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THE STABILITY OF MOTION OF SATELLITES WITH FLEXIBLE APPENDAGES

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ABSTRACT

The stability of motion of a satellite consisting of a main rigid body and three pairs of flexible booms, coinciding with the principal axes of the body in undeformed state, is under consideration. The problem formulation is a hybrid one, in the sense that some of the generalized coordinates depend on time alone and the other depend on spatial position and time. The problem is transformed into a discrete one by means of modal analysis. The motion stability is investigated by the Liapunov second method. A computer program has been written and the numerical results are displayed in the form of stability diagrams using the system properties as parameters.

CONTENTS

Introduction

The rotational motion of a torque-free rigid body is known to be stable if the rotation takes place about an axis corresponding to the maximum or minimum moment of inertia, but the motion is unstable if the rotation takes place about the axis of intermediate moment of inertia (see, for example, Reference 1, Section 6.7). If the body is not entirely rigid but possessing deformable parts, the rotational motion can be expected to exhibit different stability characteristics.

In one of the first attempts to treat rigorously distributed elastic members, the stability of motion of a spinning symmetric body which is part rigid and part elastic has been investigated by Meirovitch and Nelson (Reference 2). The mathematical formulation in Reference 2 consists of a set of ordinary differential equations for the rotational motion and another set of partial differential equations describing the elastic displacements. We shall refer to a system of both ordinary and partial differential equations as "hybrid." The hybrid system of Reference 2 has been reduced to a system consisting entirely of ordinary differential equations by means of modal analysis. The stability of the resulting discrete system has been investigated by an infinitesimal analysis and the effect of the flexible parts on the motion stability has been displayed in the form of diagrams relating various parameters of the system.

A general and rigorous method for the stability analysis of systems containing distributed elastic parts has been developed by Meirovitch (Reference 3). The method represents an extension of the Liapunov second method and works directly with the hybrid system of differential equations (in the sense defined above). As an application, the case of gravity-gradient stabilization of a satellite with flexible appendages is solved. The method has been further extended to hybrid systems possessing ignorable coordinates (Reference 4). The general theory is applied to the stability analysis of a spinning satellite resembling that of Reference 2.

The problem under investigation is related to that of Reference 4. However, whereas the mathematical model used as an illustration in Reference 4 consists of a main rigid body with a pair of booms aligned with the spin axis, the model considered here consists of a main rigid body and three pairs of booms, as shown in Figure 2. It turns out that the elastic deformations are not independent of one another, so that it is not possible to work directly with the hybrid system of equations. The formulation is reduced to a set of ordinary differential equations by modal analysis and the stability of such a set can be investigated by the Liapunov second method. Due to its generality, the problem formulation of Reference 4 is equally applicable here. The present investigation departs from that of Reference 4 in the stability analysis.

This report contains the formulation of the problem, as well as numerical results obtained by means of a computer program designed to perform the stability analysis. The program has been used to investigate the effect of changes in the parameters of the system on its stability.

General Problem Formulation

Let us consider a body of total mass m moving relative to an inertial space XYZ, as shown in Figure 1. The entire body or parts of the body are capable of small elastic deformations from a reference equilibrium position coinciding with the undeformed state of the body. Next we define two sets of body axes, the set xyz with the origin at point $\underset{\sim}{0}$ and coinciding with the principal axes of the body in the undeformed state, and the set $\xi\eta\zeta$ which is parallel to xyz but has the origin at the center of mass c of the deformed body. We note that $\xi\eta\zeta$ is not a principal set of axes. The set xyz serves as a suitable reference frame for measuring elastic deformations whereas the set $\xi\eta\zeta$ is more convenient for expressing the overall motion. The position of a typical point in the undeformed body relative to axes xyz is denoted by the vector* $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

* Vector quantities are denoted by wavy lines under the symbols.

and the elastic displacement of an element of mass dm, originally coincident with that point, by the vector $\underset{\sim}{u} = u(x,y,z,t)\underset{\sim}{i} + v(x,y,z,t)\underset{\sim}{j} + w(x,y,z,t)\underset{\sim}{k}$, where $\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}$ are unit vectors along axes x,y,z (or axes ξ,η,ζ), respectively. The radius vector from point $\underline{0}$ to c is given by $\underset{\sim}{r}_c = \frac{1}{m}\int_m (\underset{\sim}{r}+\underset{\sim}{u})\,dm = \frac{1}{m}\int_m \underset{\sim}{u}\,dm$, where we note that $\int_m \underset{\sim}{r}\,dm$ is zero by virtue of the fact that $\underline{0}$ is the center of mass of the undeformed body. All integrations involved in this report are carried over the domain occupied by the body in undeformed state, which is designated as the reference state.

From Figure 1 we conclude that the position of the mass element dm relative to the inertial space is $\underset{\sim}{R}_d = \underset{\sim}{R}_c + \underset{\sim}{r} + \underset{\sim}{u}_c$, where $\underset{\sim}{u}_c = \underset{\sim}{u} - \underset{\sim}{r}_c = u_c\underset{\sim}{i} + v_c\underset{\sim}{j} + w_c\underset{\sim}{k}$ represents the displacement vector measured with respect to axes $\xi\eta\zeta$ and $\underset{\sim}{R}_c$ is the position of the origin of these axes relative to the inertial space. Assuming that axes xyz, hence also axes $\xi\eta\zeta$, rotate with angular velocity $\underset{\sim}{\omega} = \omega_\xi\underset{\sim}{i} + \omega_\eta\underset{\sim}{j} + \omega_\zeta\underset{\sim}{k}$ relative to the inertial space, and denoting by $\dot{\underset{\sim}{u}}'_c = \dot{u}_c\underset{\sim}{i} + \dot{v}_c\underset{\sim}{j} + \dot{w}_c\underset{\sim}{k}$ the velocity of dm relative to $\xi\eta\zeta$ due to the elastic effect, it is shown in Reference 3 that the kinetic energy has the expression

$$T = \frac{1}{2}\int_m \dot{\underset{\sim}{R}}_d \cdot \dot{\underset{\sim}{R}}_d\,dm = \frac{1}{2}m\,\dot{\underset{\sim}{R}}_c \cdot \dot{\underset{\sim}{R}}_c + \frac{1}{2}\,\underset{\sim}{\omega}\cdot\underset{\sim}{J}_d\cdot\underset{\sim}{\omega} + (\underset{\sim}{\omega}\times\int_m(\underset{\sim}{r} + \underset{\sim}{u}_c))\cdot\dot{\underset{\sim}{u}}'_c\,dm + \frac{1}{2}\int_m \dot{\underset{\sim}{u}}'_c\cdot\dot{\underset{\sim}{u}}'_c\,dm \tag{1}$$

where $\underset{\sim}{J}_d$ is the inertia dyadic of the deformed body about axes $\xi\eta\zeta$. The elements of the dyadic are

$$\begin{aligned}
J_{\xi\xi} &= \int_m \left[(y+v_c)^2+(z+w_c)^2\right]dm\,, & J_{\xi\eta} &= J_{\eta\xi} = \int_m (x+u_c)(y+v_c)\,dm \\
J_{\eta\eta} &= \int_m \left[(x+u_c)^2+(z+w_c)^2\right]dm\,, & J_{\xi\zeta} &= J_{\zeta\xi} = \int_m (x+u_c)(z+w_c)\,dm \\
J_{\zeta\zeta} &= \int_m \left[(x+u_c)^2+(y+v_c)^2\right]dm\,, & J_{\eta\zeta} &= J_{\zeta\eta} = \int_m (y+v_c)(z+w_c)\,dm
\end{aligned} \tag{2}$$

The kinetic energy can be written conveniently in terms of matrix notation. If $\{\dot{R}_c\}$ is the column matrix corresponding to $\dot{\underset{\sim}{R}}_c$, $\{\omega\}$ the column matrix corresponding to $\underset{\sim}{\omega}$, and $[J]$ the symmetric matrix, whose elements are the elements of the dyadic $\underset{\sim}{J}_d$, then Eq. (1) can be rewritten in the form

$$T = \frac{1}{2}m\{\dot{R}_c\}^T\{\dot{R}_c\} + \frac{1}{2}\{\omega\}^T[J]\{\omega\} + \{K\}^T\{\omega\} + \frac{1}{2}\int_m (\dot{u}_c^2 + \dot{v}_c^2 + \dot{w}_c^2)\,dm \tag{3}$$

where $\{K\}$ is the column matrix with the elements

$$K_\xi = \int_m \left[(y+v_c)\dot{w}_c - (z+w_c)\dot{v}_c\right] dm$$

$$K_\eta = \int_m \left[(z+w_c)\dot{u}_c - (x+u_c)\dot{w}_c\right] dm \tag{4}$$

$$K_\zeta = \int_m \left[(x+u_c)\dot{v}_c - (y+v_c)\dot{u}_c\right] dm$$

The angular velocity components $\omega_\xi, \omega_\eta, \omega_\zeta$ do not represent time rates of change of certain angles but nonintegrable combinations of time derivatives of angular displacements. They are sometimes referred to as time derivatives of quasi-coordinates. Denoting by θ_i and $\dot{\theta}_i$ (i=1,2,3) the true angular displacements and their time rates of change, the angular velocity vector can be written in the matrix form $\{\omega\} = [\theta]\{\dot{\theta}\}$, where $\{\dot{\theta}\}$ is the column matrix with elements $\dot{\theta}_i$ (i=1,2,3) and $[\theta]$ is a 3x3 matrix, whose elements depend on the order of the three rotations θ_i used to produce the orientation of axes $\xi\eta\zeta$ relative to an inertial space. In view of this, the kinetic energy can be written in terms of true angular velocities as follows

$$T = \frac{1}{2}m\{\dot{R}_c\}^T\{\dot{R}_c\} + \frac{1}{2}\{\dot{\theta}\}^T[I]\{\dot{\theta}\} + \{L\}^T\{\dot{\theta}\} + \frac{1}{2}\int_m (\dot{u}_c^2 + \dot{v}_c^2 + \dot{w}_c^2)\,dm \tag{5}$$

in which the notation

$$[I] = [\theta]^T[J][\theta] \quad , \quad \{L\} = [\theta]^T\{K\} \tag{6}$$

has been adopted.

The potential energy arises primarily from two sources, namely gravity and body elasticity. The gravitational potential energy is assumed to be very small compared with the kinetic energy, or the elastic potential energy, and will be ignored. The elastic potential energy, denoted by V_{EL} and referred to at times as strain energy, depends on the nature of the elastic members and is in general a function of the partial derivatives of the elastic displacements u,v,w with respect to the spatial variables x,y,z. Since u_c, v_c, w_c differ from u,v,w by x_c, y_c, z_c, respectively, where the latter are independent of the spatial variables, V_{EL} can be regarded as depending on the partial derivatives of u_c, v_c, w_c with respect to x,y,z. We assume that V_{EL} is a function of $\partial^2 u_c/\partial x^2$, $\partial^2 u_c/\partial x \partial y$, ---, $\partial^2 w_c/\partial z^2$ but this assumption in no way affects the generality of the formulation. This particular functional dependence of V_{EL} should be regarded as mere scaffolding used in the construction of a general theory, as the final formulation is expressed in a form which involves the partial derivatives only implicitly.

The system differential equations can be obtained by means of Hamilton's principle. To this end, a brief discussion of the generalized coordinates is in order. The motion of the mass center c is generally assumed not to be affected by the motion relative to c, so that it is possible to solve for the motion of c independently of the motion relative to c. As a result, the motion of c, referred to as orbital motion, can be regarded as known. We shall confine ourselves to the case in which the first term on the right side of Eq. (5) reduces to a known constant, so that the term can be ignored. This is clearly the case when the orbit is circular, or the motion of c is uniform or zero. It follows that the system generalized coordinates are the three rotations $\theta_i(t)$ and the three elastic displacements $u_c(x,y,z,t)$, $v_c(x,y,z,t)$, $w_c(x,y,z,t)$. The elastic displacements are defined only throughout the domain D_e, namely the subdomain of D corresponding to the elastic continuum, where D is a three-dimensional domain corresponding to the entire body. The domain D_e is bounded by the surface S.

For the holonomic system at hand, Hamilton's principle has the form

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \tag{7}$$

where the motion must be such that the end conditions

$$\delta\theta_1 = \delta\theta_2 = \delta\theta_3 = \delta u_c = \delta v_c = \delta w_c = 0 \quad \text{at } t = t_1, t_2 \tag{8}$$

are satisfied. The integrand L in (7) is the Lagrangian which has the general functional form

$$L = T - V_{EL} = \int_D \hat{L}\left(\theta_i, \dot{\theta}_i, u_c, v_c, ---, \dot{w}_c, \frac{\partial^2 u_c}{\partial x^2}, \frac{\partial^2 u_c}{\partial x \partial y}, ---, \frac{\partial^2 w_c}{\partial z^2}\right) dD \tag{9}$$

in which $\hat{L}$ is the Lagrangian density.

An application of Hamilton's principle leads to the system Lagrangian equations of motion. Details of the derivation are given in Reference 3 and will not be repeated here. Instead we quote directly from Reference 3 the ordinary differential equations for the angular displacements.

$$\frac{\partial L}{\partial \theta_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) = 0 \quad , \quad i = 1,2,3 \tag{10}$$

and the partial differential equations for the elastic displacements

$$\begin{aligned}
\frac{\partial \hat{L}}{\partial u_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{u}_c}\right) + \mathcal{L}_{u_c}\left[u_c, v_c, w_c\right] + \hat{Q}_{u_c} &= 0 \\
\frac{\partial \hat{L}}{\partial v_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{v}_c}\right) + \mathcal{L}_{v_c}\left[u_c, v_c, w_c\right] + \hat{Q}_{v_c} &= 0 \\
\frac{\partial \hat{L}}{\partial w_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{w}_c}\right) + \mathcal{L}_{w_c}\left[u_c, v_c, w_c\right] + \hat{Q}_{w_c} &= 0
\end{aligned} \tag{11}$$

where Eqs. (11) must be satisfied at every point of the domain D_e. Moreover, Eqs. (11) are subject to the boundary conditions

$$\underset{\sim}{B}_j\left[u_c, v_c, w_c\right] \cdot \underset{\sim}{B}_k\left[u_c, v_c, w_c\right] = 0 \quad \text{on } S \; , \; j = 1,2; \; k = 3,4 \tag{12}$$

The differential operator vectors $\underset{\sim}{\mathcal{L}}(\mathcal{L}_{u_c}, \mathcal{L}_{v_c}, \mathcal{L}_{w_c})$, $\underset{\sim}{B}_j(B_{ju_c}, B_{jv_c}, B_{jw_c})$, and $\underset{\sim}{B}_k(B_{ku_c}, B_{kv_c}, B_{kw_c})$ are defined by the following integration by parts

$$\int_D \left[\frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x^2)} \delta\left(\frac{\partial^2 u_c}{\partial x^2}\right) + \frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x \partial y)} \left(\frac{\partial^2 u_c}{\partial x \partial y}\right) + --- + \frac{\partial \hat{L}}{\partial(\partial^2 w_c/\partial z^2)} \delta\left(\frac{\partial^2 w_c}{\partial z^2}\right) \right] dD = \int_{D_e} \underset{\sim}{\mathcal{L}}[u_c, v_c, w_c] \cdot \delta \underset{\sim}{u}\, dD_e + \underset{\sim}{B}_j[u_c, v_c, w_c] \cdot \underset{\sim}{B}_k[u_c, v_c, w_c] \Big|_S \quad , \quad j = 1,2;\ k = 3,4 \tag{13}$$

We note that the partial derivatives $\partial^2 u_c/\partial x^2$, $\partial^2 u_c/\partial x \partial y, ---, \partial^2 w_c/\partial z^2$ enter into Eqs. (11) and (12) only implicitly through the differential operator vectors $\underset{\sim}{\mathcal{L}}$, $\underset{\sim}{B}_j$, and $\underset{\sim}{B}_k$, thus lending substance to a statement made earlier regarding the generality of the formulation. The quantities $\hat{Q}_{u_c}$, $\hat{Q}_{v_c}$, $\hat{Q}_{w_c}$ represent distributed internal damping forces which depend on the elastic motion alone and not on the rotational motion. It should be pointed out that the damping forces were added afterward, as such forces cannot be treated by means of Hamilton's principle.

Introducing the generalized momenta

$$p_{\theta_i} = \frac{\partial L}{\partial \dot{\theta}_i} \quad , \quad i = 1,2,3$$

$$\hat{p}_{u_c} = \frac{\partial \hat{L}}{\partial \dot{u}_c} \quad , \quad \hat{p}_{v_c} = \frac{\partial \hat{L}}{\partial \dot{v}_c} \quad , \quad \hat{p}_{w_c} = \frac{\partial \hat{L}}{\partial \dot{w}_c} \tag{14}$$

where the latter three are momentum densities, it is shown in Reference 3 that the second-order Lagrangian equations, Eqs. (10) and (11), can be converted into twice the number of first-order Hamiltonian equations having the form

$$\dot{\theta}_i = \frac{\partial H}{\partial p_{\theta_i}} \quad , \quad \dot{p}_{\theta_i} = -\frac{\partial H}{\partial \theta_i} \quad , \quad i = 1,2,3$$

$$\left.\begin{aligned}
\dot{u}_c &= \frac{\partial \hat{H}}{\partial \hat{p}_{u_c}} \quad , \quad \dot{v}_c = \frac{\partial \hat{H}}{\partial \hat{p}_{v_c}} \quad , \quad \dot{w}_c = \frac{\partial \hat{H}}{\partial \hat{p}_{w_c}} \\
\dot{\hat{p}}_{u_c} &= -\frac{\partial \hat{H}}{\partial u_c} + \mathcal{L}_{u_c}\left[u_c, v_c, w_c\right] + \hat{Q}_{u_c} \\
\dot{\hat{p}}_{v_c} &= -\frac{\partial \hat{H}}{\partial v_c} + \mathcal{L}_{v_c}\left[u_c, v_c, w_c\right] + \hat{Q}_{v_c} \\
\dot{\hat{p}}_{w_c} &= -\frac{\partial \hat{H}}{\partial w_c} + \mathcal{L}_{w_c}\left[u_c, v_c, w_c\right] + \hat{Q}_{w_c}
\end{aligned}\right\} \text{at every point of } D_e \tag{15}$$

in which H is the Hamiltonian defined by

$$H = \sum_{i=1}^{3} p_{\theta_i} \dot{\theta}_i + \int_{D_e} (\hat{p}_{u_c} \dot{u}_c + \hat{p}_{v_c} \dot{v}_c + \hat{p}_{w_c} \dot{w}_c) \, dD_e - L \tag{16}$$

and $\hat{H}$ is the corresponding Hamiltonian density. It should be noticed here that the Hamiltonian has a hybrid form as it is a function and a functional at the same time. The equations for the elastic motion are subject to the same boundary conditions, Eqs. (12). When the kinetic energy is quadratic in the generalized velocities, the Hamiltonian reduces to the form

$$H = T + V_{EL} \tag{17}$$

which is recognized as the system total energy.

Systems with Ignorable Coordinates

In the case of a system free of external torques, such as the case under consideration, one of the angular coordinates θ_i (i=1,2,3) is absent from the Lagrangian. Then from Eqs. (10) and the first half of Eqs. (14) it follows that the system possesses a first integral of the motion in the form of the conjugate momentum. The expression of the conserved momentum may be used to eliminate from the Lagrangian the angular velocity associated with the absent angular coordinate, thus reducing the number of degrees of freedom by one. The procedure for accomplishing this is referred to as Routh's method for the ignoration of coordinates (see Reference 1, Section 2.11).

Let us assume that θ_3 is absent from the Lagrangian, so that the conjugate momentum is conserved, $p_{\theta_3} = \partial L/\partial\dot{\theta}_3 = \beta_3$ = constant. Since the potential energy does not depend on velocities, from Eq. (5) the momentum integral can be written as

$$p_{\theta_3} = \frac{\partial L}{\partial\dot{\theta}_3} = \frac{\partial T}{\partial\dot{\theta}_3} = I_{13}\dot{\theta}_1 + I_{23}\dot{\theta}_2 + I_{33}\dot{\theta}_3 + L_3 = \beta_3 \tag{18}$$

Equation (18) plays the role of a constraint equation, which can be solved for $\dot{\theta}_3$ in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$. Since the elements of the angular velocity matrix $\{\dot{\theta}\}$ in Eq. (5) can no longer be considered independent but related by (18), we can define the linear transformation

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \frac{1}{I_{33}}\begin{bmatrix} I_{33} & 0 \\ 0 & I_{33} \\ -I_{13} & -I_{23} \end{bmatrix}\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} + \frac{1}{I_{33}}\begin{Bmatrix} 0 \\ 0 \\ \beta_3 - L_3 \end{Bmatrix} = [C]\{\dot{\theta}^*\} + \{B\} \tag{19}$$

which takes Eq. (18) into account automatically. By contrast with $\{\dot{\theta}\}$, the column matrix $\{\dot{\theta}^*\}$ contains only two elements, which must be regarded as independent. Introducing Eq. (19) into (5), and disregarding the first term (assumed to be constant), we obtain

$$T = \frac{1}{2}\{\dot{\theta}^*\}^T[I^*]\{\dot{\theta}^*\} + \{L^*\}^T\{\dot{\theta}^*\} + \frac{1}{2}\frac{\beta_3^2 - L_3^2}{I_{33}} + \frac{1}{2}\int_m (\dot{u}_c^2 + \dot{v}_c^2 + \dot{w}_c^2)\,dm \tag{20}$$

where

$$[I^*] = [C]^T[I][C] = \frac{1}{I_{33}}\begin{bmatrix} I_{11}I_{33} - I_{13}^2 & I_{12}I_{33} - I_{13}I_{23} \\ I_{12}I_{33} - I_{13}I_{23} & I_{22}I_{33} - I_{23}^2 \end{bmatrix} \tag{21}$$

$$\{L^*\} = [C]^T\{L\} = \frac{1}{I_{33}}\begin{Bmatrix} I_{33}L_1 - I_{13}L_3 \\ I_{33}L_2 - I_{23}L_3 \end{Bmatrix}$$

We notice that the kinetic energy, Eq. (20), is entirely free of θ_3 and $\dot{\theta}_3$.

The elastic potential energy V_{EL} is assumed to depend only on the elastic displacements u_c, v_c, w_c and its general form will be introduced later.

Stability of Motion of a Dynamical System

Let us consider the dynamical system

$$\dot{\underset{\sim}{x}} = \underset{\sim}{X}(\underset{\sim}{x}) \qquad (22)$$

For a discrete system $\underset{\sim}{x} = \underset{\sim}{x}(t)$ represents a vector in a finite dimensional vector space S. The motion of the system can be represented as a path in that space. If Eq. (22) represents a set of canonical equations, then the motion of the dynamical system can be regarded as a succession of infinitesimal contact transformations possessing the group-property. The properties characterizing the group are as follows: 1) the identity transformation belongs to this class, 2) two successive transformations are commutative and the result is also a contact transformation, 3) two contact transformations satisfy the associative law, and 4) the inverse of a contact transformation is also a contact transformation. Hence, the motion of the system may be interpreted as a continuous mapping of the space S onto itself. For canonical systems of equations half of the elements of x represent generalized coordinates and the remaining half represent the conjugate momenta. Moreover, the space S is simply the phase space.

A solution of Eq. (22) satisfying

$$\underset{\sim}{X}(\underset{\sim}{x}) = \underset{\sim}{0} \qquad (23)$$

represents a singular point or an equilibrium position. We shall be interested in the stability of the solutions in the neighborhood of equilibrium positions. Without loss of generality, we can assume that the equilibrium point coincides with the origin so that we shall be concerned with the equilibrium of the trivial solution. Denoting the integral curve at a given time $t_0 > 0$ by $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$, and assuming that the origin is an isolated singularity, we can introduce the following definitions due to Liapunov:

a. The null solution is stable in the sense of Liapunov if any arbitrary positive ε and time t_0 there exists a $\delta(\varepsilon, t_0) > 0$ such that if the inequality

$$\|\underset{\sim}{x}_0\| < \delta \tag{24}$$

is satisfield, then the inequality

$$\|\underset{\sim}{x}(t)\| < \varepsilon \quad , \quad t_0 \leq t < \infty \tag{25}$$

is implied. If δ is independent of t_0 the stability is said to be uniform.

b. The null solution is asymptotically stable if it is Liapunov stable and in addition

$$\lim_{t \to \infty} \|\underset{\sim}{x}(t)\| = 0 \tag{26}$$

Similarly, if Eq. (26) holds, then a uniformly stable solution is said to be uniformly asymptotically stable. For autonomous systems stability is always uniform.

c. The null solution is said to be unstable if for any arbitrarily small δ and any time t_0 such that

$$\|\underset{\sim}{x}_0\| < \delta \tag{27}$$

we have at some other finite time t_1 the situation

$$\|\underset{\sim}{x}(t_1)\| = \varepsilon \quad , \quad t_1 > t_0 \tag{28}$$

To test the stability of the trivial solution, we shall use Liapunov's direct method which is based on the differential equation (22) but does not require the solution of this equation. To introduce the concepts, we confine ourselves to autonomous systems and consider a scalar function $U(\underset{\sim}{x})$ such that $U(\underset{\sim}{0}) = 0$. The total time derivative of U along a trajectory of system (22) is defined by

$$\dot{U} = \frac{dU}{dt} = \underset{\sim}{\nabla}U \cdot \dot{\underset{\sim}{x}} = \underset{\sim}{\nabla}U \cdot \underset{\sim}{X} \tag{29}$$

where $\underset{\sim}{\nabla}U$ is the gradient of the scalar function U. In the case of a hybrid system U is both a function and a functional at the same time, as the dependent variables corresponding to the distributed portion of the system appear in U in integrated form.

Next we consider the following theorems:

Theorem I - If there exists for the system (22) a positive (negative) definite function $U(\underset{\sim}{x})$ whose total time derivative $\dot{U}(\underset{\sim}{x})$ is negative (positive) semidefinite along every trajectory of (21), then the trivial solution $\underset{\sim}{x} = \underset{\sim}{0}$ is stable.

Theorem II - If the conditions of Theorem I are satisfied and if in addition the set of points at which $\dot{U}(\underset{\sim}{x})$ is zero contains no nontrivial positive half-trajectory $\underset{\sim}{x}(t)$, $t \geq t_0$, then the trivial solution is asymptotically stable.

Theorem III - If there exists for the system (22) a function $U(\underset{\sim}{x})$ whose total time derivative $\dot{U}(\underset{\sim}{x})$ is positive (negative) definite along every trajectory of (21) and the function itself can assume positive (negative) values in the neighborhood of the origin, then the trivial solution is unstable.

Theorem IV - Suppose that a function $U(\underset{\sim}{x})$ such as in Theorem III exists but for which $\dot{U}(\underset{\sim}{x})$ is only positive (negative) semidefinite and, in addition, the set of points at which $\dot{U}(\underset{\sim}{x})$ is zero contains no nontrivial positive half-trajectory $\underset{\sim}{x}(t)$, $t \geq t_0$. Suppose further that in every neighborhood of the origin there is a point $\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0$ such that for arbitrary $t_0 \geq 0$ we have $U(\underset{\sim}{x}_0) > 0 (<0)$. Then the trivial solution is unstable and the trajectories $\underset{\sim}{x}(\underset{\sim}{x}_0, t_0, t)$ for which $U(\underset{\sim}{x}_0) > 0 (<0)$ must leave the open domain $\| \underset{\sim}{x} \| < \varepsilon$ as the time t increases.

A function U satisfying any of the preceding theorems is referred to as a Liapunov function. Theorems I and III are due to Liapunov, whereas, Theorems II and IV are due to Krasovskii. A more detailed discussion of the theorems can be found in the text by L. Meirovitch (see Reference 1, Section 6.7).

The Hamiltonian as a Liapunov Function

We shall show next that under certain circumstances the Hamiltonian can be used as a Liapunov function. Taking the total time derivative of H from Eq. (16) and using Eqs. (10) and (11), as well as boundary conditions (12) and definitions (14), we obtain

$$\dot{H} = \int_{D_e} (\hat{Q}_{u_c} \dot{u}_c + \hat{Q}_{v_c} \dot{v}_c + \hat{Q}_{w_c} \dot{w}_c) dD_e \tag{30}$$

Next we assume that the damping forces are such that $\dot{H}$ is negative semidefinite

$$\dot{H} \leq 0 \tag{31}$$

Moreover, due to coupling, the forces $\hat{Q}_{u_c}$, $\hat{Q}_{v_c}$, $\hat{Q}_{w_c}$ are never identically zero at every point of the phase space but they reduce to zero at an equilibrium point. Hence, if the Hamiltonian H is positive definite at an equilibrium point, then by Theorem II, H can be regarded as a Liapunov function and the equilibrium point under consideration as asymptotically stable. On the other hand, if H is not positive definite and there are points for which it is negative, then by Theorem IV the equilibrium point is unstable.

In view of the preceding discussion, we shall consider the Hamiltonian as a Liapunov function. As indicated by Eq. (23), the equilibrium positions are those rendering the right sides of Eqs. (15) equal to zero. Hence, the equilibrium positions are the solutions of the equations

$$\frac{\partial H}{\partial p_{\theta_i}} = 0 \quad , \quad -\frac{\partial H}{\partial \theta_i} = 0 \quad , \quad i = 1,2,3$$

$$\left.\begin{aligned} &\frac{\partial \hat{H}}{\partial \hat{p}_{u_c}} = \frac{\partial \hat{H}}{\partial \hat{p}_{v_c}} = \frac{\partial \hat{H}}{\partial \hat{p}_{w_c}} = 0 \\ &-\frac{\partial \hat{H}}{\partial u_c} + \mathcal{L}_{u_c}[u_c, v_c, w_c] = 0 \\ &-\frac{\partial \hat{H}}{\partial v_c} + \mathcal{L}_{v_c}[u_c, v_c, w_c] = 0 \\ &-\frac{\partial \hat{H}}{\partial w_c} + \mathcal{L}_{w_c}[u_c, v_c, w_c] = 0 \end{aligned}\right\} \text{at every point of } D_e \tag{32}$$

To test the positive definiteness of the Hamiltonian, we use Sylvester's criterion (see Reference 1, Sec. 6.7). To this end, we represent the elastic motion by appropriate modes of vibration, derive the quadratic form associated with the Hamiltonian in the neighborhood of the equilibrium and investigate the sign properties of the Hessian matrix, namely, the matrix of the coefficients of the quadratic form.

The Stability of High-Spin Motion of a Satellite with Flexible Appendages.

The general theory developed in the preceding sections will now be used to investigate the stability of a satellite simulated by a main rigid body and six flexible thin rods, as shown in Figure 2a. In the undeformed state the body possesses principal moments of inertia A,B,C about axes x,y,z, respectively, and the rods are aligned with these axes. The body is initially spinning undeformed about axis z with angular velocity Ω_s. The domain of the elastic continuum D_e consist of three subdomains:

$$D_x : -(h_x+\ell_x) < x < -h_x , h_x < x < (h_x+\ell_x) , \quad S_x = \pm h_x, \pm (h_x+\ell_x)$$

$$D_y : -(h_y+\ell_y) < y < -h_y , h_y < y < (h_y+\ell_y) , \quad S_y = \pm h_y, \pm (h_y+\ell_y)$$

$$D_z : -(h_z+\ell_z) < z < -h_z , h_z < z < (h_z+\ell_z) , \quad S_z = \pm h_z, \pm (h_z+\ell_z)$$

Hence $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ over $D-D_e$, $\underset{\sim}{r} = x\underset{\sim}{i}$ over D_x, $\underset{\sim}{r} = y\underset{\sim}{j}$ over D_y, and $\underset{\sim}{r} = z\underset{\sim}{k}$ over D_z. Assuming only flexural transverse vibrations, it follows that

$$\underset{\sim}{u} = \underset{\sim}{u}_x = v_x\underset{\sim}{j} + w_x\underset{\sim}{k} , \quad \underset{\sim}{u}_c = \underset{\sim}{u}_{cx} = v_{cx}\underset{\sim}{j} + w_{cx}\underset{\sim}{k} , \quad \underset{\sim}{r}_c = y_c\underset{\sim}{j} + z_c\underset{\sim}{k} \text{ over } D_x$$

$$\underset{\sim}{u} = \underset{\sim}{u}_y = u_y\underset{\sim}{i} + w_y\underset{\sim}{k} , \quad \underset{\sim}{u}_c = \underset{\sim}{u}_{cy} = u_{cy}\underset{\sim}{i} + w_{cy}\underset{\sim}{k} , \quad \underset{\sim}{r}_c = x_c\underset{\sim}{i} + z_c\underset{\sim}{k} \text{ over } D_y$$

$$\underset{\sim}{u} = \underset{\sim}{u}_z = u_z\underset{\sim}{i} + v_z\underset{\sim}{j} , \quad \underset{\sim}{u}_c = \underset{\sim}{u}_{cz} = u_{cz}\underset{\sim}{i} + v_{cz}\underset{\sim}{j} , \quad \underset{\sim}{r}_c = x_c\underset{\sim}{i} + y_c\underset{\sim}{j} \text{ over } D_z$$

From Eqs. (2) we conclude that the moments and products of inertia of the deformed body have the values

$$J_{\xi\xi} = A + \int_{D_x} \rho_x (v_{cx}^2+w_{cx}^2)\, dx + \int_{D_y} \rho_y w_{cy}^2 \, dy + \int_{D_z} \rho_z v_{cz}^2 \, dz$$

$$J_{\eta\eta} = B + \int_{D_x} \rho_x w_{cx}^2 \, dx + \int_{D_y} \rho_y (u_{cy}^2 + w_{cy}^2)\, dy + \int_{D_z} \rho_z u_{cz}^2 \, dz$$

$$J_{\zeta\zeta} = C + \int_{D_x} \rho_x v_{cx}^2 \, dx + \int_{D_y} \rho_y u_{cy}^2 \, dy + \int_{D_z} \rho_z (u_{cz}^2+v_{cz}^2)\, dz \qquad (33)$$

$$J_{\xi\eta} = J_{\eta\xi} = \int_{D_x} \rho_x x v_{cx} \, dx + \int_{D_y} \rho_y y u_{cy} \, dy + \int_{D_z} \rho_z u_{cz} v_{cz} \, dz$$

$$J_{\xi\zeta} = J_{\zeta\xi} = \int_{D_x} \rho_x x w_{cx} \, dx + \int_{D_y} \rho_y u_{cy} w_{cy} \, dy + \int_{D_z} \rho_z z u_{cz} \, dz$$

$$J_{\eta\zeta} = J_{\zeta\eta} = \int_{D_x} \rho_x v_{cx} w_{cx} \, dx + \int_{D_y} \rho_y y w_{cy} \, dy + \int_{D_z} \rho_z z v_{cz} \, dz$$

where ρ_x, ρ_y, ρ_z represent mass per unit length associated with the respective rods. Moreover, the elements of the matrix $\{K\}$ in Eq. (3) have the form

$$K_\xi = \int_{D_x} \rho_x (v_{cx} \dot{w}_{cx} - \dot{v}_{cx} w_{cx}) dx + \int_{D_y} \rho_y y \dot{w}_{cy} \, dy - \int_{D_z} \rho_z z \dot{v}_{cx} \, dz$$

$$K_\eta = - \int_{D_x} \rho_x x \dot{w}_{cx} \, dx + \int_{D_y} \rho_y (w_{cy} \dot{u}_{cy} - \dot{w}_{cy} u_{cy}) dy + \int_{D_z} \rho_z z \dot{u}_{cz} \, dz \tag{34}$$

$$K_\zeta = \int_{D_x} \rho_x x \dot{v}_{cx} \, dx - \int_{D_y} \rho_y y \dot{u}_{cy} \, dy + \int_{D_z} \rho_z (u_{cz} \dot{v}_{cz} - \dot{u}_{cz} v_{cz}) dz$$

whereas the last term in Eq. (3) becomes

$$\frac{1}{2} \int_m (\dot{u}_c^2 + \dot{v}_c^2 + \dot{w}_c^2) dm = \frac{1}{2} \int_{D_x} \rho_x (\dot{v}_{cx}^2 + \dot{w}_{cx}^2) dx + \int_{D_y} \rho_y (\dot{u}_{cy}^2 + \dot{w}_{cy}^2) dy$$

$$+ \int_{D_z} \rho_z (\dot{u}_{cz}^2 + \dot{v}_{cz}^2) dz \tag{35}$$

We shall assume that the mass of the rods is symmetrically distributed, namely that $\rho(-x) = \rho(x)$, $\rho(-y) = \rho(y)$, and $\rho(-z) = \rho(z)$.

If the rotations are as shown in Figure 2b, it is not difficult to show that

$$[\theta] = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \cos\theta_1 \\ 0 & 1 & \sin\theta_1 \\ \sin\theta_2 & 0 & \cos\theta_1 \cos\theta_2 \end{bmatrix} \tag{36}$$

from which it can be concluded that θ_3 is ignorable, and the kinetic energy has the form (20). To write the kinetic energy explicitly, we need the matrices $[I]$ and $\{L\}$, which, according to Eqs. (6), have the elements

$$
\begin{aligned}
I_{11} &= c^2\theta_2\, J_{\xi\xi} - 2\, s\theta_2 c\theta_2\, J_{\xi\zeta} + s^2\theta_2\, J_{\zeta\zeta} \\
I_{22} &= J_{\eta\eta} \\
I_{33} &= c^2\theta_1 s^2\theta_2 J_{\xi\xi} + s^2\theta_1 J_{\eta\eta} + c^2\theta_1 c^2\theta_2 J_{\zeta\zeta} + 2s\theta_1 c\theta_1 s\theta_2 J_{\xi\eta} \\
&\quad + 2c^2\theta_1 s\theta_2 c\theta_2 J_{\xi\zeta} - 2s\theta_1 c\theta_1 c\theta_2 J_{\eta\zeta} \\
I_{12} &= I_{21} = -(c\theta_2 J_{\xi\eta} + s\theta_2\, J_{\eta\zeta}) \\
I_{13} &= I_{31} = c\theta_1 s\theta_2 c\theta_2 (J_{\zeta\zeta} - J_{\xi\xi}) - s\theta_1 c\theta_2 J_{\xi\eta} - c\theta_1 (c^2\theta_2 - s^2\theta_2) J_{\xi\zeta} \\
&\quad - s\theta_1 s\theta_2 J_{\eta\zeta} \\
I_{23} &= I_{32} = c\theta_1 s\theta_2 J_{\xi\eta} + s\theta_1 J_{\eta\eta} - c\theta_1 c\theta_2 J_{\eta\zeta}
\end{aligned} \tag{37}
$$

and

$$
\begin{aligned}
L_1 &= c\theta_2\, K_\xi + s\theta_2\, K_\zeta \\
L_2 &= K_\eta \\
L_3 &= -\, s\theta_2 c\theta_1\, K_\xi + s\theta_1\, K_\eta + c\theta_1 c\theta_2\, K_\zeta
\end{aligned} \tag{38}
$$

where $s\theta_i = \sin\theta_i$, $c\theta_i = \cos\theta_i$ $(i = 1,2)$.

We shall be interested in investigating the stability of the high-spin motion in which the undeformed satellite rotates with the constant angular velocity Ω_s about axis z. Hence, we consider the stability in the neighborhood of the equilibrium point

$$
\begin{aligned}
\theta_1 &= \theta_2 = u_{cy} = u_{cz} = v_{cx} = v_{cz} = w_{cx} = w_{cy} = 0 \\
\dot\theta_1 &= \dot\theta_2 = \dot u_{cy} = \dot u_{cz} = \dot v_{cx} = \dot v_{cz} = \dot w_{cx} = \dot w_{cy} = 0
\end{aligned} \tag{39}
$$

Denoting this equilibrium point by the subscript E, disregarding constant terms and terms of order higher than two, we use Eqs. (17) and (20), in conjunction with Eqs. (21), (33), (34), (35), (37), and (38), and obtain the Hamiltonian in the neighborhood of E in the form

$$H_E = \frac{1}{2}\Big\{ A\dot{\theta}_1^2 + B\dot{\theta}_2^2 + 2\dot{\theta}_1\Big(\int_{D_y} \rho_y y\dot{w}_{cy}\, dy - \int_{D_z} \rho_z z\dot{v}_{cz} dz\Big)$$

$$+ 2\dot{\theta}_2\Big(\int_{D_z} \rho_z z\dot{u}_{cz} dz - \int_{D_x} \rho_x x\dot{w}_{cx}\, dx\Big) + \Big(\frac{\beta_3}{C}\Big)^2 \Big[(C-B)\theta_1^2$$

$$+ (C-A)\theta_2^2 + 2\theta_1\Big(\int_{D_y} \rho_y y w_{cy}\, dy + \int_{D_z} \rho_z z v_{cz}\, dz\Big)$$

$$- 2\theta_2\Big(\int_{D_x} \rho_x x w_{cx}\, dx + \int_{D_z} \rho_z z u_{cz}\, dz\Big) - \int_{D_x} \rho_x v_{cx}^2\, dx$$

$$- \int_{D_y} \rho_y u_{cy}^2\, dy - \int_{D_z} \rho_z (u_{cz}^2 + v_{cz}^2) dz \Big] - \frac{1}{C}\Big(\int_{D_x} \rho_x x\dot{v}_{cx} dx$$

$$- \int_{D_y} \rho_y y\dot{u}_{cy}\, dy\Big)^2 + \int_{D_x} \rho_x (\dot{v}_{cx}^2 + \dot{w}_{cx}^2) dx + \int_{D_y} \rho_y (\dot{u}_{cy}^2 + \dot{w}_{cy}^2) dy$$

$$+ \int_{D_z} \rho_z (\dot{u}_{cz}^2 + \dot{v}_{cz}^2) dz\Big\} + V_{EL} \qquad (40)$$

where we recall that

$$u_{cy} = u_y - x_c\ ,\ v_{cz} = v_z - y_c\ ,\ w_{cx} = w_x - z_c$$

$$u_{cz} = u_z - x_c\ ,\ v_{cx} = v_x - y_c\ ,\ w_{cy} = w_y - z_c \qquad (41)$$

in which

$$x_c = \frac{1}{m}\int_{D_y} \rho_y u_y dy + \frac{1}{m}\int_{D_z} \rho_z u_z\, dz$$

$$y_c = \frac{1}{m}\int_{D_x} \rho_x v_x dx + \frac{1}{m}\int_{D_z} \rho_z v_z\, dz \qquad (42)$$

$$z_c = \frac{1}{m}\int_{D_x} \rho_x w_x dx + \frac{1}{m}\int_{D_y} \rho_y w_y\, dy$$

Assuming that the elastic potential energy is due entirely to flexure, we can write

$$V_{EL} = \frac{1}{2}\left\{ \int_{D_x} \left[EI_{v_x} \left(\frac{\partial^2 v_x}{\partial x^2}\right)^2 + EI_{w_x} \left(\frac{\partial^2 w_x}{\partial x^2}\right)^2 \right] dx \right.$$

$$+ \int_{D_y} \left[EI_{u_y} \left(\frac{\partial^2 u_y}{\partial y^2}\right)^2 + EI_{w_y} \left(\frac{\partial^2 w_y}{\partial y^2}\right)^2 \right] dy$$

$$\left. + \int_{D_z} \left[EI_{u_z} \left(\frac{\partial^2 u_z}{\partial z^2}\right)^2 + EI_{v_z} \left(\frac{\partial^2 v_z}{\partial z^2}\right)^2 \right] dz \right\} \tag{43}$$

Equation (43) can be written in a more convenient form. To this end, we recall that the boundary conditions for the clamped-free rod corresponding to the domain $h_x < x < (h_x + \ell_x)$ are

$$v_x(x,t) = \frac{\partial v_x(x,t)}{\partial x} = 0 \quad \text{at} \quad x = h_x ,$$

$$EI_{v_x} \frac{\partial^2 v_x(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left[EI_{v_x} \frac{\partial^2 v_x(x,t)}{\partial x^2} \right] = 0 \quad \text{at} \quad x = h_x + \ell_x \tag{44}$$

Similar boundary conditions can be written for the remaining rods. In view of this, expression (43) can be integrated by parts with the result

$$V_{EL} = \frac{1}{2}\left\{ \int_{D_x} \left[v_x \frac{\partial^2}{\partial x^2} \left(EI_{v_x} \frac{\partial^2 v_x}{\partial x^2}\right) + w_x \frac{\partial^2}{\partial x^2} \left(EI_{w_x} \frac{\partial^2 w_x}{\partial x^2}\right) \right] dx \right.$$

$$+ \int_{D_y} \left[u_y \frac{\partial^2}{\partial y^2} \left(EI_{u_y} \frac{\partial^2 u_y}{\partial y^2}\right) + w_y \frac{\partial^2}{\partial y^2} \left(EI_{w_y} \frac{\partial^2 w_y}{\partial y^2}\right) \right] dy$$

$$\left. + \int_{D_z} \left[u_z \frac{\partial^2}{\partial z^2} \left(EI_{u_z} \frac{\partial^2 u_z}{\partial z^2}\right) + v_z \frac{\partial^2}{\partial z^2} \left(EI_{v_z} \frac{\partial^2 v_z}{\partial z^2}\right) \right] dz \right\} \tag{45}$$

The complete expression of the Hamiltonian in the neighborhood of the equilibrium position E is obtained by inserting expression (45) into (40).

Examining the Hamiltonian, Eq. (40), and the companion equations (41), it is obvious that the elastic displacements are not independent of one another. Although it may be possible to apply the theory of Reference 3, perhaps by devising a testing function K which is known to be smaller than H and in which the elastic displacements are independent, we shall consider instead a stability analysis by modal analysis. To this end, we represent the elastic displacements by the following series

$$v_x = \sum_{i=1}^{o_x} \phi_{xoi}(x)V_{xoi}(t) + \sum_{i=1}^{e_x} \phi_{xei}(x)V_{xei}(t)$$

$$w_x = \sum_{i=1}^{o_x} \psi_{xoi}(x)W_{xoi}(t) + \sum_{i=1}^{e_x} \psi_{xei}(x)W_{xei}(t) \qquad \text{over } D_x \qquad (46a)$$

$$u_y = \sum_{i=1}^{o_y} \phi_{yoi}(y)U_{yoi}(t) + \sum_{i=1}^{e_y} \phi_{yei}(y)U_{yei}(t)$$

$$w_y = \sum_{i=1}^{o_y} \psi_{yoi}(y)W_{yoi}(t) + \sum_{i=1}^{e_y} \psi_{yei}(y)W_{yei}(t) \qquad \text{over } D_y \qquad (46b)$$

$$u_z = \sum_{i=1}^{o_z} \phi_{zoi}(z)U_{zoi}(t) + \sum_{i=1}^{e_z} \phi_{zei}(z)U_{zei}(t)$$

$$v_z = \sum_{i=1}^{o_z} \psi_{zoi}(t)\ V_{zoi}(t) + \sum_{i=1}^{e_z} \psi_{zei}(z)\ V_{zei}(t) \qquad \text{over } D_z \qquad (46c)$$

where o_x, e_x, o_y, e_y, o_z, e_z are constant integers, ϕ_{xoi}, ϕ_{xei}, ψ_{xoi}, ---, ψ_{zei} are eigenfunctions associated with the elastic rods, and V_{xoi}, V_{xei}, W_{xoi}, ---, V_{zei} are corresponding generalized coordinates, in which the letters o and e designate odd and even modes of deformation, respectively. The functions ϕ_{xoi}, ϕ_{xei}, ψ_{xoi}, --- , ψ_{xei} satisfy the relations

$$\phi_{xoi}(x) = -\phi_{xoi}(-x) = \phi_{xei}(x) = \phi_{xei}(-x)$$
$$\psi_{xoi}(x) = -\psi_{xoi}(-x) = \psi_{xei}(x) = \psi_{xei}(-x) \tag{47a}$$

$$\phi_{yoi}(y) = -\phi_{yoi}(-y) = \phi_{yei}(y) = \phi_{yei}(-y)$$
$$\psi_{yoi}(y) = -\psi_{yoi}(-y) = \psi_{yei}(y) = \psi_{yei}(-y) \tag{47b}$$

$$\phi_{zoi}(z) = -\phi_{zoi}(-z) = \phi_{zei}(z) = \phi_{zei}(-z)$$
$$\psi_{zoi}(z) = -\psi_{zoi}(-z) = \psi_{zei}(z) = \psi_{zei}(-z) \tag{47c}$$

The eigenfunctions $\phi_{x\,oi}$ constitute the solution of the eigenvalue problem defined by the differential equations

$$\frac{d^2}{dx^2}\left(EI_{vx}\frac{d^2\phi_{x\,oi}}{dx^2}\right) = \Lambda^2_{vxi}\rho_x\phi_{x\,oi} \quad , \quad i = 1,2,--- \tag{48}$$

which must be satisfied over the domain $h_x < x < h_x + \ell_x$, where $\phi_{x\,oi}$ are subject to the boundary conditions

$$\phi_{xoi}(h_x) = \left.\frac{d\phi_{x\,oi}}{dx}\right|_{x=h_x} = 0$$
$$\left.EI_{vx}\frac{d^2\phi_{x\,oi}}{dx^2}\right|_{x=h_x+\ell_x} = \left.\frac{d}{dx}\left(EI_{vx}\frac{d^2\phi_{xoi}}{dx^2}\right)\right|_{x=h_x+\ell_x} = 0 \tag{49}$$

The quantities Λ_{vxi} are the associated natural frequencies of vibration. If the rod coinciding with the positive x axis is nonuniform, the solution of the eigenvalue problem can be obtained by one of the approximate methods described in Reference 5. If the rod is uniform, the solution can be taken directly from Reference 5 (Section 5-10). Similar eigenvalue problems can be defined for ψ_{xoi}, ϕ_{yoi}, --- , ψ_{zoi}. In the sequel we shall regard all the eigenfunctions and associated eigenvalues as known.

The eigenfunctions possess the orthogonality property. Moreover, they can be normalized, so that

$$\int_{D_x} \rho_x \phi_{xoi}(x) \phi_{xoj}(x)\ dx = 2\delta_{ij}$$

$$\int_{D_x} \rho_x \phi_{xei}(x) \phi_{xej}(x)\ dx = 2\delta_{ij} \qquad i,j = 1,2, --- \tag{50}$$

$$\int_{D_x} \rho_x \phi_{xoi}(x) \phi_{xej}(x)\ dx = 0$$

where δ_{ij} is the Kronecker delta. Similar expressions can be written for the remaining eigenfunctions.

In view of the above, a typical term in expression (45) becomes

$$\int_{D_x} v_x \frac{\partial^2}{\partial x^2} \left(EI_{vx} \frac{\partial^2 v_x}{\partial x^2}\right) dx = \int_{D_x} \left(\sum_{i=1}^{o_x} \phi_{xoi} V_{xoi} + \sum_{i=1}^{e_x} \phi_{xei} V_{xei}\right)$$

$$\times \left[\sum_{j=1}^{o_x} V_{xoi} \frac{d^2}{dx^2} \left(EI_{vx} \frac{d^2\phi_{xoj}}{dx^2}\right) + \sum_{j=1}^{e_x} V_{xei} \frac{d^2}{dx^2} \left(EI_{vx} \frac{d^2\phi_{xej}}{dx^2}\right)\right] dx$$

$$= 2\left(\sum_{i=1}^{o_x} \Lambda^2_{vxi} V^2_{xoi} + \sum_{i=1}^{e_x} \Lambda^2_{vxi} V^2_{xei}\right) \tag{51}$$

Hence, the potential energy V_{EL} can be regarded as a function of the generalized coordinates.

From Eqs. (41) we conclude that the Hamiltonian depends on the displacements x_c, y_c, and z_c of the center of mass, which, in turn, depend on the elastic displacements according to Eqs. (42). Substituting Eqs. (46) into (42), we conclude that the displacements x_c, y_c, z_c depend on the generalized coordinates $U_{yoi}, U_{yei}, U_{zoi}, U_{zei}, V_{xoi}$, ---. It follows that the Hamiltonian, Eq. (40), depends on the coordinates $\theta_1, \theta_2, U_{yoi}, U_{yei}, U_{zoi}$, --- as well as their time derivatives. Hence, H_E is a quadratic form in $4(1+o_x+e_x+o_y+---+e_z)$ variables. For stability, H_E must be positive definite in these variables.

Examining expression (40), we conclude that H_E can be written as the sum of a quadratic form depending on the velocities alone and another quadratic form depending on the coordinates alone

$$H_E = H_{1E} + H_{2E} \tag{52}$$

Furthermore by using even and odd modes to represent the elastic displacements no coupling between the even and odd modes occurs. Hence, each of the testing functions H_{1E} and H_{2E} may be represented as the sum of two quadratic forms, one involving only even modes and one involving odd modes and the rigid body motion only

$$\begin{aligned} H_{1E} &= H_{1Ee} + H_{1Eo} \\ H_{2E} &= H_{2Ee} + H_{2Eo} \end{aligned} \tag{53}$$

where

$$\begin{aligned} H_{1Ee} = &\sum_{i=1}^{e_x} \sum_{j=1}^{e_x} (\delta_{ij} - 2I_{vxi}I_{vxj})\, \dot{V}_{xei}\dot{V}_{xej} \\ &+ \sum_{i=1}^{e_z} \sum_{j=1}^{e_z} (\delta_{ij} - 2I_{vzi}I_{vzj})\, \dot{V}_{zei}\dot{V}_{zej} \\ &- 4 \sum_{i=1}^{e_x} \sum_{j=1}^{e_z} I_{vxi}I_{vzj}\dot{V}_{xei}\dot{V}_{zej} \\ &+ \sum_{i=1}^{e_x} \sum_{j=1}^{e_x} (\delta_{ij} - 2I_{wxi}I_{wxj})\, \dot{W}_{xei}\dot{W}_{xej} \\ &+ \sum_{i=1}^{e_y} \sum_{j=1}^{e_y} (\delta_{ij} - 2I_{wyi}I_{wyj})\, \dot{W}_{yei}\dot{W}_{yej} \\ &- 4 \sum_{i=1}^{e_x} \sum_{j=1}^{e_y} I_{wxi}I_{wyj}\dot{W}_{xei}\dot{W}_{yej} \end{aligned}$$

$$+\sum_{i=1}^{e_y}\sum_{j=1}^{e_y}(\delta_{ij}-2I_{uyi}I_{uyj})\dot{U}_{yei}\dot{U}_{yej} + \sum_{i=1}^{e_z}\sum_{j=1}^{e_z}(\delta_{ij}-2I_{uzi}I_{uzj})\dot{U}_{zei}\dot{U}_{zej}$$

$$- 4\sum_{i=1}^{e_y}\sum_{j=1}^{e_z} I_{uyi}I_{uzj}\dot{U}_{yei}\dot{U}_{zej} \qquad (54a)$$

and

$$H_{1Eo} = \frac{1}{2}\Big\{A\dot{\theta}_1^2 + B\dot{\theta}_2^2 + 4\dot{\theta}_1(\sum_{i=1}^{o_y} J_{wyi}\dot{W}_{yoi} - \sum_{i=1}^{o_z} J_{vzi}\dot{V}_{zoi})$$

$$+ 4\dot{\theta}_2(\sum_{i=1}^{o_z} J_{uzi}\dot{U}_{zoi} - \sum_{i=1}^{o_x} J_{wxi}\dot{W}_{xoi})$$

$$+ 2\sum_{i=1}^{o_x}\sum_{j=1}^{o_x}(\delta_{ij} - \frac{2J_{vxi}J_{vxj}}{C})\dot{V}_{xoi}\dot{V}_{xoj}$$

$$+ 2\sum_{i=1}^{o_y}\sum_{j=1}^{o_y}(\delta_{ij}-\frac{2J_{uyi}J_{uyj}}{C})\dot{U}_{yoi}\dot{U}_{yoj}+\frac{8}{C}\sum_{i=1}^{o_x}\sum_{j=1}^{o_y} J_{vxi}J_{uyj}\dot{V}_{xoi}\dot{U}_{yoj}$$

$$+ 2(\sum_{i=1}^{o_z}\dot{V}_{zoi}^2 + \sum_{i=1}^{o_z}\dot{U}_{zoi}^2 + \sum_{i=1}^{o_x}\dot{W}_{xoi}^2 + \sum_{i=1}^{o_y}\dot{W}_{yoi}^2)\Big\} \qquad (54b)$$

where

$$J_{vxi} = \int_{h_x}^{h_x+\ell_x}\rho_x x\phi_{xoi}(x)\,dx \quad , \quad J_{wxi} = \int_{h_x}^{h_x+\ell_x}\rho_x x\psi_{xoi}(x)\,dx$$

$$I_{vxi} = \frac{(2m-m_x-m_z)^{1/2}}{m}\int_{h_x}^{h_x+\ell_x}\rho_x\phi_{xei}(x)\,dx \qquad (55)$$

$$I_{wxi} = \frac{(2m-m_x-m_y)^{1/2}}{m}\int_{h_x}^{h_x+\ell_x}\rho_x\psi_{xei}(x)\,dx$$

in which $m_x = 2\rho_x \ell_x$, $m_y = 2\rho_y \ell_y$, and $m_z = 2\rho_z \ell_z$. Similar expressions can be written for J_{vzi}, J_{uyi},--- and I_{yzi}, I_{uyi},---. Moreover, the conserved momentum has the value $\beta_3 = \Omega_s C$ corresponding to the equilibrium position of pure spin about the z axis in the undeformed state. Using again the normal mode expansions for the elastic displacements, we obtain for H_{2Ee} and H_{2Eo}

$$H_{2Ee} = \sum_{i=1}^{e_x} \sum_{j=1}^{e_x} \left[(\Lambda_{vxj}^2 - \Omega_s^2)\delta_{ij} + 2\Omega_s^2 I_{vxi} I_{vxj}\right] V_{xei} V_{xej}$$

$$+ \sum_{i=1}^{e_z} \sum_{j=1}^{e_z} \left[(\Lambda_{vzj}^2 - \Omega_s^2)\delta_{ij} + 2\Omega_s^2 I_{vzi} I_{vzj}\right] V_{zei} V_{zej}$$

$$+ 4\Omega_s^2 \sum_{i=1}^{e_x} \sum_{j=1}^{e_x} I_{vxi} I_{vzj} V_{xei} V_{zej}$$

$$+ \sum_{i=1}^{e_y} \sum_{j=1}^{e_y} \left[(\Lambda_{uyj}^2 - \Omega_s^2)\delta_{ij} + 2\Omega_s^2 I_{uyi} I_{uyj}\right] U_{yei} U_{yej}$$

$$+ \sum_{i=1}^{e_z} \sum_{j=1}^{e_z} \left[(\Lambda_{uzi}^2 - \Omega_s^2)\delta_{ij} + 2\Omega_s^2 I_{uzi} I_{uzj}\right] U_{zei} U_{zej}$$

$$+ 4\Omega_s^2 \sum_{i=1}^{e_y} \sum_{j=1}^{e_z} I_{uyi} I_{uzj} U_{yei} U_{zej} + \sum_{i=1}^{e_x} \Lambda_{wxi}^2 W_{xei}^2$$

$$+ \sum_{i=1}^{e_y} \Lambda_{wyi}^2 W_{yei}^2 \qquad (56a)$$

and

$$H_{2Eo} = \frac{1}{2} \Omega_s^2 \left[(C-B)\theta_1^2 + (C-A)\theta_2^2 + 4\theta_1 \left(\sum_{i=1}^{o_y} J_{wyi} W_{yoi}\right.\right.$$

$$\left.\left. + \sum_{i=1}^{o_z} J_{vzi} V_{zoi}\right) - 4\theta_2 \left(\sum_{i=1}^{o_z} J_{uzi} U_{zoi} + \sum_{i=1}^{o_x} J_{wxi} W_{xoi}\right)\right]$$

$$+ \sum_{i=1}^{o_x} (\Lambda_{vxi}^2 - \Omega_s^2) V_{xoi}^2 + \sum_{i=1}^{o_z} (\Lambda_{vzi}^2 - \Omega_s^2) V_{zoi}^2$$

$$+ \sum_{i=1}^{o_y} (\Lambda_{uyi}^2 - \Omega_s^2) U_{yoi}^2 + \sum_{i=1}^{o_z} (\Lambda_{uzi}^2 - \Omega_s^2) U_{zoi}^2$$

$$+ \sum_{i=1}^{o_x} \Lambda_{wxi}^2 W_{xoi}^2 + \sum_{i=1}^{o_y} \Lambda_{wyi}^2 W_{yoi}^2 \qquad (56b)$$

As indicated previously, the time derivative of the Hamiltonian is negative semidefinite. Hence, due to coupling, if the Hamiltonian is positive definite the equilibrium is asymptotically stable, and if the Hamiltonian can take negative values in the neighborhood of the origin the equilibrium is unstable. But by Eq.'s(52) and (53) the Hamiltonian can be written in four parts, H_{1Ee}, H_{1Eo}, H_{2Ee} and H_{2Eo}, so that for H to be positive definite it is necessary that H_{1Ee}, H_{1Eo}, --- all be positive definite. Expressions for H_{1Ee}, H_{1Eo}, ---, can be written in the general form

$$H_{1Ee} = \frac{1}{2} \sum_{i=1}^{n_e} \sum_{j=1}^{n_e} \alpha_{eij}\, q_{ei}\, q_{ej}, \quad H_{1Eo} = \frac{1}{2} \sum_{i=1}^{n_o} \sum_{j=1}^{n_o} \alpha_{oij}\, q_{oi}\, q_{oj}$$

$$H_{2Ee} = \frac{1}{2} \sum_{i=1}^{n_e} \sum_{j=1}^{n_e} \beta_{eij}\, \dot{q}_{ei}\, \dot{q}_{ej}, \quad H_{2Eo} = \frac{1}{2} \sum_{i=1}^{n_o} \sum_{j=1}^{n_o} \beta_{oij}\, \dot{q}_{oi}\, \dot{q}_{oj} \qquad (57)$$

where q_{ei} and q_{oi} are generalized coordinates and $\dot{q}_{ei}$ and $\dot{q}_{oi}$ are generalized velocities. The α_{eij}, α_{oij}, β_{eij} and β_{oij} represent constant coefficients. According to Sylvester's criterion (see Reference 1, Sec. 6.7), H_{1Ee}, H_{1Eo}, H_{2Ee} and H_{2Eo} are positive definite if conditions

$$\begin{matrix} |\alpha_{eij}| > 0 \quad , \quad |\beta_{eij}| > 0 \\ |\alpha_{oij}| > 0 \quad , \quad |\beta_{oij}| > 0 \end{matrix} \qquad i,j = 1,2,---k; k = 1,2,---,n \qquad (58)$$

are satisfied, which represents the conditions that all the principal minor determinates associated with the matrices $[\alpha_e]$, $[\alpha_o]$, $[\beta_e]$ and $[\beta_o]$ of the coefficients be positive. The matrices $[\alpha_e]$, $[\alpha_o]$, $[\beta_e]$ and $[\beta_o]$ are referred to as Hessian matrices.

Numerical Results

The general solution of the problem of a rigid satellite with three pairs of uniform rods has been programed for digital computation. A numerical solution has been obtained on an IBM 360 computer. Results are presented for the case in which the rods in the radial direction are of equal length and the satellite possesses equal moments of inertia in the x and y directions. Moreover, all rods have equal mass and stiffness properties. Figure 3 shows the spin ratio Ω_s/Λ_{min} necessary for maintaining stability as a function of the length of the radial rods for fixed values of system parameters, where Λ_{min} represents the lowest natural frequency associated with the vibration of the radial or axial rods. If the parameters of the system can be represented by a point in the region below the appropriate curve, then the equilibrium is stable. These curves show that the allowable spin ratio Ω_s/Λ_{min} must be lower than unity. The extent to which it must be lower than unity depends on the system parameters. In particular, when the relationship between the rod lengths and the system parameters is such that C approaches A the allowable spin tends to zero. Figure 4 compares the results of a two-mode approximation for the elastic displacements to those using a four-mode approximation. We observe that the region of stability corresponding to the four-mode approximation is slightly smaller than that corresponding to the two-mode approximation, which conforms with expectation. Figure 5 shows the effect of increasing the mass of the rods. As in Figure 3, the area below the appropriate curve represents stable equilibrium. The curves indicate that an increase in mass decreases the region of stability. Figure 6 shows the parameter plane Ω_s/Λ_{min} versus C_o/A_o divided into regions of stability and instability by the curves R_{Az} = constant. The symbols are defined in the Figure. Stability is possible if the parameters are such that the system is represented by a point below the appropriate R_{Az} curve. Again the ratio Ω_s/Λ_{min} depends on the system parameters and is not to be merely smaller than unity. For comparison purposes, a problem which can be regarded as a special case of the present one, in the sense that it considers only spin axis rods, has been considered; this is the problem investigated in Reference 4. Results for the four-mode approximation and those of Reference 4 are presented in Figure 7 and, as expected, they indicate that the criteria obtained in Reference 4 working directly with the hybrid system of equations are more stringent than those obtained here by means of modal analysis.

Summary and Recommendation for Future Studies

The mathematical formulation associated with the problem of the stability of motion of a satellite consisting of a main rigid body and three pairs of flexible booms has been completed. The booms are capable of bending in two orthogonal directions. Whereas the rotational motion of the body is described by generalized coordinates depending on time alone, the elastic displacements of the booms depend on spatial position and time. Because of the flexibility of the booms, the center of mass of the body is continuously shifting relative to the main rigid body. These displacements, however, do not add degrees of freedom since they can be expressed in terms of integrals involving the elastic displacements. The formulation is appreciably more complete than that of Reference 2. Assuming no external torques, one of the coordinates describing the rotational motion is ignorable.

The Liapunov second method has been chosen for the stability analysis because it is likely to yield results which can be interpreted more readily than those obtained by a purely numerical integration of the equations of motion. Due to coupling of the elastic displacements, it is not feasible to use the stability method developed by the principal investigator (see References 3 and 4). Instead, modal analysis is used to reduce the system from a hybrid to an entirely discrete one. Since the elastic vibration results in energy dissipation, according to the Liapunov second method, the equilibrium position is asymptotically stable if the Hamiltonian is positive definite and unstable if it can take negative values in the neighborhood of the equilibrium.

The equilibrium position investigated corresponds to the high-spin motion of the undeformed satellite about one of the principal axes, namely, the z axis. The constant angular velocity in that position is denoted by Ω_s. The stability of the equilibrium is investigated by means of a computer program based on Sylvester's criterion.

The formulation is quite general, in the sense that booms of arbitrary flexural stiffness and mass distribution are considered. For a numerical solution the booms are assumed uniform. Although the results presented are numerical in nature, there appears that a possibility exists for deriving closed-form criteria in terms of infinite series associated with the natural modes of the elastic booms. This possibility is presently being explored.

The fact that the booms are assumed to undergo bending places a limitation on the length of the booms vis-a-vis the flexural stiffness. If booms of relatively large length are to be considered, then the bending theory cannot be regarded as valid any longer. Whereas the length limitation on the axial booms remains, the radial booms of greater length may be regarded as strings in tension, where the tension is provided by the centrifugal forces.

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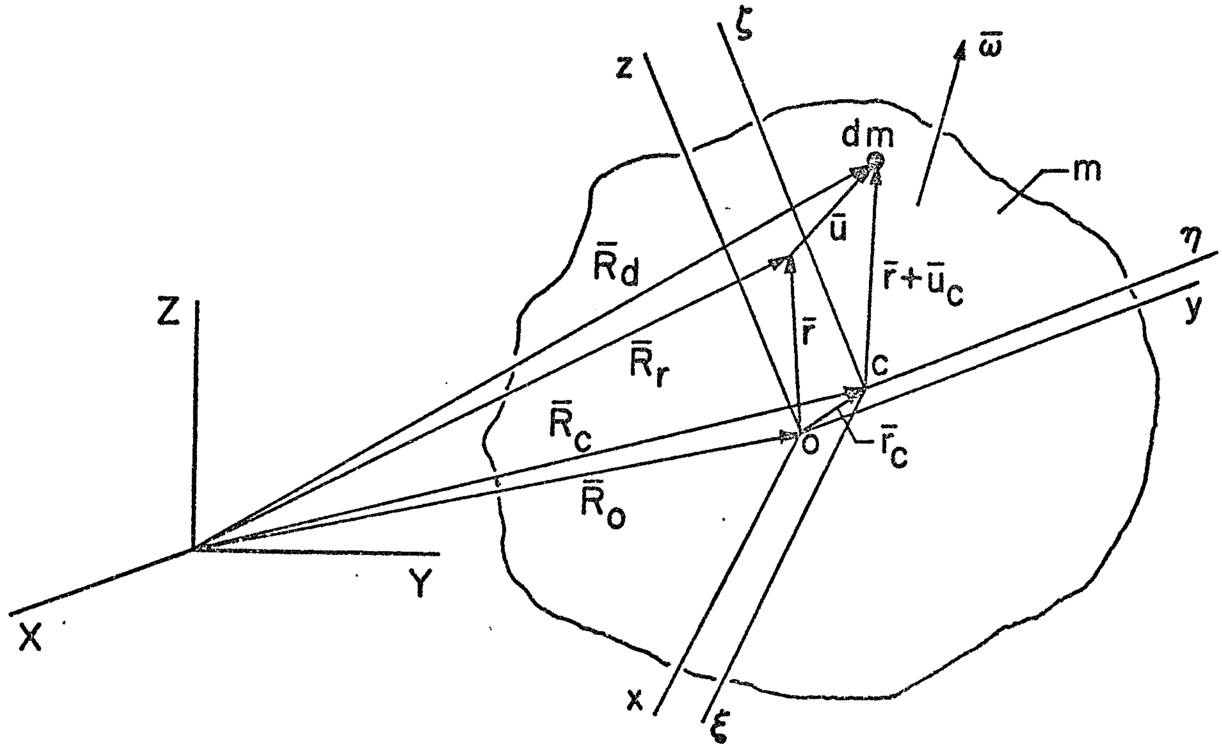
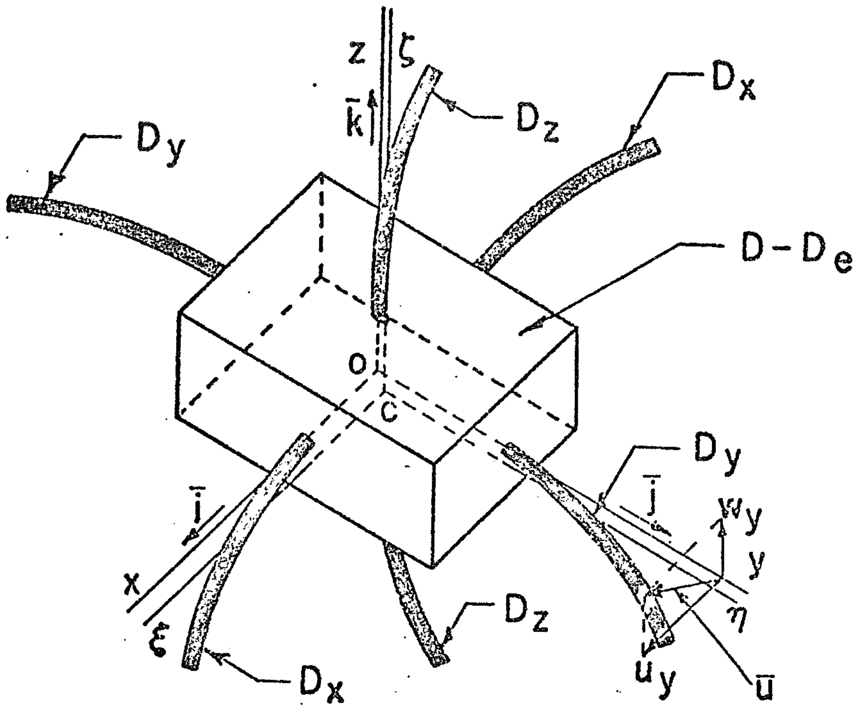
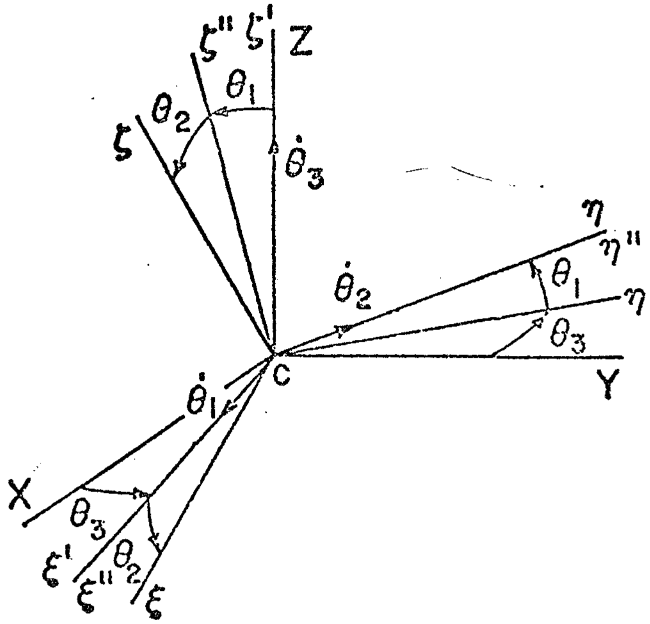


Figure 1 — The Flexible Body in an Inertial Space



a.



b.

Figure 2a — The Flexible Satellite

2b — The Satellite Rotational Motion

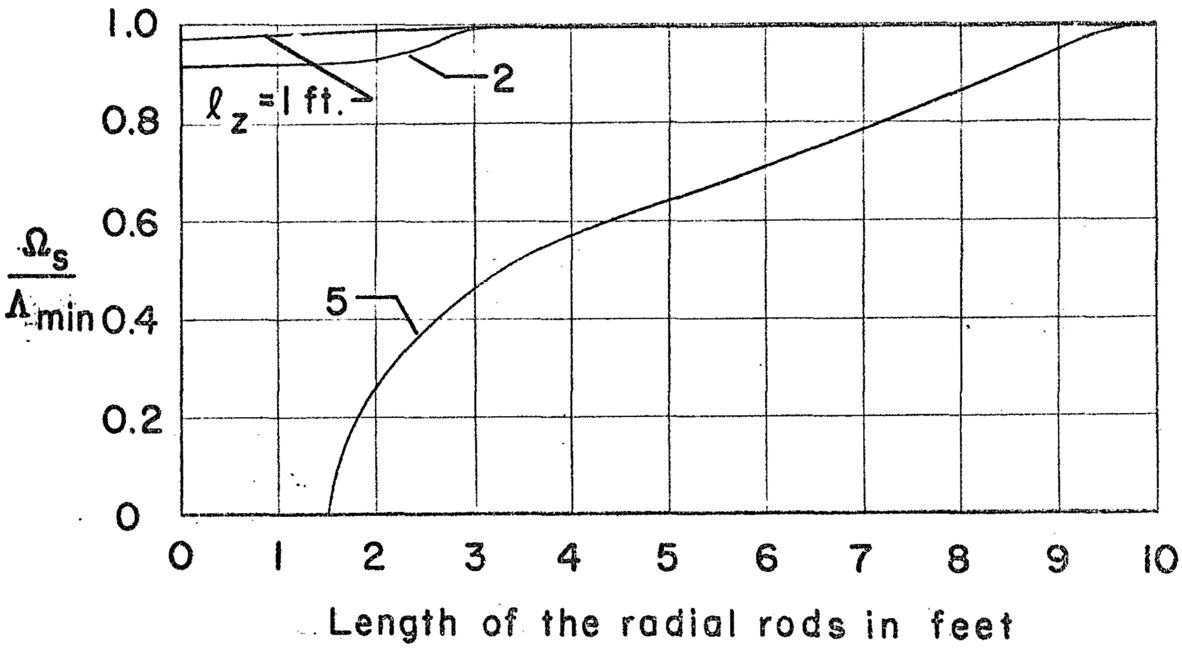


Figure 3

$h_x = h_y = h_z = 2$ feet

$C_o / A_o = 1.2$

$m = 6$ slugs

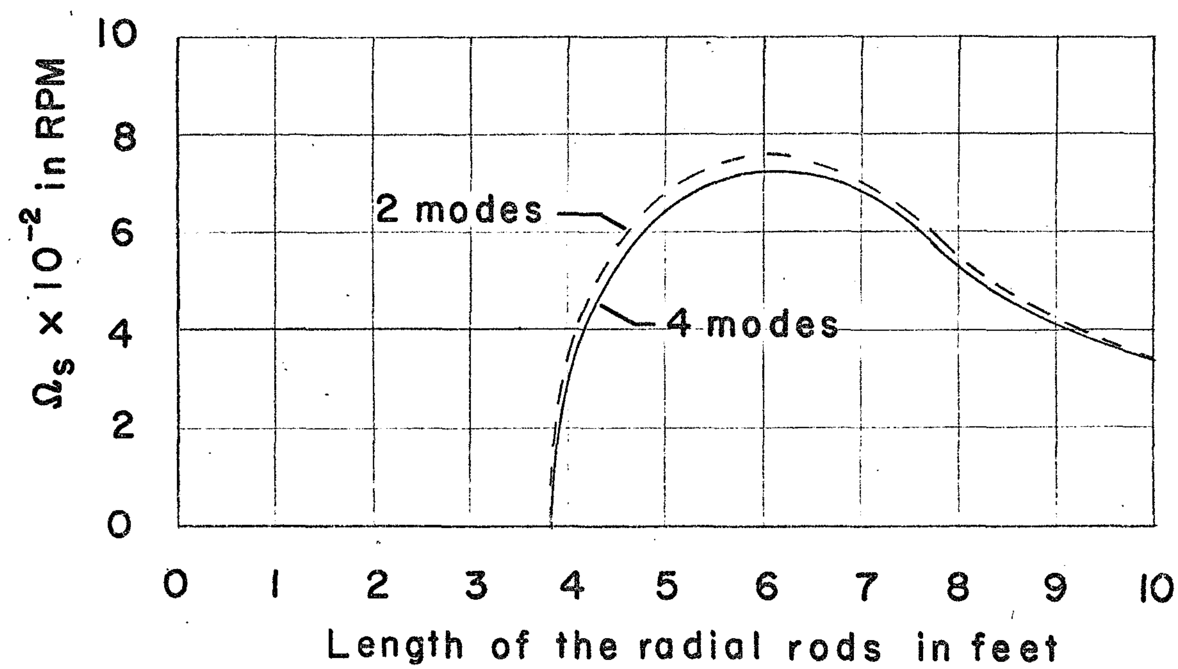


Figure 4

$C_o / A_o = 1.2$

$m = 6$ slugs

$\rho = 0.02$ slugs/ft

$h_x = h_y = h_z = 2$ feet

$EI = 2000$ lb-ft^2

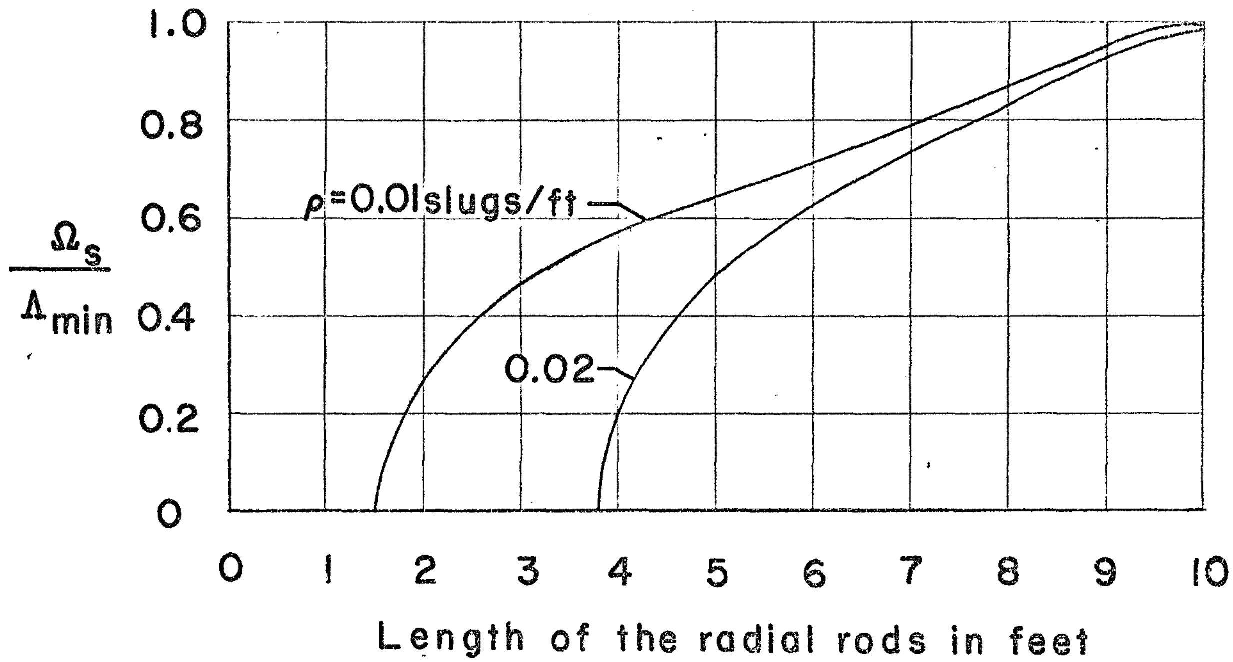


Figure 5

$C_o/A_o = 1.2$

$m = 6$ slugs

$\ell_z = 5$ feet

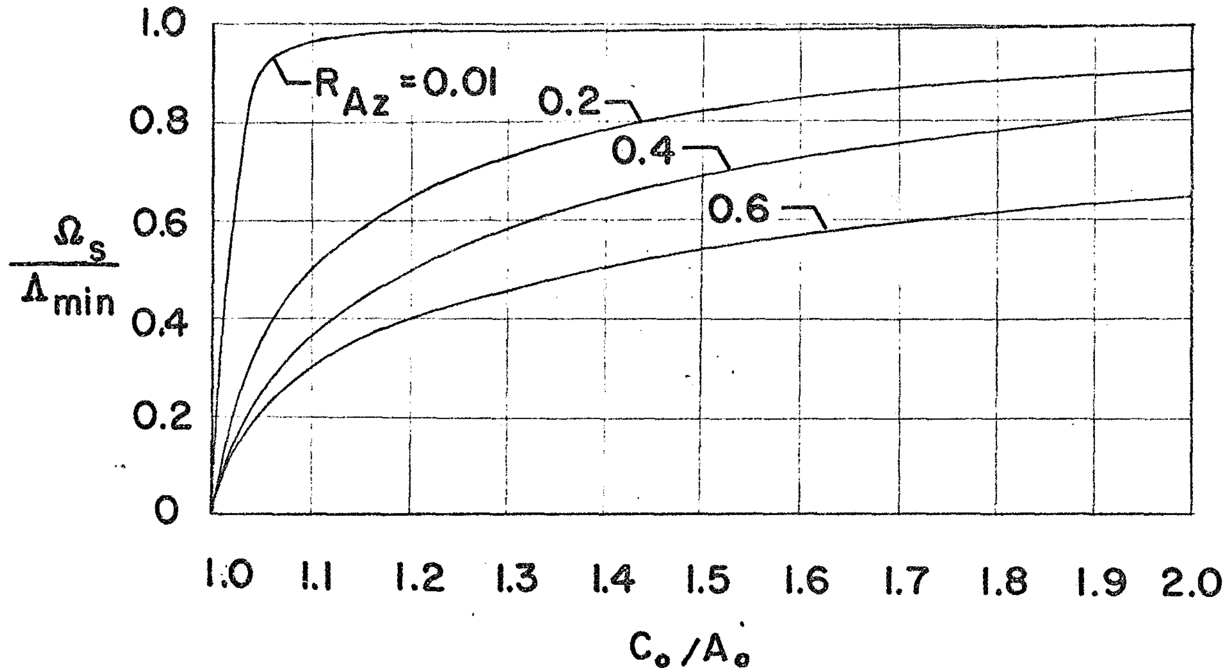


Figure 6

$$R_{Ax} = \frac{2}{A_o}\int_{h_x}^{h_x+\ell_x} \rho_x\, x^2\, dx\,, \qquad h_x = 0.1\,\ell_x$$

$$R_{Ay} = \frac{2}{A_o}\int_{h_y}^{h_y+\ell_y} \rho_y\, y^2\, dy\,, \qquad h_y = 0.1\,\ell_y$$

$$R_{Az} = \frac{2}{A_o}\int_{h_z}^{h_z+\ell_z} \rho_z\, z^2\, dz\,, \qquad h_z = 0.1\,\ell_z$$

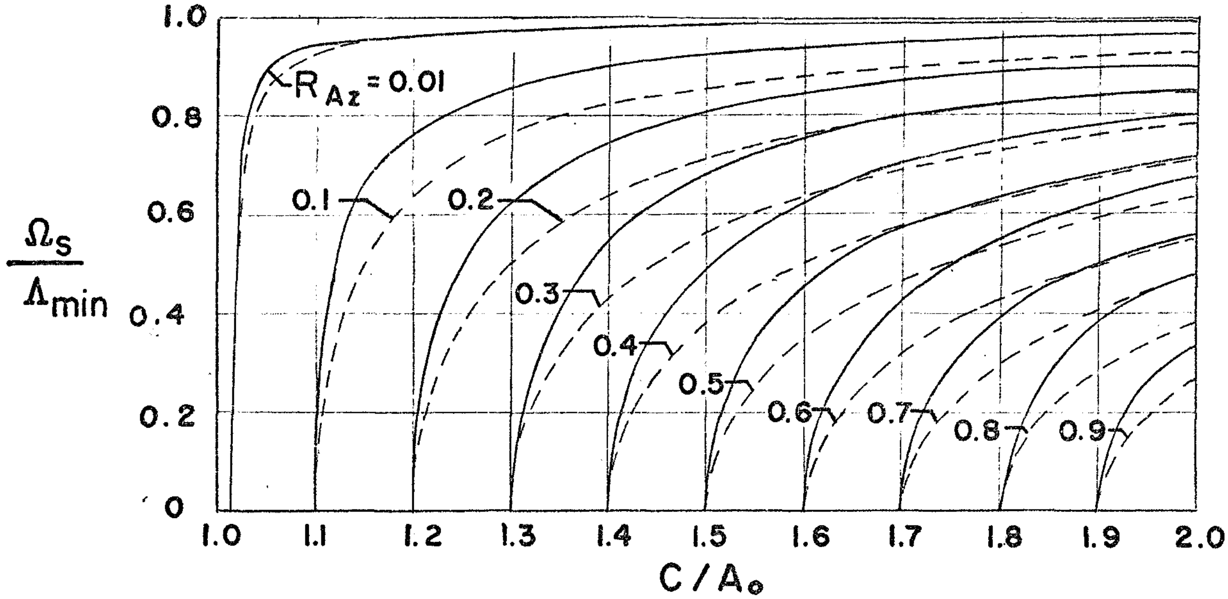


Figure 7 — Stability Regions in the Parameter Plane — $h_z = 2\ell_z$

——— Results of Present Investigation

- - - - Results of Reference 4