NONEQUILIBRIUM PLASMA BOUNDARY LAYER OVER A CATHODE

by

ROBERT KINCAID SEALS, JR.

A thesis submitted to the Graduate Faculty of
North Carolina State University of Raleigh
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING

RALEIGH

1 9 7 0

APPROVED BY:

____
Chairman of Advisory Committee

______________________________ ______________________________

______________________________ ______________________________

ABSTRACT

SEALS, ROBERT KINCAID, JR. Nonequilibrium Plasma Boundary Layer Over a Cathode. (Under the direction of HASSAN AHMAD HASSAN).

An analysis of the nonequilibrium plasma boundary layer over a cathode in the presence of a magnetic field is presented. The case considered is that where the Debye length is less than the mean free paths and the Larmor radii, and where the thermal and nonequilibrium effects are important. The solution is obtained by matching the sheath solution and the solution of the conservation equations of the plasma constituents. The transport properties for a partially ionized gas in the presence of nonequilibrium ionization and magnetic fields are obtained from appropriate equations derived by the thirteen moment method.

Local similarity is assumed, and the problem is solved numerically using an integral method on an IBM Model 360-75 digital computer. Results are presented for a range of values of magnetic fields and axial distance and for two values of wall temperature. The results indicate that the heat flux at the wall, the current density, and the electron temperature all tend to increase with increasing magnetic induction.

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Dr. H. A. Hassan, Chairman of his Advisory Committee, for his assistance and guidance during the course of this study. Gratitude is also extended to the North Carolina State University Computing Center for the use of their facilities, without which this work would have been impossible.

This research was supported in part by funds from the National Aeronautics and Space Administration under Grant NGR 34-002-048.

Finally the author wishes to thank his wife, Priscilla, and his parents for their understanding and confidence.

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

This study is concerned with the plasma boundary layer over the cathode of a plasma accelerator. The main purpose of this investigation is to gain insight into the various loss mechanisms over an electrode and to study the effects of various parameters of the flow on such quantities as the heat transfer and the current densities near the wall.

Several investigations have been made regarding the boundary layer in the internal flow of a compressible plasma. Kerrebrock (1961) considered an equilibrium magnetohydrodynamic boundary layer in an accelerator with a constant free stream temperature and cooled walls. Due to a low conductivity near the wall, he predicted a considerable amount of Joule heating which would result in a large temperature gradient and large heat fluxes. The Hall effect was neglected, and the electron temperature was assumed to be equal to that of the heavy particles. Local similarity was also used. Oates *et al.* (1962) considered an elevated electron temperature which was determined from a simple energy balance rather than from a complete electron energy equation. He found that the values of the conductivity were increased near the wall over those predicted by equilibrium theory, and thus the Joule heating was greatly reduced. Hale and Kerrebrock (1964) considered the laminar compressible boundary layer on the insulator walls of an MHD channel. They also used local similarity, but they did include the Hall effect in the Ohm's law. The nonequilibrium effect was taken into account by using a conductivity proportional to the current density rather than by accounting for the electron temperature behavior. They predicted large current concentrations that would increase the wall shear and heat transfer greatly over what would be the case for a normal boundary layer. In all of the above investigations there was no need to consider the sheath structure. Sherman and Reshotko (1969) considered the nonequilibrium boundary layer along an insulator wall. Their analysis allowed for an electron temperature different from the heavy particle temperature by taking into consideration the complete electron energy equation. However, chemical nonequilibrium effects were not included, and the Saha equation with the temperature replaced by the electron temperature was used to predict the electron number density. When the complete electron energy equation is employed, one has to consider the sheath structure to obtain the necessary boundary conditions on the electron temperature. In their work Sherman and Reshotko (1969) assumed that the electron energy flux was continuous at the sheath edge. The free molecule expression for the electron energy flux was obtained from a simple sheath model which did not take surface effects into consideration. The Hall effect and ion slip were included in the generalized Ohm's law, but all the gradients were assumed negligible. They showed that the velocity and heavy particle temperature profiles were influenced slightly by changes in the electron temperature; on the other hand, small changes in the wall temperature influenced the electron temperature significantly. Cott (1970) also considered the boundary layer over an insulator and took into consideration finite ionization rates. He assumed a cold wall which was fully-catalytic. He concluded that thermal nonequilibrium can be important in long channels, and that the Hall effect should not be neglected. However, the operation of the device considered in his study was not noticeably affected by nonequilibrium or the physics of the sheath.

In addition to the approximations previously mentioned, all of the references cited utilized flux vectors which were based on the mean free path method. The present investigation employs a model which removes all the above approximations. It incorporates the sheath model developed by Hassan (1968) which allows for emission, reflection, surface ionization, and incomplete accommodation. In addition, finite ionization and recombination rates which play an important role in plasma accelerators are taken into consideration. The flux vectors for a partially ionized gas in the presence of magnetic fields and nonequilibrium ionization are obtained from appropriate equations developed by Aliyevskiy and Zhdanov (1963) using Grad's "13-moments" method. Thus rigorous kinetic theory expressions replace the approximate expressions derived by the mean free path method.

The calculations presented here are for a plasma boundary layer over a plane cathode in the presence of a magnetic field. The boundary layer approximations are used to reduce the governing equations to a set of six simultaneous nonlinear partial differential equations. The assumption of local similarity reduces these equations to a set of ordinary nonlinear differential equations. The over-all heat flux, the electron current density, and the over-all current density are required to be continuous at the sheath edge. This requirement, along with the free stream conditions, a no-slip condition at the sheath edge, and a given wall temperature, result in a two point boundary value problem for the six simultaneous nonlinear differential equations. The solution to the problem is obtained by writing the system in integral form and solving the resulting integral equations by iteration. The computations were carried out using an IBM Model 360-75 digital computer.

The main symbols utilized throughout this thesis are presented in Appendix A.

THE INTEGRAL FORM OF THE GOVERNING EQUATIONS

This study is concerned with a plasma boundary layer over a plane cathode. The geometry of the problem is shown in Figure 1. The governing system of equations and the transport property relations are developed in Appendix B and Appendix C, respectively. The sheath solution, which gives the boundary conditions at the sheath edge, is developed in Appendix D.

The momentum equation, given by equation (34B), can be integrated formally; the result can be expressed as

$$\frac{d^2f}{d\eta^2} = (\rho\mu)^{-1}\int_0^{\eta} F_1(x)\,dx + [\rho\mu\int_0^{\eta_\delta} dx/\rho\mu]^{-1}$$

$$\{1 - \int_0^{\eta_\delta} [(\rho\mu)^{-1}\int_0^{x} F_1(y)\,dy]\,dx\} \tag{1}$$

$$\frac{df}{d\eta} = \int_0^{\eta} [(\rho\mu)^{-1}\int_0^{x} F_1(y)\,dy]\,dx + [\int_0^{\eta_\delta} dx/\rho\mu]^{-1}$$

$$\times\{1 - \int_0^{\eta_\delta} [(\rho\mu)^{-1}\int_0^{x} F_1(y)\,dy]\,dx\}\int_0^{\eta} dx/\rho\mu \tag{2}$$

$$f = \int_0^{\eta} \left(\frac{df}{d\eta}\right) d\eta \quad . \tag{3}$$

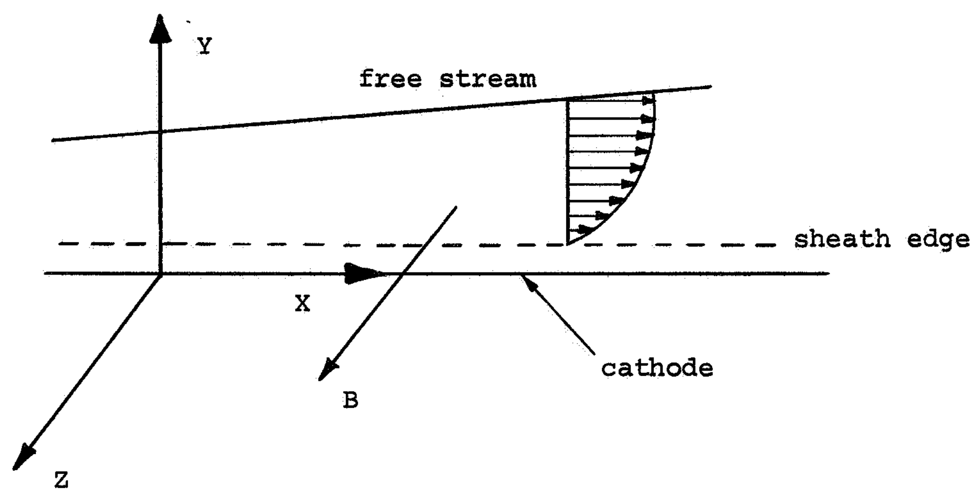

Figure 1. Schematic of geometry of problem

The boundary conditions used for this integration are

$$f = \frac{df}{d\eta} = 0 \qquad \text{at} \qquad \eta = 0$$

$$\frac{df}{d\eta} = 1 \qquad \text{at} \qquad \eta = \eta_\delta \quad . \tag{4}$$

Equation (33B), which is the electron conservation equation, can be integrated formally with respect to η once to give a relation for the electron diffusion velocity. The resulting expression is combined with the diffusion velocity relation provided by equation (30C). The combined expression is then solved for the derivative of the degree of ionization and integrated formally to give

$$\alpha = \alpha_\delta - \int_\eta^{\eta_\delta} \{\gamma_{34} - \gamma_{32} \frac{dg}{d\eta} - \gamma_{33} \frac{d\theta}{d\eta} + \frac{2.5\ p}{\rho\alpha} [\frac{je_0}{2.5 \frac{e}{m} p} - \int_0^x F_2\ (y)\ dy]\}\ dx/\gamma_{31} \quad . \tag{5}$$

The sheath solution provides the necessary boundary condition on the current density at $\eta = 0$.

If equation (35B) is combined with equation (32C), an expression for the derivative of the dimensionless temperature can be obtained. If this relation is then integrated formally using the boundary condition

$$g = g_w \qquad \text{at} \qquad \eta = 0 \tag{6}$$

the result is given by

$$g = g_w + q(0) \int_0^{\eta} (\gamma_{12})^{-1} dx + \int_0^{\eta} (\gamma_{12})^{-1} \{\gamma_{14} - \gamma_{11} \frac{d\alpha}{d\eta} - \gamma_{13} \frac{d\theta}{d\eta}$$

$$+ \int_0^{x} F_3 (y) \, dy\} \, dx \quad . \tag{7}$$

Since the neutrals are in good contact with the cathode, the temperature at the sheath edge is assumed equal to the wall temperature. The heat flux at $\eta = 0$ is determined from equation (7) and the condition that

$$g = 1 \qquad \text{at} \qquad \eta = \eta_\delta \tag{8}$$

and is given by

$$q\ (0) = \{\int_0^{\eta_\delta} (\gamma_{12})^{-1} dx\}^{-1} \{1 - g_w$$

$$- \int_0^{\eta_\delta} (\gamma_{12})^{-1} [\gamma_{14} - \gamma_{11} \frac{d\alpha}{d\eta} - \gamma_{13} \frac{d\theta}{d\eta} + \int_0^{x} F_3 (y) \, dy] \, dx\} \quad . \tag{9}$$

If a similar procedure is used with equations (36B) and (31C), an expression for the derivative of the dimensionless electron temperature can be obtained. Requiring continuity of the over-all heat flux between the boundary layer and the sheath provides the necessary boundary condition on the electron temperature at $\eta = 0$. This requirement and the condition that

$$\theta = 1 \qquad \text{at} \qquad \eta = \eta_\delta \tag{10}$$

provide the necessary boundary conditions to integrate the electron energy equation formally. The resulting expression is

$$\theta = \theta_0 + q_e(0) \int_0^{\eta} (\gamma_{23})^{-1} dx + \int_0^{\eta} (\gamma_{23})^{-1} \{\gamma_{24} - \gamma_{22} \frac{dg}{d\eta} - \gamma_{21} \frac{d\alpha}{d\eta} + \int_0^{x} F_4(y)\, dy\} dx \tag{11}$$

where

$$q_e(0) = \{\int_0^{\eta_\delta} (\gamma_{23})^{-1} dx\}^{-1} \{1 - \theta_0 - \int_0^{\eta_\delta} (\gamma_{23})^{-1} [\gamma_{24} - \gamma_{22} \frac{dg}{d\eta} - \gamma_{21} \frac{d\alpha}{d\eta} + \int_0^{x} F_4(y)\, dy]\, dx\} \quad . \tag{12}$$

In this study the over-all current density is taken to be constant throughout the boundary layer and is assumed to be equal to the sheath current density. This allows equation (38C) to be solved for the electric field strength. This completes the casting of the differential equations into integral equations.

METHOD OF SOLUTION

The solution of this problem involves solving six integral equations. These are equations (1), (2), (3), (5), (7), and (11). In order to solve these equations, several quantities must be specified. First, the free stream conditions must be given. Values of the wall temperature, the magnetic induction, the position of the station under consideration, and the various surface parameters must be specified also.

The method employed here is that of successive substitutions. Initial values of f, α, g, and θ, along with the required derivatives, must be assumed for the entire η-range of zero to η_δ. Given these initial profiles, the integral equations can be used to generate a new set of profiles. This procedure is repeated until a certain accuracy criterion is met. For this work the required accuracy was 0.5×10^{-4}.

When one employs the integral method, a value must be specified for η_δ. At this value of η, all the properties approach their free stream values, and their derivatives approach zero. An η_δ of 6 seems to meet the above requirements. However an η_δ of 5 gives almost identical values for the heat flux at the wall. Because of this and because of the increased computer storage and time requirements for an η_δ of 6, all the results presented here are for an η_δ of 5.

The actual procedure is described next. The initial profiles must be chosen first. The Blasius solution is used to give initial profiles for f and its derivatives. The initial temperature profile is determined by choosing a polynomial which meets the boundary conditions and which gives a shape similar to the temperature profiles in previous work such as that by Sherman and Reshotko (1969). Initially the electron temperature profile is chosen to be identical to that of the gas temperature. Various profiles for the degree of ionization are used initially using a trial and error procedure. It appears that the problem is fairly sensitive to the initial selection of profiles. Unless the initial profiles, particularly that of the temperature, are reasonably similar to the correct solution, the problem tends to be unstable. Considerable effort is required to obtain a solution for the first case. After obtaining any first case, however, subsequent cases can be obtained by making small increments in the variables of interest.

Given the initial profiles, the sheath solution is used to give the current density. Then the various integrals are evaluated using, in this case, a three point integration formula. Before one can proceed further, the value of θ_0 must be determined. This is done as follows. Equation (9) gives the value of the heat flux at $\eta = 0$. Using this value and requiring the value of the heat flux obtained from the sheath solution to be the same enables one to determine a value for θ_0 by some iterative procedure. This value of the electron temperature is then used in equation (12) to generate a value for the electron heat flux at $\eta = 0$ and to compute a new current density. Thus all the quantities on the right-hand sides of the integral equations can be determined. New profiles for the desired variables are obtained, and the process is repeated until convergence is reached. The derivatives of the various properties are determined from the expressions for these properties before the next iteration is performed. For example

$$\frac{dg}{d\eta} = (\gamma_{12})^{-1} \{q\ (0) + \gamma_{14} - \gamma_{11} \frac{d\alpha}{d\eta} - \gamma_{13} \frac{d\theta}{d\eta} + \int_0^\eta F_3\ (x)\ dx\} \quad . \qquad (13)$$

RESULTS AND DISCUSSION

There are numerous parameters in this problem whose effects might be of interest. However, the complexity of the problem makes it impractical to investigate all of them. In this investigation three parameters are chosen for study: the magnetic induction, the independent variable ξ, and the wall temperature.

The calculations presented here are for an argon plasma and a tungsten cathode. The values used for the ionization potential of argon and the work function of tungsten are

$$I = 15.75 \text{ electron volts} = 0.2523 \times 10^{-10} \text{ ergs}$$
$$w = 4.54 \text{ electron volts} = 0.7273 \times 10^{-11} \text{ ergs} \quad . \tag{14}$$

The cgs-esu system of units is employed throughout.

There are several parameters whose values must be stated in order to begin the calculations. The various reflection coefficients, the accommodation coefficient for electrons, and the potential drop in the sheath region are chosen as:

$$\alpha_{0,1} = \alpha_{\infty,1} = \alpha_{\infty,2} = \alpha_{\infty,3} = 0.01$$
$$\beta_{0,1} = \beta_{\infty,1} = \beta_{\infty,2} = \beta_{\infty,3} = 0.10$$
$$\gamma_2 = 0.3 \quad ; \quad (\phi_\infty - \phi_0) = 1.2484 \text{ electron volts} = 0.2 \times 10^{-11} \text{ ergs} \quad . \tag{15}$$

The free stream conditions are taken to be

$$p_\delta = p = 3.75 \text{ mm of Hg} = 5000 \text{ dynes/cm}^2$$

$$u_{x_\delta} = 45{,}000 \text{ cm/second}$$

$$T_\delta = T_{e_\delta} = 4000°K$$

$$\alpha_\delta = 0.12 \quad . \tag{16}$$

For the purposes of this investigation, the free stream conditions are assumed to be constant.

The remaining quantities that need to be assumed are the wall temperature, the magnetic induction, and the variable ξ. For constant free stream conditions, ξ is a measure of the distance along the cathode. Two wall temperatures are used: 2500°K and 2900°K. The magnetic induction is allowed to range between zero and 1000 gauss; ξ is varied between 1×10^{-6} and 2×10^{-5}, the latter value corresponding to an axial distance of approximately 1 cm.

The assumption of local similarity will be discussed first. When the heat flux at the wall has the simple representation

$$q_w = - k \frac{dT}{dy} \qquad \text{at} \qquad y = 0 \tag{17}$$

one obtains, after using the transformation introduced in equation (21B),

$$q_w = - \frac{\rho u_{x_\delta}}{\sqrt{2\xi}} k \frac{dT}{d\eta} \qquad \text{at} \qquad \eta = 0 \quad . \tag{18}$$

It can be seen that for similar solutions the heat flux at the wall will vary inversely with the square root of ξ. Thus, the plot of $\sqrt{\xi}\, q_w$

versus ξ given in Figure 2 is an indication of the accuracy of the local similarity assumption. Despite the fact that the expression for q_w given by equation (32C) is much more complicated than that given by equation (17), it is evident from Figure 2 that, for the range of parameters considered, local similarity is not a bad approximation.

Figure 3 shows the variation of the electron temperature at $\eta = 0$ with ξ. Figure 4 illustrates the variation of the current density with ξ. The behavior of both the electron temperature at the sheath edge and the current density can be explained from the requirement of continuity of the fluxes and the expressions for the current density and heat flux given by equations (1D), (2D), and (12D). Since q_w is proportional to $(T_e)^{3/2}$ and q_w decreases with increasing ξ, T_e decreases with ξ. Similarly, since the current density is proportional to $(T_3)^{1/2}$, it also decreases with increasing ξ. Because the current density is primarily made up of the electron current density, one would expect it to be negative for a cathode. However, the values of current density shown in Figure 4 are all positive, corresponding to a net flux of electrons toward the cathode. This is due to the low cathode temperature of 2500°K and the resulting low amount of thermionic emission. This also accounts for the low values of current density obtained for this wall temperature.

Figures 5 and 6 show the effect of the magnetic induction upon the electric field and the current density. The electric field tends to decrease with increasing B, while the current density increases with increasing B.

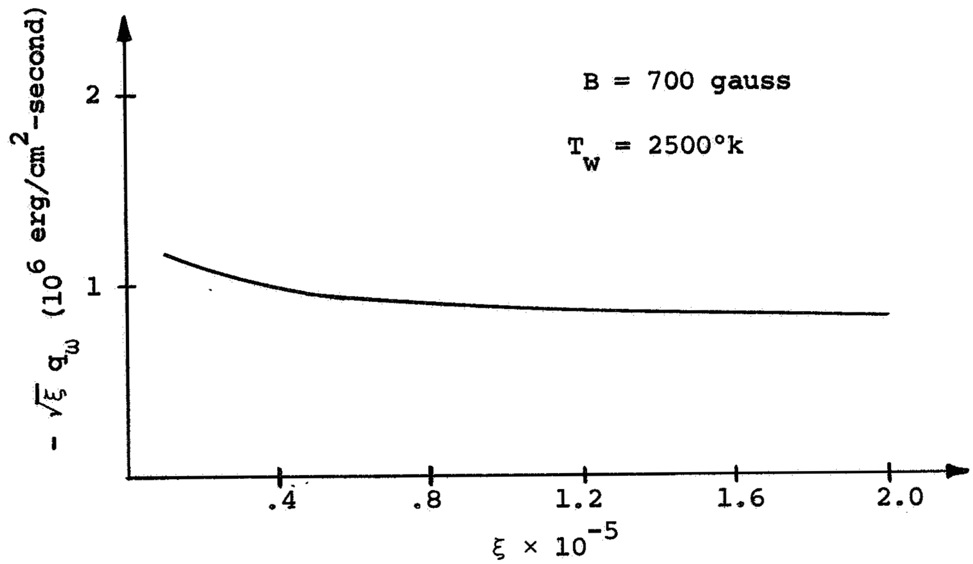

Figure 2. Heat flux at the wall versus ξ

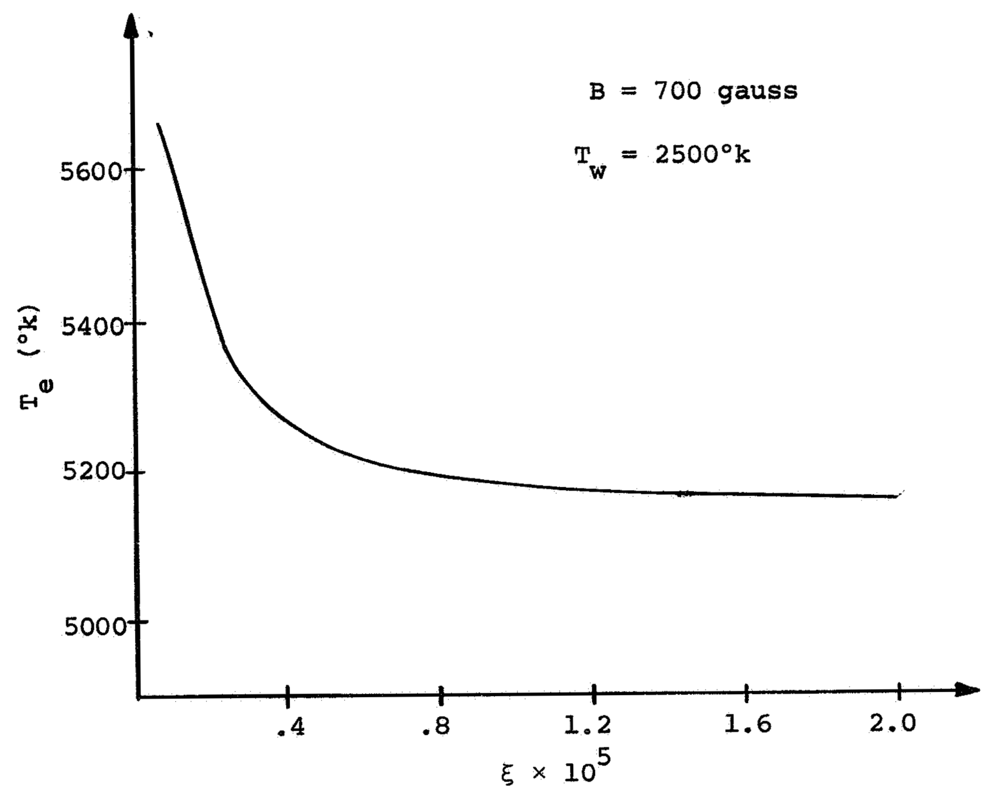

Figure 3. Electron temperature at the sheath edge versus ξ

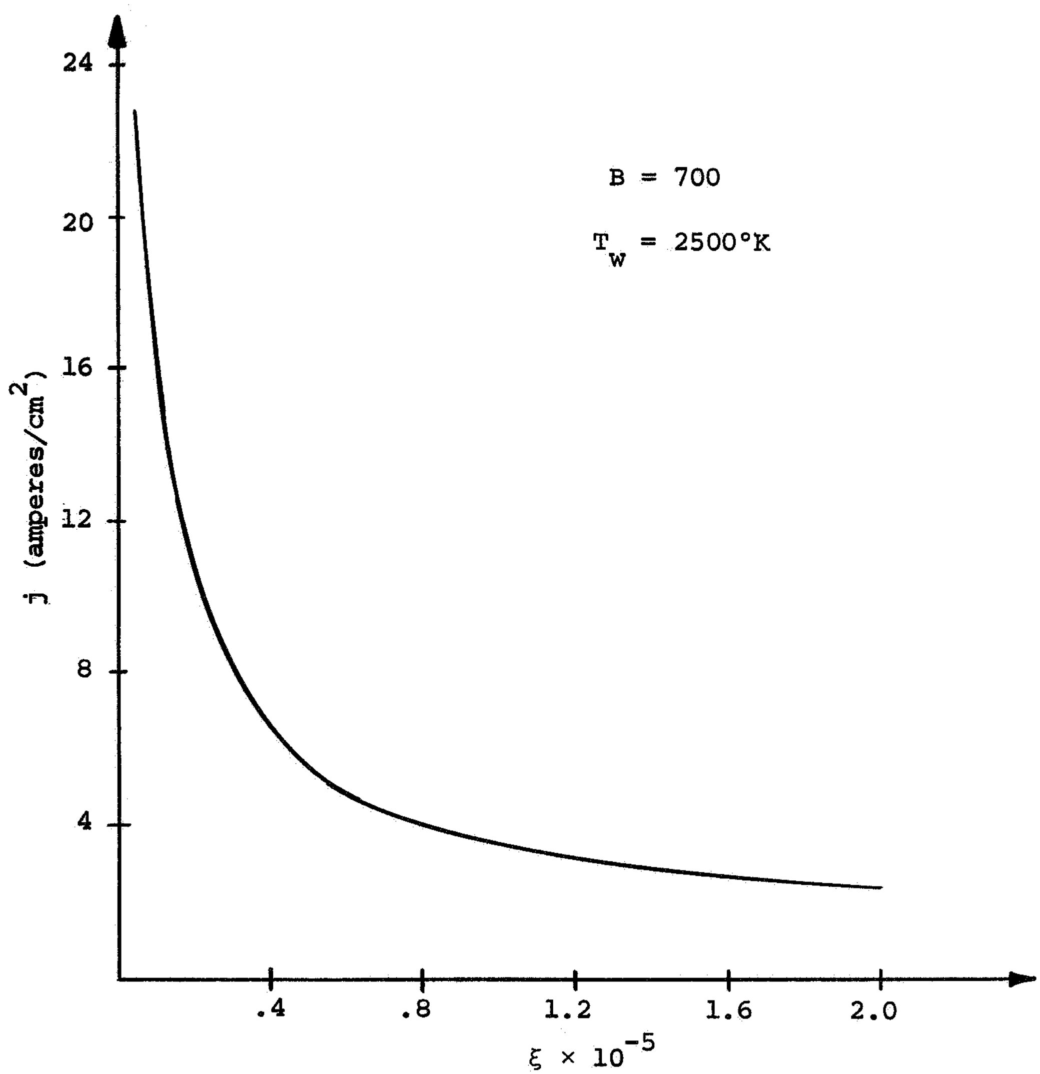

Figure 4. Current density versus ξ

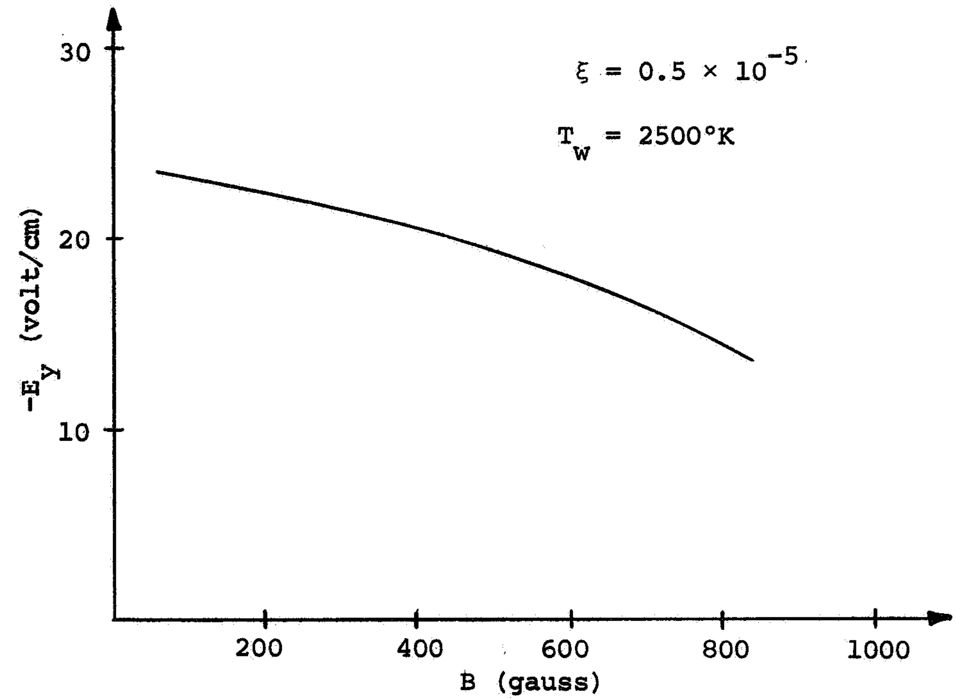

Figure 5. Electric field strength versus magnetic induction

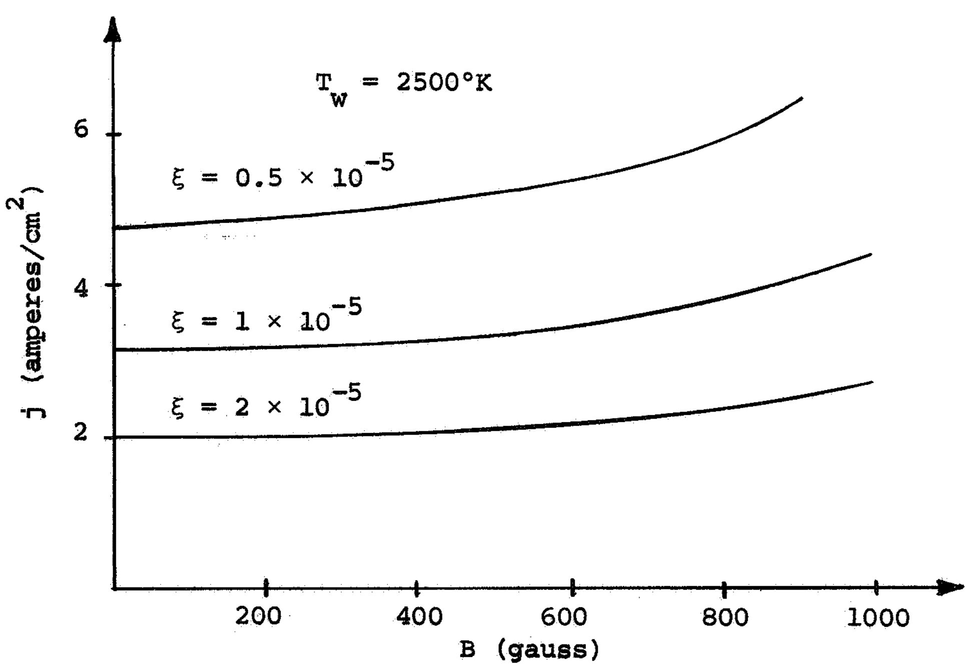

Figure 6. Current density versus magnetic induction

This effect cannot be explained by a simple Ohm's law of the type

$$j = \sigma E \tag{19}$$

because σ decreases with increasing B. Thus it is apparent that the gradients play an important role, and a simplified Ohm's law, such as that used by Sherman and Reshotko (1969) or Cott (1970), would not be adequate. With the general expressions used here, the current density increases as the magnetic induction is increased. Since the current density at the sheath edge is proportional to $(T_e)^{1/2}$, one would expect the electron temperature to increase with increasing B also. Figure 7 shows this to be the case. For small values of B, the electron temperature profile is unusual and has two inflection points. The gradient of the electron temperature is proportional to the difference between the power input to the electrons $(\vec{j}_e \cdot \vec{E})$ and the loss terms resulting from collisions with the heavy particles and heat conduction. When B is small, one cannot single out a dominating mechanism in a given region of the boundary layer. On the other hand, when B is large the power input term dominates in the region next to the sheath and losses dominate the outer region. Figure 8 shows that the heat flux at the wall also increases with increasing magnetic induction. The effect is small for the lower values of B, but it is more pronounced for values above 500 gauss. This follows from the consideration that q_w is proportional to $(T_e)^{3/2}$, and T_e increases more rapidly for large values of B. For the cases investigated here, the magnetic induction has little effect on the wall shear.

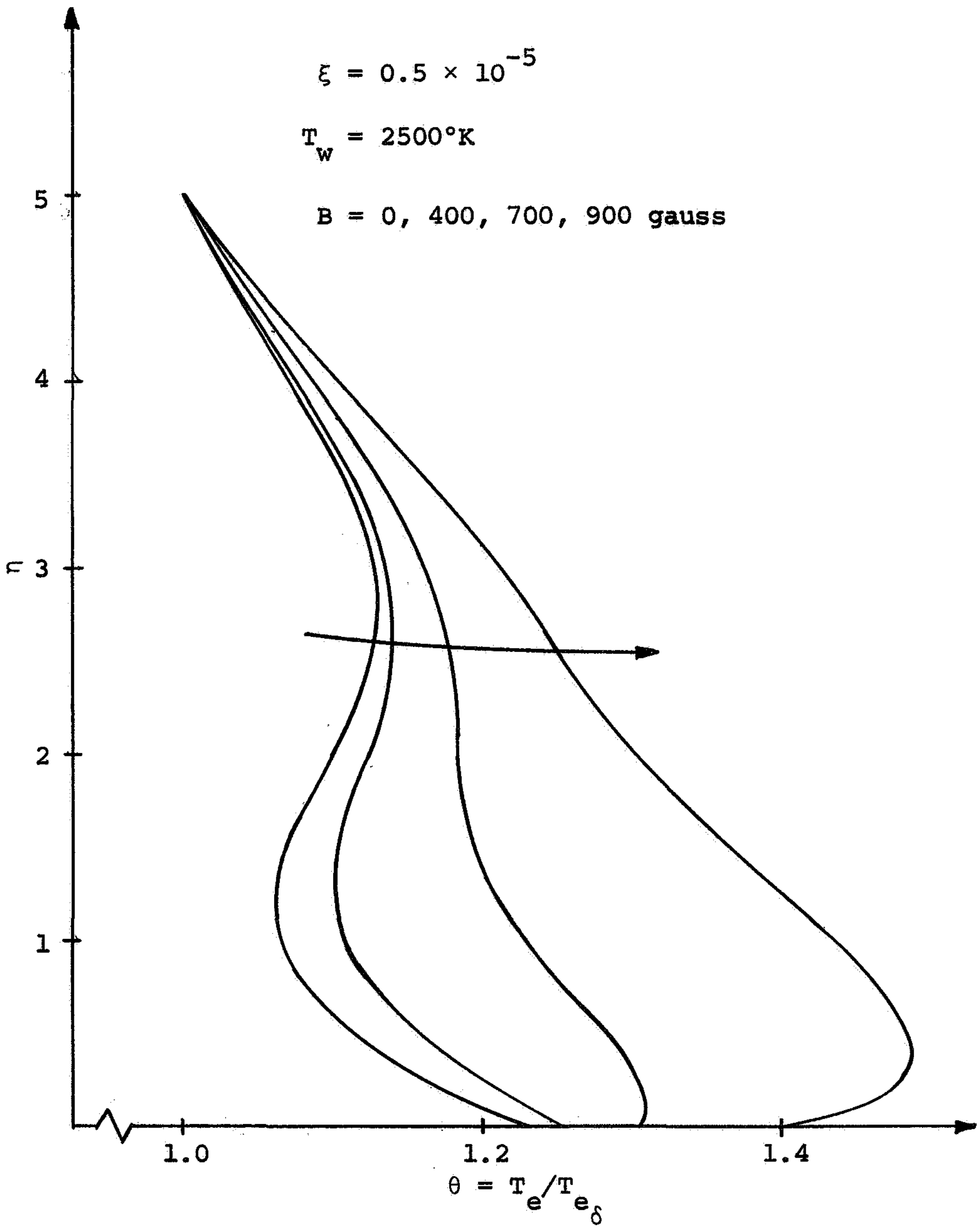

Figure 7. Transformation variable (η) versus electron temperature

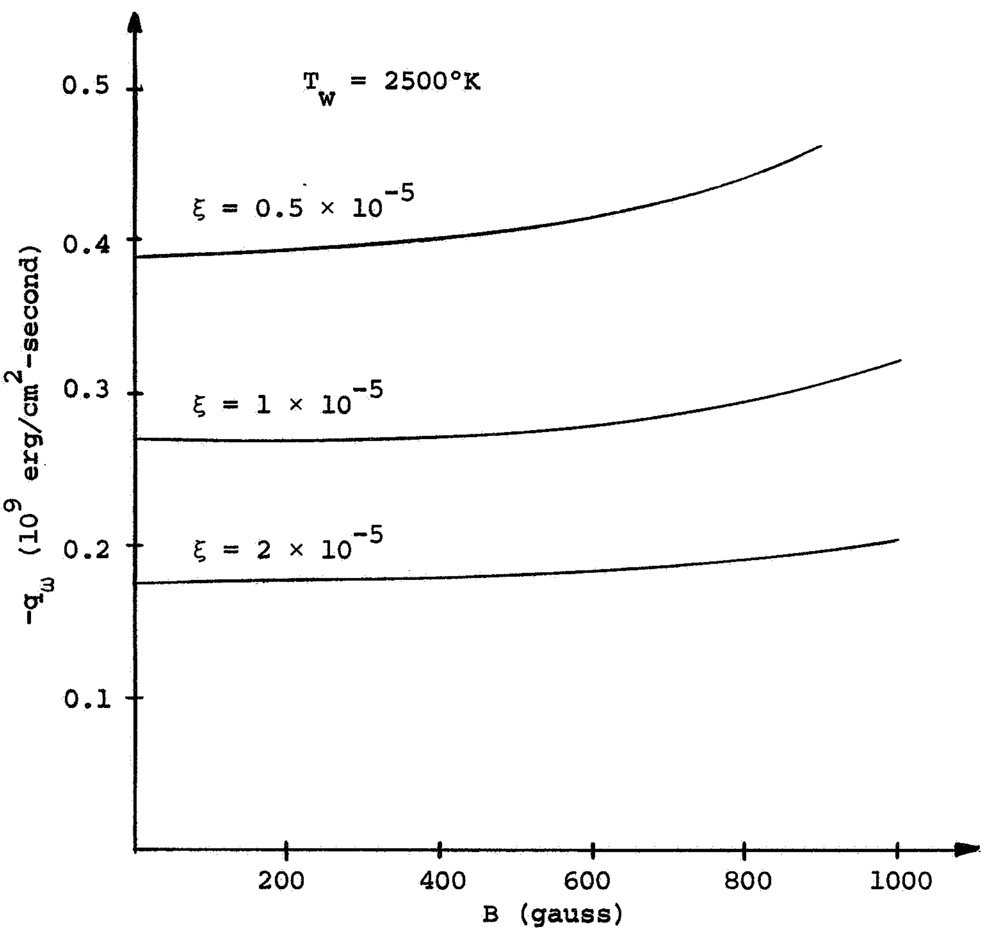

Figure 8. Heat flux at the wall versus magnetic induction

For the 2900°K cathode temperature case, more electron emission is expected. Thus the current density should be negative and greater in magnitude than for the 2500°K case, and the heat flux at the wall should show the cooling effect of emitted electrons. For comparison, the cases for which $\xi = 1 \times 10^{-5}$ and B = 700 gauss are chosen. For the lower cathode temperature the current density is 3.58 amperes/cm^2; while for the 2900°K case, the current density is -5.913 amperes/cm^2. The heat flux at the wall is 0.1009×10^9 ergs/cm^2-second for the 2900°K case as compared to -0.2829×10^9 ergs/cm^2-second for the 2500°K case. The positive value of the heat flux away from the cathode indicates that electron emission is beginning to dominate. The effect of cathode temperature upon the electron temperature profile is shown in Figure 9. For the 2900°K case the electron temperatures are considerably higher. This can be explained by looking at the simplified electron energy equation

$$\vec{j}_e \cdot \vec{E}^* - \frac{2}{3} n_e \left[\sum_{t \neq e} \left(\frac{m_e n_t}{m_t}\right) Z_{et}^{(11)} \right] \sqrt{\frac{2kT_e}{m_e}}\, 3k\,(T_e - T) = 0 \quad . \qquad (20)$$

Since $T_e \gtrsim T$ and the current density, composed primarily of the electron current density, is greater for the higher cathode temperature, examination of equation (20) shows that the electron temperature must be larger for the 2900°K case. The influence of the cathode temperature on the profiles of the temperature and degree of ionization is shown in Figures 10 and 11.

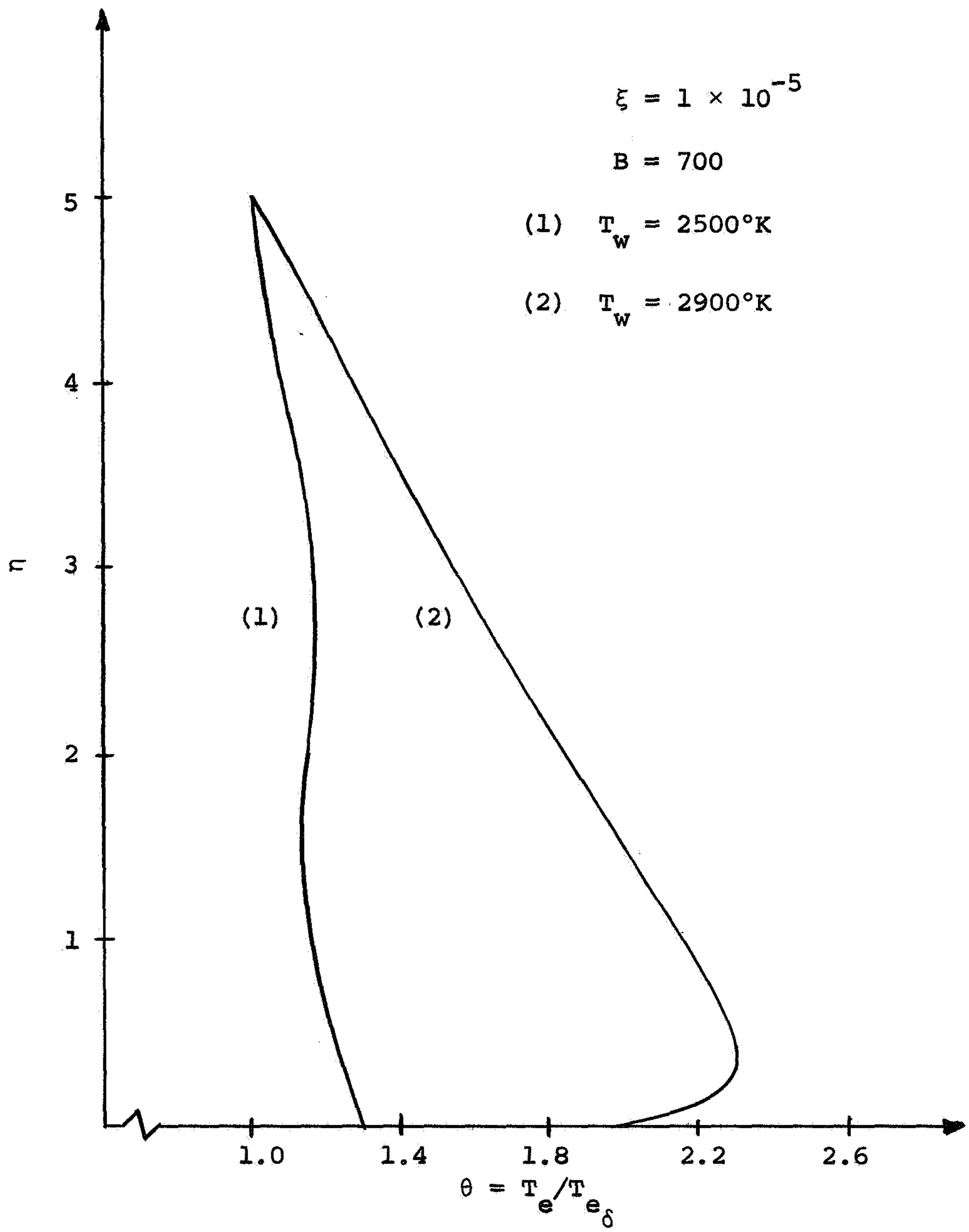

Figure 9. Transformation variable (η) versus electron temperature

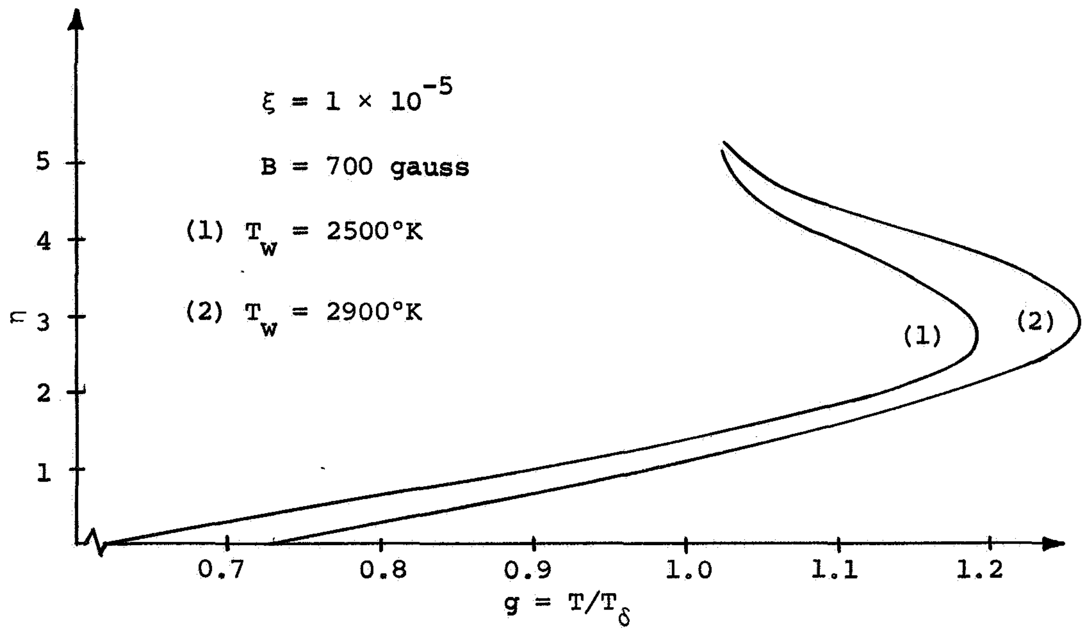

Figure 10. Transformation variable (η) versus temperature

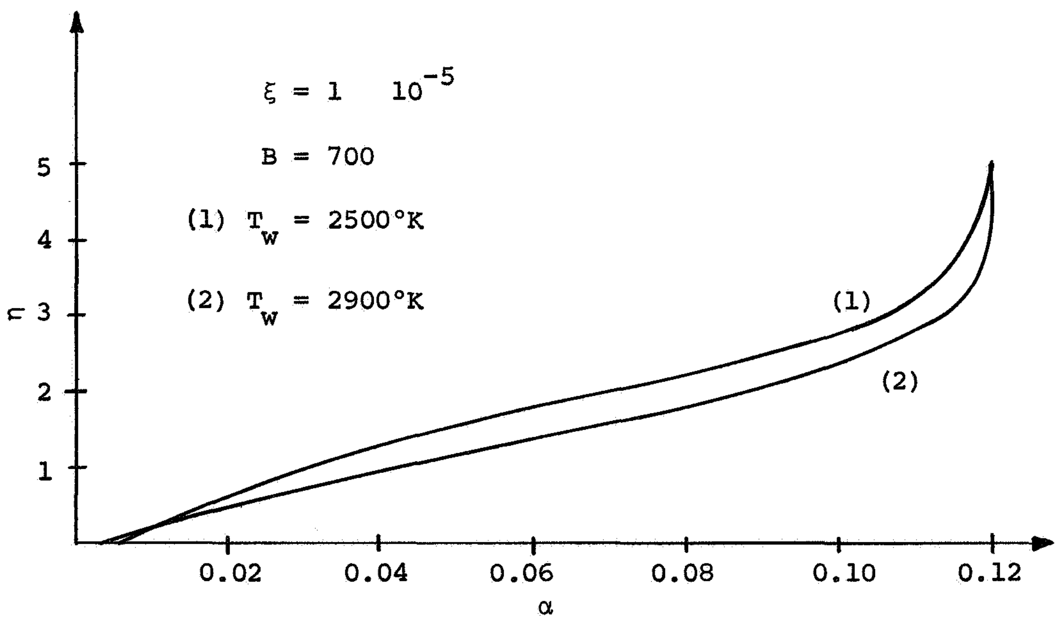

Figure 11. Transformation variable (η) versus degree of ionization

SUMMARY AND CONCLUSIONS

The assumption of local similarity appears to be a fairly good one for the cases used in this study. The quantity $\sqrt{\xi}\ q_w$ varies only slightly with ξ, indicating that local similarity is an adequate approximation. The heat flux at the wall decreases with increasing ξ because of the $(\xi)^{-1/2}$ proportionality. As a result both the current density and the electron temperature at the sheath edge decrease with increasing ξ also.

The variation of the current density with the magnetic induction indicates that gradients play an important role. The current density increases with B, while the electric field strength and conductivity decrease. This cannot be explained if gradients are ignored in Ohm's law. Since the current density increases with increasing magnetic induction, the heat flux at the wall and the electron temperature increase also. The magnetic induction has little effect on the wall shear for the cases discussed here.

For the 2500°K cathode temperature case, there is very little thermionic emission. Thus the current density is low and is dominated by electrons going toward the cathode. The heat flux is thus positive into the cathode. Raising the cathode temperature to 2900°K increases the cathode emission, and the current density becomes larger and negative, indicating electrons leaving the cathode. This has a cooling effect. The electron temperature at the sheath edge is greater for the 2900°K case.

Further possibilities for investigation are numerous. Higher cathode temperatures should be used in order to obtain higher current densities. The effects of varying some of the sheath parameters would be of interest. Also different free stream conditions could be investigated. Finally, the assumption of local similarity could be relaxed.

LIST OF REFERENCES

Aliyevskiy, M. Y., and V. M. Zhdanov. 1963. Transfer equations for a nonisothermistic multitype plasma. Zhurnal Prikladnoy Mekhaniki i Teknicheskoy Fiziki 5:11-17. Moscow.

Barbiere, D. 1951. Energy distribution, drift velocity, and temperature of slow electrons in helium and argon. The Physical Review 84(4): 653-658.

Chapman, S., and T. G. Cowling. 1960. The Mathematical Theory of Non-Uniform Gases. Cambridge University Press, New York.

Cloney, R. D., E. A. Mason, and J. T. Vanderslice. 1962. Binding energy of Ar_2 from ion scattering data. The Journal of Chemical Physics 36(4):1103-1104.

Cott, D. W. 1970. Ionizational and electron thermal nonequilibrium effects in compressible MHD boundary layers. AEDC TR 70-155. Arnold Engineering Development Center, Air Force Systems Command, USAF, Tullahoma, Tennessee.

Cramer, W. H. 1959. Elastic and inelastic scattering of low velocity ions: Ne^+ in A, A^+ in Ne, and A^+ in A. The Journal of Chemical Physics 30(3):641-642.

Hale, F. J., and J. L. Kerrebrock. 1964. Insulator boundary layers in magnetohydrodynamic channels. AIAA Journal 2(3):461-469.

Hassan, H. A. 1968. Surface effects on the structure of a collisionless sheath near an electrode. The Physics of Fluids 11(5):1085-1091.

Hirschfelder, J. O., C. F. Curtiss, and R. B. Bird. 1954. Molecular Theory of Gases and Liquids. John Wiley and Sons, Inc., New York.

Kerrebrock, J. L. 1961. Electrode boundary layers in direct-current plasma accelerators. Journal of the Aerospace Sciences 28(8): 631-643.

Oates, G. C., J. K. Richmond, Y. Aoki, and G. Grohs. 1962. Loss mechanisms of a low temperature plasma accelerator. Proceedings of the Fourth Biennial Gas Dynamics Symposium. Northwestern University Press, Evanston, Illinois.

Petschek, H., and S. Bryon. 1957. Approach to equilibrium ionization behind strong shock waves in argon. Annals of Physics 1:270-315.

Sherman, A., and E. Reshotko. 1969. The nonequilibrium boundary layer along an insulator wall. AIAA Journal 7:610-615.

Spitzer, L., Jr. 1962. Physics of Fully Ionized Gases. 2nd rev. ed. Interscience Publishers, Inc., New York.

APPENDICES

Appendix A. List of Main Symbols

A_1	quantity defined in equation (23B)
a_{st}	quantities defined in equations (10B)
B	magnetic induction
b_{st}	quantities defined in equations (10B)
E	electric field strength
e	electronic charge
f	Blasius variable defined by equations (22A) and (23A)
g	dimensionless temperature
H	specific enthalpy
h	Planck's constant
h_s	quantity defined by equation (12B)
I	Ionization potential
j	current density
k	Boltzmann's constant
m	mass
n	particle number density
p	pressure
q	heat flux
R_e	electron rate of production
T	temperature
T_e	electron temperature
u	velocity
w	work function of the cathode
W_s	species diffusion velocity
Z_{st}	collision cross section
α	degree of ionization

α_b	quantity defined by equation (23B)
$\alpha_{i,j}$	coefficient of specular reflection
$\beta_{i,j}$	coefficient of diffuse reflection
γ_{ij}	coefficients defined by equations (33B)
ξ	transformation variable defined by equation (21A)
η	transformation variable defined by equation (21A)
θ	dimensionless electron temperature
μ	coefficient of viscosity defined by equation (2B)
ψ	stream function
π	viscous stress tensor
ρ	density
ϕ	potential

Subscripts

i, e, a	ions, electrons, and neutral particles
x	axial distance
y	normal direction
w	refers to the wall
δ	refers to free stream conditions
0	refers to wall in sheath solution; otherwise, the sheath edge
1, 2, 3	ions, electrons, and neutral particles
∞	refers to sheath edge in the sheath solution

Superscripts

$\rightarrow$	refers to a vector quantity
$\overset{\rightarrow}{\rightarrow}$	refers to a tensor

Appendix B. Governing Equations

The governing equations for this problem are taken as a conservation of momentum equation, the conservation of species equations, an electron energy equation, and an over-all energy equation. The number densities of the ions and electrons are assumed to be approximately equal. If the over-all continuity equation is used, then only one conservation of species equation is required; and this is chosen to be the electron conservation equation. The equations used are in the form of Aliyevskiy and Zhdanov (1963) in which Grad's "13-moments" approach is employed. The general forms of these governing equations are given below:

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \rho_e \vec{u}_e = \dot{\rho}_e \tag{1B}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0 \tag{2B}$$

$$\rho \frac{D\vec{u}}{Dt} + \nabla p + \nabla \cdot \vec{\vec{\pi}} = (\vec{j} \times \vec{B}) \tag{3B}$$

$$\rho \frac{DH}{Dt} - \frac{Dp}{Dt} + \nabla \cdot \vec{q} + \pi_{ik} \frac{\partial u_k}{\partial x_i} = \vec{E}^* \cdot \vec{j} \tag{4B}$$

$$\frac{3}{2} \vec{u} \cdot \nabla p_e + \frac{5}{2} p_e \nabla \cdot \vec{u} + \nabla \cdot \vec{q}_e = \vec{E}^* \cdot \vec{j}_e + R_e^{(2)} \tag{5B}$$

where ρ_e and ρ are the electron and over-all densities, u_e and u are the electron and over-all velocities, $\dot{\rho}_e$ is the mass rate of production of electrons, p_e and p are the electron and over-all pressures, $\vec{\vec{\pi}}$ is the viscous stress tensor, $\vec{j}_e$ and $\vec{j}$ are the electron and over-all current densities, $\vec{B}$ is the magnetic induction, H is the specific enthalpy, and $\vec{q}_e$ and $\vec{q}$ are the electron and over-all heat flux vectors. $\vec{E}^*$ and $R_e^{(2)}$

are given by

$$\vec{E}^* = \vec{E} + \vec{u} \times \vec{B} \tag{6B}$$

$$R_e^{(2)} = -\frac{2}{3} n_e \sum_{t \neq e} \left(\frac{m_e}{m_t} n_t\right) 3k \, (T_e - T_t) \sqrt{\frac{2k\, T_e}{m_e}} \, Z_{et}^{(11)} \tag{7B}$$

where E is the electric field strength, n_t and m_t are the number density and mass of species "t", k is the Boltzmann constant, T_t is the temperature of species "t", and $Z_{st}^{(11)}$ is a collision cross section for collisions between particles "s" and "t".

Assuming steady two dimensional flow, the governing equations become

$$\frac{\partial}{\partial x} (\rho_e \, u_{e_x}) + \frac{\partial}{\partial y} (\rho_e \, u_{e_y}) = \dot{\rho}_e \tag{8B}$$

$$\frac{\partial}{\partial x} (\rho \, u_x) + \frac{\partial}{\partial y} (\rho \, u_y) = 0 \tag{9B}$$

$$\rho u_x \frac{\partial u_x}{\partial x} + \rho u_y \frac{\partial u_x}{\partial y} + \frac{\partial p}{\partial x} + \frac{\partial \pi_{xx}}{\partial x} + \frac{\partial \pi_{xy}}{\partial y} = (\vec{j} \times \vec{B}) \; x$$

$$\rho u_x \frac{\partial u_y}{\partial x} + \rho u_y \frac{\partial u_y}{\partial y} + \frac{\partial p}{\partial y} + \frac{\partial \pi_{xy}}{\partial x} + \frac{\partial \pi_{yy}}{\partial y} = (\vec{j} \times \vec{B}) \; y \tag{10B}$$

$$\rho u_x \frac{\partial H}{\partial x} + \rho u_y \frac{\partial H}{\partial y} - u_x \frac{\partial p}{\partial x} - u_y \frac{\partial p}{\partial y} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \pi_{xx} \frac{\partial u_x}{\partial x} + \pi_{xy} \frac{\partial u_y}{\partial x}$$

$$+ \pi_{xy} \frac{\partial u_x}{\partial y} + \pi_{yy} \frac{\partial u_y}{\partial y} = \vec{E}^* \cdot \vec{j} \tag{11B}$$

$$\frac{3}{2} \left(u_x \frac{\partial p_e}{\partial x} + u_y \frac{\partial p_e}{\partial y}\right) + \frac{5}{2} p_e \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \frac{\partial q_{e_x}}{\partial x} + \frac{\partial q_{e_y}}{\partial y} = \vec{E}^* \cdot \vec{j}_e + R_e^{(2)} \tag{12B}$$

where x and y refer to the axial and normal directions, respectively.

Introducing an electron diffusion velocity $\vec{W}_e$, defined as

$$\vec{W}_e = \vec{u}_e - \vec{u} \tag{13B}$$

and neglecting the x-variation of the diffusion velocity as compared to the y-variation, equation (8B) can be written as

$$\frac{\partial}{\partial x}(n_e u_x) + \frac{\partial}{\partial y}(n_e u_y) + \frac{\partial}{\partial y}(n_e W_{e_y}) = R_e \tag{14B}$$

where R_e is the rate of production of electrons. Expressions for the components of the stress tensor $\vec{\vec{\pi}}$ are obtained from Appendix C and are given below:

$$\pi_{xx} = \frac{\mu}{3}\left(2\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right)$$

$$\pi_{xy} = \frac{\mu}{2}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right)$$

$$\pi_{yy} = \frac{\mu}{3}\left(2\frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial x}\right) \quad . \tag{15B}$$

In order to simplify the governing equations, a boundary layer type analysis is used. The following dimensionless quantities are introduced:

$$x^* = x/L \quad ; \quad y^* = y/\delta \quad ; \quad u^* = u_x/u_{x_\delta} \quad ; \quad v^* = Lu_y/\delta u_{x_\delta}$$

$$\rho^* = \rho/\rho_\delta \quad ; \quad p^* = p/\rho_\delta u_{x_\delta}^2 \quad ; \quad \mu^* = \mu/\mu_\delta \quad ; \quad R_e = \rho_\delta u_{x_\delta} L/\mu_\delta \tag{16B}$$

where L is a characteristic length, δ is the boundary layer thickness, R_e is the Reynolds number, and the subscript δ refers to free stream conditions. An order of magnitude analysis is performed using these quantities, and small order terms are neglected. If one uses equations (10B) and (15B), a momentum equation of the following form is obtained:

$$\rho u_x \frac{\partial u_x}{\partial x} + \rho u_y \frac{\partial u_y}{\partial y} + \frac{dp}{dx} = \frac{\partial}{\partial y} \left(\mu \frac{\partial u_x}{\partial y}\right) + (\vec{j} \times \vec{B})_x \quad . \tag{17B}$$

If these operations are performed on equation (11B), the energy equation takes the form:

$$\rho u_x \frac{\partial H}{\partial x} + \rho u_y \frac{\partial H}{\partial y} + \frac{\partial q_y}{\partial y} = u_x \frac{dp}{dx} + \mu \left(\frac{\partial u_x}{\partial y}\right)^2 + \vec{E}^* \cdot \vec{j} \quad . \tag{18B}$$

In deriving the above expression, the x-variation of the heat flux is neglected with respect to the y-variation. In a similar manner the electron energy equation becomes

$$\frac{3}{2} \left(u_x \frac{\partial p_e}{\partial x} + u_y \frac{\partial p_e}{\partial y}\right) + \frac{5}{2} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) p_e + \frac{\partial q_{e_y}}{\partial y} = \vec{E}^* \cdot \vec{j}_e + R_e^{(2)} \quad . \tag{19B}$$

Equations (9B), (14B), and (17B) through (19B) compose the set of governing equations. If y is allowed to approach the thickness of the boundary layer, equation (17B) yields an expression for the pressure gradient:

$$\frac{dp}{dx} = -\rho_\delta \, u_{x_\delta} \frac{du_{x_\delta}}{dx} + (\vec{j} \times \vec{B})_x\Big|_\delta \quad . \tag{20B}$$

A Levy-Lees coordinate transformation of the type

$$\xi\,(x) = \int_0^x \rho_\delta \, \mu_\delta \, u_{x_\delta} \, dx \quad ; \quad \eta\,(x,y) = \frac{u_{x_\delta}}{\sqrt{2\xi}} \int_0^y \rho \, dy \tag{21B}$$

is introduced. A stream function ψ is also introduced and is defined by the following relations:

$$\rho u_x = \frac{\partial \psi}{\partial y} \quad ; \quad \rho u_y = -\frac{\partial \psi}{\partial x} \quad . \tag{22B}$$

For $\psi = \sqrt{2\xi}\, f$, equations (21B) and (22B) show that

$$u_x = u_{x_\delta} \frac{\partial f}{\partial \eta} \quad . \tag{23B}$$

The quantity $u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y}$ appears in several places. In the transformed coordinate system, it becomes

$$\rho_\delta \, u_{x_\delta}^2 \, \mu_\delta \left[\frac{\partial f}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{f}{2\xi}\frac{\partial}{\partial \eta}\right] \quad . \tag{24B}$$

A degree of ionization α is introduced and is defined by

$$\alpha = n_e/n = n_i/n = 1 - n_a/n \tag{25B}$$

where n_s is the species number density and n is the over-all number density given by

$$n = \rho/m \quad . \tag{26B}$$

Using equation (25B) and the definition of the specific enthalpy, given by

$$H = \frac{1}{\rho}\left[\frac{5}{2} p + I\, n_e\right] = \frac{5}{2}\frac{p}{\rho} + \frac{I}{m}\alpha \tag{27B}$$

where I is the ionization potential of the medium, equation (14B) transforms into

$$\frac{\partial f}{\partial \eta}\frac{\partial \alpha}{\partial \xi} - \left(\frac{\partial f}{\partial \xi} + \frac{f}{2\xi}\right)\frac{\partial \alpha}{\partial \eta} + \frac{5}{2}\frac{p}{\rho_\delta u_{x_\delta} \mu_\delta \sqrt{2\xi}}\frac{\partial}{\partial \eta}\left(\frac{\alpha\, W_{ey}}{H - \frac{I}{m}\alpha}\right) = \frac{m\, R_e}{\rho\rho_\delta\, u_{x_\delta}^2\, \mu_\delta} \quad . \tag{28B}$$

In obtaining equation (29B) use is made of the continuity equation and the assumption that the pressure is independent of η.

If equation (20B) is substituted into equation (17B), the transformed momentum equation takes the form:

$$\left(u_{x_\delta}\frac{\partial^2 f}{\partial \xi \partial \eta}\right)\frac{\partial f}{\partial \eta} - u_{x_\delta}\left(\frac{\partial f}{\partial \xi} + \frac{f}{2\xi}\right)\frac{\partial^2 f}{\partial \eta^2} = \left[\frac{\rho_\delta}{\rho} - \left(\frac{\partial f}{\partial \eta}\right)^2\right]\frac{du_{x_\delta}}{d\xi}$$

$$+ \frac{u_{x_\delta}}{\rho_\delta\, \mu_\delta\, (2\xi)}\frac{\partial}{\partial \eta}\left[\rho\mu\frac{\partial 2_f}{\partial \eta 2}\right]$$

$$+ \left[(\vec{j} \times \vec{B})_x - (\vec{j} \times \vec{B})_{x|_\delta}\right]/\rho\rho_\delta u_{x_\delta}^2 \mu_\delta \quad . \tag{29B}$$

Similarly, equation (18B) transforms into

$$\frac{\partial f}{\partial \eta}\frac{\partial H}{\partial \xi} - \left(\frac{\partial f}{\partial \xi} + \frac{f}{2\xi}\right)\frac{\partial H}{\partial \eta} = -\left(\frac{\rho_\delta u_{x_\delta}}{\rho}\frac{\partial f}{\partial \eta}\right)\frac{du_{x_\delta}}{d\xi} + \frac{u_{x_\delta}}{2\xi}\left(\frac{\rho\mu}{\rho_\delta \mu_\delta}\right)\left(\frac{\partial^2 f}{\partial \eta^2}\right)^2$$

$$+ \left[\frac{(\vec{j} \times \vec{B})_{x|_\delta}}{\rho\rho_\delta u_{x_\delta} \mu_\delta}\right]\frac{\partial f}{\partial \eta} - \frac{1}{\rho_\delta u_{x_\delta} \mu_\delta}\left[\frac{1}{\sqrt{2\xi}}\frac{\partial q_y}{\partial \eta} - \frac{\vec{E}^* \cdot \vec{j}}{\rho u_{x_\delta}}\right] \quad . \tag{30B}$$

If the definitions of electron pressure and over-all pressure

$$p_e = \alpha\,\rho\,\frac{k}{m}\,T_e$$

$$p = \rho\,\frac{k}{m}\,(\alpha T_e + T) \quad ; \quad T_i = T_a = T \tag{31B}$$

are used, the electron energy equation transforms into

$$\frac{3}{2} p \left\{\frac{\partial f}{\partial \eta} \frac{\partial}{\partial \xi} \left(\frac{\alpha T_e}{\alpha T_e + T}\right) - \left(\frac{\partial f}{\partial \xi} + \frac{f}{2\xi}\right) \frac{\partial}{\partial \eta} \left(\frac{\alpha T_e}{\alpha T_e + T}\right)\right\}$$

$$+ \frac{5}{2} \frac{k}{m} \frac{\rho p_e}{p} \left\{\frac{\partial f}{\partial \eta} \frac{\partial (\alpha T_e + T)}{\partial \xi} - \left(\frac{\partial f}{\partial \xi} + \frac{f}{2\xi}\right) \frac{\partial (\alpha T_e + T)}{\partial \eta}\right\} + \frac{p_e}{\rho} \left[\rho u_{x_\delta} \frac{du_{x_\delta}}{d\xi}\right.$$

$$\left. - \frac{(\vec{j} \times \vec{B}) \cdot x|_\delta}{\rho_\delta u_{x_\delta} \mu_\delta}\right] \frac{\partial f}{\partial \eta} + \frac{\rho}{\rho_\delta u_{x_\delta} \mu_\delta (2\xi)^{1/2}} \frac{\partial q_{e_y}}{\partial \eta} = \frac{\vec{E}^* \cdot \vec{j}_e + R_e^{(2)}}{\rho_\delta u_{x_\delta}^2 \mu_\delta} \quad . \tag{32B}$$

The assumption that the ion and neutral particle temperatures are equal is also introduced.

Assuming local similarity, where derivatives with respect to ξ are neglected, but terms containing ξ are retained, the governing equations reduce to

$$\frac{d}{d\eta} \left(\frac{\alpha W_{ey}}{H - \frac{I}{m} \alpha}\right) = F_2 (\eta) \tag{33B}$$

$$\frac{d}{d\eta} \left(\rho \mu \frac{d^2 f}{d\eta^2}\right) = F_1 (\eta) \tag{34B}$$

$$\frac{dq_y}{d\eta} = F_3 (\eta) \tag{35B}$$

$$\frac{dq_{ey}}{d\eta} = F_4 (\eta) \tag{36B}$$

where

$$F_1(\eta) = -\rho_\delta \mu_\delta f \frac{d^2 f}{d\eta^2} - \frac{2\xi}{\rho u_{x_\delta}^3} [j_y B - j_{y_\delta} B]$$

$$F_2(\eta) = [\frac{0.4 \rho_\delta u_{x_\delta} \mu_\delta f}{p\sqrt{2\xi}}] \frac{d\alpha}{d\eta} + \frac{0.4\sqrt{2\xi}\, m R_e}{\rho u_{x_\delta} p}$$

$$F_3(\eta) = \left(\frac{\rho_\delta u_{x_\delta} H_\delta f}{\sqrt{2\xi}}\right) \frac{dH}{d\eta} + \frac{\rho\mu u_{x_\delta}^3}{\sqrt{2\xi}} \left(\frac{d^2 f}{d\eta^2}\right)^2 + \frac{\sqrt{2\xi}}{\rho} [(j_{y_\delta} B) \frac{df}{d\eta}$$

$$+ \frac{E_y^* j_y}{u_{x_\delta}}]$$

$$F_4(\eta) = \frac{\rho_\delta u_{x_\delta} \mu_\delta f}{\sqrt{2\xi}} \left(\frac{k/m}{\alpha T_e + T}\right) \{ \left(\frac{5}{2}\alpha^2 T_e + \frac{3}{2}\alpha T\right) T_{e_\delta} \frac{d\theta}{d\eta}$$

$$+ \left(\frac{5}{2}\alpha T_e^2 + \frac{3}{2} T_e T\right) \frac{d\alpha}{d\eta} + \alpha T_e T_\delta \frac{dg}{d\eta}\}$$

$$+ \frac{\sqrt{2\xi}}{\rho} \left(\frac{\alpha T_e}{\alpha T_e + T}\right) \left(j_{y_\delta} B\right) \frac{df}{d\eta} + \frac{\sqrt{2\xi}}{\rho u_{x_\delta}} [E_y^* j_{e_y} + R_e^{(2)}] \quad (37B)$$

and where

$$\frac{dH}{d\eta} = 2.5 \frac{k}{m} \left(\alpha T_{e_\delta} \frac{d\theta}{d\eta} + T_\delta \frac{dg}{d}\right) + \left(2.5 \frac{k}{m} T_e + \frac{I}{m}\right) \frac{d\alpha}{d\eta}$$

$$\theta = T_e / T_{e_\delta}$$

$$g = T/T_\delta$$

$$E_y^* = E_y - u_x B \quad . \quad (38B)$$

The magnetic induction and the electric field are assumed to have the representation:

$$\vec{B} = B \hat{i}_z \quad ; \quad \vec{E} = E_y \hat{i}_y \quad . \quad (39B)$$

The electron rate of production, which appears in equation (37B), can be expressed as

$$R_e = n_e n_a f(T_e)\left[1 - \frac{n_i n_e}{n_a K}\right] \quad (40B)$$

where

$$K = \frac{2 g_i}{g_a} \left(\frac{2\pi m_e k T_e}{h^2}\right) e^{-I/kT_e} \quad (41B)$$

and (from Chapman and Cowling, 1960)

$$f(T_e) = \left(\frac{8}{\pi m_e}\right)^{1/2} (kT_e)^{-3/2} \int_I^\infty e^{-\xi/kT_e} Q_i(\xi)\xi d\xi \quad (42B)$$

where g_i and g_a are the ground state degeneracies of the ions and atoms, h is Planck's constant, and Q_i is the ionization cross section. The calculations are carried out for an argon medium. Following Petschek and Bryon (1957), the ionization cross section is assumed to have the representation

$$Q_i(\xi) = a_{ea}(\xi - I) \quad (43B)$$

where, for argon, $a_{ea} = 4.375 \times 10^{-6}$ cm^2/erg. If equations (41B) through (43B) are used in equation (40B), then the electron rate of production is given by

$$R_e = \left[\left(\frac{8I}{\pi m_e}\right)^{1/2} a_{ea}\right] \frac{\rho^2 \alpha\ (1 - \alpha)}{m} \left[\frac{I}{m} + 2\ \frac{kT_e}{m}\right] \sqrt{\frac{kT_e}{I}}\ e^{-I/kT_e}$$

$$\times \left\{1 - \frac{\frac{I}{m}\rho\ \alpha^2 \left(e^{I/kT_e}\right)}{(5.718 \times 10^{13})(1 - \alpha)(kT_e/I)^{3/2}}\right\} \quad . \tag{44B}$$

Appendix C. Transport Properties

General expressions for the diffusion velocities, the heat fluxes, and the stress tensor are obtained from Aliyevskiy and Zhdanov (1963). The viscous stress tensor $\overleftrightarrow{\pi}$ is the sum of the species stress tensors $\overleftrightarrow{\pi}_s$. If the electron stress tensor is neglected with respect to the ion and neutral stress tensors, and if the ion Larmor radius is large, then the ion stress tensor and the over-all stress tensor are given by:

$$\overleftrightarrow{\pi} = \mu' \overleftrightarrow{\varepsilon}$$

$$\overleftrightarrow{\pi} = \mu \overleftrightarrow{\varepsilon} \tag{1C}$$

where

$$\mu' = \frac{4}{3} \alpha\tau_i \{1 + \frac{0.1 \rho (1 - \alpha)}{m} \sqrt{\frac{kT}{m}} Z_{ia}^{(11)} \tau_a\}/\{\frac{m}{\rho kT} \frac{\rho\alpha(1 - \alpha)}{100 (m)^2} \left(Z_{ia}^{(11)}\right)^2 \tau_a \tau_i\}$$

$$\mu = -\frac{4}{3} \{1 - \alpha) \tau_a + \alpha\tau_i + \frac{\rho\alpha (1 - \alpha)}{5m} \sqrt{\frac{kT}{m}} Z_{ia}^{(11)} \tau_i \tau_a\}/\{\frac{m}{\rho kT}$$

$$- \frac{\rho\alpha (1 - \alpha)}{100 m^2} \left(Z_{ia}^{(11)}\right) \tau_a \tau_i\} \tag{2C}$$

and the components of $\overleftrightarrow{\varepsilon}$ are given by:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{\delta_{ij}}{3} \frac{\partial u_k}{\partial x_k} \tag{3C}$$

where δ_{ij} is the Kroneker delta. Also

$$\tau_i = [\frac{9m}{2\rho} \sqrt{\frac{m}{kT}}] / \{2.4\alpha \, Z_{ii}^{(11)} + 3.55 \, (1 - \alpha) \, Z_{ia}^{(11)} + 4 \sqrt{2} \, \alpha \sqrt{\frac{m_e T_e}{mT}} \, Z_{ei}^{(11)} \}$$

$$\tau_a = [\frac{9m}{2\rho} \sqrt{\frac{m}{kT}}] / \{3.1 \, (1 - \alpha) \, Z_{aa}^{(11)} + 3.55\alpha \, Z_{ia}^{(11)} + 4 \sqrt{2} \, \alpha \sqrt{\frac{m_e T_e}{mT}} \, Z_{ea}^{(11)} \} \tag{4C}$$

The various Z_{st} cross sections are given at the end of this Appendix.

The general expressions from which the diffusion velocities $\vec{W}_s$ and the heat fluxes $\vec{q}_s$ can be obtained are coupled and are quite complex. In order to simplify these expressions, terms of the order of $(m_e/m)^{1/2}$ or less are neglected. Also, the ion and neutral masses are assumed to be equal. With these simplifications the transport relations can be written as (Aliyevskiy and Zhdanov, 1963):

$$a_{ei} \, (\vec{W}_e - \vec{W}_i) + a_{ea} \, (\vec{W}_e - \vec{W}_a) = - \frac{a_{e_I} A_{e_i}^{(2)}}{p_e} \vec{h}_e - (\nabla p_a - \frac{\rho_a}{\rho} \nabla p) - en_e \vec{E}^* - \frac{\rho_e}{\rho} (en_i - en_e) \, \vec{E}^* - en_e \, (\vec{W}_e \times \vec{B}) \tag{5C}$$

$$a_{ea} \, (\vec{W}_a - \vec{W}_e) + a_{ai} \, (\vec{W}_a - \vec{W}_i) = - (\nabla p_a - \frac{\rho_a}{\rho} \nabla p) - \frac{\rho_a}{\rho} (en_i - en_e) \, \vec{E}^* - \frac{\rho_a}{\rho} [en_i \, (\vec{W}_i \times \vec{B} - en_e \, (\vec{W}_e \times \vec{B})] \tag{6C}$$

$$\vec{h}_e = - \tau_e^* \, \{\frac{5}{2} \frac{kp_e}{m_e} \nabla T_e + \frac{5}{2} \frac{kT_e}{m_3} [2a_{ea} \frac{m_e}{m} \left(1 - \frac{T}{T_e}\right) \vec{W}_e + a_{ei} A_{ei}^{(2)} \, (\vec{W}_e - \vec{W}_i)] + \frac{e}{m_e} (\vec{h}_e \times \vec{B}) \tag{7C}$$

$$\vec{h}_a + b_{ai}\vec{h}_i + b_{ae}\vec{h}_e = -\tau_a^* \{\frac{5}{2}\frac{kp_a}{m}\nabla T + \frac{5}{2}\frac{kT}{m}[2a_{ea}\left(1 - \frac{T_e}{T}\right)\vec{W}_a + \frac{m_e T}{mT_e} a_{ea} A_{ae}^{(7)} (\vec{W}_a - \vec{W}_e)]\} \quad (8C)$$

$$\vec{h}_i + b_{ia}\vec{h}_a + b_{ie}\vec{h}_e = -\tau_i^* \{\frac{5}{2}\frac{kp_i}{m}[\nabla T - \left(\frac{2e}{5kp_i}\right)\left(\vec{\pi}_i \cdot E^*\right)] + \frac{5}{2}\frac{kT}{m}[2a_{ea}\left(1 - \frac{T_e}{T}\right)\vec{W}_i + \frac{m_e T}{mT_e} a_{ei} A_{ie}^{(7)} (\vec{W}_i - \vec{W}_e)] - \frac{e}{m}(\vec{h}_i \times \vec{B})\} \quad (9C)$$

where

$$a_{ei} = \frac{2\sqrt{2}}{3}\frac{\rho^2\alpha^2 m_e}{m^2}\sqrt{\frac{kT_e}{m_e}}\, Z_{ei}^{(11)}$$

$$a_{ea} = \frac{2\sqrt{2}}{3}\frac{\rho^2\,\alpha(1-\alpha)\, m_e}{m^2}\sqrt{\frac{kT_e}{m_e}}\, Z_{ea}^{(11)}$$

$$a_{ai} = \frac{2}{3}\frac{\rho^2\,\alpha(1-\alpha)}{m}\sqrt{\frac{kT}{m}}\, Z_{ia}^{(11)}$$

$$b_{ai} = -\,0.32227\,\frac{\rho\,(1-\alpha)}{m}\sqrt{\frac{kT}{m}}\,\tau_a^*\, Z_{ia}^{(11)}$$

$$b_{ae} = -\frac{8\sqrt{2}}{15}\frac{\rho\,(1-\alpha)}{m}\left(\frac{m_e}{m}\right)^2\sqrt{\frac{kT_e}{m_e}}\{8.8745 - 6.4575\left(\frac{T}{T_e}\right)\}\,\tau_a^*\, Z_{ea}^{(11)}$$

$$b_{ia} = 0\ 0.3227\,\frac{\rho\alpha}{m}\sqrt{\frac{kT}{m}}\,\tau_i^*\, Z_{ia}^{(11)}$$

$$b_{ie} = -\frac{8\sqrt{2}}{15}\frac{\rho\alpha}{m}\left(\frac{m_e}{m}\right)^2\sqrt{\frac{kT_e}{m_e}}\{2 - 4.25\,\frac{T}{T_e} + 5.625\left(\frac{T}{T_e}\right)^2\}\,\tau_i^*\, Z_{ei}^{(11)}$$

$$A_{ei}^{(2)} = -\frac{3}{5} \quad ; \quad A_{ae}^{(7)} = -0.8\ (2.417)\ \frac{T_e}{T} \left(1 - \frac{T_e}{T}\right) \quad ; \quad A_{ie}^{(7)} = -1.8 + 1.2\ \frac{T_e}{T} \tag{10C}$$

and where

$$\tau_e^* = \frac{3m}{2\rho} \sqrt{\frac{m_e}{kT_e}} \Big/ \{2.6385\alpha\ Z_{ei}^{(11)} + \sqrt{2}\ (1 - \alpha)\ Z_{ea}^{(11)}\}$$

$$\tau_i^* = \frac{3m}{2\rho} \sqrt{\frac{m}{kT}} \Big/ \{0.8\ Z_{ei}^{(11)} + 1.5166\ (1 - \alpha)\ Z_{ea}^{(11)}\}$$

$$\tau_a^* = \frac{3m}{2\rho} \sqrt{\frac{m}{kT}} \Big/ \{1.0332\ (1 - \alpha)\ Z_{aa}^{(11)} + 1.5166\alpha\ Z_{ia}^{(11)} + 3\sqrt{2} \sqrt{\frac{m_e T_e}{mT}}\ \alpha\ Z_{ea}^{(11)}\} \ . \tag{11C}$$

The quantity $\vec{h}_s$ is related to the desired transport properties by the relation

$$\vec{q}_s = \vec{h}_s + \frac{5}{2} p_s \vec{W}_s \ . \tag{12C}$$

The diffusion velocities also satisfy

$$\rho_e \vec{W}_e + \rho_i W_i + \rho_a \vec{W}_a = 0 \ . \tag{13C}$$

If one eliminates $\vec{W}_a$ from equation (6C) by means of equation (12C) and takes the dot and cross products of the resulting expression with respect to $\vec{B}$ to eliminate $\vec{W}_i \times B$, the ion diffusion velocity can be written as:

$$\vec{W}_i = \vec{G}_i + A_i \vec{W}_e + C_i\ (\vec{W}_e \cdot \vec{B})\ \vec{B} + D_i\ \left(\vec{W}_e \times \frac{\vec{B}}{B}\right) \ . \tag{14C}$$

Similarly, using equation (14C) in equation (13C) results in

$$\vec{W}_a = \vec{G}_a + A_a\vec{W}_e + C_a\ (\vec{W}_e \cdot \vec{B})\ \vec{B} + D_a\ \left(\vec{W}_e \times \frac{\vec{B}}{B}\right)\ . \qquad (15C)$$

By taking the dot and cross products of equation (7C) with respect to $\vec{B}$, the term $\vec{h}_e \times \vec{B}$ can be eliminated. Combining the resulting expression with equation (14C) yields

$$\vec{h}_e = \vec{x}_e + \alpha_e\vec{W}_e + \phi_e\ (\vec{W}_e \cdot \vec{B})\ \vec{B} + \theta_e\ \left(\vec{W}_e \times \frac{\vec{B}}{B}\right)\ . \qquad (16C)$$

Equation (8C) is used to eliminate $\vec{h}_a$ in equation (9C), and then the resulting equation is manipulated as previously described to eliminate $\vec{h}_i \times \vec{B}$. Equations (14C) through (16C) are then combined with this espression to give

$$\vec{h}_i = \vec{x}_i + \alpha_i\vec{W}_e + \phi_i\ (\vec{W}_e \cdot \vec{B})\ \vec{B} + \theta_i\ \left(\vec{W}_e \times \frac{\vec{B}}{B}\right)\ . \qquad (17C)$$

Combining equations (15C) through (17C) with equation (8C) gives

$$\vec{h}_a = \vec{x}_a + \alpha_a\vec{W}_e + \phi_a\ (\vec{W}_e \cdot \vec{B})\ \vec{B} + \theta_a\ \left(\vec{W}_e \times \frac{\vec{B}}{B}\right)\ . \qquad (18C)$$

Everything is now expressed in terms of the electron diffusion velocity. If equations (14C) through (16C) are used in equation (2C), the following expression for the electron diffusion velocity can be obtained:

$$\vec{W}_e = (1 + \alpha_b^2)^{-1}\ \{R + \alpha_b\ [\frac{\alpha_b}{B^2}\ (\vec{R} \cdot \vec{B})\ \vec{B} - \frac{1}{B}\ (\vec{R} \times \vec{B})]\}\ . \qquad (19C)$$

Equations (14C) through (19C) now provide the necessary relationships for $\vec{W}_s$ and $\vec{q}_s$ in a general form.

The problem of interest here allows the various properties to vary in the axial and normal directions and assumes a magnetic induction in the transverse direction. As a result, any dot products with $\vec{B}$ are zero. Considering this and using equation (19C) in equations (14C) through (18C), the following general relations are obtained:

$$\vec{W}_s = \vec{G}_s + (1 + \alpha_b^2)^{-1} \{(A_s + \alpha_b \theta_s)\ \vec{R} - \left(\alpha_b A_s - D_s\right)\left(\vec{R} \times \frac{\vec{B}}{B}\right)\} \qquad s = i, a \qquad (20C)$$

$$\vec{h}_s = \vec{x}_s + (1 + \alpha_b^2)^{-1} \{(\alpha_s + \alpha_b \theta_s)\ \vec{R} - \left(\alpha_b \alpha_s - \theta_s\right)\left(\vec{R} \times \frac{\vec{B}}{B}\right)\} \qquad s = e, i, a \qquad (21C)$$

The expressions which are of interest here are those for the electron and ion diffusion velocities, the electron heat flux, and the over-all heat flux. Because of their complexity, equations (14C) and (19C) through (21C) indicate the procedure employed in deriving expressions for these quantities, while omitting the details. As an illustration of this complexity, the complete for of the electron diffusion velocity relation is given below:

$$\begin{aligned}\vec{W}_e = {} & D_{e_1} [\nabla p_a - (1 - \alpha)\ \nabla p] + D_{e_2} [\nabla p_a - (1 - \alpha)\ \nabla p] \times \frac{\vec{B}}{B} \\ & + [A_1\ (1 + \alpha_b^2)]^{-1} \{- \frac{3}{2} \frac{k}{m_e} [\frac{a_{e_i} \tau_e^*}{1 + (\omega_e \tau_e^*)^2}] [(1 + \alpha_b \omega_e \tau_e^*)\ \nabla T_e \\ & + (\omega_e \tau_e^* - \alpha_b) (\nabla T_e \times \frac{\vec{B}}{B}) - \nabla p_e - \frac{e}{m} \rho \alpha \vec{E}^* - \alpha_b\ (\nabla p_e + \frac{e}{m} \rho \alpha \vec{E}^*) \times \frac{\vec{B}}{B}\}\end{aligned} \qquad (22C)$$

where

$$b_2 = \frac{e}{m} \rho\alpha \ (1 - \alpha) \ B \ / \ g_{ie} \qquad ; \qquad \omega_e = -e \ B/m_e$$

$$g_{ie} = (a_{ai} + \alpha a_{ea}) \ / \ (1 - \alpha) \qquad ; \qquad g_{ei} = [(1 - \alpha) \ a_{ea} + \frac{m_e}{m} \alpha a_{ai}]/(1 - \alpha)$$

$$A_1 = \left(- \frac{3}{5} \frac{a_{ei}}{p_e}\right) \alpha_e + a_{ei} + a_{ea} - \left(a_{ei} - \frac{\alpha a_{ea}}{1 - \alpha}\right)\left(b_2^2 - \frac{g_{ei}}{g_{ie}}\right) \ / \ \left(1 + b_2^2\right)$$

$$\alpha_b = (A_1)^{-1} \{\frac{e}{m} \rho\alpha \ B - \frac{3}{5} \frac{a_{ei}}{p_e} \theta_e + \left(a_{ei} - \frac{\alpha a_{ea}}{1 - \alpha}\right)\left(1 + \frac{g_{ei}}{g_{ie}}\right) b_2 \ / \ (1 + b_2^2)\}$$

$$\alpha_e = [- \frac{\tau_e^*}{1 + (\omega_e \tau_e^*)^2}]\{\frac{5kT_e}{m_e} \left(1 - \frac{T}{T_e}\right) a_{ea}$$

$$- \frac{3}{2} \frac{kT_e}{m_e} a_{ei} \left(1 + \frac{g_{ei}}{g_{ie}}\right)\left(1 - b_2 \omega_e \tau_e^*\right) \ / \ \left(1 + b_2^2\right)\}$$

$$\theta_e = [\frac{\tau_e^*}{1 + (\omega_e \tau_e^*)^2}]\{\frac{3}{2} \frac{kT_e}{m_e} a_{ei} \ b_2 \ \left(1 + \frac{g_{ei}}{g_{ie}}\right) \ / \ \left(1 + b_2^2\right)$$

$$- \omega_e \tau_e^* \ [\frac{5kT_e}{m_e} \left(1 - \frac{T}{T_e}\right) a_{ea} - \frac{3}{2} \frac{kT_e}{m_e} a_{ei} \ \left(1 + \frac{g_{ei}}{g_{ie}}\right) \ / \ \left(1 + b_2^2\right)\}$$

$$D_{e_1} = [A_1 \ (1 + \alpha_b^2)]^{-1} \ \{\left(1 + b_2 \alpha_b\right)\left(a_{ei} - \frac{\alpha a_{ea}}{1 - \alpha}\right)$$

$$- \frac{0.9 \ a_{eI}^2}{\rho\alpha \ (m_e/m)} \ [\frac{\tau_e^*}{1 + (\omega_e \tau_e^*)^2}] (1 + b_2 \alpha_b - b_2 \omega_e \tau_e^* + \alpha_b \omega_e \tau_e^*)\}$$

$$D_{e_2} = [A_1 \ (1 + \alpha_b^2)]^{-1} \ \{\left(b_2 - \alpha_b\right)\left(a_{ei} - \frac{\alpha a_{ea}}{1 - \alpha}\right)$$

$$- \frac{0.9 \ a_{ei}^2}{\rho\alpha \ (m_e/m)} \ [\frac{\tau_e^*}{1 + (\omega_e \tau_e^*)^2}] \left(b_2 - \alpha_b + \omega_e \tau_e^* \ (1 - b_2 \alpha_b)\right)]\} \ .$$

(23C)

A similar equation for $\vec{W}_i$ can be obtained from equation (20C). If equations (12C) and (16C) are used, an expanded equation for the electron heat flux can be written as:

$$\vec{q}_e = D_{e_7} [\nabla p_a - (1 - \alpha) \nabla p] + D_{e_8} [\nabla p_a - (1 - \alpha) \nabla p] \times \frac{\vec{B}}{B} + D_{e_9} \nabla T_e + D_{e_{10}} \left(\nabla T_e \times \frac{\vec{B}}{B}\right) - [A_1 (1 + \alpha_b^2)]^{-1} \{c_{e_1} (\nabla p_e + \frac{e}{m} \rho\alpha\vec{E}^*) + c_{e_2} (\nabla p_e + \frac{e}{m} \rho\alpha\vec{E}^*) \times \frac{\vec{B}}{B}\} \quad . \tag{24C}$$

The over-all heat flux is given by

$$\vec{q} = \sum_s \vec{q}_s = \sum_s \left(\vec{h}_s + \frac{5}{2} p_s \vec{W}_s\right) \qquad s = e, i, a \quad . \tag{25C}$$

If equation (13C) is used in equation (25C) and small order terms are neglected, it can be shown that

$$\vec{q} = (\sum_s \vec{h}_s) + \frac{5}{2} p_e \vec{W}_e \qquad s = e, i, a \quad . \tag{26C}$$

After expansion this equation can be written in a form like that of equation (24C):

$$\vec{q} = c_1 [\nabla p_a - (1 - \alpha) \nabla p] + c_2 [\nabla p_a - (1 - \alpha) \nabla p] \times \frac{\vec{B}}{B} + c_3 \nabla T_e + c_4 \left(\nabla T_e \times \frac{\vec{B}}{B}\right) + c_5 \nabla T + c_6 \left(\nabla T \times \frac{\vec{B}}{B}\right) - [A_1 (1 + \alpha_b^2)]^{-1} \{(\sum_s c_{s_1})(\nabla p_e + \frac{e}{m} p\alpha\vec{E}^*) - (\sum_2 c_{s_2})(\nabla p_e + \frac{e}{m} \rho\alpha\vec{E}^*) \times \frac{\vec{B}}{B} + \frac{e}{m} b_1 (1 - b_{ai}) \tau_i^* \{\vec{E}^* \cdot \vec{\vec{\pi}}_i + \left(\frac{\omega_i \tau_i^*}{1 - b_{ia} b_{ai}}\right) \left(\vec{E}^* \cdot \vec{\vec{\pi}}_i\right) \times \frac{\vec{B}}{B}\} \quad . \tag{27C}$$

If one examines the governing equations in Appendix B, it can be seen that only the normal, or y, components of the various transport properties are needed. In further simplifying the equations, axial derivatives of properties are neglected in comparison to normal derivatives except in the case of the pressure. In order to have the final equations in terms of derivatives of the temperature, the electron temperature, and the degree of ionization, it is necessary to eliminate the derivatives of p_a and p_e in favor of derivatives of the desired properties. The neutral and electron pressures are defined as

$$p_a = n_a kT = \frac{(1 - \alpha)\, pT}{\alpha T_e + T}$$

$$p_e = n_e kT_e = \frac{\alpha T_e p}{\alpha T_e + T} \qquad (28C)$$

where p is the gas pressure, which is defined by

$$p = \sum_s n_s kT_s = \rho \frac{k}{m} (\alpha T_e + T) \quad . \qquad (29C)$$

Taking the derivatives of equations (28C) with respect to y and using the coordinate transformation and dimensionless quantities introduced in Appendix B, the normal components of the flux vectors can be obtained in the following form:

$$W_{e_y} = \gamma_{31} \frac{d\alpha}{d_\eta} + \gamma_{32} \frac{dg}{d_\eta} + \gamma_{33} \frac{d\theta}{d_\eta} - \gamma_{34} \qquad (30C)$$

$$q_{e_y} = \gamma_{21} \frac{d\alpha}{d_\eta} + \gamma_{22} \frac{dg}{d_\eta} + \gamma_{23} \frac{d\theta}{d_\eta} - \gamma_{24} \qquad (31C)$$

$$q_y = \gamma_{11} \frac{d\alpha}{d_\eta} + \gamma_{12} \frac{dg}{d_\eta} + \gamma_{13} \frac{d\theta}{d_\eta} - \gamma_{14} \tag{32C}$$

where the coefficients are given by

$$\gamma_{31} = - \frac{\rho u_{x_\delta}}{\sqrt{2\xi}} [\frac{pT}{(\alpha T_e + T)^2}] \{(T + T_e) D_{e_1} + T_e / A_1 (1 + \alpha_b^2)\}$$

$$\gamma_{32} = \frac{\rho u_{x_\delta} T_s}{\sqrt{2\xi}} [\frac{p\alpha T_e}{(\alpha T_e + T)^2}] \{(1 - \alpha) D_{e_1} + 1 / A_1 (1 + \alpha_b^2)\}$$

$$\gamma_{33} = - \frac{\rho u_{x_\delta} T_{e_\delta}}{\sqrt{2\xi}} \{\frac{3}{2} \frac{k}{m_e} [\frac{a_{ei} \tau_e^*}{1 + (\omega_e \tau_e^*)^2}] (1 + \alpha_b \omega_e \tau_e^*) / A_1 (1 + \alpha_b^2)$$

$$+ [\frac{p\alpha T}{(\alpha T_e + T)^2}] [1 - \alpha) D_{e_1} + 1 / A_1 (1 + \alpha_b^2)]\}$$

$$\gamma_{34} = \left(- \frac{\alpha T_e}{\alpha T_e + T}\right) \{(1 - \alpha) D_{e_2} - \alpha_b / A_1 (1 + \alpha_b^2)\} j_{y_\delta} B$$

$$+ \frac{e}{m} \rho \alpha E_y^* / A_1 (1 + \alpha_b^2)$$

$$\gamma_{21} = - \frac{\rho u_{x_\delta}}{\sqrt{2\xi}} [\frac{pT}{(\alpha T_e + T)^2}] \{(T + T_e) D_{e_7}$$

$$+ T_e (\frac{5}{2} p_e + \alpha_e + \alpha_b \theta_e) / A_1 (1 + \alpha_b^2)\}$$

$$\gamma_{22} = \frac{\rho u_{x_\delta} T_\delta}{\sqrt{2\xi}} [\frac{p\alpha T_e}{(\alpha T_e + T)^2}] \{1 - \alpha) D_{e_7}$$

$$+ (\frac{5}{2} p_e + \alpha_e + \alpha_b \theta_e) / A_1 (1 + \alpha_b^2)\}$$

$$\gamma_{23} = \frac{\rho u_{x_\delta} T_{e_\delta}}{\sqrt{2\xi}} \{D_{e_9} - \frac{p\alpha T}{(\alpha T_e + T)^2} [(1 - \alpha) D_{e_7}$$

$$+ (\frac{5}{2} p_e + \alpha_e + \alpha_b \theta_e) / A_1 (1 + \alpha_b^2)]\}$$

$$\gamma_{24} = - \frac{\alpha T_e}{\alpha T_e + T} \{(1 - \alpha) D_{e_8} - (A_1 (1 + \alpha_b^2)]^{-1} [\alpha_b (\frac{5}{2} p_e + \alpha_e)$$

$$- \theta_e]\} j_{y_\delta} B + [A_1 (1 + \alpha_b^2)]^{-1} \{\frac{e}{m} \rho\alpha (\frac{5}{2} p_e + \alpha_e + \alpha_b \theta_e) E_y^*\}$$

$$\gamma_{11} = - \frac{\rho u_{x_\delta}}{\sqrt{2\xi}} [\frac{pT}{(\alpha T_e + T)^2}] (T + T_e) c_1$$

$$+ T_e [\frac{5}{2} p_e + \sum_s (\alpha_s + \alpha_b \theta_s)] / A_1 (1 + \alpha_b^2)\}$$

$$\gamma_{12} = \frac{\rho u_{x_\delta} T_\delta}{\sqrt{2\xi}} \{c_5 + \frac{p\alpha T_e}{(\alpha T_e + T)^2} [(1 - \alpha) c_1$$

$$+ \{\frac{5}{2} p_e + \sum_s (\alpha_s + \alpha_b \theta_s)\} / A_1 (1 + \alpha_b^2)]\}$$

$$\gamma_{13} = \frac{\rho u_{x_\delta} T_{e_\delta}}{\sqrt{2\xi}} \{c_3 - \frac{p\alpha T}{(\alpha T_e + T)^2} [(1 - \alpha) c_1$$

$$+ \{\frac{5}{2} p_e + \sum_2 (\alpha_s + \alpha_b \theta_s)\} / A_1 (1 + \alpha_b^2)]\}$$

$$\gamma_{14} = (- \frac{\alpha T_e}{\alpha T_e + T}) \{(1 - \alpha) c_2 - [A_1 (1 + \alpha_b^2)]^{-1} [\alpha_b (\frac{5}{2} p_e + \sum_s \alpha_s)$$

$$- \sum_s \theta_s]\} j_{y_\delta} B + (A_1 (1 + \alpha_b^2)]^{-1} \frac{e}{m} \rho\alpha \{\frac{5}{2} p_e + \sum_s (\alpha_s + \alpha_b \theta_s)\} E_y^*$$

$$- (1 - b_{ia} b_{ai})^{-1} b_1 (1 - b_{ai}) (\frac{e}{m} \tau_i^*)^2 E_y^* B \mu' (\frac{\rho u_{x_\delta}^2}{\sqrt{2\xi}}) \frac{d^2 f}{d\eta^2} \quad . \qquad (33C)$$

The quantities appearing in equations (33C) which have not been previously defined are given below:

$$D_{e_7} = [g_{ie}\ (1 + b_2^2)]^{-1} \{[(\alpha_e + \frac{5}{2} p_e)(1 + b_2\alpha_b) + \theta_e\ (\alpha_b - b_2)]\Big(a_{ei}$$

$$- \frac{\alpha a_{ea}}{1 - \alpha}\Big) / [A_1\ (1 + \alpha_b^2)] - \frac{3}{2} \frac{kT_e}{m_e} [\frac{a_{ei}\tau_e^*}{1 + (\omega_e\tau_e^*)^2}] [1 - b_2\omega_e\tau_e^*$$

$$+ \frac{0.6\ a_{ei}}{p_e} \Big(\{\alpha_e + \frac{5}{2} p_e\} \{1 - b_2\omega_e\tau_e^* + b_2\alpha_b + \alpha_b\omega_e\tau_e^*\}$$

$$+ \theta_e \{\alpha_b - b_2\alpha_b\omega_e\tau_e^* - b_2 - \omega_e\tau_e^*\}\Big) / A_1\ (1 + \alpha_b^2)]\}$$

$$D_{e_8} = [g_{ie}\ (1 + b_2^2)]^{-1} \{[(\alpha_e + \frac{5}{2} p_e)(b_2 - \alpha_b) + \theta_e\ (1 + b_2\alpha_b)]\Big(a_{ei}$$

$$- \frac{\alpha a_{ea}}{1 - \alpha}\Big) / [A_1\ (1 + \alpha_b^2)] - \frac{3}{2} \frac{kT_e}{m_e} [\frac{a_{ei}\tau_e^*}{1 + (\omega_e\tau_e^*)^2}] [b_2 + \omega_e\tau_e^*$$

$$+ \frac{0.6\ a_{ei}}{p_e} \Big(\{\alpha_e + \frac{5}{2} p_e\} \{b_2 - \alpha_b + \omega_e\tau_e^* + b_2\alpha_b\omega_e\tau_e^*\}$$

$$+ \theta_e \{1 + b_2\alpha_b + \alpha_b\omega_e\tau_e^* - b_2\omega_e\tau_e^*\}) / A_1\ (1 + \alpha_b^2)]\}$$

$$D_{e_9} = - \frac{5kp_e}{2m_e} [\frac{\tau_e^*}{1 + (\omega_e\tau_e^*)^2}] \{1 + \frac{0.6\ a_{eI}}{p_e} [(\alpha_e + \frac{5}{2} p_e)(1 + \alpha_b\omega_e\tau_e^*)$$

$$+ \theta_e\ (\alpha_b - \omega_e\tau_e^*)] / [A_1\ (1 + \alpha_b^2)]\}$$

$$c_1 = [g_{ie}\ (1 + b_2^2)]^{-1}\ \{-\ \frac{3}{2}\ \frac{kT_e}{m_e}\ [\frac{a_{ei}\tau_e^*}{1 + (\omega_e\tau_e^*)^2}]\ (1 - b_2\omega_e\tau_e^*)\,(1 - b_{ae})$$

$$+ \frac{5kT}{m}\ \left(1 - \frac{T_e}{T}\right)\left(\frac{\alpha a_{ea}\tau_a^*}{1 - \alpha}\right) + (1 - b_{ai})\ A_7 + [A_1\ (1 - \alpha_b^2)]^{-1}\ [(\frac{5}{2}\ p_e$$

$$+ \sum_s \{\alpha_s + \alpha_b\theta_s\})\,(a_{ei} - \frac{\alpha a_{ea}}{1 - \alpha} - \frac{0.9\ a_{ei}^2}{\rho\alpha\ m_e/m}\ \frac{\tau_e^*}{1 + (\omega_e\tau_e^*)^2}\ \{1 - b_2\omega_e\tau_e^*\})$$

$$+ \left(\frac{5}{2}\ p_e\alpha_b + \sum_s \{\alpha_b\alpha_s - \theta_s\}\right)\left(b_2\ \{a_{ei} - \frac{\alpha a_{ea}}{1 - \alpha}\}\right.$$

$$\left. - \frac{0.9\ a_{ei}^2}{\rho\alpha\ m_e/m}\ \{\frac{\tau_e^*}{1 + (\omega_e\tau_e^*)}\}\{b_2 + \omega_e\tau_e\}\right)]\}$$

$$c_2 = [g_{ie}\ (1 + b_2^2)]^{-1}\ \{-\ \frac{3}{2}\ \frac{kT_e}{m_e}\ [\frac{a_{ei}\tau_e^*}{1 + (\omega_e\tau_e^*)^2}]\ (b_2 + \omega_e\tau_e^*)\,(1 - b_{ae})$$

$$+ \frac{5kT}{m}\ \left(1 - \frac{T_e}{T}\right)\ b_2 a_{ea}\tau_a^* + (1 - b_{ai})\ A_8$$

$$+ [A_1\ (1 + \alpha_b^2)]^{-1}\ [\left(\frac{5}{2}\ p_e + \sum_s\ (\alpha_s + \alpha_b\theta_s)\right)\left(b_2\ \{a_{ei} - \frac{\alpha a_{ea}}{1 - \alpha}\}\right.$$

$$\left. - \frac{0.9\ a_{ei}^2}{\rho\alpha\ m_e/m}\ \{\frac{\tau_e^*}{1 + (\omega_e\tau_e^*)^2}\}\{b_2 + \omega_e\tau_e^*\}\right)$$

$$- \left(\frac{5}{2}\ p_e\ \alpha_b + \sum_s\ \{\alpha_b\alpha_s - \theta_s\}\right)\left(a_{ei} - \frac{\alpha a_{ea}}{1 - \alpha}\right.$$

$$\left. - \frac{0.9\ a_{ei}^2}{\rho\alpha\ m_e/m}\ \{\frac{\tau_e^*}{1 + (\omega_e\tau_e^*)^2}\}\{1 - b_{2\ e\ e}^{\ \ \ *}\}\right)]\}$$

$$c_3 = -\frac{5kp_e}{2m_e}\left[\frac{\tau^*_e}{1+(\omega_e\tau^*_e)^2}\right]\{1 - b_{ae} + b_1\ (1-b_{ai})(b_{ia}b_{ae} - b_{ie})\Big(1 - \frac{\omega_e\omega_i\tau^*_e\tau^*_i}{1-b_{ia}b_{ai}}\Big) + \frac{0.6\ a_{ei}}{p_e}\ [(\frac{5}{2}\ p_e + \Sigma\ \alpha_s)(1 + \alpha_b\omega_e\tau^*_e) + (\sum_s \theta_s)(\alpha_b - \omega_e\tau^*_e)]\ /\ [A_1\ (1+\alpha_b^2)]\}$$

$$c_5 = \frac{5k}{2m}\{b_1\ (1-b_{ai})(b_{ia}\tau^*_a p_a - \tau^*_i p_i) - p_a\tau^*_a\}$$

$$\alpha_i = b_1\ (b_{ia}b_{ae} - b_{ie})\ [\alpha_e - \Big(\frac{\omega_i\tau^*_i}{1-b_{ia}b_{ai}}\Big)\ \theta_e] - \frac{5kT}{m}\Big(1-\frac{T_e}{T}\Big)\ b_1\ \Big(\frac{\alpha}{1-\alpha}\ b_{ia}\tau^*_a a_{ea} + \tau^*_i a_{ei}\Big)\ [b_2^2 - \frac{g_{ei}}{g_{ie}} + b_2\ \Big(\frac{\omega_i\tau^*_i}{1-b_{ia}b_{ai}}\Big)\Big(1+\frac{g_{ei}}{g_{ie}}\Big)]\ /\ (1+b_2^2)$$

$$\theta_i = b_1\ (b_{ia}b_{ae} - b_{ie})\ [\theta_e + \Big(\frac{\omega_i\tau^*_i}{1-b_{ia}b_{ai}}\Big)\ \alpha_e] - \frac{5kT}{m}\Big(1-\frac{T_e}{T}\Big)\ b_1\ \Big(\frac{\alpha}{1-\alpha}\ b_{ia}\tau^*_a a_{ea} + \tau^*_i a_{ei}\Big)\ [-b_2\ \Big(1+\frac{g_{ei}}{g_{ie}}\Big) + \Big(\frac{\omega_i\tau^*_i}{1-b_{ia}b_{ai}}\Big)\Big(b_2^2 - \frac{g_{ei}}{g_{ie}}\Big)]\ /\ (1+b_2^2)$$

$$\alpha_a = \frac{5kT}{m}\Big(1-\frac{T_e}{T}\Big)\Big(b_2^2 - \frac{g_{ei}}{g_{ie}}\Big)\Big(\frac{\alpha a_{ea}\tau^*_a}{1-\alpha}\Big)\ /\ (1+b_2^2) - b_{ae}\alpha_e - b_{ai}\alpha_i$$

$$\theta_a = -\frac{5kT}{m}\Big(1-\frac{T_e}{T}\Big)\ b_2\ \Big(1+\frac{g_{ei}}{g_{ie}}\Big)\Big(\frac{\alpha a_{ea}\tau^*_a}{1-\alpha}\Big)\ /\ (1+b_2^2) - b_{ae}\theta_e - b_{ai}\theta_i$$

(34C)

where

$$A_7 = -\frac{3}{2}\frac{kT_e}{m_e}\left[\frac{a_{ei}\tau_e^*}{1+(\omega_e\tau_e^*)^2}\right] b_1\,(b_{ia}b_{ae} - b_{ie})\,\{1 - b_2\omega_e\tau_e^*$$

$$- \left(\frac{\omega_i\tau_i^*}{1 - b_{ia}b_{ai}}\right)\left(b_2 + \omega_e\tau_e^*\right)\}$$

$$- \frac{5kT}{m}\left(1 - \frac{T_e}{T}\right) b_1 \left(1 - \frac{b_2\omega_i\tau_i^*}{1 - b_{ia}b_{ai}}\right)\left(a_{ei}\tau_i^* + \frac{\alpha}{1-\alpha}a_{ea}b_{ia}\tau_a^*\right) \qquad (34C)$$

$$A_8 = -\frac{3}{2}\frac{kT_e}{m_e}\left[\frac{a_{ei}\tau_e^*}{1+(\omega_e\tau_e^*)^2}\right] b_1\,(b_{ia}b_{ae} - b_{ie})\,\{b_2 + \omega_e\tau_e^*$$

$$+ \left(\frac{\omega_i\tau_i^*}{1 - b_{ia}b_{ai}}\right)\left(1 - b_2\omega_e\tau_e^*\right)\}$$

$$- \frac{5kT}{m}\left(1 - \frac{T_e}{T}\right) b_1 \left(b_2 + \frac{\omega_i\tau_i^*}{1 - b_{ia}b_{ai}}\right)\left(a_{ei}\tau_i^* + \frac{\alpha}{1-\alpha}a_{ea}b_{ia}\tau_a^*\right) \quad .$$

(35C)

Expressions for the electron current density j_{e_y} and the over-all current density j_y are needed also. These quantities are defined as

$$j_{e_y} = -\,e\,n_e\,W_{e_y} = -\frac{e}{m}\rho\alpha\,W_{e_y} \qquad (36C)$$

$$j_y = \sum_s j_{s_y} = \frac{e}{m}\rho\alpha\,(W_{i_y} - W_{e_y}) \quad . \qquad (37C)$$

Equations (30C) and (36C) combine to give the desired expression for the electron current density. If equation (20C) is expanded with s = i using the same procedure was was used in obtaining equation (19C), an expression for W_{i_y} can be obtained. If this expression is combined with equations (30C) and (37C), the over-all current is given by

$$j_y = \frac{e}{m}\rho\alpha\left(\frac{\rho u_{x_\delta}}{\sqrt{2\xi}}\right)\{(D_{i_1} - D_{e_1})\frac{dp_a}{d\eta} + [A_1\ (1 + \alpha_b^2)(1 + b_2^2)]^{-1}\{\left(1 + \frac{g_{ei}}{g_{ie}}\right)\left(1 + b_2\alpha_b\right)\frac{dp_e}{d\eta} + \frac{3}{2}\frac{k}{m_e}\left[\frac{a_{ei}\tau_e^*}{1 + (\omega_e\tau_e^*)^2}\right]\left[\left(1 + \frac{g_{ei}}{g_{ie}}\right)\left(1 + \alpha_b\omega_e\tau_e^*\right) + b_2\ (1 + \frac{g_{ei}}{g_{ie}})\left(\alpha_b - \omega_e\tau_e^*\right]T_{e_\delta}\frac{d\theta}{d\eta}\}\}$$

$$+ \frac{e}{m}\rho\alpha\left(\frac{\alpha T_e}{\alpha T_e + T}\right)\{(1 - \alpha)(D_{i_2} - D_{e_2}) - \left(1 + \frac{g_{ei}}{g_{ie}}\right)\left(b_2 - \alpha_b\right) / [A_1\ (1 + \alpha_b^2)(1 + b_2^2)]\}\ j_{y_\delta}\ B$$

$$+ \{\left(\frac{e}{m}\rho\alpha\right)^2 \left(1 + \frac{g_{ei}}{g_{ie}}\right)\left(1 + b_2\alpha_b\right) / [A_1\ (1 + \alpha_b^2)(1 + b_2^2)]\}\ E_y^* \tag{38C}$$

where

$$\frac{dp_s}{d\eta} = \frac{\sqrt{2\xi}}{\rho u_{x_\delta}}\frac{dp_s}{dy}\ . \tag{39C}$$

Various ratios of cross sections are used in deriving the desired forms of the governing equations and transport property equations. For the sake of simplicity, the ratios for collisions involving neutrals are taken to be those appropriate for Maxwellian molecules and are given as (Chapman and Cowling, 1960)

$$\frac{Z_{st}^{(12)}}{Z_{st}^{(11)}} = \frac{5}{2} \quad ; \quad \frac{Z_{st}^{(13)}}{Z_{st}^{(11)}} = \frac{35}{4} \quad ; \quad \frac{Z_{st}^{(22)}}{Z_{st}^{(11)}} = \frac{Z_{st}^{(23)}}{Z_{st}^{(11)}} = 2.583 \quad . \tag{40C}$$

However, the $Z_{st}^{(11)}$ expressions employed in the computations are curve fits of experimental data. For collisions between two charged particles, coulomb interactions are assumed, and in this case (Chapman and Cowling, 1960)

$$\frac{Z_{st}^{(12)}}{Z_{st}^{(11)}} = 1 \quad ; \quad \frac{Z_{st}^{(13)}}{Z_{st}^{(11)}} = 2 \quad ; \quad \frac{Z_{st}^{(22)}}{Z_{st}^{(11)}} = \frac{Z_{st}^{(23)}}{Z_{st}^{(11)}} = 2 \quad . \tag{41C}$$

The cross sections $Z_{st}^{(11)}$ are given next. Following Spitzer (1962) $Z_{ie}^{(11)}$ for argon can be written as

$$Z_{ie}^{(11)} = \left(2.96 \times 10^{-16}\right)\left(\frac{I}{kT_e}\right)^2 \ell n \left\{\left(3.441 \times 10^6\right)\left(\frac{kT_e}{I}\right)^3 \Big/ \left(\rho\alpha \frac{I}{m}\right)\right\} \quad cm^2 \quad . \tag{42C}$$

Using data from Cramer (1959) and the curve fit suggested by Cloney *et al.* (1962), $Z_{ia}^{(11)}$ can be written as

$$Z_{ia}^{(11)} = 16.383 \times 10^{-15} \{1 - [\ell n \left(7.88 \frac{kT}{I}\right)] [0.1302 - 0.00648 \, \ell n \left(7.88 \frac{kT}{I}\right)]\} \quad cm^2 \quad . \tag{43C}$$

Using data from Barbiere (1951) and a best straight line curve fit, $Z_{ea}^{(11)}$ can be written as

$$Z_{ea}^{(11)} = 0.283 \times 10^{-16} \{5.82 + 4.4504 \left(\frac{T_e}{10^4} - 1.27655\right)\} \quad cm^2 \quad . \qquad (44C)$$

The neutral-neutral collisions are assumed to follow the Lennard-Jones potential. At the high temperatures of interest here, the repulsive part of the potential dominates; and thus, following Hirschfelder *et al.* (1954), $Z_{aa}^{(11)}$ for argon is given by

$$Z_{aa}^{(11)} = 2.528 \times 10^{-15} \{ \left(\frac{kT}{I}\right)^{-1/6} \} \quad cm^2 \quad . \qquad (45C)$$

Appendix D. Sheath Solution

The sheath solution used here considers a collisionless cathode sheath consisting of ions, electrons, and neutral particles. These particles may be incident, diffusely reflected, specularly reflected, trapped, or emitted. The derivation is that of Hassan (1968) and will not be repeated here. In this work, however, the temperature of the ions and neutral particles in the sheath is taken equal to the wall temperature. Also the minimum drift velocity of the ions is taken to be zero. Letting the subscripts 1, 2 and 3 refer to ions, electrons, and neutrals, respectively, the current densities in the sheath are given by

$$j_1 = -e \sqrt{\frac{kT_0}{2\pi m}} \{a_1 (1 - \alpha_{\infty,1}) - (b_1 + d_1) e^{-a}\} \tag{1D}$$

$$j_2 = e \sqrt{\frac{kT_0}{2\pi m}} \{a_2 (1 - \alpha_{\infty,2} - \beta_{\infty,2}) \sqrt{\theta_{i,2}}\, e^{-a/\theta_{i,2}} - d_2\} \tag{2D}$$

where

$$a = e (\phi_\infty - \phi_0) / kT_0$$

$$\theta_{i,2} = T_{e_\infty} / T_0$$

$$\theta_{d,2} = \theta_{i,2} + \gamma_2 (1 - \theta_{i,2}) \tag{3D}$$

and where $\alpha_{\infty,j}$ and $\beta_{\infty,j}$ are the coefficients of specular and diffuse reflection for particles "j" coming from the sheath edge, γ_2 is the accommodation coefficient for electrons, $(\phi_\infty - \phi_0)$ is the potential drop across the sheath, m and m_e are the masses of an atom and an electron, e is the electronic charge, and k is Boltzmann's constant. The subscripts i and d refer to incident and diffusely reflected particles. The quantities d_1 and d_2 are given by

$$d_1 = \frac{\nu}{k} \{a_1 \, [(1 - \alpha_{\infty,1})(1 - \alpha_{0,1} - \beta_{0,1}) + e^{-a} \, (\beta_{0,1} \{1 - \alpha_{\infty,1}\}$$

$$- \beta_{\infty,1} \{1 - \alpha_{0,1}\})]$$

$$+ a_3 \, [1 - \alpha_{\infty,3} - \beta_{\infty,3}][1 - \alpha_{0,1} - \beta_{0,1} (1 - e^{-a})]\}$$

$$d_2 = 4 \left(\frac{2\pi m_e kT_0}{h^2}\right)^{3/2} e^{-ew/kT_0} \tag{4D}$$

where

$$k = 1 - \alpha_{0,1} - \beta_{0,1} (1 - e^{-a}) + \nu \, (1 - \alpha_{0,1} - \beta_{0,1}) \, e^{-a}$$

$$\nu = \{2 \, (1 + \text{erf} \sqrt{a}) \, e^{e(I - w)/kT_0}\}^{-1} \tag{5D}$$

and where h is Planck's constant, I is the ionization potential of the medium, and w is the work function of the cathode. The sign convention is such that particle flux away from the cathode is considered positive. The over-all current density can be found by adding equations (1D) and (2D).

The heat flux for a particular species "j" is the sum of the heat fluxes relating to those particles of species "j" which are incident upon the cathode, those which are specularly and diffusely reflected, and those which are emitted. Thus

$$q_j = -q_{i,j} + q_{s,j} + q_{d,j} + q_{e,j} \tag{6D}$$

where i, s, d, and e refer to incident, specularly reflected, diffusely reflected, and emitted particles, respectively. Following the procedure of Hassan (1968)

$$q_{i,j} = \int \frac{F_{ij}}{m_j} \varepsilon_j \, d\varepsilon_j + (kT_{i,j} - e_j\phi) \, \dot{n}_j \tag{7D}$$

where $\dot{n}_j$ is the flux of particles "j". Using the distribution functions of Hassan (1968), equations (6D) and (7D) combine to give the following relations:

$$q_1 = - kT_0 \sqrt{\frac{kT_0}{2\pi m}} \{a_1 \, [2 + e \, (\phi_\infty - \phi)/kT_0] \, (1 - \alpha_{\infty,1}$$
$$- b_1 \, e^{-a} \, [2 + e \, (\phi_\infty - \phi)/kT_0] - d_1 \, e^{-a} \, [2 + e \, (\phi_\infty - \phi)/kT_0]\} \tag{8D}$$

$$q_2 = - kT_0 \sqrt{\frac{kT_0}{2\pi m}} \{a_2 \, [2\theta_{i,2} + e \, (\phi - \phi_0)/kT_0] (1 - \alpha_{\infty,2}) \sqrt{\theta_{i,2}} \, e^{-a/\theta_{i,2}}$$
$$- \sqrt{\theta_{d,2}} \, [2\theta_{d,2} + e \, (\phi - \phi_0)/kT_0] \, b_2$$
$$- [2 + e \, (\phi - \phi_0)/kT_0] \, d_2\} \tag{9D}$$

$$q_3 = - 2kT_0 \sqrt{\frac{kT_0}{2\pi m}} \{(1 - \alpha_{\infty,3}) \, a_3 - b_3 - d_3\} \tag{10D}$$

where ϕ is the potential and where

$$d_3 = d_1/\nu$$

$$b_1 = \lambda_1 a_1 + \lambda_3 a_3 \quad ; \quad b_3 = \beta_{\infty,3} a_3$$

$$b_2 = a_2 \, \{\beta_{\infty,2} \sqrt{\theta_{i,2}/\theta_{d,2}} \, e^{-a/\theta_{i,2}}\}$$

$$\lambda_1 = \{1 - \alpha_{0,1} - \beta_{0,1} (1 - e^{-a})\}^{-1} \{\beta_{\infty,1} (1 - \alpha_{0,1})$$

$$+ \frac{\nu}{k} \beta_{0,1} (1 - e^{-a}) [(1 - \alpha_{\infty,1})(1 - \alpha_{0,1} - \beta_{0,1})$$

$$+ e^{-a} (\beta_{0,1} \{1 - \alpha_{\infty,1}\} - \beta_{\infty,1} \{1 - \alpha_{0,1}\})]\}$$

$$\lambda_3 = \frac{\nu}{k} \beta_{0,1} (1 - e^{-a})(1 - \alpha_{\infty,3} - \beta_{\infty,3}) \tag{11D}$$

If one sets $\phi = \phi_\infty$ at the sheath edge and $\phi = \phi_0$ at the wall, equations (8D) through (10D) give the following:

$$q_{1_\omega} = - kT_0 \sqrt{\frac{kT_0}{2\pi m}} \{(1 - \alpha_{\infty,1})(2 + a)\ a_1 - e^{-a}\ (2 + a)(b_1 + d_1)\}$$

$$+ j_1\ (I - w)$$

$$q_{2_\omega} = - 2kT_0 \sqrt{\frac{kT_0}{2\pi m_3}} \{(1 - \alpha_{\infty,2})(\theta_{i,2})^{3/2}\ e^{-a/\theta_{i,2}}\ a_2$$

$$- (\theta_{d,2})^{3/2}\ b_2 - d_2\} - j_2 w$$

$$q_{3_\omega} = - 2kT_0 \sqrt{\frac{kT_0}{2\pi m}} \{(1 - \alpha_{\infty,3})\ a_3 - (b_3 + d_3)\} \tag{12D}$$

$$q_{1,\infty} = - 2kT_0 \sqrt{\frac{kT_0}{2\pi m}} \{(1 - \alpha_{\infty,1})\ a_1 - e^{-a}\ (b_1 + d_1)\}$$

$$q_{2,\infty} = - kT_0 \sqrt{\frac{kT_0}{2\pi m_e}} \{(1 - \alpha_{\infty,2})\ \sqrt{\theta_{i,2}}\ (2\theta_{i,2} + a)\ e^{-a/\theta_{i,2}}\ a_2$$

$$- \sqrt{\theta_{d,2}}\ (2\theta_{d,2} + a)\ b_2 - (2 + a)\ d_2\}$$

$$q_{3,\infty} = q_{3_\omega} \tag{13D}$$

where the quantities j_1 (I - w) and $j_2 w$ are the surface contributions to the heat flux.

All the expressions presented thus far depend upon three undetermined quantities: a_1, a_2, and a_3. These quantities are determined by specifying the following:

$$\frac{n_{1\infty}}{n_{2\infty}} = 1 \quad , \quad \frac{n_{2\infty}}{n_{3\infty}} = \frac{\alpha}{1 - \alpha} \tag{14D}$$

$$p = kT_0 \, \Sigma \, n_{i,j} \, \theta_{i,j} \tag{15D}$$

where $n_{i,j}$ are the various number densities for species "j", α is the degree of ionization at the sheath edge, and p is the pressure at the sheath edge. Using the expressions of Hassan (1968) for the number densities, equations (14D) and (15D) give

$$a_1 = n_{2\infty} \left(x_{33} - \frac{1 - \alpha}{\alpha} x_{31}\right) / \left(x_{11} x_{33} - x_{13} x_{31}\right) \tag{16D}$$

$$a_3 = n_{2\infty} \left(\frac{1 - \alpha}{\alpha} x_{11} - x_{13}\right) / \left(x_{11} x_{33} - x_{13} x_{31}\right) \tag{17D}$$

$$\begin{aligned} a_2 = \{ & y_{22} + x_{22} \left[y_{11} \left(x_{33} - \frac{1 - \alpha}{\alpha} x_{31}\right) + y_{33} \left(\frac{1 - \alpha}{\alpha} x_{11} \right.\right. \\ & \left.\left. - x_{13}\right)\right] / \left(x_{11} x_{33} - x_{13} x_{31}\right\}^{-1} \left\{\frac{p}{kT_0}\right. \\ & - \frac{d_2}{2} e^a \left(\text{erfc} \sqrt{a}\right) \left[1 + y_{11} \left(x_{33} - \frac{1 - \alpha}{\alpha} x_{31}\right)\right. \\ & \left. + y_{33} \left(\frac{1 - \alpha}{\alpha} x_{11} - x_{13}\right)\right] / \left(x_{11} x_{33} - x_{13} x_{31}\right) \end{aligned} \tag{18D}$$

where

$$n_{2\infty} = a_2 \{\text{erf} \sqrt{a/\theta_{i,2}} + 0.5 \, (1 + \alpha_{\infty,2}) \text{ erfc} \sqrt{a/\theta_{i,2}}$$

$$+ 0.5 \, \beta_{\infty,2} \sqrt{\theta_{i,2}/\theta_{d,2}} \; e^{a\left(\frac{1}{\theta_{d,2}} - \frac{1}{\theta_{i,2}}\right)} \; (\text{erfc} \sqrt{a/\theta_{d,2}})\}$$

$$+ 2 \left(\frac{2\pi m_e kT_0}{h^2}\right)^{3/2} e^{(a - ew/kT_0)} \text{ erfc} \sqrt{a}$$

$$x_{11} = 0.5 \, \{1 + \alpha_{\infty,1} + \lambda_1 e^{-a} + \nu d_1 \, e^{-a}\}$$

$$x_{31} = 0.5 \, \{\lambda_3 + \nu d_3\} \, e^{-a} \qquad ; \qquad x_{13} = 0.5 \, d_1$$

$$x_{33} = 0.5 \, \{1 + \alpha_{\infty,3} + \beta_{\infty,3} \sqrt{\theta_{i,3}/\theta_{d,3}} + \sigma_3\}$$

$$y_{11} = 0.5 \, \{1 + \alpha_{\infty,1} + \lambda_1 \, e^{-a} + \sigma_1 \, (\nu_e^{-a} + 1)\}$$

$$y_{22} = \theta_{i,2} \, (\text{erf} \sqrt{a/\theta_{i,2}}) + 0.5 \, \{(1 + \alpha_{\infty,2}) \, \theta_{i,2} \, (\text{erfc} \sqrt{a/\theta_{i,2}})$$

$$+ \beta_{\infty,2} \sqrt{\theta_{i,2}\theta_{d,2}} \; e^{a\left(\frac{1}{\theta_{d,2}} - \frac{1}{\theta_{i,2}}\right)} \left(\text{erfc} \sqrt{a/\theta_{d,2}}\right)\}$$

$$y_{33} = 0.5 \, \{\lambda_3 \, e^{-a} + 1 + \alpha_{\infty,3} + \beta_{\infty,3} + \sigma_3 \, (\nu_e^{-a} + 1)\} \tag{13D}$$

The quantities σ_1, σ_2, and σ_3 are given by Hassan (1968).