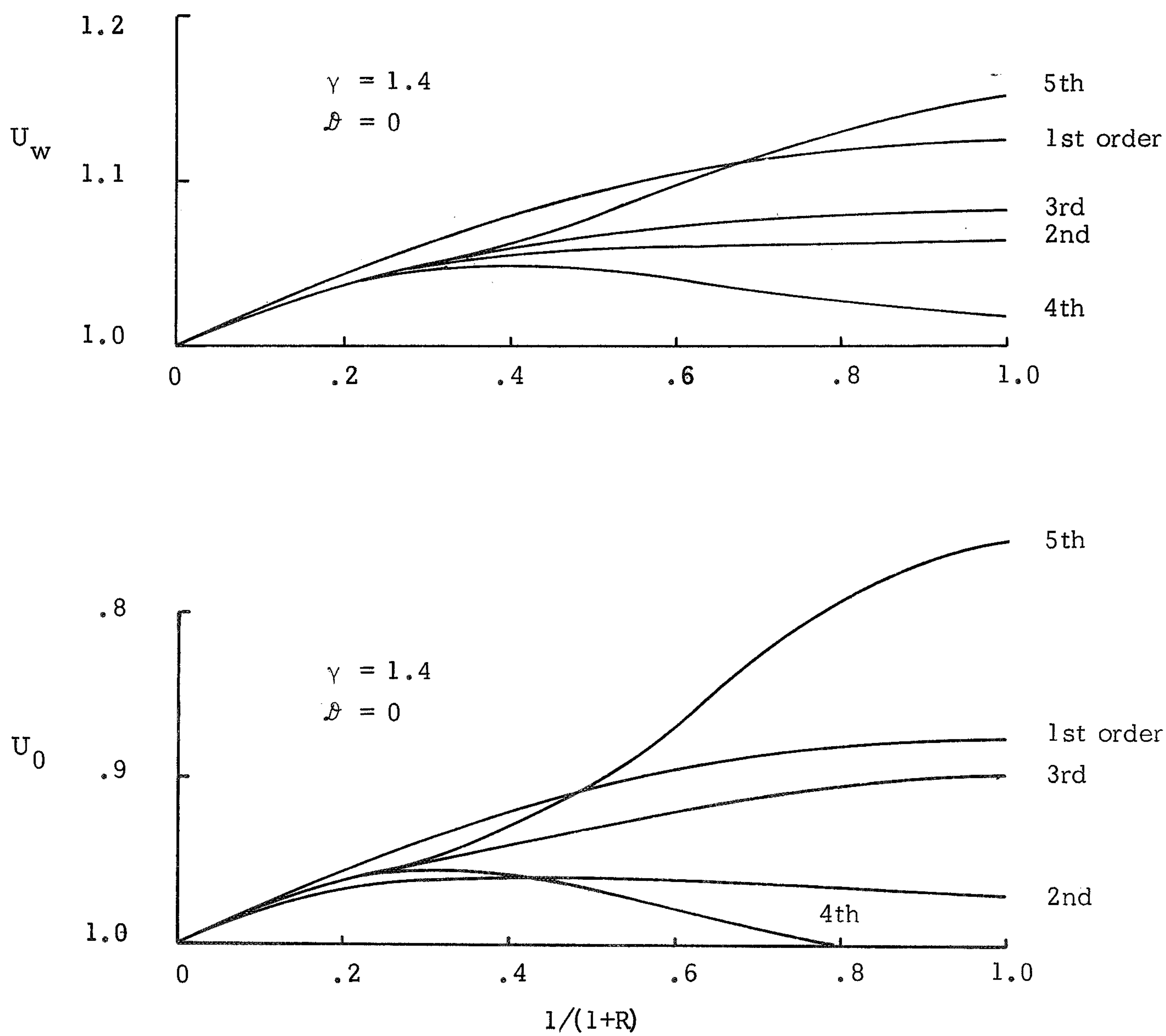



Figure 2. Throat Wall and Axis Velocities
$\epsilon = (1+2R)^{\frac{1}{2}}/(1+R)$ Toroidal Coordinates

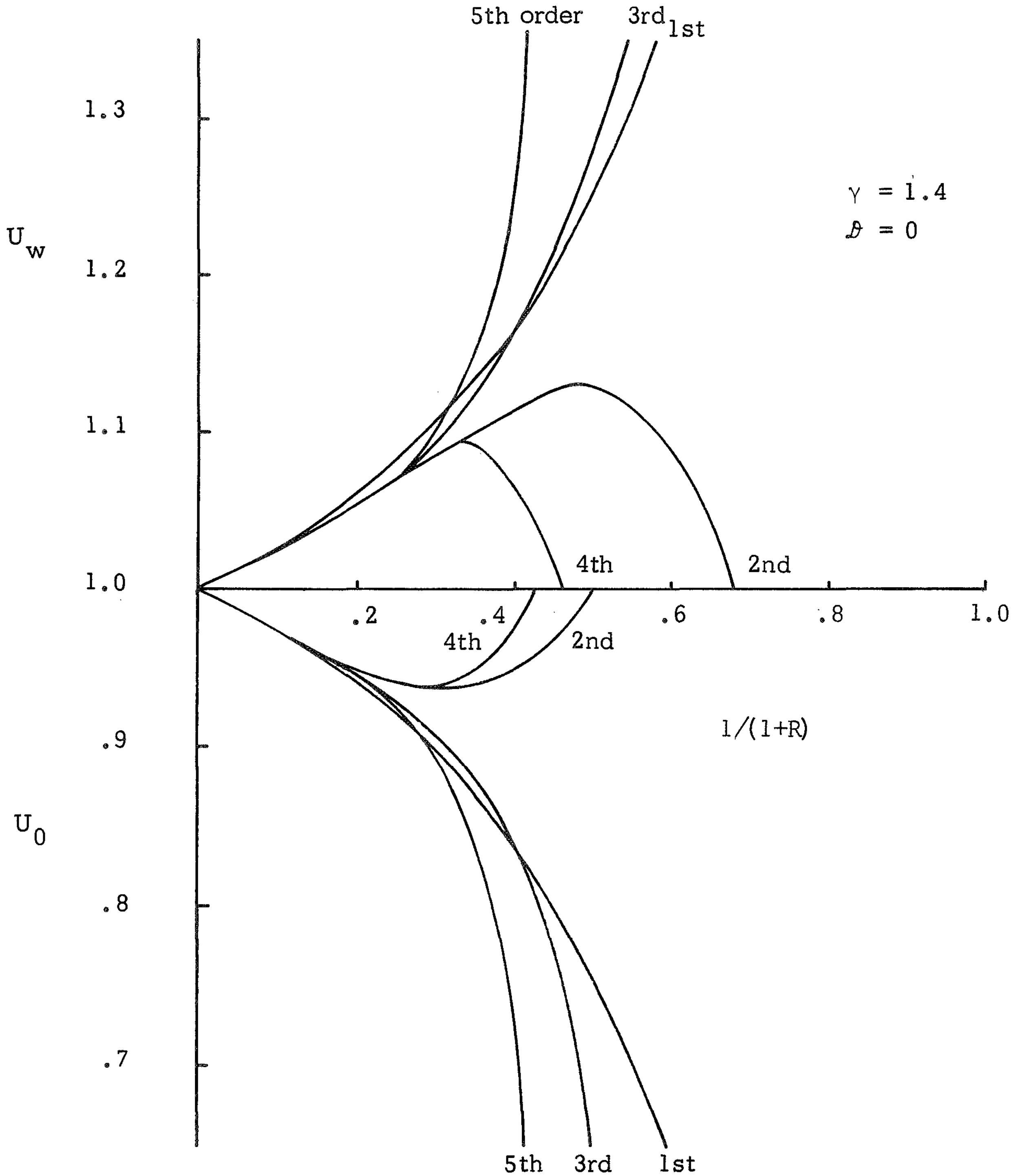



Figure 3. Throat Wall and Axis Velocities
$\epsilon = 1/R^{\frac{1}{2}}$, Cylindrical Coordinates.

latter solutions were so poor, can be attributed to the fact that they implicitly contain $\bar{\eta}_w = \eta_w/\epsilon$ as a parameter and $\bar{\eta}_w \to \infty$ as $R \to 0$.

Figure 4 shows the third order results for $u_w$ from Figures 1-3 compared to experimentally measured values. Also shown in Figure 4 are the results of Reference 10. It can be seen that as R becomes smaller all of the theoretical results begin to diverge from the experimental data. The results obtained using the method of Reference 10 show excellent agreement with the data for R down to approximately .4, however, this must be considered fortuitous in view of the following. It is claimed in Reference 10 that the solution presented therein represented the solution in toroidal coordinates (with $\epsilon = 1/(1+R)^{1/2}$) transformed back into cylindrical coordinates. The current results show that contention to be false. A reexamination of the results of Reference 10 also show that the proposed series do not satisfy the differential equations of motion in cylindrical coordinates. It appears then that the method of Reference 10 is actually an empiricism which agrees quite well with the data. While this method must now be viewed in a different light, it shouldn't inhibit its use, since it still represents a useful and unique engineering tool for cheaply and accurately (within its limits) calculating transonic flows.

Having extended Hall's method of solution to fifth order, it was possible to extend the method presented in Reference 10 to higher orders, just to see what would happen. It turns out that the fourth order results that are obtained are significantly worse than third order and the fifth order results are seriously divergent.

The results obtained in this study, as outlined above, refute the contention of Reference 10 that the inability of previous expansion solutions to yield good solutions for small R was a limitation imposed by the coordinate system, rather than a fundamental limitation of the method itself. Further reflection upon this matter has given rise to the following explanation for the inapplicability of expansion methods for small R. All of the expansion methods including the present solution, assume that the transonic solution is completely determined by the local geometry of nozzle throat. Experimental evidence, and theroretical results obtained by other means, show that the

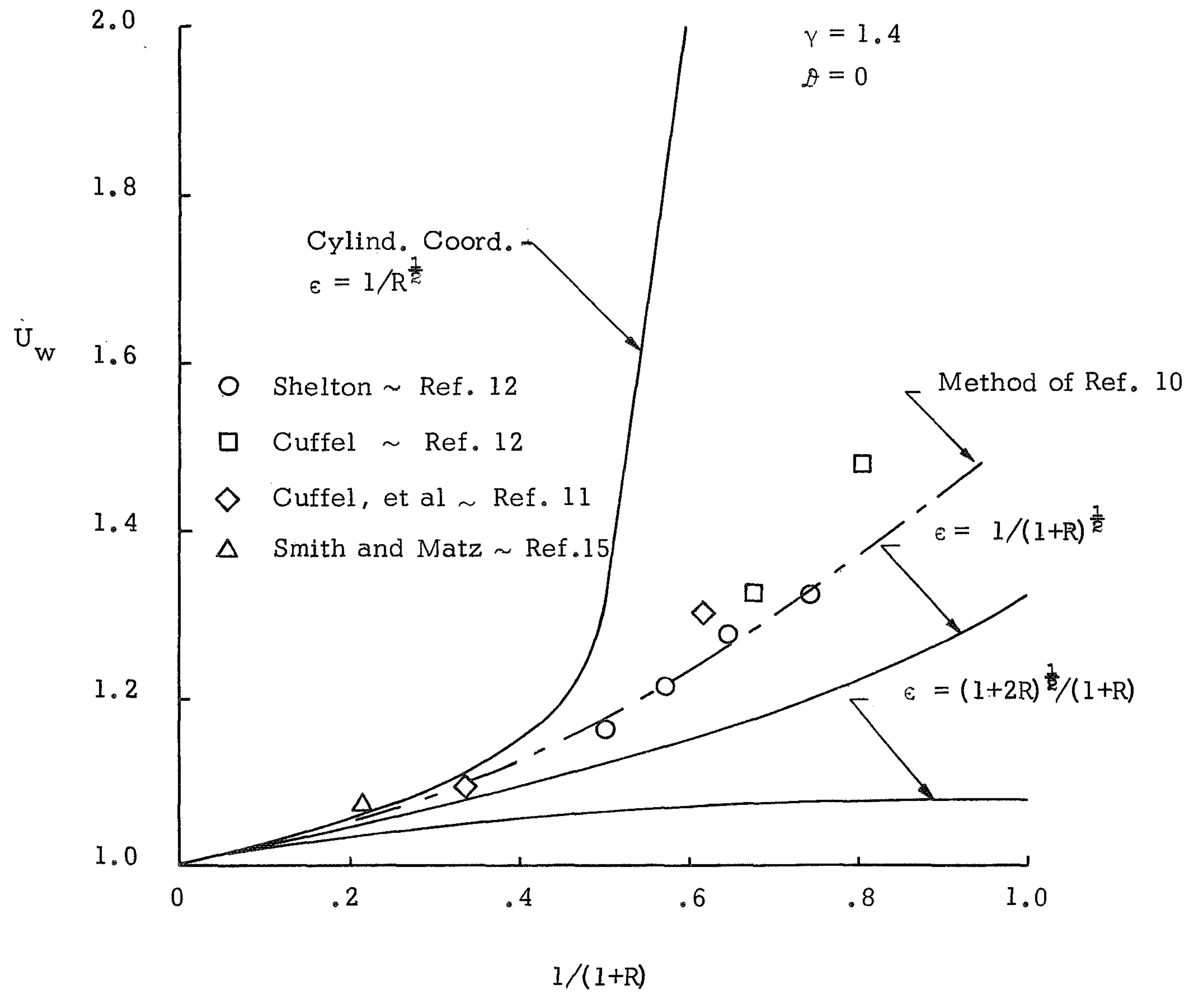



Figure 4. Third Order Throat Wall Velocity

throat geometry does essentially determine the transonic flow, for large to medium values of R. For small R, however, there is some evidence (Ref. 8) that the upstream nozzle geometry begins to noticeably affect the transonic flow, while for $R \approx 0$ the local throat geometry essentially disappears and the transonic flow cannot possibly be treated as a local expansion problem (see Reference 9). Thus, it appears that the expansion methods fail for small R because a basic premise upon which they rely begins to degenerate as $R \to 0$, and at $R = 0$ the premise becomes untenable.

Table 1 also contains the results of a calculation that was carried out to assess the effect of variable gamma on the transonic flow. This effect is incorporated in the parameter, $\mathcal{B}$, and represents the change in gamma due to real gas effects only, and does not account for specific heat variations due to flow striations which can occur in real engines. $\mathcal{B}$ is given by

$$\mathcal{B} = \frac{1}{\gamma^*(\gamma^* + 1)} \left. \frac{d\gamma}{d\bar{P}} \right|_*$$

and sample calculations were carried out which indicated that $\mathcal{B}$ should be less than 0.05 for the conditions of interest in rocket engines. The calculation shown in Table 1 was performed with $\mathcal{B} = 0.05$ and as such should represent an approximate upper bound on the effect of variable, $\gamma$. Again, it is difficult to assess the differences between the solutions with $\mathcal{B} = 0$ and $\mathcal{B} = 0.05$ from the velocity coefficients of Table 1, so Table 2 comparing $u_o$ and $u_w$ for the two solutions is presented. From Table 2 it can be seen that the effect of variable $\gamma$ is quite small even at a value of R and at orders for which the solution is diverging. In the range where the solutions are most applicable, i.e., $R \geq 1.5$, the effect of variable gamma appear to be completely negligible.

TABLE - 2

Comparison of $U_0$ and $U_w$ With and Without The Effect of Variable Gamma

$U_0$

| Order of Solution | D = 0 | | D = 0.05 | |
|---|---|---|---|---|
| | R = 2 | R = .625 | R = 2 | R = .625 |
| 1 | .9167 | .8462 | .9167 | .8462 |
| 2 | .9441 | .9395 | .9445 | .9411 |
| 3 | .9282 | .8396 | .9277 | .8354 |
| 4 | .9442 | 1.0261 | .9457 | 1.0437 |
| 5 | .9187 | .4780 | .9154 | .3943 |

$U_w$

| Order of Solution | D = 0 | | D = 0.05 | |
|---|---|---|---|---|
| | R = 2 | R = .625 | R = 2 | R = .625 |
| 1 | 1.0833 | 1.1538 | 1.0833 | 1.1538 |
| 2 | 1.0700 | 1.1083 | 1.0693 | 1.1061 |
| 3 | 1.0773 | 1.1544 | 1.0773 | 1.1563 |
| 4 | 1.0693 | 1.0619 | 1.0682 | 1.0507 |
| 5 | 1.0829 | 1.3531 | 1.0846 | 1.4015 |

## IV. SUBSONIC SOLUTION

The combination of the equations of motion being elliptic in subsonic flow, of mixed type in transonic flow, and the throat choked flow singularity, seriously complicates the task of obtaining numerical subsonic-transonic solutions in rocket nozzles. Attempts to solve this problem have either encountered insurmountable numerical difficulties, or frequently, complex methods have been developed which can achieve solutions only at the cost of large amounts of computer time. Recognition of the need for a relatively simple and economical subsonic-transonic method has led us to develop the following new approach to the problem.

It is fairly well known that the nature of the transonic flow in the region of the throat, including the mass flux for choked flow, is governed almost completely by the local geometry and is essentially free of upstream influence from the convergent section.[+] In view of this fact, it is suggested that the subsonic and transonic solutions be obtained separately, in the following manner. First, a transonic solution is obtained either with the method presented herein, or another method which depends only upon the local geometry. With a known transonic solution, the problem of solving the elliptic subsonic equations is simplified in two related ways. First, the transonic solution determines the proper choked flow mass flux, thereby eliminating the need for lengthy iterations of the subsonic numerical method in order to integrate through the throat singularity. Second, the transonic solution can be used to generate a subsonic "start line," so to speak, thereby providing boundary conditions for the subsonic flow on a completely closed contour.

When approached in the above manner, the subsonic regime should be amenable to solution in a fraction of the time currently required. A properly conceived and executed relaxation technique appears to be ideally suited to the task and one such approach is outlined below.

+Recent results of References 8 and 9, appear to indicate that for small normalized throat radii of curvature, upstream influences become apparent.

In cylindrical coordinates, the equations describing the flow in the subsonic region of the nozzle are (from (13) and (14))

$$\frac{\partial v}{\partial z} = \frac{\partial u}{\partial r} \tag{79}$$

$$(a^2 - u^2)\frac{\partial u}{\partial z} - 2uv\frac{\partial u}{\partial r} + (a^2 - v^2)\frac{\partial v}{\partial r} + \frac{a^2 v}{r} = 0 \tag{80}$$

where $a^2$ is known explicitly in terms of velocity for an ideal gas, and implicitly, through the equation of state and Bernoulli's Equation, for a real gas, u, v, and a have been normalized by the critical sound speed, $a^*$, and z and r by the throat radius, $r^*$. The boundary conditions are, in general

a. On a solid boundary

$$\vec{v} \cdot \vec{n} = 0$$

b. On the center line

$$v = 0 \tag{81}$$

c. At $z = -\infty$

$$v(r) = 0$$

d. On the transonic "start line"

$$u(r, z) = u_{tr}$$

where $u_{tr}$ is computed from the transonic expansion solution.

The reasons for specifying the upstream boundary condition (c in the above) as shown are not immediately obvious. A given rocket nozzle is finite in length and one might, at first, feel that the proper boundary condition should be uniform parallel flow at the head end, or some other station in the combustion chamber, with the velocity selected so as to match the known choked mass flux (from the transonic solution). There are, however, several faults with such a boundary condition, two of which are as follows: uniform parallel flow implies that $v$, $\partial v/\partial r$ and $\partial u/\partial r$ all equal zero, however, equations (79) and (80), then imply that $\partial u/\partial z$ and $\partial v/\partial z$ also equal zero, which in turn implies the erroneous conclusion that the velocity will remain uniform and parallel as long as the nozzle cross-sectional area remains constant. Secondly, the uniform parallel flow boundary condition constrains the mass flux into the solution domain to be equal to the mass flux out, for all time. As a result, the mass M inside the domain remains constant. The value of M corresponding to the proper solution of the equations of motion is not known a priori, and, in general, the initial guess from which the relaxation solution proceeds will yield an inconsistent value for M.

The boundary condition, $v = 0$, however, does not lead to either of these paradoxical results. The velocity profiles may vary even in a straight channel and the mass flux in, at the upstream boundary is not constrained, so that during the relaxation procedure mass can flow into or out of the domain until the proper value is achieved. The reason for specifying the boundary condition at $-\infty$ rather than at a finite distance is touched on by Moretti (Ref. 13) and can be heuristically stated as follows: the boundary conditions on the axis and on solid walls are fixed, and the downstream boundary condition is set by the transonic flow solution (or by the throat singularity if other techniques are used); fixing the remaining boundary condition, $v = 0$, at a finite distance may not yield a solution compatible with the equations of motions. Using the current technique, this would show up as differences between the values of v along the transonic start line as found by the transonic and subsonic solutions.

The equations of motion may be written in many different forms by employing potential functions, stream functions, changes in independent variables, etc. Bearing in mind that the solution is to be sought via the method of relaxation, the advantages and drawbacks of the alternative formulations were considered. It was concluded that changes in the dependent variables did not result in simplifications significant enough to warrant their use. However, computationally, it is convenient to map as many of the physical boundaries on to constant transformed coordinate lines as possible. This type of mapping reduces the amount of special differencing required at the physical boundaries. Without specifying particular forms of the transformations, it is assumed that

a. $x = x(r, z)$

b. $y = y(r, z)$

c. $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial z} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial z}$ (82)

d. $\dfrac{\partial}{\partial r} = \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial}{\partial y}\dfrac{\partial y}{\partial r}$

The above transformations can be made to allow the z coordinate to be mapped into a finite region and the wall boundary onto a constant coordinate line. This mapping leaves only the region near the throat to be handled in a special manner.

Substitution of equation (82) into (79) and (80) yields

$$v_x x_z + v_y y_z = u_y y_r + u_x x_r \tag{83}$$

$$(a^2 - u^2)\left\{u_x x_z + u_y y_z\right\} - 2uv\left(u_y y_r + u_x x_r\right) + (a^2 - v^2)\left(v_y y_r + v_x x_r\right) + \frac{a^2 v}{r} = 0 \tag{84}$$

where the subscripts denote partial differentiation with respect to the subscripted variable.

Prescribing appropriate transforms for r and z into x and y will allow constant mesh spacing to be used in the transformed plane. Second order central difference formulas can then be used to evaluate the derivatives in equations (83) and (84). The applicable central difference formulas are

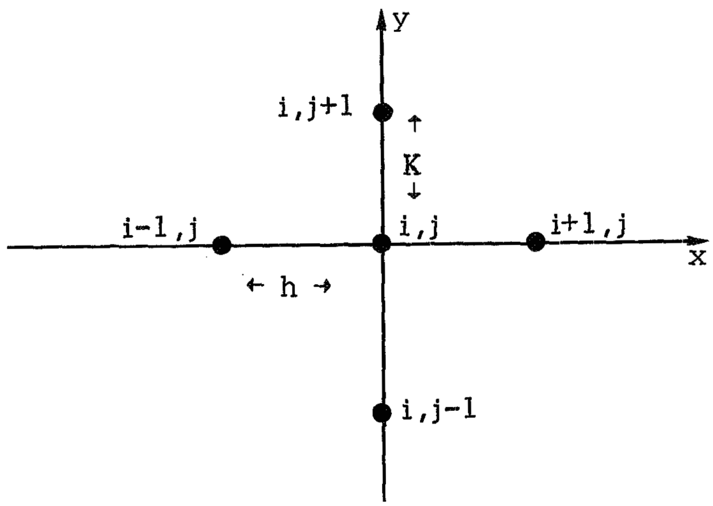
$$\frac{\partial u}{\partial x} \approx \frac{\delta u}{\delta x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\,\Delta x}$$

$$\frac{\partial u}{\partial y} \approx \frac{\delta u}{\delta y} = \frac{u_{i,j+1} - u_{i,j-1}}{2\,\Delta y}$$

$$\frac{\partial v}{\partial x} \approx \frac{\delta v}{\delta x} = \frac{v_{i+1,j} - v_{i-1,j}}{2\,\Delta x} \tag{85}$$

$$\frac{\partial v}{\partial y} \approx \frac{\delta v}{\delta y} = \frac{v_{i,j+1} - v_{i,j-1}}{2\,\Delta y}$$

with the diagram below illustrating the mesh spacing.




Substitution of the difference analogs into equations (83) and (84) gives

$$x_z \left(v_{i+1,j} - v_{i-1,j}\right) + y_z \frac{h}{K} \left(v_{i,j+1} - v_{i,j-1}\right)$$
$$= x_r \left(u_{i+1,j} - u_{i-1,j}\right) + y_r \frac{h}{K} \left(u_{i,j+1} - u_{i,j-1}\right) \tag{86}$$

$$\left\{\left(a^2_{i,j} - u^2_{i,j}\right) x_z - 2 u_{i,j} v_{i,j} x_r\right\} \left(u_{i+1,j} - u_{i-1,j}\right)$$
$$+ \left\{\left(a^2_{i,j} - u^2_{i,j}\right) y_z - 2 u_{i,j} v_{i,j} y_r\right\} \frac{h}{K} \left(u_{i,j+1} - u_{i,j-1}\right)$$
$$+ \left(a^2_{i,j} - v^2_{i,j}\right) x_r \left(v_{i+1,j} - v_{i-1,j}\right) + \frac{h}{K} \left(a^2_{i,j} - v^2_{i,j}\right) y_r \left(v_{i,j+1} - v_{i,j-1}\right) \tag{87}$$
$$+ 2h \frac{a^2_{i,j} v_{i,j}}{r} = 0$$

Since equation (86) does not involve values of $u_{i,j}$ or $v_{i,j}$, a point by point relaxation solution of equations (86) and (87) is impossible. However, a solution can be found if $v_{i,j}$ is evaluated by numerically integrating equation (86), i.e.,

$$v_{i,j} = \int_0^{x_{i,j}} (u_y y_r - v_y y_z)\, dx$$

or if all the values of u and v are coupled, and solved for, along a coordinate line. The latter method is known as line relaxation and has several advantages over point by point techniques. They are:

1. All the values of u and v along a coordinate line are solved for simultaneously and therefore the domain of influence of each point is extended.

2. The differenced equations can be solved for values which include the derivatives of the functions instead of just the functionals at a given point. This type of differencing gives the best assurance that the differenced equations are indeed analogs of the differential equations.

When equations (86) and (87) are cast in the line difference form, a banded matrix is generated involving values of u and v along the line x = constant or y = constant. The line difference equations in the y direction are:

$$y_r \frac{h}{K} u_{i,j-1} - y_z \frac{h}{K} v_{i,j-1} + - y_r \frac{h}{K} u_{i,j+1} + y_z \frac{h}{K} v_{i,j+1} = x_z \left(v_{i+1,j} - v_{i-1,j}\right) + x_r\left(u_{i+1,j} - u_{i-1,j}\right) \tag{89}$$

which may be conveniently written as

$$A_1 u_{i,j-1} + A_2 v_{i,j-1} + A_3 u_{i,j} + A_4 v_{i,j} + A_5 u_{i,j+1} + A_6 v_{i,j+1} = R_1 \tag{90}$$

where

$$
\begin{aligned}
A_1 &= y_r\, h/K \\
A_2 &= -y_z\, h/K \\
A_3 &= A_4 = 0 \\
A_5 &= -A_1 \\
A_6 &= -A_2 \\
R_1 &= x_z\left(v_{i+1,j} - v_{i-1,j}\right) + x_r\left(u_{i+1,j} - u_{i-1,j}\right)
\end{aligned}
\tag{91}
$$

Equation (87) is conveniently written as

$$
B_1\, u_{i,j-1} + B_2\, v_{i,j-1} + B_3\, u_{i,j} + B_4\, v_{i,j} + B_5\, u_{i,j+1} + B_6\, v_{i,j+1} = R_2 \tag{92}
$$

where

$$
\begin{aligned}
B_1 &= -\frac{h}{K}\left\{\left(a^2_{i,j} - u^2_{i,j}\right) y_z - 2\, u_{i,j}\, v_{i,j}\, y_r\right\} \\
B_2 &= -\frac{h}{K}\, y_r\left(a^2_{i,j} - v^2_{i,j}\right) \\
B_3 &= -u_{i,j}\, x_z\left(u_{i+1,j} - u_{i-1,j}\right) - v_{i,j}\, x_r\left(u_{i+1,j} - u_{i-1,j}\right) \\
B_4 &= 2\, h\,\frac{a^2_{i,j}\, v_{i,j}}{r} - u_{i,j}\, x_r\left(u_{i+1,j} - u_{i-1,j}\right) - v_{i,j}\, x_r\left(v_{i+1,j} - v_{i-1,j}\right) \\
B_5 &= -B_1 \\
B_6 &= -B_2 \\
R_2 &= a^2_{i,j}\, x_z\left(u_{i+1,j} - u_{i-1,j}\right) + a^2_{i,j}\, x_r\left(v_{i+1,j} - v_{i-1,j}\right)
\end{aligned}
\tag{93}
$$

The line difference equations for the y = constant line are similar to equations (90) and (92). In order to actually solve equations (90) and (92), two relations which either specify u and v at the boundaries, or relate their values to those at adjacent points, are needed. Since the boundary conditions can only fix one velocity component, or derivative on each boundary, the other relation must be found by using one of the differential equations in finite difference form. Without going into the details of the numerics, the boundary conditions for the difference equations are:

a. On a solid boundary

$$\vec{v} \cdot \vec{n} = 0 \quad \rightarrow \quad u_w \sin \theta + v_w \cos \theta = 0 \qquad \theta = \tan^{-1} \frac{dr_w}{dz}$$

and $\quad v_y \, y_z - u_y \, y_r = v_x \, x_z \quad$ which in finite difference form yields a relation between $u_w$, $v_w$ and adjacent points.

b. On the centerline

$$v = 0$$

and the irrotational equation which implies $\frac{\partial u}{\partial y} = 0$

c. at $z = -\infty$

$$v = 0$$

and the momentum equation which implies $\frac{\partial u}{\partial x} = 0$

d. On the "start line"

$$u = u_{tr}$$

and the irrotational equation which relates v on the boundary to the velocity values at neighboring points.

Central difference quotients cannot, in general, be used to evaluate derivatives on the boundaries. Therefore, in order to retain second order accuracy at the boundaries, three point forward (or backward) difference quotients and interpolation formulas should be used.

In order to begin the relaxation procedure, initial values must be assigned at each mesh point. These can be calculated from one-dimensional theory up to a specified axial station and then interpolated to fair smoothly into the start line, or, if necessary, more sophisticated starting procedures can be devised. With known initial values at each point, equations (90) and (92) can be solved line by line, using the usual techniques for inverting banded matricies, until a complete set of new values at each point has been calculated. The calculation is then repeated until the new and old values of the velocities at each point differ by less than an assigned error criterion. Various modified forms of the above procedure, such as over and under-relaxation and the method of alternating displacement (see Ref. 14), have been developed which, in many cases, significantly increase the solution convergence rate. If one proceeds properly, many of these variations can be easily tested to find the one most suitable for the current problem.

The technique outlined above holds promise of being able to provide combined subsonic-transonic solutions much more economically than other currently available methods, and efforts to implement it appear to be warranted. However, since the transonic solutions obtained herein are not accurate enough for small radii of curvature nozzles, the above technique will be limited to nozzles having normalized radii of curvature greater than about one, unless accurate local transonic expansion solutions can be found for small R, in the future.

## V. SUMMARY AND CONCLUSIONS

The transonic equations of motion for a converging-diverging nozzle, including the effect of variable gamma, have been solved in toroidal coordinates using a combination of an asymptotic small parameter expansion and a double coordinate expansion. The series expansions were carried out in general for nth order terms so that high order solution could be found recursively.

Various related solutions were obtained using different expansion parameters and coordinate normalizations, however, all of these efforts failed to yield a series solution which was convergent for small R. After an initial region of convergence all of the series begin to diverge in a manner typical of asymptotic expansions. The degree of divergence and the value of R where it begins is a function of the expansion parameter utilized and the order of the solution. These results refute the contentions of Reference 10 in regards to the applicability of expansion techniques to nozzles with small throat radii of curvature. It is currently felt that the failure of expansion techniques for small R is due to the following reason. The expansion solutions assume that the local throat geometry completely determines the transonic flow field and that there is no significant influence from the upstream flow. This assumption is certainly wrong at R = 0 where there is no throat geometry to determine the flow, and recent evidence from several sources suggests that upstream influence on the transonic region becomes more significant as R gets smaller. Thus, the expansion methods probably fail due to a breakdown in one of the premises upon which they are based.

An expansion solution which included the effect of variable gamma (for a homogeneous unstriated flow) was also calculated, and it appears that the effect of variable gamma in the transonic region is negligible. The analysis and resultant computer program were also modified slightly to enable them to extend the method of Hall to higher orders by solving the equations in cylindrical coordinates. This enabled the technique proposed in Reference 10 to be extended, and the results were found to grow progressively worse for higher orders.

A novel, and potentially useful method (although it is probably limited to $R > 1$ in view of the previous conclusions) for calculating the subsonic portion of the flow is also described. The method is based on the assumption that a local transonic expansion solution can be used to generate a subsonic "start line" and eliminate the need to iterate to satisfy the mass flow singularity at the throat.

## VI. REFERENCES


1. Hall, I. M., "Transonic Flow in Two-Dimensional and Axially-Symmetric Nozzles," Quarterly Journal of Mechanics and Applied Mechanics, Vol. XV, Pt. 4, 1962, pp. 487-508.

2. Sauer, R., "General Characteristics of the Flow Through Nozzles at Near Critical Speeds," TM1147, June 1967, NACA.

3. Oswatitsch, K. and Rothstein, W., "Flow Pattern in a Converging-Diverging Nozzle," TM1215, March 1949, NACA.

4. Mendelson, R. S., "A General Transonic Flow Analysis for Axially-Symmetric Rocket Nozzles," TR HSM-R037, Feb. 1964, Chrysler Corp. Space Division, Huntsville, Ala.

5. Sims, J. L., "Calculation of Transonic Flow," TMX-53081, Oct. 1964, NASA.

6. Kliegel, J. R. and Quan, V., "Convergent-Divergent Nozzle Flows," Rept. 02874-6002-R000, Dec. 1966, TRW Systems, Redondo Beach, Calif.

7. Saunders, L. M., "Numerical Solution of the Flow Field In The Throat Region of A Nozzle," NASA CR82601, Aug. 1966, Brown Engineering Co., Huntsville, Ala.

8. Prozan, R. J., "Transonic Flow In A Converging-Diverging Nozzle", Lockheed Missile and Space Center, Huntsville, Ala., Report No. D162177, May 1970.

9. Wehofer, S., and Moger, W. C., "Transonic Flow In Conical Convergent and Convergent-Divergent Nozzles with Nonuniform Inlet Conditions," AIAA Paper No. 70-635 presented at 6th Propulsion Joint Specialists Conference, San Diego, Calif., June 15-19, 1970.

10. Kliegel, J. R., Levine, J. N., "Transonic Flow In A Small Throat Radius of Curvature Nozzles," AIAA Journal Vol. 7, No. 7, July 1969, pp. 1375-78.

11. Cuffel, R. F., Back, L. H., and Massier, P. F., "The Transonic Flowfield in a Supersonic Nozzle with Small Throat Radius of Curvature," AIAA Journal, Vol. 7, No. 7, July 1969, pp. 1364-1366.

12. Cuffel, R. F., Private Communication.

13. Moretti, G., "The Importance of Boundary Conditions In The Numerical Treatment of Hyperbolic Equations. Polytechnic Institute of Brooklyn, Department of Aerospace Engineering, Report No. 68-34, Nov. 1968.

14. Fox, L., Numerical Solution of Ordinary and Partial Differential Equations, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962.

15. Smith, R. E., and Matz, R. J., "A Theoretical Method of Determining Discharge Coefficients for Venturi's Operating at Critical Flow Conditions," Journal of Basic Engineering, Dec. 1962.

APPENDIX A

This appendix contains a brief derivation of the coordinate and velocity transformations, the metrics and their derivatives in toroidal coordinates, and the general orthogonal coordinate forms of the divergence, curl and gradient operators.

Toroidal Coordinate Transformation

Let x, y, z be the usual cartesian coordinates, r, z, $\Phi$, the usual cylindrical coordinates and $\xi$, $\eta$, $\psi$ the toroidal coordinates. Then if the complex variable, $\rho$, is defined as

$$\rho = r + iz \tag{A-1}$$

$$r = \frac{\rho + \rho^{*+}}{2} \qquad z = \frac{i}{2}(\rho^* - \rho) \tag{A-2}$$

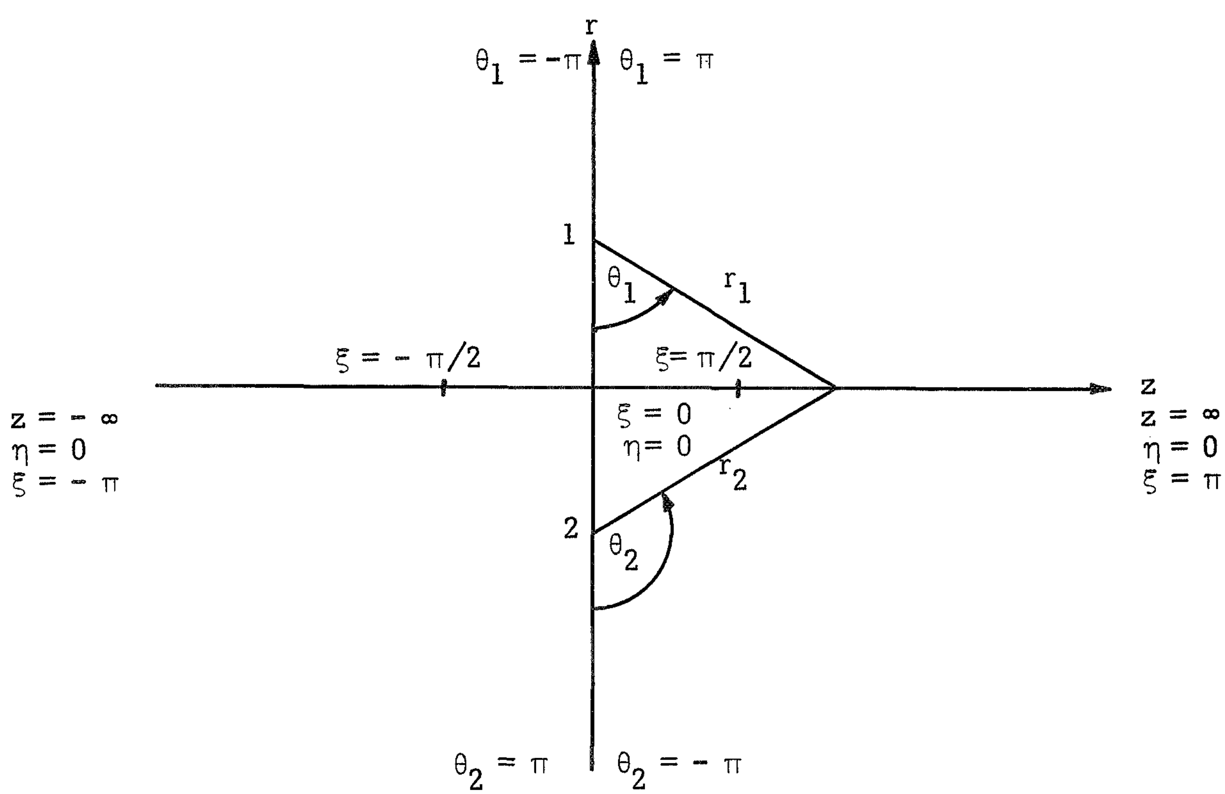



Figure A-1

+In this Appendix, an asterisk denotes the complex conjugate.

The coordinates of a point with respect to points 1 and 2 of Figure A-1 are:

$$\rho - a = - r_1 \, e^{-i\theta_1}$$
$$\rho^* + a = r_2 \, e^{-i\theta_2} \tag{A-3}$$

The toroidal coordinates $\xi$, $\eta$ are defined as

$$\eta = \ln \frac{r_2}{r_1}$$
$$\xi = \theta_1 - \theta_2 \tag{A-4}$$

Using (A-3) and (A-4) it can be shown that

$$\frac{\rho}{a} = \frac{e^{\eta} - e^{-\eta} + 2i \sin \xi}{e^{\eta} + e^{-\eta} + 2 \cos \xi} = \frac{\sinh \eta + i \sin \xi}{\cosh \eta + \cos \xi} \tag{A-5}$$

which, when combined with equation (A-2) yields

$$r = \frac{a \sinh \eta}{\cos \xi + \cosh \eta}$$
$$z = \frac{a \sin \xi}{\cos \xi + \cosh \eta} \tag{A-6}$$

Using the previous results, the equations of the coordinate lines are found to be

$$\xi = \text{constant} \qquad r^2 + (z + a \cot \xi)^2 = a^2 \csc^2 \xi \tag{A-7}$$

$$\eta = \text{constant} \qquad (r - a \coth \eta)^2 + z^2 = a^2 \operatorname{csch}^2 \eta \tag{A-8}$$

The $\xi$ = constant lines are circles with centers at $z = - a \cot \xi$ and radii equal to $a \csc \xi$; while the $\eta$ = constant lines are circles with centers at $r = a \coth \eta$ with radii of $a \operatorname{csch} \eta$.

If the throat wall is taken to be part of a circle of radius $R_w$, and if the throat radius is $r^*$, it follows from (A-7) and (A-8) that

$$R = \frac{1}{\cosh \eta_w - 1} \tag{A-9}$$

$$a = r^* \; (1 + 2R)^{\frac{1}{2}} \tag{A-10}$$

$$\eta_w = \frac{1}{2} \; \ln \left[ \frac{1 + \dfrac{(1 + 2R)^{\frac{1}{2}}}{1 + R}}{1 - \dfrac{(1 + 2R)^{\frac{1}{2}}}{1 + R}} \right] \tag{A-11}$$

where $R = R_w/r^*$ is the nondimensional throat wall radius of curvature.

Metrics

In addition to the coordinate transformation, the metrics, $h_1$, $h_2$, $h_3$, in toroidal coordinates, and their derivatives, are also required. The third toroidal coordinate, $\psi$, is defined as

$$y/x = \tan \psi$$

or

$$x = r \cos \psi \qquad y = r \sin \psi \tag{A-12}$$

The metrics are given by

$$\begin{aligned} h_1 &= \left( \frac{\partial x}{\partial \xi}^2 + \frac{\partial y}{\partial \xi}^2 + \frac{\partial z}{\partial \xi}^2 \right)^{\frac{1}{2}} \\ h_2 &= \left( \frac{\partial x}{\partial \eta}^2 + \frac{\partial y}{\partial \eta}^2 + \frac{\partial z}{\partial \eta}^2 \right)^{\frac{1}{2}} \\ h_3 &= \left( \frac{\partial x}{\partial \psi}^2 + \frac{\partial y}{\partial \psi}^2 + \frac{\partial z}{\partial \psi}^2 \right)^{\frac{1}{2}} \end{aligned} \tag{A-13}$$

The required derivatives can be found using (A-6) and (A-12) and, after much simplification, lead to

$$h_1 = h_2 = \frac{a}{\cos \xi + \cosh \eta} \tag{A-14}$$

$$h_3 = \frac{a \sinh \eta}{\cos \xi + \cosh \eta} \tag{A-15}$$

The following derivatives of the metrics are required in order to find the curl, gradient and divergence in toroidal coordinates.

$$h_{1_\xi} = \frac{\sin \xi}{\cos \xi + \cosh \eta} \; h_1$$

$$h_{1_\eta} = - \frac{\sinh \eta}{\cos \xi + \cosh \eta} \; h_1$$

$$h_1 h_3 = \frac{a^2 \sinh \eta}{(\cos \xi + \cosh \eta)^2} \tag{A-16}$$

$$(h_1 h_3)_\xi = \frac{2 \sin \xi}{\cos \xi + \cosh \eta} \; h_1 \, h_3$$

$$(h_1 h_3)_\eta = \left[ \coth \eta - \frac{2 \sinh \eta}{\cos \xi + \cosh \eta} \right] h_1 \, h_3$$

$$= \left[ \frac{1 + \cosh \eta \, \cos \xi - \sinh^2 \eta}{\sinh \eta \, (\cos \xi + \cosh \eta)} \right] h_1 \, h_3$$

Curl, Divergence and Gradient in General Curvilinear Coordinates

In order to write the equations of motion in toroidal coordinates, the curl, divergence and gradient operators must be defined. In general othogonal coordinates $x_1$, $x_2$, $x_3$ with metrics $h_1$, $h_2$, $h_3$ and unit vectors $\vec{\alpha}_1$, $\vec{\alpha}_2$, $\vec{\alpha}_3$; a general vector $\vec{A}$ is

$$\vec{A} = A_1 \ \vec{\alpha}_1 + A_2 \vec{\alpha}_2 + A_3 \vec{\alpha}_3$$

The divergence, gradient and curl of $\vec{A}$ are

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[ (h_2 \ h_3 \ A_1)_{x_1} + (h_1 \ h_3 \ A_2)_{x_2} + (h_1 \ h_2 \ A_3)_{x_3} \right] \tag{A-17}$$

$$\nabla = \frac{1}{h_1} \frac{\partial A_1}{\partial x_1} + \frac{1}{h_2} \frac{\partial A_2}{\partial x_2} + \frac{1}{h_3} \frac{\partial A_3}{\partial x_3} \tag{A-18}$$

$$\begin{aligned} \nabla \times \vec{A} = {} & \frac{\vec{\alpha}_1}{h_2 h_3} \left[ (h_3 \ A_3)_{x_2} - (h_2 \ A_2)_{x_3} \right] + \frac{\vec{\alpha}_2}{h_3 h_1} \left[ (h_1 \ A_1)_{x_3} - (h_3 \ A_3)_{x_1} \right] \\ & + \frac{\vec{\alpha}_3}{h_1 h_2} \left[ (h_2 \ A_2)_{x_1} - (h_1 \ A_1)_{x_2} \right] \end{aligned} \tag{A-19}$$

To find these operators in toroidal coordinates, set $x_1 = \xi$, $x_2 = \eta$, $x_3 = \psi$ and use the metrics and their derivatives given by (A-14)-(A-16). As a result of the axial symmetry of the present problem $\partial/\partial x_3 = \partial/\partial \psi = 0$.

Transformation of the Velocities to Cylindrical Coordinates

The direction cosines between toroidal and cartesian coordinates are given by Table A-1 below.

TABLE A-1

| | x | y | z |
|---|---|---|---|
| $\xi$ | $\frac{1}{h_1}\frac{\partial x}{\partial \xi}$ | $\frac{1}{h_1}\frac{\partial y}{\partial \xi}$ | $\frac{1}{h_1}\frac{\partial z}{\partial \xi}$ |
| $\eta$ | $\frac{1}{h_2}\frac{\partial x}{\partial \eta}$ | $\frac{1}{h_2}\frac{\partial y}{\partial \eta}$ | $\frac{1}{h_2}\frac{\partial z}{\partial \eta}$ |
| $\psi$ | $\frac{1}{h_3}\frac{\partial x}{\partial \psi}$ | $\frac{1}{h_3}\frac{\partial y}{\partial \psi}$ | $\frac{1}{h_3}\frac{\partial z}{\partial \psi}$ |

Using the metrics and the tranformations x, y, z $\rightarrow \xi$, $\eta$, $\psi$, it is found that if $\sigma$ is used to denote the direction cosine between the subscripted axes, then

$$\sigma_{\xi,x} = \left[\frac{\sinh\eta \sin\xi}{(\cos\xi + \cosh\eta)}\right] \cos\psi$$

$$\sigma_{\xi,y} = \left[\frac{\sinh\eta \sin\xi}{(\cos\xi + \cosh\eta)}\right] \sin\psi$$

$$\sigma_{\xi,z} = \cos\xi + \frac{\sin^2\xi}{(\cos\xi + \cosh\eta)} \qquad \text{(A-20)}$$

$$\sigma_{\eta,x} = \left[\cosh\eta - \frac{\sinh^2\eta}{(\cos\xi + \cosh\eta)}\right] \cos\psi$$

$$\sigma_{\eta,y} = \left[\cosh\eta - \frac{\sinh^2\eta}{(\cos\xi + \cosh\eta)}\right]\sin\psi$$

(A-20) Cont.

$$\sigma_{\eta,z} = -\frac{\sin\xi\sinh\eta}{(\cos\xi\ \sinh\eta)}$$

In cylindrical coordinates, $r^2 = x^2 + y^2$, so the components in the r direction are given by

$$\left(\sigma^2_{\xi,x} + \sigma^2_{\xi,y}\right)^{\frac{1}{2}}$$

(A-21)

$$\left(\sigma^2_{\eta,x} + \sigma^2_{\eta,y}\right)^{\frac{1}{2}}$$

Then if $v_r$ and $v_z$ are used to denote the velocities in the r and z directions, respectively, then

$$v_r = u\,\frac{\sinh\eta\sin\xi}{(\cos\xi + \cosh\eta)} + v\left[\cosh\eta - \frac{\sinh^2\eta}{(\cos\xi + \cosh\eta)}\right]$$

(A-22)

$$v_z = u\left[\cos\xi + \frac{\sin^2\xi}{(\cos\xi + \cosh\eta)}\right] - v\,\frac{\sin\xi\sinh\eta}{(\cos\xi + \cosh\eta)}$$

APPENDIX B

Since the transonic equations are to be solved by an expansion technique, it is desirable to have all of the variables of order unity. A general derivation of the proper scale transformations and series forms is outlined below.

The following general scaling and series forms are assumed:

$$\bar{\xi} = \frac{\xi}{\epsilon^{a}} \qquad \bar{\eta} = \frac{\eta}{\epsilon^{b}} \tag{B-1}$$

$$\begin{aligned} u' &= \epsilon^{c_1} u_1 (\bar{\xi}, \bar{\eta}) + \epsilon^{c_2} u_2 (\bar{\xi}, \bar{\eta}) + \ldots \\ v' &= \epsilon^{d_1} v_1 (\bar{\xi}, \bar{\eta}) + \epsilon^{d_2} v_2 (\bar{\xi}, \bar{\eta}) + \ldots \end{aligned} \tag{B-2}$$

(Note: $\bar{\xi}$, $\bar{\eta}$, $u_1$, $u_2$ . . ., $v_1$, $v_2$ . . . are then all of order unity) where $\epsilon$ is the expansion parameter. As discussed in the text, $\text{Lim}\ \epsilon \doteq \frac{1}{R^{\frac{1}{2}}}$, and $\eta_w = 0(\epsilon)$; therefore, from equation (B-1),

$$b = 1 \tag{B-3}$$

To determine a, $c_1$, $c_2$, $d_1$, $d_2$, equations (B-1)-(B-3) are substituted into equations (22) and (23). The unknown coefficients are found by requiring first the lowest order terms, and then the next lowest order terms to yield nontrivial solutions. The following expansions will be needed:

$$\begin{aligned} &\cosh x = 1 + \frac{x^2}{2} + \ldots \qquad &&\cos x = 1 - \frac{x^2}{2} + \ldots \\ &\sinh x = x + \frac{x^3}{6} + \ldots &&\sin x = x - \frac{x^3}{6} + \ldots \\ &\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \ldots \end{aligned} \tag{B-4}$$

The lowest order terms from equation (22) are: (Note: In the following, the constant arithmetic coefficients are ignored since they do not affect the ordering of terms).

$$\epsilon^{d_1 - a} \, v_{1_{\bar{\xi}}} + \epsilon \, \bar{\eta} + \epsilon^{c_1 - 1} \, u_{1_{\bar{\eta}}} = 0 \qquad \text{(B-5)}$$

The lowest order terms from equation (23) are:

$$\epsilon^{2c_1 - a} \, u_1 \, u_{1_{\bar{\xi}}} + \epsilon^{d_1 - 1} \, v_{1_{\bar{\eta}}} + \epsilon^{a} \bar{\xi} + \frac{\xi^{d_1 - 1}}{\bar{\eta}} \, v_1 = 0 \qquad \text{(B-6)}$$

Since the boundary conditions are homogeneous

$$\begin{aligned} \bar{\eta} &= 0 \qquad v' = 0 \\ & \qquad\quad \frac{\partial u'}{\partial \bar{\eta}} = 0 \\ \bar{\eta} &= \bar{\eta}_w \qquad v' = 0 \end{aligned} \qquad \text{(B-7)}$$

at least one nonhomogeneous term must remain in the lowest order equations; otherwise the trivial solution, u' = constant and v' = 0 results. Thus, equating powers of $\epsilon$, the conditions

$$d_1 - a = 1 = c_1 - 1$$

and

$$2c_1 - a = a = d_1 - 1 \qquad \text{(B-8)}$$

must be satisfied. These conditions lead to

$$a = 2, \; c_1 = 2, \; d_1 = 3 \qquad \text{(B-9)}$$

In order to check for the possible occurrence of fractional intermediate powers of $\epsilon$ in the velocity expansions, the second order terms in the expansion of equations (22) and (23) have been examined.

Equation (22) gives:

$$\epsilon^{d_2 - 2} \, v_{2_{\bar{\xi}}} + \epsilon^3 \, \bar{\eta} \, u_1 + \epsilon^3 \, \bar{\eta}^3 + \epsilon^{c_2 - 1} \, u_{2_{\bar{\eta}}} = 0 \qquad \text{(B-10)}$$

for which the condition

$$d_2 - 2 = c_2 - 1 = 3 \tag{B-11}$$

must be satisfied. Thus,

$$\begin{aligned} d_2 &= 5 \\ c_2 &= 4 \end{aligned} \tag{B-12}$$

The second order terms from equation (23) give exactly the same result.

Thus, from (B-3) and (B-9) it can be seen that the toroidal coordinates scale as

$$\overline{\xi} = \frac{\xi}{\epsilon^2} \quad \text{and} \quad \overline{\eta} = \frac{\eta}{\epsilon} \tag{B-13}$$

and (B-9) and (B-12) show that the velocity series should be written as

$$\begin{aligned} u' &= \epsilon^2 u_1(\overline{\xi}, \overline{\eta}) + \epsilon^4 u_2(\overline{\xi}, \overline{\eta}) + \ldots \\ v' &= \epsilon^3 v_1(\overline{\xi}, \overline{\eta}) + \epsilon^5 v_2(\overline{\xi}, \overline{\eta}) + \ldots \end{aligned} \tag{B-14}$$

APPENDIX C

The $D_{K_i}$'s appear in the momentum equation (23) and contain velocity expansions and the products of velocity expansions, such as

$$u' = \sum_{K=1}^{\infty} \epsilon^{2K} u_K \qquad v' = \sum_{K=1}^{\infty} \epsilon^{2K+1} v_K \tag{C-1}$$

$$u'^2 = \sum_{K=2}^{\infty} \epsilon^{2K} \sum_{n=1}^{K=1} u_{K-n} u_n = \nu(K-2) \sum_{K=0}^{\infty} \epsilon^{2K} \sum_{n=1}^{K-1} u_{K-n} U_n \tag{C-2}$$

$$v'^2 = \sum_{K=3}^{\infty} \epsilon^{2K} \sum_{n=1}^{K-2} v_{K-1-n} v_n = \nu(K-3) \sum_{K=0}^{\infty} \epsilon^{2K} \sum_{n=1}^{K=2} v_{K-1-n} v_n \tag{C-3}$$

The expansions of the trigonometric and hyperbolic functions require that the series expansion of $\eta_w$ be raised to integral powers.

$$\eta_w = \epsilon \sum_{n=0}^{\infty} a_n \epsilon^{2n} = \epsilon R \tag{C-4}$$

$$R^N = \sum_{n=0}^{\infty} \epsilon^{2n} A_{N,n} \tag{C-5}$$

The $A_{N,n}$ are given in equation (45).

Using equation (C-5) the trigonometric and hyperbolic functions can be expressed as

$$\sinh \eta = \sinh \eta_w \bar{\eta} = \sum_{P=1}^{\infty} \epsilon^{2P-1} B_{P_1}$$

$$\cosh \eta = \cosh \eta_w \bar{\eta} = \sum_{P=0}^{\infty} \epsilon^{2P} B_{P_2} \tag{C-6}$$

$$\sin \xi = \sin \epsilon \eta_w \bar{\xi} = \sum_{P=1}^{\infty} \epsilon^{2P} B_{P_3}$$

$$\cos \xi = \cos \epsilon \eta_w \bar{\xi} = \sum_{P=0}^{\infty} \epsilon^{2P} B_{P_4}$$

$$\coth \eta = \coth \eta_w \bar{\eta} = \sum_{P=0}^{\infty} \epsilon^{2P-1} B_{P_5} \qquad \text{(C-6) Cont.}$$

where the $B_P$'s are given by equation (41).

The velocity derivatives require the velocity series to be divided by the $\eta_w$ series and can be conveniently written as

$$u'_{\xi} = \sum_{P=0}^{\infty} \epsilon^{2P} C_{P_3} \qquad v'_{\xi} = \sum_{P=1}^{\infty} \epsilon^{2P-1} C_{P_1}$$

$$u'_{\eta} = \sum_{P=1}^{\infty} \epsilon^{2P-1} C_{P_4} \qquad v'_{\eta} = \sum_{P=1}^{\infty} \epsilon^{2P} C_{P_2} \qquad \text{(C-7)}$$

The $C_P$'s are given in equation (42).

The E variables result from the various products of velocity derivatives and trigonometric or hyperbolic functions.

APPENDIX D

The $\overline{B}$, $\overline{C}$, $\overline{D}$, and $\overline{E}$ variables (equations 57, 59, 60, 62) are derived from the B, C, D and E's (equations 41-44), respectively, by expanding the latter in powers of $\overline{\xi}$, $\overline{\eta}$ and collecting terms. The two sets of variables are related as follows.

$$B_{P_1} = \sum_{m=1}^{P} \overline{B}_{P,m_1} \eta^{(-1+2m)}$$

$$B_{P_2} = \sum_{m=0}^{P} \overline{B}_{P,m_2} \overline{\eta}^{2m}$$

$$B_{P_3} = \sum_{m=0}^{P-1} \overline{B}_{P,m_3} \overline{\xi}^{(P-m)} \tag{D-1}$$

$$B_{P_4} = \sum_{m=0}^{P} \overline{B}_{P,m_4} \overline{\xi}^{(P-m)}$$

$$B_{P_5} = \sum_{m=0}^{P} \overline{B}_{P,m_5} \overline{\eta}^{(2m-1)}$$

$$C_{P_1} = \sum_{m=0}^{P-1} \sum_{n=0}^{P-1-m} \overline{C}_{P,m,n_1} \overline{\xi}^{m} \overline{\eta}^{(2n+1)}$$

$$C_{P_2} = \sum_{m=0}^{P} \sum_{n=0}^{P-m} \overline{C}_{P,m,n_2} \overline{\xi}^{m} \overline{\eta}^{2n} \tag{D-2}$$

$$C_{P_3} = \sum_{m=0}^{P} \sum_{n=0}^{P-m} \overline{C}_{P,m,n_3} \overline{\xi}^{m} \overline{\eta}^{2n}$$

$$C_{P_4} = \sum_{m=0}^{P-1} \sum_{n=1}^{P-m} \overline{C}_{P,m,n_4} \, \overline{\xi}^{m} \, \overline{\eta}^{(2n-1)} \qquad \text{(D-2) Cont.}$$

The $\overline{C}_{P,m,n}$'s, themselves, make use of the following relations for the velocity derivatives.

$$u_{N_{\overline{\xi}}} = \sum_{m=1}^{N} \sum_{n=0}^{N-m} a_{N,m,n} \, m \, \overline{\xi}^{m-1} \, \overline{\eta}^{2n}$$

$$u_{N_{\overline{\eta}}} = \sum_{m=0}^{N-1} \sum_{n=1}^{N-m} a_{N,m,n} \, 2n \, \overline{\xi}^{m} \, \overline{\eta}^{(2n-1)}$$

$$\text{(D-3)}$$

$$v_{N_{\overline{\xi}}} = \sum_{m-1}^{N} \sum_{n=0}^{N-m} b_{N,m,n} \, m \, \overline{\xi}^{m-1} \, \overline{\eta}^{(2n+1)}$$

$$v_{N_{\overline{\eta}}} = \sum_{m=0}^{N} \sum_{n=0}^{N-m} b_{N,m,n} \, 2n+1 \, \overline{\xi}^{m} \, \overline{\eta}^{2n}$$

$$E_{Q_1} = \sum_{K=0}^{Q} \sum_{L=0}^{Q} \overline{\xi}^{K} \, \overline{\eta}^{2L} \, \overline{E}_{Q,K,L_1}$$

$$E_{Q_2} = \sum_{K=0}^{Q} \sum_{L=0}^{Q} \overline{\xi}^{K} \, \overline{\eta}^{2L} \, \overline{E}_{Q,K,L_2}$$

$$\text{(D-4)}$$

$$E_{Q_3} = \sum_{K=0}^{Q-1} \sum_{L=0}^{Q-1} \overline{\xi}^{K} \, \overline{\eta}^{(2L+1)} \, \overline{E}_{Q,K,L_3}$$

$$E_{Q_4} = \sum_{K=0}^{Q-1} \sum_{L=0}^{Q-1} \overline{\xi}^{K+1} \, \overline{\eta}^{2L} \, \overline{E}_{Q,K,L_4}$$

$$E_{Q_5} = \sum_{K=0}^{Q} \sum_{L=0}^{Q-K} \bar{\xi}^{K} \bar{\eta}^{2L} \bar{E}_{Q,K,L_5} \qquad \text{(D-4) Cont.}$$

The $\bar{D}$'s are related to the D's through the equations for the products of velocities (which are expressed in terms of the F's). All of the velocity multiplications are of one of the three following types.

$$u_N \, u_M = \sum_{Q=0}^{M+N} \sum_{P=0}^{N+M-Q} \bar{\xi}^{Q} \bar{\eta}^{2P} F_{N,M,Q,P_1}$$

$$v_N v_M = \sum_{Q=0}^{M+N} \sum_{P=1}^{N+M+1-Q} \bar{\xi}^{Q} \bar{\eta}^{2P} F_{N,M,Q,P_2} \qquad \text{(D-5)}$$

$$u_N \, v_M = \sum_{Q=0}^{M+N} \sum_{P=0}^{N+M-Q} \bar{\xi}^{Q} \bar{\eta}^{(2P+1)} F_{N,M,Q,P_3}$$

## APPENDIX E

### Computer Program

The philosophy used in writing the transonic computer program was to make the program listing correspond as closely as possible to the equations of section II of this report. To this end, each of the functions in the equations were programmed as FØRTRAN functions using the following naming conventions:

a) functions beginning with lower case letters are pretended with the letter s for small. Capital letters were left unchanged

b) the number of arguments to the function is always the last character of the function name

c) for numbered functions names the number immediately follows the letter identifying the function and the letter A separates the function number and the number of arguments

hence:

| | | |
|---|---|---|
| $\overline{E}_{Q,K,L_1}$ | becomes | E1A3(Q, K, L) |
| $b_{P,M,N}$ | becomes | SB3(P, M, N) |

Since zero indexing was required, a dynamic storage allocation technique known as bucketing was used to compute indexes and also to conserve storage. The use of the bucket also allowed most functions to be evaluated only once.

The following gives a brief description of the subroutines and functions used in the program:

Program TRANSØN

Main program which controlled overall logic

Subroutine INPUTM

Reads the input data

Subroutine GETADD

Calculates the indexes for each array in the bucket

Subroutine INIT

Calculates constants and initializes some variables

Function FAC

Returns the factorial of its argument

Function SA1

Returns $a_i$

Function SE1

Returns $e_i$

Function A2

Returns $A_{i,j}$

Function LØK

Computes the position in the bucket of 3 dimensional variables

Subroutine DEBUG

Supplies some Namelist debug print out

Function B1A2

| Entry Point | Returns |
|---|---|
| B1A2 | $B_{i,j_1}$ |
| B2A2 | $B_{i,j_2}$ |
| B3A2 | $B_{i,j_3}$ |
| B4A2 | $B_{i,j_4}$ |
| B5A2 | $B_{i,j_5}$ |

Function SA3

Returns $a_{i,j,k}$

Function SB3

Returns $b_{i,j,k}$

Function DELTA

Returns $\delta(s)$

Function SB1

Returns $b_i$

Function C1A3

| Entry point | Returns |
|---|---|
| C1A3 | $\overline{C}_{i,j,k_1}$ |
| C2A3 | $\overline{C}_{i,j,k_2}$ |
| C3A3 | $\overline{C}_{i,j,k_3}$ |
| C4A3 | $\overline{C}_{i,j,k_4}$ |

Function S2

Returns $\overline{S}_{i,j}$

Function S3

Returns $\overline{S}_{i,j,k}$

Function D1A3

| Entry Point | Returns |
|---|---|
| D1A3 | $\overline{D}_{i,j,k_1}$ |
| D2A3 | $\overline{D}_{i,j,k_2}$ |
| D3A3 | $\overline{D}_{i,j,k_3}$ |
| D4A3 | $\overline{D}_{i,j,k_4}$ |
| D5A3 | $\overline{D}_{i,j,k_5}$ |
| D6A3 | $\overline{D}_{i,j,k_6}$ |

Function F1A4

| Entry Point | Returns |
|---|---|
| F1A4 | $F_{N,M,Q,P_1}$ |
| F2A4 | $F_{N,M,Q,P_2}$ |
| F3A4 | $F_{N,M,Q,P_3}$ |

Function E1A3

| Entry Point | Returns |
|---|---|
| E1A3 | $\bar{E}_{Q,K,L_1}$ |
| E2A3 | $\bar{E}_{Q,K,L_2}$ |
| E3A3 | $\bar{E}_{Q,K,L_3}$ |
| E4A3 | $\bar{E}_{Q,K,L_4}$ |
| E5A3 | $\bar{E}_{Q,K,L_5}$ |

Function EP1A3

| Entry Point | Returns |
|---|---|
| EP1A3 | $\bar{E}'_{Q,K,L_1}$ |
| EP2A3 | $\bar{E}'_{Q,K,L_2}$ |
| EP5A3 | $\bar{E}'_{Q,K,L_5}$ |

Subroutine CØEFF

Generates the coefficients matrix for the transonic solution.

Subroutine RHSIDE

Controls the calculation of the right hand sides of the transonic equations

Subroutine MØMEN

Calculates the right hand sides of the momentum equations

Subroutine IRRØT

Calculates the right hand sides of irrotational equations

Subroutine INVRT

Inverts the coefficient matrix

Subroutine SØLN

Calculates the coefficients in the solution to the transonic equations

The computer program input is standard FØRTRAN IV NAMELIST. Familiarity with this standard input procedure is assumed.

The input list of variables are as follows:

$DATA

GAMMA = , ratio of specific heats

D = , $\mathcal{D} = \dfrac{1}{\gamma(\gamma+1)} \quad \dfrac{d\gamma}{d\bar{P}}\Big|_*$

PMAX = , maximum order of solution desired

RCURV = , throat radius of curvature, only used for EFLAG > 2, but a value must always be input.

EFLAG = , boundary condition flag

EFLAG = 1, $\epsilon = \dfrac{(1+2R)^{\frac{1}{2}}}{1+R}$, $\bar{\eta} = \eta/\eta_w$

EFLAG = 2, $\epsilon = 1/(1+R)$, $\eta = \eta/\eta_w$

EFLAG = 3, $\epsilon = \dfrac{(1+2R)^{\frac{1}{2}}}{1+R}$, $\eta = \eta/\epsilon$

EFLAG = 4, $\epsilon = 1/(1+R)$, $\eta = \eta/\epsilon$

for EFLAG $\geq 3$, RCURV is used

$END

One note on conversion, different FØRTRAN IV compilers treat multiple entry points to function subprograms differently. The CDC 6000 series, RUN compiler uses the following conventions:

a) a value is assigned to every entry point of a function subprogram

b) the argument list for each entry point is implied to be identical with that of the main entry point. Hence, each entry point must be called with the same number of argument as the main entry point, but that argument list must only appear on the main entry point

Output from the program consists of the coefficients of the velocity expansions up to order PMAX. The output is in the form A(I,J,K) B(I,J,K) which correspond to $a_{i,j,k}$ and $b_{i,j,k}$ in equation (51). A sample of the output (up to third order) for the following input conditions is given below.

INPUT: GAMMA = 1.4, D = 0.05, EFLAG = 2.0, PMAX = 5,
RCURV = 0.25

OUTPUT:

| | | | |
|---|---|---|---|
| A( 2, 0, 0)= | .25069 | B( 2, 0, 0)= | .48158 |
| A( 2, 0, 1)= | -.71167 | B( 2, 0, 1)= | -.90501 |
| A( 2, 0, 2)= | .33500 | B( 2, 0, 2)= | .42343 |
| A( 2, 1, 0)= | -.40832 | B( 2, 1, 0)= | -1.0067 |
| A( 2, 1, 1)= | .44544 | B( 2, 1, 1)= | 1.0067 |
| A( 2, 2, 0)= | -5.29101E-03 | B( 2, 2, 0)= | 0. |
| A( 3, 0, 0)= | -.45340 | B( 3, 0, 0)= | -1.1143 |
| A( 3, 0, 1)= | 1.7442 | B( 3, 0, 1)= | 2.9409 |
| A( 3, 0, 2)= | -1.4934 | B( 3, 0, 2)= | -2.5575 |
| A( 3, 0, 3)= | .41796 | B( 3, 0, 3)= | .73091 |
| A( 3, 1, 0)= | .90869 | B( 3, 1, 0)= | 3.2794 |
| A( 3, 1, 1)= | -2.5721 | B( 3, 1, 1)= | -5.5021 |
| A( 3, 1, 2)= | 1.1495 | B( 3, 1, 2)= | 2.2226 |
| A( 3, 2, 0)= | -.12147 | B( 3, 2, 0)= | -2.0101 |
| A( 3, 2, 1)= | .24735 | B( 3, 2, 1)= | 2.0101 |
| A( 3, 3, 0)= | .42070 | B( 3, 3, 0)= | 0. |

```
      PROGRAM TRANSON(TAPE1,INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      COMMON LA(100),B(20000)
      COMMON/INDEXS/PMAX,NSTOR
      COMMON/POINTS/NPMAX
      INTEGER PMAX
      KSTART=0
C
 10   CALL INPUTM(KSTART)
C
      CALL GETADD
C
      CALL INIT
      CALL DEBUG
C
      NP=(PMAX+1)*(PMAX+2)
      DO 100 I=2,PMAX
      IP1=LA(32)+1
      IP2=LA(33)+1
      IP3=LA(34)+1
      CALL COEFF(B(IP1),B(IP3),B(IP2),I,NP)
C
      CALL SOLN(B(IP1),B(IP3),B(IP2),I,NP)
 100  CONTINUE
      KSTART=1
      GO TO 10
C
      CALL OUTS
C
      END
```

```
      SUBROUTINE INPUTM(KS)
      COMMON/WALLBC/ETAW, RCURV, EPSIL
      COMMON/EPSFLG/EFLAG
      COMMON/GAMS/GAM,G1,G2,G3,D
      COMMON/INDEXS/PMAX,NSTOR
      INTEGER PMAX
      NAMELIST/DATA/GAMMA,D,EFLAG,A100,A101,A110,B100,B101,PMAX
     1  , RCURV
      IF(KS.NE.0)GO TO 100
      RCURV=10.0
 100  READ(5,DATA)
      PMAX=MAX0( PMAX,3)
      WRITE(6,DATA)
      GAM=GAMMA
      RETURN
      END
```

```
      SUBROUTINE GETADD
      COMMON/INDEXS/PMAX,NSTOR
      COMMON LA(100),B(1)
      INTEGER PMAX
      DIMENSION  IB(1)
      EQUIVALENCE (IB(1),B(1))
      DATA NCALC/5HNCALC/
C
C        THIS ROUTINE CALCS THE STARTING INDEX IN THE B ARRAY FOR
C        THE STORAGE OF VALUES OF EACH OF THE FUNCTIONS
C
      NSTOR=PMAX+3
C
C        FUNCTION     NUM OF ARGS       LA(I)      FORTRAN NAME
C        SMALL A            1         I= 1         SA1
C        SMALL E            1            2         SE1
C        SMALL B            1            3         SB1
C        CAP A              2            4         CA2
C        CAP B BAR 1        2            5         CBB1A2
C        CAP B BAR 2        2            6         CBB2A2
C        CAP B BAR 3        2            7         CBB3A2
C        CAP B BAR 4        2            8         CBB4A2
C        CAP B BAR 5        2            9         CBB5A2
C        SBAR3              3           10         CSB3
C        SBAR2              2           11         CSB2
C        CAP C BAR 1        3           12         CCB1A3
C        CAP C BAR 2        3           13         CCB2A3
C        CAP C BAR 3        3           14         CCB3A3
C        CAP C BAR 4        3           15         CCB4A3
C        CAP D BAR 1        3           16         CDB1A3
C        CAP D BAR 2        3           17         CDB2A3
C        CAP D BAR 3        3           18         CDB3A3
C        CAP D BAR 4        3           19         CDB4A3
C        CAP D BAR 5        3           20         CDB5A3
C        CAP D BAR 6        3           21         CDB6A3
C        CAP E BAR 1        3           22         CEB1A3
C        CAP E BAR 2        3           23         CEB2A3
C        CAP E BAR 3        3           24         CEB3A3
C        CAP E BAR 4        3           25         CEB4A3
C        CAP E BAR 5        3           26         CEB5A3
C        SMALL A            3           27         SA3
C        SMALL B                 3           28         SB3
C        CAP E PRIME 1      3           29         EP1A3
C        CAP E PRIME 2      3           30         EP2A3
C        CAP E PRIME 5      3           31         EP5A3
C        COEFF MATRIX       NA          32         NA
C        SOLUTION VECTOR    NA          33         NA
C        R.H. SIDE NA        34         NA
      L1=PMAX+1
      L2=L1*L1
      L3=L2*L1
C
```

```
      LA(1)=0
      LA(2)=LA(1)+NSTOR
      LA(3)=LA(2)+NSTOR
      LA(4)=LA(3)+NSTOR
      LA(5)=LA(4)+L2
      LA(6)=LA(5)+L2
      LA(7)=LA(6)+L2
      LA(8)=LA(7)+L2
      LA(9)=LA(8)+L2
      LA(10)=LA( 9)+L2
      LA(11)=LA(10)+L3
      LA(12)=LA(11)+L2
      DO 50 I=13,31
      LA(I)= LA(I-1)+L3
50    CONTINUE
      LA(32)=LA(31)+L3
      LA(33)=LA(32)+(L1*(L1+1))**2
      LA(34)=LA(33)+L1*(L1+1)
      LMAX=35
      LA(35)=LA(34)+L1*(L1+1)
      NMAX=LA(LMAX)
      DO 500 I=1,NMAX
      IB(I)=NCALC
500   CONTINUE
      RETURN
      END
```

```
      SUBROUTINE INIT
      COMMON/EPSFLG/EFLAG
      COMMON/GAMS/GAM,G1,G2,G3,D
      COMMON/INDEXS/PMAX,NSTOR
      INTEGER PMAX,ALPHA
      COMMON  LA(100),B(1)
      COMMON/POINTS/NPMAX
      COMMON/WALLBC/ETAW, RCURV, EPSIL
      DIMENSION T(50),RS(50)
      NAMELIST/BUG/T,RS,PMAX,NPMAX
      NSTOR=NSTOR-1
      GP1=GAM+1.0
      GM1=GAM-1.0
      G1= GM1/GP1 +D
      G2= 2.0/GP1
      G3= 2.0*( (GAM-2.0)/GP1  +D)
      CON1=SQRT(1.0+2.0*RCURV)
      CON2=1.0+RCURV
      EPSIL=1.0
      IF(EFLAG.EQ.3.) EPSIL=CON1/CON2
      IF(EFLAG.EQ.4.)EPSIL=1.0/SQRT(CON2)
      ETAW=0.5*ALOG((CON2+CON1)/(CON2-CON1))/EPSIL
      LE=LA(2)
      B(LE+1)= 1.0
      B(1)=1.0
```

```
      SQRT2=SQRT(2.0)
      IF(EFLAG .EQ. 2.0) B(1)=SQRT2
      N=0
      K1=1
      K2=LE+1
      WRITE(6,900)N,B(K1),B(K2)
C
      DO 100 I=1,NSTOR
      K1=I+1
      K2=LE+I+1
      N=I
C
C          CALC E S
C
      SUM=0.0
      NM1=N-1
      DO 20 J=1,N
      M=J-1
      SUM= SUM + B(LE+J)/FAC( 2*(N-M)+1)
 20   CONTINUE
      B(K2)=  1.0/FAC(2*N) -SUM
C
C          BRANCH AND CALC SMALL A S
C
      IF(EFLAG .GT.1.) GO TO 50
      B(K1)=  1.0/(1.0 +2.0*FLOAT(N) )
      WRITE(6,900)N,B(K1),B(K2)
      GO TO 100
 50   SUM=0.0
      XN=N
      B(K1)=   2.0**N*SQRT2/(2.0*XN +1.0)
      DO 70 J=1,N
      ALPHA=J-1
      XA=ALPHA
      PROD=1.0
      NMA= N-ALPHA
      DO 60  K=1,NMA
      XJ=K-1
      PROD=PROD*( XA -( XJ-0.5) )
 60   CONTINUE
      SUM= SUM  + SQRT2*2.0**ALPHA*(-.5)**(N-ALPHA)/((2.0* ALPHA+1.0)*
     1    FAC(N-ALPHA) )*PROD
 70   CONTINUE
      B(K1)=B(K1) +SUM
      IF(EFLAG .GT. 2.0) B(K1) = 0.0
      WRITE(6,900)N,B(K1),B(K2)
 100  CONTINUE
      LB=LA(3)
      B0=1.0/B(1)
      B(LB+1)=B0
C
      DO 200 I=1,NSTOR
      N=I
      SUM=0.0
      DO 180 J=1,N
      M=J-1
      K1= N-M +1
      K2=LB+ M+1
      SUM=SUM + B(K1)*B(K2)
 180  CONTINUE
```

```
      B(K2+1)= -B0*SUM
      WRITE(6,910)N,B(K2+1)
  200 CONTINUE
      LCA=LA(4)
C
C         CALC A(0,N) THRU A(2,N)
C
      NPMAX=PMAX+1
      DO 250 J=1,NPMAX
      NC=J-1
      K0= NC*NPMAX +LCA+1
      B(K0)=1.0
      IF( EFLAG .GT. 2.0) B(K0)=0.0
      K1= NC*NPMAX+ LCA+2
      B(K1)= B(J)
      K2=NC*NPMAX +LCA+3
      N=NC
      SUM=0.0
      NP1=N+1
      DO 240 K=1,NP1
      M=K-1
      SUM=SUM +SA1(N-M)*SA1(M)
  240 CONTINUE
      B(K2)= SUM
      WRITE(6,920) NC,B(K0),B(K1),B(K2)
  920 FORMAT(10X,2HN=,I4, 9H  A(0,N)=,G17.5,9H  A(1,N)=,G17.5,8H  A(2,N)
     1   1H=,G17.5)
  250 CONTINUE
C
C         CALC  A(3,N)  THRU  A(N,N)
C
      DO 400 N=3,PMAX
      DO 360 J=1,NPMAX
      SUM=0.0
      LN=J-1
      DO 350 K=1,J
      I=K-1
      SUM=SUM+SA1(LN-I)*A2(N-1,I)
  350 CONTINUE
      IBA= LN*NPMAX + N+1 +LCA
      B(IBA)=SUM
      WRITE(6,930) N,LN,SUM
  360 CONTINUE
  400 CONTINUE
  930 FORMAT(10X, 2HA(,I2,1H,I2,2H)=,G17.5)
      DO 500 I=1,NPMAX
      IP=I-1
      DO 450 J=1,NPMAX
      M=J-1
      IF((IP-M).LT. 0) GO TO 500
C         CALC POINTERS
      IB1= M*NPMAX +IP +1 + LA(5)
      IB2= M*NPMAX +IP +1 + LA(6)
      IB3= M*NPMAX +IP +1 + LA(7)
      IB4= M*NPMAX +IP +1 + LA(8)
      IB5= M*NPMAX +IP +1 + LA(9)
C         B1
      IF( M .EQ. 0) GO TO 420
      B(IB1)= A2(2*M -1,IP-M)/FAC(2*M-1)
C         B2
  420 B(IB2)= A2(2*M,IP-M)/FAC(2*M)
```

```
C           B3
      DEL=DELTA(IP-1-M)
      B(IB3)=0.0
      IF(DEL.NE. 0.0) B(IB3)= A2(IP-M,M)
C           B4
      DEL=DELTA(IP-M)
      B(IB4)=0.0
      IF(DEL .NE. 0.0)B(IB4)=DEL*A2(IP-M,M)/FAC(IP-M)
C           B5
      B(IB5)= SE1(M)*A2(2*M-1,IP-M)
  450 CONTINUE
  500 CONTINUE
C
C           CALC 1 ST ORDER SOLN
C
      SQRTAL=SQRT((1.0+D)/G2)
      IF(EFLAG.EQ.2.0)GO TO 550
      CON=1.0
      IF(EFLAG.GT.2) CON=ETAW**2
      A100=-CON/8.0
      A101=.25
      A110=1.0/SQRTAL/SQRT2
      B110=0.0
      B101=SQRTAL/SQRT2/8.0
      B100=-B101*CON
      GO TO 600
  550 A100=-.25
      A101=.5
      A110=1.0/SQRTAL
      B101=0.25*SQRTAL
      B100=-B101
      B110=0.0
C
C           PUT IN FIRST ORDER SOLN
C
  600 IP=LOK(1,0,0,27)
      B(IP)=A100
      IP=LOK(1,0,1,27)
      B(IP)=A101
      IP=LOK(1,1,0,27)
      B(IP)=A110
      IP=LOK(1,0,0,28)
      B(IP)=B100
      IP=LOK(1,0,1,28)
      B(IP)=B101
      IP=LOK(1,1,0,28)
      B(IP)=B110
      RETURN
  910 FORMAT(1H0,20X,*B(* I2,*)=*G17.5)
  900 FORMAT(10X,*N=*I5,* A(N)=*G17.5,* E(N)=*G17.5)
      END
```

```
      FUNCTION FAC(M)
C
C          FAC RETURNS WITH M FACTORIAL
C          USES STERLINGS APPROX AFTER 14 FACTORIAL
C
       DIMENSION F(15)
       DATA F/1.,1.,2.,6.,24.,120.,720.,5040.,40320.,362880.,3628800.,
     1     39916800.,4790016.0E+2,62270208.0E+2,871782912.0E+2/
C
       IF( M.LT. 0)GO TO 800
       IF( M.GT.14)GO TO 500
       FAC=F(M+1)
  500  N=M
       RETURN
       XN=N
       FAC= SQRT(6.2831530718*XN)*XN**N*EXP(-XN)
       RETURN
  800  PRINT 900,M
       FAC=1.0
  900  FORMAT(1H0,5X,110(1H*)/30X,34HERROR MESSAGE FROM FACTORIAL FUNCT,
     1 10HION, FAC   /30X,  12HARGUMENT WAS,I10,5X,16HFAC WAS SET TO 1/
     26X,110(1H*) )
       RETURN
       END




       FUNCTION SA1(INDX)
C           SA1= SMALL A WITH 1 ARGUMENT
       COMMON LA(100),B(1)
       IP= INDX +1
       SA1= B(IP)
       RETURN
       END
       FUNCTION SE1(INDX)
C           SE1= SMALL E WITH 1 ARGUMENT
       COMMON LA(100),B(1)
       IP= LA(2) +INDX +1
       SE1= B(IP)
       RETURN
       END
```

```
      FUNCTION A2(N,J)
C          A2 = CAP A WITH 2 ARGUMENTS
      COMMON LA(100),B(1)
      COMMON/POINTS/NPMAX
      IF( J.LT. 0) GO TO 800
      IF(N.EQ.-1)GO TO 300
      IF( N.LT. 0) GO TO 800
      IP= J*NPMAX+ N +1 +LA(4)
      A2= B(IP)
      RETURN
  300 A2= SB1(J)
      RETURN
  800 CONTINUE
C
C          FORCE TRACE BACK
C
      PRINT 900,I,J
  900 FORMAT(1H0,21HFROM A2 - N AND J ARE,2I10)
      Z=-10.0
      Q= SQRT(Z)
      CALL EXIT
      RETURN
      END
```

```
      FUNCTION LOK(I,J,K,L)
      COMMON LA(100)
C
C          FIND LOCATIONS IN BUCKET FOR 3-D ARRAYS
C
      COMMON/POINTS/NPMAX
C
      IP=NPMAX*(NPMAX*K+J)+I+1+LA(L)
C
C          ERROR CHECK
C
      IF(IP.LE.LA(L) .OR. IP.GT.LA(L+1))GO TO 800
      LOK=IP
      RETURN
  800 PRINT 900,I,J,K,L
C          FORCE TRACE BACK
      Z=-10.0
      G=SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,5X,25HFROM LOK I,J,K AND L ARE,4I10)
      END
```

```
      SUBROUTINE DEBUG
       NAMELIST/IDBUG/B1A00,B1A11,B1A21,B1A22,B2A10,B2A00,B2A11,
     1     B3A21,B3A11,B3A10,B3A20
     2     ,B4A00,B4A11,B4A10,B5A00,B5A10,B5A11
     3      ,A01
     4     ,F1A1100,F1A1101,F1A1102,F1A1110,F1A1111,F1A1120
     5     ,S00,S10,S11,S000,S100,S101,S110
     6     ,A100,A101,A110,B100,B101,B110
     7     ,D1A100,D1A200,D1A101,D1A201,D1A202,D1A110,D1A210
     8,D2A100,D2A101,D2A110,D3A100,D3A101,D3A110,
     9D4A100,D4A101,D4A110,D6A100,D6A101,D6A110,
     *E3A100,E5A100,E5A101,E5A110,E5A111,
     1EP1A100,EP1A101,EP1A110,EP1A000,EP1A111,E2A100,E2A101,E2A110,
     2E2A111,E2PA200,E2PA201,E2PA202,E2PA210,E2PA220,E5PA200,E5PA201,
     3E5A202,E5PA210,E5PA211,E5PA220,E2PA211
     4    ,C1A100,C2A100,C2A000,C2A110,C3A000,C4A101,C2A101
C
       B1A11=B1A2(1,1)
       B1A21=B1A2(2,1)
       B1A22=B1A2(2,2)
C
       B2A00=B2A2(0,0)
       B2A10=B2A2(1,0)
       B2A11=B2A2(1,1)
C
       B3A10=B3A2(1,0)
       B3A11=B3A2(1,1)
       B3A20=B3A2(2,0)
       B3A21=B3A2(2,1)
C
C
       B4A00=B4A2(0,0)
       B4A11=B4A2(1,1)
       B4A10=B4A2(1,0)
C
       B5A00=B5A2(0,0)
       B5A10=B5A2(1,0)
       B5A11=B5A2(1,1)
C
       F1A1100=F1A4(1,1,0,0)
       F1A1101=F1A4(1,1,0,1)
       F1A1102=F1A4(1,1,0,2)
       F1A1110=F1A4(1,1,1,0)
       F1A1111=F1A4(1,1,1,1)
       F1A1120=F1A4(1,1,2,0)
C
       S00=S2(0,0)
       S10=S2(1,0)
       S11=S2(1,1)
C
       S000=S3(0,0,0)
       S100=S3(1,0,0)
       S101=S3(1,0,1)
       S110=S3(1,1,0)
C
       B100=SB3(1,0,0)
       B101=SB3(1,0,1)
       B110=SB3(1,1,0)
C
       A100=SA3(1,0,0)
       A101=SA3(1,0,1)
```

```
      A110=SA3(1,1,0)
C
      A01=A2(0,1)
C
      D1A100=D1A3(1,0,0)
      D1A101=D1A3(1,0,1)
      D1A110=D1A3(1,1,0)
      E5PA202=EP5A3(2,0,2)
      E5PA200=EP5A3(2,0,0)
      E5PA201=EP5A3(2,0,1)
      E5PA210=EP5A3(2,1,0)
      E5PA211=EP5A3(2,1,1)
      E5PA220=EP5A3(2,2,0)
      D2A100=D2A3(1,0,0)
      D2A101=D2A3(1,0,1)
      D2A110=D2A3(1,1,0)
      D3A100=D3A3(1,0,0)
      D3A101=D3A3(1,0,1)
      D3A110=D3A3(1,1,0)
      D4A100=D4A3(1,0,0)
      D4A101=D4A3(1,0,1)
      D4A110=D4A3(1,1,0)
      D6A100=D6A3(1,0,0)
      D6A101=D6A3(1,0,1)
      D6A110=D6A3(1,1,0)
C
      E3A100=E3A3(1,0,0)
      E5A100=E5A3(1,0,0)
      E5A101=E5A3(1,0,1)
      E5A110=E5A3(1,1,0)
      E5A111=E5A3(1,1,1)
C
      EP1A100=EP1A3(1,0,0)
      EP1A101=EP1A3(1,0,1)
      EP1A110=EP1A3(1,1,0)
      EP1A000=EP1A3(0,0,0)
      EP1A111=EP1A3(1,1,1)
C
      E2A100=E2A3(1,0,0)
      E2A101=E2A3(1,0,1)
      E2A110=E2A3(1,1,0)
      E2A111=E2A3(1,1,1)
C
      E2PA200=EP2A3(2,0,0)
      E2PA201=EP2A3(2,0,1)
      E2PA202=EP2A3(2,0,2)
      E2PA210=EP2A3(2,1,0)
      E2PA220=EP2A3(2,2,0)
      E2PA211=EP2A3(2,1,1)
C
      C1A100=C1A3(1,0,0)
      C2A100= C2A3(1,0,0)
      C2A000= C2A3(0,0,0)
      C2A101=C2A3(1,0,1)
      C2A110= C2A3(1,1,0)
      C3A000= C3A3(0,0,0)
      C4A101= C4A3(1,0,1)
      WRITE(6,IDBUG)
      RETURN
      END
```

```
      FUNCTION B1A2(I,J)
      COMMON LA(100),B(1)
      COMMON/POINTS/NPMAX
      DATA XNCALC/5HNCALC/
      IB=LA(5)
      ASSIGN 10 TO NPATH
      GO TO 600
   10 B1A2= TERM
      RETURN
C
      ENTRY B2A2
      IB=LA(6)
      ASSIGN 20 TO NPATH
      GO TO 600
   20 B2A2= TERM
      RETURN
C
      ENTRY B3A2
      IB= LA(7)
      ASSIGN 30 TO NPATH
      GO TO 600
   30 B3A2=TERM
      RETURN
C
      ENTRY B4A2
      IB=LA(8)
      ASSIGN 40 TO NPATH
      GO TO 600
   40 B4A2= TERM
      RETURN
C
      ENTRY B5A2
      IB=LA(9)
      ASSIGN 50 TO NPATH
      GO TO 600
   50 B5A2= TERM
      RETURN
  600 IF( I.LT.0 .OR. J.LT. 0 ) GO TO 800
      IP= J*NPMAX +I +1 +IB
      TERM=B(IP)
      IF( TERM .EQ. XNCALC)GO TO 800
      GO TO NPATH
  800 PRINT 900,I,J
C
C                       FORCE TRACE BACK
C
      Z=-10.
      W=SQRT(Z)
      CALL EXIT
  900 FORMAT(1H0,31HFROM B WITH 2 ARGS, I AND J ARE,2I10)
      RETURN
      END
```

```
      FUNCTION SA3(I,J,K)
C          SMALL A WITH 3 ARGUMENTS
      COMMON LA(100),B(1)
      COMMON/POINTS/NPMAX
C
      DATA XNCALC/5HNCALC/
      IF( I.LT.0 ,OR, J,LT, 0 ,OR, K,LT, 0)GO TO 800
      IP=LOK(I,J,K,27)
      SA3= B(IP)
      IF( SA3 .NE, XNCALC) RETURN
  800 PRINT 900, I,J,K
C
C          FORCE  TRACE BACK
C
      Z=-10.0
      Q=SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,21H  FROM SA3 I,J,K  ARE,3I10)
      END
```

```
      FUNCTION SB3(I,J,K)
C          SMALL B WITH 3 ARGUMENTS
      COMMON LA(100),B(1)
      COMMON/POINTS/NPMAX
C
      DATA XNCALC/5HNCALC/
      IF( I.LT.0 ,OR, J,LT, 0 .OR. K,LT, 0)GO TO 800
      IP=LOK(I,J,K,28)
      SB3= B(IP)
      IF( SB3 .NE, XNCALC) RETURN
  800 PRINT 900, I,J,K
C
C          FORCE  TRACE BACK
C
      Z=-10.0
      Q=SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,21H  FROM SB3 I,J,K  ARE,3I10)
      END
```

```
      FUNCTION DELTA(NUM)
      N=IABS(NUM)
      M=N/2
      IN=N-   M *2
      ND=0
      IF(IN.NE.0)GO TO 10
      IN= M -(M/2)*2
      ND=-1
      IF(IN.EQ, 0)ND=1
   10 DELTA= ND
      RETURN
      END
```

```
      FUNCTION SB1(INDX)
C          SB1=SMALL B WITH 1 ARGUMENT
      COMMON LA(100),B(1)
      IP= LA(3) + INDX +1
      SB1= B(IP)
      RETURN
      END
```

```
      FUNCTION C1A3(P,M,N)
C          C1A3= CAP C 1 WITH 3 ARGS
      INTEGER P
      COMMON/POINTS/NPMAX
      COMMON LA(100),B(1)
      DATA XNCALC/5HNCALC/
C
      NC=1
      LC=12
      GO TO 700
   10 NSTART= M+N+1
      XMP1=M+1
      SUM=0.0
C
      DO 50 I=NSTART,P
      SUM= SUM +  SB1(P-I)*XMP1*SB3(I,M+1,N)
   50 CONTINUE
      B(IP)=SUM
  100 C1A3=SUM
      RETURN
C
      ENTRY C2A3
C
      NC=2
      LC=13
      GO TO 700
  110 NSTART= M+N+1
      NEND= P+1
      SUM=0.0
      TNP1= 2*N+1
C
      DO 150 J=NSTART,NEND
      I=J-1
      IF( I.LT. 1)GO TO 150
      SUM= SUM + SB1(P-I)*TNP1*SB3(I,M,N)
  150 CONTINUE
      B(IP)=SUM
  200 C2A3=SUM
      RETURN
C
      ENTRY C3A3
      NC=3
      LC=14
```

```
      GO TO 700
  210 XMP1= M+1
      NSTART=M+N+1
      NPP1= P+1
      SUM=0.0
C
      DO 250 I=NSTART,NPP1
      SUM=SUM + SB1(NPP1-I)*XMP1*SA3(I,M+1,N)
  250 CONTINUE
      B(IP)=SUM
  300 C3A3=SUM
      RETURN
C
      ENTRY C4A3
      NC=4
      LC=15
      GO TO 700
  310 NSTART=M+N+1
      NEND=P+1
      SUM=0.0
      TN=2*N
C
      DO 350 J=NSTART,NEND
      I=J-1
      SUM=SUM + TN*SB1(P-I)*SA3(I,M,N)
  350 CONTINUE
      B(IP)=SUM
  400 C4A3= SUM
      RETURN
  700 IF( P.LT.0 .OR. M.LT. 0 .OR. N.LT. 0) GO TO 800
      IP=LOK(P,M,N,LC)
      SUM= B(IP)
      IF( SUM.EQ. XNCALC)GO TO(10,110,210,310),NC
      GO TO(100,200,300,400),NC
  800 PRINT 900,NC,P,M,N
C
C         FORCE TRACE BACK
C
      Z=-10.0
      Q= SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,10X, 6HFROM C,I2,   20HA3 - P,M, AND N ARE ,3I10)
      END
```

```
      FUNCTION S2(R,Q)
      INTEGER R,Q
      COMMON LA(100),B(1)
      COMMON/POINTS/NPMAX
      DATA XNCAL/5HNCALC/
C
C         CAP S WITH 2 ARGS
C
      IF( R.LT.0 .OR. Q.LT.0) GO TO 800
      IP= Q*NPMAX +R +1 +LA(11)
      S2=B(IP)
      IF( S2 .NE. XNCAL)RETURN
C
      NQ=Q+1
      NR=R+1
C
      SUM=0.0
      DO 100 I=1,NQ
      N=I-1
      DO 90 K=1,NR
      J=K-1
      IF( J.LT. N) GO TO 90
      IF( R+N-J-Q  .LT. 0 )GO TO 90
      SUM= SUM + B5A2(R-J,Q-N)* B2A2(J,N)
   90 CONTINUE
      B(IP)=SUM
  100 CONTINUE
      S2=SUM
      RETURN
  800 PRINT 900,R,Q
C
C         FORCE TRACE BACK
C
      Z=-10.0
      Q=SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,10X,22H FROM S2 - R AND Q ARE,2I10)
      END
```

```
      FUNCTION  S3(R,S,M)
      INTEGER R,S
      COMMON LA(100),B(1)
      COMMON/POINTS/NPMAX
      DATA XNCALC/5HNCALC/
C
C         CAP S WITH 3 ARGS
C
      IF( R.LT. 0 .OR. S.LT.0 .OR.  M.LT. 0)GO TO 800
      IP=LOK(R,S,M,10)
      S3= B(IP)
      IF( S3.NE. XNCALC) RETURN
C
      NEND= R-S-M+1
      SUM=0.0
      DO 100 I=1,NEND
```

```
      N=I-1
      SUM=SUM +B5A2(R-S-N,M)* B4A2(N+S,N)
  100 CONTINUE
      B(IP)=SUM
      S3=SUM
      RETURN
  800 PRINT 900 ,R,S,M
C
C         FORCE  TRACE BACK
C
      Z=-10.0
      Q=SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,10X,36HFROM S WITH 3 ARGS - R,S, AND M ARE ,3I10)
      END
```

```
      FUNCTION D1A3(T,U,V)
C          CAP  D  WITH 3 ARGS
      COMMON/GAMS/GAM,G1,G2,G3,D
      COMMON/POINTS/NPMAX
      COMMON LA(100),B(1)
      INTEGER T,U,V
      DATA XNCALC/5HNCALC/
C         D1
      ND=1
      LD=16
      GO TO 700
   10 NE1=T-1
      NE2=T-2
      OPD=1.0+D
      SUM1=0.0
      SUM2=0.0
      IF(T-2)80,20,20
   20 DO 30 J=1,NE1
      SUM1=SUM1 + F1A4(T-J,J,U,V)
   30 CONTINUE
      IF( T.LT. 3)GO TO 80
      IF( V.LT. 1)GO TO 80
      IF( T-U .LT. 1 )GO TO 80
      DO 40 J=1,NE2
      SUM2= SUM2 +F2A4(T-1-J,J,U,V)
   40 CONTINUE
C
   80 B(IP)= -OPD*( 2*SA3(T,U,V) + SUM1) -G1*SUM2
  100 D1A3= B(IP)
      RETURN
C         D2
      ENTRY D2A3
      ND=2
      LD=17
      GO TO 700
  110 NE1=T-2
      NE2=T-1
      SUM1=0.0
      SUM2=0.0
      IF( T.LT. 3)GO TO 150
      IF( V.LT. 1)GO TO 150
      IF(T-U .LT. 1 ) GO TO 150
C
      DO 130 J=1,NE1
```

```
      SUM1= SUM1 + F2A4(T-1-J,J,U,V)
  130 CONTINUE
  150 IF(T.LT. 2)GO TO 180
C
      DO 160 J=1,NE2
      SUM2= SUM2 +F1A4(T-J,J,U,V)
  160 CONTINUE
C
  180 B(IP)= -(1.0+D)*SUM1-2.0*G1*SA3(T,U,V) -G1*SUM2
  200 D2A3= B(IP)
      RETURN
C         D3
      ENTRY D3A3
C
      ND=3
      LD=18
      GO TO 700
  210 NEND=T-1
      SUM=0.0
      IF(T.LT. 2) GO TO 250
C
      DO 240 J=1,NEND
      SUM=SUM + F3A4(T-J,J,U,V)
  240 CONTINUE
  250 B(IP)= SB3(T,U,V)  +SUM
  300 D3A3=B(IP)
      RETURN
C         D4
      ENTRY D4A3
C
      ND=4
      LD=19
      GO TO 700
  310 SUM1=0.0
      SUM2=0.0
      IF(T.LT. 2)GO TO 370
      NEND=T-1
      DO 320 J=1,NEND
      SUM1= SUM1 + F1A4(T-J,J,U,V)
  320 CONTINUE
      IF(T.LT.3)GO TO 370
      IF(V.LT.1)GO TO 370
      IF( T-U .LT. 1)GO TO 370
      NEND= T-2
C
      DO 340 J=1,NEND
      SUM2= SUM2 + F2A4(T-J-1,J,U,V)
  340 CONTINUE
C
  370 B(IP)= -G3*SA3(T,U,V) -3.0*G1*SUM1 -G1*SUM2
  400 D4A3=B(IP)
      RETURN
C         D5
      ENTRY D5A3
C
      ND=5
      LD=20
      GO TO 700
  410 SUM1= 0.0
      SUM2= 0.0
```

```
      IF(T.LT.2) GO TO 480
      NEND=T-1
C
      DO 420 J=1,NEND
      SUM1=SUM1 +F1A4(T-J,J,U,V)
 420  CONTINUE
C
      IF(T.LT.3)GO TO 480
      IF(V.LT.1)GO TO 480
      IF(T-U .LT. 1 ) GO TO 480
      NEND=T-2
C
      DO 440 J=1,NEND
      SUM2=SUM2 +F2A4(T-J-1,J,U,V)
 440  CONTINUE
 480  B(IP)= -G1*SUM1 -G1 *SUM2
 500  D5A3= B(IP)
      RETURN
C         D6
      ENTRY D6A3
      ND=6
      LD=21
      GO TO 700
 510  SUM1=0.0
      SUM2=0.0
      IF( T .LT. 2) GO TO 580
      NEND= T-1
C
      DO 520 J=1,NEND
      SUM1= SUM1 +  F1A4(T-J,J,U,V)
 520  CONTINUE
      IF(T.LT. 3) GO TO 580
      IF(V.LT. 1) GO TO 580
      IF( T-U .LT. 1)GO TO 580
      NEND= T-2
C
      DO 540 J=1,NEND
      SUM2= SUM2 + F2A4(T-J-1,J,U,V)
 540  CONTINUE
 580  B(IP)=-G1*( 2.0*SA3(T,U,V)+SUM1+SUM2)
 600  D6A3=B(IP)
      RETURN
 700  IF( T.LT. 0 .OR. U.LT. 0 .OR. V.LT. 0)GO TO 800
C
C         CALC POINTER AND OK ON NO CALC
C
      IP=LOK(T,U,V,LD)
      IF( B(IP) .EQ. XNCALC)GO TO(10,110,210,310,410,510),ND
      GO TO(100,200,300,400,500,600),ND
 800  PRINT 900,ND,T,U,V
C
C         FORCE TRACE BACK
C
      Z=-10.0
      Q=SQRT(Z)
      CALL EXIT
      RETURN
 900  FORMAT(1H0,10X, 7H FROM D,I2,  30HWITH 3 ARGS - T,U, AND V ARE
     1    3I10)
      END
```

```
      FUNCTION F1A4(N,M,Q,P)
C          F S WITH 4 ARGUMENTS
      COMMON LA(100),B(1)
      INTEGER Q,P
C
      NF=1
      GO TO 600
 10   NQ=Q+1
      SUM=0.0
      NP=P+1
C
      DO 100  I=1,NQ
      LM=I-1
C
      DO 90  J=1,NP
      LN=J-1
      IF(N-LM-LN)90,20,20
 20   IF(M-Q+LM-P+LN)90,30,30
 30   SUM=SUM + SA3(N,LM,LN)*SA3(M,Q-LM,P-LN)
 90   CONTINUE
 100  CONTINUE
      F1A4=SUM
      RETURN
C
      ENTRY F2A4
      NF=2
      GO TO 600
 110  NQ=Q+1
      NP=P
      SUM=0.0
C
      DO 200 I=1,NQ
      LM=I-1
      DO 190 J=1,NP
      LN= J-1
      IF(N-LM-LN)190,120,120
 120  IF(M-Q+LM-P+1+LN)190,130,130
 130  SUM=SUM + SB3(N,LM,LN)*SB3(M,Q-LM,P-LN-1)
 190  CONTINUE
 200  CONTINUE
      F2A4=SUM
      RETURN
C
      ENTRY F3A4
      NF=3
      GO TO 600
 210  NQ=Q+1
      NP=P+1
      SUM=0.0
C
      DO 300 I=1,NQ
      LM=I-1
      DO 290 J=1,NP
      LN=J-1
      IF(N-LM-LN)290,220,220
 220  IF(M-Q+LM-P+LN)290,230,230
 230  SUM= SUM+ SA3(N,LM,LN)*SB3(M,Q-LM,P-LN)
 290  CONTINUE
 300  CONTINUE
      F3A4=SUM
```

```
      RETURN
  600 IF( N.LT. 0 .OR. M.LT.0 .OR. Q.LT.0 .OR. P.LT.0)GO TO 800
      IF(NF-2)10,110,210
  800 PRINT  900,NF,N,M,Q,P
C
C         FORCE TRACE BACK
C
      Z=-10.
      Q=SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,10X, 6HFROM F,I2,30H   WITH 4 ARGS N,M,Q AND P ARE,
     1   4I10)
      END
```

```
      FUNCTION  E1A3(Q,K,L)
C
C         CAP E1 WITH 3 ARGS
C
      COMMON LA(100),B(1)
      COMMON/POINTS/NPMAX
      INTEGER Q,R
      INTEGER D,C
C
      DATA XNCALC/5HNCALC/
C
      NE=1
      LE=22
      GO TO 700
   10 SUM1=0.0
      SUM2=0.0
      NE1=Q-L+1
C
      DO 50 I=1,NE1
      M=I-1
      IF( K-M .LT. 0)GO TO 50
      NE2=Q-M-L+1
C
      DO 40 J=1,NE2
      R=J-1
      IF(Q-R-M .LT. 0)GO TO 40
      IF(M+R-K .LT. 0)GO TO 40
      SUM1= SUM1 + C3A3(Q-R,M,L)*B4A2(R,M+R-K)
   40 CONTINUE
   50 CONTINUE
C
      IF( Q-K-L .LT. 0)GO TO 80
      NE1= Q-K+1
      NE2= L+1
      DO 70 I=1,NE1
      R=I-1
      DO 60 J=1,NE2
      M=J-1
      IF(Q-R-K-M .LT. 0)GO TO 60
      IF( R-L+M .LT. 0) GO TO 60
      SUM2= SUM2 + C3A3(Q-R,K,M)*B2A2(R,L-M)
   60 CONTINUE
   70 CONTINUE
   80 B(IP)= SUM1 +SUM2
  100 E1A3= B(IP)
      RETURN
```

```
C
      ENTRY E2A3
C         E2A3
C         E2
      NE=2
      LE=23
      GO TO 700
  110 SUM1=0.0
      SUM2=0.0
      NE1=Q-L+1
C
      DO 150 I=1,NE1
      M=I-1
      IF(K-M .LT. 0)GO TO 155
C
      NE2= Q-M-L +1
      DO 140 J=1,NE2
      R=J-1
      IF( Q-R-1 .LT. 0) GO TO 140
      IF( Q-R-M .LT. 0) GO TO 140
      IF( M+R-K .LT. 0) GO TO 140
      SUM1=SUM1 + C2A3(Q-R,M,L)*B4A2(R,M+R-K)
  140 CONTINUE
  150 CONTINUE
  155 IF(Q-K-L .LT. 0) GO TO 180
      NE1=Q-K+1
C
      DO 170 I=1,NE1
      R=I-1
      IF(Q-R-1.LT. 0) GO TO 180
C
      NE2=L+1
      DO 160 J=1,NE2
      M=J-1
      IF( Q-R-K-M .LT. 0)GO TO 160
      IF(R-L+M .LT. 0) GO TO 160
      SUM2= SUM2 + C2A3(Q-R,K,M)*B2A2(R,L-M)
  160 CONTINUE
  170 CONTINUE
  180 B(IP)= SUM1 +SUM2
  200 E2A3=B(IP)
      RETURN
C
      ENTRY E3A3
C         CAP E3
      NE=3
      LE=24
      GO TO 700
  210 SUM1=0.0
      SUM2=0.0
      NE1=Q-L
C
      DO 250 I=1,NE1
      M=I-1
      IF(K-M .LT. 0)GO TO 255
      NE2= Q-M-L
      DO 240 J=1,NE2
      R=J-1
      IF( Q-R-M-1 .LT. 0)GO TO 240
      IF( M+R-K .LT. 0) GO TO 240
```

```
            SUM1= SUM1 + B4A2(R,M+R-K)*( C4A3(Q-R,M,L+1) + C1A3(Q-R,M,L) )
   240      CONTINUE
   250      CONTINUE
   255      IF( Q-1-K-L .LT. 0)GO TO 280
            NE1= Q   -K
            DO 270 I=1,NE1
            R=I-1
            NE2=L+1
            DO 260 J=1,NE2
            M=J-1
            IF( Q-1-R-K-M .LT.0 )GO TO 260
            IF( R-L+M .LT. 0)GO TO 260
            SUM2= SUM2 +B2A2(R,L-M)*( C4A3(Q-R,K,M+1) + C1A3(Q-R,K,M) )
   260      CONTINUE
   270      CONTINUE
   280      B(IP)= SUM1+SUM2
   300      E3A3= B(IP)
            RETURN
C
            ENTRY E4A3
C                CAP E 4
            NE=4
            LE= 25
            GO TO 700
   310      SUM1=0.0
            NE1=Q-L
            DO 350 I=1,NE1
            M=I-1
            IF(K-M .LT. 0)GO TO 350
            NE2=Q-M-L
C
            DO 340 J=1,NE2
            R=J-1
            IF(Q-R-2 .LT. 0)GO TO 340
            IF(Q-R-M-1 .LT. 0)GO TO 340
            IF(M+R-K .LT. 0) GO TO 340
            SUM1= SUM1 + SA3(Q-R-1,M,L)*B3A2(R+1,M+R-K)
   340      CONTINUE
   350      CONTINUE
   355      B(IP)= SUM1
   400      E4A3= B(IP)
            RETURN
            ENTRY E5A3
C                CAP E 5
            NE=5
            LE=26
            GO TO 700
   410      SUM1=0.0
            SUM2=0.0
            SUM3=0.0
            IF( Q-K-L .LT. 0)GO TO 435
            NE1=Q
            NE2=K+1
            NE3=L+1
            DO 430 I=1,NE1
            R=I-1
            DO 425  J=1,NE2
            C=J-1
            DO 420 I3=1,NE3
            D=I3-1
```

```
      IF( Q-R-C-D .LT. 0)GO TO 420
      IF(R-K+C-L+D .LT. 0)GO TO 420
      SUM1= SUM1 + S3(R,K-C,L-D)*SB3(Q-R,C,D)
  420 CONTINUE
  425 CONTINUE
  430 CONTINUE
  435 NE1=Q-K+1
      NE2=L+1
C
      DO 450 I=1,NE1
      R=I-1
      IF( Q-1-R .LT. 0)GO TO 450
C
      DO 440 J=1,NE2
      D=J-1
      IF(Q-R-K-D .LT. 0)GO TO 440
      IF(R-L+D .LT. 0)GO TO 440
      SUM2= SUM2 + S2(R,L-D)*SB3(Q-R,K,D)
  440 CONTINUE
  450 CONTINUE
      NE1=Q-K
      NE2=L
      IF( L-1 .LT. 0)GO TO 480
C
      DO 470 I=1,NE1
      R=I-1
      IF( Q-2-R .LT. 0)GO TO 480
C
      DO 460 J=1,NE2
      D=J-1
      IF( Q-1-R-K-D .LT. 0)GO TO 460
      IF( R+1-L+D .LT. 0)GO TO 460
      SUM3= SUM3 + B1A2(R+1,L-D)* SB3(Q-1-R,K,D)
  460 CONTINUE
  470 CONTINUE
  480 B(IP)= SUM1 +SUM2  -2.0*SUM3
  500 E5A3=B(IP)
      RETURN
  700 IF( Q.LT. 0 .OR. K.LT. 0 .OR. L .LT. 0 )GO TO 800
      IP=LOK(Q,K,L,LE)
      IF( B(IP) .EQ.XNCALC)GO TO(10,110,210,310,410),NE
      GO TO(100,200,300,400,500),NE
  800 PRINT 900, NE,Q,K,L
C
C         FORCE TRACE BACK
C
      Z=-10.0
      QQ=SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,10X, 7H FROM E,I2,  15H Q,K, AND L ARE,3I10)
      END
```

```
      FUNCTION EP1A3(Q,K,L)
C         CAP E PRIME WITH 3 ARGS
      INTEGER      R,D,C,Q
      COMMON LA(100),B(1)
      COMMON/POINTS/NPMAX
      DATA XNCALC/5HNCALC/
C
      NEP=1
      LEP=29
      GO TO 700
 10   SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      NE1=Q-L+1
      DO 30 I=1,NE1
      M=I-1
      IF(K-M,LT, 0)GO TO 30
      NE2= Q-M-L
      DO 20 R=1,NE2
      IF( Q-R-M-L .LT. 0)GO TO 20
      IF( M+R-K .LT. 0)GO TO 20
      SUM1=SUM1 + C3A3(Q-R,M,L)*B4A2(R,M+R-K)
 20   CONTINUE
 30   CONTINUE
      IF( Q-K-L .LT. 0)GO TO 80
      NE1=Q-K
      NE2= L+1
      DO 50 R=1,NE1
      DO 40 I=1,NE2
      M=I-1
      IF(Q-R-K-M .LT. 0)GO TO 40
      IF( R-L+M .LT. 0)GO TO 40
      SUM2=SUM2 + C3A3(Q-R,K,M)*B2A2(R,L-M)
 40   CONTINUE
 50   CONTINUE
      IF(Q-K-L-1 .LT. 0) GO TO 80
      XKP1=K+1
      NSTART= 1+K+L
      DO 60 I=NSTART,Q
      SUM3=SUM3+ SB1(Q-I+1)*XKP1*SA3(I,K+1,L)
 60   CONTINUE
 80   B(IP)= SUM1 +SUM2 + 2.0*SUM3
 100  EP1A3= B(IP)
      RETURN
C
      ENTRY EP2A3
C         CAP E PRIME 2
      NEP=2
      LEP=30
      GO TO 700
 110  SUM1=0.0
      SUM2=0.0
      NE1=Q-L+1
C
      DO 150 I=1,NE1
      M=I-1
      IF(K-M .LT. 0)GO TO 155
C
      NE2= Q-M-L
      DO 140 R=1,NE2
      IF( Q-R-1 .LT. 0) GO TO 140
      IF( Q-R-M-L .LT. 0) GO TO 140
      IF( M+R-K .LT. 0) GO TO 140
      SUM1=SUM1 + C2A3(Q-R,M,L)*B4A2(R,M+R-K)
 140  CONTINUE
 150  CONTINUE
 155  IF(Q-K-L .LT. 0) GO TO 180
      NE1=Q-K
```

```
C
      DO 170 R=1,NE1
      IF(Q-R-1,LT, 0) GO TO 180
C
      NE2=L+1
      DO 160 J=1,NE2
      M=J-1
      IF( Q-R-K-M .LT. 0)GO TO 160
      IF( R-L+M .LT. 0) GO TO 160
      SUM2= SUM2 + C2A3(Q-R,K,M)*B2A2(R,L-M)
  160 CONTINUE
  170 CONTINUE
  180 B(IP)= SUM1 +SUM2
  200 EP2A3=B(IP)
      RETURN
C
      ENTRY EP5A3
C         CAP EPRIME 5
      NEP=5
      LEP=31
      GO TO 700
  410 SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      IF( Q-K-L .LT. 0)GO TO 435
      NE1=Q -1
      NE2=K+1
      NE3=L+1
      DO 430 R=1,NE1
      DO 425  J=1,NE2
      C=J-1
      DO 420 I3=1,NE3
      D=I3-1
      IF( Q-R-C-D .LT. 0)GO TO 420
      IF(R-K+C-L+D .LT. 0)GO TO 420
      SUM1= SUM1 + S3(R,K-C,L-D)*SB3(Q-R,C,D)
  420 CONTINUE
  425 CONTINUE
  430 CONTINUE
  435 NE1=Q-K
      NE2=L+1
C
      DO 450 R=1,NE1
      IF( Q-1-R .LT. 0)GO TO 450
C
      DO 440 J=1,NE2
      D=J-1
      IF(Q-R-K-D .LT. 0)GO TO 440
      IF(R-L+D .LT. 0)GO TO 440
      SUM2= SUM2 + S2(R,L-D)*SB3(Q-R,K,D)
  440 CONTINUE
  450 CONTINUE
      NE1=Q-K
      NE2=L
      IF( L-1 .LT. 0)GO TO 480
C
      DO 470 I=1,NE1
      R=I-1
      IF( Q-2-R .LT. 0)GO TO 480
C
```

```
      DO 460 J=1,NE2
      D=J-1
      IF( Q-1-R-K-D .LT. 0)GO TO 460
      IF( R+1-L+D .LT. 0)GO TO 460
      SUM3= SUM3 + B1A2(R+1,L-D)* SB3(Q-1-R,K,D)
  460 CONTINUE
  470 CONTINUE
  480 B(IP)= SUM1 +SUM2  -2.0*SUM3
  500 EP5A3=B(IP)
      RETURN
  700 IF( Q.LT. 0 .OR. K.LT. 0 .OR. L .LT. 0 )GO TO 800
      IP=LCK(Q,K,L,LEP)
      IF(  B(IP) .EQ. XNCALC) GO TO(10,110,800,800,410),NEP
      GO TO(100,200,800,800,500),NEP
  800 PRINT 900,NEP,Q,K,L
C
C         FORCE TRACE BACK
C
      Z=-10.0
      QQ=SQRT(Z)
      CALL EXIT
      RETURN
  900 FORMAT(1H0,10X, 8HFROM E PI2,  15H Q,K, AND L ARE,3I10)
      END
```

```
      SUBROUTINE COEFF(CF,RHS,X,P,NP)
      INTEGER P,G,H
      COMMON/GAMS/GAM,G1,G2,G3,D
      COMMON/WALLBC/ETAW, RCURV, EPSIL
      COMMON/EPSFLG/EFLAG
      DIMENSION CF(NP,NP),RHS(1),X(1)
C
C         THIS ROUTINE CALCULATES THE COEFF MATRIX
C
      A0=SA1(0)
      A110= SA3(1,1,0)
      NBB= (P+1)*(P+2)/2
      DO 10 I=1,NP
      RHS(I)=0.0
      X(I)=0.0
      DO 10 J=1,NP
      CF(J,I)=0.0
   10 CONTINUE
C
C         IRROTATIONAL EQNS
C
      NROW=0
      DO 100 N=1,P
      K=N-1
      NEND= P-K
      DO 90 M=1,NEND
      L=M-1
```

```
      NROW=NROW +1
      NCOL= L*(P+1)+ N + 1 - L*(L-1)/2  +NBB
      CF(NROW,NCOL)= 2.0*FLOAT(N)/AO
      NCOL= M*(P+1) +K+1 - M*(M-1)/2
      CF(NROW,NCOL)=-4.0*FLOAT(M)/AO
  90  CONTINUE
 100  CONTINUE
C
C          MOMENTUM EQN
C
      NE1=P+1
      DO 300 I=1,NE1
      G=I-1
      NE2= P-G+1
      DO 250 J=1,NE2
      H=J-1
      NROW=NROW+1
      NCOL=  H*(P+1)+ G+1 -H*(H-1)/2
      CF(NROW,NCOL)= -4.0*(1.0+D)*A110/AO
      NCOL=NCOL +NBB
      CF(NROW,NCOL)= 4.0*G2*FLOAT(J)/AO
C
      DO 220 M=1,I
      K= M-1
      IF( P-1-K .LT. 0 )GO TO 220
C
      DO 200 N=1,J
      L=N-1
      IF( P-1-L .LT. 0)GO TO 200
      IF( 1+K+L-G-H .LT. 0)GO TO 200
      IF( P-1-K-L .LT. 0) GO TO 200
      NCOL= L*(P+1) + M+1 - L*(L-1)/2
      TERM=-4.0*(1.0+D)*FLOAT(M)*SA3(1,G-K,H-L)/AO
      CF(NROW,NCOL)=TERM+CF(NROW,NCOL)
 200  CONTINUE
 220  CONTINUE
 250  CONTINUE
 300  CONTINUE
      NEND=P+1
      DO 400 I=1,NEND
      M=I-1
      NE2= P-M+1
      NROW=NROW+1
C
      DO 350 J=1,NE2
      N=J-1
      NCOL= N*(P+1) + M +1 -  N*(N-1)/2 +NBB
      CF(NROW,NCOL)=1.0
      IF(EFLAG.GT.2.0)CF(NROW,NCOL)=ETAW**(2*N+1)
 350  CONTINUE
 400  CONTINUE
C
C          GET  R. H. SIDE
C
      CALL RHSIDE(RHS,P)
      DO 700 I=1,NROW
      WRITE(6,900)I,(K,CF(I,K),K=1,NROW)
      WRITE(6,910)RHS(I)
  700 CONTINUE
 900  FORMAT(1H0,28HCOEFF MATRIX AND RHS FOR ROW,I5/
     1    (10X,5(I5,G16.5)))
```

```
  910 FORMAT(10X,4HRHS=,G17.6)
      RETURN
      END
      SUBROUTINE RHSIDE(RHS,P)
      INTEGER P
      DIMENSION RHS(1)
      NMAX=(P+1)**2
      NBB=P*(P+1)/2
      NROW=0
      CALL IRROT(RHS,P,NROW)
      IF(NROW .NE. NBB)GO TO 800
      CALL MOMEN(RHS,P,NROW)
      IF(NROW .NE. NMAX)GO TO 800
      RETURN
  800 PRINT 900, NROW,P,NBB,NMAX,(I,RHS(I),I=1,NROW)
      CALL EXIT
  900 FORMAT(1H1,10X,20HIMPROPER NUM OF ROWS,
     1     20H NROW,P,NBB,NMAX ARE,4I7/10X,
     2     16HTHE RH SIDES ARE/(10X,I5,G17.8) )
      END
```

```
      SUBROUTINE MOMEN(RHS,P,NROW)
C
C          ROUTINE TO CALC R.H. SIDE OF MOMENT EQN
C
      DIMENSION RHS(1)
      INTEGER P,T,H,G
      COMMON/GAMS/GAM,G1,G2,G3,D
      NAMELIST/BUGS1/ SUM1,SUM2,SUM3,SUM4,SUM5,SUM6,
     1     SUM7,SUM8,TERM2,TERM3
C
      NEND1=P+1
C
      DO 1000 I1=1,NEND1
      G=I1-1
      NEND2=P-G+1
      DO 980 I2=1,NEND2
      H=I2-1
      NROW=NROW+1
      SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      SUM4=0.0
      SUM5=0.0
      SUM6=0.0
      SUM7=0.0
      SUM8=0.0
      NE1=P-1
      NE2=G+1
      DO 100 T=2,NE1
      IF(P-3 .LT. 0) GO TO 100
```

```
          DO 80 J=1,NE2
          K=J-1
          IF(P-T-K .LT. 0)GO TO 100
          NE3=H+1
          DO 60 I=1,NE3
          L=I-1
          IF(P-T-L .LT. 0)GO TO 60
          IF(T-G+K-H+L .LT. 0)GO TO 60
          SUM1=SUM1 + E1A3(P-T,K,L)*D1A3(T,G-K,H-L)
   60     CONTINUE
   80     CONTINUE
  100     CONTINUE
          IF(P-2 .LT. 0)GO TO 155
          NE1=P-1
          DO 150 J=1,NE1
          SUM2=SUM2+F1A4(P-J,J,G,H)
  150     CONTINUE
          SUM2=(1.0+D)*SUM2
  155     IF(P-3 .LT. 0)GO TO 201
          IF(H-1 .LT. 0)GO TO 201
          IF(P-1-G .LT. 0)GO TO 201
          NE2=P-2
          SUM22=0.0
          DO 190 J=1,NE2
          SUM22=SUM22+F2A4(P-1-J,J,G,H)
  190     CONTINUE
  200     SUM2=SUM2+G1*SUM22
  201     NE1=G+1
          NE2=H+1
          DO 300 I=1,NE1
          K=I-1
          IF(P-1-K .LT. 0)GO TO 300
          DO 280 J=1,NE2
          L=J-1
          IF(P-1-L .LT. 0)GO TO 280
          IF(1+K+L-G-H .LT. 0)GO TO 280
          SUM3=SUM3+SA3(1,G-K,H-L)*EP1A3(P-1,K,L)
  280     CONTINUE
  300     CONTINUE
C
C              4 TH SUM
C
          IF(P-1-G-H .LT. 0) GO TO 400
          NST1=G+H+1
          NE1=P
          DO 380 J=NST1,NE1
          I=J-1
          IF(I-1 .LT. 0)GO TO 380
          SUM4=SUM4+SB1(P-I)*SB3(I,G,H)*FLOAT(2*H+1)
  380     CONTINUE
  400     SUM4=2.0*SUM4+EP2A3(P,G,H)
C
C              5 TH SUM
C
          NE1=P-1
          NE2=G+1
          NE3=H+1
C
          DO 500 T=1,NE1
          DO 480 I=1,NE2
          K=I-1
```

```
            IF(P-T-K .LT. 0)GO TO 480
      C
            DO 460 J=1,NE3
            L=J-1
            IF(P-T-L .LT. 0)GO TO 460
            IF(T-G+K-H+L .LT. 0)GO TO 460
            SUM5=SUM5 +E2A3(P-T,K,L)*D2A3(T,G-K,H-L)
           1      +E5A3(P-T,K,L)*D6A3(T,G-K,H-L)
        460 CONTINUE
        480 CONTINUE
        500 CONTINUE
      C
      C        6 TH SUM
      C
            IF(P-1-G .LT. 0)GO TO 601
            IF(H-1 .LT. 0)GO TO 601
            NE1=P-1
            NE2=G+1
            NE3=H
      C
            DO 600 T=1,NE1
            DO 580 I=1,NE2
            K=I-1
            IF(P-T-1-K .LT. 0)GO TO 580
            DO 560 J=1,NE3
            L=J-1
            IF(P-T-1-L .LT. 0)GO TO 560
            IF(T-G+K-H+1+L .LT. 0)GO TO 560
            SUM6=SUM6+E3A3(P-T,K,L)*D3A3(T,G-K,H-1-L)
        560 CONTINUE
        580 CONTINUE
        600 CONTINUE
      C
      C
        601 TERM2=0.0
            IF(G-1.LT.0)GO TO 710
            IF(H.EQ.0)TERM2=2.0*G2*B3A2(P,P-G)
      C
      C        7 TH SUM
      C
            NE1=P-H
            NE2=P
            DO 700 I=1,NE1
            K=I-1
            NST2=H+K+1
            IF(G-K .LT. 0)GO TO 700
            IF(P-1-H-K .LT. 0)GO TO 700
            DO 680 J=NST2,NE2
            T=J-1
            IF(T-1 .LT. 0)GO TO 680
            IF(K+P-T-G .LT. 0)GO TO 680
            SUM7=SUM7+D4A3(T,K,H)*B3A2(P-T,K+P-T-G)
        680 CONTINUE
        700 CONTINUE
      C
      C        8 TH SUM
      C
        710 IF(P-3.LT.0)GO TO 810
            IF(G-1.LT.0)GO TO 810
            IF(P-1-G.LT.0)GO TO 810
            NE1=P-2
```

```
      NE2=G
      NE3=H+1
C
      DO 800 T=1,NE1
      DO 780 I=1,NE2
      K=I-1
      IF(P-T-1-K.LT.0)GO TO 780
C
      DO 760 J=1,NE3
      L=J-1
      IF(P-T-1-L .LT. 0)GO TO 760
      IF(T-G+1+K-H+L .LT. 0)GO TO 760
      SUM8=SUM8+D5A3(T,G-1-K,H-L)*E4A3(P-T,K,L)
  760 CONTINUE
  780 CONTINUE
  800 CONTINUE
  810 TERM3=EP5A3(P,G,H)
C
C          COLLECT TERMS
C
  910 RHS(NROW)=-SUM1+2.0*SA3(1,1,0)*SUM2/SA1(0)
     1     +2.0*(1.0+D)*SUM3-G2*SUM4-SUM5
     2     +G2*SUM6-TERM2-2.0*SUM7
     3     -2.0*SUM8-G2*TERM3
  980 CONTINUE
 1000 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE IRROT(RHS,P,NROW)
      DIMENSION RHS(1)
      INTEGER P,R,T,G,H
C
C          CALC RIGHT HAND SIDES
C
      BPM1=SB1(P-1)
      DO 1000 I1=1,P
      K=I1-1
      NEND2=P-K
      DO 980 I2=1,NEND2
      L=I2-1
      TTLP1=2*(L+1)
      NROW=NROW+1
      SUM1=0.0
      SUM2=0.0
      SUM3=0.0
      SUM4=0.0
      SUM5=0.0
      SUM6=0.0
      SUM7=0.0
      SUM8=0.0
      IF( K-1 .LT.0)GO TO 101
      IF( P-3 .LT.0)GO TO 101
      NE2=P-1
      NE1=P-1-L
```

```
      DO 100 I=1,NE1
      M=I-1
      IF( K-1-M .LT. 0) GO TO 100
      NST=M+L+1
      DO 80 J=NST,NE2
      R=J-1
      IF(R-1 .LT. 0)GO TO 80
      IF(P-1-R+M-K .LT. 0)GO TO 80
      SUM1=SUM1 + B3A2(P-1-R,P-1-R+M-K)*SB3(R,M,L)
   80 CONTINUE
  100 CONTINUE
C
C          2 ND SUM
C
  101 NE1=P-L
C
      DO 200 J=1,NE1
      M=J-1
      IF(K-M .LT. 0)GO TO 200
      IF(P-M-L-2.LT.0)GO TO 200
      NST=M+L+1
      NE2=P-1
      DO 180 R=NST,NE2
      SUM22=0.0
      IF(P-R+M-K .LT. 0)GO TO 180
      NST2=M+L+1
      DO 160 I=NST2,R
      SUM22=SUM22+FLOAT(M+1)*SB1(R-I)*SB3(I,M+1,L)
  160 CONTINUE
      SUM2=SUM2+B4A2(P-R,P-R+M-K)*SUM22
  180 CONTINUE
  200 CONTINUE
C
C          3 RD SUM
C
      NSTART=K+L+1
      NEND=P-1
      BPM1=SB1(P-1)
      DO 300 I=NSTART,NEND
      IF(P-K-L-2 .LT. 0) GO TO 300
      SUM3=SUM3+BPM1*SB3(I,K+1,L)
  300 CONTINUE
C
C          4 TH SUM
C
      NE1=L+1
      NE2=P-1
      NST2=K+1
      DO 400 J=1,NE1
      M=J-1
      IF(P-K-2.LT.0)GO TO 400
      DO 380 R=NST2,NE2
      SUM44=0.0
      IF(R-1-K-M .LT. 0) GO TO 380
      IF(P-R-L+M .LT. 0) GO TO 380
C
      NST3=K+M+1
      DO 350 I=NST3,R
      SUM44=SUM44+SB1(R-I)*FLOAT(K+1)*SB3(I,K+1,M)
  350 CONTINUE
      SUM4=SUM4+B2A2(P-R,L-M)*SUM44
```

```
  380 CONTINUE
  400 CONTINUE
C
C         5 TH SUM
C
      NE1=P-L
      NE2=P-1
      DO 500 J=1,NE1
      M=J-1
      IF(K-M .LT. 0)GO TO 500
      IF(P-M-L-2.LT.0)GO TO 500
      NST2=M+L+1
      DO 480 R=NST2,NE2
      SUM55=0.0
      IF(P-R+M-K .LT. 0)GO TO 480
      NST3=M+L+1
      DO 450 I=NST3,R
      SUM55=SUM55+SB1(R-I)*TTLP1*SA3(I,M,L+1)
  450 CONTINUE
      SUM5 = SUM5+B4A2(P-R,P-R+M-K)*SUM55
  480 CONTINUE
  500 CONTINUE
C
C         6 TH SUM
C
      NST1=K+L+1
      NE1=P-1
      DO 600 I=NE1,NST1
      IF(P-K-L-2 .LT. 0) GO TO 600
      SUM6 = SUM6 +BPM1*TTLP1*SA3(I,K,L+1)
  600 CONTINUE
C
C         7 TH SUM
C
      NE1=L+1
      NE2=P-1
      NST2=K+1
      DO 700 M=1,NE1
C
      IF(P-K-2.LT.0)GO TO 700
      DO 680 R=NST2,NE2
      SUM77=0.0
      IF(R-K-M .LT. 0)GO TO 680
      IF(P-R-L-1+M .LT. 0)GO TO 680
      NST3=K+M
      TWOM=2*M
      DO 650 I=NST3,R
      IF(R-I .LT. 0) GO TO 650
      SUM77 = SUM77+SB1(R-I)*TWOM*SA3(I,K,M)
  650 CONTINUE
      SUM7 = SUM7+B2A2(P-R,L+1-M)*SUM77
  680 CONTINUE
  700 CONTINUE
C
C         8 TH SUM
C
      IF(K.EQ.0) SUM8=B1A2(P,L+1)
      NE1=L+1
      NE2=P
      NST2=K+1
C
```

```
      M(K)=J
  120 CONTINUE
      JROW=L(K)
      IF(L(K)-K)135,135,125
  125 DO 130 I=1,N
      HOLD=-A(K,I)
      A(K,I)=A(JROW,I)
  130 A(JROW,I)=HOLD
  135 ICOL=M(K)
      IF(M(K)-K)145,145,137
  137 DO 140  J=1,N
      HOLD=-A(J,K)
      A(J,K)=A(J,ICOL)
  140 A(J,ICOL)=HOLD
  145 IF(A(K,K))147,143,147
  143 TEST =1.
      GO TO 235
  147 DO 155 IC=1,N
      IF(IC-K)150,155,150
  150 A(IC,K)=A(IC,K)/(-A(K,K))
  155 CONTINUE
      DO 165 I=1,N
      DO 165 J=1,N
  156 IF(I-K)157,165,157
  157 IF(J-K)160,165,160
  160 A(I,J)=A(I,K)*A(K,J)+A(I,J)
  165 CONTINUE
      DO 175 JR=1,N
  168 IF(JR-K)170,175,170
  170 A(K,JR)=A(K,JR)/A(K,K)
  175 CONTINUE
      DETER=DETER*A(K,K)
      A(K,K)=1.0/A(K,K)
  180 CONTINUE
      K=N
  200 K=K-1
      IF(K)235,235,203
  203 I=L(K)
      IF(I-K)220,220,205
  205 DO 210 J=1,N
      HOLD=A(J,K)
      A(J,K)=-A(J,I)
  210 A(J,I)=HOLD
  220 J=M(K)
      IF(J-K)200,200,225
  225 DO 230 I=1,N
      HOLD=A(K,I)
      A(K,I)=-A(J,I)
  230 A(J,I)=HOLD
      GO TO 200
  235 DO 240 I=1,N
      DO 240 J=1,N
      A(I,J)= A(I,J)/R(J)/C(I)
 240  CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SOLN(CF,RHS,X,P,NP)
      DIMENSION CF(NP,NP),RHS(1),X(1),LL(144),MM(144),CC(144)
      COMMON LA(100),B(1)
      INTEGER P
C
C          ROUTINE TO CALC SOLN VECTOR AND MAP
C          IT INTO THE SA3 ARRAY
C
      NORDER=(P+1)*(P+2)
      CALL INVRT(CF,LL,MM,X,CC,NP,NORDER,DET)
C
C          CALC SOLN VECTOR
C
      DO 100 I=1,NORDER
      SUM=0.0
      DO 90 J=1,NORDER
      SUM=SUM+CF(I,J)*RHS(J)
   90 CONTINUE
      X(I)=SUM
  100 CONTINUE
C
C          MAP SOLN VECTOR
C
      N=P+1
      NN=N+1
      NBB=N*NN/2
      DO 200 II=1,N
      NN=NN-1
      I=II-1
      DO 150 JJ=1,NN
      J=JJ-1
      NC1=J*N+I+1-J*(J-1)/2
      NC2=NC1+NBB
      LOC1=LOK(P,I,J,27)
      LOC2=LOK(P,I,J,28)
      B(LOC1)=X(NC1)
      B(LOC2)=X(NC2)
      WRITE(6,900)P,I,J,X(NC1),P,I,J,X(NC2)
  150 CONTINUE
  200 CONTINUE
  900 FORMAT(1H0,5X, 2HA(,I2,1H,,I2,1H,,I2,2H)=,G16.5,
     1    5X, 2HB(,I2,1H,,I2,1H,,I2,2H)=,G16.5)
      RETURN
      END
$DATA
 GAMMA = 1.4,
 D = 0.05,
 EFLAG = 2.,
 PMAX = 5,
 RCURV = 0.25,
 $END
```

## APPENDIX F

### DISTRIBUTION LIST FOR FINAL REPORT

### CONTRACT NAS7-756

| Copies | Recipient | Designee |
|---|---|---|
| | NASA Headquarters<br>Washington, D.C. 20546 | |
| 1 | Contracting Officer | X |
| 1 | Patent Office | X |
| | NASA Lewis Research Center<br>21000 Brookpark Road<br>Cleveland, Ohio 44135 | |
| 1 | Office of Technical Information | X |
| 1 | Contracting Officer | X |
| | NASA Manned Spacecraft Center<br>Houston, Texas 77001 | |
| 1 | Office of Technical Information | X |
| 1 | Contracting Officer | X |
| 1 | Patent Office | X |
| | NASA Marshall Space Flight Center<br>Huntsville, Alabama 35812 | |
| 2 | Office of Technical Information, MS-IP | X |
| 1 | Purchasing Office, PR-EC | X |
| 1 | Patent Office, M-Pat | X |
| 1 | Technology Utilization Office, MS-T | X |
| | NASA Pasadena Office<br>4800 Oak Grove Drive<br>Pasadena, California 91103 | |
| 1 | Patents and Contracts Management | X |
| 3 | Contracting Officer, F. Abbott | X |
| | Jet Propulsion Laboratory<br>4800 Oak Grove Drive<br>Pasadena, California 91103 | |
| 20 | R. F. Cuffel | X |
| 3 | Chief, Liquid Experimental Engineering, RPL<br>Office of Advanced Research and Technology<br>NASA Headquarters<br>Washington, D.C. 20546 | X |
| 1 | Director, Technology Utilization Division<br>Office of Technology Utilization<br>NASA Headquarters<br>Washington, D.C. 20546 | X |

| Copies | Recipient | Designee |
|---|---|---|
| 20 | NASA Scientific and Technical Information Facility<br>P.O. Box 33<br>College Park, Maryland 20740 | X |
| 1 | Director, Launch Vehicles and Propulsion, SV<br>Office of Space Science and Applications<br>NASA Headquarters<br>Washington, D.C. 20546 | X |
| 1 | Director. Advanced Manned Missions, MT<br>Office of Manned Space Flight<br>NASA Headquarters<br>Washington, D.C. 20546 | X |
| 1 | Mission Analysis Division<br>NASA Ames Research Center<br>Moffett Field, California 24035 | X |

<u>NASA Field Centers</u>

| Copies | Recipient | Designee |
|---|---|---|
| 2 | Ames Research Center<br>Moffett Field, California 94035 | Hans M. Mark |
| 1 | Goddard Space Flight Center<br>Greenbelt, Maryland 20771 | Merland L. Moseson<br>Code 620 |
| 2 | Jet Propulsion Laboratory<br>California Institute of Technology<br>4800 Oak Grove Drive<br>Pasadena, California | Henry Burlage, Jr.<br>Propulsion Div. 38 |
| 2 | Langley Research Center<br>Langley Station<br>Hampton, Virginia 23365 | Ed Cortwright<br>Director |
| 2 | Lewis Research Center<br>21000 Brookpark Road<br>Cleveland, Ohio 44135 | Dr. Abe Silverstein<br>Director |
| 2 | Marshall Space Flight Center<br>Huntsville, Alabama 35812 | Hans G. Paul<br>Code R-P & VED |
| 2 | Manned Spacecraft Center<br>Houston, Texas 77001 | J. G. Thibodaux, Jr.<br>Chief, Prop. & Power Div. |
| 2 | John F. Kennedy Space Center, NASA<br>Cocoa Beach, Florida 32931 | Dr. Kurt H. Debus |

| Copies | Recipient | Designee |
|---|---|---|
| | **Government Installations** | |
| 1 | Aeronautical Systems Division<br>Air Force Systems Command<br>Wright-Patterson Air Force Base<br>Dayton, Ohio 45433 | D. L. Schmidt<br>Code ASRCNC-2 |
| 1 | Air Force Missile Test Center<br>Patrick Air Force Base, Florida | L. J. Ullian |
| 1 | Space and Missile Systems Organization<br>Air Force Unit Post Office<br>Los Angeles, California 90045 | Col. Clark<br>Technical Data Center |
| 1 | Arnold Engineering Development Center<br>Arnold Air Force Station<br>Tullahoma, Tennessee 37388 | Dr. H. K. Doetsch |
| 1 | Bureau of Naval Weapons<br>Department of the Navy<br>Washington, D.C. 20546 | J. Kay<br>RTMS-41 |
| 1 | Defense Documentation Center Headquarters<br>Cameron Station, Building 5<br>5010 Duke Street<br>Alexandria, Virginia 22314<br>Attn: TISIA | |
| 1 | Headquarters, U.S. Air Force<br>Washington, D.C. 20546 | Col. C. K. Stambaugh<br>AFRST |
| 1 | Picatinny Arsenal<br>Dover, New Jersey 07801 | I. Forsten, Chief<br>Liquid Propulsion Lab |
| 2 | Air Force Rocket Propulsion Laboratory<br>Research and Technology Division<br>Air Force Systems Command<br>Edwards, California 93523 | RPRPD/Mr. H. Main |
| 1. | U.S. Army Missile Command<br>Redstone Arsenal<br>Alabama 35809 | Mr. Walter Wharton |
| 1 | U.S. Naval Ordnance Test Station<br>China Lake<br>California 93557 | Code 4562<br>Chief, Missile Pro-<br>pulsion Division |
| 1 | Air Force Missile Development Center<br>Holloman Air Force Base<br>New Mexico 88330 | Maj. R. E. Bracken |

| Copies | Recipient | Designee |
|---|---|---|
| | Industry Contractors | |
| 1 | Aerojet-General Corporation<br>P.O. Box 1947<br>Technical Library, Building 2015, Department 2410<br>Sacramento, California 95809 | R. Stiff |
| 1 | Aerospace Corporation<br>2400 East El Segundo Boulevard<br>P.O. Box 95085<br>Los Angeles, California 90045 | John G. Wilder<br>MS-2293 |
| 1 | Atlantic Research Corporation<br>Edsall Road and Shirley Highway<br>Alexandria, Virginia 22314 | Dr. Ray Friedman |
| 1 | Bell Aerospace Company<br>P.O. Box 1<br>Buffalo, New York 14240 | W. M. Smith |
| 1 | Bellcom<br>955 L'Enfant Plaza, S.W.<br>Washington, D.C. | H. S. London |
| 1 | Boeing Company<br>P.O. Box 3707<br>Seattle, Washington 98124 | J. D. Alexander |
| 1 | Wright Aeronautical Division<br>Curtiss-Wright Corporation<br>Wood-Ridge, New Jersey 07075 | G. Kelley |
| 1 | Republic Aviation Corporation<br>Fairchild Hiller Corporation<br>Farmingdale, Long Island, New York | Library |
| 1 | General Dynamics, Convair Division<br>Library & Information Services (128-00)<br>P.O. Box 1128 | |
| 1 | Missile and Space Systems Center<br>General Electric Company<br>Valley Forge Space Technology Center<br>P.O. Box 8555<br>Philadelphia, Pennsylvania | F. Mezger<br>F. E. Schultz |
| 1 | Grumman Aircraft Engineering Corporation<br>Bethpage. Long Island<br>New York 11714 | Joseph Gavin |

| Copies | Recipient | Designee |
|---|---|---|
| 1 | Honeywell, Inc.<br>Aerospace Division<br>2600 Ridgway Road<br>Minneapolis, Minnesota | Mr. Gordon Harms |
| 1 | Hughes Aircraft Company<br>Aerospace Group<br>Centinela and Teale Streets<br>Culver City, California 90230 | E. H. Meier<br>V.P. and Div. Mgr.<br>Research & Development<br>Division |
| 1 | Walter Kidde and Company, Inc.<br>Aerospace Operations<br>567 Main Street<br>Belleville, New Jersey | R. J. Hanville<br>Dir. of Research Engr. |
| 1 | Lockheed Missiles and Space Company<br>Attn: Technical Information Center<br>P.O. Box 504<br>Sunnyvale, California 94088 | J. Guill |
| 1 | Lockheed Propulsion Company<br>P.O. Box 111<br>Redlands, California 92374 | H. L. Thackwell |
| 1 | The Marquardt Corporation<br>16555 Saticoy Street<br>Van Nuys, California 91409 | Howard McFarland |
| 1 | Denver Division<br>Martin Marietta Corporation<br>P.O. Box 179<br>Denver, Colorado 80201 | Dr. Morganthaler<br>A. J. Kullas |
| 1 | Astropower Laboratory<br>McDonnell-Douglas Aircraft Company<br>2121 Paularino<br>Newport Beach, California 92663 | Dr. George Moc<br>Director, Research |
| 1 | McDonnel-Douglas Aircraft Corporation<br>P.O. Box 516<br>Municipal Airport<br>St. Louis, Missouri 63166 | R. A. Herzmark |
| 1 | Missile and Space Systems Division<br>McDonnell-Douglas Aircraft Company<br>3000 Ocean Park Boulevard<br>Santa Monica, California 90406 | Mr. R. W. Hallet<br>Chief Engineer<br>Adv. Space Technology |
| 1 | Space and Information Systems Division<br>North American Rockwell<br>12214 Lakewood Boulevard<br>Downey, California 90241 | Library |

| Copies | Recipient | Designee |
|---|---|---|
| 1 | Rocketdyne (Library 586-306)<br>6633 Canoga Avenue<br>Canoga Park, California 91304 | Dr. R. J. Thompson<br>S. F. Iacobellis |
| 1 | Northrop Space Laboratories<br>3401 West Broadway<br>Hawthorne, California 90250 | Dr. William Howard |
| 1 | Rocket Research Corporation<br>520 South Portland Street<br>Seattle, Washington 98108 | Foy McCullough, Jr. |
| 1 | Sunstrand Aviation<br>2421 11th Street<br>Rockford, Illinois 61101 | R. W. Reynolds |
| 1 | TRW Systems Group<br>TRW Incorporated<br>One Space Park<br>Redondo Beach, California 90278 | G. W. Elverum |
| 1 | Thiokol Chemical Corporation<br>Huntsville Division<br>Huntsville, Alabama 35807 | John Goodloe |
| 1 | Research Laboratories<br>United Aircraft Corporation<br>400 Main Street<br>East Hartford, Connecticut 06108 | Erle Martin |
| 1 | Hamilton Standard Division<br>United Aircraft Corporation<br>Windsor Locks, Connecticut 06096 | Mr. R. Hatch |
| 1 | United Technology Center<br>587 Methilda Avenue<br>P.O. Box 358<br>Sunnyvale, California 94088 | Dr. David Altman |
| 1 | Florida Research and Development<br>Pratt and Whitney Aircraft<br>United Aircraft Corporation<br>P.O. Box 2691<br>West Palm Beach, Florida 33402 | R. J. Coar |
| 1 | Lockheed Missiles and Space Company<br>Huntsville Research and Engineering Center<br>4800 Bradford Drive<br>Huntsville, Alabama 35807 | Mr. R. Prozan |

| Copies | Recipient | Designee |
| --- | --- | --- |
| | CPIA | |
| 1 | Chemical Propulsion Information Agency<br>Applied Physics Laboratory<br>8621 Georgia Avenue<br>Silver Spring, Maryland 20910 | Tom Reedy |