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DIFFERENTIAL CORRECTION METHODS IN SPACECRAFT ATTITUDE DETERMINATION

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ABSTRACT

Differential correction can be applied in certain cases, to the determination of the attitude of space vehicles. Choosing the parameter sets and prediction functions are two of the most critical considerations.

Parameter sets should be complete and independent. They may include instrument calibration constants, moments of inertia, attitude control system specifications, residual magnetic moments, drag moments, as well as the parameters which describe the motion itself. Numerous quantities besides those directly specifying the motion may, therefore, be calculated using this approach.

Prediction functions are used to predict the observed signals from the attitude sensing devices. Although the foundation of prediction function is a coordinate transformation, this simple foundation usually needs varying amounts of augmentation depending on the complexity of the sensing instruments. When the Euler coordinate transformation is used as the foundation of a prediction function, the results given in standard treatises on mechanics are readily adapted to the problem of attitude determination. For example, the Euler angles are given as continuous functions of time for passive torque-free rigid spacecraft.

In the balanced case the equations are simple. For nonbalanced bodies the expressions for the Euler angles are complicated, involving elliptic functions and quadratures. In the presence of torques the forms of the solutions are generally not known and in such cases numerical integration of the inhomogeneous equations of motion is required. But if the inhomogeneous terms are sufficiently small or periodic, the resulting motion may be modelled with empirical formulae and numerical integration may not be necessary.

One of the advantages of the approach considered here is the absence of any requirement concerning the simultaneity of instrumental observations. This means that the three components of each vector, such as the solar and magnetic

vectors, need not be known simultaneously or even nearly so. Moreover there is neither a requirement as to orthogonality of vector-component-sensing instrument axes nor a requirement that all three components of a vector be measured. Sometimes it is sufficient to know merely the time coordinate of a sensor's output to establish correct attitude.

This approach permits simple solutions to several problems posed by hardware malfunctions, hardware deterioration, residual magnetic moments, and other signal degrading phenomena. It also permits simple solutions to problems posed by sensors whose output signals are somehow singular, such as pulse signals, for example. Two useful devices for dealing with "difficult" sensor signals are "functional replacement" and "artificial sensors."

I. PREFACE

This discussion is concerned with certain specific questions arising in the application of least squares differential correction to the problem of spacecraft attitude determination. That is to say, the attitude determination problem is stated as a least square problem in several variables to which the method of differential correction is applied. The mathematical principles involved are identified and the correct variables are chosen and discussed. It is not intended to present either an explanation of the motion of space vehicles or an exposition of the methods of differential correction per se. Nor is it intended to fully discuss sensors, coordinate systems, least squares curve fitting, numerical techniques, error analysis, and other subjects germane to the problem. These subjects are discussed briefly only when necessary to sustain the train of thought. References have been included for the reader interested in further clarification.

The nomenclature employed conforms to the following conventions. Scalar quantities are shown in lower case Roman except for those having special significance which are denoted by letters of the Greek alphabet. Roman capitals are used for vectors and matrices. Absolute differentiation with respect to time is in some cases denoted by the popular dot over the variable in question.

The following symbols have permanent significance throughout this paper and represent concepts with which the reader is assumed to have some familiarity.

L	-	angular momentum
M	-	moment of force (torque)
I	-	inertia tensor
B	-	geomagnetic field
S	-	solar line-of-sight vector
α	-	right ascension
δ	-	declination

φ, θ, ψ - the Euler angles

Ω - instantaneous angluar velocity

∇ - gradient operator:

$$\nabla = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\} \text{ , or sometimes}$$

$$\nabla = \{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4} \quad \ldots\}.$$

Δ - a prefix denoting a small quantity

$\tilde{T}$ - transpose of a matrix T

$\{E\}$ - statistical expectation of a random variate E

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II. INTRODUCTION

A. Empiricism and Attitude Determination

The determination of the orientation of a space vehicle may be accomplished with varying degrees of empiricism as well as speed and accuracy. The most direct empirical approach is to plot the data and measure amplitudes, phase relationships, mean values, and apparent frequencies. From these quantities it is often possible to establish the orientation of vectors, such as the geomagnetic field and solar line of sight vectors, with respect to the body-fixed system of coordinates. Algebraic methods using independent ephemeris knowledge of the magnitude and direction of such vectors can then be applied to obtain the orientation of the vehicle [43, 44, 58]. Since only two observations of non-collinear vectors suffice to establish orientation at any given instant of time, the algebraic method, when automated, is the fastest way to compute attitude. When computed in this way, the attitude is defined as a discrete sequence; however, each calculation is subject to errors, mostly those of instrumental origin. Obviously filtering according to some suitable empirical formula should then be applied to attenuate the fluctuations. Presmoothing of raw data is sometimes preferable but hazardous owing to the nonlinear relationship between sensor outputs and orientation angles [28, 53, 68, 69]. (In [68] see page 303.)

B. Some Optimization Methods

When speed can be somewhat sacrificed, the smoothing is better accomplished with the aid of dynamical and statistical knowledge. An approach to attitude determination employing a minimum variance statistical filter (Kalman filter) is reported in reference [28]. This approach is well suited to real time applications because experimental data are processed in a stepwise fashion (hence the name "sequential estimation"). Processing occurs in a manner called minimum variance estimation so that this method constantly provides "best" estimate of the system parameters based on all accumulated data [28, 37, 38, 46, 53, 68].

The method discussed here is based on the classical optimization procedure of differential correction which has been applied since the eighteenth century to the computation of orbital elements [14, 53, 55]. In this approach it is preferable to have an abundance of observations evenly distributed over an extended time interval. One endeavors to "fit" these observations with a set of formulae which represent our knowledge of the dynamics and measuring devices.

C. The Model

Such a set of formulae are called a model or a prediction function. (See pages 47, 73, reference [46]). With a model one endeavors to simulate with the utmost fidelity the important behavior of the spacecraft. One might then say that the problem is to devise a predictor which actually predicts. The laws of motion should, therefore, be applied with care. The parameters which define the model are adjusted until a certain measure of agreement is achieved between predicted and observed values.

D. Correction of the Model

Initially a model is built based on the best available knowledge, both theoretical and laboratory measurements. Then a series of adjustments or corrections to the model is performed until the required agreement is obtained. While judicious application of trial and error could succeed, systematic correction methods are available. The most powerful modern methods come under the headings of gradient methods [14, 17, 23, 32, 46, 51], relaxation methods [17, 40], and differential correction methods [41, 46, 51, 54]. Sometimes the latter are called Taylor methods [32, 33].

E. Differential Correction

Differential correction methods have evolved from Newton's method for calculating the roots of a polynomial and are a generalization of his scheme for solving functional (rather than function) equations in several variables and higher order derivatives. As in Newton's method, the corrections are estimated with the help of the derivatives. Under modestly favorable conditions these corrections converge rapidly. The parameters suitable in applying differential correction to attitude determination, their properties, and the problems of convergence are considered below.

F. The Model as a Vector Transfer Function

The model will be treated below as a mathematical operator denoted by the symbol $\mathfrak{F}$. In mathematical terms, we say that a function $\mathfrak{F}$ is sought which, operating upon certain space environmental variables, predicts the observed data to a high degree of approximation.

As illustrated in figure 1, the observed data may derive from more than one sensor. Evidently $\mathfrak{F}$ is a vector operator and hence requires a vector differential correction process. The application of such a process, moreover, places certain analytical requirements on $\mathfrak{F}$. These requirements are also discussed in this paper.

Since we deal with signals from a piece of hardware, we should allow the model to be viewed from the point of view of impulse and transfer functions [18]. For the spacecraft and the telemetry it generates can be considered an open loop system whose input is the space environment (radiation, plasma, gravity, geomagnetism) Refer to figure 1. (Closed loop attitude controls do not affect the open loop status of our model unless the controls are a function of the said telemetry and are transmitted from a ground station on a real time basis.) Furthermore, since a spacecraft is a complex electromechanical device usually capable of functioning in several distinct modes, it is clear that the form of the operator $\mathfrak{F}$ may be time dependent.

G. An Elementary Example

As an elementary example, suppose we wish to determine the attitude of a rigid spacecraft equipped with a trio of perfectly linear, orthogonally mounted x, y, z magnetometers and suppose B is the geomagnetic field vector in the vicinity of the spacecraft. Let f_x, f_y, f_z represent the predicted output functions of the three magnetometers, respectively. Now, since all three sensors are to be equal partners in this attitude determination task, their output functions are considered as components of a vector so that we define the vector function

$$F \equiv \{f_x, f_y, f_z\} . \tag{II.1}$$

Recalling that F is the function of the predictor operator, we have

$$F = \mathfrak{F}(B). \tag{II.2}$$

Equation II.2 is to be regarded as the definition of a predictor operator. [In the more complicated cases considered below, $\mathfrak{F}$ will likewise be defined as an operator relating the predicted sensor output functions to various physical quantities of the space environment.]

Several properties of this operator are apparent upon consideration of equation II.2. First, its vector nature is illustrated. Although F happens to be a three-dimensional (cartesian) vector in this case, the dimensionality of the predictor operator is in general determined by the number of sensor output functions being predicted. Second, the arguments may be vectors and scalars. Also, since in this example F is the geomagnetic field expressed in the body-fixed coordinate system, it follows from equation II.2 that $\mathfrak{F}$ is merely a (memory-less) coordinate transformation. The simplicity of the operator in this example derives from the fact that the three sensors were assumed to be ideal, i.e., linear, with zero bias and unit gain with instantaneous response, and orthogonal. In practical applications, however, departures from the ideal are important to the attitude determination problem and it is for this reason that the predictor concept is approached here in a generalized fashion. The approach to be described resembles the method of "separation of variables" employed in partial differential equations.

H. Separation of Variables

The heart of any prediction function in attitude determination is, of course, a coordinate transformation. But, as pointed out, our predictor must account for the characteristics of both the hardware and the motion. Hence the prediction function is written as the product

$$\mathfrak{F} \equiv \mathcal{G}\,\mathcal{K}\,\mathcal{E}. \qquad \text{II.3}$$

This expression shows the three fundamentally different functions performed by $\mathfrak{F}$. The first, symbolized by $\mathcal{E}$, is to express mathematically the motion of the vehicle's coordinate axes by means of an orthogonal coordinate transformation. The second function, symbolized by $\mathcal{K}$, accounts for the position or mounting of the sensing instruments. (In a typical case the operation $\mathcal{K}$ represents a vector "dot" product involving some vector, such as B, and the unit vector collinear with the instrument "sensitive axis".) The purpose of the two operators $\mathcal{K}\mathcal{E}$, then, is to predict the ideal or perfect input functions being sensed by the instruments. For instance, the quantity

$$f = \mathcal{K}\,\mathcal{E}\,S \equiv K \cdot \mathcal{E}\,S \qquad \text{II.4}$$

could be the theoretical amount of solar radiation reaching the sensitized surface of a solar cell whose surface normal is given, in the body frame of reference, by the unit vector K. Hence $\mathcal{K}\,\mathcal{E}$ will be called the "ideal predictor." The third function, symbolized by $\mathcal{G}$, is to account for the instrument transfer functions, residual electric and magnetic moments, and any other signal degrading effects operating on the ideal signal f.

It will be seen presently that the three operators on the right hand side of II.3 are independent in the sense that they have no common arguments. This "separation of variables" helps to simplify the attitude determination problem. In Section VI it will also be seen that ill-defined normal equations are likely to result from attempts to apply differential correction with a set of parameters which include arguments of two or more of the aforementioned operators.

I. The Dependences

Consider now the parameter dependence of these operators. Clearly $\mathcal{G}$ depends on the instrument physical design parameters such as calibration constants and time delays. Let us denote these parameters by c_1, c_2, c_3, ... and let τ represent the predicted sensor output function. A typical sensor behaves according to the expression

$$\tau = \sum_{i=1}^{n} c_i \zeta^{i-1} \qquad \text{II.5}$$

where ζ is a function of f, the ideal signal as in equation II.4. (For instance, $\zeta = f$ or $\zeta = \sin(f)$.) Rewriting II.5 with the help of operator notation,

$$\tau = \mathcal{G}(c_1, c_2, c_3, \ldots t) \; f. \qquad \text{II.6}$$

Let us identify in a similar fashion, the arguments of $\mathcal{H}$. Suppose γ_1, γ_2, γ_3, ... are the various parameters, such as direction cosines for instance, which define the orientation of the sensor sensitive axis or surface with respect to the vehicle's coordinate axes. (We include among these parameters those required to handle geometric shadowing.) Then, for a solar cell sensor,

$$f = \mathcal{H}(\gamma_1, \gamma_2, \ldots t) \; S' \qquad \text{II.7}$$

where S' is the sun's vector also expressed in the body-fixed system. Notice that the parameter time is included in II.6 and II.7. One reason was given in II.F. Other reasons will become evident when the problems of sensor deterioration and changing space environment are considered. Refer to Section VI.

Finally let us try to identify the arguments of the ideal predictor which has already been recognized as a coordinate transformation. There are various ways to do this since there are various ways to express (parametrize) coordinate transformations. The method herein adopted and discussed below uses the angles α, δ, φ, θ, ψ. Hence we write

$$S' = \mathcal{E}(\alpha, \delta, \varphi, \theta, \psi) S \qquad \text{II.8}$$

where the time is not shown because its role is strictly implicit, in contrast to the preceding cases.

Equations II.6, II.7, II.8 show the separated parameter dependences. Observe that these parameters are not necessarily constants. They are not, therefore, necessarily suited for roles as parameters of differential correction. As explained in section VI, moreover, it is sometimes impossible to isolate the arguments of $\mathcal{G}$ from those of $\mathcal{E}$. This situation arises when one or more of the sensor output signals τ employed in the attitude determination problem occurs inside an active attitude feedback control loop on board the space vehicle. In such cases the angles α, δ, φ, θ, and ψ are obviously affected by τ, i. e. they are functions of τ. Numerical integration of the equations of motion may then be the only alternative in calculating the predicted functions.

J. The Ideal and Modified Prediction Functions

To the best of our ability to design it, the operator $\mathcal{G}$ modifies the ideal signal f in the same way that, through their nonlinear and delayed response, real sensors modify the ideal quantities they are designed to detect. If we were to limit ourselves to perfect sensors whose outputs were vector components along the vehicle coordinate axes, $\mathcal{E}$ would be a satisfactory prediction function and there would be no need for $\mathcal{G}\mathcal{H}$. Hence $\mathcal{G}\mathcal{H}\mathcal{E}$ is said to be the "modified prediction function". Before $\mathcal{G}$ and $\mathcal{H}$ are discussed, the fundamentally important coordinate transformation $\mathcal{E}$ is discussed.

K. The Auxiliary Reference Frame

Coordinate transformations are, of course, expressible (parametrized) in diverse ways. In this approach $\mathcal{E}$ is rewritten in the form

$$\mathcal{E} = \mathcal{A}\mathcal{L}. \qquad \text{II.9}$$

$\mathcal{L}$ is simply a transformation from the inertial coordinate system of our choice to some other system whose only purpose is to simplify calculations. For example, in the case of a simple spinning spacecraft, such a system is the angular momentum coordinate system. This trihedral is so called because its z axis is collinear with the spin angular momentum, L, and is generally an inertial system (or nearly so) [52]. (The vector L is constant unless it is forced to change under the influence of external torques or through the reaction of escaping matter [1, 2, 3, 4].) In this case, if the base reference frame is the usual vernal equinox system, then

$$\mathcal{L} = \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha\sin\delta & -\sin\alpha\sin\delta & \cos\delta \\ \cos\alpha\cos\delta & \sin\alpha\cos\delta & \sin\delta \end{bmatrix} . \qquad \text{II.10}$$

(This expression is derived in Appendix **D.**)

For gravity gradient stabilization, however, the auxiliary system might be aligned with the gradient of gravity [28] while for a vehicle with active attitude control devices, the auxiliary system is the one to which the control laws refer. Consult Section IV.G.1. In each case the onus of the auxiliary reference frame is to simplify the mathematics.

L. The Euler Transformation

Having thus defined the transformation $\mathcal{L}$, we may identify G as a transformation from an inertial system whose z axis is aligned with L to vehicle coordinates. This is the same purpose for which the Euler transformation is employed in standard treatises on mechanics.

The Euler angles are illustrated in figure 2. The reader is cautioned to observe the lack of unanimity in the definition of these angles. In this discussion the modern version of reference [1] is employed. The equations governing the evolution of the Euler angles as functions of time are considered below and are also discussed in the references.

Writing $\mathcal{E}$ as the product of $\mathcal{A}\,\mathcal{L}$ has several advantages. When a spacecraft is stabilized primarily by means of large angular momentum, the "fast" variables φ, θ, and ψ are segregated from the "slow" variables α and δ. Computational efficiency is thereby improved. Moreover, it may be justifiable to carry out the prediction of forced motion by means of a series of force-free approximations or "osculating solutions" (to borrow a phrase from the literature of orbit determination). Thus it will be seen in section IV that the form $\mathcal{A}\,\mathcal{L}$ is well suited to the determination of the osculating motion and also to the application of the technique called "rectification" [53].

Similar arguments can be made for non-spinning vehicles. In each case, the arguments α and δ are the right ascension and declination of the z coordinate axis of an auxiliary reference frame to which one can conveniently refer when using Euler's transformation.

The form thus chosen for $\mathcal{E}$ also affords one of the most convenient and plausible of methods for visualizing the motion of rigid bodies. We are able, moreover, to borrow from the wealth of existing treatises on the dynamics of motion with scarcely any modifications beyond matters of definition. References [1, 2, 3, 4] are recommended.

The Euler transformation is obtained from the product of three successive and ordered rotations corresponding to the angles φ, θ, and ψ, respectively. Thus if

$$\mathcal{B} \equiv \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{II.11a}$$

$$
\mathcal{C} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \text{ and} \qquad \text{II.11b}
$$

$$
\mathcal{D} \equiv \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ then} \qquad \text{II.11c}
$$

$$
\mathcal{A} = \mathcal{B}\,\mathcal{C}\,\mathcal{D}. \qquad \text{II.12}
$$

Carrying out the multiplications, we have

$$
\mathcal{A} = \begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}
$$

II.13

III. THE LEAST SQUARES ESTIMATION OF THE ATTITUDE

A. Squared Error Risk Function

Let us number the attitude sensing devices on board a space vehicle from 1 through m. Then the least square definition of the attitude problem may be stated as follows: find the minimum quantity

$$q = \sum_{j}^{m} \sum_{i}^{n^j} w^j(t_i)(\tau^j(t_i) - y^j_i)^2 , \qquad \text{III.1}$$

where

y^j_i = i^{th} observation obtained from j^{th} sensor,

$\tau^j(t_i)$ = i^{th} predicted output for j^{th} sensor,

$w^j(t_i)$ = weight factor for y^j_i,

t_i = i^{th} observation time,

m = total number of sensors,

n_j = total number of observations for j^{th} sensor.

Because the observations are random variates (they possess a statistical distribution), the attitude determination problem is an estimation problem. That is to say, the objective is to estimate certain parameters and establish measures of confidence. Since it is possible to regard the estimation of parameters from the powerful standpoint of decision theory [46], we do not hesitate to mold our problem at once in the dies of that theory. For this reason q is called the "risk" function.

According to equations III.1, the risk can be thought of as a mean squared error function. In section VI the reader will find the risk more formally defined as the expected value of the "loss" where the loss is a measure of demerit assigned to a given observation. As our "loss", therefore, we have adopted the squared error. This type of loss function is not the only one that can be defined but is the most convenient for the present application [46]. Nor is our definition of the least squares loss unique - refer to [59].

Notice that since the outputs from the sensing devices are considered to be scalar quantities, each with its own sample time t_i, no assumption has been made about the simultaneity of observations. The value of this feature is apparent upon considering the problem of establishing the magnitude and direction of the magnetic field, for example, from an orthogonal trio of magnetometers which are sampled at different times by means of a commutator. If we depend on knowing the field vector in body coordinates, we encounter difficulties for rapidly spinning and, especially, tumbling vehicles. In employing equation III.1, however, there is no need for us to know the on-board direction and magnitude of the field; only components along arbitrary (though non-collinear) axes sampled arbitrarily need be known. Nor is there need to know all three components.

Assumptions and restrictions placed on the function q arise below where Taylor's expansion formula is applied. It will then become evident that certain kinds of sensors require extra care when included in a least square formulation.

The weight functions w_{ij} serve several purposes and their importance cannot be overemphasized. First, the weight functions are used to regulate the weight (importance) of the observations. When a given observation is considered "good" or "important", it should be weighted high and vice-versa. Second, the weight functions help to balance the amplitudes of the various signals. The outputs from solar sensors are apt to be numbers between -1 and +1, while a magnetometer will normally read several hundred gammas (10^{-5} Gauss). For a proper balance, the solar sensors should then be weighted proportional to the field if their contribution to attitude is considered on equal footing with the magnetometers. For complete flexibility, therefore, the weight functions are shown as functions of the time as well as of the sensor index j [64].

It is argued (Chapter 5, reference [46] that the "best" weighting matrix, in the sense of minimum system parameter variance, is the inverse observational error covariance matrix. This choice results in the minimum variance estimate, but, unfortunately, exact knowledge of observational errors is seldom available.

Let us note, parenthetically, that the use of summation signs to sum over sensors is not to be interpreted literally. The group of sensors designated by the summation are not necessarily a fixed physical group. In automatic attitude determination, these groups are defined (formed) at will, depending on circumstances, as an aid in program organization. In the shadow of the earth, for example, the group of sensors would not include solar sensors while in the sunlight it would. This illustrates the time dependence of $\mathfrak{F}$.

B. Matrix vs. Scalar Notation

The notation simplifies if the matrix product is employed to indicate summation over the observed data points [39, 46, 54]. Suppose the following definitions are made:

$$\tilde{Y}^j \equiv \{ y^j_1 \; y^j_2 \ldots \} \qquad \text{III. 2a}$$

$$\tilde{Y} \equiv \{ \tilde{Y}^1 \; \tilde{Y}^2 \ldots \} \qquad \text{III. 2b}$$

$$\tilde{T}^j \equiv \{ \tau^j_1 \; \tau^j_2 \ldots \} \qquad \text{III. 2c}$$

$$\tilde{T} \equiv \{ \tilde{T}^1 \; \tilde{T}^2 \ldots \} \qquad \text{III. 2d}$$

$$W^j \equiv \begin{bmatrix} w_1^j & 0 & \cdots \\ 0 & w_2^j & \cdots \\ . & . & \cdots \end{bmatrix} \qquad \text{III. 2e}$$

$$W \equiv \begin{bmatrix} W^1 & . & \cdots \\ . & W^2 & \cdots \\ . & . & \cdots \end{bmatrix} \qquad \text{III. 2f}$$

$$\Phi \equiv \tilde{W}W \qquad \text{III. 2g}$$

$$D \equiv T - Y \qquad \text{III. 2h}$$

Equation III. 1 can now be written

$$q = (\tilde{T} - \tilde{Y})\, \Phi\, (T - Y), \text{ or} \qquad \text{III. 3a}$$

$$q = \tilde{D}\, \Phi D. \qquad \text{III. 3b}$$

This method of notation is convenient but it has two disadvantages: i) it may be confusing in the presence of cartesian vectors such as B and S which are ubiquitous in this report and ii) it does not promote understanding of the mechanization of the computations shortly to be considered. Consequently we do not adhere to the matrix formulation whenever this understanding may be jeopardized. Hence, in the remainder of this paper, the symbols Σ_i and Σ_j will always mean summation over data points and over sensors, respectively.

C. The System Parameters

Let us now introduce the parameters or variables of our problem. These parameters are designated by the array

$$U \equiv \{u_1, u_2, u_3, \ldots\}$$

The quantities u are the arguments of the attitude problem. Their role in the expression for the squared error appear through τ, the predicted sensor output:

$$\tau^j (t_i) = \mathfrak{F} B$$

where

$$\mathfrak{F} = \mathfrak{F}(U, t).$$

That is, the attitude determination problem consists in adjusting U so that q is minimized. Although, as pointed out in the introduction, there are several applicable and powerful approaches to this optimization problem, our differential correction method is a second order Newton iteration scheme employing Taylor's expansion formula in several variables.

D. The Computational Scheme

1. Iteration Functions

As is often done in solving nonlinear problems, we employ the concept of an "iteration function." Briefly, if U is a vector and Ξ is a vector function of equal dimension, then the recursive relation

$$U_{i+1} = U_i - \Xi (U_i) \qquad \text{III.4}$$

is called an iteration function. That is, an iteration function is a recipe for obtaining a series. See references [8, 9, 10, 46, 55].

The order of an iteration function is related to its convergence properties. For example, if near convergence the rate of change of q (U) is of second order in the rate of change of U, the iteration function is said to be of second order. It is evident that the order of an iteration function is the order of the first nonvanishing gradient of q (U) near convergence.

In this discussion we limit ourselves to the "second order one point [8]" Newton iteration function and briefly introduce it in the following paragraphs in order to help the discussion of Lagrange multipliers.

2. The Normal Equations

As far as the method is concerned, it is irrelevant that our problem is an attitude problem. What is relevant is the fact that the quantity q given by equation III.1 is a function of u_1, u_2, u_3 .... and that the u's are to be so adjusted that q is minimized. If q is regarded as a hypersurface in the variables u, then one may think of the minimization problem geometrically as that of finding the lowest point on the surface.

The procedure for locating the minimum is to assume approximate starting values for the parameters u. This set of values is designated by U_0. Then the differential corrections

$$\Delta U \equiv \{ \Delta u_1, \Delta u_2, \Delta u_3, \dots \} \qquad \text{III.5}$$

are defined so that an improved value of U is given by

$$U_1 = U_0 + \Delta U \qquad \text{III.6}$$

It is tacitly assumed that U_1 is an improvement over U_0 if $q(U_1) < q(U_0)$. Hence the next step is to calculate ΔU. In differential correction that is accomplished with the aid of Taylor's expansion formula for vector functions [26, 46] and the standard procedure for minimax problems [7, 16, 17, 20, 23, 46, 51, 65]. First, Taylor's formula states that

$$\tau(U_1) = \tau(U_0) + \Delta\widetilde{U} \nabla \tau(U_0) + \ldots \qquad \text{III.7}$$

where small quantities of order higher than the first have been neglected. Substituting expression III.7 into III.1, we have

$$q = \sum_j \sum_i \left(\left(\tau_i^j (U_0) - y_i^j + \Delta\widetilde{U} \nabla \tau^j (U_0) \right) w_i^j \right)^2 \qquad \text{III.8}$$

In this expression it is understood that τ^j is computed for t_i, the observation time corresponding to y_i^j.

In the neighborhood of a minimum, the first order gradient of q vanishes. That is

$$\nabla q = \frac{\partial q}{\partial \Delta u_k} = 0; \; k = 1, 2, 3, \ldots n_u \qquad \text{III.9}$$

where n_u is the number of parameters u. Performing the differentiation, we have

$$\frac{\partial q}{\partial \Delta u_k} = 2 \sum_i \sum_j w_i^j \left(\epsilon_i^j + \Delta\widetilde{U} \nabla \tau_i^j \right) \frac{\partial \tau_i^j}{\partial u_k} = 0; \; k = 1, 2, 3, \ldots n. \qquad \text{III.10}$$

where ϵ, called residual, is defined as

$$\epsilon_i^j \equiv \tau^j (t_i) - y_i^j . \qquad \text{III.11}$$

In order to obtain an expression for the iteration function Ξ, let us derive the normal equations in matrix form. For brevity define the gradient of a vector as

$$\nabla S \equiv \begin{bmatrix} \frac{\partial S_1}{\partial u_1} & \frac{\partial S_1}{\partial u_2} & \cdots \\ \frac{\partial S_2}{\partial u_1} & \frac{\partial S_2}{\partial u_2} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}, \qquad \text{III.12}$$

so that, to a second order approximation,

$$T(U) = T(U_0) + \nabla T \Delta U. \qquad \text{III.13}$$

Hence, in view of equation III.2h,

$$D = T_0 + \nabla T \Delta U - Y, \qquad \text{III.14}$$

where

$$T_0 \equiv T(U_0). \qquad \text{III.15}$$

For convenience, define the residual vector as

$$E \equiv T_0 - Y. \qquad \text{III.16}$$

With the help of these definitions, the error function III.3b becomes

$$q = (\widetilde{E} + \Delta \widetilde{U} \nabla \widetilde{T}) \Phi (E + \nabla T \Delta U). \qquad \text{III.17}$$

The normal equations are obtained upon differentiating with respect to ΔU:

$$0 = 2(\nabla \widetilde{T} \Phi E + \nabla \widetilde{T} \Phi \nabla T \Delta U) \qquad \text{III.18}$$

Upon transposing we have:

$$\nabla \widetilde{T} \Phi \nabla T \Delta U = - \nabla \widetilde{T} \Phi E. \qquad \text{III.19}$$

Our iteration function may therefore be written as:

$$\Xi = (\nabla \tilde{T} \Phi \nabla T)^{-1} \nabla \tilde{T} \Phi E. \qquad \text{III.20}$$

Equations III.10 or III.19 are a system of n linear equations in n unknowns, namely Δu_k. The array solution, ΔU, if it exists, can then be applied as prescribed by equation III.6 and the foregoing calculation is repeated again and again. When circumstances are favorable, this recursive process will converge to the required minimum.

The subject of convergence is too large to discuss here, as may be inferred from the references dealing with numerical techniques. Some practical comments pertaining to convergence in attitude determination problems are offered in Appendix B, however. For the present, suffice it to say that various iteration functions have differing regions of convergence and hence certain methods will converge when others fail. Gradient methods [17, 46] (the method of steepest descent) and the method of conjugate directions [17, 31] exemplify techniques directed at increasing the region of convergence. None can match the computational efficiency of the Newton iteration function near convergence, however. For a method combining the best qualities of the gradient method with those of the Newton function, see references [32 and 33].

3. The Notation of Gauss

It is convenient at this point to introduce simplified notation due to Gauss [47]. Define the quantities

$$(aa)^j \equiv \sum_i \frac{\partial \tau_i^j}{\partial u_1} w_i^{j\,2} \frac{\partial \tau_i^j}{\partial u_1} \qquad \text{III.22a}$$

$$(ab)^j \equiv \sum_i \frac{\partial \tau_i^j}{\partial u_1} w_i^{j\,2} \frac{\partial \tau_i^j}{\partial u_2} \qquad \text{III.22b}$$

$$(ac)^j \equiv \ldots\ldots\ldots\ldots \qquad \text{III.22c}$$

and

$$(a\epsilon)^j \equiv \sum_i \epsilon_i^j w_i^{j\,2} \frac{\partial \tau_i^j}{\partial u_1} \qquad \text{III. 22d}$$

Next, suppose there is but one sensor. Hence the j designator may be dropped. Now the normal equations may be expressed as

$$\begin{bmatrix} (aa) & (ab) & (ac) & \cdots \\ (ba) & (bb) & (bc) & \cdots \\ (ca) & (cb) & (cc) & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \Delta U = - \begin{bmatrix} (a\epsilon) \\ (b\epsilon) \\ (c\epsilon) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}. \qquad \text{III. 23}$$

If the matrix of coefficients is denoted by N and the column matrix on the right by Γ, the normal equations are

$$N \Delta U = -\Gamma. \qquad \text{III. 24a}$$

In the presence of several sensors, the normal equations are succinctly expressed as

$$\left(\sum_j N^j \right) \Delta U = -\sum_j \Gamma^j. \qquad \text{III. 24b}$$

For convenience below, let us note that

$$N = \nabla \tilde{T} \Phi \nabla T \qquad \text{III.25a}$$

$$\Gamma = \nabla \tilde{T} \Phi E, \qquad \text{III.25b}$$

as can be readily seen by inspection of III.19.

4. The Normal Equations and Generalized Vector Spaces

An interesting interpretation of equations III.22 and III.23 is achieved with generalized linear vector spaces [16, 46]. (In Appendix E the reader will find the concept of a linear vector space concisely defined.) The vector manifold presently to be discussed as an aid in understanding the normal equations is also applied advantageously in the probabilistic approach to error estimation. This approach is considered in section VI.

Our vector manifold is considered generated by the random variates y_i^j (a random variate is a variable for which a probability density function is said to exist [66]) defined in section III.A. It is immediately apparent that, since these experimental observations represent highly nonlinear functions of the system parameters, their probability density functions are dependent on the index i as well as j. That is to say, if the variates y_i^j are taken as the components of a random vector

$$Y \equiv \begin{bmatrix} Y^1 \\ Y^2 \\ Y^3 \\ \vdots \end{bmatrix} \qquad \text{III.26a}$$

where

$$Y^j \equiv \begin{bmatrix} y_1^j \\ y_2^j \\ \vdots \end{bmatrix}, \qquad \text{III.26b}$$

then each element (component) of Y has a generally distinct probability density function.

We have thus defined a vector space having one dimension per random variate (observable) so that the dimensionality is Σn_j. Evidently the predicted functions τ_i^j are the components of a vector in this space and, consequently, the derivatives of this vector, $\partial T/\partial u_k$ also belong to the same space.

Let us denote the derivatives of T as follows:

$$\frac{\partial T}{\partial u_1} = A, \qquad \text{III.27a}$$

$$\frac{\partial T}{\partial u_2} = B, \qquad \text{III.27b}$$

... etc.

Hence the quantities (aa), (ab), defined earlier, are regarded as inner products:

$$(aa) = A \cdot A \qquad \text{III.28a}$$

$$(ab) = A \cdot B \qquad \text{III.28b}$$

...

If the mean values of A, B, ... vanish, then, in the statistical sense, $A \cdot A$, $A \cdot B$, ... are interpreted as sample covariances [19]. Terms like $A \cdot A$, $B \cdot B$, ... are called autocovariances and, if normalized, they are considered correlations. Normalization is achieved by first defining the "norm" (length) of a vector:

$$|A| \equiv (A \cdot A)^{\frac{1}{2}}. \qquad \text{III.29}$$

The autocorrelation of A is

$$\mu_{aa} \equiv (aa)/|A|^2, \qquad \text{III.30a}$$

and the crosscorrelation of A and B is

$$\mu_{ab} \equiv (ab)/(|A|\ |B|), \text{ etc.} \qquad \text{III.30b}$$

....

(Notice that $\mu_{aa} = 1$ and $-1 \leq \mu_{ab} \leq +1$.)

Consider also the inverse covariances obtained by inverting N:

$$(aa)^* \equiv N^{-1}(1, 1), \tag{III.31a}$$

$$(ab)^* \equiv N^{-1}(1, 2) \tag{III.31b}$$

. . . .

These covariances may also be normalized. The resulting correlations are perhaps more interesting since they provide a direct measure of the correlation between system parameters:

$$\sigma_{aa} \equiv (aa)^* / |A^*|^2 \tag{III.32a}$$

$$\sigma_{ab} \equiv (ab)^* / (|A^*|\ |B^*|) \tag{III.32b}$$

. . . .

where

$$|A^*| \equiv [(aa)^*]^{\frac{1}{2}}, \text{ etc.} \tag{III.33}$$

Pursuing further the vector interpretation, the quantity

$$\mu_{a\epsilon} \equiv (A \cdot E) / |A| \tag{III.34}$$

is the projection of the residual E on the derivative vector A.

Now, if in applying a scheme as outlined in section III.C.1, 2, convergence is not achieved, the geometric and vector interpretations may be helpful in correcting the difficulty. The magnitudes of the σ's and μ's calculated in the first iteration contain strong hints as to the nature of the difficulty.

For example, if the correlations μ_{kl}, $1 \neq k$, are near +1 or -1, then the parameters u_k and u_l may be poorly chosen. That is, they may be correlated, or at least they give rise to skewed angle derivative vectors. Skewed conditioning (ill-conditioning) is discussed in references [20] and [46]. Even in the absence of skew-conditioning, difficulties may be due to exceptionally large projection of the residual vector on one or more of the derivative vectors. This situation can arise legitimately (see Appendix B) and calls for special handling. Suspicion toward computational or analytical errors should, of course, be aroused by identically vanishing correlations. Situations giving rise to perfectly correlated parameters are discussed in section VI while Appendices B and E are recommended for further discussion concerning the ideas presented here.

5. Existence of the Inverse

Existence of N^{-1} is, of course, a necessary condition for the application of our computational scheme. Unfortunately it is not a sufficient condition to guarantee convergence to the absolute minimum nor does it guarantee that the iteration process will converge at all. (See Section 7.5, reference [46]. See also references [32, 33].) In the preceding section, certain correlation coefficients are defined which are useful in determining if the normal equations are well posed. The best circumstances for inverting N are those in which σ_{kl} (or μ_{kl}); $k \neq l$ are small. See reference [53], page 89. As will be seen later, the matrix will not be singular if the attitude determination problem is well posed according to the precautions to be developed in Section VI.

Suppose two system parameters, say u_1 and u_2, give rise to derivatives that are correlated (proportional). That is to say, suppose that

$$\frac{\partial \tau_i}{\partial u_2} = k \frac{\partial \tau_i}{\partial u_1}; \quad i = 1, 2, 3, \ldots n \qquad \text{III.35a}$$

or $$B = kA. \quad \text{III. 35b}$$

Then $$(bb) = B \cdot B = k^2 A \cdot A \quad \text{III. 36a}$$

$$(ab) = B \cdot A = k A \cdot A \quad \text{III. 36b}$$

$$\cdots\cdots\cdots \quad \text{III. 36c}$$

or $$(bb) = k^2 (aa), \quad \text{III. 37a}$$

$$(ab) = k (aa), \quad \text{III. 37b}$$

$$\cdots\cdots\cdots \quad \text{III. 37c}$$

If III. 27a, b, c ... are substituted into III. 34, it is immediately apparent that the first two rows (or columns) are proportional with proportionality k and the determinant vanishes [16].

The relationship III. 35 means that parameters u_1 and u_2 give rise to derivatives whose ratio remains constants for all observation times. While this may seem a remote possibility when the parameters are independent, observe that condition III. 35 is satisfied whenever a derivative vanishes altogether. (In this instance it is immediately clear that N contains a column and a row of zeros.) A case in point is that of a dynamically balanced vehicle having only one magnetometer collinear with its axis of dynamical symmetry. Should one attempt to determine the three Euler angles and two rates, the normal equations become singular because $\partial \tau / \partial \dot{\psi} = 0$. This will become clearer when the balanced vehicle is discussed below. Proportional derivatives may arise in a number of ways. Some specific cases are discussed in Section VI.

E. Constraints

1. The Meaning of Constraint

The case of the dynamically balanced vehicle also serves to illustrate an elementary problem with side conditions or constraints. It is explained later that, for such a vehicle, the two Euler rates are functionally related. This functional relationship is the constraint in the attitude problem and can be employed to remove the singularity of N mentioned above. But more important is the fact that the constraint equation defines a path on the hypersurface $\underline{q}$ to which the "iteration path" ought to be confined in order to achieve the fastest convergence. Even when, because of the appropriate mounting sensors, the normal equations are nonsingular, failure to account for a constraint makes it more likely that a false minimum will be attained. Figure 4 shows the output of the x magnetometer of the dynamically balanced EPE-D (S3-C) satellite during precessional motion. The curve shows the presence of two superimposed sinusoids with angular rates $\dot{\varphi}$ and $\dot{\psi}$, respectively. Without knowledge of the constraint equation or other data, we could fit this curve employing

$$\varphi = \dot{\varphi}\, t + \varphi_0 \qquad \text{III.38a}$$

$$\psi = \dot{\psi}\, t + \psi_0 \qquad \text{III.38b}$$

or

$$\varphi = \dot{\psi}\, t + \varphi_0 \qquad \text{III.39a}$$

$$\psi = \dot{\varphi}\, t + \psi_0 . \qquad \text{III.39b}$$

The second set of equations lead to the "false minimum."

It is also clear that the constraint equation can be used advantageously in guiding the iteration path by reducing the number of independent variables. This is accomplished by solving the constraint for one variable and applying the chain rule of differentiation. Let u, v, and w be the three parameters in a certain attitude problem. Then, in Gauss' notation, the normal equations are:

$$\begin{bmatrix} (aa) & (ab) & (ac) \\ (ba) & (bb) & (bc) \\ (ca) & (cb) & (cc) \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix} = - \begin{bmatrix} (a\epsilon) \\ (b\epsilon) \\ (c\epsilon) \end{bmatrix} . \qquad \text{III.40}$$

Let a constraint relation be

$$h\,(u,\ v,\ w) = 0 . \qquad \text{III.41}$$

If it is possible to solve III.41 for, say w, then the expressions

$$\frac{d\tau}{du} = \frac{\partial \tau}{\partial u} + \frac{dw}{du}\frac{\partial \tau}{\partial w} \qquad \text{III.42a}$$

$$\frac{d\tau}{dv} = \frac{\partial \tau}{\partial v} + \frac{dw}{dv}\frac{\partial \tau}{\partial w} \qquad \text{III.42b}$$

$$dw = \frac{\partial w}{\partial u}\, du + \frac{\partial w}{\partial v}\, dv \qquad \text{III.42c}$$

may be used to reduce the system of three equations III.40 to a system of only two equations:

$$\begin{bmatrix} (aa)' & (ab)' \\ (ba)' & (bb)' \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} = - \begin{bmatrix} (a\epsilon)' \\ (b\epsilon)' \end{bmatrix} , \qquad \text{III.43}$$

where

$$(aa)' \equiv \sum_i \left(\frac{\partial \tau}{\partial u} + \frac{dw}{du}\frac{\partial \tau}{\partial w} \right)^2 , \qquad \text{III.44a}$$

$$(ab)' \equiv \Sigma \left(\frac{\partial \tau}{\partial u} + \frac{dw}{du}\frac{\partial \tau}{\partial u} \right) \left(\frac{\partial \tau}{\partial v} + \frac{dw}{dv}\frac{\partial \tau}{\partial v} \right), \qquad \text{III.44b}$$

$$(a\epsilon)' \equiv \Sigma\, \epsilon_i \left(\frac{\partial \tau}{\partial u} + \frac{dw}{du}\frac{\partial \tau}{\partial w} \right). \qquad \text{III.44c}$$

This system of 2 normal equations is obviously more efficient than the larger system of 3 equations. When III.43 is solved, Δu and Δv will be the best corrections possible under our straight forward Newton iteration approach. For other approaches with better results, refer to Appendix B.

It is interesting to note that if equations III.42 are substituted into all three expressions III.40, the determinant vanishes identically--the resulting three equations have only two unknowns. That is, it is nonsense to work with constrained parameters if the differentiation is carried out exactly and, conversely, it is necessary to work with constraints when differentiation is not done exactly. This is illustrated in section V.G.2.

2. The Dynamically Balanced Vehicle

As a simple illustration, consider the case of an unperturbed dynamically balanced spinning spacecraft. Once the orientation of the angular momentum is specified, say by means of α and δ, the Euler parameters φ, θ, and ψ and their rates $\dot{\varphi}$ and $\dot{\psi}$ satisfy the relations [1, 44]:

$$\varphi = \dot{\varphi} t + \varphi_0 \qquad \text{III.45a}$$

$$\theta = \theta_0 \qquad \text{III.45b}$$

$$\psi = \dot{\psi} t + \psi_0. \qquad \text{III.45c}$$

As discussed again below, the attitude relative to the angular momentum is solved when the five constants φ_0, θ_0, ψ_0, $\dot{\varphi}$, $\dot{\psi}$ are determined and, as demonstrated with the EPE-D satellite, fitting equations III.45 to the instrument outputs is feasible provided it is recognized that these five constants are dependent. It is readily shown [44] that the following equation of constraint exists:

$$\dot{\psi} = \dot{\varphi} \frac{(a - c)}{a} \cos \theta_0. \qquad \text{III.46}$$

In this case we are eliminating $\dot{\psi}$ from the array of parameters. Following the chain rule as in expressions III.42, the complete set of total derivatives becomes

$$\frac{d}{d\alpha} = \frac{\partial}{\partial \alpha} \qquad \text{III.47a}$$

$$\frac{d}{d\delta} = \frac{\partial}{\partial \delta} \qquad \text{III.47b}$$

$$\frac{d}{d\varphi_0} = \frac{\partial}{\partial \varphi_0} \qquad \text{III.47c}$$

$$\frac{d}{d\theta_0} = \frac{\partial}{\partial \theta_0} + \left(\frac{d\dot{\psi}}{d\theta_0}\right) \frac{\partial}{\partial \dot{\psi}} \qquad \text{III.47d}$$

$$\frac{d}{d\psi_0} = \frac{\partial}{\partial \psi_0} \qquad \text{III.47e}$$

$$\frac{d}{d\dot{\varphi}} = \frac{\partial}{\partial \dot{\varphi}} + \left(\frac{d\dot{\psi}}{d\dot{\varphi}}\right) \frac{\partial}{\partial \dot{\psi}} \, . \qquad \text{III.47f}$$

In view of equation III.46, we have

$$\frac{d\dot{\psi}}{d\theta_0} = -\dot{\varphi}\, \frac{(a-c)}{a} \sin\theta_0 \qquad \text{III.48a}$$

$$\frac{d\dot{\psi}}{d\dot{\varphi}} = \frac{(a-c)}{a} \cos\theta_0 \, . \qquad \text{III.48b}$$

The motion of dynamically balanced rigid body may now be determined employing the parameters

$$U = \{ \alpha , \delta , \varphi_0, \theta_0, \psi_0, \dot{\varphi} \} \qquad \text{III.49}$$

and the gradients as given in equations III.47 and III.48.

Further commentary here would anticipate the discussion of the balanced spacecraft as a special case of the general problem discussed below in section V.G.1.

F. Differential Correction With Constraints

The foregoing example was presented to illustrate the problem of a constraint that can be removed by reducing the number of variables. As shown below in connection with the nonbalanced spacecraft, it is often impossible to solve the equation of constraint. One may then resort to the method of undetermined Lagrange multipliers if the constraint relation can be stated in a way which renders it compatible with the least squares normal equations. Compatibility is assured when the constraints are holonomic [1, 65], that is if they are of the form

$$h\,(u_1,\, u_2,\, \ldots) = 0\,. \qquad \text{III.50}$$

The method of undetermined multipliers consists of introducing new variables λ sufficient in number to offset the over-determining effect of the constraints. The equations of condition are then derived from the fact that the gradients of q and each of the constraint relationships h must be collinear (linearly dependent) [7]. Hence we wish to minimize the composite function

$$q^* = \sum_i (\tau - y)^2 + \sum_{r=1}^{m} \lambda_r h_r. \qquad \text{III.51}$$

The approach here is similar to that employed in deriving the normal equations III.40. We must now keep in mind that the parameter array U is augmented. Thus

$$\tilde{U}^* = \{u_1,\, u_2,\, \ldots\, u_l;\, \lambda_1,\, \ldots\ldots\, \lambda_m\}. \qquad \text{III.52}$$

$$\Delta\tilde{U}^* = \{\Delta u_1,\, \Delta u_2,\, \ldots\Delta u_l,\, \Delta\lambda_1\ldots\Delta\lambda_m\}. \qquad \text{III.53}$$

Evidently the $\Delta\lambda$'s are to be equal partners with the Δu's in the expansion of q in a Taylor series.

For simplicity let

$$\tilde{\Lambda} = \{\lambda_1, \lambda_2, \dots \lambda_m\} \qquad \text{III.54a}$$

$$\tilde{H} = \{h_1, h_2, \dots h_m\} \qquad \text{III.54b}$$

$$\tilde{U}^* = \{\tilde{U}, \tilde{\Lambda}\} \qquad \text{III.54c}$$

$$\Delta\tilde{U}^* = \{\Delta\tilde{U}, \Delta\tilde{\Lambda}\}. \qquad \text{III.54d}$$

Hence we can write

$$T = T(U_0) + \nabla T \Delta U \qquad \text{III.55a}$$

$$H = H(U_0) + \nabla H \Delta U \qquad \text{III.55b}$$

so that q^* may be expressed as

$$q^* = (\tilde{E} + \Delta\tilde{U}\nabla\tilde{T})\,\Phi\,(E + \nabla T\Delta U) + (\tilde{\Lambda} + \Delta\tilde{\Lambda})\,(H + \nabla H\Delta U). \qquad \text{III.56}$$

Then the first l normal equations are

$$0 = 2\nabla\tilde{T}\Phi E + 2\nabla\tilde{T}\Phi\nabla T\Delta U + \tilde{\Lambda}\nabla H + \Delta\tilde{\Lambda}\nabla H \qquad \text{III.57a}$$

and the $l + 1^{\underline{th}}$ through $l + m^{\underline{th}}$ equations are

$$0 = H_0 + \nabla H \Delta U. \qquad \text{II.57b}$$

Equations III.57 are a system of $l + m$ equations in $l + m$ unknowns namely ΔU^*. To illustrate, the three variable system III.30 with one constraint like III.31 would read:

$$
\begin{bmatrix}
2(aa) & 2(ab) & 2(ac) & \frac{\partial h}{\partial u} \\
2(ba) & 2(bb) & 2(bc) & \frac{\partial h}{\partial v} \\
2(ca) & 2(cb) & 2(cc) & \frac{\partial h}{\partial w} \\
\frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta u \\ \Delta v \\ \Delta w \\ \Delta\lambda
\end{bmatrix}
= -
\begin{bmatrix}
2(a\epsilon) \\ 2(b\epsilon) \\ 2(c\epsilon) \\ 0
\end{bmatrix}
-
\begin{bmatrix}
\lambda \frac{\partial h}{\partial u} \\ \lambda \frac{\partial h}{\partial v} \\ \lambda \frac{\partial h}{\partial w} \\ h
\end{bmatrix}
\qquad \text{III.58}
$$

These equations provide a method of solution to system III.40 in the presence of a constraint. This system has four instead of two independent variables and is called an augmented system. Because of the increased number of variables, the method of Lagrange multipliers should obviously be avoided if possible. Another reason to avoid this approach is the need to estimate initial values for the λ's. This quantity relates the relative absolute magnitudes of the gradients of the constituent functions in q^*, the augmented error function. The λ's may sometimes be available in analytical form. Refer to page 44 of Goldstein's text [1]. Generally one must calculate the λ's employing special techniques. Computer simulation, for example, may provide tabular functions of λ.

IV. THE EULER ANGLES AS A FUNCTION OF TIME

A. Introduction - The Equations of Motion

Figure 2 illustrates a suitable definition of the three Euler angles. They serve to define the orientation of one frame of reference with respect to another. Notice that when $\theta = 0, \varphi$ is not defined. This circumstance can be troublesome if ignored [28, 40, 62]. For an example dealing with this case, see [41]. Refer also to Section IV. G. 1 below. These angles play a key role in attitude determination for they are the arguments of the ideal operator $\mathfrak{G}$. See Section II. L. Hence it is important to obtain the Euler angles as a function of time. For a moving vehicle these expressions are obtained upon solving the Euler differential equations of motion which are derived from the fundamental law of angular momentum:

$$\frac{dL}{dt} = M . \qquad \text{IV.1}$$

Refer to [1, 2, 3, 4].

B. The State Vector Form

In order to apply numerical integration techniques it is desirable to put equation IV.1 in the "state variable" form [28, 37, 38, 46, 61, 65]. (This can be done in several ways as discussed in [1, 2, 3, 4, 40].) The state variable form is

$$\frac{dV}{dt} = F(V(t),\ t) \qquad \text{IV.2}$$

where V is the array of six variables and F is a vector function with six components. (When t appears only through V(t) as in the force - free case, the system IV.2 is said to be autonomous [4, 61].)

It is shown in the references [1, 2, 3, and 4] that when the orientations of all vectors are expressed in the moving (body-fixed) system of coordinates, equation IV.1 becomes

$$a\dot{\Omega}_x = \Omega_y\ \Omega_z\ (b-c) + M_x \qquad \text{IV.3a}$$

$$b\dot{\Omega}_y = \Omega_z\ \Omega_x\ (c-a) + M_y \qquad \text{IV.3b}$$

$$c\dot{\Omega}_z = \Omega_x\ \Omega_y\ (a-b) + M_z \qquad \text{IV.3c}$$

where for convenience we have set $I_x = a$, $I_y = b$, $I_z = c$.

It is also shown in the references [4, 45] that

$$\dot{\varphi} = \Omega_x \sin\psi \cos\theta + \Omega_y \cos\psi \cos\theta \qquad \text{IV.3d}$$

$$\dot{\theta} = \Omega_x \cos\psi - \Omega_y \sin\psi \qquad \text{IV.3e}$$

$$\dot{\psi} = \Omega_x \sin\psi \cot\theta - \Omega_y \cos\psi \cot\theta + \Omega_z\ . \qquad \text{IV.3f}$$

C. The Force-Free Case

Equations IV.3a through f are in state variable form (IV.2) and are a system six first order nonlinear simultaneous differential equations. The solutions are available in special cases. For example, if M = 0, references 2, 3, and 4 give

$$\Omega_x(t) = p = p_0 \text{ cn } (s,\ k) \qquad \text{IV.4a}$$

$$\Omega_y(t) = q = q_0 \text{ sn } (s,\ k) \qquad \text{IV.4b}$$

$$\Omega_z(t) = r = r_0 \text{ dn } (s,\ k) \qquad \text{IV. 4c}$$

$$\cos\ \theta = cr/u \qquad \text{IV.4d}$$

$$\cos\ \psi = bq/ap \qquad \text{IV.4e}$$

$$\varphi = \mu \int_{t_0}^{t} \frac{2\ \upsilon - cr^2}{\mu^2 - c^2 r^2}\ dt' \qquad \text{Iv.4f}$$

$$s = (t - t_0)\ \sigma \qquad \text{IV.4g}$$

$$\sigma^2 = \frac{(b-c)\ (2a\upsilon - \mu^2)}{abc} \qquad \text{IV.4h}$$

$$k^2 = \frac{(a-b)\ (\mu^2 - 2c\upsilon)}{(b-c)\ (2a\upsilon - \mu^2)} \qquad \text{IV.4i}$$

$$\mu^2 = (ap_0)^2 + (bq_0)^2 + (cr_0)^2 \qquad \text{IV.4j}$$

$$2\upsilon = ap^2{}_0 + bq^2{}_0 + cr^2{}_0\ . \qquad \text{IV.4k}$$

In these expressions p_0, q_0, r_0 are constants; μ and υ are the absolute magnitudes of L and the kinetic energy of rotation, respectively; cn, sn, and dn are the Jacobi elliptic functions [21, 56, 57, 67]; k is the modulus of the elliptic functions; a, b, and c are the three principal moments of inertia.

To derive these equations an inertial reference frame whose z axis is collinear with the angular momentum is chosen [4]. This choice engenders no loss of generality but it is responsible for our need to work with an intermediate reference frame which we have previously designated as the "auxiliary reference frame."

As given by these equations, the Euler angles describe the general motion of a passive rigid spacecraft subject to no external forces. No assumption regarding dynamical balance has been made. Substitution of expressions IV.4 into II.11 and II.13, therefore, enables one to predict (calculate) the attitude of a spacecraft of the type stated.

D. The Constraints of Energy and Angular Momentum

Consider equations IV.4j and k. These expressions are the equations of three-dimensional ellipsoids, namely the momental and energy ellipsoids [3]. Since μ and υ are constant in the passive torque-free case, these expressions would also be ture if p_0, q_0, r_0 were replaced by p(t), q(t), r(t) and, hence, may be regarded as constraints. When the system parameters include p_0, q_0, r_0, these equations may be carried as constraint relations in the form

$$h_1(U) = {p_0}^2 a + {q_0}^2 b + {r_0}^2 c - 2\upsilon = 0, \qquad \text{IV.5.a}$$

$$h_2(U) = (p_0 a)^2 + (q_0 b)^2 + (r_0 c)^2 - \mu = 0 \qquad \text{IV.5.b}$$

If μ and υ are not known accurately, they must, of course, be included in the array of system parameters. Refer to Section V. A similar argument can be made for expressions IV.4.h and i. The advantages of dealing with equations of constraint may outweigh the disadvantages induced by the Lagrange multipliers. For, as will be observed in Section V.G.2, the calculations of analytical derivatives $\partial\tau/\partial u$ may be simplified.

E. A Note Concerning the Assumptions

The choice of the inertial z axis collinear with L deserves a word of caution. According to relations IV.4 which results from this choice, the orientation of the vehicle is defined with respect to a special coordinate system, previously called the angular momentum system. (See section IIK) Hence, the angles given by expressions IV.4d, IV.4e, IV.4f are not necessarily equivalent to those in the equations of motion IV.3. The latter system of equations may refer to an arbitrary frame of reference, such as the equatorial vernal equinox system. The significance of this observation becomes apparent upon considering initial conditions. In system IV.3, which is in "state variable form", we may arbitrarily assign six initial values of the six components of the state vector. In applying the formulas IV.4 however, it is not possible to assign values arbitrarily to the said state vector. For in employing these formulas, we are not dealing with a system of differential equations but a solution to such a system. Hence the assumptions employed in the solution must be respected.

For example, if the angular velocity Ω is taken as an initial condition, then the three Euler angles are uniquely determined. This means that the orientation of the vehicle is determined with respect to the vector L. It remains, therefore, to supply more information in order to define the orientation of L in the inertial coordinate system of our choice. Two parameters are necessary and sufficient for this purpose. Finally, one more parameter must be specified in order to define the orientation of the angular momentum reference frame with respect to a fixed rotation about L. These considerations are important in choosing an appropriate, independent, and complete set of parameters of the motion. Specific examples are given in Section V.

F. Force-Free Dynamical Balance

If the spacecraft is dynamically balanced, that is if two principal moments of inertia are equal, the foregoing expressions simplify. Without loss of generality, let c be the unique moment of inertia. Then the modulus k of the elliptic functions vanishes and these functions degenerate into ordinary circular functions. See [21], paragraph 16.6. Hence we may write:

$$\Omega_x = p_0 \cos(s) \qquad \text{IV.6a}$$

$$\Omega_y = q_0 \sin(s) = p_0 \sin(s) \qquad \text{IV.6b}$$

$$\Omega_z = r_0 \qquad \text{IV.6c}$$

$$\theta = \theta_0 \qquad \text{IV.6d}$$

$$\varphi = \dot{\varphi}(t-t_0) \qquad \text{IV.6e}$$

$$\cot \psi = \cot(s) \qquad \text{IV.6f}$$

$$s = (t-t_0)(a-c)\, r_0/a \qquad \text{IV.6g}$$

$$k = 0 \qquad \text{IV.6h}$$

where p_0, q_0, r_0, θ_0, t_0, $\dot{\varphi}$ are constants. Equation IV.6f may be replaced by:

$$\psi = \dot{\psi}(t - t_0) \qquad \text{IV.7a}$$

where

$$\dot{\psi} = \dot{\varphi}(a-c)(\cos\theta_0)/a \qquad \text{IV.7b}$$

Equation IV.7b is derived in [2]; section 64, and in [44], page 5. There are several additional interesting relationships in this case. Refer to [2]; section 64, [3]; section 69, and [4]; sections 5 and 6.

The equations necessary to construct a predictor for calculating the attitude of a rigid spacecraft subject to no torque, be it balanced or not, have now been stated. Needless to say, when torques are small these equations are still useful provided one does not attempt to fit sensor output data defined over too large an interval of time. This follows from the fact that the force-free parameters of the motion actually drift in time under the influence of the torque. No such fixed set of parameters could, therefore, be expected to provide a satisfactory fit to the raw data over large intervals of time. Parametrization of the attitude determination problem is, in this instance, better envisioned in terms of the parameters which characterize the driving terms M.

G. Inhomogeneous Cases

In the presence of torques it is often necessary to integrate the equations of motion IV.3 numerically. Integration is possible when the nature of the torques is known so that mathematical expressions for the driving terms can be written. These expressions may be functions of time, the six parameters of the motion and their time derivatives, engineering or calibration constants (design parameters), and well defined control signals. A well defined control signal is one which can be mathematically reconstructed as a function of time. This does not imply that they must be continuously differentiable functions of time. Like the driving terms, the control signals are often characterized by the square wave; they are "on" or "off". Due care must be given to the integration of the equations of motion subject to driving terms with discontinuous time derivatives.

In the examples to follow, we discuss several types of driving terms. Quantities are identified according to their suitability for numerical integration, i.e., according as to whether or not they are well defined.

Example 1. An Active Control System

Consider an attitude control system designed to align the body x axis with the solar line of sight. Let the body-fixed coordinate system be designated x, y, z and, for convenience, let the inertial reference frame X, Y, Z be chosen so that the solar line of sight is the X axis. The control law could be stated as:

$$x \times X = 0 . \qquad \text{IV.8}$$

In order for this expression to hold, a control torque must be applied to the space vehicle which tends to make this expression true. Such a torque might be:

$$M = -k_1 x \times X + H \qquad \text{IV.9}$$

where k_1 is a proportionality factor and H represents a damping torque. (Without damping or momentum removal, as it is sometimes called, the resulting motion would be oscillatory, of course). A damping torque of mathematical simplicity is provided by a set of gas jets activated by rate sensing gyros. The rate sensing gyros provide electrical signals proportional to the components of the vehicle angular velocity Ω. In order to neturalize Ω, the gas jets should exert a torque

$$H = -k_2 \Omega \qquad \text{IV.10}$$

where k_2 is also a constant. Our control torque may now be expressed

$$M = k_1 x \times X - k_2 \Omega. \qquad \text{IV.11}$$

In order to integrate M, it is expressed in the same coordinates as system IV.3, the vehicle coordinate system. Hence

$$M = -k_1 \begin{bmatrix} 0 \\ -X_3 \\ X_2 \end{bmatrix} - k_2 \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \qquad \text{IV.12}$$

(In this expression Ω_1 , the roll rate, could be ignored insofar as damping the control torque is concerned.) Now every quantity in this expression is well defined. k_1 and k_2 are design constants; Ω_1, Ω_2, Ω_3 are measured by gyros; X_2 and X_3 are the y, z components of solar radiation in the vehicle system and may be measured by two cosine solar sensors. (The components of Ω need not be telemetered or known for, as it happens, they are parameters of motion and are automatically well defined.)

A Note Concerning the "Gimbal Lock" and Auxiliary Reference Frame.

It is clear that the auxiliary reference frame in this problem is chosen so that solar radiation is parallel to the X or the Y axes. (The Z axis is excluded because, at acquisition, the correct attitude results in a vanishing second Euler angle. This implies that the Euler line of nodes is ill-defined (see Figure 2) and the equations of motion IV.3 cannot be integrated as is evident upon inspection of the coefficients in system IV.3d, e, f. (This occurrence is sometimes called the "gimbal lock" effect [28].) The foregoing choice of the auxiliary reference frame results in the simplest expressions possible for the driving terms. It may also be said to coincide with the terminal "angular momentum system" defined for spin stabilized vehicles. See Section II.k.

Example 2. A Passive Torque Model

A space vehicle with at least two distinct principal moments of inertia experiences a torque of gravitational origin. In the case of a central inverse square gravitational field and for a spacecraft of arbitrary shape but negligible size, this torque may be expressed as:

$$M_x = 3\gamma^2 \, m \, z' \, y' (c - b)/R^5 \qquad \text{IV.13a}$$

$$M_y = 3\gamma^2 \, m \, x' \, z' (a - c)/R^5 \qquad \text{IV.13b}$$

$$M_z = 3\gamma^2 \, m \, y' x' (b - a)/R^5 \qquad \text{IV.13c}$$

where γ m is the gravitational constant of the attracting body , R is the orbit radius vector; x' , y' , z' , its components in the body frame of reference. See chapters 18, 19 and 20, reference[4].

Expressing R in the body frame of reference by means of the coordinate transformation $\mathcal{E}$, we have

$$R' = \mathcal{E} \, R = \mathcal{A} \, \mathcal{L} \, R, \qquad \text{IV.14}$$

For example, with the help of equations II.10 and II .13.

$$\begin{aligned} x' = \; & (\cos\psi \cos\varphi - \cos\theta \sin\psi)(-x \sin\alpha + y \cos\alpha) \\ & + (\cos\psi \sin\varphi + \cos\theta \cos\varphi)(-x \cos\alpha \sin\delta - y \sin\alpha \sin\delta + z \cos\delta) \\ & + (\sin\psi \sin\theta)(x \cos\alpha \cos\delta + y \sin\alpha \cos\delta + z \sin\delta). \end{aligned} \qquad \text{IV.15}$$

In a similar manner, the reader may obtain expressions for y' and z' .

Thus, substituting in equations IV.13, we have a force model describing the torque acting on a vehicle in a central gravitational field. Every term in equation IV.15 is well defined: φ, θ, ψ are the Euler angles of equations IV.3 and x, y, z are available from independent knowledge of the ephemeris.

In the event that several attracting centers must be taken into account, their associated torques M are additive. In near earth orbits, the noncentral gravitational terms may acquire importance and the principle of linear superposition (addition) also applies to them [4].

Notice that equations IV.13 have an interesting interpretation when a = b. That is, in the case of dynamical symmetry, the gravitational torque about the axis of dynamical symmetry vanishes. Then, if one recalls that the vector cross product of the body symmetry axis and R is

$$\hat{k} \times R' = \begin{bmatrix} -y' \\ x' \\ 0 \end{bmatrix}, \qquad \text{IV.16}$$

we can say that equations IV.13 become

$$M = 3\, m\gamma^2 \,(a - c)\, z' \hat{k} \times R'/R^5 \qquad \text{IV.17}$$

which indicates that the instantaneous torque is normal to the plane containing the radius vector R and the axis of dynamical symmetry.

Equation IV.17 shows how "gravity capture" works: the stable orientation of a nonspinning spacecraft is one for which R and k are collinear. If, on the other hand, the spacecraft spins about its axis of symmetry, the projection of R on this axis is constant causing it to precess about R.

In near earth orbits the vector R cannot be considered constant, of course, and hence, for such orbits, the overall effect of the gravitational torque is more complicated. The net effect of this torque is obtained upon averaging over the period of an orbital revolution. See the article by R. J. Naumman in [71]. It is found that the rotational angular momentum precesses about the orbital angular momentum (or the normal to the orbit plane). In the absence of spin, i. e., for "gravity capture", the stable orientation is one for which the principal axis of greatest moment of inertia is aligned with the orbital angular momentum; the axis of least inertia, with the orbit radius vector. Consult [4, 29, 35, 36, 42, 70, 71, 72, 73, 74, 75, 76, 79].

Two important observations can be made about this type of perturbation. First, the gravitational effects are apt to be negligible in the presence of other forces such as aerodynamic, magnetic, and control forces [70, 71]. Second, because they are conservative, gravitational forces do not net work on a rigid spacecraft. This means that the total combined kinetic energy and angular momentum of the orbit and the attitude are conserved. The orbit attitude problems are, strictly speaking, no longer independent but are said to be coupled as a single 12-degree-of-freedom problem. Refer to [42]. Furthermore, because of their weakness and conservative nature, it is clear that 1) their effects on the motion about the center of mass should be periodic, 2) the first integrals [1, 2, 3, 4, 5] should be recognizable, though perturbed [4, 70], 3) the number of first integrals unchanged, and 4) the number of constants of integration unchanged. These observations are helpful in designing a prediction function without numerical integration. Refer to section V.E.3.

Example 3. <u>Torque Model for the On-board Flywheel</u>

—The driving terms $\dot{M}$ in equations IV.3 are now presented for the case of a vehicle possessing momentum storage devices such as flywheels and tape reels. The vehicle is otherwise assumed to be rigid. Here again, the basic principle is the constancy of the total angular momentum vector, L.

Let the main body inertia tensor and angular velocity be I and Ω respectively, Likewise, denote the inertia tensor and angular velocity of the k^{th} flywheel by I^k and Ω^k, respectively. These quantities are all referred to one coordinate reference frame, such as the main body coordinates. The total angular momentum is the sum of the angular momenta:

$$L = I\Omega + \sum_k I^k \, \Omega^k \qquad \text{IV.18}$$

Hence

$$\frac{dL}{dt} = I\dot{\Omega} + \Omega \times I\Omega + \Sigma(I^k\dot{\Omega}^k + \Omega^k \times I^k\Omega^k) \qquad \text{IV.19}$$

If we invoke the principle of vector addition for angular velocities [1], then

$$\Omega^k = \Omega + \omega^k \qquad \text{IV.20}$$

which states that the angular velocity of flywheel k is the vector sum of the angular velocity of the main body and the velocity, ω^k , of the wheel with respect to the main body. Substituting IV.20 into IV.19, we have

$$\frac{dL}{dt} = (I + \Sigma I^k)\dot{\Omega} + \Omega \times (I + \Sigma I^k)\Omega + \Sigma I^k \dot{\omega}^k$$

$$+ \Omega \times \Sigma I^k \omega^k + \omega^k \times \Sigma I^k \Omega + \omega^k \times \Sigma I^k \omega^k. \qquad \text{IV.21}$$

Let us identify the coefficient of $\dot{\Omega}$ as the modified inertia tensor. Then, except for the last four terms, equations IV.21 resemble equations IV.3. It follows that the driving terms are provided by the said four terms.

In applying these equations, however, we should point out that the inertia tensor which results from

$$I^* = I + \Sigma I^k \qquad \text{IV.22}$$

is not necessarily diagonal. Hence, the equations which would result form the vector relation IV.21 would be complicated. For example, if I is diagonal, then the first equation reads:

$$\frac{dL_1}{dt} = I_1\dot{\Omega}_1 + \Sigma(I^k_{11}\dot{\Omega}_1 + I^k_{12}\dot{\Omega}_2 + I^k_{13}\dot{\Omega}_3)$$

$$+ \Omega_2\Omega_3(I_3 - I_2) + \text{etc.} \qquad \text{IV.23}$$

This system is not convenient for numerical integration since the derivatives of the state variables are not isolated. Evidently it is necessary to find a reference frame in the main body which diagonalizes I*. When this is done, we obtain:

$$\frac{dL_1}{dt} = I_1^* \dot{\Omega}_1 + \Omega_2 \; \Omega_3 (I_3^* - I_2^*) + (\Sigma \; I^k \; \dot{\omega}^k)_1^*$$

$$+ \; (\Omega \times \Sigma I^k \omega^k)_1^* + \text{ etc.} \qquad \text{IV.24}$$

Strictly speaking the I^k s are now in a different reference frame, and, like I*, they should also have asterisks. If the masses of the flywheels are small compared to that of the main body, there will be little difference between the two coordinate systems. It could be that the axes of the wheels are parallel, or at least nearly so, to the assembled vehicle body principal axes. Hence, it may sometimes be justifiable to ignore from the start the non-diagonal terms in equation IV.23.

In equations IV.24 the ω's and $\dot{\omega}$'s are known from the tachometer telemetry for these are the wheel rates and accelerations with respect to the main body. The driving terms are left in vector notation for brevity.

Several examples of vehicles with on-board torques have been solved analytically so that numerical integration may be avoided. Consult Chapter 10 in reference [4]. The case of an on-board tape recorder has been integrated with analog computers by personnel at Allied Research. See reference [45].

H. Additional Material on Torques

Additional material about forces acting on spacecraft may be found in the following references:

Gravitational forces: [4, 29, 35, 36, 42, 53, 70, 71, 72, 73, 74, 75, 76, 78, 79]

Magnetic forces: [34, 40, 45, 49, 53, 61, 78, 79]

Electric forces: [53, 77, 78, 79]

Aerodynamic forces: [53, 70, 71, 78, 80 volume VIII and XI)]

Self-Excitation: [4, 78, 79, 80]

Radiation: [53]

V THE SYSTEM PARAMETERS

A. Properties

We have seen that the differential correction method leads to a set of linear simultaneous equations requiring a set of independent parameters. See Section III.C. There are two additional conditions which should be satisfied. The computational scheme implies that the parameters are constants of the motion. Furthermore, since it is sought to find the minimum of a function $q\ (u_1, u_2, \ldots)$, one must insure that a sufficient number of parameters have been taken into account. Should an important parameter be overlooked, the computational scheme cannot be expected to locate the true minimum. Thus we have our third property: the parameters should be a complete set. Examples of complete and independent parameter sets are discussed below.

B. Cascading

It is not implied, however, that all parameters ought to be adjusted simultaneously. If a given parameter is known accurately, it may be held constant during the computation. In actual practice this procedure is sometimes indeed necessary. At first, those parameters known accurately are held constant in order to relax (obtain approximate values for) others not so accurately known. Then the procedure is repeated, allowing freedom to all parameters for final vernier adjustment. This technique is sometimes called "cascading."

C. Constants of Integration and the First Integrals

The parameters have thus been found to be a complete and independent set of constants of the motion appearing in the expression for the squared error function q. That is, they appear in the predictor operator, $\mathfrak{F}$.

Notice that since the predictor embodies the solutions of the differential equations of motion, the constants of integration are candidates for out set of parameters U.

It follows that, in the torque-free case at least, the first integrals are likewise elligible. (First integrals are defined in [5] and discussed in [1, 2, 3, 4].) Whether one chooses to work with the six initial conditions to the equations of motion or the independent first integrals, depends on the circumstances. Initial conditions are easier to estimate and visualize. But the physical interpretation of the equations of motion is most natural in terms of the first integrals.

To illustrate in the absence of an external torque, L is constant and the expression for angular momentum L is a first integral [1, 4]. Should our predictor need to account for a small external torque M, it is advantageous to interpret the resulting motion in terms of the fundamental dynamical law of angular momentum.

$$\frac{dL}{dt} = M \cdot \qquad \text{V.1a}$$

This equation is approximated by the finite difference form

$$\frac{\Delta L}{\Delta t} = M \cdot \qquad \text{V.1b}$$

Now consider the attitude problem divided into several parts each covering a (possibly overlapping) time period Δt so small that ΔL is also small. The solutions L are then obtained as a discrete sequence of "osculating solutions" which can be smoothed. This method provides both the attitude of the spacecraft and an empirical determination of M. Conversely, if M is known, the force-free parameters of the motion may be expressed as function of time in order to construct a closed form prediction function which continuously "follows" the forced motion. Such a prediction function might be called one of "constant osculation." Refer below in section V.E.3 for an illustration.

D. On the Number of Independent Parameters

It is known [1] that six arbitrary (independent) constants exactly determine the unperturbed motion of a rigid body about its center mass. (This is the same number of constants that determine the orbit.) This is not to say, however, that an attitude problem may not include more than six parameters. While in the zero-torque case there cannot be more than six parameters of the orientation, there are numerous other constants of the motion which may be included in our set of parameters. In other words, it is possible to choose quantities to be adjusted besides merely those associaged with the motional degrees of freedom of a rigid body. The two independent ratios between the three principal moments of inertia, for example, which serve to define the distribution of mass, maybe included. Thus, the number of degrees of freedom in the definition of the least square formulation of the problem may be equal to or greater than six; the degrees of freedom of the differential equations of motion.

Notice that the number of parameters adjusted in the example discussed with equations III.45 was six. Had we overlooked the equation of constraint, seven orientation parameters would have appeared. The existence of a constraint could, therefore, have been inferred.

Eugene Leimanis [4] states that the equations of motion have but five independent first integrals or arbitrary constants of integration. This statement is reconciled with the foregoing discussion in Appendix A.

The reader may be interested to note the distinction between various types of degrees of freedom appearing in this paper. First, we have the three degrees of freedom of a rigid body [1, 3] about its center of mass. Second, there are said to be six degrees of freedom in the differential equations of motion. Third, there may be six or more degrees of freedom in the statement of the least squares problem. That is, q may be thought of as a function of six or more parameters.

E. Specific Parameter Sets

1. General Case

a. Inhomogeneous Equations of Motion

We state several possible sets of independent parameters suitable for our Eulerian definition of the attitude problem. The simplest set is the set of six initial components of the "state vector" V in equation IV.2. This amounts to stating the initial angular velocity vector in body coordinates and the three Euler angles at that instant of time. This approach is most convenient when equations of motion IV.3 are to be integrated numerically. In applying this set of initial conditions, we should note that the assumption that L is collinear with the inertial z axis is not made. (This assumption is made to simplify the integration. See page 17, reference [4]. Therefore, the Euler angles φ_0, θ_0, and ψ_0 in equations IV.4 are not equivalent to those in equations IV.3. In this case we have:

$$u_1 = \Omega_1 (0) \qquad \text{V.2a}$$

$$u_2 = \Omega_2 (0) \qquad \text{V.2b}$$

$$u_3 = \Omega_3 (0) \qquad \text{V.2c}$$

$$u_4 = \varphi_0 \qquad \text{V.2d}$$

$$u_5 = \theta_0 \qquad \text{V.2e}$$

$$u_6 = \psi_0 \qquad \text{V.2f}$$

b. Homogeneous Equations of Motion

In the event we wish to apply the torque-free solutions IV.4, a suitable set of parameters is given by the following set:

$$u_1 = \Omega_1(0) = p\,(0) \qquad \text{V.3a}$$

$$u_2 = \Omega_2\,(0) = q\,(0) \qquad \text{V.3b}$$

$$u_3 = \Omega_3\,(0) = r\,(0) \qquad \text{V.3c}$$

$$u_4 = \alpha \qquad \text{V.3d}$$

$$u_5 = \delta \qquad \text{V.3e}$$

$$u_6 = \beta \qquad \text{V.3f}$$

In contrast to the preceeding case, we have refrained from specifying the Euler angles because they are uniquely determined by equations IV.4 once the angular velocity is defined. As already explained in section IV.E, although they dynamical aspects of the motion are defined with respect to L, it is also possible to arbitrarily assign the orientation of L as well as a fixed rotation β about L. Hence α, δ, and β are the last three parameters.

The parameter set V.3 is equivalent to the following set:

$$u_1 = \Omega_1\,(0) \qquad \text{V.4a}$$

$$u_2 = \Omega_3\,(0) \qquad \text{V.4b}$$

$$u_3 = t_0 \qquad \text{V.4c}$$

$$u_4 = \alpha \qquad \text{V.4d}$$

$$u_5 = \delta \qquad \text{V.4e}$$

$$u_6 = \beta \qquad \text{V.4f}$$

In this set of parameters, the time t_0 is adjusted independently. This amounts to seeking the instant of time when the angular velocity lies in the vehicle x, z plane. It is erroneous, therefore, to select both Ω_2 and t_0 as parameters of the motion because they are related by the equation

$$\Omega_2 \ (t_0) = 0 . \qquad \text{V.5}$$

Refer to Page 20 in the work by Eugene Leimanis [4].

Parameters of special physical interest are the first integrals of equations IV.2. (For the definition of a first integral consult page8, reference [5].)

In the homogeneous case, two first integrals are the expressions for the kinetic energy of rotation, υ, and the magnitude, μ , of the angular momentum:

$$2\,\upsilon = a\,p^2 + b\,q^2 + c\,r^2 \qquad \text{V.6a}$$

$$\mu^2 = (a\,p)^2 + (b\,q)^2 + (c\,r)^2 \qquad \text{V.6b}$$

where p, q, and r are the components of Ω.

If υ and μ are specified, expressions of V.6 are two equations in three unknowns p, q, and r. With one more equation, the unknowns would be determined and the motion with respect to the angular momentum would, therefore, also be determined. The third equation could be:

$$q\ (t_0) = \Omega_2\,(t_0) = 0. \qquad \text{V.7}$$

It follows that t_0 is a third parameter. Arguing as before, the remaining three parameters are those specifying the orientation of the angular momentum reference frame. Thus, we have a possible set of parameters:

$$u_1 = \upsilon \qquad \text{V.8a}$$

$$u_2 = \mu \qquad \text{V.8b}$$

$$u_3 = t_0 \qquad \text{V.8c}$$

$$u_4 = \alpha \qquad \text{V.8d}$$

$$u_5 = \delta \qquad \text{V.8e}$$

$$u_6 = \beta \quad . \qquad \text{V.8f}$$

(First integrals can sometimes be found also for the nonhomogeneous case [4, 70]. In [70] see page 47.)

2. Dynamical Balance

In this case the preceding parameter sets are also applicable. But if the equations of motion are homogeneous, i.e. if $M = \vec{0}$, then the Euler angles are given by the simple equations IV.6 and IV.7. Hence the most plausible set of parameters becomes:

$$u_1 = \varphi_0 \qquad \text{V.9a}$$

$$u_2 = \theta_0 \qquad \text{V.9b}$$

$$u_3 = \psi_0 \qquad \text{V.9c}$$

$$u_4 = \dot{\varphi} \qquad \text{V.9d}$$

$$u_5 = \alpha \qquad \text{V.9e}$$

$$u_6 = \delta \qquad \text{V.9f}$$

As stated above, $\dot{\varphi}$ and $\dot{\psi}$ are related by the equation of constraint; equation III.46. It is sufficient to include only one rate in the array U, the other rate being computed from the equation of constraint.

3. A Predictable Inhomogeneous Case

We wish to find a parameter set suitable for the determination of the attitude of a spinning space vehicle subject to torques having a known overall effect upon the motion. For instance, gravitational torque causes regular precession of L about the orbit normal P for earth centered circular orbits. We would like, therefore, to calculate the coordinates of L, α (t) and δ (t), in order to construct our prediction function $\mathcal{E} = \mathcal{A}\mathcal{L}$ (t). We use the coordinate transformation Γ which transforms a vector L from the vernal equinox system to the orbit oriented system (see Appendix D). Let

$$N \equiv \Gamma L \qquad \text{V.10}$$

$$N_1 \equiv (N_x^2 + N_y^2)^{\frac{1}{2}}. \qquad \text{V.11}$$

Then, for circular orbits, the rate of precession ω of L about P is constant so that

$$N = \begin{bmatrix} N_1 \cos(\omega t + \beta) \\ N_1 \sin(\omega t + \beta) \\ N_z \end{bmatrix}. \qquad \text{V.12}$$

We can now write

$$L = \Gamma^{-1} N \qquad \text{V.13}$$

$$\alpha(t) = \arctan\ (L_y / L_x) \qquad \text{V.14b}$$

$$\delta(t) = \arcsin\ (L_z / |L|). \qquad \text{V.14c}$$

If the functions V. 14 are used with a set of parameters like V. 3, 4, 5, 8 or 9, the resulting prediction function is one of "constant osculation." Such a set enables us, in the absence of other perturbations, to construct a prediction function without recourse to numerical integration.

The rate of precession ω can be calculated from the fact that

$$\omega \times L = dL/dt = \text{average} \{M\} \qquad \text{V. 15}$$

where M is the gravitational torque IV. 17. The phase angle β is determined from geometrical considerations. For example, if L is known at some instant of time, say t_0, then

$$\beta = \arctan (N_y/N_x) - \omega t_0 . \qquad \text{V. 16}$$

F. The Gradients

Expressions for an ideal attitude prediction function $\mathcal{E}$ based on the Euler transformation have now been stated. In the torque-free case the Euler angles have been given as continuous functions of time. We consider next the gradients which are needed in the computational scheme of differential correction. This scheme requires knowledge of the gradients of the quantities τ which represent the predicted values of observed variables. Refer to section III.D. According to that discussion, we require

$$\nabla\tau = \{ \frac{\partial \tau}{\partial u_1}, \frac{\partial \tau}{\partial u_2}, \ldots \}.$$

The primary purpose of this section is to exhibit in detail the differentiation needed to solve the force-free attitude problem based on the ideal prediction function $\mathcal{E}$ for both the balanced and nonbalanced cases. One way to do this is to write down the explicit form of $\mathcal{E}$ and proceed undaunted with the laborious differentiation. We wish, however, to steer the approach toward a system capable of handling more complicated attitude problems. The present discussion will, moreover, suggest a foundation for the mechanization of the general attitude problem.

Let $\mathfrak{F}$ be an operator and S' its formal (vector) operand both of which may be functions of the "system parameter" U. As in section II.H, we have

$$\tau(U) = \mathfrak{F}(U) \otimes S'(U) \qquad \text{V.17}$$

where $\otimes$ emphasizes the operator-operand relationship. In analogy with the definition of ∇ (see Preface), let the symbol $\Box$ denote partial differentiation of $\mathfrak{F}$ with respect to its formal operand so that the following equation shall be true

$$\nabla\tau \equiv [\nabla\mathfrak{F}] \otimes S' + \Box\mathfrak{F} \otimes \nabla S'. \qquad \text{V.18}$$

When an operator is differentiated, a new operator is formed. Hence it is convenient to define the new operators

$$\mathfrak{F}_1 \equiv \Box\, \mathfrak{F}, \qquad \text{V.19a}$$

$$\mathfrak{F}_2 \equiv [\nabla \mathfrak{F}] \qquad \text{V.19b}$$

so that

$$\nabla \tau = \mathfrak{F}_1 \otimes \nabla S' + \mathfrak{F}_2 \otimes S'. \qquad \text{V.20}$$

Notice that if the system parameters are arranged so that those affecting S' appear before those which affect $\mathfrak{F}$ alone, then equation V.20 becomes

$$\nabla \tau = \mathfrak{F}_1 \otimes \nabla S' + \begin{bmatrix} 0 \\ \\ \mathfrak{F}_2 \otimes S' \end{bmatrix}. \qquad \text{V.21}$$

When there are no modification parameters the second member vanishes. That is to say, the first member represents the ideal differentiation while the second member alone accounts for nonideal effects.

In view of equations II.3, 4, 6, we have

$$\nabla \tau \equiv \mathcal{G}_1 \otimes \nabla f + \mathcal{G}_2 \otimes f \qquad \text{V.22a}$$

$$\nabla f \equiv \mathcal{K}_1 \otimes \nabla S' + \mathcal{K}_2 \otimes S' \qquad \text{V.22b}$$

$$\nabla S' \equiv \mathcal{E}_1 \otimes \nabla S + \mathcal{E}_2 \otimes S \qquad \text{V.23}$$

Equations V.22, 23 suggest the way in which the differentiation of nonideal attitude problems can be analyzed. In most cases several terms will reduce to zero or, alternatively, to the identity operation. We are interested here mainly in calculating the last term in equation V.23, the only one needed to handle a force-free spacecraft equipped with ideal sensors. (The first term in this equation can be nonzero for a force-free spacecraft equipped with optical, infrared, or other pulsed output instruments. Refer to section VI.E)

G. Calculation of the Gradients of the Ideal Prediction Function

1. Dynamical Balance

We consider first the case of dynamical balance because of its simplicity and because it serves as a model for the laborious case of a nonbalanced body which follows. This discussion is limited to the parametrization recommended in section V.E.2. For simplicity, the derivative of $\mathcal{E}$ with respect to a certain variable u is denoted by $\mathcal{E}_u$ and the derivative of a transformation, say $\mathfrak{B}$, with respect to its only argument is denoted by $\mathfrak{B}'$. Recalling equations II.10, II.11, II.12, we have:

$$
\nabla \mathcal{E} = \begin{bmatrix} \mathcal{E}_\alpha \\ \mathcal{E}_\delta \\ \mathcal{E}_\varphi \\ \mathcal{E}_\theta \\ \mathcal{E}_\psi \\ \mathcal{E}_{\dot\varphi} \end{bmatrix} = \begin{bmatrix} \mathfrak{A}\,\mathcal{L}_\alpha \\ \mathfrak{A}\,\mathcal{L}_\delta \\ \mathfrak{B}\,C\,\mathfrak{D}'\,\mathcal{L} \\ [\mathfrak{B}\,C' + \frac{d\dot\psi}{d\theta}\,\mathfrak{B}_{\dot\psi}\,C]\,\mathfrak{D}\,\mathcal{L} \\ \mathfrak{B}'\,C\,\mathfrak{D}\,\mathcal{L} \\ [\mathfrak{B}\,C\,\mathfrak{D}_{\dot\varphi} + \frac{d\dot\psi}{d\dot\varphi}\,\mathfrak{B}_{\dot\psi}\,C\,\mathfrak{D}]\,\mathcal{L} \end{bmatrix} \quad \begin{matrix} \text{V.24a} \\ \text{V.24b} \\ \text{V.24c} \\ \text{V.24d} \\ \text{V.24e} \\ \text{V.24f} \end{matrix}
$$

Equations V.24a, b, c, and e are trivial to evaluate. In equations V.24d and V.24f, however, we have used the chain rule. The reason for the chain rule is explained in sections III.E and III.F.

In order to evaluate V.24d and V.24f explicitly, recall the equation relating $\dot\varphi$ and $\dot\psi$, namely equation III.46 or IV.7b. Upon differentiation we obtain

$$\frac{d\dot{\psi}}{d\theta} = -\dot{\varphi}\sin\theta\,(a-c)\,/\,a \qquad \text{V.25a}$$

$$\frac{d\dot{\psi}}{d\dot{\varphi}} = \cos\theta\,(a-c)\,/\,a. \qquad \text{V.25b}$$

Furthermore, in view of equation IV.6e, we may write

$$\frac{d\mathfrak{D}}{d\dot{\varphi}} = \frac{d\varphi}{d\dot{\varphi}}\,\mathfrak{D}' = t\mathfrak{D}' \qquad \text{V.26a}$$

and similarly, in view of equation IV.7a,

$$\frac{d\mathfrak{B}}{d\dot{\psi}} = t\,\mathfrak{B}'. \qquad \text{V.26b}$$

As far as the six parameters of the orientation are concerned, we have now stated all the gradients needed in the case of dynamical balance without torques.

2. Nonbalanced Force-Free Case

To begin with, observe that if our array of system parameters is like V.2, then the following gradients are trivial:

$$\mathcal{E}_{\alpha} = \mathfrak{A}\,\mathfrak{L}'_{\alpha} \qquad \text{V.27a}$$

$$\mathcal{E}_{\delta} = \mathfrak{A}\,\mathfrak{L}_{\delta} \qquad \text{V.27b}$$

$$\mathcal{E}_{\varphi} = \mathfrak{B}\,\mathfrak{C}\,\mathfrak{D}'\,\mathfrak{L} \qquad \text{V.27c}$$

$$\mathcal{E}_{\theta} = \mathfrak{B}\,\mathfrak{C}'\,\mathfrak{D}\,\mathfrak{L} \qquad \text{V.27d}$$

$$\mathcal{E}_{\psi} = \mathfrak{B}'\,\mathfrak{C}\,\mathfrak{D}\,\mathfrak{L} \qquad \text{V.27e}$$

$$\mathcal{E}_{\beta} = \mathfrak{A}\,\mathfrak{B}'_{1}\,\mathfrak{L} \qquad \text{V.27f}$$

where $\mathcal{B}_1$ is a transformation like II.11a with β in place of ψ. But if our array of system parameters contains p_0, q_0, r_0, or t_0, the derivatives are very much more complicated. For in this case the force-free expressions for the Euler angles are given by equations IV.4. Consider, for example, the total derivative of θ with respect to r_0:

$$\theta_{r_0} = -\csc\theta(\mathrm{dn}(s)\ \frac{c}{\mu} + r_0\ \frac{c}{\mu}\ [\ \frac{\partial s}{\partial\sigma}\frac{d\sigma}{dr_0} + \frac{\partial s}{\partial k}\ \frac{dk}{dr_0}\ ]\ d\,n'(s) - r(s)\ \frac{c}{\mu^2}\ \frac{d\mu}{dr_0})$$

The most expedient approach seems to be to regard σ, k, μ, υ as independent parameters so that the chain rule can be relaxed.

We propose, therefore, to adopt expressions IV.4h through IV.4k as constraints. For simplicity the subscripts "0" are deleted from the ensuing equations whenever there is no confusion between the system parameter, say p_0, and the corresponding function p(s). Hence our arrays of constraints and system parameters are, respectively,

$$h_1 \equiv \sigma^2 - \sigma^{*2} = 0 \qquad \text{V.29a}$$

$$h_2 \equiv k^2 - k^{*2} = 0 \qquad \text{V.29b}$$

$$h_3 \equiv \mu^2 - \mu^{*2} = 0 \qquad \text{V.29c}$$

$$h_4 \equiv 2\upsilon - 2\upsilon^* = 0 \qquad \text{V.29d}$$

$$\underset{\cdot}{H} \equiv \{h_1, h_2, h_3, h_4\} \qquad \text{V.29e}$$

and

$$\widetilde{U} \equiv \{p,\ r,\ t,\ \alpha,\ \delta,\ \beta,\ \sigma,\ k,\ \mu,\ \upsilon,\ \widetilde{\Lambda}\} \qquad \text{V.30}$$

where the array

$$\tilde{\Lambda} \equiv \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \qquad \text{V.31}$$

is the array of Lagrange multipliers. In equations V.29, the asterisk is to remind us that the symbol in question represents the appropriate expression from the set IV.4h, i, j, and k.

Our aim is to set up explicit expressions for equations III.57a, b which are the normal equations in the presence of constraints. Since in the torque-free case $\mathcal{L}$ is not a function of p, q, r, and t, its derivatives are trivial and therefore we concentrate our attention on the Euler transformation $\mathcal{C}$. Let u stand for any one of our system parameters. Upon differentiating this transformation, we obtain

$$\mathcal{C}_u = \mathcal{B}\mathcal{C}\mathcal{D}'\varphi_u + \mathcal{B}\mathcal{C}'\mathcal{D}\,\theta_u + \mathcal{B}'\mathcal{C}\mathcal{D}\,\psi_u. \qquad \text{V.32}$$

Evidently we are to calculate the derivatives of the Euler angles with respect to each parameter in the system V.30. We are also required to calculate the derivatives of H. The differentiation is simplified if we adopt the following definitions:

$$w \equiv 2a\upsilon^* - \mu^{*2} \qquad \text{V.33a}$$

$$x \equiv \mu^{*2} - 2c\upsilon^* \qquad \text{V.33b}$$

$$y \equiv 2\upsilon^* - cr^2 \qquad \text{V.33c}$$

$$z \equiv \mu^{*2} - c^2 r^2 \qquad \text{V.33d}$$

$$J(m, n) \equiv \int_{t_0}^{t} \frac{zm - yn}{z^2}\, dt \qquad \text{V.33e}$$

Notice that the partial derivatives of w, x, y, z satisfy the following relationships:

$$w_p = w_t = x_r = x_t = 0 \qquad \text{V.34a}$$

$$w_r = 2cr\,(a-c) \qquad \text{V.34b}$$

$$x_p = 2ap\,(a-c) \qquad \text{V.34c}$$

$$y_p = 2ap(s) \qquad \text{V.34d}$$

$$y_r = 2cr\,(s)\,(1 - dn(s)) \qquad \text{V.34e}$$

$$z_p = ay_p \qquad \text{V.34f}$$

$$z_r = cy_r \qquad \text{V.34g}$$

$$y_s = 2cr_0 r \frac{d}{ds}\, dn(s) \qquad \text{V.34h}$$

$$z_s = cy_s \qquad \text{V.34i}$$

$$y_k = y_s \frac{ds}{dk} \qquad \text{V.34j}$$

$$z_k = cy_k. \qquad \text{V.34k}$$

The derivatives of the elliptic functions are:

$$\frac{d}{du}\, cn(s) = -\, sn \cdot dn \cdot s_u \qquad \text{V.36a}$$

$$\frac{d}{du}\, sn(s) = cn \cdot dn \cdot s_u \qquad \text{V.36b}$$

$$\frac{d}{du}\, dn(s) = -k^2 \cdot sn \cdot cn \cdot s_u \qquad \text{V.36c}$$

where the symbol u stands for p_0, r_0, t_0, k. Refer to [21, 56, 57, 67]. The partial derivatives of s are readily obtained from equation IV.4g.

The nonvanishing derivatives of s are:

$$s_{t_0} = -\sigma \qquad \text{V.37a}$$

$$s_{\sigma} = -t_0. \qquad \text{V.37b}$$

(It is worthwhile to notice here that the derivatives of s would have been singularly complicated had k, υ, and μ not been taken as independent parameters. For example, the total derivative of s with respect k is given by Arthur Cayley [56, paragraph 74] as

$$\frac{ds}{dk} = \frac{1}{k\,k'^2}\,(e - k'^2 f) - \frac{k \cdot sn \cdot cn}{k'^2} \qquad \text{V.38}$$

where k' is the complementary modulus of the elliptic functions, e and f are the Jacobi elliptic integrals of the second and first kinds, respectively, taken from t_0 to t.)

We are now ready to write the derivatives of the Euler angles given by IV.4. The required derivatives are:

$$\theta_p = \theta_\upsilon = 0 \qquad \text{V.39a}$$

$$\theta_r = -\csc\theta \cdot \frac{c}{\mu} \cdot dn(s) \qquad \text{V.39b}$$

$$\theta_u = -\csc\theta \cdot \frac{c}{\mu} \cdot r_s\, s_u;\; u = t_0,\ \sigma \qquad \text{V.39c}$$

$$\theta_\mu = \csc\theta \cdot \frac{c}{\mu^2} \cdot r \qquad \text{V.39d}$$

$$\psi_p = \csc^2 \psi \cdot \frac{b}{a} \cdot q\, p^{-2} \cdot \mathrm{cn}(s) \qquad \text{V.39e}$$

$$\psi_r = \psi_\mu = \psi_\upsilon = 0 \qquad \text{V.39f}$$

$$\psi_u = -\csc^2 \psi \cdot \frac{b}{a} \cdot \left(\frac{pq_s - qp_s}{p^2}\right) s_u ;\ u = t_0,\ \sigma \qquad \text{V.39g}$$

$$\varphi_u = \mu\ J\,(y_u,\ z_u);\ u = p,\ r \qquad \text{V.39h}$$

$$\varphi_{t_0} = \mu\left(\frac{y(0)}{z(0)} + J\,(y_s,\ z_s)\ s_{t_0} \right) \qquad \text{V.39i}$$

$$\varphi_\sigma = \mu J\,(y_s,\ z_s)\, s_\sigma \qquad \text{V.39j}$$

$$\varphi_\mu = \frac{\varphi}{\mu} + \mu\ J\,(y_\mu,\ z_\mu) \qquad \text{V.39k}$$

$$\varphi_\upsilon = \mu\ J\,(y_\upsilon,\ z_\upsilon)\,. \qquad \text{V.39l}$$

This completes the expressions required to evaluate equation V.32, namely the derivatives of the Euler transformation.

In order to apply equations III.57, we are further asked to evaluate the matrix ∇H. Again we avail ourselves of the definitions V.33 and the results V.34, Then the matrix ∇H is:

$\frac{\partial H}{\partial}$ \ H	p_0	r_0	t_0	α	δ	β	σ	k	μ	υ	λ_1	λ_2	λ_3	λ_4
h_1	0	$-2r_0 \frac{(b-c)(a-c)}{ab}$	0	0	0	0	2σ	0	0	0	0	0	0	0
h_2	$2ap_0 \frac{(c-a)(a-b)}{w\,(b-c)}$	$2cr_0 \frac{x(c-a)\,(a-b)}{w^2(b-c)}$	0	0	0	0	0	$2k$	0	0	0	0	0	0
h_3	$-2\,a^2 p_0$	$-2\,c^2 r_0$	0	0	0	0	0	0	2μ	0	0	0	0	0
h_4	$-2\,ap_0$	$-2\,cr_0$	0	0	0	0	0	0	0	2	0	0	0	0

V.40

This completes the derivatives needed to carry out differential correction with constraints in the case of the force-free rigid spacecraft.

VI. THE NONIDEAL ATTITUDE PROBLEM AND SPECIAL TOPICS

A. Preface

The preceding chapters describe the basic needs for solving an attitude problem defined in terms of the Euler angles, the Euler coordinate transformation, Gauss least squares, and Newton differential correction. In this chapter, the effect of nonideal sensor output functions are considered. This subject is distinct from the former because it is concerned with sensors and the ways in which their properties affect the computations. It is not concerned with our understanding of the motion itself or the definitive prediction of it. Thus it is conceivable to have near perfect knowledge of the dynamics of the motion and yet be unable to determine the motion because of unsatisfactory knowledge of sensor functions. The present discussion, therefore, is concerned with the "what," "when," "where" instead of the "why" and "how." Other topics of special interest are also considered.

Sensor output functions are nonideal for several reasons i) they are subject to errors, ii) they are often complicated (nonlinear) functions of the quantities they are supposed to measure, and iii) they can lead to intractable mathematical expressions. These difficulties are illustrated in these paragraphs and methods for dealing with them are suggested. But first we consider how nonideal factors affect the normal equations.

B. The Normal Equations in the Presence of Modification Parameters

Let $\widetilde{U} = \{u_1, u_2, \ldots u_6\}$ represent the array of six system parameters, such as those discussed in section V.E, which define the force-free motion of the spacecraft and define the array $C \equiv \{c_1, c_2, \ldots c_m\}$ of m modification parameters which characterize the operator $\mathfrak{G}$ or $\mathfrak{H}$. Our new array of system parameters becomes

$$\widetilde{U}^* \equiv \{\widetilde{U}, \widetilde{C}\} \qquad \text{VI.1}$$

and the gradient operator becomes

$$\widetilde{\nabla} \equiv \{\widetilde{\nabla}_u, \widetilde{\nabla}_c\}. \qquad \text{VI.2}$$

We wish to obtain the normal equations III.24 when the modification parameters are included in the computations. For the sake of brevity, let us recall equations III.25:

$$N \equiv \nabla_u \widetilde{T} \Phi \nabla_u T \qquad \text{VI.3}$$

$$\Gamma \equiv \nabla_u \widetilde{T} \Phi E. \qquad \text{VI.4}$$

In a similar fashion, let us define

$$M \equiv \nabla_c \widetilde{T} \Phi \nabla_c T \qquad \text{VI.5}$$

$$\Lambda \equiv \nabla_c \widetilde{T} \Phi E \qquad \text{VI.6}$$

$$P \equiv \nabla_u \widetilde{T} \Phi \nabla_c T. \qquad \text{VI.7}$$

Drawing on the arguments leading to equation III.24, we may write the modified normal equations as

$$\begin{bmatrix} N & P \\ \widetilde{P} & M \end{bmatrix} \begin{bmatrix} \Delta U \\ \Delta C \end{bmatrix} = - \begin{bmatrix} \Gamma \\ \Lambda \end{bmatrix}. \qquad \text{VI.8}$$

Recalling the relationships V.21, 22, 23 and keeping in mind the partioning shown in equation VI.1, let us calculate these arrays. For instance,

$$\begin{bmatrix} \Gamma \\ \Lambda \end{bmatrix} = \left(\begin{bmatrix} \mathcal{G}_1 \mathcal{K}_1 \mathcal{E}_2 \widetilde{S} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{G}_2 \widetilde{F} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{G}_1 \mathcal{K}_2 \widetilde{S} \end{bmatrix} + \mathcal{G}_1 \mathcal{K}_1 \mathcal{E}_1 \begin{bmatrix} \nabla_u \widetilde{S} \\ \nabla_c \widetilde{S} \end{bmatrix} \right) \Phi E \qquad \text{VI.9}$$

and the corresponding expression for the matrix of coefficients in VI.8 is obtained upon replacing E in this expression with the transposed premultiplier of Φ. As already pointed out in section V.F, the first term in VI.9 represents the ideal problem in which our knowledge of the sensor transfer functions is assumed perfect. The second and third terms are concerned with the correction of sensor transfer functions. The fourth term takes into account the possibility that the "sensor operand" S may not be a "pure" environmental variable but may be a function of U^*. Refer to section VI.E. Notice that the partitioning of variables recommended by equations VI.1, 2 leads to the "separation of variables" illustrated in VI.9. The accuracy of the results are, of course, independent of the chosen method of partitioning, but our method leads to simplification in the mechanization of the calculations.

C. The Modified Prediction Function

Consider the problem of predicting the signals from sensors whose characteristics deviate appreciably from the ideal. The most common problems are those concerned with calibrations (zero point bias, slope, non-linearity), misalignment of sensor axes, time delays, residual magnetic moments, and the like. The modifier $\mathcal{G}$ (see equation II.3) represents the mathematical operations that simulate the operation of the non-ideal sensor. Hence $\mathcal{G}$ is modelled in a similar sense as the torque terms. Parameters appearing in these models can be included in the array U and corrected as though they were parameters of the motion. Several examples are presented here in order to illustrate how the procedure works.

Example 1 - Linear Response

Suppose a given magnetometer has the following calibration curve:

$$\tau = \mathcal{G} \otimes f \equiv mf + b. \qquad \text{VI.10}$$

(This expression defines the operator $\mathcal{G}$.)

Here τ is the predicted transmitted signal having the units of pure number. Such a calibration curve contains two parameters: the slope m and the intercept b or bias. The calibration here is viewed in reverse, of course, for one would normally think of a calibration curve as

$$f = (\tau - b)/m. \qquad \text{VI.11}$$

But, owing to the resulting mathematical symmetry, we prefer to deal in "telemetry space," to coin an expression. In this way the predicted field f is transformed by means of $\mathcal{G}$ into dimensionless telemetry counts compatible with the vehicle's actual observed telemetry. In this example f is given by an expression like II.4 with B in place of S and where the significance of $\mathcal{K}$ and K are unchanged.

Since we are interested in adjusting b and m, our array of parameters, equation VI.1, becomes

$$\widetilde{U}^* \equiv \{u_1, u_2, \ldots u_6, b, m\}. \qquad \text{VI.12}$$

Then, from equations II.4 and VI.10, it follows that

$$\mathcal{G}_1 \otimes \nabla f = m \nabla f \qquad \text{VI.13a}$$

$$\mathcal{G}_2 \otimes f = \{1, f\} \qquad \text{VI.13b}$$

$$\mathcal{K}_1 \otimes \nabla B' = K \cdot \nabla \mathcal{E} B \qquad \text{VI.13c}$$

$$\mathcal{K}_2 = 0. \qquad \text{VI.13d}$$

Substituting into equations VI.9, we find that the normal equations are

$$\begin{bmatrix} m^2 \nabla \tilde{F} \Phi \nabla F & m \nabla \tilde{F} \Phi 1 & m \nabla \tilde{F} \Phi F \\ \cdot & \text{trace } \Phi & \tilde{1} \Phi F \\ \cdot & \cdot & \tilde{F} \Phi F \end{bmatrix} \begin{bmatrix} \Delta U \\ \Delta b \\ \Delta m \end{bmatrix} = - \begin{bmatrix} \nabla \tilde{F} \Phi E \\ \tilde{1} \Phi E \\ \tilde{F} \Phi E \end{bmatrix} \qquad \text{VI.14}$$

where 1 represents a column vector of 1's and where the periods are reminders of the fact that the matrix N^* is symmetric.

Example 2 - Nonlinear Response

Suppose next that the sensor in the preceding example has a nonlinear response so that the calibration curve VI.10 is replaced by

$$\tau = \mathcal{G} \otimes f \equiv \sum_{r=0}^{m} c_r (f)^r. \qquad \text{VI.15}$$

Then

$$\mathcal{G}_1 \otimes \nabla f \equiv [\sum_{r=0}^{m} r c_r (f)^{r-1}] \nabla f \qquad \text{VI.16a}$$

$$\mathcal{G}_2 \otimes f \equiv \{1, f, (f)^2, \ldots (f)^m\} \qquad \text{VI.16b}$$

and the normal equations are

$$
\begin{bmatrix}
\Sigma^2 \nabla\tilde{F}\Phi\nabla F & \Sigma\nabla\tilde{F}\Phi 1 & \Sigma\nabla\tilde{F}\Phi F & \Sigma\nabla\tilde{F}\Phi F^2 & \cdots \\
\cdot & \mathrm{trace}\,\Phi & \tilde{1}\Phi F & \tilde{1}\Phi F^2 & \cdots \\
\cdot & \cdot & \tilde{F}\Phi F & \tilde{F}\Phi F^2 & \cdots \\
\cdot & \cdot & \cdot & \tilde{F}^2\Phi F^2 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdots
\end{bmatrix}
\cdot
\begin{bmatrix}
\Delta U \\ \Delta c_1 \\ \Delta c_2 \\ \Delta c_3 \\ \cdot \\ \cdot \\ \cdot
\end{bmatrix}
= -
\begin{bmatrix}
\Sigma\nabla\tilde{F}\Phi E \\ \tilde{1}\Phi E \\ \tilde{F}\Phi E \\ \tilde{F}^2\Phi E \\ \cdot \\ \cdot \\ \cdot
\end{bmatrix}
\qquad \text{VI. 17}
$$

where the symbol Σ represents the factor in front of ∇f in VI.16a and terms like F^2 represent column vectors like $\{f^2(t_i)\}$.

Equations VI.17 enable us to adjust, in a least square sense, the various coefficients in the power expansion VI.15.

Example 3 - Residual Magnetic Dipole Moment

When the space vehicle possesses a residual magnetic moment, each magnetometer detects both the field due to the moment and that due to the environment according to the principle of linear superposition. (We exclude from this discussion those magnetometers insensitive to constant fields.) If B^e and B^p denote the fields due to the environment and the magnetic moment P, respectively, then the resulting field sensed by the magnetometer is

$$B = B^e + B^p. \qquad \text{VI.18}$$

Let us limit the discussion to a dipole moment, denoted by P, whose direction and magnitude are to be determined and let J denote the position vector of the magnetometer with respect to the dipole. (That is, let the origin of body coordinates be at the dipole.) Then the magnetometer is in a potential field [26, 27]

$$\phi(J) = \epsilon \frac{P \cdot J}{J^3} \qquad \text{VI.19}$$

where ϵ is a constant which reduces to 1 in the Gaussian system of units [26]. Hence we shall drop it from the rest of this discussion.

Taking the cartesian gradient of ϕ, we have the field

$$B^p = -\nabla_J \phi = -\frac{P}{J^3} + 3\frac{(P \cdot J)}{J^5} J \qquad \text{VI.20}$$

so that the component of B^p actually sensed by the magnetometer is

$$B^p \cdot K = -\frac{P \cdot K}{J^3} + 3\frac{(P \cdot J)(J \cdot K)}{J^5}. \qquad \text{VI.21}$$

Since our aim is to determine P, our array of system parameters is

$$U^* = \{u_1, u_2, \dots u_6, P_x, P_y, P_z\}. \qquad \text{VI.22}$$

For brevity, define the vector Q with components X, Y, Z, so that

$$Q \equiv \{X, Y, Z\} \equiv -\frac{K}{J^3} + 3\frac{(J \cdot K)\,J}{J^5}. \qquad \text{VI.23}$$

Evidently we have

$$\tau = \mathcal{G} \otimes f \equiv f - \frac{P \cdot K}{J^3} + 3\frac{(P \cdot J)(J \cdot K)}{J^5}, \qquad \text{VI.24}$$

$$\mathcal{G}_1 \otimes \nabla f \equiv \nabla f \qquad \text{VI 25}$$

$$\mathcal{G}_2 \otimes f \equiv Q. \qquad \text{VI.26}$$

Substituting once again into VI.9, we have the normal equations:

$$\begin{bmatrix} \widetilde{\nabla F}\Phi\nabla F & \widetilde{\nabla F}\Phi 1X & \widetilde{\nabla F}\Phi 1Y & \widetilde{\nabla F}\Phi 1Z \\ \\ \cdot & \tilde{1}\Phi 1X^2 & \tilde{1}\Phi 1XY & \tilde{1}\Phi 1XZ \\ \cdot & \cdot & \tilde{1}\Phi 1Y^2 & \tilde{1}\Phi 1YZ \\ \cdot & \cdot & \cdot & \tilde{1}\Phi 1Z^2 \end{bmatrix} \begin{bmatrix} \Delta U \\ \\ \Delta P_x \\ \Delta P_y \\ \Delta P_z \end{bmatrix} = - \begin{bmatrix} \widetilde{\nabla F}\Phi E \\ \\ X\tilde{1}\Phi E \\ Y\tilde{1}\Phi E \\ Z\tilde{1}\Phi E \end{bmatrix}. \qquad \text{VI.27}$$

This system of normal equations enables us to use the magnetometers to compute the attitude of a space vehicle despite the perturbing effect of a magnetic dipole.

D. Singular Normal Equations

In section III.E it is pointed out that the normal equations are singular when a parameter gives rise to an identically vanishing gradient or when two or more parameters result in proportional gradients. These singularities can disappear when there are two or more sensors contributing to the raw data. Suppose that we have a two-sensor, two-parameter system. In Gauss' notation, let the matrix of coefficients associated with the first sensor be

$$N^{(1)} = \begin{bmatrix} (aa) & (ab) \\ (ba) & (bb) \end{bmatrix}. \qquad \text{VI.28}$$

(See equation III.23.) Suppose that the parameters give rise to proportional gradients so that, as shown in equations III.37, $N^{(1)}$ can be written as

$$N^{(1)} = (aa) \begin{bmatrix} 1 & k \\ k & k^2 \end{bmatrix}. \qquad \text{VI.29}$$

where k is the proportionality factor. Now suppose that the second sensor likewise gives rise to a singular matrix of coefficients $N^{(2)}$:

$$N^{(2)} = \begin{bmatrix} (xx) & (xy) \\ (yx) & (yy) \end{bmatrix}, \qquad \text{VI.30}$$

$$N^{(2)} = (xx) \begin{bmatrix} 1 & \kappa \\ \kappa & \kappa^2 \end{bmatrix}. \qquad \text{VI.31}$$

The complete system is

$$(N^{(1)} + N^{(2)})\, \Delta U = -\Gamma^{(1)} - \Gamma^{(2)}. \qquad \text{VI.32}$$

It is immediately apparent that $N^{(1)} + N^{(2)}$, though the sum of two singular matrices, is not generally singular.

In the examples to follow a sensor is called a z-sensor if its axis of sensitivity coincides with the body z-axis. We wish, moreover, to recall equation II.13 which describes the motion of the body axes with respect to the auxiliary frame. (See section II.K.)

Example 1 - Vanishing Gradient

An obvious example of a vanishing gradient leading to a singular matrix of coefficients is $\partial \tau / \partial \psi$ for linear or nonlinear z-sensors. This is evident from equation II.13. Hence any attempt to determine ψ or $\dot{\psi}$ directly from the raw data using a z-sensor alone will fail. (This angle and spin rate could be obtained indirectly from, say, knowledge of the angular momentum, velocity, kinetic energy, and the principal moments of inertia. See equations IV.4.)

Example 2 - Proportional Gradients

Consider also a force-free spacecraft dynamically balanced about its z-axis and equipped solely with a linear z-solar sensor. From equations II.4, II.9, and VI.10, we have

$$\tau = mf + b \qquad \text{VI.33}$$

$$f = [\mathfrak{a}]_3 \mathfrak{L} S \qquad \text{VI.34}$$

where the 3 identifies the 3rd row of the Euler transformation. For simplicity let the components of the vector $\mathfrak{L} S$ be $\{X, Y \ Z\}$. Thus, in view of II.13,

$$f = \sin\theta (X \sin\varphi - Y \cos\varphi) + Z \cos\theta \qquad \text{VI.35}$$

where θ is a constant (See section IV.F). It is easily shown that

$$f_\theta \equiv \frac{\partial f}{\partial \theta} = (f - Z \cos \theta) \cot \theta - Z \sin \theta . \qquad \text{VI.36}$$

Hence

$$f_\theta / f = \cot \theta - Z/(f \sin \theta). \qquad \text{VI.37}$$

This shows that f_θ (and consequently τ_θ) correlates with τ_m and τ_b if $Z = 0$, i. e., if solar radiation is normal to the angular momentum. In this instance, an attempt to adjust any two of the three parameters θ, bias b, or slope m, results in a singular matrix of coefficients.

Consider further the case of vanishing inclination θ:

$$\lim_{\theta \to 0} (f/f_\theta) = f \sin \theta / (f \cos \theta - z). \qquad \text{VI.38}$$

Thus we see that, for a balanced vehicle with vanishing inclination, the z sensor slope and bias cannot be determined solely from the output of the said sensor.

This example illustrates the possibility that parameters taken from two or more operators result in proportional derivatives. Inspection of equation VI.24, for instance, also reveals that an attempt to adjust at once the bias level of a magnetometer and the residual magnetic moment can lead to proportional derivatives and singular normal equations.

Example 3 - Inertial Platform OAO

Suppose we wish to apply least square differential correction to an "inertial platform" spacecraft (like the Orbiting Astronomical Observatory) with vanishing angular momentum. Clearly, the gradients associated with the output of any one sensor are constants (in so far as the sensor operands are constants) and therefore lead to singular matrix of coefficients. But, as explained by equation VI.32, the singularity disappears when other sensors are included.

E. Analytical Requirements for $\mathfrak{F}$

Because Taylor's formula is employed to expand τ in terms of the parameters, it is clear that our expression τ should be continuous and should possess continuous derivatives up to the highest order appearing in the expansion. These requirements must be met in the neighborhood of the correct point of convergence. As observed in section V.F, S itself is not generally a function of the parameters. Hence the stated conditions apply to $\mathfrak{F}$. The resulting error function will then be well behaved in the same domain.

1. Functional Replacement

Unfortunately the functions encountered in attitude determination do not always conform to these requirements. One approach to this difficulty is that of "functional replacement." This strategy is one of recognizing that the actual non-analytic sensor output may be approximated by another suitable analytic function. In order to do this it is merely necessary to insure that i) the shape of the new curve yield zero average squared error near convergence and ii) the derivatives and, above all, their algebraic signs contribute correctly toward the task of minimizing the squared error. Another approach is to disregard the sensor output function altogether and regard its time coordinate as the new function. This type of functional substitution is most natural in dealing with sensors which record events as opposed to those which measure intensive quantities. If the time coordinate replaces the function or ordinate, then the new abscissa is the "event number." This outlines the technique by which infra-red and digital counter outputs can be employed in a differential correction scheme without the extensive preprocessing which is otherwise needed to transform their pulsed outputs into suitable analytical quantities.

Sensors whose outputs are characterized by the word "event" are referred to as "event sensors" and their outputs will be referred to as "event functions." Several examples are offered to illustrate the suggested method of approach.

Example 1 - The Tuned Oscillator Magnetometer

Consider a magnetometer whose output is characterized by the equation

$$\tau = 0 \qquad \text{VI.40a}$$

$$\tau = k_1 \, |K \cdot B'| \, |K \times B'|/|B|^2 - k_2 \qquad \text{VI.40b}$$

whichever is greater [50] k_1 and k_2 are constants. If ρ denotes the angle between the sensor axis K and the magnetic field B, then

$$\tau = 0 \qquad \text{VI.41a}$$

$$\tau = k_1 \, |\cos \rho| \, \sin \rho - k_2 \qquad \text{VI.41b}$$

whichever is greater.

This function possesses discontinuous derivatives at several places. See Figure 5. The simplest way to handle this case is to screen raw data to remove all the $\tau = 0$ values. Next, the function VI.41b is replaced by

$$\tau = k_1 \cos \rho \sin \rho - k_2 \qquad \text{VI.42}$$

where negative τ are retained. Then, in order to insure that the signs of the derivatives and those of the errors ϵ combine effectively, the following test is performed: If ρ is in the range $[0, \pi/4]$, ϵ is set equal to ϵ . That is, it is left unchanged. In the range $[\pi/4, \pi/2]$, the "replaced" function is reflected in the x axis and, therefore, the sign of ϵ is reversed. The intervals $[\pi/2, 3\pi/4]$ and $[3\pi/4, \pi]$ are handled in the same way.

Example 2 - An Event Sensor Function (Time as a Function)

Consider an infrared scanner on board a dynamically balanced vehicle spinning about its axis of dynamical symmetry. The line of sight of the scanner axis can be imagined to describe a smooth surface as it sweeps the sky. Ideally this surface is a cone. If the sensor is mounted exactly perpendicular to the axis of rotation, the cone becomes a plane. At certain times, this imaginary surface will intersect the horizon. When the sensor performs properly, an electrical pulse is generated each time the horizon crosses the field of view of the scanner. The time intervals between pulses convey information, though indirectly, as to the orientation of the spin axis and the spin phase. Computation of attitude from this information can involve a considerable amount of intermediate calculations [44]. Sometimes it may also involve preprocessing and transformation of raw data [49].

The principle advanced here is to rely on raw data as much as possible. But the signal generated by the scanner is an "event function" and is not suitable for differential correction since it is not, in principle, differentiable. The time of the pulse signal, however, is a piecewise differentiable function of the parameters of the motion. To see this, refer to Figure 6. A small continuous displacement in, say α, will cause a similar displacement in the intersection of the horizon and the imaginary cone.

The problem, therefore, is to find an analytical expression for the pulse times in terms of the parameters of the motion. We proceed as follows. First make the following definitions:

ρ	$\equiv$	orbit radius vector	VI.43a
L	$\equiv$	spin axis vector	VI.43b
ϵ	$\equiv$	half angle subtended by the earth at the vehicle	VI.43c
X	$\equiv$	a cartesian vector in space with components x, y, z.	VI.43d
η	$\equiv$	the angle between L and the sensor axis.	VI.43e

The origin of coordinates is chosen at the vehicle center of mass and the dimensions of the latter are considered negligible. Hence the earth is located at $-\rho$ and its horizon defines a cone with apex at the origin. The expression for this cone is

$$h(x, y, z) = h(X) = \frac{-X \cdot \rho}{|X||\rho|} = \cos \epsilon . \qquad \text{VI.44}$$

Similarly, the expression for the cone described by the sensor line of sight is:

$$g(x, y, z) = g(X) = \frac{X \cdot L}{|X||L|} = \cos \eta . \qquad \text{VI.45}$$

These two imaginary conic surfaces have a common apex at the spacecraft center of mass.

The vectors X satisfying equations VI.44 and VI.45 simultaneously are the intersections of these surfaces when, or if, they intersect. In order to solve these equations simultaneously, a third condition is required. Such a condition might be that $x^2 + y^2 + z^2 = 1$. Hence our system of equations can be written as:

$$-\frac{X \cdot \rho}{|X|} = |\rho| \cos \epsilon \qquad \text{VI.46a}$$

$$\frac{X : L}{|X|} = |L| \cos \eta \qquad \text{VI.46b}$$

$$X \cdot X = 1 \qquad \text{VI.46c}$$

Let $$\rho \equiv \{ -p, -q, -r \} \qquad \text{VI.47a}$$

$$L \equiv \{a, b, c\} \qquad \text{VI.47b}$$

$$\upsilon \equiv |\rho| \cos \epsilon \qquad \text{VI.47c}$$

$$\xi \equiv |L| \cos \eta. \qquad \text{VI.47d}$$

Hence we have

$$px + qy = \upsilon - rz \qquad \text{VI.48a}$$

$$ax + by = \xi - cz \qquad \text{VI.48b}$$

$$(1 - x^2 - y^2) = z^2. \qquad \text{VI.48c}$$

The determinant of the first pair is

$$\Delta = pb - qa. \qquad \text{VI.49}$$

Then

$$x = \begin{vmatrix} \upsilon - rx & q \\ \xi - cz & b \end{vmatrix} / \Delta \qquad \text{VI.50a}$$

$$y = \begin{vmatrix} p & \upsilon - rz \\ a & \xi - cz \end{vmatrix} / \Delta \qquad \text{VI.50b}$$

$$z = \pm (1 - x^2 - y^2)^{\frac{1}{2}} \qquad \text{VI.50c}$$

When VI.50a and VI.50b are substituted into VI.50c, the standard method of solving quadratics yields solutions for z whence x and y may then be obtained via equations VI.50a and VI.50b.

We have thus obtained the predicted direction of the sensor axis at the instants of time when the horizon is intersected. These solutions are functions of the known orbital position vector and the estimated spin direction which is given by:

$$L_1 = \cos\delta \cos\alpha \qquad \text{VI.51a}$$

$$L_2 = \cos\delta \sin\alpha \qquad \text{VI.51b}$$

$$L_3 = \sin\delta. \qquad \text{VI.51c}$$

In order to predict the pulse times, the predicted sensor line of sight X' is computed as a function of time according to

$$X'(t) = K \cdot \mathcal{E}(U(t)). \qquad \text{VI.52}$$

As $\mathcal{E}$ is advanced in sizeable time increments, the inner product

$$X'(t) \cdot X \qquad \text{VI.53}$$

is surveiled. When it approaches its maximum value of +1, the step size is reduced somewhat and the time of the maximum is obtained. (Linear interpolation could conceivably be accurate enough to establish the crossing time.)

This outlines an approach to the calculation of an error function based on the squared time differences between observed and predicted pulses. No essential transformation of raw data has been assumed. Certain precautions would be required. This method expects reasonably good estimates of the orientation to begin with. Ambiguities arising in connection with the use of I. R. scanners would still have to be resolved prior to differential correction. Consult references [44] and [49].

2. An Artificial Function and Differentiable Sensor Operand

Instead of replacing the "event sensor function" by its time coordinate, there is another approach which is more interesting. In this next method, the sensor and its function are replaced by an artificial sensor and an artificial function. We now avail ourselves of the fact that the intersections X of the two conic surfaces are normal to the cartesian gradients of these surfaces everywhere along the said intersections. This means that if the spacecraft were equipped with a device capable of measuring the vector inner product between its line of sight and the gradient of the imaginary cone h, it would register zero at the same time that an I. R. scanner generates its pulse. We propose, therefore, to replace the original nonanalytic square wave by an ordinary ideal cosine function, τ . It is implied, however, that the only times when we have any raw knowledge of the new artifical function τ is when $\tau = 0$. Hence the predicted values themselves are the residuals.

The cartesian gradient of our surface is obtained as follows:

$$\nabla h = \left\{ \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z} \right\} \qquad \text{VI.54}$$

where

$$h = px + qy + rz = \upsilon \qquad \text{VI.55}$$

$$\nabla h = - \frac{\rho}{|X|} + \frac{(\rho \cdot X)}{|X^3|} X \qquad \text{VI.56}$$

which is evidently normal to X as claimed. The vector ∇h is now taken as that part of the vehicle environment which is detected. Detection is accomplished by a hypothetical sensor whose output is

$$f = K \cdot \mathcal{E} \nabla h \qquad \text{VI.57}$$

which now replaces the I. R. scanner square wave. The new function is seen to be similar to that of, say, a magnetometer. Refer to equation II.4. Here ∇h plays the role of B. Thus, the original pulse output of the I. R. scanner is replaced by a vector inner product which satisfies the analytical conditions for differential correction. The new function f is, in fact, differentiable in terms of elementary functions. In calculating these derivatives, we observe a simple precaution: the new operand, ∇h, is a predicted vector function and admits differentiation according to equation V.23.

The differentiation is as follows:

$$\frac{df}{du} = K \cdot \frac{d\mathcal{E}}{du} \nabla h + K \cdot \mathcal{E} \frac{d\nabla h}{du}. \qquad \text{VI.58}$$

The first member is recognized as the usual differentiation discussed in sections V.G, F. The second member accounts for the fact that ∇h is a predicted vector dependent on the right ascension and declination of the spin axis. Suppose that, without loss of generality, the gradient ∇h is taken at $|X| = 1$. Then

$$\nabla h = -\rho + (\rho \cdot X) X \qquad \text{VI.59}$$

$$\frac{d\,\nabla h}{du} = \rho \cdot \left(\frac{dX}{du} \right) X + (\rho \cdot X) \frac{dX}{du} \qquad \text{VI.60}$$

In order to calculate $\frac{d\,\nabla h}{du}$, it is evident that dX/du is to be evaluated. This calculation may be carried out by first differentiating equations VI.46 and then solving dX/du. Carrying out the differentiation and maintaining $|X| = 1$, we have

$$-\frac{d}{du}(X \cdot \rho) = -\frac{dX}{du} \cdot \rho = 0 \qquad \text{VI.61a}$$

$$\frac{d}{du}(X \cdot L) = \frac{dX \cdot L}{du} + X \cdot \frac{dL}{du} = 0 \qquad \text{VI.61b}$$

$$\frac{d}{du}(X \cdot X) = 2X \cdot \frac{dX}{du} = 0 \qquad \text{VI.61c}$$

That is,

$$-\frac{dX}{du} \cdot \rho = 0 \qquad \text{VI.62a}$$

$$\frac{dX}{du} \cdot L = -X \cdot \frac{dL}{du} \qquad \text{VI.62b}$$

$$\frac{dX}{du} \cdot X = 0 \quad . \qquad \text{VI.62c}$$

This is a system of three linear equations in three unknowns, $\frac{dx}{du}$, $\frac{dy}{du}$, and $\frac{dz}{du}$. The determinant is:

$$|\Delta| = \begin{vmatrix} p & q & r \\ L_1 & L_2 & L_3 \\ x & y & z \end{vmatrix} . \qquad \text{VI.63}$$

Thus

$$\frac{dX}{du} = \left(X \cdot \frac{dL}{du}\right)\left(X \times \rho \right)/\Delta. \qquad \text{VI.64}$$

Substituting into equation VI.60, we can now proceed with differential correction. This is an example of a nonvanishing sensor operand gradient discussed in sections V.F and VI.B.

F. Mechanization of the Calculations

Granted that a given attitude problem is to be solved by means of least square differential correction, we wish to know the principal steps in the calculations. It is assumed that we have the various algorithms or "subprograms" needed to calculate trigonometric functions, Jacobi elliptic functions, quadratures, matrix inversion, and the like. We also assume below that we have certain subprograms especially useful in attitude determination. One such subprogram is a generalized vector differential correction algorithm. Another important program is a generalized n-degree-of-freedom integration package for n-coupled first order differential equations.

Still more specialized subsystems are those needed to calculate the sensor transfer functions $\mathfrak{G}$ and the geometric operator $\mathfrak{K}$. In the discussion to follow, the statement "CALL $\mathfrak{G}$ " and "CALL $\mathfrak{K}$ " are understood to mean that the designated operation is called for and performed on the appropriate operand. It is also understood that these subprograms test a "status indicator" L in order to determine whether the primitive operation or one of the derivative operations is sought. That is, the statement "CALL $\mathfrak{G}$" results in $\mathfrak{G}_0 \otimes f$, $\mathfrak{G}_1 \otimes f$, or $\mathfrak{G}_2 \otimes \nabla f$, depending on whether $L = 0, 1, 2$, respectively (we are saying that $\mathfrak{G}_0$ is identical with $\mathfrak{G}$).

It is assumed, finally, that the participating sensors and parameters are selected and that environmental data are available for the entire time interval under consideration.

We look at the computational process first from a high level, namely the differential correction loop level. The important steps, illustrated in Figure 7, are as follows: The first is calculation of the predicted vector function T(I, J) for all I, J. The indices I and J refer to ith observation time and jth component (sensor), respectively. The second step is calculation of the gradients $\nabla T(I,J,K)$ where K pertains to the kth parameter. The third step is to solve the normal equations; the fourth step, to test for convergence.

We consider next the processes taking place within the first box, figure 7. The object is to calculate the sensor output functions T(I,J) corresponding to $\tau^j(t_i)$. Prior to the start of the I loop (time), the appropriate sensor (J) related data are set up for ready and efficient access. Then, if the nature of the circumstances require, the sensor status and environment are tested for each sample time. This step is necessary, for instance, to account for an eclipse. It is, more generally speaking, a test for the possibility that the operators $\mathfrak{G}$ and $\mathfrak{K}$ and their operands may have suffered a change that entails a new logical flow. The first order of business in calculating the sensor functions is to predict the attitude itself. This is accomplished by calculating the five angles α, δ, φ, θ, ψ.

Five distinct possibilities exist for calculating these angles, depending on the type of motion. The type of motion is designated by the index M. See Figure 8. There are three force-free cases: i) force-free simple spin (M = 1), ii) force-free balanced precessional motion (M = 2), and iii) force-free nonbalanced general motion. The case of forced but predictable motion, like that discussed in section V.E.3, is handled in case M = 4. On the other hand, all situations calling for numerical integration of the differential equations of motion are handled in case M = 5. When the angles are computed, the attitude is obtained upon substituting into equations II.10 and II.13. Then it is a simple matter to transform the environmental variable S into the body system of coordinates by means of $S' = \mathcal{E}S$, namely equation II.8.

The next important consideration is the geometry of the sensor mounting. This problem is handled by the operator $\mathfrak{K}$ operating on S'. See equation II.7. With L = 0, CALL $\mathfrak{K}$ gives us the ideal sensor input, f.

Hence the next step is CALL $\mathcal{G}$ which obtains the sensor output function τ in a telemetry counts. See equation II.6.

The next important process, box 3 in Figure 7, is calculation of the gradients. The simplest method for calculating gradients in all but the first few types of motion (M = 1, 2), is to estimate them by finite differences. If we choose to calculate gradients from analytical expressions, we follow the "separation of variables" guidelines of section V.F and VI.B, which lead to modular processing illustrated in Figure 9. Calculations proceed from top to bottom along a column, the column chosen depending on the type of parameter being processed. The left-most column is for the fundamental parameters of the motion, discussed in section V.E, such as α_0, δ_0, φ_0, θ_0, ψ_0, $\mathring{\varphi}$, a, b, c, p_0, q_0, r_0, μ, υ, t_0. The second column leads to the gradient of τ with respect to γ_1, γ_2, γ_3 ..., namely the "mounting" or geometric parameters affecting $\mathcal{K}$. Similarly, the third column leads to the gradient of τ with respect to sensor calibration constants c_1, c_2, c_3... .

We have mentioned the most important considerations. Questions concerning the organization of raw data, access to environmental data, preprocessing, and output calculations belong to the data processing problem which the author is considering separately. One of the most interesting aspects of the latter task is the mechanization of sensor data (sensor definition).

G. Optimization and Error Estimation

1. Preface

Because we have observational data in excess of the minimum needed to solve the problem, we wish to find the best estimate of the system parameters. Hence, the task is a dual one: to design a good prediction function $\mathfrak{F}(U, t)$ and to optimize the estimate of U. The purpose of the following paragraphs is to outline a definition of the optimization problem from the standpoint of probability and to consider measures of confidence. In order to avoid the proliferation of subscripts, the following conventions are adopted. The symbol U is reserved for the true system parameters which are always unknown to the observer. His estimate of U is designated by W. V, on the other hand, refers to the entire range of possibilities for the system parameters. It is said that W is the estimate of U, the estimand. Quantities like dV stand for $dv_1 . dv_2 . . .$. Since our problem inherently involves arrays, our argument is in matrix form and we speak of U as the "system parameter" and E as "the error", etc. Finally we let Z be defined as V-U.

2. Methods of Approach

In Section III, the least square observational error is adopted as the criterion for the optimization of the estimate of U. Although it can be accepted as an ad hoc postulate, it can also be related to more fundamental approaches under special conditions. (Refer to "maximum liklihood" and "minimum variance" estimation methods [46, 81, 82]).

3. Definition of Loss, Risk, Covariance

a. The Loss

The "loss" is a measure of disapproval assigned to V on the basis of an observation. The square of the observational error is such a function. Other types of loss can be defined but there are several reasons why the squared error is used most frequently. (See pages 11, 12, 13, 17, 18, 19 and 136, reference [46].) Notice that, although the loss has been defined in terms of observational error, it is nevertheless a criterion of optimality for V. Hence, if we wish to estimate the mean value U of a random but otherwise constant signal V, the loss is

$$\ell(V,U) \equiv (V-U)^2 \qquad \text{VI.65a}$$

while in problems where the parameter is not directly observed the loss becomes

$$\ell(V,U) \equiv (\mathfrak{F}(V) + E - \mathfrak{F}(U))^2 \qquad \text{VI.65b}$$

$$\equiv (Y-\mathfrak{F}(U))^2. \qquad \text{VI.65c}$$

$$\equiv \ell(Y,U) \qquad \text{VI.65d}$$

b. The Risk

Let us now define a quantity r, called the risk (sometimes cost) as the expected value of the loss with respect to the entire range of possible observational error:

$$r(U) \equiv \int \ell(V, U)\, \rho(V, U)\, dV \qquad \text{VI.66a}$$

where $\rho(V, U)$ is the (joint) probability that the observation V will occur when the true system parameter is U. Of course, when the observable Y is a function of V, then

$$r(U) \equiv \int \ell(Y, U)\, \rho(Y, U)\, dY. \qquad \text{VI.66b}$$

(In the following discussion, no further mention need be made of the special loss VI.65a since the nonlinearity of our problem demands the more general form VI.65b.

Because exact knowledge of U is impossible, the experimenter can neither compute the loss nor the risk. Even with the help of U he would still find that his knowledge of ρ is approximate, at best. On the other hand, he can estimate them on the basis of experimental data.

Notice that the loss and risk functions can be considered as functions of Y, W instead of Y, U. Then one of the purposes of the theory is to show that, under certain simple conditions, r (W) has an absolute minimum when W = U. The estimation problem is, consequently, equivalent to the minimization of r (W) . (The argumentation for the method of "maximum liklihood" is similar. In that method a function, called the liklihood, is maximized.) Subtle and lengthly arguments are needed for a full discussion [46, 81, 82]

By the time we consider minimization of the risk, we are coerced into several assumptions. First we must assume that, if the observed data were free of errors to begin with, then the iteration function Ξ (see Section III.D.1.) would converge on U. . Our second assumption follows a fortiori: the prediction function $\mathfrak{F}$ is "perfect" in the sense that the random errors have zero means. That is

$$Y = T + E \qquad \text{VI.67a}$$

$$T \equiv \mathfrak{F}(U, t) \qquad \text{VI.67b}$$

$$\{\text{ Expected value of } E\} \equiv \{E\} = 0. \qquad \text{VI.67c}$$

In order to carry out specific calculations it is often necessary to assume further that E_i is gaussian (normal) and that E_i and E_j satisfy

$$\{E_i, E_j\} = 0 \; ; i \neq j, \qquad \text{VI.68}$$

i.e. uncorrelated. For the time being, however, we assume merely equation VI.67c.

c. Covariance

We now define the covariance matrix C as follows

$$C_{ij} \equiv \{E_i , E_j\} \; ; i, j = 1, 2, 3, \ldots n \qquad \text{VI.69}$$

where n is the number of observations. Notice that, if E_i , i = 1, 2, 3, ..., are normal variates, then the error matrix E has the multivariate normal distribution

$$\rho\ (E) = \frac{1}{(2\pi)^{n/2}} \quad |\Phi|^{\frac{1}{2}} \qquad \exp\left(-\frac{\tilde{E}\,\Phi E}{2}\right). \qquad \text{VI.70}$$

(In this instance the diagonal elements of C are sometimes called "the minimum variances.) We have defined Φ here as the inverse of C.

4. Optimization Procedure

The method for optimizing W (or minimizing r) has been described in Section III. We first estimate r (W):

$$r\ (W) = (\tilde{T}\ (W) - \tilde{Y})\ \Phi(\ T\ (W) - Y\ ).$$

The second step is to assume that T is analytic in the neighborhood of U and that the linear terms of Taylor's formula are an adequate representation. The resulting expressions are linear in ΔW and are solved by standard methods. Considered as a function, the Ξ algorithm transforms a random variable $\dot{Y}$ into another random variable W. (We say that W is a random variable of a nonrandom function of a random variable.) In order to calculate confidence measures, we wish to know the statistical distribution of W .

5. The Sampling Distribution of W .

Perhaps the most useful result provided by the theory of statistical estimation is that the sampling distribution of W is asymptotically normal under certain mild requirements. If it is granted that E is a normal variate, then this claim is easy to make. Regarding W as an analytic function of $E = T - Y$, we have by Taylor's formula,

$$W = U + \nabla_E W \, E + \ldots \qquad \text{VI.71a}$$

Thus W is a linear transformation of a normal vector (for sufficiently small error dispersion). According to Peter Swerling [82] , the expression for ∇W is

$$\nabla W = (\nabla \tilde{T} \, \Phi \nabla T)^{-1} \quad \nabla \tilde{T} \, \Phi \, . \qquad \text{VI.71b}$$

(This is the author's version of Swerling's equation (7)).

In order to write the expression for the probability density of W , we calculate (or estimate) the covariance matrix D of W :

$$D = \{ (W-U), (W-U) \} \, .$$

This calculation can be found in [46] and [81] for the maximum liklihood estimation which is, for normal variates, equivalent to least squares. We state without proof that

$$D = \nabla \tilde{T} \, \Phi \nabla T \qquad \text{VI.72}$$

It follows that, for normal E , the probability density of W obeys

$$\rho\,(W - U) = \frac{|\Theta|^{\frac{1}{2}}}{(2\,\pi)^{k/2}} \exp \frac{(-(\widetilde{W} - \widetilde{U})\,\Theta\,(W - U))}{2}$$

$$\Theta \equiv D^{-1} \qquad \text{VI.73}$$

where k is the number of elements (system parameters) in W.

6. Multidimensional Confidence Region

We now ask for the equation of a curve (in the k-dimensional Euclidean space spanned by W) which gives a constant probability, say c. This curve is defined when equation VI.73 is set equal to c. This request is tantamount to asking for the equation

$$\widetilde{X}\ \Theta\ X = q \qquad \text{VI.74}$$

where $X \equiv W - U$

and $q \equiv -2 \ln (c\,(2\,\pi)^{k/2} / |\Theta|^{\frac{1}{2}})$.

Equation IV.74 is the equation of a k-dimensional ellipse, sometimes called the error ellipsoid. If the parameters give rise to perfectly uncorrelated gradients (if Θ happens to be diagonal) then the semiaxes of the ellipse are given by the diagonal elements. Their inverses are sometimes called parameter variances [33]. When q is not diagonal it may be diagonalized by means of a principal axis transformation [46].

7. Measures of Confidence

Notice that if X is assumed to be a normal k-dimensional variate, it follows that q is a chi-square variate with k degrees-of-freedom. Hence, with the help of the chi-square tables of marginal probability distributions, we can compute the "confidence probability" that U lies inside the error ellipsoid defined by $q = q_1$; that is, the "probability" that U lies inside the ellipsoid equal to

$$\text{prob.}\ (q \le q_1) = \int_0^{q_1} \chi_k^2\ (q')\, dq'$$

$$= \gamma\left(\frac{k}{2}, \frac{q_1}{2}\right)/\Gamma(k/2), \qquad \text{VI. 75}$$

where Γ and γ are the gamma and incomplete gamma functions,. respectively. This expression gives a measure of confidence to the hypothesis that the estimand U lies somewhere inside the error ellipsoid defined by q_1 centered at W Consult Section 10.5 in [46]. The reader will find the chi-square distribution tabulated in [84], table XIV, page 286.

As already stated, the diagonal elements of Θ^{-1} are sometimes regarded as parameter variances. This and other measures of confidence are given in [33]. But perhaps the best single measure of confidence is simply a plot showing the predicted curves superimposed over the observed data. See Figures 4 and 5.

Finally, notice that equations VI. 71 are to be regarded as "parameter sensitivity equations" for they relate the rate of change of W with respect to the observational error E. References [46] and [82] are recommended for further discussion.

H. Special Problems

1. Distinction between the Laboratory and the Dynamical Body-Fixed Frames of References (An Application of the Geometric Operator).

A subtle application of $\mathcal{H}$ is as follows. Consider the case of a non-balanced rigid body, such as POGO. There exists a certain frame of reference fixed in the body in which the inertia tensor I is diagonal. Thus

$$I = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} . \qquad \text{VI.76}$$

There is another reference frame also fixed in the body, to which the sensor mountings are related. The first system is of dynamical importance because of equation III.76, the second system is of geometrical importance because it is the one employed in bench measurements and the only one in which the sensor mountings are known. But these two frames are not necessarily the same. Thus a discrepancy arises if the prediction function estimates the sensor output as expected from the geometrical location of the sensor when the orientation of the vehicle is given by $\mathcal{E}(\alpha, \delta, \varphi, \theta, \psi)$ where

$$\varphi = \varphi(p, q, r, t_0, a, b, c) \qquad \text{VI.77a}$$

$$\theta = \theta(p, q, r, t_0, a, b, c) \qquad \text{VI.77b}$$

$$\psi = \psi(p, q, r, t_0, a, b, c). \qquad \text{VI.77c}$$

That is, φ, θ, ψ give the orientation of the dynamical coordinate system, not the geometrical system as desired. This discrepancy may be dealt with in the following way.

The two aforementioned coordinate systems are related by a constant transformation, say $\mathcal{C}$, which is expressible as a product of three simple rotations. (Three parameters exactly determine a general rotation matrix.) Let the sensor's axis K' in the geometric (laboratory) system be related to the said axis K in the dynamical system according to

$$K' = \mathcal{C}\,K. \qquad \text{VI.78}$$

The three parameters which define the fixed rotation $\mathcal{C}$ are included in the array of system parameters and are adjusted in the differential correction scheme in a manner analogous to the correction of the three Euler angles φ_0, θ_0, and ψ_0. The geometric operator then becomes

$$\mathcal{K} = K' \cdot \mathcal{C}. \qquad \text{VI.79}$$

In this expression K' is measured in the laboratory prior to launch. Hence we have an approach to the problem posed by inexact knowledge of the orientation of the principal axes of inertia. Because of the similarity in the definitions of $\mathcal{C}$ and the Euler transformation $\mathcal{G}$, certain precautions are needed to avoid singular normal equations arising from proportional gradients as discussed in section VI.D.

2. Closed Attitude Control Feedback Loop

Example 1, section IV.G, exemplifies a closed loop attitude control system. The Euler angles are obtained by means of numerical integration of the inhomogeneous equations of motion. The driving terms are functions of the solar sensor outputs which are, in turn, functions of geometric and calibration constants. Consequently we cannot in this instance claim independence of $\mathcal{G}$ and $\mathcal{K}$. This presents no extra difficulties in calculating gradients, however, because no attempt would be made to calculate them analytically - gradients are calculated numerically as shown in figure 9.

The derivatives can be computed by the straight-forward method of perturbations which is implied in the flowchart, in that figure. When the prediction function is time consuming to compute, however, it is desirable to devise a way to avoid the double computations. This problem is especially pressing when the prediction function is generated by integration of the differential equations of motion. Several approaches are available. For example, one may capitalize on the results of previous iterations to obtain an estimate of the derivative. (Refer to the methods of (i) secants [22, 24, 30], (ii) regula falsi [54], (iii) Whittaker [54], and Muller [54].) An ingenious approach to this problem in the case of numerical integration of equations of motion subject to constraints is reported in [39].

3. A Ground Based Sensor

Because of the nonuniform nature of antennae patterns, the intensity of the telemetry received from a space vehicle varies with its orientation. This variation can be used to infer the motion in much the same fashion as on-board instruments are used [83].

APPENDIX A

ON THE NUMBER OF INDEPENDENT CONSTANTS OF THE MOTION

A rigid body with one point fixed is known to possess three degrees freedom. See page 93, reference [1]. (This is the same number of degrees of freedom as in translational motion.) Because the fundamental laws of motion lead to a second degree differential equation for each degree of freedom, six arbitrary (independent) constants of integration appear. Hence, six parameters are required to define the motion of a rigid body with one point fixed.

It is known [4], however, that the Euler-Poisson equations of motion for a rigid body with one point fixed have but five arbitrary constants of integration. We wish now to reconcile this fact with the foregoing remarks.

Let us state the dynamical equations according to the Lagrangian formulation. In this formulation one constructs a function h, called "the Lagrangian," [1] possessing certain properties. One of these properties is that h is a function of the three parameters, x_i, associated with the three degrees of freedom of motion. According to Lagrange [1], the differential equations of motion are:

$$\frac{\partial h}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial h}{\partial \dot{x}_i}\right) = r_i;\ i = 1,\ 2,\ 3. \qquad \text{A-1}$$

In this expression r_i is a force function which appears when not accounted for in the design of the Lagrangian. Equation (A-1) leads to a system of three simultaneous second order equations:

$$\ddot{x}_i = f_i (X, \dot{X}, t);\ i = 1,\ 2,\ 3. \qquad \text{A-2}$$

where $X \equiv \{x_1, x_2, x_3\}$. System (A-2) can be replaced by the "state variable system,"

$$\dot{w}_i = g_i\ (W, t); \ i = 1, 2, \ldots 6 \qquad \text{A-3}$$

where $W \equiv \{x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3\}$.

From system (A-3) it is clear that six arbitrary constants of integration are admitted, namely the six initial conditions $w_i(t_0)$; $i = 1, 2, \ldots 6$.

In the event that there exists a constraint relation between the w's, however, the number of arbitrary constants of integration would be correspondingly reduced. As we shall observe below, this is the case with the Euler-Poisson equations.

Let

$$w_1 = \Omega_1 \qquad \text{A-4a}$$

$$w_2 = \Omega_2 \qquad \text{A-4b}$$

$$w_3 = \Omega_3 \qquad \text{A-4c}$$

$$w_4 = \text{roll} \qquad \text{A-4d}$$

$$w_5 = \text{pitch} \qquad \text{A-4e}$$

$$w_6 = \text{yaw} \qquad \text{A-4f}$$

Then we have a system like (A-3) with six arbitrary constants of integration. Our equations IV.3 are equivalent to this system.

On the other hand, the Euler-Poisson equations are derived in a different fashion. In this case, as before, the three Euler equations IV.3a are adopted. But instead of defining the state variable W, three equations, known as the Poisson kinematical equations, are appended to the Euler equations. These are derived as follows. Let K be a constant vector. It follows (see reference [1], chapter 4) that, in the body system of coordinates,

$$\frac{d}{dt} K + \Omega \times K = 0. \qquad \text{A-5}$$

Hence we have the six Euler-Poisson equations:

$$\frac{dL}{dt} + \Omega \times L = M \qquad \text{A-6a}$$

$$\frac{dK}{dt} + \Omega \times K = 0. \qquad \text{A-6b}$$

This system of six first order simultaneous differential equations is not in state variable form. Moreover, since K was assumed a constant vector, we have a constraint relation:

$$K_1^{\ 2} + K_2^{\ 2} + K_3^{\ 2} = |K|^2. \qquad \text{A-7}$$

This constraint reduces the number of independent constants to five. (It is also possible to show that, according to Jacobi's "theory of the last multiplier", once we have obtained four independent first integrals of system (A-6), the fifth may be obtained by means of a quadrature. See reference [4], chapter 1, and reference [5], chapter 8. Also note that a first integral is a constant of the motion. See reference [1], page 47 and reference [5], page 8.)

It is worthwhile to understand how two systems of equations of motion can lead to different numbers of arbitrary parameters. In equations IV.3, the orientation of the vehicle is uniquely defined with respect to the inertial space. In the Euler-Poisson system, however, the orientation of the vehicle is defined with respect to some fixed direction K. No information regarding the outside world is contained in equations (A-6) save for the direction cosines of K in the vehicle coordinate system. This means that these equations admit an ambiguity with respect to rotations about K. Hence the Euler-Poisson equation alone would not suffice in determining the attitude of a vehicle with respect to, say, the vernal equinox system. If, for example, K was chosen as the north star line of sight, an attitude determination predictor based on the Euler-Poisson equations of motion would require that the right ascension of L be adjusted together with the initial values of Ω_1, Ω_2, Ω_3, K_1, K_2, K_3, while carrying A-7 as a constraint.

In designing predictors and choosing the parameters of the motion, therefore, one must note what information each parameter provides and avoid both the duplication of information and the ambiguity of insufficient information. For example, as we have seen, the Euler-Poisson variables are ambiguous with respect to rotations about K. On the other hand, in section V.E we had chosen the initial conditions p_0, q_0, and r_0, as our initial conditions. These variables uniquely define φ , θ, and ψ . (They define the kinetic energy of rotation and the angular momentum as well. See Section IV.D.) Hence, the orientation of the vehicle is thus far defined with respect to the angular momentum vector. As explained in Section IV.E, two parameters, such as α and δ , may be employed to define the direction of this vector. The last parameter must, therefore, be a fixed rotation β about L.

APPENDIX B

SOME PROBLEMS OF CONVERGENCE

The discussion in this paper has been largely concerned with posing well defined minimization attitude determination problems. When this has been accomplished, the problems of convergence are likely to be important. The following comments are offered to the prospective practioner as an appendix since the subject of convergence is not the main issue in this paper. The subject of convergence is extensive. Consult references [9, 12, 20, 22, 23, 24, and 46.]

From the practical standpoint, the outstanding questions are (i) what reasonable precautions must be taken to assure convergence? (ii) when convergence has been achieved, is it a minimum, a maximum, a global minimum, or a global [23] maximum? (iii) when convergence seems unattainable, how can the cure be prescribed?

If it is granted that the problem is well posed, the first question is primarily the question of initial conditions--initial conditions must be inside the region of convergence. Now the minima of highly nonlinear functions are apt to be at the bottom of long, narrow, and curving trough-like depressions. It is no surprise, therefore, to discover the inadequacy of Newton's method in which Taylor's formula is truncated after the first order gradients. For if we regard the computed corrections ΔU as a string of vectors pointing along the direction of travel, it may happen that some of these ΔU's are too large to follow the bends in the said trough. For an approach to this problem, see page 103 in [46].

In attitude determination problems, the steepest gradients encountered are those associated with angular velocities for these derivatives contain time as a factor. Refer to equations V.26a, b. Hence, one should endeavor to determine angular velocities as accurately as possible prior to initiating differential correction.

A minimization problem may be well posed and the initial conditions may be inside the region of converge and yet convergence will not necessarily obtain. This is the case when the contribution to the error function from a certain variable or sensor is highly attenuated. It is, so to speak, too weak. The associated gradient may be insignificant in comparison to those effects arising from the "noise level" in other sensors. This type of problem is readily overcome by means of weighting factors.

The preceding situation has a strong resemblance to the ill-posed problem where it is sought to determine attitude solely from solar data. The solution to such a problem is mathematically ambiguous with respect to rotations about the solar line of sight. In such a situation, the error function does not have a well defined minimum.

There are some well defined error functions which are only piece-wise continuous with respect to certain parameters of the motion. From a geometrical point of view, it is clear that a broken error surface could defeat a scheme to locate its minima. During the convergence process, the argument U of $q(U)$ must not stray into a region where $q(U)$ and its first derivatives are not defined.

Consider the case of a stably spinning weather satellite which is equipped with horizon pulse sensors. As already pointed out, the pulse itself is not suitable for differential correction. It is the time of the pulse which conveys the useful information. Consider then an error function which is a sum of the squared time differences between the predicted and observed pulse times. Such a function is a well defined error function, being continuous with continuous first derivatives, in the neighborhood of the solution or correct attitude. If the parameters of the motion α and δ, which define the orientation of the predicted spin axis wander too far from their correct values, then the predicted line of sight of the I. R. sensor will fail to intersect the horizon. Clearly, the error function is then ill defined. It is then said to be piece-wise defined with respect to α and δ . Each such type of problem can be handled individually by observing straight forward precautions.

Since they are often functions of orientation angles (α, δ, φ_0, θ_0, ψ_0), the error functions of attitude determination are likely to be periodic. For example, α and $\alpha \pm 2n\pi$ are equivalent if n is an integer. An interesting example is the pair (α, δ) and the equivalent pair ($\alpha \pm \pi$, $\pi - \delta$). Innumerable such examples can readily be found. This means that when non-orientation parameters are well known (and preferably held constant), there is a 100% likelihood that convergence will take place. Moreover, it requires a small number of iterations, usually less than five, for the argument U to "land" inside a region of fast convergence. This is true even when all five orientation phase angles are being adjusted. In constrast to the preceding example, there exists a type of periodicity which arises when a sinusoidal signal is sampled uniformly. It is known [19, 27] that a uniformly sampled sinusoid has a mathematically ambiguous frequency. This phenomenon, sometimes called aliasing [19], may be encountered when processing the outputs of solar sensors [41]. No iteration scheme alone can ascertain when an aliased frequency has been located--independent considerations must be invoked. See Figure 10.

As mentioned, the second order Newton method is prone to diverge in highly nonlinear problems owing to its tendency to "overshoot" and its poor "cornering." The method of gradients (steepest descent) is an example of an attempt to circumvent these difficulties. Unfortunately their convergence properties are poor. In references [32] and [33] the method of gradients and the Newton method are effectively combined to provide the best performance of which each is capable. Other techniques include the method of conjugate directions, [11] the variable metric method, [31] the random walk method, [40] and the method of relaxation [40, 46]. For a comprehensive survey of methods, consult references [25], [46], and [65].

APPENDIX C

NON-EULERIAN METHODS OF DEFINITION

The discussion in this paper has adhered to the Eulerian representation of the motion, i. e. to the use of Euler angles. It should be mentioned that the entire discussion could be applied to several other methods of representing the motion of rigid bodies.

Roll, Pitch, and Yaw

It is often desirable to employ the angles sometimes called yaw, pitch, and roll. In this case the transformation $\mathcal{L}$ remains as before. Instead of the Euler transformation, $\mathcal{a}$, however, one would employ a transformation obtained as a product of the three successive rotations defined by the said angles. Since the Euler transformation is constructed in like manner, the distinction between the two methods is negligible until it is desired to compute the angles as a function of time. In this situation one would see whether the definitions of the angles conform to alternate definitions of Euler angles. If so, the analysis would carry over. Otherwise the methods employed in deriving the formulas IV.4 could be imitated. Alternate definitions of Euler angles are shown in references [2 and 3].

Euler Symmetrical Parameters and Cayley - Klein Parameters

Other methods of specifying the orientation of rigid bodies are available. Of particular interest are those avoiding the difficulties which accompany the use of Euler angles. These difficulties are i) the equations of motion IV.3 are not symmetrical, ii) when $\theta = 0$, we have the "gimbal lock effect" in which the line of nodes is ill-defined and the coefficients in the equations of motion are singular. and iii) trigonometric functions must be evaluated. These difficulties are overcome with the aid of Euler's four symmetrical parameters [3, 40] (sometimes appearing as Hamilton quaternions [42, 63]) or with Cayley-Klein parameters [1, 42]. Unfortunately these parameter sets exceed the number of rotational degrees of freedom and therefore lead to the use of Lagrange multipliers [62]. Ambiguities arising in connection with the Cayley-Klein parameters are explained in reference [1]. One could, of course, also employ the nine direction cosines of the vehicle's axes [40]. Since they satisfy several constraint relationships, they present special difficulties [61].

APPENDIX D

BASIC TRANSFORMATION BETWEEN TWO INERTIAL SYSTEMS OF COORDINATES

We wish to obtain the coordinate transformation relating an inertial (cartesian) system to a second (cartesian) system for which merely the z axis direction is specified. This direction is specified by two angles. The first two Euler rotations fulfill these requirements and, hence, the desired expression is obtained by multiplying equations II.12b and II.12c. The result is

$$\Gamma \equiv C\mathscr{D} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi\cos\theta & \cos\varphi\cos\theta & \sin\theta \\ \sin\varphi\sin\theta & -\cos\varphi\sin\theta & \cos\theta \end{bmatrix}. \qquad \text{D.1}$$

Referring to figure 3, it is clear that, if the first system is a vernal equinox equatorial system, the right ascension and declination are related to the first two Euler angles according to

$$\alpha = \varphi - \pi/2 \qquad \text{D.2a}$$

$$\delta = \pi/2 - \theta. \qquad \text{D.2b}$$

Substituting these expressions into D.1 we obtain equation II.4.

Notice that equation D.1 occurs often in the literature of celestial mechanics and orbit theory. To see this, replace φ with Ω and θ with i; the longitude of the ascending node and the orbital inclination, respectively.

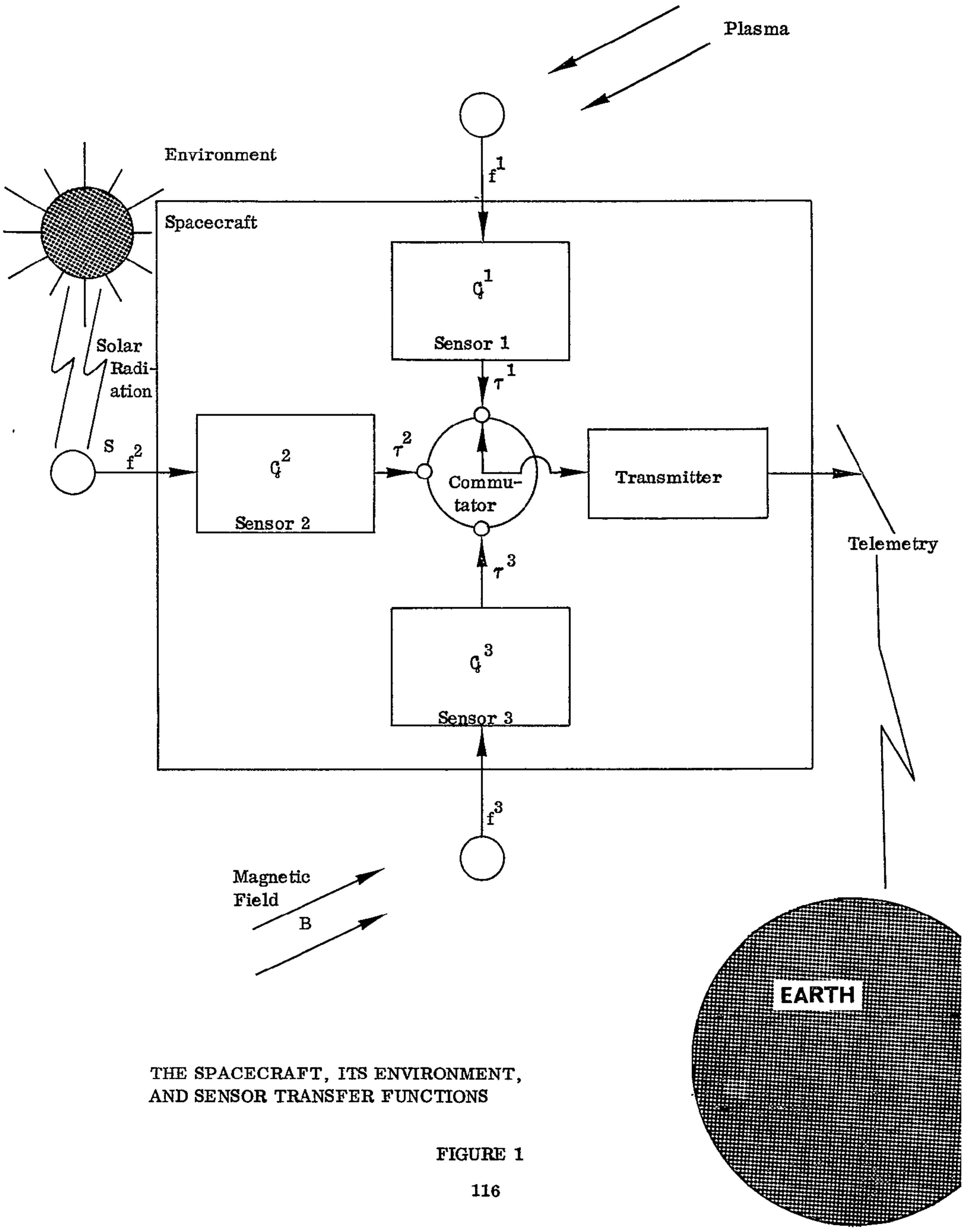
APPENDIX E

DEFINITION OF LINEAR VECTOR MANIFOLD

Let Ξ be a ring containing the multiplicative identy 1, the elements of Ξ being a, b, c ... Then a class of objects X, Y, Z ... are called a linear vector manifold or vector space M if and only if for any X in M and a in Ξ:

1. $a\ X$ is in M
2. $(ab)X = a(bX)$
3. $(a + b)X = aX + bX$
4. $a(X + Y) = aX + aY$
5. $X + Y = Y + X$
6. $X + (Y + Z) = (X + Y) + Z$
7. $X + 0 = X$
8. $X + (-X) = 0$.

Properties 5, 6, 7, 8 are recognized as the definition of a group with the binary operator of vector addition. That is to say, M is a commutative group with the null vector 0 and $-X$ acting as the inverse of $+X$. See Reference [66].

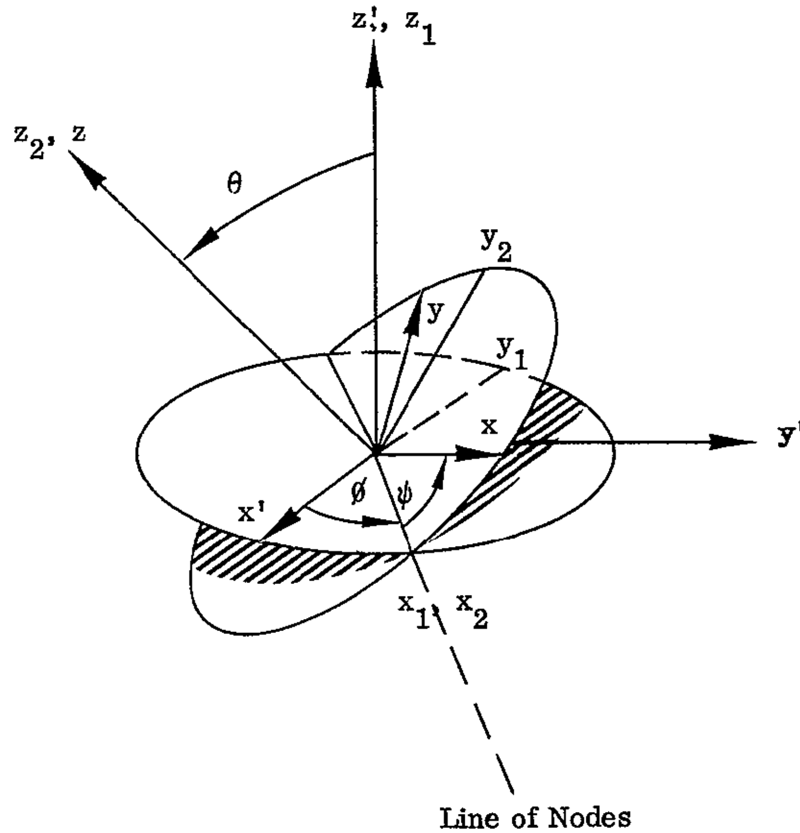


THE SPACECRAFT, ITS ENVIRONMENT,
AND SENSOR TRANSFER FUNCTIONS

FIGURE 1

TRANSFORMATION OF AXES

BY EULERIAN ANGLES



x', y', z'	Initial axes
x_1, y_1, z_1	Intermediate axes
x_2, y_2, z_2	Intermediate axes
x, y, z	Transformed Axes

FIGURE 2

THE Γ TRANSFORMATION

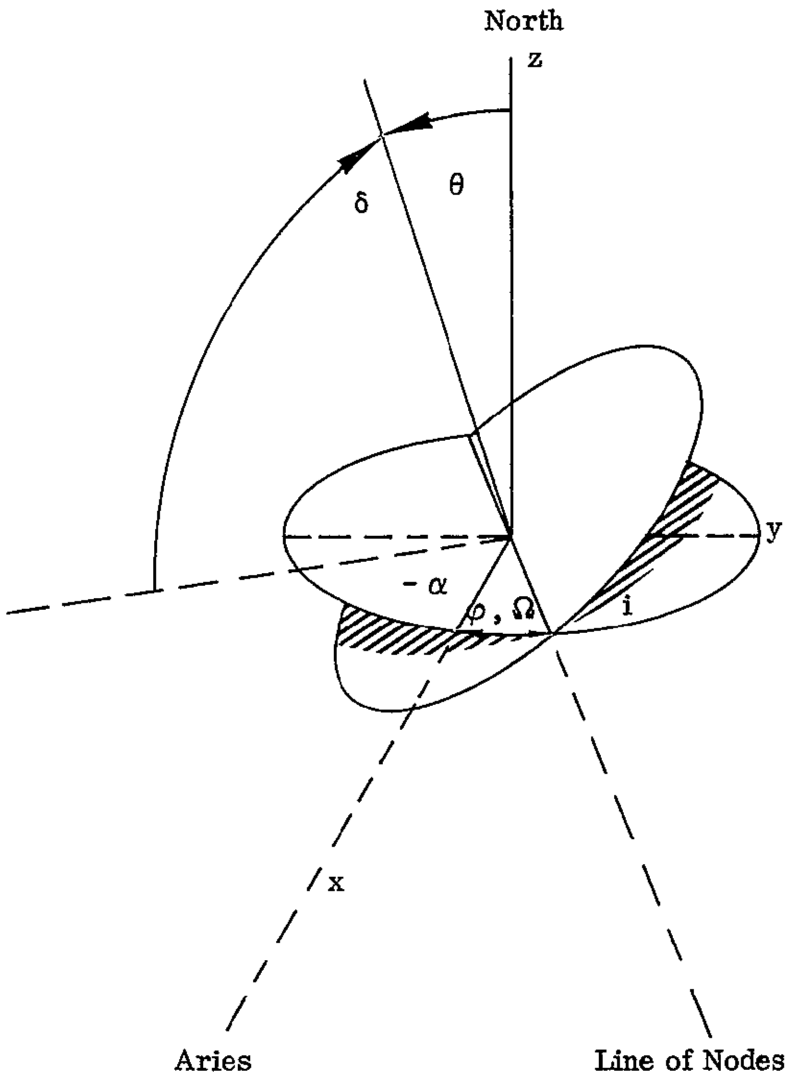
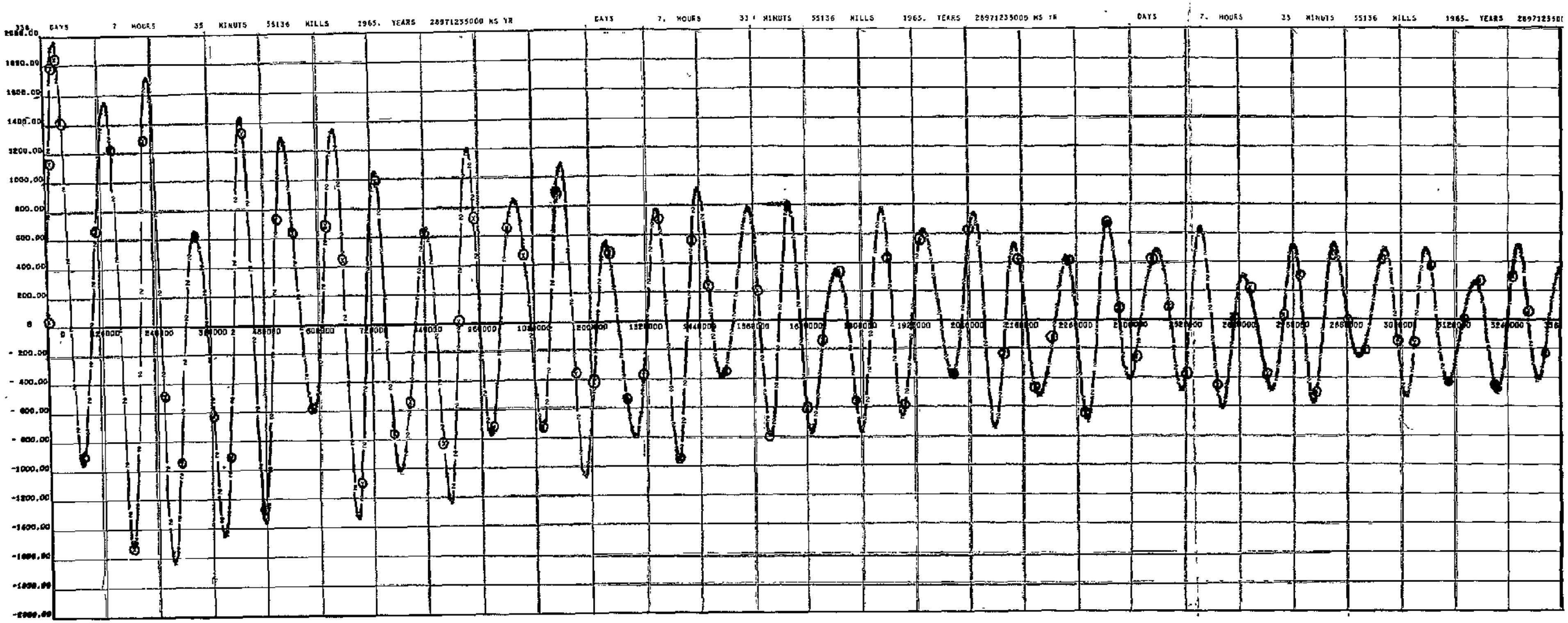


FIGURE 3



EPE-D X MAGNETOMETER DURING PRECESSION

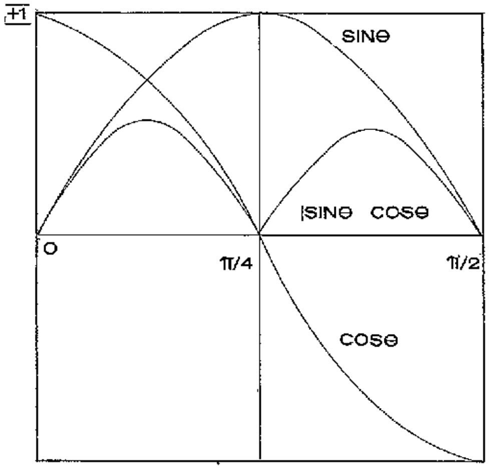
DASHED LINE=PREDICTED FUNCTION

FIGURE 4

FOLDOUT FRAME 2

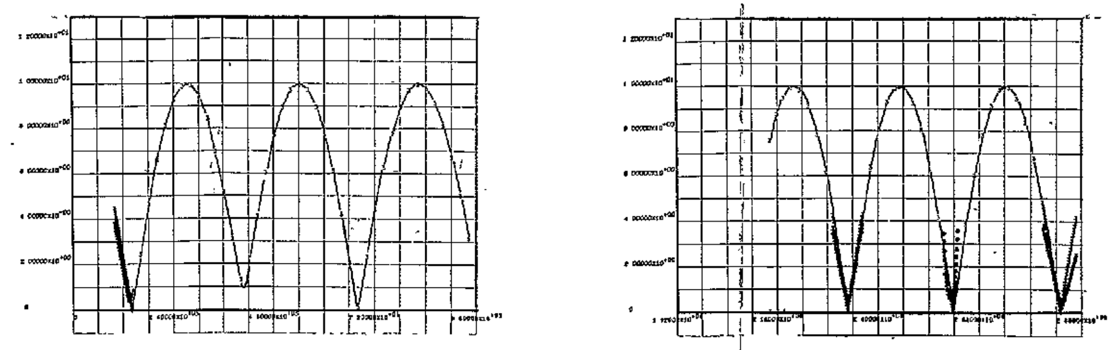
FOLDOUT FRAME 1

A



FUNCTION $\overline{SIN\Theta \cdot COS\Theta}$

B



POGO OSCILLATOR AMPILITUDE VS PREDICTED AMPLITUDE
DASHED LINE=PREDICTED FUNCTION

C

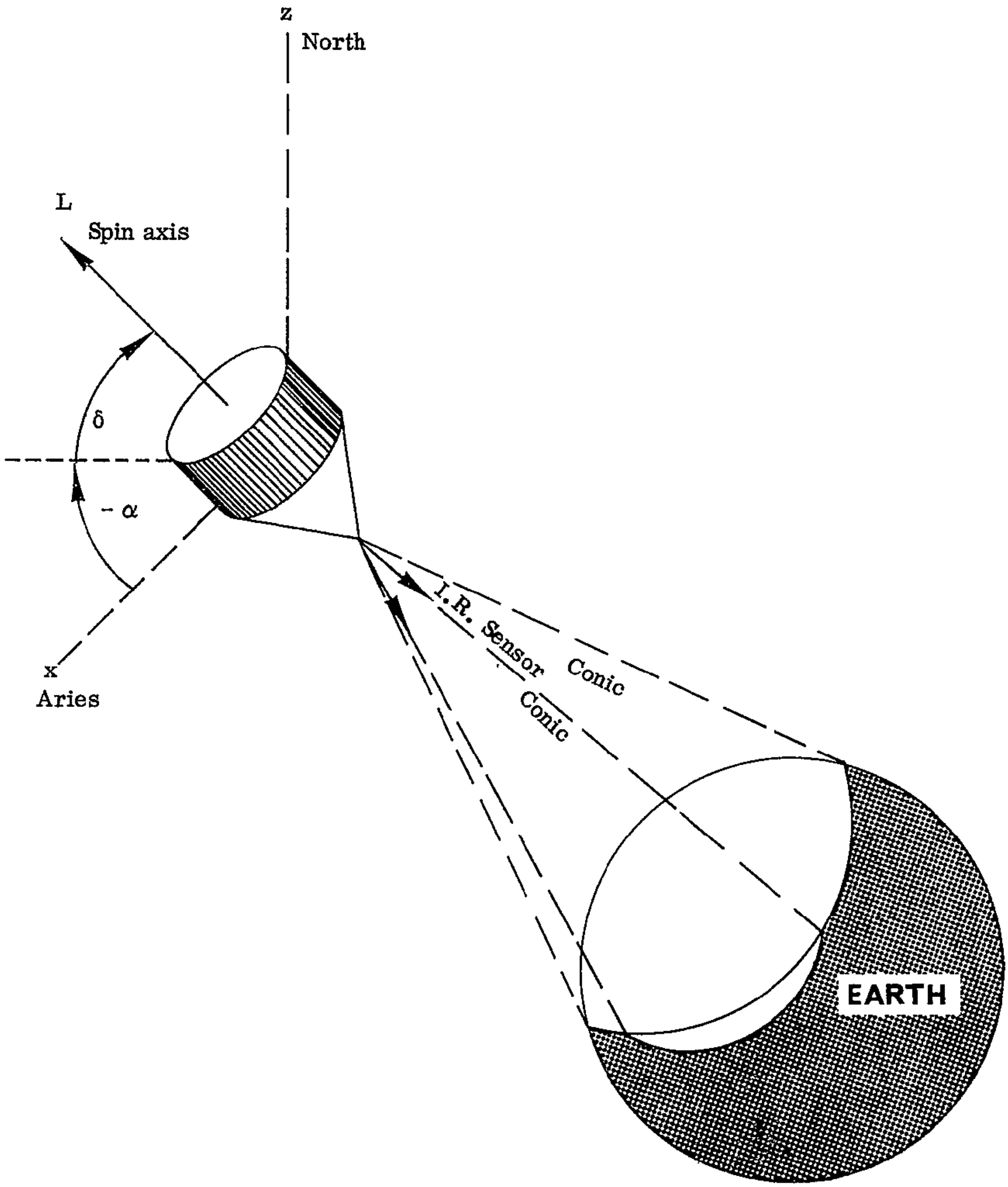
POGO RAW OSCILLATOR AMPLITUDE

CONTINUATION

FIGURE 5

FOLDOUT FRAME 2

FOLDOUT FRAME



SPINNING SPACECRAFT WITH IR SENSORS

FIGURE 6

NEWTON DIFFERENTIAL CORRECTION PROCEDURE
IN SEVERAL VARIABLES FOR VECTOR FUNCTIONS

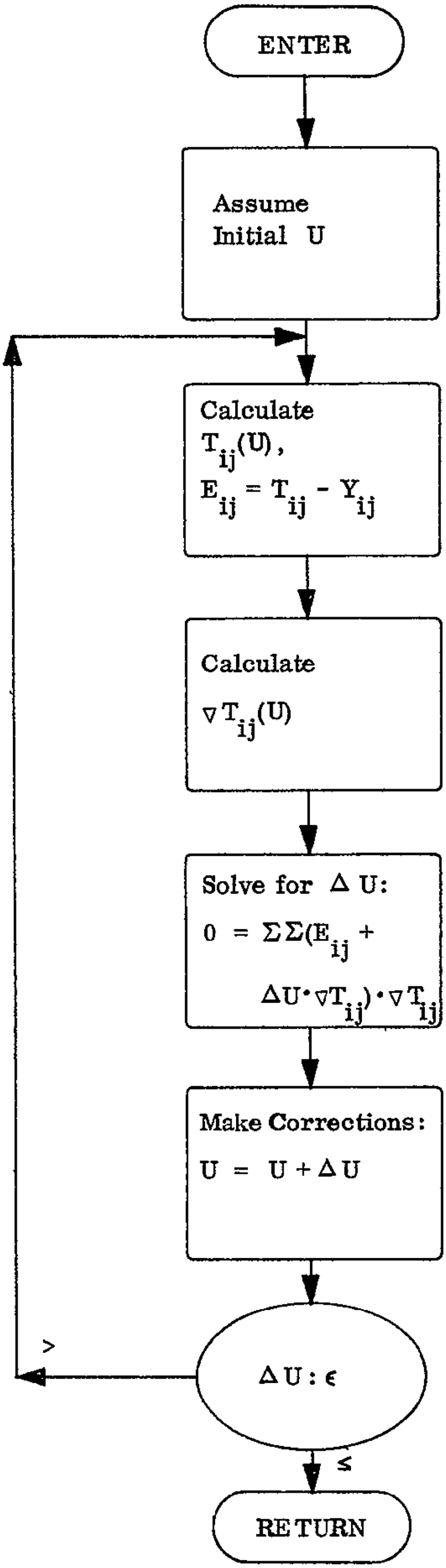
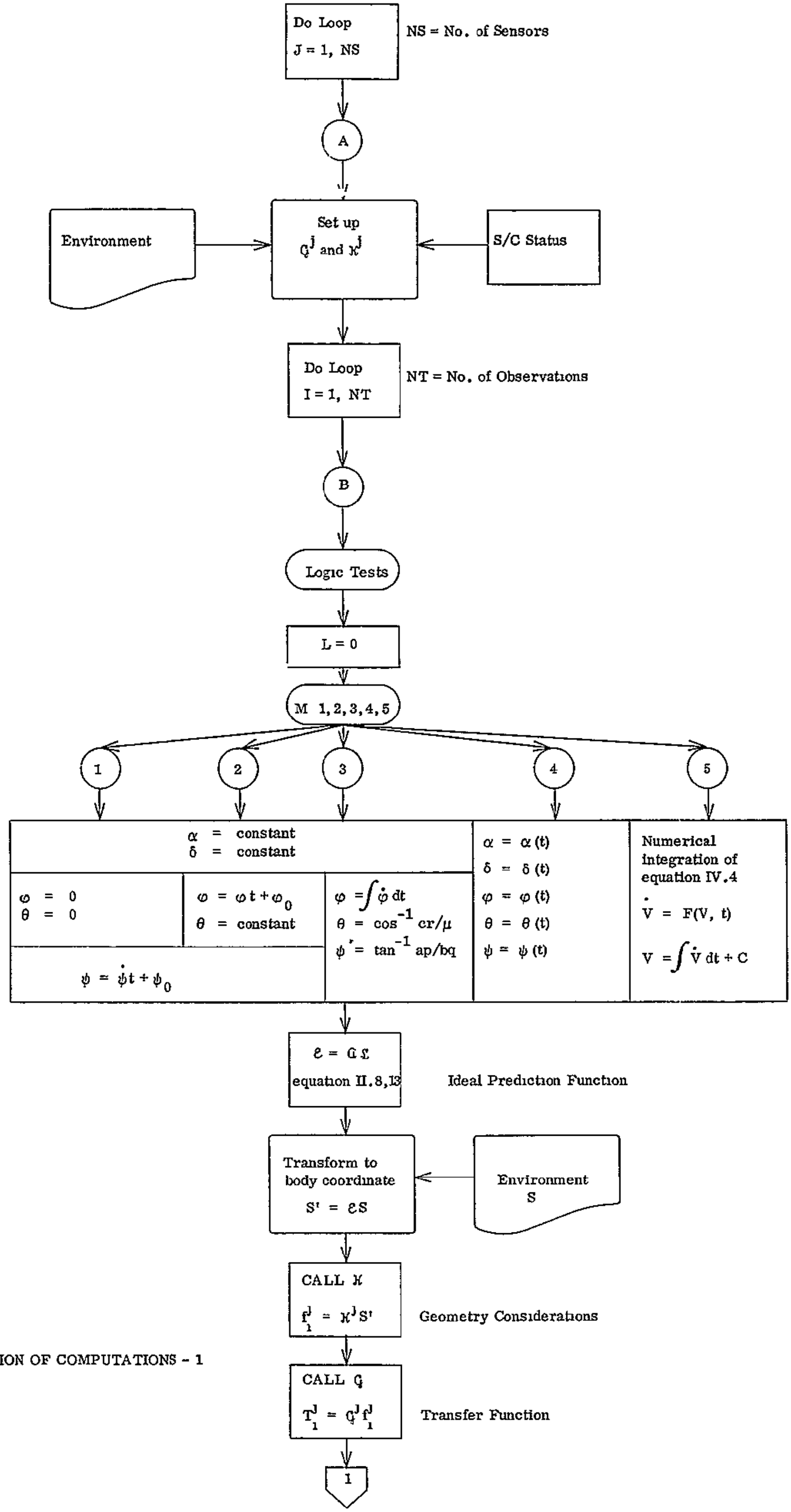
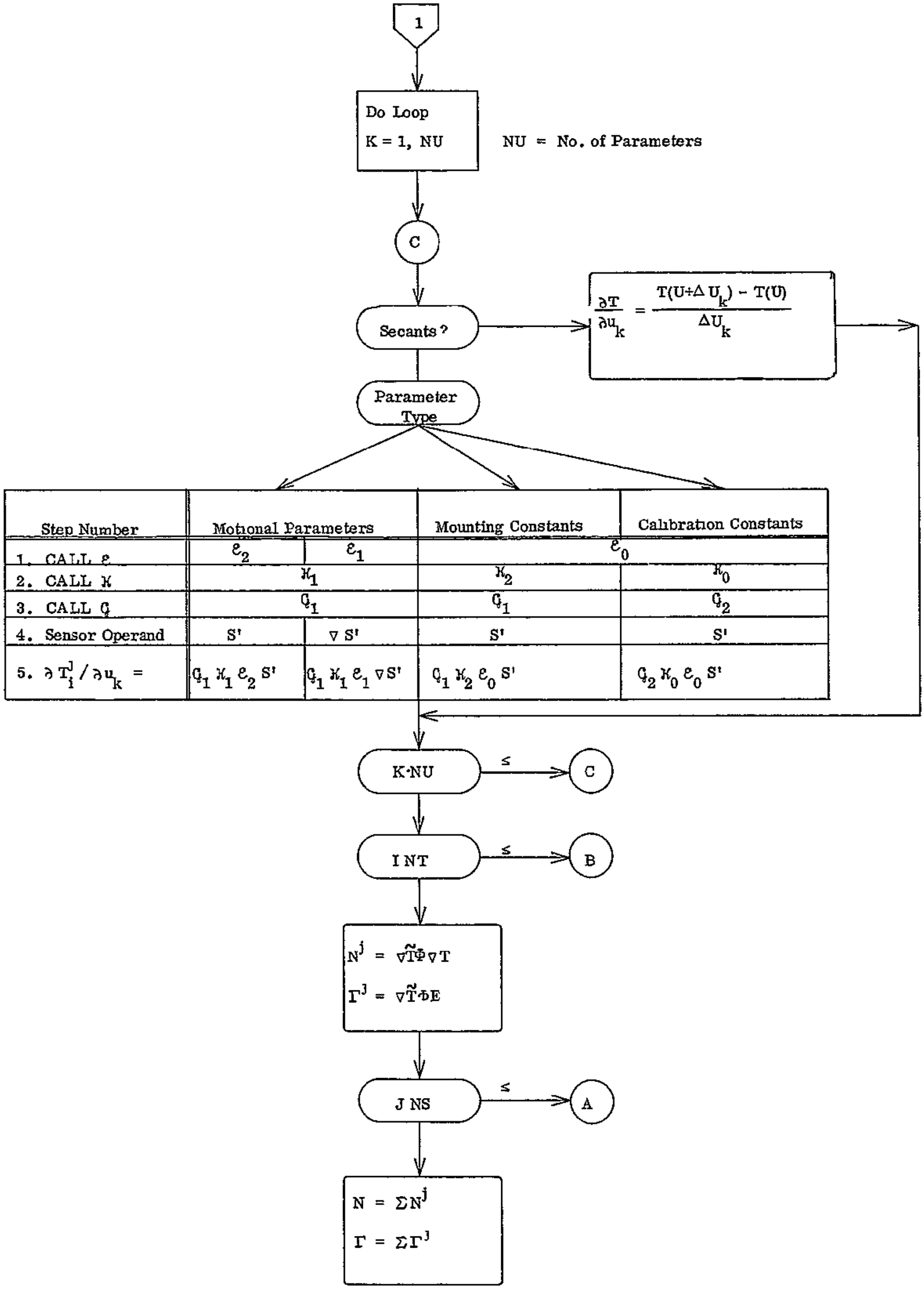


FIGURE 7



MECHANIZATION OF COMPUTATIONS - 1

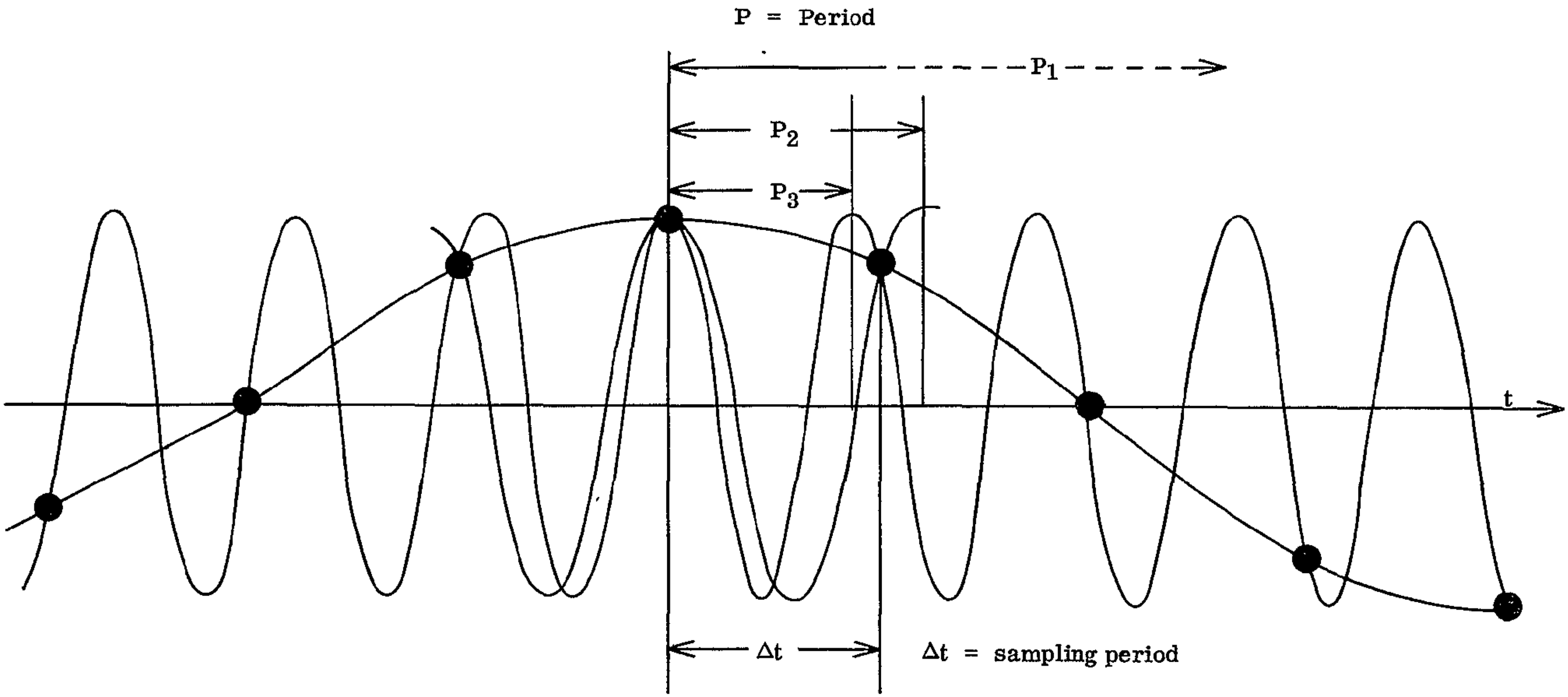
FIGURE 8



Step Number	Motional Parameters		Mounting Constants	Calibration Constants
1. CALL $\mathcal{E}$	$\mathcal{E}_2$	$\mathcal{E}_1$	$\mathcal{E}_0$	
2. CALL $\mathcal{K}$	$\mathcal{K}_1$		$\mathcal{K}_2$	$\mathcal{K}_0$
3. CALL $\mathcal{G}$	$\mathcal{G}_1$		$\mathcal{G}_1$	$\mathcal{G}_2$
4. Sensor Operand	S'	∇ S'	S'	S'
5. $\partial T_i^j / \partial u_k$ =	$\mathcal{G}_1 \mathcal{K}_1 \mathcal{E}_2 S'$	$\mathcal{G}_1 \mathcal{K}_1 \mathcal{E}_1 \nabla S'$	$\mathcal{G}_1 \mathcal{K}_2 \mathcal{E}_0 S'$	$\mathcal{G}_2 \mathcal{K}_0 \mathcal{E}_0 S'$

MECHANIZATION OF COMPUTATIONS - 2

FIGURE 9



FREQUENCY AMBIGUITY FOR UNIFORMLY SAMPLED SINUSOIDS

FIGURE 10

IX. REFERENCES

1. Goldstein, H.: "Classical Mechanics." Addison-Wesley, 1953.

2. Ames, J. S., and Murnaghan, F. D.: "Theoretical Mechanics." Dover, 1953.

3. Whittaker, E. T.: "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies." Dover, 1944.

4. Leimanis, E.: "The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point." Springer-Verlag, 1965.

5. Forsyth, A. R.: "A Treatise on Differential Equations." Macmillan and Company, 1951.

6. Ford, L. R.: "Differential Equations." McGraw-Hill, 1933.

7. Kaplan, Wilfred,: "Advanced Calculus," Addison-Wesley, 1952.

8. Traub, J. F.: "Iterative Methods for the Solution of Equations."

9. Todd, John: "Survey of Numerical Analysis," McGraw-Hill, 1962.

10. Beckett, R. and Hurt, J.: "Numerical Calculations and Algorithms," McGraw-Hill, 1967.

11. Householder, A. S.: "Principles of Numerical Analysis," McGraw-Hill, 1953.

12. Scarborough, J. B.: "Numerical Mathematical Analysis," Johns Hopkins, 195

13. Hamming, Richard W.: "Numerical Methods for Scientists and Engineers," McGraw-Hill, 1962.

14. Baker, R. M. L., and Makemson, M. W.: "An Introduction to Astrodynamics," Academic Press, 1960.

15. Blackman, R. B.: "Linear Data-Smoothing and Prediction in Theory and Practice," Addison-Wesley, 1965.

16. Bellman, Richard: "Introduction to Matrix Analysis," McGraw-Hill, 1960.

17. Faddeev, D. K. and Faddeeva, V. N.: "Computational Methods of Linear Algebra," Freeman, 1963.

18. Rogers, A. E., and Connolly, T. W.,: "Analog Computation in Engineering Design," McGraw-Hill, 1960.

19. Blackman, R. B., and Tukey, J. W.: "The Measurement of Power Spectra," Dover, 1958.

20. Lanczos, Cornelius: "Applied Analysis," Prentice Hall, 1964.

21. Abramowitz M. and Stegun, I. A.: "Handbook of Mathematical Functions," N. B. S. Applied Mathematics Series 55, 1965.

22. Redish, K. A.: "An Introduction to Computational Methods."

23. Hadley, G.: "Nonlinear and Dynamic Programming," Addison-Wesley, 1964.

24. Wendroff, Burton: "Theoretical Numerical Analysis."

25. Spang, H. A.: "A Review of Minimization Techniques for Nonlinear Functions," SIAM Review, 4, (1962).

26. Scott, W. T.: "The Physics of Electricity and Magnetism," John Wiley, 1959.

27. Guillemin, E. A.: "The Mathematics of Circuit Analysis," John Wiley, 1959.

28. Farrel; James L.: "Attitude Determination by Kalman Filtering," NASA CR-598, 599.

29. Wheeler, P. C., Nishinaga, R. G., Zaremba, J. G., and Williams, H. L.,: "Evaluation of a Semi-Active Gravity Gradient System," NASA CR-593, 4.

30. Wolfe, P.: "The Secant Method for Simultaneous Nonlinear Equations," Communications of the ACM 2(1959).

31. Davidon, W. C.: "Variable Metric Method for Minimization," ANL 5990.

32. Marquardt, Donald W.: "An Algorithm for Least-Squares Estimation of Nonlinear Parameters," J.S.I.A.M. 11 No. 2, 431, June 1963.

33. Daniels, Walter E. Jr.,: "General Least-Squares With Statistics" University of Maryland, Department of Physics and Astronomy, T. R. 579.

34. Wheeler, Phillip C.: "Magnetic Attitude Control of Rigid, Axially Symmetric, Spinning Satellites in Circular Earth Orbits," NASA CR-313.

35. "Feasibility Study of Passively Oriented Lenticular Satellites," Final Report, General Electric Corp., Contract NAS 5-2324, NASA CR-53938.

36. Roberson, R. E.: "Gravitational Torque on a Satellite Vehicle," J. Franklin Institute. 265, (1958).

37. Battin, Richard H.: "A Statistical Optimizing Navigation Procedure for Space Flight," ARS Journal 32, No. 11 (1962).

38. "Sequential Position and Covariance Estimation (SPACE) Trajectory Program Analytical Manual," Sperry Rand Systems Group, Report No. AB-21-0001, NASA Contract NAS 5-3509.

39. Fein, A. E.: "A Differential Correction Procedure for Fitting Dynamically Constrained Curves to Physical Data," Westinghouse Report MDE-6626, May 6, 1966.

40. Sabroff, A., Farrenkopf, R., Frew, A., and Gran, M.: "Investigation of the Acquisition Problem in Satellite Attitude Control," TRW Technical Report AFFDL-TR-65-115, December 1965.

41. Chapman, G. A., and Nowotny, A. J., et al: "Satellite Telemetry Reduction Integrated Processing System (STRIPS)," Computer Sciences Corporation Report for NASA Contract WD-9758-34/35-1, May 1966.

42. Lundquist, C. A., and Naumann, R. J.: "Orbital and Rotational Motion of a Rigid Satellite," JPL Seminar Proceedings.

43. Davenport, P. B.: "A Vector Approach to the Algebra of Rotations with Applications," NASA-X-546-65-437, 1965.

44. Albus, J. S.: "Attitude Determination of Spin Stabilized Spacecraft Using a Digital Aspect Sensor," NASA-X-631-65-13.

45. Vrablik, G. R. and Kent, E. F.: "Studies of the Vanguard II (1959 Alpha) Satellite, Part I," Allied Research Associates, Inc., Nov. 1960.

46. Deutsch, R.: "Estimation Theory," Prentice Hall, 1965.

47. Shchigolev, B. M.: "Mathematical Analysis of Observations."

48. Churchill, R. V.: "Fourier Series and Boundary Value Problems," McGraw Hill, 1941.

49. Blumberg, M. H., et al: "Attitude Determination System," NASA-X-547-64-180, June 1964.

50. Farthing, Harry (NASA-Goddard): personal communication.

51. Grove, Wendell E.: "Brief Numerical Methods."

52. Crocker, J. P.: "OSO-E Buffer Simulation System," Computer Sciences Corporation Report for NASA Contract NAS 5-9758, August 1966.

53. Baker, R. M. L.: "Astrodynamics, Applications and Advanced Topics," Academic Press, 1967.

54. Henrici, Peter: "Elements of Numerical Analysis," Wiley, 1967.

55. Brouwer, D. and Clemence, G. M.: "Methods of Celestial Mechanics," McGraw-Hill, 1961

56. Cayley, A.: "An Elementary Treatise on Elliptic Functions," Dover, 1961.

57. Greenhill, A. G.: "The Applications of Elliptic Functions," Dover, 1959.

58. Ott, R. H., Jr. et al; "The Determination of the Orientation of a Uniformly Rotating Vehicle Utilizing the Outputs from Solar Sensors and a Lateral Magnetometer," NASA-X-721-66-95.

59. Wahba, G.: "Problem 65-1, A Least Squares Estimate of a Satellite Attitude," SIAM Review 7, 409, 1965.

60. "OSO Aspect System Manual," TM65-1, July 15, 1965, Ball Brothers Research.

61. Redisch, W. M.: "Equilibrium Analysis and Its Applications to Attitude Control Systems," NASA TR-R-270, August 1967.

62. Steulpnagel, J.: "On the Parametrization of the Three-Dimensional Rotation Group," SIAM Review 6 4(1964).

63. Robinson, A. C.: "On the Use of Quaternions in Simulation of Rigid Body Motion," WADC TR 58-17.

64. Chapman, G. A. and Nowotny, A. J. et al: "EOGO Attitude Determination Program (S49R)," Computer Sciences Corporation, June 1965.

65. Leitman, G.: "Optimization Techniques with Applications to Aerospace Systems," Academic Press, 1962.

66. Korn, G. A. and Korn, T. M.: "Mathematical Handbook for Scientists and Engineers," McGraw-Hill, 1961.

67. Jeffreys, H. and Jeffreys, B. S.: "Methods of Mathematical Physics," Cambridge, 1956.

68. Deutsch, R.: "Orbital Dynamics of Space Vehicles," Prentice Hall, 1963.

69. Lecar, M.: "A Method for Estimating Residuals in Orbital Theory," NASA TN D-493, (1961).

70. Kurnosova, L. V. (editor): "Artificial Earth Satellites," Plenum Press, 1960.

71. Roy, Maurice (editor): "Dynamics of Satellites," Paris Symposium, 1962.

72. Roberson, R. E. and Tatistcheff, : "Potential Energy of a Small Body in a Gravitational Field," J. Franklin Institute (pp.209 - 214) 262 (1956).

73. Ketchum, H. B.: "Advances in Astronautical Sciences," volume 1, Plenum Press, (pp. 31-41).

74. Nidey, R. A.: "Gravitational Torque on a Satellite of Arbitrary Shape," ARS 30 (pp. 203-4) (1960).

75. Frick, R. M. and Garber, T. B.: "General Equations of a Satellite in a Gravitational Field," Rand Report RM-2527, December 1959.

76. DeBra, D. B.: "The Effect of Aerodynamic Forces on Satellite Attitude," J. of the Astronautical Sciences 6, (1959) pp.40ff

77. Smith, G. L.: "Effects of Eddy Current Torques on Spin Motions of an Earth Satellite", NAST TN D-2198.

78. Ball, K. J and Osborne, G. F.: "Space Vehicles Dynamics", Oxford University Press, 1967.

79. Singer, F.: "Torques and Attitude Sensing in Earth Satellites", Academic Press, 1964.

80. General Dynamics Corporation: "Dynamic Stability of Space Vehicles" NASA TN CR-942 (Volume VIII) and TN CR-945 (Volume XI).

81. Wilks, S. S.: "Mathematical Statistics", John Wiley and Sons, 1962.

82. Swerling, P.: "Parameter Estimation Accuracy Formulas, " IEEE, Transactions PGIT, IT-10 (October 1964), 302-314.

83. NASA TN D-608 (page 188)

84. Burington, R. S. and May, D. C.: "Handbook of Probability and Statistics with Tables," Handbook Publishers, 1953.

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