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TECHNICAL REPORT

INVESTIGATION OF ZERO-CROSSINGS AS INFORMATION CARRIERS

for

National Aeronautics & Space Administration
Electronics Research Center

under

NASA Grant NGR 33-006-040

Prepared by

Arthur L. Fawe, Graduate Student
Robert R. Boorstyn, Associate Professor
Department of Electrical Engineering
Polytechnic Institute of Brooklyn

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ABSTRACT

INVESTIGATION OF ZERO-CROSSINGS AS INFORMATION CARRIERS

by

Arthur L. Fawe

Adviser: Robert R. Boorstyn

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We consider the transmission of a signal bandlimited to (- W, + W) by the zero-crossings of the optimum signal

$$y(t) = \sum_{k=-\infty}^{+\infty} (-1)^k \, |x(k/2W)| \, \frac{\sin(2\pi\, Wt - k\pi)}{2\pi\, Wt - k\pi}$$

We describe a computer algorithm to estimate y(t) from its zero-crossing in a finite time interval.

We show that as the channel bandwidth increases the output signal-to-noise-ratio a_o for the clipped version of the optimum signal tends to an exponential function of the channel signal-to-noise ratio a_c

$$a_o \rightarrow \frac{\sqrt{2\pi\, a_c}}{4} \exp\,(a_c/2)$$

A similar behavior is obtained for x(t) itself.

We analyse the effect of the channel $H_c(\omega) = \sin(\omega T)/\omega T$. "Finite" channel bandwidth introduces an additional term, linear in a_c. This term is related to the appearance of extra zeros in the 2T interval of time about the zero-crossings of the optimum signal; the exponential term is related to extra zeros occuring outside this interval at low channel signal-to-noise ratio.

Directions for further research on this problem are: nonlinear memoryless transformation of $x(t)$ to control the distribution of the zero-crossings of the optimum signal $y(t)$, nonlinear memoryless transformation of $y(t)$ and filtering at the receiver input to minimize the zero-crossings displacement, optimization of the zero-crossings detector, and estimation of the optimum signal from its zero-crossings.

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List of symbols.

$x(t)$	system input
x_k	$= x(k/2W)$, k-th sample of $x(t)$
$y_i(t)$	output of the i-th intermediate stage of the communication system. In particular $y_1(t)$ is the optimum signal associated to $x(t)$ or $x(t)$ differentiated or $x(t)$ plus a narrowband process centered about W.
a	signal-to-mean-square-error ratio
a_c	channel signal-to-noise ratio
a_o	output signal-to-noise ratio
W	highest frequency of the signal $x(t)$
Ω	$2\pi W$
B	channel bandwidth for the transmission of the optimum signal
T	half the time interval in which the computer algorithm gives an estimate of the signal $x(t)$ (in III-2 and IV-2 however T has been used for the half integration time of the channel $h_c(t) = p_T(t)$)
b	bandwidth expansion factor for optimum signals
β	bandwidth expansion factor for general signals ($\beta = b + 1$)
$h_c(t)$	channel impulse response
$H_c(\omega)$	channel transfer function
N_o	power density of white noise
λ	zero-crossing rate (or density) of a signal = lim (number of zero-crossings in time interval T/T) for $T\to\infty$
λ_n	normalized zero-crossing rate for a signal bandlimited to $(-W, +W)$: $\lambda_n = \lambda/2W$
n	number of differentiations of the signal $x(t)$
γ	ratio of narrow-band process power to signal power
α	ω_o/Ω = ratio of half-bandwidth of narrow-band process to center frequency
z_k	zero-crossing of the optimum process between $k/2W$ and $(k+1)/2W$

Z	random variable which takes the values +1 of -1, each with probability 1/2
$P(.)$	probability of the event described in the parentheses
$f_x(.)$	density function of the random variable x
$F_x(.)$	distribution function of the random variable x
$E(.)$	expectation
$R_{xy}(t_1, t_2)$	crosscorrelation function of the two random processes x(t) and y(t)
$r_{xy}(t_1, t_2)$	normalized crosscorrelation function $= R_{xy}(t_1, t_2)/\sigma_x(t_1)\sigma_y(t_2)$
$S_{xy}(\omega)$	when the processes are stationary, Fourier transform of $R_{xy}(t_1 - t_2)$
σ_x^2	variance of the random variable x
$\Phi_x(.)$	characteristic function of the random variable x
$p_\Omega(\omega)$	function to 1 if $\lvert\omega\rvert \leq \Omega$, 0 otherwise
$sgn(t)$	function equal to -1 if $t < 0$, 0 if $t=0$, and +1 if $t > 0$
$\delta(t)$	Dirac impulse function
$\psi_k(t)$	$= \dfrac{\sin(\Omega t - k\pi)}{(\Omega t - k\pi)}$
$Si(t)$	sine integral $= \displaystyle\int_0^t \frac{\sin(u)}{u} du$
$Q(t)$	$= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_t^\infty \exp\left(-\frac{u^2}{2}\right) du$. This function is used rather than the distribution function itself (usually denoted by $\Phi(t)$) to avoid confusion with the characteristic function
δ_{kl}	Kronecker symbol = 1 if k = l, 0 otherwise
$(.)^*$	conjugate of of the quantity
$\lvert.\rvert$	absolute value of the quantity
$\longleftrightarrow$	denotes a pair of Fourier transforms
$p_{\Omega,\omega_o}(\omega)$	function equal to 1 if $\Omega - \omega_o \leq \lvert\omega\rvert \leq \Omega + \omega_o$, 0 otherwise

INTRODUCTION

Signal transmission by means of zero-crossings takes its root in a phenomena, clipped speech intelligibility, reported in the literature some 20 years ago[1]. Advantage has been taken of this property for the communication of speech signals[2-5]. Clipped speech makes possible the transmission of intelligible speech through channels of poor quality: typically for the same intelligibility clipped speech needs about 1/4 of the channel capacity required for normal speech[6]. In this work, we investigate the transmission of the bandlimited signal (in general) by means of a set of related zero-crossings.

In Chapter 1 we describe the main statistical properties of a subclass of bandlimited processes (called optimum) characterized by one zero-crossing in each Nyquist interval (it will appear that the zero-crossings of this subclass are the related zero-crossings we are looking for). The ideal clipper destroys all information but the zero-crossings of its input; thus we give a description of this device by the transform method. We define a measure of communication systems performances; we show that in our case, since the bandwidth expansion occurs by clipping the optimum process, the suitable types of modulation give the same performances as a direct transmission of the signal. We advise the reader to start by Chapter II and to come to the appropriate sections of Chapter I as the understanding of the material requires it.

In general a bandlimited signal is not completely defined by its zero-crossings. Thus the set of these points contains less information than the signal itself. In order to use the zero-crossings as information carriers we should therefore find the mapping which maximize the amount of information in the new set of zero-crossings. But, at the present time, even for Gaussian processes, we do not have a nice analytic form for the distribution of the zero-crossings interval. Thus the approach just mentioned cannot be used. We rather process in an heuristic way by investigating in Chapter II various ways of increasing the zero-crossings <u>rate</u>. The two first methods (differentiation and addition of a sine wave) have a common property: as the zero-crossings rate increases, the spacing between two successive zeros becomes more regular; we show that these techniques fail to give a signal completely defined by its zero-crossings. From the theorems discovered by

Titchmarsh(7) and Polya'(8) we derive a third technique (mapping into an optimum process) which satisfies the above requirement.

At this point we have a bandlimited signal completely defined by its zero-crossings and related in a known way to the original signal. The next step is the investigation of the effect of noise and filtering on these zero-crossings. This is undertaken in Chapter III. We first derive the properties of random square waves transmission for large bandwidth ($\frac{B}{W} \to \infty$) and additive Gaussian noise. For finite channel bandwidth we find an approximate expression when the channel transfer function is $H_c(\omega) = \frac{\sin \omega T}{\omega T}$ (this channel has no exact finite bandwidth; however it has a filtering action and its bandwidth, although somewhat arbitrary, can be defined).

Finally, in Chapter IV we assemble the various parts of the system: the mapping of the signal into an optimum process, its transmission through the channel, the estimation of the optimum process from the received zero-crossings by means of a computer algorithm and the inverse mapping into the estimate of the signal. The overall performances are then related to the result found in the previous chapter.

CHAPTER I

TOOLS OF THIS RESEARCH

1. Description of signal clipping.

We shall investigate the transmission of a signal by a set of related zero-crossings. Thus the ideal clipper appears an an essential element in our system and we need a convenient representation of this device.

Let y(t) and z(t) be the ideal clipper input and output respectively.

$$z(t) = \begin{cases} +1 & \text{if } y(t) > 0 \\ -1 & \text{if } y(t) < 0 \end{cases} \tag{I-1-1}$$

or

$$z(t) = \operatorname{sgn}[y(t)] \tag{I-1-2}$$

$$\operatorname{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

we can write

$$z(t) = \frac{1}{j\pi} \int_{-\infty}^{+\infty} \omega^{-1} e^{j\omega y(t)} d\omega \tag{I-1-3}$$

Let $f(y_1, y_2; t_1, t_2)$ be the joint density function of $y(t_1)$ and $y(t_2)$ (denoted by y_1 and y_2 respectively in the following), and $\Phi(\omega_1, \omega_2; t_1, t_2)$ the joint characteristic function

$$\Phi(\omega_1, \omega_2; t_1, t_2) = E\left\{e^{j(\omega_1 y_1 + \omega_2 y_2)}\right\} \tag{I-1-4}$$

Then (since we consider real processes only) the cross-correlation function between the clipper input and output is

$$\begin{aligned} R_{yz}(t_1, t_2) &= E\left\{y(t_1) z(t_2)\right\} \\ &= \frac{1}{j\pi} \int_{-\infty}^{+\infty} \omega_2^{-1} E\left\{y_1 e^{j\omega_2 y_2}\right\} d\omega_2 \\ &= -\frac{1}{\pi} \int_{-\infty}^{+\infty} \omega_2^{-1} \left[\frac{\partial \Phi}{\partial \omega_1}\right]_{\omega_1 = 0} d\omega_2 \end{aligned} \tag{I-1-5}$$

since $\left[\frac{\partial \Phi}{\partial \omega_1}\right]_{\omega_1 = 0} = jE\left\{y_1 e^{j\omega_2 y_2}\right\}$ (I-1-6)

For $t_1 = t_2$ (I-1-5) leads to

$$R_{yz}(t_1, t_1) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \omega^{-1} \frac{d\Phi}{d\omega}\, d\omega \tag{I-1-7}$$

where Φ is now the characteristic function of the random variable y(t)

$$\Phi(\omega; t) = E\left\{e^{j\omega y(t)}\right\} \tag{I-1-8}$$

Similarly the autocorrelation of the clipper output is

$$\begin{aligned} R_{zz}(t_1, t_2) &= E\left\{z(t_1)\, z(t_2)\right\} \\ &= -\frac{1}{\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (\omega_1\omega_2)^{-1}\, \Phi(\omega_1, \omega_2; t_1, t_2)\, d\omega_1 d\omega_2 \end{aligned} \tag{I-1-9}$$

Formulae (I-1-5, I-1-7 and I-1-9) make sense only if in the integrand the pole at the origin cancels with a zero in the other factor. However if this does not happen we can still get a correct result by the use of differentiation on both sides of these relations followed by appropriate integration (the constant of integration being derived from the known initial conditions). To illustrate this point let us investigate the autocorrelation of the output of a clipper when the input is a stationary Gaussian process. The answer is known to be given by the arc-sine law

$$R_{zz}(\tau) = \frac{2}{\pi}\sin^{-1} r_y(\tau) \tag{I-1-10}$$

where $r_y(\tau) = R_y(\tau)/\sigma_y^2$ (I-1-11)

Taking account of the stationarity we write (I-1-9) as

$$D_\tau R_{zz}(\tau) = -\frac{1}{\pi^2}\iint_{-\infty}^{+\infty} (\omega_1\omega_2)^{-1}\, D_\tau\left[e^{-\sigma_y^2 \frac{\omega_1^2 + \omega_2^2 + 2r_y(\tau)\omega_1\omega_2}{2}}\right] d\omega_1 d\omega_2 \tag{I-1-12}$$

where the differential operator $D_\tau = d/d\tau$. We get in this way a relation where the pole at the origin disappears

$$R'_{zz}(\tau) = \frac{1}{\pi^2} R'_{yy}(\tau) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\sigma_y^2 \frac{\omega_1^2+\omega_2^2+2r_y\omega_1\omega_2}{2}} d\omega_1 d\omega_2 \qquad \text{(I-1-13)}$$

where R' stands for $dR/d\tau$. With the change of variables

$$\omega_i \sigma_y = \frac{u_i}{\sigma_y \sqrt{(1-r_y^2)}} \qquad (i = 1, 2) \qquad \text{(I-1-14)}$$

we get

$$R'_{zz}(\tau) = \frac{2}{\pi} \frac{r'_y(\tau)}{\sqrt{1-r_y^2(\tau)}} \qquad \text{(I-1-15)}$$

By integration

$$R_{zz}(\tau) = \frac{2}{\pi} \sin^{-1} [r_y(\tau)] + C \qquad \text{(I-1-16)}$$

But

$$R_{zz}(0) = 1 \text{ and therefore } C = 0.$$

The technique we have described implies no restriction on the non-linearity involved (except the existence of its Fourier transform). It implies also no restriction on the input process (thus this result is not restricted to the Gaussian case). Therefore we have here a very general tool.

2. Statistical description of optimum signals

In chapter II we shall need the properties of a special class of processes, introduced in our Master's thesis, which we called "optimum signals"(9). For convenience we recall first the definition. Let x(t) be a signal bandlimited to $(-\Omega, +\Omega)$ (deterministic or random at the present time)

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{(I-2-1)}$$

or

$$S_x(\omega) = \int_{-\infty}^{+\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau \qquad \text{(I-2-2)}$$

vanishes outside $(-\Omega, +\Omega)$. In both cases the sampling theorem holds, e.g.

$$x(t) = \sum_{k=-\infty}^{+\infty} x\left(\frac{k}{2W}\right) \frac{\sin(\Omega t - k\pi)}{(\Omega t - k\pi)} = \sum_{k=-\infty}^{+\infty} x_k\, \psi_k(t) \qquad \text{(I-2-3)}$$

In the random case we must understand this equality in the mean square sense.

The optimum signal, $y(t)$, is derived from $x(t)$ in the following way: we leave the sample amplitudes unchanged but we change their polarities so that the following equality is true

$$\operatorname{sgn}\left[y\left(\frac{k}{2W}\right) y\left(\frac{k-1}{2W}\right)\right] = -1 \quad \text{for all } k \qquad \text{(I-2-4)}$$

Thus

$$y(t) = \sum_{k=-\infty}^{+\infty} (-1)^k |x_k|\, \psi_k(t) \qquad \text{(I-2-5)}$$

It follows that between two consecutive samples $y(t)$ has at least one zero-crossing. In the deterministic case Polyá has shown that there is only one zero-crossing (appendix A). In the random case there is no proof that such a property exists ; we will just say that in the simulation part of this work we were unable to find an interval with more than one zero-crossing and we are tempted to say that "there is only one zero-crossing in each Nyquist interval with probability one". The occurrence of a zero-crossing in each interval is the basis of the optimum signal properties.

To find the cross-correlation function between the optimum process and its clipped version section 1 shows that we need the second-order characteristic function, $x(t)$ will be assumed stationary in the wide sense, Gaussian, zero-mean with a power spectrum uniform in $(-\Omega, +\Omega)$. Then the samples x_k are independent, Gaussian, zero-mean random variables with identical distribution. It will be useful* to multiply each realization of y(t) by the following random variable, independent of the x_k's ;

$$Z = +1 \text{ or } -1 \text{ with probability } 1/2 \tag{I-2-12}$$

This will not affect the optimality of y(t), and the sign of y(t) is already undetermined when the zero-crossings are known. Thus

$$y(t) = Z \sum_{k=-\infty}^{+\infty} (-1)^k \, |x_k| \, \psi_{k(t)} \tag{I-2-13}$$

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* Otherwise the means will not be zero. Indeed

$$E\left\{y(t)\right\} = \sigma_x \sqrt{\frac{2}{\pi}} \sum_{k=-\infty}^{+\infty} (-1)^k \, \psi_k(t) \tag{I-2-6}$$

But

$$\sum_{k=-\infty}^{+\infty} (-1)^k \psi_k(t) \longleftrightarrow p_\Omega(\omega) \frac{1}{2W} \sum_{k=-\infty}^{+\infty} \exp\left[-jk\left(\frac{\omega}{2W} + \pi\right)\right] \tag{I-2-7}$$

The finite sum

$$\frac{1}{2W} \sum_{k=-N}^{+N} \exp\left[-jk\left(\frac{\omega}{2W} + \pi\right)\right] = \frac{1}{2W} \frac{\sin\left[\left(N + \frac{1}{2}\right)\left(\frac{\omega}{2W} + \pi\right)\right]}{\sin\left[\frac{1}{2}\left(\frac{\omega}{2W} + \pi\right)\right]} \tag{I-2-8}$$

is periodic with period 2Ω. On the other hand

$$\lim_{N \to \infty} \frac{\sin\left[N + \frac{1}{2})\left(\frac{\omega}{2W} + \pi\right)\right]}{\omega + 2\pi W} = \pi \, \delta \, (\omega + \Omega) \tag{I-2-9}$$

and $2\pi \, \delta \, (\omega + \Omega) \dfrac{(\omega + \Omega)/4W}{\sin[\omega + \Omega)/4W]} = 2\pi \, \delta \, (\omega + \Omega)$ (I-2-10)

Therefore (I-2-8) is a sequence of impulses of strength 2π and (I-2-7) is equal to

$$\pi \, [\delta \, (\omega - \Omega) + \delta \, (\omega + \Omega)] \longleftrightarrow \cos \, (\Omega t) \tag{I-2-11}$$

Thus $E\left\{y(t)\right\} = \sigma_x \sqrt{\frac{2}{\pi}} \cos \Omega t$ with (I-2-5) as definition of y(t).

It is easy to see that at each sampling time y(t) is a normally distributed random variable with mean zero and same variance σ_x^2 as x(t).

First we consider the first-order characteristics function of y(t).

$$\Phi_y(\omega;t) = E\left\{e^{j\omega y(t)}\right\}$$

$$= E\left\{\exp\left[j\omega Z \sum_{k=-\infty}^{+\infty} (-1)^k |x_k| \psi_k(t)\right]\right\} \qquad \text{(I-2-14)}$$

Since Z is independent of the x_k's

$$\Phi_y(\omega;t) = \frac{1}{2}\left[E\left\{\exp\left[j\omega \sum_{k=-\infty}^{+\infty} (-1)^k |x_k| \psi_k(t)\right]\right\} + \right.$$

$$\left. E\left\{\exp\left[-j\omega \sum_{k=-\infty}^{+\infty} (-1)^k |x_k| \psi_k(t)\right]\right\}\right]$$

$$= \text{Real}\, E\left\{\exp\left[j\omega \sum_{k=-\infty}^{+\infty} (-1)^k |x_k| \psi_k(t)\right]\right\} \qquad \text{(I-2-15)}$$

Thus $\Phi_y(\omega;t)$ is a real function; this comes from the fact that the random variable Z makes the density of y(t) symmetrical. Since the x_k's are independent

$$\Phi_y(\omega;t) = \text{Real} \prod_{k=-\infty}^{+\infty} E\left\{\exp\left[j(-1)^k \omega |x_k| \psi_k(t)\right]\right\} \qquad \text{(I-2-16)}$$

For t = n/2W we get

$$\Phi_y(\omega;n/2W) = \text{Real}\; E\left\{\exp\left[j(-1)^n \omega |x_n|\right]\right\}$$

$$= E\left\{\exp(j\omega x_n)\right\}$$

$$= \exp\left(-\frac{\sigma_x^2 \omega^2}{2}\right) \qquad \text{(I-2-17)}$$

as expected. Otherwise since*

* I. S. Gradshteyn and I. M. Ryzhik, "Tables of Integrals, Series, and Products," Ac. Press, section 3.896 (p. 480).

$$\int_0^\infty e^{-au^2} \sin(bu)du = \frac{b}{2a} \exp\left(-\frac{b^2}{4a}\right) {}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{b^2}{4a}\right) \qquad \text{(I-2-18)}$$

where ${}_1F_1(a;b;z)$ is Kummer's function

$${}_1F_1(a;b;z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \ldots \qquad \text{(I-2-19)}$$

and

$$\int_0^\infty e^{-au^2} \cos(bu)\, du = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(-\frac{b^2}{4a}\right) \qquad \text{(I-2-20)}$$

(I-2-16) becomes

$$\Phi_y(\omega;t) = \text{Real} \prod_{k=-\infty}^{+\infty} e^{-\frac{\omega_k^2}{2}}\left[1 + j(-1)^k\sqrt{\frac{2}{\pi}}\, \omega_k\, {}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{\omega_k^2}{2}\right)\right] \qquad \text{(I-2-21)}$$

where $\omega_k = \omega\, \sigma_x \psi_k(t)$ (I-2-22)

But

$$\exp\left(-\frac{1}{2}\sum_{k=-\infty}^{+\infty} \omega_k^2\right) = \exp\left[-\frac{\omega^2\sigma_x^2}{2}\sum_{k=-\infty}^{+\infty} \psi_k^2(t)\right] \qquad \text{(I-2-23)}$$

and since for two bandlimited functions f(t) and g(t) with finite energy and same bandwidth W

$$\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \frac{1}{2W}\sum_{k=-\infty}^{+\infty} f\left(\frac{k}{2W}\right) g\left(t-\frac{k}{2W}\right) \qquad \text{(I-2-24)}$$

we get

$$\frac{1}{2W}\sum_{k=-\infty}^{+\infty}\left[\frac{\sin(\Omega t-k\pi)}{(\Omega t-k\pi)}\right]^2 = \int_{-\infty}^{+\infty}\left[\frac{\sin\Omega(t-\tau)}{\Omega(t-\tau)}\right]^2 d\tau = \frac{1}{2W} \qquad \text{(I-2-25)}$$

and

$$\exp\left(-\frac{1}{2}\sum_{k=-\infty}^{+\infty}\omega_k^2\right) = \exp\left(-\frac{\omega^2\sigma_x^2}{2}\right) \qquad \text{(I-2-26)}$$

and finally

$$\Phi_y(\omega;t) = \exp\left(-\frac{\omega^2\sigma_x^2}{2}\right)\text{Real}\prod_{k=-\infty}^{+\infty}\left[1 + j\sqrt{\frac{2}{\pi}}(-1)^k\omega_k\, {}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{\omega_k^2}{2}\right)\right]$$

(I-2-27)

The second-order characteristic function follows immediately from the above result:

$$\Phi_y(\omega_1, \omega_2; t_1, t_2) = E\left\{ e^{j\left[\omega_1 y(t_1) + \omega_2 y(t_2)\right]} \right\}$$

$$= E\left\{ \exp\left[jZ \sum_{k=-\infty}^{+\infty} (-1)^k |x_k| \left[\omega_1 \psi_k(t_1) + \omega_2 \psi_k(t_2)\right] \right] \right\} \tag{I-2-28}$$

i.e., the same result as (I-2-8) except the replacement of $\omega\psi_k(t)$ by $\omega_1\psi_k(t_1) + \omega_2\psi_k(t_2)$. Therefore the second order characteristic function is given by (I-2-21) where

$$\omega_k = \sigma_x\left[\omega_1\psi_k(t_1) + \omega_2\psi_k(t_2)\right] \tag{I-2-29}$$

Now the first factor becomes

$$\exp\left(-\frac{1}{2}\sum_{k=-\infty}^{+\infty}\omega_k^2\right) = \exp\left\{-\frac{\sigma_x^2}{2}\left[\omega_1^2 \sum_{k=-\infty}^{+\infty}\psi_k^2(t_1)\right.\right.$$

$$\left.\left. + 2\omega_1\omega_2 \sum_{k=-\infty}^{+\infty}\psi_k(t_1)\psi_k(t_2) + \omega_2^2 \sum_{k=-\infty}^{+\infty}\psi_k^2(t_2)\right]\right\}$$

$$= \exp\left\{-\frac{\sigma_x^2}{2}\left[\omega_1^2 + 2\,\omega_1\omega_2 r_x(t_2 - t_1) + \omega_2^2\right]\right\} \tag{I-2-30}$$

by the use of (I-2-24)*.

Therefore the characteristic function (first or second order) of the optimum process can be written as the product of the original characteristic function and the factor

$$\text{Real} \prod_{k=-\infty}^{+\infty}\left[1 + j\sqrt{\frac{2}{\pi}}\,(-1)^k \omega_k \, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{\omega_k^2}{2}\right)\right] \tag{I-2-31}$$

---

* where $r_x(\tau) = \frac{\sin\Omega\tau}{\Omega\tau}$ is the normalized correlation function of the process $x(t)$.

where

$$\omega_k = \sigma_x \omega \psi_k(t) \quad \text{or} \quad \sigma_x\left[\omega_1 \psi_k(t_1) + \omega_2 \psi_k(t_2)\right] \tag{I-2-32}$$

respectively, so that this factor appears as the perturbation of the characteristic function of x(t) due to the mapping.

We shall also derive the auto-correlation function of the optimum process.

$$R_y(t_1, t_2) = E\left\{y_1(t_1) y_2(t_2)\right\}$$

$$= \sum_{k,\, l=-\infty}^{+\infty} (-1)^{k+1} E\left\{|x_k x_l|\right\} \psi_k(t_1) \psi_l(t_2)$$

$$= \sigma_x^2 \sum_{k=-\infty}^{+\infty} \psi_k(t_1) \psi_k(t_2) + \frac{2}{\pi} \sigma_x^2 \sum_{\substack{k \neq l \\ k,\, l=-\infty}}^{+\infty} (-1)^{k+1} \psi_k(t_1) \psi_l(t_2)$$

$$= \sigma_x^2 \left[(1 - \frac{2}{\pi}) \sum_{k=-\infty}^{+\infty} \psi_k(t_1) \psi_k(t_2) + \frac{2}{\pi} \sum_{k,\, l=-\infty}^{+\infty} (-1)^{k+1} \psi_k(t_1) \psi_l(t_2)\right]$$

$$= \sigma_x^2 \left[(1 - \frac{2}{\pi}) \frac{\sin \Omega (t_2 - t_1)}{\Omega(t_2 - t_1)} + \frac{2}{\pi} \cos(\Omega t_2) \cos(\Omega t_1)\right] \tag{I-2-33}$$

or with $t_2 - t_1 = \tau$ and $t_1 = t$

$$R_y(\tau; t) = \sigma_x^2 \left[(1 - \frac{2}{\pi}) \frac{\sin(\Omega\tau)}{\Omega\tau} + \frac{1}{\pi} \cos(\Omega \tau) + \frac{1}{\pi} \cos \Omega(2t + \tau)\right] \tag{I-2-34}$$

Particularly the variance is given by

$$\sigma_y^2(t) = \sigma_x^2 \left[(1 - \frac{1}{\pi}) + \frac{1}{\pi} \cos(2\Omega t)\right] \tag{I-2-35}$$

i.e., the variance (and the correlation function) are modulated at twice the highest frequency in the signal x(t). At a sampling time the variance is maximum and equal to the variance of x(t); at the mid-point of a Nyquist interval the variance achieves its minimum $\sigma_x^2 (1 - \frac{2}{\pi})$ which shows that the process y(t) is "pinched" in the center of each Nyquist interval. Taking the Fourier transform of (I-2-34) we get

$$S_y(\omega;t) = \sigma_x^2 \left\{ (1-\frac{2}{\pi}) \frac{\pi}{\Omega} p_\Omega(\omega) + \left[\delta(\omega-\Omega) + \delta(\omega+\Omega) \right] (1+e^{j2\omega t}) \right\}$$

$$= \sigma_x^2 \left[(1-\frac{2}{\pi}) \frac{\pi}{\Omega} p_\Omega(\omega) + \delta(\omega-\Omega)(1+e^{j2\Omega t}) + \delta(\omega+\Omega)(1+e^{-j2\Omega t}) \right] \quad \text{(I-2-36)}$$

i.e., the power spectrum of the optimum process has a component identical to that of x(t) and two modulated delta-function at $\pm\Omega$. It is clear that these components are correlated (for the samples sign to alternate).

These results can be obtained from (I-2-27) and (I-2-28) since

$$\Phi(\omega;t) \to 1 - \frac{\omega^2\sigma^2(t)}{2} \text{ when } \omega \to 0 \quad \text{(I-2-37)}$$

$$\Phi(\omega_1,\omega_2;t_1,t_2) \to 1 - \frac{1}{2}\left[\omega_1^2\sigma^2(t_1) + 2\omega_1\omega_2 R(t_1,t_2) + \omega_2^2\sigma^2(t_2)\right] \text{ when } \omega_1 \text{ and } \omega_2 \to 0 \quad \text{(I-2-38)}$$

for zero-mean processes. Since ${}_1F_1 \to 1$ when the argument tends to zero (I-2-27) becomes for small values of the argument

$$\Phi_y(\omega;t) \approx (1-\frac{\omega^2\sigma_x^2}{2})(1+\frac{1}{\pi}\sum_{k=-\infty}^{+\infty}\omega_k^2)\cos\left[\sqrt{\frac{2}{\pi}}\sum_{k=-\infty}^{+\infty}(-1)^k\omega_k\right]$$

$$\approx (1-\frac{\omega^2\sigma_x^2}{2})(1+\frac{1}{\pi}\omega^2\sigma_x^2)\left[1-\frac{1}{\pi}\omega^2\sigma_x^2\cos^2(\Omega t)\right]$$

$$\approx 1 - \frac{\omega^2}{2}\sigma_x^2\left[(1-\frac{2}{\pi}) + \frac{2}{\pi}\cos^2(\Omega t)\right] \quad \text{(I-2-39)}$$

Therefore

$$\sigma_y^2(t) = \sigma_x^2\left[(1-\frac{1}{\pi}) + \frac{1}{\pi}\cos(2\Omega t)\right] \quad \text{(I-2-35)}$$

which checks with (I-2-35). Similarly

$$\Phi(\omega_1,\omega_2;t_1,t_2) \approx 1 - \frac{\sigma_x^2}{2}\left[(1-\frac{2}{\pi})[\omega_1^2 + 2\omega_1\omega_2 r_x(\tau) + \omega_2^2] + \frac{2}{\pi}[\omega_1\cos(\Omega t_1) + \omega_2\cos(\Omega t_2)]^2\right] \quad \text{(I-2-40)}$$

and by (I-2-35) and (I-2-38) we get

$$R_y(t_1, t_2) = \frac{\sigma_x^2}{2}\left[(1 - \frac{2}{\pi})\, r_x(\tau) + \frac{2}{\pi}\cos(\Omega t_1)\cos(\Omega t_2)\right] \qquad \text{(I-2-33)}$$

which is identical to (I-2-33).

3. Signal-to-mean-squared-error ratio at the output of a system

In chapter II we shall investigate how close a signal x(t) can be recovered after some transformation and passage through an ideal clipper. The system is followed at least by an amplifier with appropriate gain A and more generally by a filter h(t)(Fig. I-1). The final output which is an approximation of x(t) will be denoted by $\hat{x}(t)$. We shall use the following measure of the closeness of x(t) and $\hat{x}(t)$

$$a = \max_{p_i}\ \min_{t_1}\ \max_{t_2} \frac{E\left\{x^2(t_1)\right\}}{E\left\{[x(t_1) - \hat{x}(t_2)]^2\right\}} \qquad \text{(I-3-1)}$$

where the first maximum is taken over the set of parameters p_i describing the system. Maximization over t_2 for a given t_1 provides a means to take account of a possible time shift in the system. There is no restriction on the

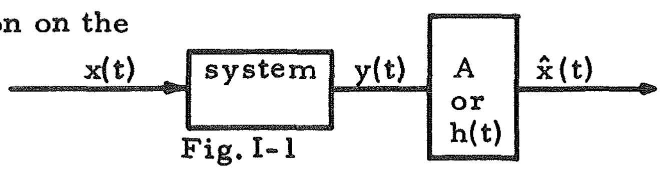

Fig. I-1

system which may be nonlinear and perturbed by noise. Assuming zero-mean processes $E\left\{\hat{x}^2(t_2)\right\} = \sigma_{\hat{x}}^2(t_2)$ and $E\{x^2(t_1)\} = \sigma_x^2(t_1)$, and (I-3-1) may be written

$$a = \max_{p_i}\ \min_{t_1}\ \max_{t_2}\left[1 + \frac{\sigma_{\hat{x}}^2(t_2)}{\sigma_x^2(t_1)} - 2\,\frac{R_{x\hat{x}}(t_1, t_2)}{\sigma_x^2(t_1)}\right]^{-1} \qquad \text{(I-3-2)}$$

We can at least maximize a by an amplifier placed after the system; for this its gain should be

$$A = \frac{R_{xy}(t_1, t_2)}{\sigma_y^2(t_2)} \qquad \text{(I-3-3)}$$

and

$$a = \max_{p_i} \min_{t_1} \max_{t_2} \left[1 - \frac{R_{xy}^2(t_1, t_2)}{\sigma_x^2(t_1)\sigma_y^2(t_2)} \right]^{-1} \tag{I-3-4}$$

When the system introduces no time delay $t_1 = t_2$; if furthermore the processes involved are stationary the quantity $\underline{a}$ is independent of time; then (I-3-4) becomes

$$a = \max_{p_i} \left[1 - \frac{R_{xy}^2(0)}{\sigma_x^2 \sigma_y^2} \right]^{-1} \tag{I-3-5}$$

We can do better than (I-3-4) by replacing the amplifier with appropriate gain by a filter. Because of its mathematical tractability we shall consider the Wiener-Kolmogoroff filter which given the data y(t) (the output of the system) gives the linear-minimum-mean-square-error estimation of the signal x(t). We shall recall the theory of the unrealizable filter[10, Ch. 11]. Linear estimation implies that $\hat{x}(t)$ derive from y(t) by an expression of the type

$$\hat{x}(t_1) = \int_{-\infty}^{+\infty} h(t_1, u)\, y(u)\, du \tag{I-3-6}$$

The mean-square-error $E\left\{ \left[x(t_1) - \hat{x}(t_1) \right]^2 \right\}$ achieves its minimum when the weight $h(t_1, u)$ is such that the error is orthogonal to the data,* e. g.

$$E\left\{ \left[x(t_1) - \hat{x}(t_1) \right] y(\sigma) \right\} = 0 \quad \text{for} \quad -\infty < \sigma < \infty \tag{I-3-7}$$

and it is easily shown that the minimum mean-square-error is given by

$$E\left\{ \left[x(t_1) - \hat{x}(t_1) \right]^2 \right\} = E\left\{ \left[x(t_1) - \hat{x}(t_1) \right] x(t_1) \right\} \tag{I-3-8}$$

Therefore the Wiener-Kolmogoroff filter is the solution of

$$R_{xy}(t_1, \sigma) = \int_{-\infty}^{+\infty} h(t_1, u)\, R_y(u, \sigma)\, du \tag{I-3-9}$$

---

* In the unrealizable case (I-3-6) takes account of a possible time shift in the system since the weight h(t, u) is chosen to minimize the error (in the mean-square sense) and all data (from $-\infty$ to $+\infty$) are used to build the estimate.

and the minimum mean-square-error is given by

$$E\left\{ [x(t_1) - \hat{x}(t_1)]^2 \right\} = R_x(t_1,t_1) - \int_{-\infty}^{+\infty} h(t_1,u)\, R_{xy}(t_1,u)\, du \qquad \text{(I-3-10)}$$

Therefore with the Wiener-Kolmogoroff filter (I-3-1) becomes

$$a = \max_{p_i} \min_{t_1} \left[1 - \frac{\int_{-\infty}^{+\infty} h(t_1,u)t_1,u)\, R_{xy}(t_1,u)du}{R_x\,(t_1,\, t_1)} \right]^{-1} \qquad \text{(I-3-11)}$$

For stationary processes (I-3-11) simplifies in

$$a = \max_{p_i} \left[1 - \frac{\int_{-\infty}^{+\infty} h(t_1-u)R_{xy}(t_1-u)du}{\sigma_x^2} \right]^{-1}$$

$$= \max_{p_i} \left[1 - \frac{\int_{-\infty}^{+\infty} h(u)\, R_{xy}(u)du}{\sigma_x^2} \right]^{-1} \qquad \text{(I-3-12)}$$

while (I-3-9) becomes

$$R_{xy}(t_1 - \sigma) = \int_{-\infty}^{+\infty} h(t_1-u)\, R_y(u-\sigma)\, du$$

or

$$R_{xy}(t) = \int_{-\infty}^{+\infty} h(t-u)\, R_y(u)\, du \qquad \text{(I-3-13)}$$

Finally by the use of the Fourier transform of the quantities involved we can write

$$H(\omega) = \frac{S_{xy}(\omega)}{S_y(\omega)} \qquad \text{(I-3-14)}$$

$$a = \max_{p_i} \left[1 - \frac{\int_{-\infty}^{+\infty} H(\omega)\, S_{xy}^{*}(\omega)\, d\omega}{2\pi\, \sigma_x^2} \right]^{-1}$$

$$= \max_{p_i} \left[1 - \frac{\int_{-\infty}^{+\infty} \frac{|S_{xy}(\omega)|^2}{S_y(\omega)}\, d\omega}{2\pi\, \sigma_x^2} \right]^{-1} \qquad \text{(I-3-15)}$$

In chapter IV we shall simulate a communication system on the computer. In this case we cannot define the quality of the receiver output in terms of expectations. Thus we shall use a definition of the signal-to-mean-square-error ratio using integrals over the time domain:

$$a(T) = \max_{p_i} \min_{t_1} \max_{t_2} \frac{\frac{1}{T}\int_{t_1}^{t_1+T} x^2(t)\, dt}{\frac{1}{T}\int_{t_1}^{t_1+T} [x(t) - \hat{x}(t_2)]^2 dt} \qquad \text{(I-3-16)}$$

(where t_2 is to be understood as a function of t) and

$$a = \lim_{T \to \infty} a(T) \qquad \text{(I-3-17)}$$

For our purpose in that chapter since we shall consider the passage of a stationary process (integrals become independent of t_1 for $T \to \infty$) through a system with zero time delay ($t_2 = t$) and fixed parameters (no max over the p_i's) (I-3-17) becomes

$$a = \lim_{T \to \infty} \frac{\frac{1}{T}\int_0^T x^2(t)\, dt}{\frac{1}{T}\int_0^T [x(t) - \hat{x}(t)]^2\, dt} \qquad \text{(I-3-18)}$$

The optimum amplifier gain is here given by

$$A = \lim_{T \to \infty} \frac{\frac{1}{T}\int_0^T x(t)\, y(t) dt}{\frac{1}{T}\int_0^T y^2(t) dt} \tag{I-3-19}$$

and

$$a^{-1} = 1 - \lim_{T \to \infty} \frac{\left[\frac{1}{T}\int_0^T x(t)y(t)dt \right]^2}{\frac{1}{T}\int_0^T x^2(t)dt \, \frac{1}{T}\int_0^T y^2(t)\, dt} \tag{I-3-20}$$

4. Relation between signal-to-mean-square-error ratio and signal-to-noise ratio.

In section 3 we have considered the mean-square-error as a measure of the system performance. The quantity $a = \dfrac{\text{signal power}}{\text{mean-square-error}}$ has the desirable property that it is monotonic with the channel signal-to-noise ratio a_c. But as it can be seen from (I-3-4) and (I-3-11) when the channel signal-to-noise ratio tends to zero $\underline{a}$ tends to one. Thus for comparison with other communication systems in term of quality of the output vs. quality of the channel the quantity $\underline{a}$ would lead to erroneous conclusions since the output signal-to-noise ratio should tend to zero with the channel signal-to-noise ratio. Lawton[11] has proposed the following theoretical definition for the output signal-to-noise ratio

$$a_o = \frac{r_{xy}^2(0)}{1 - r_{xy}^2(0)} \tag{I-4-1}$$

with

$$r_{xy}(0) = \frac{E\{x(t)y(t)\}}{\sigma_x \sigma_y} \tag{I-4-2}$$

and where $x(t)$ is the signal to be transmitted and $y(t)$ the output of the system. This definition assumes that the input and output processes are zero-mean (otherwise for high channel signal-to-noise ratio a_o would not tend to infinity); it also assumes that the processes are stationary and that no delay occurs in the system (or that the output is properly shifted in time for comparison with the signal). With a_o defined as above we can see that

for $a_c \to 0$ $\quad r_{xy}(o) \to 0 \quad$ and $\quad a_o \to 0$

$a_c \to \infty$ $\quad r_{xy}(o) \to 1 \quad$ and $\quad a_o \to \infty$

as desired. The definition also satisfies the requirement that if the "communication system" merely consists of an addition of zero-mean noise independent of the signal (Fig. I-2) the output signal-to-noise ratio is equal to the channel signal-to-noise ratio indeed

$$a_i = \sigma_x^2 / \sigma_n^2 \tag{I-4-3}$$

$$r_{xy}^2(0) = \frac{\sigma_x^4}{\sigma_x^2(\sigma_x^2 + \sigma_n^2)} = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2} \tag{I-4-4}$$

$$a_o = \frac{r_{xy}^2(0)}{1 - r_{xy}^2(0)} = \frac{\sigma_x^2}{\sigma_n^2} \tag{I-4-5}$$

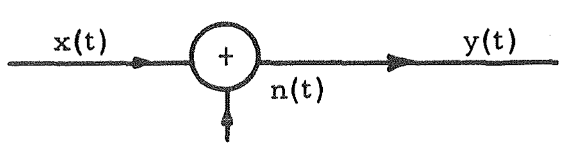

Fig. I-2

Finally the definition leads to a result that is independent of the level of the system output.

From (I-3-5) we find that a_o and a are related by the simple relationship

$$a_o = a - 1 \tag{I-4-6}$$

Therefore for nonstationary processes and communication systems with time delay followed by the optimum amplifier (I-3-4) leads to

$$a_o^{-1} = \min_{p_i} \max_{t_1} \min_{\tau} \frac{\sigma_x^2(t_1)\,\sigma_y^2(t_1+\tau)}{R_{xy}^2(t_1, t_1+\tau)} - 1 \tag{I-4-7}$$

For nonstationary processes and communication systems with or without time delay followed by the unrealizable Wiener-Kolmogoroff filter (I-3-11) leads to

$$a_o^{-1} = \min_{p_i} \max_{t_1} \frac{R_x(t_1, t_1)}{\int_{-\infty}^{+\infty} h(t_1, u)\, R_{xy}(t_1, u)\, du} - 1 \tag{I-4-8}$$

and finally if the processes are stationary the previous formula becomes

$$a_o^{-1} = \min_{p_i} \frac{2\pi\sigma_x^2}{\int_{-\infty}^{+\infty} \frac{|S_{xy}(\omega)|^2}{S_y(\omega)}\, d\omega} - 1 \tag{I-4-9}$$

5. Comparison of the performance of the communication system with and without modulation.

We intend to show here that the system may be investigated without taking modulation into account. Indeed we shall send a bandlimited signal by the clipped version of the associated optimum process which means a bandwidth expansion before modulation. This expansion will give the noise immunity and therefore no further bandwidth increase (by means of modulation) will be necessary. Thus double-sideband suppressed carrier (DSB-SC), and single-sideband (SSB) are suitable types of modulation for our problem.

We shall assume the noise gaussian, wide-sense stationary, and zero-mean. Then the noise outside $(\omega_c - \Omega, \omega_c + \Omega)$ or, for instance, $(\omega_c, \omega_c + \Omega)$ (where ω_c is the carrier angular frequency) is independent of the noise in that band and therefore cannot help in estimating y(t), the signal sent. The signal to be demodulated now can be written respectively as

$$y(t)\sqrt{2}\ \cos(\omega_c t) + n(t) \tag{I-5-1}$$

$$y(t)\sqrt{2}\ \cos(\omega_c t) - \check{y}(t)\sqrt{2}\ \sin(\omega_c t) + n(t)$$

where $y(t)$ is the clipped version of the optimum process filtered by the channel, $\check{y}(t)$ is the Hilbert transform of $y(t)$, and $n(t)$ can be written as

$$n(t) = \sqrt{2}\ n_c(t)\cos(\omega_c t) - n_s(t)\sqrt{2}\sin(\omega_c t) \tag{I-5-2}$$

(the noise $n(t)$ is obviously not the same in the two cases: in SSB its bandwidth and power are reduced by a factor 2).

In DSB-SC and SSB if the phase of the carrier is known we get, multiplying by $\sqrt{2}\cos(\omega_c t)$ and filtering,

$$y(t) + n_c(t) \tag{I-5-3}$$

In DSB-SC the channel signal-to-noise ratio is given by

$$a_c = \frac{\sigma_y^2}{\sigma_{n_c}^2 + \sigma_{n_s}^2} \tag{I-5-4}$$

and the signal-to-noise ratio after demodulation is

$$a_o = \frac{\sigma_y^2}{\sigma_{n_s}^2} \tag{I-5-5}$$

therefore we get a 3 dB improvement. In SSB

$$a_i = \frac{\sigma_y^2 + \sigma_{\check{y}}^2}{\sigma_{n_c}^2 + \sigma_{n_s}^2} \tag{I-5-6}$$

$$a_o = \frac{\sigma_y^2}{\sigma_{n_c}^2} \tag{I-5-7}$$

and therefore $a_c = a_o$. We note that any phase difference between the carrier and the local oscillator entails a loss of performance.

Therefore with the types of modulation suitable for our problem the performances are easily derived from the performances under the assumption that transmission takes place in the baseband.

CHAPTER II

ESTIMATION OF A STATIONARY, BANDLIMITED, GAUSSIAN, ZERO-MEAN SIGNAL FROM A STREAM OF RELATED ZERO-CROSSINGS

In this chapter, as well as in the following ones, we assume that the input x(t) of the communication system is a stationary, bandlimited, Gaussian, zero-mean process, with a flat power spectrum in $(-\Omega, +\Omega)$. Such a process has a zero-crossing rate $\lambda = \frac{2W}{\sqrt{3}}$ (12-p.193). A band-limited process is defined (in the mean-square sense) from its samples taken at the rate 2W per second; just by intuition we might expect that the zero-crossings do not carry enough information to define the signal completely; in section 1 we prove that this is indeed true.

The aim now would be to find the mapping such that the information carried by the new set of zero-crossings is maximum. This approach cannot be used because of our lack of knowledge on the distribution of the zero-crossings interval. We use a less ambitious approach, namely we shall increase the zero crossing rate (one of the factors which determine the amount of information carried by the zero-crossing stream) and then check whether the new process is better defined by its zero-crossings than the original signal itself.

We first consider multiple differentiation of the signal, and addition of a sine wave with the highest frequency Ω (Sections 2 and 3). In these two sections the signal x(t) is estimated by the optimum unrealizable (Wiener-Kolmogoroff) filter from the zero-crossings of its mapping. In both cases we get a decrease of the output quality with the increase of the zero-crossing rate.

Section 4 is an attempt to estimate the signal x(t) from the zero-crossings of its associated optimum signal by means of optimum linear filtering. Here we run into analytical difficulties.

In section 5 we state two theorems due to Titchmarsch and Polyá. These theorems, valid when the signal has a Fourier transform vanishing outside some interval, give the maximum zero-crossing rate to be expected and the conditions under which the signal (then called optimum) is completely defined by its zero-crossings. We extend these theorems to bandlimited stochastic processes, and we give an algorithm for the estimation of the optimum process from its zero-crossings.

1. Some preliminary results

Let us first consider the simplest system we can think about: no manipulation of the signal and clipping followed by an amplifier to minimize the mean-square-error (Fig. II-1).
By (I-3-5)

$$a = \left[1 - r^2_{y_1 y_2}(0) \right]^{-1} \qquad \text{(II-1-1)}$$

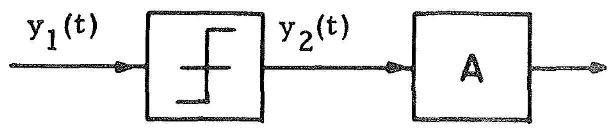

Fig. II-1 Signal Clipping

and by (I-1-7)

$$R_{y_1 y_2}(0) = \frac{\sigma^2_{y_1}}{\pi} \int_{-\infty}^{+\infty} \exp\left(- \frac{\sigma^2_{y_1} \omega^2}{2}\right) d\omega = \sqrt{\frac{2}{\pi}}\, \sigma_{y_1} \qquad \text{(II-1-2)}$$

where $y_1(t)$ and $y_2(t)$ are the clipper input and output. Thus

$$a = \left(1 - \frac{2}{\pi}\right)^{-1} = 2.76 \qquad \text{(II-1-3)}$$

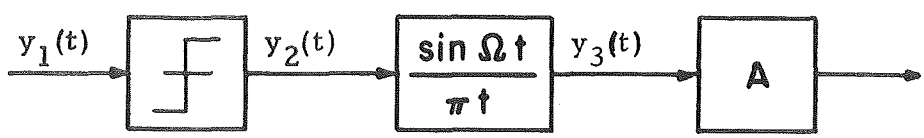

Fig. II-2 Signal Clipping and Ideal Filtering

A first improvement is obtained by an ideal low-pass filter(with the same bandwidth as the signal itself) between the clipper and the amplifier (Fig. II-2). The system introduces no time delay; therefore we apply (I-3-5). We denote by $y_3(t)$ the output of the ideal filter. From (I-1-5) we get

$$R_{y_1 y_2}(\tau) = \frac{R_{y_1}(\tau)}{\pi} \int_{-\infty}^{+\infty} \exp\left(-\frac{\sigma_{y_1}^2 \omega_2^2}{2}\right) d\omega = \frac{R_{y_1}(\tau)}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}} \qquad \text{(II-1-4)}$$

Extension of this result to a broad class of processes and nonlinear transformation is given in appendix B. Now

$$S_{y_1 y_3}(\omega) = \frac{S_{y_1}(\omega)}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}}\; p_{\Omega}^{*}(\omega) = \frac{S_{y_1}(\omega)}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}} \qquad \text{(II-1-5)}$$

$$R_{y_1 y_3}(0) = \frac{1}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}}\; \frac{1}{2\pi} \int_{-\Omega}^{\Omega} S_{y_1}(\omega)\, d\omega = \sigma_{y_1} \sqrt{\frac{2}{\pi}} \qquad \text{(II-1-6)}$$

We also have

$$\sigma_{y_3}^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{y_2}(\omega)\, p_{\Omega}(\omega)\, d\omega = \int_{-\infty}^{+\infty} R_{y_2}(\tau) \frac{\sin(\Omega\tau)}{\pi\tau}\, d\tau \qquad \text{(II-1-7)}$$

which for a Gaussian zero-mean signal becomes

$$\sigma_{y_3}^2 = \frac{2}{\pi^2} \int_{-\infty}^{+\infty} \sin^{-1}\left[r_{y_1}(\tau)\right] \frac{\sin(\Omega\tau)}{\tau}\, d\tau \qquad \text{(II-1-8)}$$

Thus by (I-3-5)

$$a^{-1} = 1 - \pi \left[\int_{-\infty}^{+\infty} \sin^{-1}\left[r_{y_1}(\tau)\right] \frac{\sin(\Omega\tau)}{\tau}\, d\tau \right]^{-1} \qquad \text{(II-1-9)}$$

Finally for a signal with flat power spectrum, e.g.

$$r_{y_1}(\tau) = \frac{\sin(\Omega\tau)}{\Omega\tau} \qquad \text{(II-1-10)}$$

and with the dimensionless variable $t = \Omega\tau$

$$a^{-1} = 1 - \pi \left[\int_{-\infty}^{+\infty} \sin^{-1}\left[\frac{\sin(t)}{t}\right] \frac{\sin(t)}{t}\, dt \right]^{-1} \qquad \text{(II-1-11)}$$

The value of the above integral is 3.89[13]. Therefore

$$a = \left[1 - \frac{\pi}{3.89}\right]^{-1} = 5.21 \qquad \text{(II-1-12)}$$

A last improvement (if we require linear filtering and directly apply the signal to the clipper) is obtained by a Wiener-Kolmogoroff filter. We already know that

$$S_{y_1y_2}(\omega) = \frac{S_{y_1}(\omega)}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}} \qquad \text{(II-1-13)}$$

Applying (I-3-15)

$$a^{-1} = 1 - \frac{1}{\pi^2 \sigma^4_{y_1}} \int_{-\Omega}^{\Omega} \frac{S^2_{y_1}(\omega)}{S_{y_2}(\omega)} \, d\omega \qquad \text{(II-1-14)}$$

and since

$$S_{y_2}(\omega) = \frac{2}{\pi} \int_{-\infty}^{+\infty} \sin^{-1} [\, r_{y_1}(\tau)\,] \cos(\omega\tau)\, d\tau \qquad \text{(II-1-15)}$$

$$a^{-1} = 1 - \frac{\int_{-\Omega}^{+\Omega} S^2_{y_1}(\omega) \left[\int_{-\infty}^{+\infty} \sin^{-1} [\, r_{y_1}(\tau)\,] \cos(\omega\tau)\, d\tau \right]^{-1} d\omega}{2\pi\sigma^4_{y_1}} \qquad \text{(II-1-16)}$$

Again for a signal with flat power spectrum

$$S_{y_1}(\omega) = \frac{\pi}{\Omega} \sigma^2_{y_1} \, p_\Omega(\omega) \qquad \text{(II-1-17)}$$

and with the dimensionless variables $t = \Omega\tau$ and $u = \omega/\Omega$

$$a^{-1} = 1 - \frac{\pi}{2} \int_0^1 \left[\int_0^\infty \sin^{-1} \left[\frac{\sin(t)}{t} \right] \cos(ut)\, dt \right]^{-1} du \qquad \text{(II-1-18)}$$

Since [13]

$$1.886 \leq \int_0^\infty \sin^{-1} \left(\frac{\sin t}{t} \right) \cos(u\, t) dt \leq 1.982 \qquad \text{for } |u| < 1 \qquad \text{(II-1-19)}$$

We can write

$$4.85 < a < 6.02 \qquad \text{(II-1-20)}$$

Actually we are doing better than with the ideal low-pass filter and we can replace (II-1-20) by

$$5.21 < a < 6.02 \qquad \text{(II-1-21)}$$

Thus in this case it does not pay to replace the ideal filter by the optimum filter (this is expected since (II-1-19) shows that the spectrum of the clipped signal is almost flat in $(-\Omega, +\Omega)$: the main function of the filter is to remove the frequencies generated outside this interval).

2. Signal differentiation, clipping, and Wiener filtering.

Differentiation of the signal increases the zero-crossing rate at the clipper input, and therefore one of the factors which determine the amount of the information available after clipping to recover the signal. Thus we consider the following scheme (Fig. II-3)

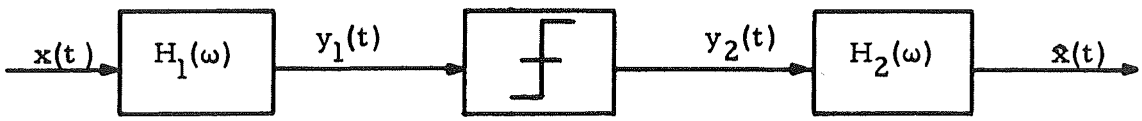

Fig. II-3 Differentiation, Clipping and Wiener Filtering

where $H_1(\omega) = (j\omega)^n$ (n differentiators) and $H_2(\omega)$ is the Wiener filter

$$H_2(\omega) = \frac{S_{xy_2}(\omega)}{S_{y_2}(\omega)} \qquad \text{(II-2-1)}$$

Since $y_1(t)$ is still Gaussian we have

$$R_{y_2}(\tau) = \frac{2}{\pi} \sin^{-1} r_{y_1}(\tau) \longleftrightarrow S_{y_2}(\omega) \qquad \text{(II-2-2)}$$

From (II-1-4)

$$R_{y_1 y_2}(\tau) = \frac{R_{y_1}(\tau)}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}} \qquad \text{(II-2-3)}$$

Thus

$$S_{xy_2}(\omega) = \frac{S_{y_1}(\omega)}{H_1(\omega)} \frac{1}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}} = S_x(\omega) H_1^*(\omega) \frac{1}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}} \qquad \text{(II-2-4)}$$

and

$$H_2(\omega) = \frac{S_x(\omega)H_1^*(\omega)}{S_{y_2}(\omega)} \frac{1}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}} \tag{II-2-5}$$

The structure of $H_2(\omega)$ is better understood when we write

$$H_2(\omega) = \frac{1}{\sigma_{y_1}} \sqrt{\frac{2}{\pi}} \frac{S_{y_1}(\omega)}{S_{y_2}(\omega)} H_1^{-1}(\omega) \tag{II-2-6}$$

which is a filter with transfer function proportional to $S_{y_1}(\omega)/S_{y_2}(\omega)$ followed by the inverse of the preemphasis filter. Now by (I-3-15)

$$a^{-1} = 1 - \frac{1}{2\pi\sigma_x^2} \int_{-\infty}^{+\infty} \frac{|S_{xy_2}(\omega)|^2}{S_{y_2}(\omega)} d\omega$$

$$= 1 - \frac{1}{\pi^2 \sigma_x^2 \sigma_{y_1}^2} \int_{-\infty}^{+\infty} \frac{S_x^2(\omega)\,|H_1(\omega)|^2}{S_{y_2}(\omega)} d\omega \tag{II-2-7}$$

Up to here we did not specify $H_1(\omega)$ and $S_x(\omega)$. Now we specialize to

$$H_1(\omega) = (j\omega)^n \tag{II-2-8}$$

and

$$S_x(\omega) = \frac{\pi}{\Omega} \sigma_x^2 p_\Omega(\omega) \tag{II-2-9}$$

(Gaussian signal with flat power spectrum in $(-\Omega, +\Omega)$ and n differentiations to increase the zero-crossing rate). For this case (II-2-7) becomes

$$a^{-1} = 1 - \frac{2}{\Omega} \frac{\int_0^\Omega \omega^{2n} S_{y_2}^{-1}(\omega)\, d\omega}{\int_0^\Omega \omega^{2n}\, d\omega} \tag{II-2-10}$$

where

$$S_{y_2}(\omega) \longleftrightarrow \frac{2}{\pi} \sin^{-1} r_{y_1}(\tau) \tag{II-2-11}$$

$$R_{y_1}(\tau) = \frac{\sigma_x^2}{\Omega} \int_0^{\Omega} \omega^{2n} \cos(\omega\tau)\, d\omega \qquad \text{(II-2-12)}$$

Therefore

$$a^{-1} = 1 - \frac{2(2n+1)}{\Omega^{2n+2}} \int_0^{\Omega} \omega^{2n} \left[\frac{2}{\pi} \int_{-\infty}^{+\infty} \sin^{-1} \left(\frac{\int_0^{\Omega} \omega^{2n} \cos(\omega\tau)\, d\omega}{\Omega^{2n+1}} (2n+1) \right) \cos(\omega\tau)\, d\tau \right]^{-1} d\omega \qquad \text{(II-2-11)}$$

or, with the dimensionless quantities $u = \omega/\Omega$ (in the first integral), $v = \omega/\Omega$ (in the second), and $t = \Omega\tau$

$$a^{-1} = 1 - \frac{(2n+1)\pi}{2} \int_0^{1} u^{2n} \left[\int_0^{\infty} \cos(ut) \sin^{-1}[r_{y_1}(t;n)]\, dt \right]^{-1} du \qquad \text{(II-2-12)}$$

where

$$r_{y_1}(t;n) = (2n+1) \int_0^{1} v^{2n} \cos(vt)\, dv \qquad \text{(II-2-13)}$$

When we let $n = 0$ in (II-2-12) (no differentiation of the signal before clipping) we get

$$a^{-1} = 1 - \frac{\pi}{2} \int_0^{1} \left[\int_0^{\infty} \cos(ut) \sin^{-1}\left(\frac{\sin(t)}{t}\right) dt \right]^{-1} du \qquad \text{(II-2-14)}$$

a result which checks with (II-1-18)

The normalized correlation function $r_{y_1}(t)$ of the n-th derivative of the process $x(t)$ can be computed by the following recurrence formula

$$r_{y_1}(t;n) = (2n+1) \left[\frac{\sin(t)}{t} + \frac{2n}{t^2} \left[\cos(t) - r_{y_1}(t;n-1) \right] \right] \qquad \text{(II-2-15)}$$

which we get after two integration by parts.

Finally since we can write (II-2-12) as

$$a^{-1} = 1 - \frac{(2n+1)\pi}{2} \int_0^1 u^{2n} \left[\int_0^\infty \cos(ut) \left\{ \sin^{-1} \left[r_{y_1}(t;n) \right] - r_{y_1}(t;n) + r_{y_1}(t;n) \right\} dt \right]^{-1} du \qquad \text{(II-2-16)}$$

we get

$$a^{-1} = 1 - \int_0^1 u^{2n} \left[u^{2n} + \frac{2}{(2n+1)\pi} \int_0^\infty \cos(ut) \left\{ \sin^{-1} \left[r_{y_1}(t;n) \right] - r_{y_1}(t;n) \right\} dt \right]^{-1} du \qquad \text{(II-2-17)}$$

which is more convenient for computation since $\sin^{-1}[r_{y_1}(t,n)] - r_{y_1}(t,n)$ tends more rapidly to zero than $\sin^{-1}[r_{y_1}(t,n)]$ itself.

The zero-crossing rate at the clipper input is easily found since $y_1(t)$ is Gaussian, and

$$R_{y_1}(\tau) = \frac{\sigma_x^2}{\Omega} \int_0^\Omega \omega^{2n} \cos(\omega\tau)\, d\omega \qquad \text{(II-2-18)}$$

shows that the condition $R'_{y_1}(0) = 0$ is satisfied. For this case the zero-crossing rate is given by(10-Section 14-4)

$$\lambda = \frac{1}{\pi}\sqrt{-r''_{y_1}(0)} = \frac{1}{\pi}\sqrt{\frac{\int_0^\Omega \omega^{2n+2}\, d\omega}{\int_0^\Omega \omega^{2n}\, d\omega}} = \frac{\Omega}{\pi}\sqrt{\frac{2n+1}{2n+3}} \qquad \text{(II-2-19)}$$

The following table gives the normalized zero-crossing rate

$$\lambda_n = \frac{\lambda}{2W} = \sqrt{\frac{2n+1}{2n+3}} \qquad \text{(II-2-20)}$$

n	0	1	2	3	4	5
λ_n	.578	.775	.845	.883	.904	.920

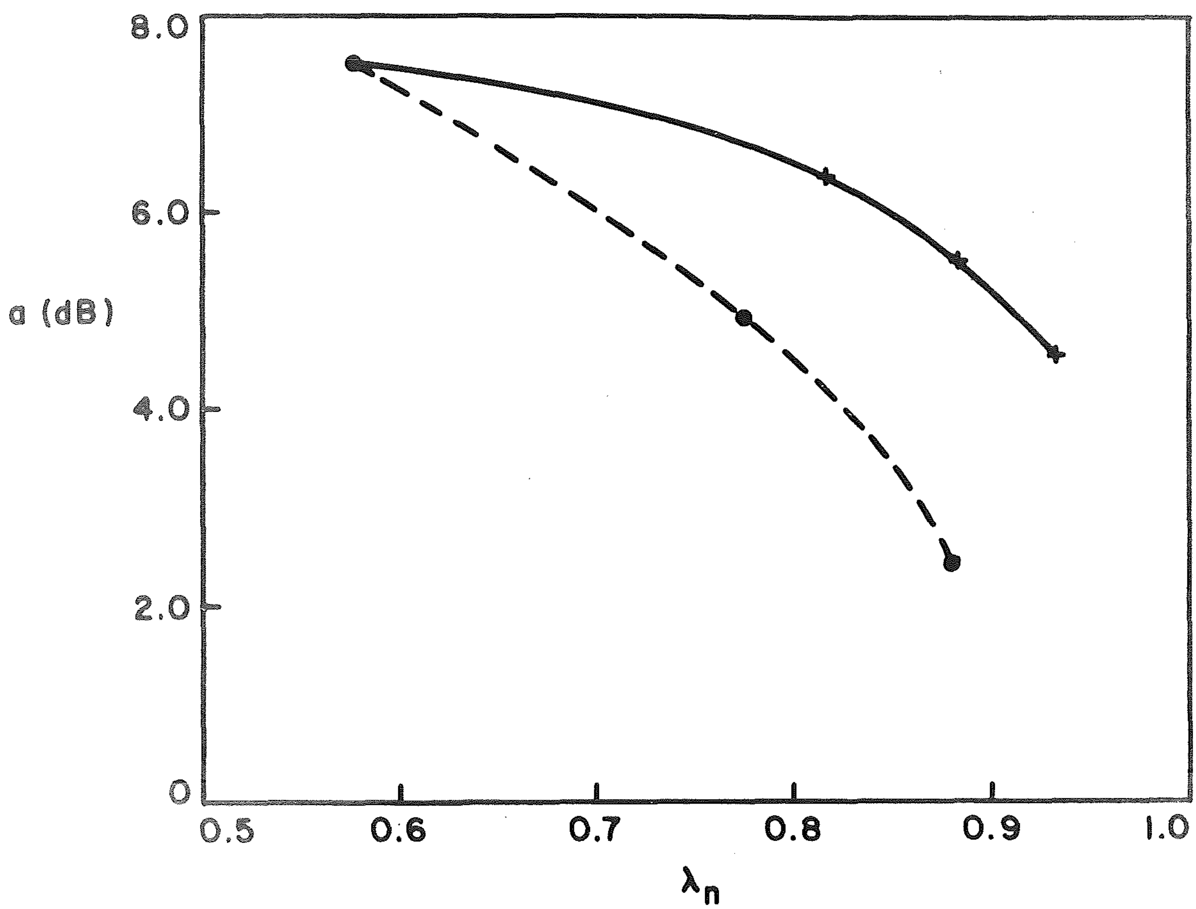

Fig. II-4

Signal-to-mean-square-error ratio for the estimation of the signal from a set of related zero-crossings vs. the normalized zero-crossing rate $\lambda_n = \lambda / 2W$ (o : differentiation of the signal; + : addition of a Gaussian, narrow-band process with center frequency W , and bandwidth $\frac{\omega_o}{\pi} \rightarrow 0$).

(II-2-17) has been evaluated on a computer. The results are plotted (dots) on Fig. II-4 vs. the normalized zero-crossing rate rather than the number of differentiations. The dashed curve has been drawn merely for convenience. As long as we consider linear estimation differentiation, although it increases the zero-crossing rate, does not decrease the mean-squared-error. However differentiation also decreases the variance of the zero-crossings interval and therefore the result is not completely surprising.

3. Addition of a narrow band Gaussian process, clipping, and Wiener filtering.

Another way to increase the zero-crossing rate at the clipper input is the addition of the sine wave $s(t) = A \sin \Omega t$ to the signal $x(t)$ (Fig. II-5). It is obvious that this technique as well as differentiation leads to a ZC rate at the clipper input as close to 2W as we wish. Here however the theoretical investigation would not be so straightforward because we lose the Gaussian character of the clipper input. Thus we shall consider the addition of a narrow-band Gaussian process centered about the frequency W.

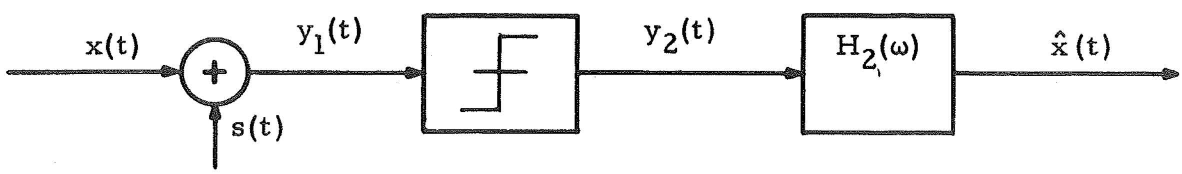

Fig. II-5 Addition of $A(t) \sin(\Omega t + \theta)$, Clipping, and Wiener Filtering

Now the clipper input

$$y_1(t) = x(t) + A(t) \sin(\Omega t + \theta) \qquad \text{(II-3-1)}$$

where θ has uniform distribution in $(0, 2\pi)$ and $A(t)$ has Rayleigh distribution with parameter σ_s^2. $x(t) + s(t)$ is Gaussian with mean zero, variance $\sigma_x^2 + \sigma_s^2$, and autocorrelation function $R_{y_1}(\tau) = R_x(\tau) + R_s(\tau)$ since $x(t)$ and $s(t)$ are independent. Therefore

$$R_{y_2}(\tau) = \frac{2}{\pi} \sin^{-1}[r_{y_1}(\tau)] \longleftrightarrow S_{y_2}(\omega) \qquad \text{(II-3-2)}$$

with

$$r_{y_1}(\tau) = \frac{R_x(\tau) + R_s(\tau)}{\sigma_x^2 + \sigma_s^2} \qquad \text{(II-3-3)}$$

On the other hand

$$R_{xy_2}(\tau) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \omega_2^{-1} \left[\frac{\partial \Phi_x}{\partial \omega_1}\right]_{\omega_1=0} e^{-\frac{\sigma_s^2 \omega_2^2}{2}} \, d\,\omega_2$$

$$= \frac{R_x(\tau)}{\pi} \int_{-\infty}^{+\infty} \exp\left[-\frac{(\sigma_x^2+\sigma_s^2)\,\omega_2^2}{2}\right] d\,\omega_2$$

$$= \frac{R_x(\tau)}{\sqrt{\sigma_x^2+\sigma_s^2}} \sqrt{\frac{2}{\pi}} \qquad \text{(II-3-4)}$$

Thus the closeness between the output $\hat{x}(t)$ of the Wiener filter and the signal $x(t)$ is given by (I-3-15), i.e.,

$$a^{-1} = 1 - \frac{1}{2\pi\sigma_x^2} \int_{-\infty}^{+\infty} \frac{|S_{xy_2}(\omega)|^2}{S_{y_2}(\omega)} \, d\,\omega \qquad \text{(II-3-5)}$$

where

$$S_{xy_2}(\omega) = \frac{S_x(\omega)}{\sqrt{\sigma_x^2 + \sigma_s^2}} \sqrt{\frac{2}{\pi}} \qquad \text{(II-3-6)}$$

Therefore, with the dimensionless parameter $\gamma = \sigma_s^2/\sigma_x^2$

$$a^{-1} = 1 - \frac{1}{\pi^2 \sigma_x^4 (1+\gamma)} \int_{-\infty}^{+\infty} S_x^2(\omega)\, S_{y_2}^{-1}(\omega)\, d\omega \qquad \text{(II-3-7)}$$

We now specialize to the following case: $S_x(\omega)$ flat in $|\omega| \leq \Omega$, i.e.,

$$S_x(\omega) = \sigma_x^2 \frac{\pi}{\Omega} p_\Omega(\omega) \qquad \text{(II-3-8)}$$

and $S_s(\omega)$ flat in $\Omega - \omega_o \leq |\omega| \leq \Omega + \omega_o$, i.e.,

$$R_s(\tau) = \sigma_s^2 \frac{\sin(\omega_o \tau)}{\omega_o \tau} \cos \Omega\tau \tag{II-3-9}$$

$$S_s(\omega) = \gamma\, \sigma_x^2 \frac{\pi}{2\omega_o} p_{\Omega, \omega_o}(\omega) \quad . \tag{II-3-10}$$

where we use the notation

$$p_{\Omega, \omega_o}(\omega) = \begin{array}{ll} 1 & \text{if } \Omega - \omega_o \leq |\omega| \leq \Omega + \omega_o \\ 0 & \text{otherwise} \end{array} \tag{II-3-11}$$

For this case

$$a^{-1} = 1 - \frac{2}{\Omega^2(1+\gamma)} \int_0^{\Omega} S_{y_2}^{-1}(\omega)\, d\omega \tag{II-3-12}$$

where

$$S_{y_2}(\omega) \leftrightarrow \frac{2}{\pi} \sin^{-1}\left[\frac{\frac{\sin \Omega\tau}{\Omega\tau} + \gamma \frac{\sin\omega_o\tau}{\omega_o \tau} \cos \Omega\tau}{1+\gamma} \right] \tag{II-3-13}$$

When $\omega_o = 0$, $s(t) = A \sin(\Omega t + \theta)$ where θ is a random variable with uniform distribution in $(0, 2\pi)$ and A a random variable with Rayleigh distribution. In this case

$$S_{y_2}(\omega) \leftrightarrow \frac{2}{\pi} \sin^{-1}\left[\frac{\frac{\sin \Omega\tau}{\Omega\tau} + \gamma \cos \Omega\tau}{1+\gamma} \right] \tag{II-3-14}$$

A periodic component appears in the clipper output. Since this component is irrelevant to the information we seek from $y_2(t)$ it can be eliminated by an appropriate filter. We shall denote the output of this filter by $y_3(t)$. Now a is given by (II-3-5) with y_2 replaced by y_3. Obviously

$$S_{xy_3}(\omega) = S_{xy_2}(\omega) \tag{II-3-15}$$

(Eq. (II-3-4) shows that $S_{xy_2}(\omega)$ is free of δ-functions). To find $S_{y_3}(\omega)$ we shall need the following theorem:

Theorem. -If a nonlinear device is such that the correlation function of its output $y(t)$ can be written as some function $f(.)$ of the correlation function of its input $x(t)$, then for an input sum of a periodic process (whose correlation $R_p(\tau)$ has period $T = \frac{2\pi}{\Omega}$) and a process with continuous power spectrum (correlation $R_c(\tau)$) independent of the former, i.e.,

$$R_x(\tau) = R_p(\tau) + R_c(\tau) \tag{II-3-16}$$

the correlation function of the output of an ideal filter band-stop at the multiples of the fundamental frequency of the periodic process is given by

$$f[R_p(\tau) + R_c(\tau)] - f[R_p(\tau)] \tag{II-3-17}$$

<u>Proof.</u> We expand the function $f(.)$ in a Taylor series:

$$f[R_p(\tau) + R_c(\tau)] = f[R_p(\tau)] + R_c(\tau) f'[R_p(\tau)] + \dots \tag{II-3-18}$$

The Fourier transform of the first term leads to a succession of impulses in the frequency domain at $k\Omega$, ($k = 0, \pm 1, \pm 2 \dots$). Therefore if we remove these impulses by a band-stop filter we are left with the autocorrelation function (II-3-17). Applying this theorem to (II-3-14) we get

$$S_{y_3}(\omega) \leftrightarrow R_{y_3}(\tau) = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{\frac{\sin\Omega\tau}{\Omega\tau} + \gamma\cos\Omega\tau}{1+\gamma}\right) - \sin^{-1}\left(\frac{\gamma\cos\Omega\tau}{1+\gamma}\right)\right] \tag{II-3-19}$$

Therefore with the dimensionless quantities

$$a = \omega_o/\Omega, \qquad t = \Omega\tau, \qquad u = \omega/\Omega \tag{II-3-20}$$

(II-3-12) becomes

$$a^{-1} = 1 - \frac{\pi}{2(1+\gamma)}\int_0^1 \left[\int_0^\infty \cos(ut)\sin^{-1}\left(\frac{\frac{\sin t}{t} + \gamma\frac{\sin(at)}{at}}{1+\gamma}\right)dt\right]^{-1} du \tag{II-3-21}$$

and for the case $\omega_o = 0$

$$a^{-1} = 1 - \frac{\pi}{2(1+\gamma)} \int_0^1 \left\{ \int_0^\infty \cos(ut) \left[\sin^{-1}\left(\frac{\frac{\sin t}{t} + \gamma \cos t}{1+\gamma} \right) - \sin^{-1}\left(\frac{\gamma \cos t}{1+\gamma} \right) \right] dt \right\}^{-1} du \tag{II-3-22}$$

The process $y_1(t)$ is Gaussian, and $R'_{y_1}(0) = 0$ since $R'_x(0)$ and $R'_s(0)$ both vanish. Therefore the zero-crossing rate is given by

$$\lambda = \frac{1}{\pi} \sqrt{-r''_{y_1}(0)} \tag{II-3-23}$$

with $-R''_{y_1}(0) = \frac{1}{\pi} \int_0^\infty \omega^2 S_{y_1}(\omega)\, d\omega$

$$= \frac{1}{\pi} \left[\int_0^\Omega \omega^2 S_x(\omega)\, d\omega + \int_{\Omega - \omega_o}^{\Omega + \omega_o} \omega^2 S_s(\omega)\, d\omega \right]$$

$$= \frac{\sigma_x^2}{3\Omega} \left\{ \Omega^3 + \frac{\gamma}{2a} \left[(\Omega + \omega_o)^3 - (\Omega - \omega_o)^3 \right] \right\}$$

$$= \frac{\sigma_x^2}{3} \left[\Omega^3 + \frac{\gamma \omega_o}{a} (3\Omega^2 + \omega_o^2) \right] \tag{II-3-24}$$

Finally

$$\lambda_n = \frac{\lambda}{2W} = \sqrt{\frac{1+\gamma(3+a^2)}{3(1+\gamma)}} \tag{II-3-25}$$

As a check we find at the limits the expected results:

$\gamma \to 0$ (no narrow-band process added): $\lambda_n \to 1/\sqrt{3}$
$\gamma \to \infty$ (narrow-band process alone) and $a = 0$: $\lambda_n \to 1$

The following is a table of the normalized zero-crossing rate as a function of the power of the narrow band process for $a = 0$.

γ (dB)	0	3	6	9	12
λ_n	.817	.882	.931	.963	.982

The numerical results corresponding to (II-3-22) (i.e., addition of a sine process) are the + on Fig. II-4. We encounter the same problem as in Section 2: in spite of a zero-crossing rate increase there is no improvement of the estimate of the signal. But the drawback of the technique is also the same: the variance of the zero-crossings interval decreases as the power of the sine wave increases.

Finally we underline that in both sections we have considered the linear optimum filter. Nonlinear filtering would obviously give better performances. However, in the light of the previous results the nonlinear approach does not seem promising since at a zero-crossing rate close to 2W the quality of the linear estimate is very poor.

4. Mapping of the Signal into the Optimum Process, Clipping, and Wiener Filtering

The optimum process associated to a bandlimited signal has a zero-crossing rate equal to (at least) 2W. In the light of the sampling theorem the transformation into this process might very well be the mapping we are looking for (by the two previous techniques we can only approach the rate 2W). Thus we consider the linear estimation of $x(t)$ from the zero-crossings of its optimum process. The results obtained in I-2 for the characteristic function of an optimum process however indicates that this approach leads to serious analytical difficulties which will be outlined here.

We first show that the two problems of optimum linear filtering of $y_2(t)$ to recover $x(t)$ or $y_1(t)$ have the same solution (Fig. II-6).

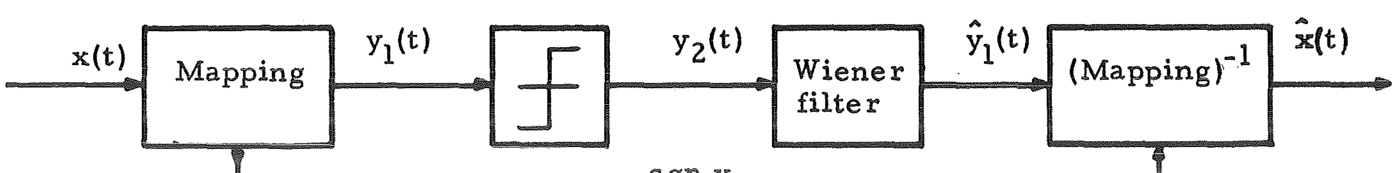

Fig. II-6

We have seen in I-2 that the optimum process can be written as

$$y_1(t) = Z \sum_{k=-\infty}^{+\infty} (-1)^k \, |x_k| \, \psi_k(t) \tag{II-4-1}$$

We shall presume that we are able to achieve a small mean-squared-error; then $\hat{y}_1(t)$ is itself optimum and we can write

$$\hat{y}_1(t) = Z \sum_{k=-\infty}^{+\infty} (-1)^k \, |\hat{x}_k| \, \psi_k(t) \tag{II-4-2}$$

If we minimize $E\left\{\left[y_1(t)-\hat{y}_1(t) \right]^2\right\}$ the following quantity is minized

$$\lim \frac{1}{2T} \int_{-T}^{+T} E\left\{\left[y_1(t)-\hat{y}_1(t) \right]^2\right\} = E\left\{(|x_k| - |\hat{x}_k|)^2\right\} \tag{II-4-3}$$

But x_k and $\hat{x}_k$ have the same sign and $E\left\{(x_k| - |\hat{x}_k|)^2\right\}$ is also minimized. Now

$$E\left\{(x_k-\hat{x}_k)^2\right\} = \lim \frac{1}{2T} \int_{-T}^{+T} E\left\{\left[x(t)-\hat{x}(t)\right]^2\right\} dt \tag{II-4-4}$$

Therefore $E\left\{\left[x(t) - \hat{x}(t)\right]^2\right\}$ itself is minimized.

$y_1(t)$ is nonstationary and $\underline{a}$ is given by (I-3-11) i.e.,

$$a^{-1} = \max_t 1 - \frac{\int_{-\infty}^{+\infty} h(t,u) R_{y_1 y_2}(t,u)\, du}{\sigma^2_{y_1}(t)} \tag{II-4-5}$$

where h is given by

$$R_{y_1 y_2}(t_1,t_2) = \int_{-\infty}^{+\infty} h(t_1,u) R_{y_2}(u,t_2)\, du \tag{II-4-6}$$

Thus the problem reduces to finding $R_{y_1y_2}(t_1,t_2)$ and $R_{y_2}(t_1,t_2)$. By (I-1-15)

$$R_{y_1y_2}(t_1,t_2) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \omega_2^{-1} \left[\frac{\partial \Phi_{y_1}}{\partial \omega_1}\right]_{\omega_1 = 0} d\omega_2 \tag{II-4-7}$$

where(+)

$$\Phi_{y_1} = \Phi_x \text{ Real} \prod_{k=-\infty}^{+\infty} \left[1 + j\sqrt{\frac{2}{\pi}}(-1)^k w_{k\,1}F_1\left(\frac{\omega_k^2}{2}\right) \right] \tag{II-4-8}$$

$$\omega_k = \sigma_x \left[\omega_1 \psi_k(t_1) + \omega_2 \psi_k(t_2) \right] \tag{II-4-9}$$

Thus

$$R_{y_1y_2}(t_1,t_2) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \omega_2^{-1} \left\{ \frac{\partial\Phi_x}{\partial\omega_1} \text{ Real} \prod_{k=-\infty}^{+\infty} [1+j\sqrt{\frac{2}{\pi}}(-1)^k\omega_{k\,1}F_1(\frac{\omega_k^2}{2})] \right.$$

$$\left. + \Phi_x \text{Real} \frac{\partial}{\partial\omega_1} \prod_{k=-\infty}^{+\infty} [1+j\sqrt{\frac{2}{\pi}}(-1)^k\omega_{k\,1}F_1(\frac{\omega_k^2}{2})] \right\}_{\omega_1=0} d\omega_2$$

$$= -\frac{1}{\pi}\int_{-\infty}^{+\infty} \omega_2^{-1} \left\{ -R_x(t_1-t_2)\,\omega_2\,\Phi_{y_1}(\omega_2;t_2) \right.$$

$$\left. + \exp(-\frac{\sigma_x^2\omega_2^2}{2})\left[\text{Real} \frac{\partial}{\partial\omega_1} \prod_{k=-\infty}^{+\infty} [1+j\sqrt{\frac{2}{\pi}}(-1)^k\omega_{k\,1}F_1(\frac{\omega_k^2}{2})] \right]_{\omega_1=0} \right\} d\omega_2$$

$$= 2R_x(t_1-t_2)f_{y_1}(0;t_2) - \frac{1}{\pi}\int_{-\infty}^{+\infty}\omega_2^{-1}\exp(-\frac{\sigma_x\omega_2^2}{2})\text{Real}\sum_{\ell=-\infty}^{+\infty} j\sqrt{\frac{2}{\pi}}(-1)^\ell\sigma_x\psi_1(t_1)\Big[{}_1F_1(\frac{\sigma_x^2\omega_2^2\psi_1^2(t_2)}{2})$$

$$+(\sigma_x^2\omega_2^2\psi_1^2(t_2)\,{}_1F_1'(\frac{\sigma_x^2\omega_2^2\psi_1^2(t_2)}{2})\Big] \prod_{\substack{k\neq\ell \\ k=-\infty}}^{+\infty} [1+j\sqrt{\frac{2}{\pi}}(-1)^k\sigma_x\omega_2\psi_k(t_2)\,{}_1F_1(\frac{\sigma_x^2\omega_2^2\psi_k^2}{2})] \tag{II-4-10}$$

The second term cannot be further reduced. $R_{y_2}(t_1,t_2)$ is given by

$$D\,R_{y_2}(t_1,t_2) = -\frac{1}{\pi^2}\iint_{-\infty}^{+\infty}(\omega_1\omega_2)^{-1}\,D\,\Phi_{y_1}(\omega_1,\omega_2;t_1,t_2)d\omega_1\,d\omega_2 \tag{II-4-11}$$

(+) For convenience we drop the arguments of the characteristic functions as well as the two parameters of the hypergeometric function.

where D is the appropriate differential operator. With $D=d/d(t_1-t_2)$ we get a first term

$$4R_x'(t_1-t_2)\, f_{y_1}(0,0;t_1,t_2) \qquad \text{(II-4-12)}$$

Again the second term cannot be put in a nice form.

5. Mapping of the signal into an optimum process, clipping, and estimation by means of a computer algorithm.

It is well-known from the sampling theorem that 4WT samples in an interval (-T, +T) define a signal (deterministic or random) bandlimited to (-W, +W). When the Fourier transform of the signal exists a theorem due to Titchmarsh[7] proves that when all zeros of the complex function

$$x_T(\theta) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} X(\omega)\, e^{j\omega\theta}\, d\omega \qquad \theta = t + j\tau \qquad \text{(II-5-1)}$$

are real the zero-crossing rate of x(t) is equal to 2W, and the signal is completely defined (except for a scale factor) by its zero-crossings z_k:

$$x(t) = x(0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{z_k}\right). \qquad \text{(II-5-2)}$$

On the other hand Polya[8 and Appendix A] shows that when the samples of the signal alternate in sign, e.g.

$$\operatorname{sgn}\left[x\left(\frac{k}{2W}\right)x\left(\frac{k+1}{2W}\right)\right] = -1 \quad \text{for all } k \qquad \text{(II-5-3)}$$

x(t) has one zero-crossings in each Nyquist interval. This is more than required by Titchmarsh's theorem, but it is a very useful property when the zero-crossings are known in a finite time interval only.

Do these results hold for a bandlimited stochastic process? Let us consider the following argument. Since the signal is bandlimited we can write

$$x(t) = \sum_{k=-\infty}^{+\infty} x_k\, \psi_k(t) \qquad \text{(II-5-4)}$$

in the mean-square sense. Or for $T \to \infty$

$$x(t) \rightarrow \sum_{k=-2WT}^{2WT} x_k \, \psi_k(t) \qquad \text{for} \quad |t| \leq T \tag{II-5-5}$$

The zero-crossings z_k determine the samples x_k if and only if the zero-crossings rate is equal to $2W$ (for instance if $x(t)$ has one zero-crossings in each Nyquist interval), indeed

$$\sum_{k=-2WT}^{2WT} x_k \, \psi_k (z_\ell) = 0 \qquad \ell = -2WT, \ldots, 2WT \tag{II-5-6}$$

(where T is assumed large) is a set of $4WT$ equations in $4WT$ unknowns (the ratios x_k / x_o). Therefore if the condition is satisfied the zero-crossings define the process in the mean-square sense*.

Therefore the optimum process defined in I-2 is completely defined by its zero-crossings. If $x(t)$ has a zero-crossing rate smaller than $2W$ the mapping into an optimum signal is straightforward: sample $x(t)$, change the sign where necessary to satisfy the condition of Polya's theorem and low-pass filter to get a function with the same bandwidth as $x(t)$.

Previously we designed an algorithm to find an estimate of an optimum signal in an interval of time from the zero-crossings in that interval[9]. We recall the principle of the algorithm. Since it is usual for a communication engineer to expand in a Fourier series a signal truncated in time we write the estimate

$$\hat{y}_1(t) = \sum_{k=-N}^{+N} C_{N-k} \, e^{jk\omega_o t} \qquad (\omega_o = \pi/T; \ |t| \leq T) \tag{II-5-7}$$

Since the signal is optimum the interval contains $4WT$ zeros; from the zero-crossings location we are able to compute the coefficients in (II-5-7) if we let $N = 2WT$ and require that $y_1(t)$ and $\hat{y}_1(t)$ have the same zeros. With the change of variable+

$$q = \exp(j\,\omega_o t) \tag{II-5-8}$$

* We ignore the ~~un~~derlying mathematical problems

+ Since $|t| \leq T$ and $\omega_o = \pi/T$ this is a one-to-one mapping between the real variable t and the complex variable q.

we can write

$$\sum_{i=1}^{2N} (q-q_i) = \sum_{k=-N}^{+N} \frac{C_{N-k}}{C_0} q^{N+k} \qquad \text{(II-5-9)}$$

where q_i is the value taken by q at the i-th zero-crossings. From (II-5-9) we derive the 2N equations

$$\sum_{i=1}^{2N} q_i = -\frac{C_1}{C_0}$$

$$\sum_{i=1,\, j>i}^{2N} q_i q_j = +\frac{C_2}{C_0} \qquad \text{(II-5-10)}$$

$$\sum_{i=1,\ k>j>i}^{2N} q_i q_j q_k = -\frac{C_3}{C_0}$$

etc.

We note that the algorithm determined $\hat{y}_1(t)$ except for a scale factor. An ambiguity of sign also exists[(+)]. In general the algorithm will give a complex approximation, but experience shows that the imaginary part is small (some 5 percent of the real part) and therefore we finally use the estimate

$$\hat{y}_1(t) = \text{Real} \sum_{k=-N}^{+N} C_{N-k} \exp(jk\,\omega_o t) \quad \text{for } |t| \leq T \qquad \text{(II-5-11)}$$

The scale-and-sign factor is defined by (I-3-19), i.e.,

$$A = \frac{\int_{-T}^{+T} y_1(t)\hat{y}_1(t)\, dt}{\int_{-T}^{+T} \hat{y}_1^{\,2}(t)\, dt} \qquad \text{(II-5-12)}$$

and the signal-to-mean-square -error ratio by (I-3-20)

$$a^{-1} = 1 - \frac{\left[\int_{-T}^{+T} y_1(t)\, \hat{y}_1(t)\, dt\right]^2}{\int_{-T}^{+T} y_1^2(t)\, dt \int_{-T}^{+T} \hat{y}_1^2(t)\, dt} \qquad \text{(II-5-13)}$$

+ However for the general bandlimited signal the ambiguity of sign disappears because we have to transmit the samples sign in addition to the associated optimum signal.

In our Master's thesis we considered intervals of time with 6 zero-crossings and reached values of a of some 20 dB. In recent investigations increasing the number of zeros to 30, using double precision in the computation, and using a guard time band at the ends of the interval (in other words by computing (II-5-13) for the central part of the interval) we were able to reach 33 dB[+]. These results will be described in detail in chapter IV.

To close this section we describe closely related results obtained by Voelcker[14]. The starting point is the analytic signal:

$$x_V(t) = x(t) + j\check{x}(t) = |x_V(t)| \exp[j\Phi(t)]$$
$$= \frac{1}{\pi}\int_0^{\Omega} X(\omega)\, e^{j\omega t}\, d\omega \qquad \text{(II-5-14)}$$

where $\check{x}(t)$ and $X(\omega)$ are the Hilbert and Fourier transform of $x(t)$ respectively:

$$\check{x}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x(\tau)}{t-\tau}\, d\tau \qquad \text{(II-5-15)}$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau \qquad \text{(II-5-16)}$$

The function

$$x_V(\theta) = \frac{1}{\pi}\int_0^{\Omega} X(\omega)\, e^{j\omega\theta}\, d\omega \quad (\theta = t + j\tau) \qquad \text{(II-5-17)}$$

is free of singularities in the finite complex plane for finite energy signals (an entire function). At infinity such a function is of the order of $\exp(k|\theta|)$, k some constant,

$$x_V(\theta) = 0\,[\exp(k|\theta|)] \qquad \text{(II-5-18)}$$

+ In this section we have considered estimation of an optimum process from its zeros. It is easy to show, since in this context the sign of the samples of $x(t)$ itself are exactly known, that a is practically the same for $x(t)$ and $y_1(t)$ if T is large.

Next we consider the integral

$$I = \int_C \frac{d\ln x_V(\theta)}{d\theta} \frac{d\theta}{\theta(\theta - t)} \qquad \text{(II-5-19)}$$

where C is the infinite circle. Equation (II-5-18) shows that this integral vanishes. Therefore the sum of the residues at the poles of the integrand is zero. This yields

$$\frac{d\Phi(t)}{dt} = \left[\frac{d\Phi(t)}{dt}\right]_{t=0} - \sum_k \frac{r_k \tau_k}{t_k^2 + \tau_k^2} + \sum_k \frac{r_k \tau_k}{(t-t_k)^2 + \tau_k^2} \qquad \text{(II-5-20)}$$

$$\frac{d\ln|x_V(t)|}{dt} = \left[\frac{d\ln|x_V(t)|}{dt}\right]_{t=0} + \sum_k \frac{r_k t_k}{t_k^2 + \tau_k^2} + \sum_k \frac{r_k(t-t_k)}{(t-t_k)^2 + \tau_k^2} \qquad \text{(II-5-21)}$$

where $\theta_k = t_k + j\tau_k$ is the position of the k-th zero of $x_V(\theta)$ and r_k its order. (II-5-20) and (II-5-21) show that in the bandlimited case the envelope and phase (and therefore the signal $x(t) = |x_V(t)| \cos \Phi(t)$ itself) are determined by the zeros of the analytic signal.

At this point we note that the function $x_T(\theta)$ considered by Titchmarsh is also entire since

$$|x_T(\theta)|^2 \leq (2\pi)^{-2} \int_{-\Omega}^{\Omega} |X(\omega)|^2 \, d\omega \int_{-\Omega}^{\Omega} e^{-2\omega\tau} \, d\omega$$

$$= \frac{E}{2\pi} \frac{\sinh(2\Omega\tau)}{\tau} \qquad \text{(II-5-22)}$$

where E is the signal energy. Therefore (II-5-20 and 21) are applicable to $x_T(\theta)$ and as already said the zeros of $x_T(\theta)$ define $x(t)$ completely. The advantage of $x_T(\theta)$ is that the real zeros of the entire function are the zero-crossings of the signal (which, in general, is not true for the analytic signal $x_V(\theta)$); zero-crossings are directly observable while the zeros of the analytic signal are not physical quantities (to find these zeros Voelcker proposes the factorization of the Fourier series representation of $x_V(\theta)$). To find the signal from its zero-crossings Voelcker has proposed a nonlinear device called "Real-Zero Interpolator". We shall describe the mathematics behind the device . The purpose of the real-zero interpolator is to recover a signal which has real zeros of order one only (and more generally the part of the signal which corresponds to its real zeros). It is therefore an interesting alternative to the computer algorithm we shall use in this work. Applying (II-5-20 and 21) to $x_T(t)$ and ignoring the constant terms we get[+]

$$\Phi(t) = \lim_{\tau \to o} \sum_k \int \frac{\tau}{(t-t_k)^2 + \tau^2} dt$$

$$= \lim_{\tau \to o} \sum_k \tan^{-1}\left(\frac{t-z_k}{\tau}\right) \qquad (\tau > o) \qquad \text{(II-5-23)}$$

where z_k is the k-th zero-crossing of the signal. Thus

$$\frac{d\Phi(t)}{dt} = \pi \sum_k \delta(t-z_k) \qquad \text{(II-5-24)}$$

and

$$\frac{d\ln |x_T(t)|}{dt} = \sum_k (t-z_k)^{-1}$$

$$= [d\Phi(\check{t})/dt] \qquad \text{(II-5-25)}$$

Thus from the knowledge of the zero-crossings we generate the derivative of the phase. A device which approximates the Hilbert transform gives the derivative of the logarithm of the envelope. An integrator followed by a nonlinear device with exponential characteristic gives the envelope of the signal. On the other hand (II-5-24) shows that $\Phi(t)$ increases by π at each zero-crossing therefore x(t) itself can be recovered by multiplication of the envelope by the clipped version of x(t) (known from the zero-crossings). Voelcker and recently Sekey[15] have shown the feasibility of this technique, while however going in the direction opposite to the purpose here, namely

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+ (II-5-20) shows that $\frac{d\Phi(t)}{dt} = 0$ everywhere except at the position of a zero-crossing. The limiting process is necessary to get the amplitude of the impulses of the phase derivative at these points.

the transmission of 2-levels signals with a bandwidth saving (and consequently an increased vulnerability to noise).

6. Conclusions.

In II-2 and II-3 we have seen that the estimate of a signal derived from a set of related zero-crossings cannot be improved by two linear techniques. In II-4 we have seen that optimum linear filtering of the random square wave which carries the zero-crossings of the optimum process related to the signal leads to analytical problems. By contrast the approach considered in II-5 gives excellent results. However in this case we could not derive an expression for the signal-to-mean-square-error-ratio vs T when the signal to be transmitted is random (such an expression can be derived for the waveform sin(t)/t [9-p. 27]).

Thus in the following chapters we shall consider the transmission of a bandlimited process by means of its associated optimum process. We underline that the failure encountered in II-3 and II-4 does not mean differentiation or addition of a narrow-band gaussian process are not worth to be considered; indeed we have investigated linear filtering only. Thus further research in this direction should deal with nonlinear filtering. We think that the success of the mapping and algorithm technique is essentially related to the nonlinear nature of the transformations involved.

CHAPTER III

PROPERTIES OF THE TRANSMISSION OF A RANDOM SQUARE WAVE.

We investigate here the second point pertinent to the transmission of a signal by means of a related set of zero-crossings. After clipping of the optimum signal we have to transmit a random rectangular wave; does this result in an advantage at the transmission point of view?

We shall first consider the case $B/W \to \infty$, where B is the channel bandwidth. Next we shall take the effect of bandwidth into account. We shall assume the channel noise $n(t)$ Gaussian, zero-mean, additive, independent of the signal.

To estimate $y_2(t)$(+) from the received signal we consider the following scheme (Fig. III-1): an ideal clipper followed by an amplifier with gain properly chosen to minimize the mean-squared-error. This scheme has the advantage of eliminating a great deal of noise in a simple way.

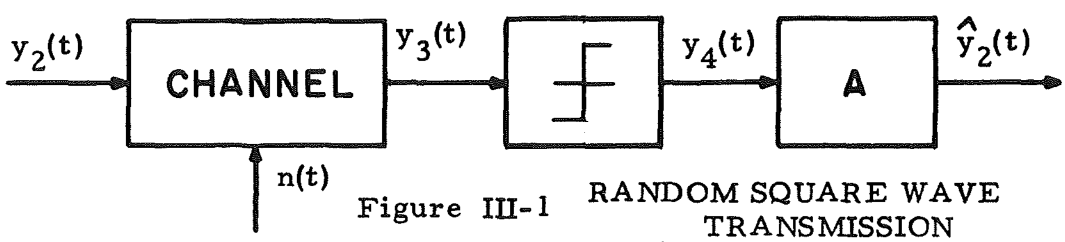

Figure III-1 RANDOM SQUARE WAVE TRANSMISSION

This is not the best nonlinear receiver however, but it exhibits a very interesting property. We shall not attempt to optimize the structure in some way (for instance by a Wiener filter between channel and clipper). because of the analytical difficulties involved.

By (I-3-4) we have

$$a = \min_{t_1} \max_{t_2} \left[1 - R^2_{y_2 y_4}(t_1, t_2) \right]^{-1} \qquad \text{(III-1)}$$

Where for the sake of generality we have assumed $y_2(t)$ and $n(t)$ non-stationary.(++)

(+) To be consistent with previous chapter the random square wave is denoted by $y_2(t)$ and has fixed unit amplitude.

(++) Again we note however that $|y_2(t)| = 1$ and $E\{n(t)\} = 0$.

1. Asymptotic properties ($B/W \to \infty$)

As $\frac{B}{W}$ increases the impulse response of the channel tends to $\delta(t)$ and

$$y_3(t) \to y_2(t) + n(t) \qquad \text{(III-1-1)}$$

From

$$y_4(t) = \frac{1}{j\pi} \int_{-\infty}^{+\infty} \omega^{-1} \exp\left\{ j\omega \left[y_2(t) + n(t) \right] \right\} d\omega \qquad \text{(III-1-2)}$$

$$R_{y_2 y_4}(t_1, t_2) = \frac{1}{j\pi} \int_{-\infty}^{+\infty} \omega_2^{-1} E\left\{ y_2(t_1) \exp\left[j\omega_2 y_2(t_2) \right] \right\} E\left\{ \exp\left[j\omega_2 n(t_2) \right] \right\} d\omega_2$$

$$= -\frac{1}{\pi} \int_{-\infty}^{+\infty} \omega_2^{-1} \left[\frac{\partial \Phi_{y_2}(\omega_1, \omega_2; t_1, t_2)}{\partial \omega_1} \right]_{\omega_1 = 0} \Phi_n(\omega_2; t_2)\, d\omega_2 \qquad \text{(III-1-3)}$$

Let us denote by $P_{ij}(t_1, t_2)$ $(i, j = +, -)$ the transition probabilities of the random square wave $y_2(t)$, e.g.

$$P_{+-}(t_1, t_2) = P\left[y_2(t_1) = +1 \text{ and } y_2(t_2) = -1 \right] \qquad \text{(III-1-4)}$$

etc.

The density function of the optimum process $y_1(t)$ is symmetrical, therefore $y_2(t)$ is zero-mean since

$$P_{++}(t_1, t_2) + P_{+-}(t_1, t_2) = P\left[y_2(t_1) = +1 \right] = 1/2 \qquad \text{(III-1-5)}$$

$$P_{-+}(t_1, t_2) + P_{--}(t_1, t_2) = P\left[y_2(t_1) = -1 \right] = 1/2 \qquad \text{(III-1-6)}$$

$$P_{++}(t_1, t_2) + P_{-+}(t_1, t_2) = P\left[y_2(t_2) = +1 \right] = 1/2 \qquad \text{(III-1-7)}$$

$$P_{+-}(t_1, t_2) + P_{--}(t_1, t_2) = P\left[y_2(t_2) = -1 \right] = 1/2 \qquad \text{(III-1-8)}$$

Therefore

$$P_{+-}(t_1, t_2) = P_{-+}(t_1, t_2) \qquad \text{(III-1-9)}$$

and

$$P_{++}(t_1, t_2) = P_{--}(t_1, t_2) \qquad \text{(III-1-10)}$$

The correlation function of $y_2(t)$ is

$$R_{y_2}(t_1, t_2) = P_{++} + P_{--} - P_{+-} - P_{-+}$$

$$= 2(P_{++} - P_{+-})$$

$$= 4P_{++}(t_1, t_2) - 1 \qquad \text{(III-1-11)}$$

Therefore the P's can be expressed in terms of the correlation function

$$P_{++}(t_1, t_2) = \frac{1}{4}\left[1 + R_{y_2}(t_1, t_2)\right] \qquad \text{(III-1-12)}$$

$$P_{+-}(t_1, t_2) = \frac{1}{4}\left[1 - R_{y_2}(t_1, t_2)\right] \qquad \text{(III-1-13)}$$

And the characteristic function of $y_2(t)$ can be written as

$$\Phi_{y_2}(\omega_1, \omega_2; t_1, t_2) = E\left\{\exp\left[j\left(\omega_1 y_2(t_1) + \omega_2 y_2(t_2)\right)\right]\right\}$$

$$= P_{++}(t_1, t_2)\, e^{j(\omega_1 + \omega_2)} + P_{--}(t_1, t_2)\, e^{-j(\omega_1 + \omega_2)}$$

$$+ P_{+-}(t_1, t_2)\, e^{j(\omega_1 - \omega_2)} + P_{-+}(t_1, t_2)\, e^{-j(\omega_1 - \omega_2)}$$

$$= 2P_{++}(t_1, t_2) \cos(\omega_1 + \omega_2) + 2P_{+-}(t_1, t_2) \cos(\omega_1 - \omega_2) \qquad \text{(III-1-14)}$$

By (III-1-12) and (III-1-13)

$$\Phi_{y_2}(\omega_1, \omega_2; t_1, t_2) = \cos(\omega_1) \cos(\omega_2)$$

$$- R_{y_2}(t_1, t_2) \sin(\omega_1) \sin(\omega_2) \qquad \text{(III-1-15)}$$

Therefore

$$R_{y_2y_4}(t_1, t_2) = \frac{R_{y_2}(t_1, t_2)}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(u)}{u} e^{-\frac{u^2}{2a_c(t_2)}} du \qquad \text{(III-1-16)}$$

where $a_c(t_2)$ is the channel signal-to-noise ratio at time t_2

$$a_c(t_2) = \sigma_n^{-2}(t_2) \qquad \text{(III-1-17)}$$

and by Parseval's formula

$$R_{y_2y_4}(t_1, t_2) = R_{y_2}(t_1, t_2) \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{a_c(t_2)}} e^{-\frac{u^2}{2}} du \qquad \text{(III-1-18)}$$

When $a_c \to \infty$

$$R_{y_2y_4}(t_1, t_2) \to R_{y_2}(t_1, t_2) \qquad \text{(III-1-19)}$$

as expected since in this case $y_4(t) \to y_2(t)$.
From (III-1) and (III-1-17) we get finally the performance of the system

$$a^{-1} = \max_{t_1} \min_{t_2} \left\{ 1 - R_{y_2}^2(t_1, t_2) \frac{2}{\pi} \left[\int_0^{\sqrt{a_c(t_2)}} e^{-\frac{u^2}{2}} du \right]^2 \right\} \qquad \text{(III-1-20)}$$

For practical considerations $t_2 - t_1$ is constant. On the other hand maximization of $\underline{a}$ over t_2 would require knowledge of $a_c(t)$. The best choice if the channel signal-to-noise ratio is not measured is $t_1 = t_2$. Therefore

$$a(t_1)^{-1} = 1 - \left[\sqrt{\frac{2}{\pi}} \int_0^{\sqrt{a_c(t_1)}} e^{-\frac{u^2}{2}} du \right]^2 \qquad \text{(III-1-21)}$$

Or with the Q-function defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du \tag{III-1-22}$$

$$a(t_1)^{-1} = 1 - \left[1 - 2Q\left(\sqrt{a_c(t_1)}\right)\right]^2$$

$$= 4Q\left(\sqrt{a_c(t_1)}\right)\left[1 - Q\left(\sqrt{a_c(t_1)}\right)\right] \tag{III-1-23}$$

$$a_o(t_1) = a(t_1) - 1 = \frac{\left[1 - 2Q\left(\sqrt{a_c(t_1)}\right)\right]^2}{4Q\left(\sqrt{a_c(t_1)}\right)\left[1 - Q\left(\sqrt{a_c(t_1)}\right)\right]} \tag{III-1-24}$$

When $x \to \infty$

$$Q(x) \to \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{x^2}{2}} \tag{III-1-25}$$

Thus for high channel signal-to-noise ratio

$$a(t_1)^{-1} \to 4Q\left(\sqrt{a_c(t_1)}\right) \to \frac{4}{\sqrt{2\pi a_c(t_1)}} e^{-\frac{a_c(t_1)}{2}} \tag{III-1-26}$$

and the output signal-to-noise ratio behaves asymptotically as

$$a_o(t_1) \to a(t_1) \to \frac{\sqrt{2\pi\, a_c(t_1)}}{4} e^{\frac{a_c(t_1)}{2}} \tag{III-1-27}$$

Therefore we get a very fast improvement (essentially exponential) of the output signal-to-noise ratio with the channel signal-to-noise ratio. In Chapter IV it will be shown that this property remains true for the complete communication system.

For purpose of comparison with this result we shall consider the performance for optimum linear filtering of the channel output. From

(I-3-9 and 11)

$$a^{-1} = 1 - \frac{\int_{-\infty}^{+\infty} h(t_1, u)\, R_{y_2 y_3}(t_1, u)\, du}{R_{y_2}(t_1, t_1)} \qquad \text{(III-1-28)}$$

with h given by

$$R_{y_2 y_3}(t_1, t_2) = \int_{-\infty}^{+\infty} h(t_1, u)\, R_{y_3}(u, t_2)\, du \qquad \text{(III-1-29)}$$

Since $y_2(t)$ and $n(t)$ are independent and zero-mean

$$R_{y_2 y_3}(t_1, t_2) = R_{y_2}(t_1, t_2) \qquad \text{(III-1-30)}$$

$$R_{y_3}(t_1, t_2) = R_{y_2}(t_1, t_2) + R_n(t_1, t_2) \qquad \text{(III-1-31)}$$

therefore h is solution of the integral equation

$$R_{y_2}(t_1, t_2) = \int_{-\infty}^{+\infty} h(t_1, u)\left[R_{y_2}(u, t_2) + R_n(u, t_2)\right] du \qquad \text{(III-1-32)}$$

and

$$a^{-1} = 1 - \frac{\int_{-\infty}^{+\infty} h(t_1, u)\, R_{y_2}(t_1, u)\, du}{R_{y_2}(t_1, t_1)} \qquad \text{(III-1-33)}$$

We shall consider the special case (generalisation of $a_c = S_{y_2}(\omega)/S_n(\omega)$)

$$R_n(t_1, t_2)\, a_c(t_2) = R_{y_2}(t_1, t_2) \qquad \text{(III-1-34)}$$

Then

$$R_{y_2}(t_1, t_2) = \left[1 + a_c^{-1}(t_2)\right] \int_{-\infty}^{+\infty} h(t_1, u)\, R_{y_2}(u, t_2)\, du \qquad \text{(III-1-35)}$$

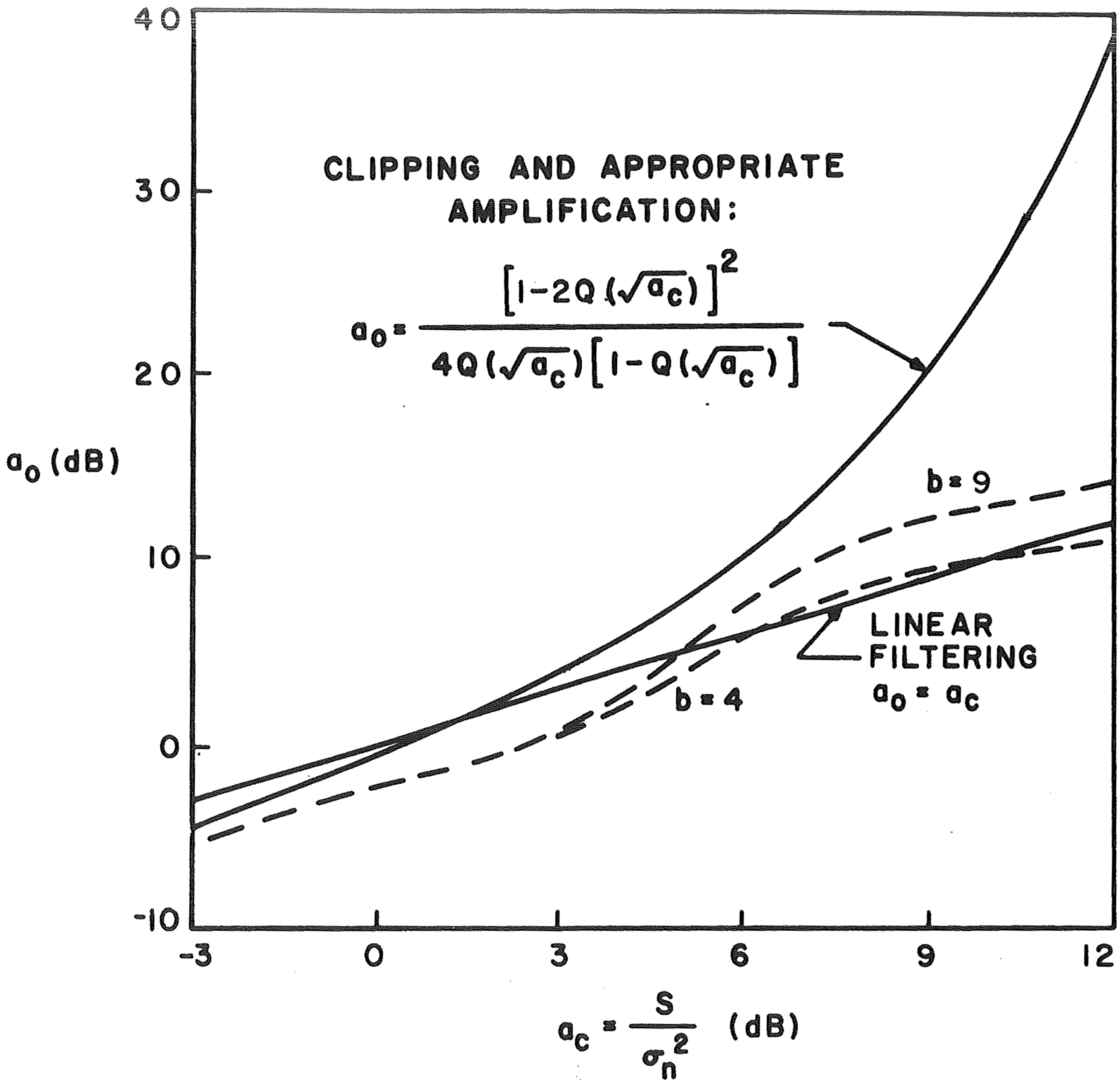

Fig. III-2

Estimation of the clipped optimum process

Plain curve : clipping and amplification (asymptotic behavior: $\frac{B}{W} \rightarrow \infty$)

Dotted curve : clipping and amplification for the channel

$$H_c(\omega) = \frac{\sin \omega T}{\omega T} \quad \left(b = \frac{B}{W} = \frac{1}{4WT}\right).$$

and

$$a^{-1} = 1 - \left[1 + a_c^{-1}(t_1)\right]^{-1} = \left[1 + a_c(t_1)\right]^{-1} \tag{III-1-36}$$

Therefore

$$a_o = a_c \tag{III-1-37}$$

i.e. a noise with correlation function given by (III-1-34) is the worse which can be encountered since the optimum linear filter cannot improve the signal-to-noise ratio. (III-1-24) which has been derived for nonlinear "filtering" is exact here since we can let $B = \infty$ without getting an infinite amount of noise in the receiver. As we can see on Fig. III-2 the point where our simple nonlinear receiver behaves better than the Wiener filter in the worse type of noise corresponds to a channel signal-to-noise ratio of 1.5 dB. For low channel signal-to-noise ratio the scheme is worse by a factor $2/\pi$. However, some improvement are possible if we allow a more complex receiver structure, a Wiener filter followed by the clipper for instance.

Finally it is worthwhile to mention that (III-1-24) only depends on the total amount of noise which gets into the receiver and not on its spectral distribution; also it is actually independent of the statistics of the signal from which the random square wave $y_2(t)$ is derived. If they are known one might take advantage of these to improve the performance of the receiver.

2. Effect of the channel $H_c(\omega) = \frac{\sin \omega T}{\omega T}$ on clipped optimum signals

In general the channel output can be written

$$y_3(t) = \int_{-\infty}^{+\infty} h_c(t - \tau) y_2(\tau)\, d\tau + n(t) \tag{III-2-1}$$

When we specialize to $h_c(t) = \frac{1}{2T} p_T(t)^{(+)}$ we get

(+)$H_c(\omega) = \frac{\sin(\omega T)}{\omega T}$. This channel although a nonphysical one has one of the features of an actual channel, namely it is not perfectly bandlimited. Its bandwidth can be defined, for instance, by $\int_{-\infty}^{+\infty} H_c^2(\omega)\, d\omega = 4\pi B$ which leads to $B = 1/4T$. The noise power density is $\frac{N_o}{2} \left[\sin(\omega T)/\omega T\right]^2$ and the noise power is $\sigma_n^2 = N_o B$ if we assume white noise at the input.

$$y_3(t) = \frac{1}{2T} \int_{t-T}^{t+T} y_2(\tau)\, d\tau + n(t) = y_c(t) + n(t) \tag{III-2-2}$$

On the other hand

$$R_{y_2 y_4}(t_1, t_2) = \frac{1}{j\pi} \int_{-\infty}^{+\infty} \omega_2^{-1}\, E\left\{ y_2(t_1) e^{j\omega_2 [y_c(t_2) + n(t_2)]} \right\} d\omega_2$$

$$= -\frac{1}{\pi} \int_{-\infty}^{+\infty} \omega_2^{-1} \left[\frac{\partial \Phi_{y_2 y_c}}{\partial \omega_1} \right]_{\omega_1 = 0} e^{-\frac{\sigma_n^2 \omega_2^2}{2}}\, d\omega_2 \tag{III-2-3}$$

where we denote by $y_c(t)$ the channel output in the noise-free conditions.

We shall make the following hypothesis: for all t the random square wave process $y_2(t)$ has only one zero-crossing in the time interval $(t-T, t+T)$(++). Then if a zero-crossing occurs at $t+\tau$ $(-T \leq \tau \leq T)$ we have

$$y_c(t) = \frac{\tau}{T}\, \mathrm{sgn}\left[y_2(t-T)\right]$$

$$= \frac{\tau}{T}\, y_2(t-T) \tag{III-2-4}$$

otherwise

$$y_c(t) = y_2(t-T) = y_2(t) \tag{III-2-5}$$

Let us denote by $p_{ij}(t_1, t_2 + \tau)d\tau$ (i, j= +, -) the probability densities defined by the following

$$p_{+-}d\tau = P\left\{ y_2(t_1) = +1 \text{ and } y_2(t) \text{ has a downward zero-crossing between } t_2 + \tau \text{ and } t_2 + \tau + d\tau \right\} \tag{III-2-6}$$

$$p_{++}d\tau = P\left\{ y_2(t_1) = +1 \text{ and } y_2(t) \text{ has an upward zero-crossing between } t_2 + \tau \text{ and } t_2 + \tau + d\tau \right\} \tag{III-2-7}$$

(++) Since $y_2(t)$ is the clipped version of an optimum process there is only one zero-crossing in each Nyquist interval. Therefore the hypothesis is true for each Nyquist interval $(k/2W, (k+1)/2W)$ when t is in $(\frac{k}{2W} + T, \frac{k+1}{2W} - T)$. If t belongs to $(\frac{k}{2W}, \frac{k}{2W} + T)$ or $(\frac{k+1}{2W} - T, \frac{k+1}{2W})$ we note that the hypothesis will not be true (and this will not happen for all these values of t) if the interval between zero-crossings of 2 successive Nyquist intervals is smaller than 2T. Anticipating experimental results which will be described in chapter IV we quote that for B=5W (2T=1/10W) this probability is already as small as .01. For B=2.5W the probability is .04.

and similarly for p_{-+} and p_{--}. Then

$$\Phi_{y_2 y_c} = E\left\{ e^{j\left[\omega_1 y_2(t_1) + \omega_2 y_c(t_2)\right]} \right\}$$

$$= \int_{-T}^{+T} \left\{ p_{+-}\, e^{j\left[\omega_1 + \omega_2 \frac{\tau}{T}\right]} + p_{-+}\, e^{j\left[-\omega_1 - \omega_2 \frac{\tau}{T}\right]} \right.$$

$$\left. + p_{++}\, e^{j\left[\omega_1 - \omega_2 \frac{\tau}{T}\right]} + p_{--}\, e^{j\left[-\omega_1 + \omega_2 \frac{\tau}{T}\right]} \right\} d\tau$$

$$+ P_{++0}\, e^{j(\omega_1 + \omega_2)} + P_{--0}\, e^{-j(\omega_1 + \omega_2)}$$

$$+ P_{+-0}\, e^{j(\omega_1 - \omega_2)} + P_{-+0}\, e^{-j(\omega_1 - \omega_2)} \qquad \text{(III-2-8)}$$

where

$$P_{++o} = P\left\{ y_2(t_1) = +1,\ y_2(t_2) = +1, \text{ and no zero-crossing occurs in in } (t_2 - T,\ t_2 + T) \right\} \qquad \text{(III-2-9)}$$

etc.

Since each realization of an optimum process has been multiplied by the random variable Z (±1 with probability 1/2) we have complete symmetry and therefore

$$p_{+-} = p_{-+} \tag{III-2-10}$$

$$p_{++} = p_{--} \tag{III-2-11}$$

$$P_{+-o} = P_{-+o} \tag{III-2-12}$$

$$P_{++o} = P_{--o} \tag{III-2-13}$$

On the other hand

$$\int_{-T}^{o} p_{++}\, d\tau + \int_{o}^{T} p_{+-}\, d\tau = P\Big\{ y_2(t_1) = +1,\ y_2(t_2) = +1, \text{ and }$$
$$y_2(t) \text{ has a zero-crossing in } (t_2-T, t_2+T)\Big\}$$
$$= P_{++} - P_{++o} \tag{III-2-14}$$

$$\int_{-T}^{o} p_{+-}\, d\tau + \int_{o}^{T} p_{++} d\tau = P\Big\{ y_2(t_1) = +1,\ y_2(t_2) = -1, \text{ and }$$
$$y_2(t) \text{ has a zero-crossing in } (t_2-T,\ t_2+T)\Big\}$$
$$= P_{+-} - P_{+-o} \tag{III-2-15}$$

Therefore

$$\Phi_{y_2 y_c} = 2\int_{-T}^{+T} \Big[p_{+-} \cos\big(\omega_1 + \omega_2 \tfrac{\tau}{T}\big) + p_{++} \cos\big(\omega_1 - \omega_2 \tfrac{\tau}{T}\big) \Big]\, d\tau$$
$$+ 2\Big[P_{++} - \Big(\int_{-T}^{o} p_{++}\, d\tau + \int_{o}^{T} p_{+-} d\tau \Big)\Big] \cos(\omega_1 + \omega_2)$$
$$+ 2\Big[P_{+-} - \Big(\int_{-T}^{0} p_{+-} d\tau + \int_{o}^{T} p_{++}\, d\tau \Big)\Big] \cos(\omega_1 - \omega_2) \tag{III-2-16}$$

and

$$\left[\frac{\partial \Phi_{y_2 y_c}}{\partial \omega_1}\right]_{\omega_1=0} = -2\int_{-T}^{+T} (p_{+-}-p_{++})\sin(\omega_2 \tfrac{\tau}{T})\, d\tau$$

$$-2\left[(P_{++}-P_{+-}) + \int_{-T}^{o} (p_{+-}-p_{++})\, d\tau - \int_{o}^{T} (p_{+-}-p_{++})\, d\tau\right] \sin(\omega_2) \qquad \text{(III-2-17)}$$

Now

$$R_{y_2 y_4}(t_1,t_2) = \frac{2}{\pi}\int_{-T}^{+T} (p_{+-}-p_{++})\left[\int_{-\infty}^{+\infty} \frac{\sin(\omega_2 \frac{\tau}{T})}{\omega_2} e^{-\frac{\sigma_n^2 \omega_2^2}{2}} d\omega_2\right] d\tau$$

$$+\frac{2}{\pi}\left[P_{++}-P_{+-} + \int_{-T}^{o} (p_{+-}-p_{++})\, d\tau - \int_{o}^{T} (p_{+-}-p_{++})\, d\tau\right]\cdot$$

$$\int_{-\infty}^{+\infty} \frac{\sin(\omega_2)}{\omega_2} e^{-\frac{\sigma_n^2 \omega_2^2}{2}} d\omega_2 \qquad \text{(III-2-18)}$$

By Parseval's formula and (III-1-12 and 13) we get

$$R_{y_2 y_4}(t_1,t_2) = 2\sqrt{\frac{2}{\pi}}\left\{\int_{-T}^{+T} (p_{+-}-p_{++})\left[\int_{o}^{\frac{\tau}{T\sigma_n}} e^{-\frac{u^2}{2}} du\right] d\tau\right.$$

$$\left.+\left[\frac{R_{y_2}(t_1,t_2)}{2} + \int_{-T}^{o} (p_{+-}-p_{++})\, d\tau - \int_{o}^{T} (p_{+-}-p_{++})\, d\tau\right]\int^{\frac{1}{\sigma_n}} e^{-\frac{u^2}{2}} du\right. \qquad \text{(III-2-19)}$$

i.e., the cross-correlation function of channel input and output is completely defined by the autocorrelation function of the random square wave and the two quantities p_{+-} and p_{++}. To check (III-2-19) we let $T \to 0$: the first term vanishes as well as the last part of the second and

$$R_{y_2 y_4}(t_1,t_2) \to R_{y_2}(t_1,t_2)\sqrt{\frac{2}{\pi}}\int_{o}^{\frac{1}{\sigma_n}} e^{-\frac{u^2}{2}} du \qquad \text{(III-2-20)}$$

i.e., as expected, the result found for the zero time-delay channel investigated in section 1 (equation III-1-18). On the other hand if we let $\sigma_n \to 0$

$$R_{y_2y_4}(t_1,t_2) \to 2\Big[-\int_{-T}^{o} (p_{+-}-p_{++})\, d\tau + \int_{o}^{T} (p_{+-}-p_{++})\, d\tau$$

$$+ \frac{R_{y_2}(t_1,t_2)}{2} + \int_{-T}^{o} (p_{+-}-p_{++})\, d\tau - \int_{o}^{T} (p_{+-}-p_{++})\, d\tau \Big]$$

$$= R_{y_2}(t_1,t_2) \qquad \text{(III-2-21)}$$

as expected since when the channel is noise free there is no zero-crossing displacement (with the hypothesis of at most one zero-crossing in any time interval of width 2T).

To go further we must find the two quantities p_{+-} and p_{++}. Since $y_2(t)$ is the clipped version of the optimum process $y_1(t)$ we can obviously write

$p_{+-}d\tau = P\Big\{ y_1(t_1) > 0$, and $y_1(t)$ has a downward zero-crossing between $t_2 + \tau$ and $t_2 + \tau + d\tau \Big\}$, and similarly for p_{++}. For convenience we shall write $x_1 = y_1(t_1)$, $x_2 = y_1(t_2 + \tau)$, and $x_3 = \left[\frac{dy_1(t)}{dt}\right]_{t=t_2+\tau}$.

Then(12 - pp. 190-191)

$$p_{+-} = -\int_{x_1=o}^{\infty} dx_1 \int_{x_3=-\infty}^{o} x_3 f(x_1, 0, x_3)\, dx_3 \qquad \text{(III-2-22)}$$

$$p_{++} = +\int_{x_1=o}^{\infty} dx_1 \int_{x_3=o}^{\infty} x_3 f(x_1, 0, x_3)\, dx_3 \qquad \text{(III-2-23)}$$

Except for gaussian processes these quantities cannot be derived at the present time. Thus we shall derive p_{+-} and p_{++} as if $y_1(t)$ was Gaussian, with autocorrelation function and variance given by (I-2-33 and 35). Shortly we shall be able to quote a result which shows to what extent this is justified. With this hypothesis x_1, x_2, and x_3 have joint characteristic function

$$\Phi = \exp\Big[-\frac{1}{2}(\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + \sigma_3^2\omega_3^2 + 2R_{12}\omega_1\omega_2 + 2R_{13}\omega_1\omega_3 + 2R_{23}\omega_2\omega_3)\Big] \qquad \text{(III-2-24)}$$

(We shall use the characteristic function rather than the density function itself to avoid inversion of a 3x3 matrix). In (III-2-24) the coefficients are given by

$$\sigma_1^2 = \sigma_x^2 \left[1 - \frac{2}{\pi} \sin^2 (t_1)\right]$$

$$\sigma_2^2 = \sigma_x^2 \left[1 - \frac{2}{\pi} \sin^2 (t_2+\tau)\right]$$

$$R_{12} = R_{y_1}(t_1, t_2+\tau) = \sigma_x^2 \left[\left(1 - \frac{2}{\pi}\right) \frac{\sin(t_2+\tau - t_1)}{(t_2+\tau - t_1)} + \frac{2}{\pi} \cos (t_2+\tau) \cos(t_1)\right]$$

$$R_{13} = R_{y_1 y_1'}(t_1, t_2+\tau) = \qquad \text{(III-2-25)}$$

$$= \Omega \sigma_x^2 \left[\left(1 - \frac{2}{\pi}\right) \frac{(t_2+\tau - t_1)\cos(t_2+\tau - t_1) - \sin(t_2+\tau - t_1)}{(t_2+\tau - t_1)^2} - \frac{2}{\pi} \sin(t_2+\tau) \cos(t_1)\right]$$

$$R_{23} = R_{y_1 y_1'}(t_2+\tau, t_2+\tau) = -\sigma_x^2 \Omega \frac{2}{\pi} \sin(t_2+\tau) \cos(t_2+\tau)$$

$$\sigma_3^2 = R_{y_1' y_1'}(t_2+\tau, t_2+\tau) = \Omega^2 \sigma_x^2 \left[\left(1 - \frac{2}{\pi}\right) \frac{1}{3} + \frac{2}{\pi} \sin^2 (t_2+\tau)\right]$$

For convenience we have dropped Ω in these expressions. Thus in the following each time we write t it actually means Ωt. Now for bandwidth expansion factor of, say, at least 2.5

$$2.5\, W \leq B = (4T)^{-1} \qquad \text{(III-2-26)}$$

and

$$\Omega |\tau| \leq \pi/5 \qquad \text{(III-2-27)}$$

we can use the approximation

$$\sin(\Omega\tau) \approx \Omega\tau \qquad \text{(III-2-28)}$$

$$\cos(\Omega\tau) \approx 1 - \frac{(\Omega\tau)^2}{2} \qquad \text{(III-2-29)}$$

On the other hand there is no time delay in the system, therefore we let $t_1 = t_2$ in (III-1) and actually we need $p_{+-}(t_1, t_1+\tau)$ and $p_{++}(t_1, t_1+\tau)$. Thus (III-2-25) simplifies in

$$\sigma_1^2 = \sigma_x^2 \left[1 - \frac{2}{\pi} \sin^2(t_1) \right]$$

$$\sigma_2^2 = \sigma_x^2 \left[1 - \frac{2}{\pi} \sin^2(t_1) - \frac{4}{\pi} \tau \sin(t_1) \cos(t_1) \right]$$

$$\sigma_3^2 = \Omega^2 \sigma_x^2 \left[(1 - \frac{2}{\pi}) \frac{1}{3} + \frac{2}{\pi} \sin^2(t_1) + \frac{4}{\pi} \tau \sin(t_1) \cos(t_1) \right]$$

$$R_{12} = \sigma_x^2 \left[1 - \frac{2}{\pi} \sin^2(t_1) - \tau \frac{2}{\pi} \sin(t_1) \cos(t_1) \right] \tag{III-2-30}$$

$$R_{13} = \Omega \sigma_x^2 \left\{ - \frac{2}{\pi} \sin(t_1) \cos(t_1) - \tau \left[(1 - \frac{2}{\pi}) \frac{1}{3} + \frac{2}{\pi} \cos^2(t_1) \right] \right\}$$

$$R_{23} = - \Omega \sigma_x^2 \left[\frac{2}{\pi} \sin(t_1) \cos(t_1) + \tau \frac{2}{\pi} [\cos^2(t_1) - \sin^2(t_1)] \right]$$

Now (III-2-22) becomes

$$p_{+-} = - \left(\frac{1}{2\pi} \right)^3 \int_{x_1=0}^{\infty} \int_{x_3=-\infty}^{0} \iiint_{-\infty}^{+\infty} x_3 \, \Phi(\omega_1, \omega_2, \omega_3) e^{-j(\omega_1 x_1 + \omega_3 x_3)} d\omega_1 d\omega_2 d\omega_3 dx_1 dx_3 \tag{III-2-31}$$

Integrating over ω_2 we get

$$p_{+-} = - \left(\frac{1}{2\pi} \right)^2 \frac{1}{\sigma_2 \sqrt{2\pi}} \int_{x_1=0}^{\infty} \int_{x_3=-\infty}^{0} \iint_{-\infty}^{+\infty} x_3 e^{-j(\omega_1 x_1 + \omega_3 x_3)} d\omega_1 \, d\omega_3 dx_1 \, dx_3 \, .$$

$$\exp \left\{ - \frac{1}{2} \left[\left(\sigma_1^2 - \frac{R_{12}^2}{\sigma_2^2} \right) \omega_1^2 + \left(\sigma_3^2 - \frac{R_{23}^2}{\sigma_2^2} \right) \omega_3^2 + 2 \left(R_{13} - \frac{R_{12} R_{23}}{\sigma_2^2} \right) \omega_1 \omega_3 \right] \right\} \tag{III-2-32}$$

or with the following notations

$$A^2 = \sigma_3^2 - \frac{R_{23}^2}{\sigma_2^2} \qquad \text{(III-2-33)}$$

$$B = \frac{R_{13}\,\sigma_2^2 - R_{12}R_{23}}{\sigma_2^2\,\sigma_3^2 - R_{23}^2} \qquad \text{(III-2-34)}$$

$$C^2 = \left(\sigma_1^2 - \frac{R_{12}^2}{\sigma_2^2}\right) - \frac{\left[R_{13} - \frac{R_{12}\,R_{23}}{\sigma_2^2}\right]^2}{\sigma_3^2 - \frac{R_{23}^2}{\sigma_2^2}} \qquad \text{(III-2-35)}$$

$$p_{+-} = -\left(\frac{1}{2\pi}\right)^2 \frac{1}{\sigma_2\sqrt{2\pi}} \int_{x_1=o}^{\infty} \int_{x_3=-\infty}^{o} \iint_{-\infty}^{+\infty} x_3 \exp\left[-j(\omega_1 x_1 + \omega_3 x_3)\right].$$

$$\exp\left\{-\frac{1}{2}\left[C^2\omega_1^2 + A^2\,(B\omega_1 + \omega_3)^2\right]\right\} d\omega_1 d\omega_3 dx_1 dx_3 \qquad \text{(III-2-36)}$$

Integrating over ω_3 we get

$$p_{+-} = -\left(\frac{1}{2\pi}\right)^2 \frac{1}{A\sigma_2} \int_{x_1=o}^{\infty} \int_{x_3=-\infty}^{o} \int_{-\infty}^{+\infty} x_3 \exp\left(-\frac{x_3^2}{2A^2}\right) \exp\left[-\frac{C^2\omega_1^2}{2} - j\omega_1(x_1 - Bx_3)\right] d\omega_1 dx_1 dx_3 \qquad \text{(III-2-37)}$$

Integrating over ω_1 we get

$$p_{+-} = -\frac{1}{(2\pi)^{\frac{3}{2}}\,\sigma_2 AC} \int_{x_3=-\infty}^{o} \int_{x_1=o}^{\infty} x_3 \exp\left(-\frac{x_3^2}{2A^2}\right) \exp\left[-\frac{(x_1 - Bx_3)^2}{2C^2}\right] dx_1 dx_3$$

(III-2-38)

Integrating over x_1 we get

$$p_{+-} = -\frac{1}{2\pi\sigma_2 A}\int_{x_3=-\infty}^{0} x_3 \exp\left(-\frac{x_3^2}{2A^2}\right) Q\left(-\frac{B}{C}x_3\right) dx_3 \qquad \text{(III-2-39)}$$

From (III-2-23) we get immediately

$$p_{++} = +\frac{1}{2\pi\sigma_2 A}\int_{x_3=0}^{\infty} x_3 \exp\left(-\frac{x_3^2}{2A^2}\right) Q\left(-\frac{B}{C}x_3\right) dx_3 \qquad \text{(III-2-40)}$$

We can check these results in the following way

$2(p_{+-}+p_{++})\, d\tau = P\left\{ y_1(t) \text{ has a zero-crossing between } t_1+\tau \text{ and } t_1+\tau+d\tau \right\}$

$$= \frac{d\tau}{\pi\sigma_2 A}\int_{0}^{\infty} x_3 \exp\left(-\frac{x_3^2}{2A^2}\right)\left[Q\left(-\frac{B}{C}x_3\right)+Q\left(\frac{B}{C}x_3\right)\right] dx_3 \qquad \text{(III-2-41)}$$

$$= \frac{d\tau}{2\pi\sigma_2 A}\int_{-\infty}^{\infty} |x_3| \exp\left(-\frac{x_3^2}{2A^2}\right) dx_3 \qquad \text{(III-2-42)}$$

On the other hand Rice gives for this probability[12-p.190]

$$d\tau \int_{-\infty}^{+\infty} |x_3|\, f(0, x_3)\, dx_3 \qquad \text{(III-2-43)}$$

where f is the joint density of x_2 and x_3; we write it down at once:

$$f(x_2, x_3) = \frac{1}{2\pi\sigma_2\sigma_3\sqrt{1-\dfrac{R_{23}^2}{\sigma_2^2\sigma_3^2}}} \exp\left[-\frac{\dfrac{x_2^2}{\sigma_2^2} - 2\dfrac{R_{23}}{\sigma_2^2\sigma_3^2}x_2x_3 + \dfrac{x_3^2}{\sigma_3^2}}{2\left(1-\dfrac{R_{23}^2}{\sigma_2^2\sigma_3^2}\right)}\right] \qquad \text{(III-2-44)}$$

where the parameters are given by (III-2-30). Therefore

$$f(0,x_3) = \frac{1}{2\pi\sigma_2 A} \exp\left(-\frac{x_3^2}{2A^2}\right) \qquad \text{(III-2-45)}$$

If we use this expression in (III-2-43) we get (III-2-42).

(III-2-42) gives us a relation between p_{+-} and p_{++}, namely

$$p_{+-} + p_{++} = \frac{A}{2\pi\sigma_2} = \frac{\sqrt{1-r_{23}^2}}{2\pi} \frac{\sigma_3}{\sigma_2} \qquad \text{(III-2-46)}$$

Going back to (III-2-38) we can write

$$p_{+-} - p_{++} = -\frac{1}{(2\pi)^{\frac{3}{2}}\sigma_2 AC} \int_{x_1=o}^{\infty} e^{-\frac{x_1^2}{2C^2}\left(1+\frac{A^2B^2}{C^2}\right)^{-1}} dx_1 \,.$$

$$\int_{x_3=-\infty}^{+\infty} x_3\, e^{-\frac{1}{2A^2}\left(1+\frac{A^2B^2}{C^2}\right)\left(x_3 - \frac{A^2B}{C^2}\frac{x_1}{1+\frac{A^2B^2}{C^2}}\right)^2} dx_3$$

$$= -\frac{1}{2\pi\sigma_2 C} \frac{A^2B}{C^2}\left(1+\frac{A^2B^2}{C^2}\right)^{-\frac{3}{2}} \int_{x_1=o}^{\infty} x_1 e^{-\frac{x_1^2}{2C^2}\left(1+\frac{A^2B^2}{C^2}\right)^{-1}} dx_1$$

$$= -\frac{A^2B}{2\pi\sigma_2 C} \frac{1}{\sqrt{1+\frac{A^2B^2}{C^2}}} \qquad \text{(III-47→51)}$$

and from (III-2-35 through 37) we get:

$$p_{+-} - p_{++} = -\frac{1}{2\pi\sigma_2^2} \frac{\sigma_2^2 R_{13} - R_{12}R_{23}}{\sqrt{\sigma_1^2\sigma_2^2 - R_{12}^2}} \qquad \text{(III-2-52)}$$

There is no time delay thus we let $t_1 = t_2 = t$. From (III-2-30) we get (neglecting the terms of order higher than one)

$$\sigma_2^2 R_{13} - R_{12}R_{23} = \Omega\sigma_x^4 \tau \left\{ \frac{4}{\pi^2} \sin^2 t \cos^2 t - \right.$$

$$\left. \left(1-\frac{2}{\pi}\sin^2 t\right)\left[\left(1-\frac{2}{\pi}\right)\frac{1}{3} + \frac{2}{\pi}\sin^2 t\right] \right\}$$

$$= -\Omega\, \sigma_x^4\, \frac{\tau}{3} (1 - \frac{2}{\pi})\, (1 + \frac{4}{\pi} \sin^2 t\,) \qquad \text{(III-2-53)}$$

and

$$\sigma_2^{-2} = \sigma_x^2\, (1 - \frac{2}{\pi} \sin^2 t)^{-1}\, (1 + \tau \frac{4}{\pi}\, \frac{\sin t \cos t}{1 - \frac{2}{\pi} \sin^2 t}) \qquad \text{(III-2-54)}$$

To evaluate the square root we must keep the terms up to the order two; from (III-2-25) we get

$$\sigma_1^2\, \sigma_2^2 - R_{12}^2 = \sigma_x^4\, \{\, (1 - \frac{2}{\pi} \sin^2 t)\, [\, 1 - \frac{2}{\pi} (\sin t + \tau \cos t - \frac{\tau^2}{2} \sin t)^2\,]$$

$$-[\, (1 - \frac{2}{\pi})\, (1 - \frac{\tau^2}{6}) + \frac{2}{\pi}\, (\cos^2 t - \tau \sin t \cos t - \frac{\tau^2}{2} \cos^2 t)\,]^2\, \}$$

$$= \sigma_x^4\, \tau^2\, \{\, (1 - \frac{2}{\pi} \sin^2 t)\, [\, \frac{2}{\pi} \sin^2 t + \frac{1}{3} (1 - \frac{2}{\pi})\,] - \frac{4}{\pi^2} \sin^2 t \cos^2 t\, \}$$

$$= \sigma_x^4\, \tau^2\, \frac{1}{3} (1 - \frac{2}{\pi})\, (1 + \frac{4}{\pi} \sin^2 t)$$

or

$$\sqrt{\sigma_1^2\, \sigma_2^2 - R_{12}^2} = \sigma_x^2\, |\tau| \sqrt{\frac{1}{3} (1 - \frac{2}{\pi})\, (1 + \frac{4}{\pi} \sin^2 t)} \qquad \text{(III-2-55)}$$

As a check we can see that the coefficient of $\sigma_x^4\, \tau^2$ is always positive as expected since we have to take the square root. Now (III-2-52) becomes

$$p_{+-} - p_{++} = W \frac{\sqrt{\frac{1}{3} (1 - \frac{2}{\pi})\, (1 + \frac{4}{\pi} \sin^2 \Omega t)}}{1 - \frac{2}{\pi} \sin^2 \Omega t}\, (1 + \Omega \tau \frac{4}{\pi}\, \frac{\sin \Omega t \cos \Omega t}{1 - \frac{2}{\pi} \sin^2 \Omega t})\ \mathrm{sgn}\, \tau \qquad \text{(III-2-56)}$$

Now we can return to (III-2-19). We shall consider two cases. For low channel signal-to-noise ratio ($\sigma_n > 1$)

$$R_{y_2 y_4}(t, t) \cong 2 \sqrt{\frac{2}{\pi}}\, \{\, \int_{-T}^{+T} (p_{+-} - p_{++})\, \frac{\tau}{T \sigma_n}\, d\tau + [\, \int_{-\tau}^{0} (p_{+-} - p_{++}) d\tau$$

$$- \int_0^T (p_{+-} - p_{++})\, d\tau\,]\, \frac{1}{\sigma_n}\, \} + \sqrt{\frac{2}{\pi}}\, \frac{1}{\sigma_n} \qquad \text{(III-2-57)}$$

Then by (III-2-56)

$$R_{y_2y_4}(t, t) = \frac{1}{\sigma_n} \sqrt{\frac{2}{\pi}} \left[1 - \frac{1}{2b} \frac{\sqrt{\frac{1}{3}(1 - \frac{2}{\pi})(1 + \frac{4}{\pi}\sin^2 \Omega t}}{(1 - \frac{2}{\pi}\sin^2 \Omega t)} \right] \quad \text{(III-2-58)}$$

where

$$b = \frac{B}{W} \quad \text{(III-2-59)}$$

is the bandwidth expansion factor for optimum signals. By (III-1) we get

$$a^{-1} = \max_t \left\{ 1 - \frac{1}{\sigma_n^2} \frac{2}{\pi} \left[1 - \frac{1}{2b} d(t) \right] \right\} \quad \text{(III-2-60)}$$

where

$$d(t) = \frac{\sqrt{\frac{1}{3}(1 - \frac{2}{\pi})(1 + \frac{4}{\pi}\sin^2 \Omega t)}}{(1 - \frac{2}{\pi}\sin^2 \Omega t)} \quad \text{(III-2-61)}$$

The maximum of d(t) is 1.44, therefore

$$a = 1 + \frac{1}{\sigma_n^2} \frac{2}{\pi} (1 - \frac{1.44}{b}) \quad \text{(III-2-62)}$$

and

$$a_o = \frac{1}{\sigma_n^2} \frac{2}{\pi} (1 - \frac{1.44}{b}) \quad \text{(III-2-63)}$$

On the other hand if we average a^{-1} over t in (III-2-60) (which corresponds to averaging the mean-square-error at the receiver output) we get (average of d(t) is .767)

$$a_{ave} = 1 + \frac{1}{\sigma_n^2} \frac{2}{\pi} (1 - \frac{.767}{b}) \quad \text{(III-2-64)}$$

and

$$a_{o,ave} = \frac{1}{\sigma_n^2} \frac{2}{\pi} (1 - \frac{.767}{b}) \quad \text{(III-2-65)}$$

We still need the expression of the channel signal-to-noise ratio (which is not σ_n^{-2} because of the finite bandwidth). By (III-2-4 and 5) the signal power at the channel output is

$$E\{y_c^2(t)\} = \int_{-T}^{+T} \frac{\tau^2}{T^2}\, P\{y_2(t) \text{ has a zero-crossing between } t+\tau \text{ and } t+\tau+d\tau\} + 1\cdot P\{y_2(t) \text{ has no zero-crossing in } (t-T,\ t+T)\}$$

$$= T^{-2}\int_{-T}^{+T} 2(p_{+-}+p_{++})\,\tau^2\, d\tau + 1 - \int_{-T}^{+T} 2(p_{+-}+p_{++})\, d\tau \qquad \text{(III-2-66)}$$

By (III-2-30 and 46)

$$2(p_{+-}+p_{++}) = \frac{A}{\pi\sigma_2}$$

$$= 2Wd(t)\left[1 + \frac{8}{\pi}\tau\, \frac{\sin\Omega t\, \cos\Omega t\,(1+\frac{1}{\pi}\sin^2\Omega t)}{(1+\frac{4}{\pi}\sin^2\Omega t)(1-\frac{2}{\pi}\sin^2\Omega t)}\right] \qquad \text{(III-2-67)}$$

Therefore

$$E\ y_c^2(t) = 1 - 2Wd(t)\frac{4T}{3}$$

$$= 1 - \frac{2}{3}\frac{d(t)}{b} \qquad \text{(III-2-68)}$$

$d(t)$ is plotted on Fig. III-3 for the first half of a Nyquist interval (the function is even about the center). Its average is .767 and therefore the average signal power at the channel output is

$$S = 1 - .511\, b^{-1} \qquad \text{(III-2-69)}$$

Now we shall define the channel signal-to-noise ratio as (+)

$$a_c = \frac{S}{N_o W} \qquad \text{(III-2-70)}$$

$$= \frac{S}{\sigma_n^2}\, b = \frac{1}{\sigma_n^2}(b - .511)$$

Then (III-2-62) becomes

$$a_o = \frac{2}{\pi}\ \frac{1 - 1.44\, b^{-1}}{b - .511}\ a_c\,. \qquad \text{(III-2-71)}$$

(+) Such a definition was not possible in section 1 because of the "infinite" bandwidth.

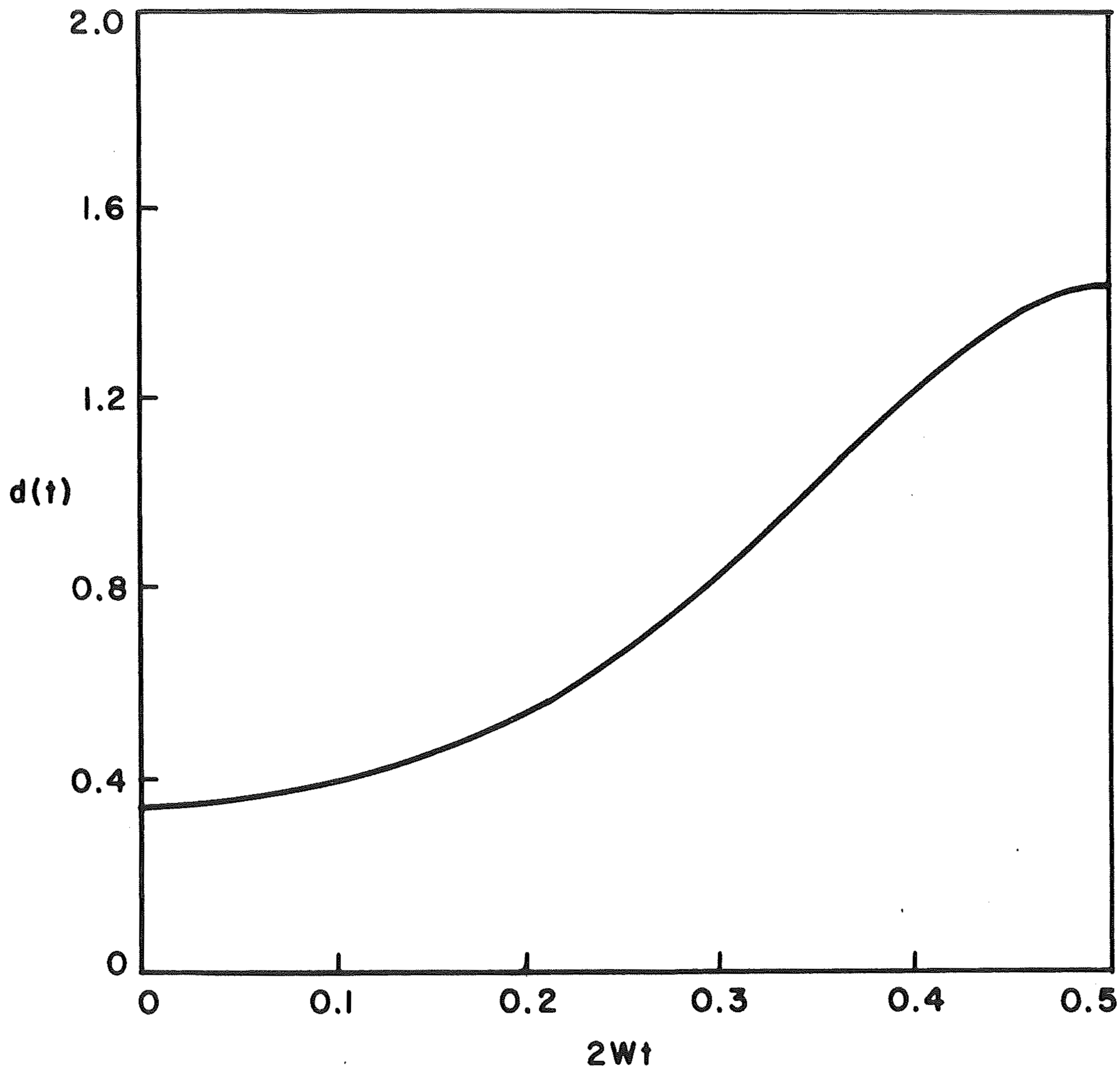

Fig. III-3 Normalized instantaneous zero-crossing density of the optimum process

$$d(t) = \frac{\lambda(t)}{2W} = \frac{\sqrt{\frac{1}{3}\left(1-\frac{2}{\pi}\right)\left(1+\frac{4}{\pi}\sin^2 \Omega t\right)}}{1-\frac{2}{\pi}\sin^2 \Omega t}$$

and (III-2-64)

$$a_{o,ave} = \frac{2}{\pi} \frac{a_c}{b - .511} \left(1 - \frac{.767}{b}\right) \qquad \text{(III-2-72)}$$

The output signal - to noise ratio is plotted on fig. III-4 for b=4 and 9. (III-2-71 and 72 gives essentially the same results).

Going back to (III-2-67) we can see that the instantaneous zero-crossing density is 2Wd(t). Thus its average is .767(2W). But we know that the zero-crossing density for optimum signals should be 2W. This discrepancy follows from out main hypothesis in this section: actually an optimum signal is not a Gaussian process. The difference between these two results (23 %) is in some way a measure of the relevance of the hypothesis.

Finally we consider the large channel signal-to-noise ratio case ($\sigma_n < 1$). From (III-2-19)

$$R_{y_2 y_4}(t,t) = 1 - 2Q(\sigma_n^{-1}) + 2\left\{ \int_{-T}^{+T} (p_{+-} - p_{++}) \left[1 - 2Q\left(\frac{\tau}{T\sigma_n}\right)\right] d\tau \right. \qquad \text{(III-2-73)}$$

$$\left. + \left[\int_{-T}^{0} (p_{+-} - p_{++}) d\tau - \int_0^T (p_{+-} - p_{++}) d\tau\right]\left[1 - 2Q(\sigma_n^{-1})\right] \right\}$$

and by (III-2-56)

$$R_{y_2 y_4}(t,t) = 1 - 2Q(\sigma_n^{-1}) - \frac{2d(t)}{b}\left[\sigma_n \int_0^{\frac{1}{\sigma_n}} Q(u)du - Q(\sigma_n^{-1})\right] \qquad \text{(III-2-74)}$$

Integrating by parts

$$\int_0^{\frac{1}{\sigma_n}} Q(u)du = \frac{1}{\sigma_n} Q(\sigma_n^{-1}) + \frac{1}{\sqrt{2\pi}} \int_0^{\frac{1}{\sigma_n}} u \exp\left(-\frac{u^2}{2}\right) du$$

$$= \sigma_n^{-1} Q(\sigma_n^{-1}) + \frac{1}{\sqrt{2\pi}}\left[1 - \exp(-\sigma_n^{-2})\right] \qquad \text{(III-2-75)}$$

Therefore since

$$Q(x) \approx \frac{1}{\sqrt{2\pi}\, x} \exp\left(-\frac{x^2}{2}\right) \qquad \text{(III-2-78)}$$

$$R_{y_2 y_4}(t,t) \approx 1 - \sqrt{\frac{2}{\pi}}\, \sigma_n \exp\left(-\frac{1}{2\sigma_n^2}\right)\left[1 - \frac{d(t)}{b}\right] - \frac{d(t)}{b}\, \sigma_n \sqrt{\frac{2}{\pi}} \qquad \text{(III-2-79)}$$

Averaging over time for comparison with experimental results we get

$$a^{-1} \approx 2\sqrt{\frac{2}{\pi}}\,\sigma_n \left[\left(1 - \frac{.767}{b}\right) e^{-\frac{1}{2\sigma_n^2}} + \frac{.767}{b} \right] \tag{III-2-80}$$

and by (III-2-70)

$$a^{-1} = \frac{2}{b}\sqrt{\frac{2}{\pi}\,\frac{b-.511}{a_c}} \left[(b - .767)\, e^{-\frac{a_c}{2(b-.511)}} + .767 \right] \tag{III-2-81}$$

where $a_c = S/N_oW$

As $b \rightarrow \infty$

$$a^{-1} \rightarrow 2\sqrt{\frac{2}{\pi}}\,\sigma_n\, e^{-\frac{1}{2\sigma_n^2}}$$

i.e., (III-1-27) as expected.

$a_o = a - 1$ is plotted on Fig. III-4 again for $b = 4$ and 9. For comparison with the asymptotic behavior a_o is also plotted vs S/N_oB on Fig. III-2.

In this section we have considered the transmission of the clipped version of an optimum signal through the channel $\frac{\sin \omega T}{\omega T}$. We have assumed white, additive, Gaussian noise, independent of the signal, at the channel input. We have shown that the hypothesis of one zero-crossing in any time interval of width 2T was reasonable for the bandwidth expansion considered. These hypothesis leads to an expression for the cross-correlation function between clipped optimum signal and output of the clipper which depends on the correlation function of the clipped optimum signal and the probabilities of the optimum process to take a positive or negative value at t, and to have a downward or upward zero-crossing in the interval $(t_2 + \tau,\ t_2 + \tau + d\tau)$. To go further we have used these probabilities for Gaussian processes as an approximation. In the expression for a^{-1} "finite" channel bandwidth adds a term in $a_c^{-\frac{1}{2}}$ to the exponential term obtained in Section 1.

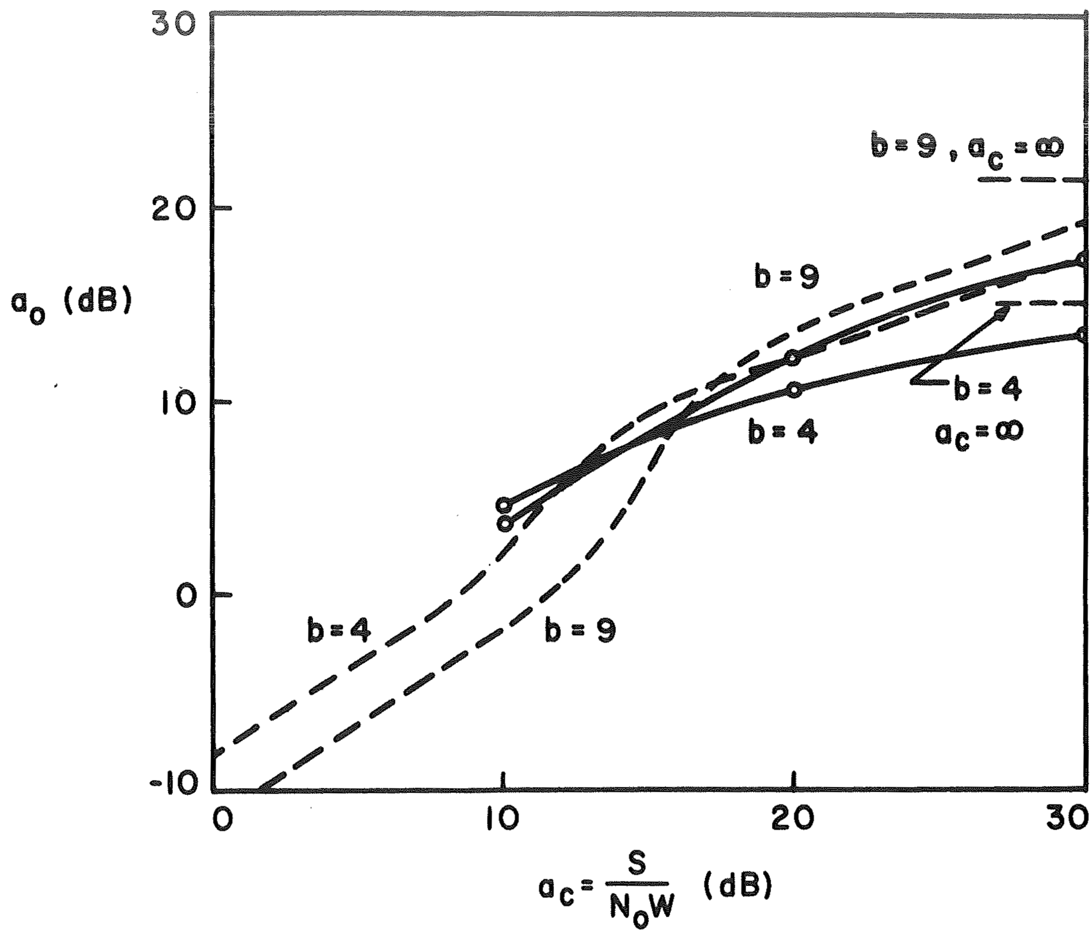

Fig. III-4

Output signal-to-noise ratio for the transmission of the clipped optimum process. $b = B/W$.

Plain curves: experimental results for $H_c(\omega) = \pi\, p_{2\pi B}(\omega)$

Dashed curves: theoretical results for $H_c(\omega) = \dfrac{\sin \omega T}{\omega T}$

(Parameters for the experimental results: $N_4 = 10$, $N_5 = 5$ (b=9) or 12 (b=4), $N_6 = 250$, $L = 8$).

3. Effect of the channel $H_c(\omega) = \pi\, p_{2\pi B}^{(\omega)}$ on clipped optimum signals

For the previous channel the probability of a zero-crossing displacement in the noise-free conditions is small provided $B \gg W$. The channel we consider now displaces all the zero-crossings by an amount decreasing as the bandwidth increases.

Here the channel output is

$$y_3(t) = \int_{-\infty}^{+\infty} y_2(\tau) \frac{\sin 2\pi B(t-\tau)}{(t-\tau)}\, d\tau + n(t) \qquad \text{(III-3-1)}$$

$y_2(t)$ is the clipped version of the optimum process

$$y_1(t) = Z \sum_{k=-\infty}^{+\infty} (-1)^k |x_k| \psi_k(t) \tag{III-3-2}$$

It has a zero-crossing between two consecutive samples. If we denote by z_k the zero-crossing between k/2W and (k+1)/2W

$$y_2(t) = Z(-1)^{k+1} \quad \text{for} \quad z_k < t < z_{k+1} \tag{III-3-3}$$

Therefore

$$y_3(t) = Z \sum_{k=-\infty}^{+\infty} (-1)^{k+1} \int_{z_k}^{z_{k+1}} \frac{\sin 2\pi B(t-\tau)}{t-\tau} d\tau + n(t)$$

$$= Z \sum_k (-1)^{k+1} \left\{ Si\left[2\pi B(z_{k+1}-t)\right] - Si\left[2\pi B(z_k - t)\right] \right\} + n(t) \tag{III-3-4}$$

$$= 2Z \sum_k (-1)^k Si\left[2\pi B(z_k - t)\right] + n(t) \tag{III-3-5}$$

Now

$$R_{y_2 y_4}(t_1, t_2) = \frac{1}{j\pi} \int_{-\infty}^{+\infty} \omega^{-1} E\left\{ y_2(t_1) e^{2j\omega Z \sum_{k=-\infty}^{+\infty} (-1)^k Si[2\pi B(z_k - t_2)]} \right\} e^{-\frac{\sigma_n^2 \omega^2}{2}} d\omega \tag{III-3-6}$$

or with the characteristic function

$$\Phi(\omega', \omega_k; t_1, t_2) = E\left\{ \exp\left[j\left(\omega' y_2(t_1) + \sum_{k=-\infty}^{+\infty} \omega_k Si\ 2\pi B(z_k - t_2) \right)\right]\right\} \tag{III-3-7}$$

$$R_{y_2 y_4}(t_1, t_2) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega^{-1} \left\{ \left[\frac{\partial \Phi(\omega', \omega_k)}{\partial \omega'} \right]_{\substack{\omega' = 0 \\ \omega_k = 2\omega(-1)^k}} + \left[\frac{\partial \Phi(\omega', \omega_k)}{\partial \omega'} \right]_{\substack{\omega' = 0 \\ \omega_k = 2\omega(-1)^{k+1}}} \right\} e^{-\frac{\sigma_n^2 \omega^2}{2}} \omega \tag{III-3-8}$$

The density required to solve the problem is as in Section 2 of the type

$$f[y_1(t_1), y_1(t_k), y_1'(t_k); k = -\infty, \ldots, +\infty] \tag{III-3-9}$$

where actually for practical purpose the number of indices k must be finite.

Going back to (III-3-4) it appears that we have to consider at least the zero-crossing in the interval which contains t and the zero-crossing in the interval to the left or to the right. This already requires the density

$$f[y_1(t_1), y_1(t_k), y_1'(t_k), y_1(t_{k\pm 1}), y_1'(t_{k\pm 1})]$$

and, say,

$$\frac{2\pi B}{2W} = \pi b \approx 15 \quad \text{or} \quad b \approx 5 \tag{III-3-10}$$

This last condition is required to neglect the other terms in the series (III-3-4)(+). Therefore in conditions similar to those encountered in Section 2 we need the density of the optimum process at three different instants of time and its derivatives at two of these instants. Thus the analysis is even more difficult and will not be considered here.

However we can quote experimental results for this case. In chapter IV we shall see that the communication of a signal (Gaussian, zero-mean, bandlimited with a flat power spectrum) by the clipped version of the optimum process can be easily simulated when the channel is $\pi p_{2\pi B}(\omega)$. In the simulation we also compare the two random square waves $y_2(t)$ and $y_4(t) = \hat{y}_2(t)$ at the transmitter and receiver (see Fig. IV-2). The signal-to-mean-square-error ratio for the random square wave transmission is given by

$$a^{-1} = 1 - \left[\frac{1}{T}\int_0^T y_2(t)\,\hat{y}_2(t)\,dt\right]^2 \tag{III-3-11}$$

(+) for $t > 15$ the difference between maxima and minima of $Si(t)$ is less than 5% of $Si(\infty) = \frac{\pi}{2}$

As described in (IV-4d) the device which follows the channel is more than a clipper: it detects one zero in each Nyquist interval. Then

$$\int_o^T y_2(t)\,\hat{y}_2(t) = \sum_k \left(1 - 2\left|z_{ki} - z_{ko}\right|\right) \qquad \text{(III-3-12)}$$

where the second index indicates that z_k is the position of the zero-crossing in the k-th interval at the system input or output, and 2W has been set equal to 1 without loss of generality. Finally since T = N (number of zero-crossings in the time interval T)

$$a^{-1} = 1 - \left(1 - \frac{2}{N}\sum_{k=1}^{N}\left|z_{ki} - z_{ko}\right|\right)^2 \qquad \text{(III-3-13)}$$

The experimental results are plotted on Fig. III-4.

CHAPTER IV.

PERFORMANCES OF THE COMMUNICATION SYSTEM.

In this chapter we shall investigate the properties of the communication system shown in fig. IV-2. We shall again assume that the signal to be sent, $x(t)$, is a stationary, zero-mean, bandlimited, Gaussian process with flat power spectrum in $(-W, +W)$. $x(t)$ is first mapped into an optimum signal $y_1(t)$. Fig. IV-3 shows a possible implementation of the black-box labeled "mapping". The clipped version of the optimum process is sent through the channel $H_{c1}(\omega) = \frac{\sin \omega T}{\omega T}$ for which we have just given the analysis.* In the receiver a computer estimates the optimum process from the received zero-crossings by means of the algorithm previously described. Finally by inverse mapping an estimate of the original signal is found. Fig. IV-4 gives a block-diagram for the inverse mapping. The samples sign required for the inverse mapping are sent by means of the signal $z_1(t) = \sum \mathrm{sgn}\,(x_k)\psi_k(t)$ through the channel $H_{c2}(\omega) = p_\Omega(\omega)\Omega^{-1}$**. A_1 and A_2 are two amplifiers which set the two transmitted signals at an appropriate level. The ratio A_1/A_2 should be such that the mean-square-error at the receiver output is minimized while the total power of the two signals is fixed. Fig. IV-5 suggests a way to generate the signal $z_1(t)$ which carries the sample signs.

We shall first derive a relation (for high channel signal-to-noise ratio) between $R_{y_2y_4}(t,t)$ and $R_{x\hat{x}}(t,t)$ for $t=k/2W$. $R_{y_2y_4}(t,t)$ has been derived in III-2. From these results we shall be able to find the signal-to-noise ratio at the receiver output for optimum signals, then to generalize to any bandlimited signal.

1. Fundamental relation for high channel signal-to-noise ratio.

$y_2(t)$ is the clipped version of the optimum signal $y_1(t)$, therefore we can write

$$y_2(t) = \frac{1}{\pi j} \int_{-\infty}^{+\infty} \omega^{-1} e^{j\omega y_1(t)} d\omega \qquad \text{(IV-1-1)}$$

---

* However, $H_{c1}(\omega) = \pi\, p_{2\pi B}(\omega)$ will be assumed in the experimental investigation.

**Since the two signals $y_2(t)$ and $z_1(t)$ have to be multiplexed before modulation, this is justified if $z_1(t)$ is allowed to occupy the low frequency part of the complete channel.

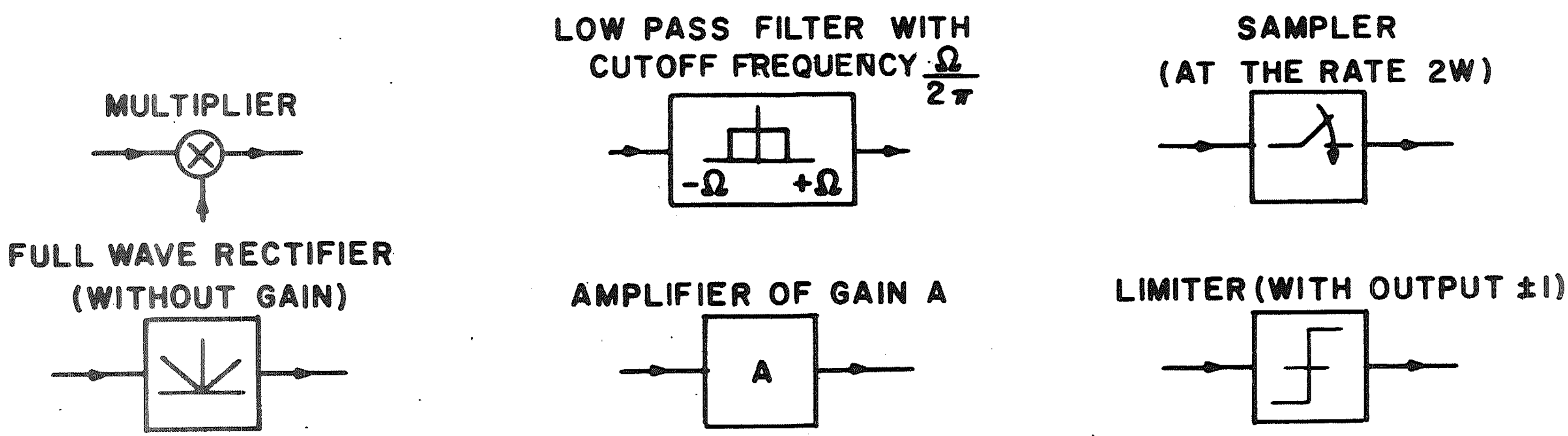

Fig. IV - 1. Symbols used in Figure IV - 2 thru 5

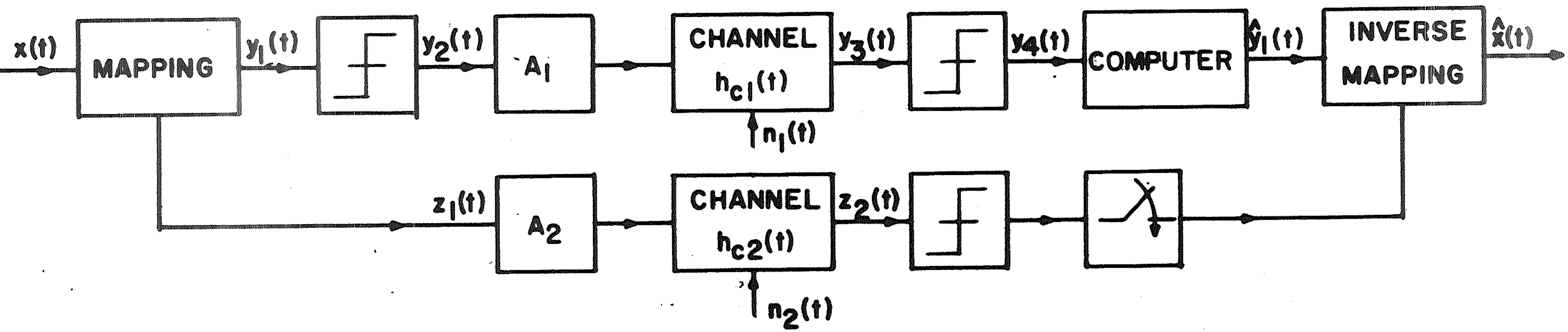

Fig. IV - 2. The Communication System

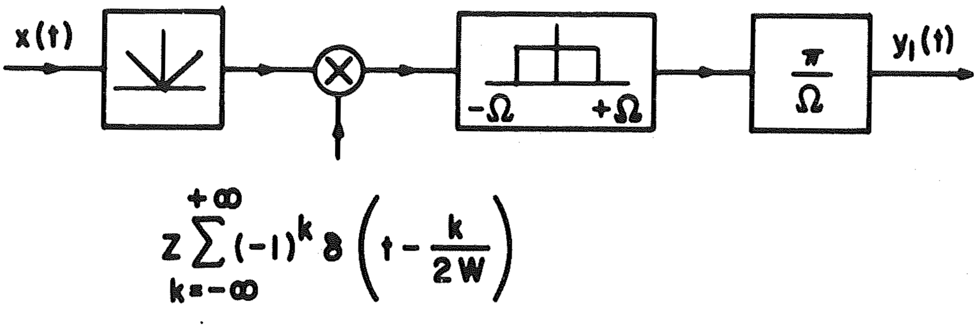

Fig. IV - 3. Mapping

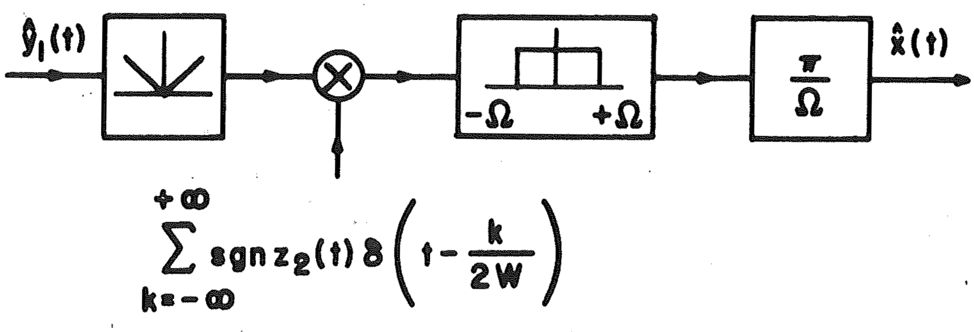

Fig. IV - 4. Inverse Mapping

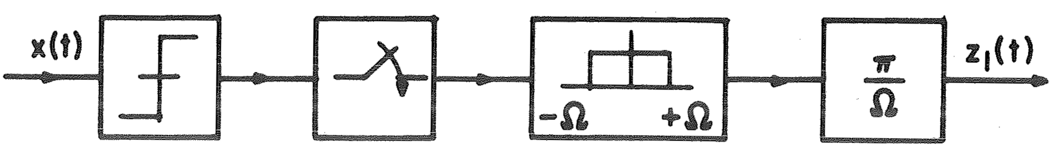

Fig. IV - 5. Generation of the Signal which Carries the Sample Sign

Similarly $y_4(t)$ is the clipped version of the approximation $\hat{y}_1(t)$ given by the computer ($y_4(t)$ and $y_1(t)$ have the same zero-crossings) therefore*

$$y_4(t) = \frac{1}{\pi j} \int_{-\infty}^{+\infty} \omega^{-1} e^{j\omega y_1(t)} d\omega \qquad \text{(IV-1-2)}$$

It follows that

$$R_{y_2 y_4}(t_1, t_2) = -\frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\omega_1 \omega_2)^{-1} \Phi_{y_1 \hat{y}_1}(\omega_1, \omega_2) d\omega_1 d\omega_2 \qquad \text{(IV-1-3)}$$

where $\Phi_{y_1\hat{y}_1}(\omega_1, \omega_2)$ is the joint characteristic function of the random variables $y_1(t_1)$ and $\hat{y}_1(t_2)$. We shall have to remember that in this expression ω^{-1} must be interpreted as a distribution since we have represented the clipper by a Fourier transform(16 - Appendix 1). (IV-1-3) is valid for any pair (t_1, t_2). However, to go further we must consider $t_1 = t_2 = k/2W$. In I-2 we have seen that $y_1(k/2W)$ are normally distributed random variables with mean zero and variance σ_x^2. On the other hand, for high channel signal-to-noise ratio $\hat{y}_1(t)$ is a close approximation of $y_1(t)$, therefore $\hat{y}_1(k/2W)$ is almost Gaussian. We shall assume in the following that the two random variables $y_1(k/2W)$ and $\hat{y}_1(k/2W)$ are jointly Gaussian, i.e.,

$$\Phi_{y_1\hat{y}_1}(\omega_1, \omega_2) = \exp\left[-\frac{\sigma_{y_1}^2 \omega_1^2 + 2R_{y_1\hat{y}_1}(t_1, t_2)\omega_1\omega_2 + \sigma_{\hat{y}_1}^2 \omega_2^2}{2}\right] \qquad \text{(IV-1-4)}$$

Then plugging (IV-1-4) in (IV-1-3) and expanding $\exp[-R_{y_1\hat{y}_1}(k/2W, k/2W)\omega_1\omega_2]$ in series we get

$$R_{y_2 y_4}(k/2W, k/2W) = -\frac{1}{\pi^2} \sum_{n=0}^{\infty} \frac{\left[-2r_{y_1\hat{y}_1}(k/2W, k/2W)\right]^n}{n!} \left[\int_{-\infty}^{+\infty} u^{n-1} e^{-u^2} du\right]^2$$

The first term is actually the only one which requires the interpretation of u^{-1} as a distribution:

$$\int_{-\infty}^{+\infty} u^{-1} \exp(-u^2) du = 2\int_{-\infty}^{+\infty} u \ln|u| \exp(-u^2) du = 0 \qquad \text{(IV-1-6)}$$

---

* We can write $y_4(t) = (\pi j)^{-1} \int_{-\infty}^{+\infty} \omega^{-1} \exp(j\omega s(t)) d\omega$ for any signal $s(t)$ which has same zero-crossings as $y_4(t)$. However, this is useful only if $s(t)$ is a close approximation of $y_1(t)$.

Then with $n = 2k + 1$ (all terms with n even vanish)

$$R_{y_2y_4} = \frac{2}{\pi^2} \sum_{k=0}^{\infty} 2^{2k} \frac{r^{2k+1}_{y_1\hat{y}_1}}{(2k+1)!} \left[\int_0^{\infty} u^{k-\frac{1}{2}} e^{-u} du\right]^2 = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{[(2k-1)!!]^2}{(2k+1)!} r^{2k+1}_{y_1\hat{y}_1}$$

$$= \frac{2}{\pi} \sin^{-1}(r_{y_1\hat{y}_1}) \text{ for all } t_1 = t_2 = k/2W \qquad \text{(IV-1-7)}$$

or

$$r_{y_1\hat{y}_1}(k/2W, k/2W) = \sin\left[\frac{\pi}{2} R_{y_2y_4}(k/2W, k/2W)\right] \qquad \text{(IV-1-8)}$$

Therefore from an hypothesis certainly appropriate for high channel signal-to-noise ratio we have been able to derive a relation between an already known quantity and the cross-correlation function between the communication system input and output (when the input belongs to the class of optimum signals). Now since the system has zero time-delay

$$a^{-1}(k/2W) = 1 - r^2_{y_1\hat{y}_1}(k/2W,\ k/2W)$$

$$= \cos^2\left[\frac{\pi}{2} R_{y_2y_4}(k/2W, k/2W)\right] \qquad \text{(IV-1-9)}$$

But at high channel signal-to-noise ratio

$$R_{y_2y_4}(t, t) \approx 1 \qquad \text{(IV-1-10)}$$

and therefore

$$a(k/2W) \cong \frac{4}{\pi^2}\left[1 - R_{y_2y_4}(k/2W, k/2W)\right]^{-2} \text{ for all } k \qquad \text{(IV-1-11)}$$

At this point we should try to find the minimum of the signal-to-mean-square-error ratio. Actually we cannot carry this out. On the other hand when we are dealing with experimental results we have to define a by (see I-3-20)

$$a^{-1} = 1 - \frac{\left[\int_0^T y_1(t)\hat{y}_1(t)dt\right]^2}{\int_0^T y_1^2(t)dt \int_0^T \hat{y}_1^2(t)dt} \qquad \text{(IV-1-12)}$$

(compared to (I-3-20) we have dropped the limit for $T \to \infty$ since, as we shall see, in IV-4, we cannot really achieve very large value of T). Thus a theoretical definition more appropriate for purpose of comparison with experimental data is

$$a^{-1} = \lim_{T \to \infty} \frac{\frac{1}{T}\int_o^T E\left\{\left[y_1(t) - A\hat{y}_1(t)\right]^2\right\} dt}{\frac{1}{T}\int_o^T E\left\{y_1^2(t)\right\} dt} \tag{IV-1-13}$$

By (I-2-13) and from the fact that $y_1(t)$ and $y_1(t)$ are both bandlimited signals

$$a^{-1} = \frac{E\left\{\left[y_{1k} - A\hat{y}_{1k}\right]^2\right\}}{\sigma_x^2 (1 - \frac{1}{\pi})} \tag{IV-1-14}$$

Therefore $\underline{a}$ is maximized by the same A as $a(k/2W)$ and*

$$a_{y_1} \cong (1 - \frac{1}{\pi}) \frac{4}{\pi^2} \left[1 - R_{y_2 y_4}(k/2W, k/2W)\right]^{-2} \tag{IV-1-15}$$

The final goal is to find the performances of the system for the general bandlimited signals. Thus in section 2 and 3 we shall need the expression of $\underline{a}$ for x(t) when there is no error in the transmission of the samples sign:

$$a^{-1} = \frac{E\left\{\left[x(t) - A\hat{x}(t)\right]^2\right\}}{\sigma_x^2} \tag{IV-1-16}$$

Since x(t) and $\hat{x}(t)$ are stationary and bandlimited, we can write

$$a^{-1} = \lim_{T \to \infty} \frac{T^{-1}\int_o^T E\left\{\left[\sum_k x_k \psi_k(t) - A\hat{x}_k \psi_k(t)\right]^2\right\}}{\sigma_x^2}$$

$$= \frac{E\left\{(x_k - A\hat{x}_k)^2\right\}}{\sigma_x^2} \tag{IV-1-17}$$

---

* The subscript associated with $\underline{a}$ underline that this expression is for optimum signals.

Or with the optimum $A = E\{x_k \hat{x}_k\} / E\{\hat{x}_k^2\}$

$$a^{-1} = 1 - \frac{E\{\hat{x}_k^2\}}{E\{x_k^2\}} A^2 = 1 - \frac{E\{\hat{y}_{1k}^2\}}{E\{y_{1k}^2\}} A^2 \qquad \text{(IV-1-18)}$$

If there is no error in the transmission of the samples sign, the gain A is exactly the same as for the optimum processes, i.e.,

$$A = \frac{E\{y_{1k}\hat{y}_{1k}\}}{E\{\hat{y}_{1k}^2\}} \qquad \text{(IV-1-19)}$$

and therefore

$$a_x = \frac{4}{\pi^2}\left[1 - R_{y_2 y_4}(k/2W,\ k/2W)\right]^{-2} \qquad \text{(IV-1-20)}$$

Thus, we have found a relation (IV-1-18) between the cross-correlation of the two random square waves in the system and the cross-correlation of the optimum process and its estimate (this relation holds only at the sampling times and for high channel signal-to-noise ratio). From this relation we have been able to derive the signal-to-mean-square-error ratio for the optimum process and for the general bandlimited signals (however, not taking into account the effect of error in the transmission of the samples sign for the last case).

2. Asymptotic properties

In III-1 we have found that if the channel noise is Gaussian, zero-mean additive, independent of the signal and if the channel bandwidth is such that we can assume negligible distortion of the random square wave $y_2(t)$

$$R_{y_2 y_4}(t, t) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{a_c}} e^{-\frac{u^2}{2}}\, du$$

$$= 1 - 2Q(\sqrt{a_c}) \qquad \text{(IV-2-1)}$$

(Here we have assumed the noise stationary in the wide sense, i.e., the channel signal-to-noise ratio independent of time.) From (IV-1-15) the output signal-to-noise ratio achieved for the class of optimum signals is

$$a_{o,y_1} = a_{y_1} - 1 \approx a_{y_1} \approx \left(1 - \frac{1}{\pi}\left[\pi Q\left(\sqrt{a_c}\right)\right]\right)^{-2}$$

$$\approx \left(1 - \frac{1}{\pi}\right) \frac{2}{\pi} a_c \exp(a_c) = .434\, a_c \exp(a_c) \qquad \text{(IV-2-2)}$$

or in decibels

$$a_{o,y_1}(\text{dB}) = -3.63 + a_c(\text{dB}) + 4.34\, a_c \qquad \text{(IV-2-3)}$$

(IV-2-3) is plotted on fig. IV-6. The exact formula*

$$a_{o,y_1} = \left(1 - \frac{1}{\pi}\right) \cos^{-2}\left\{\frac{\pi}{2}\left[1 - 2Q\left(\sqrt{a_c}\right)\right]\right\} - 1 \qquad \text{(IV-2-4)}$$

does not give significant differences with (IV-2-2): at a_c = 3 dB the difference is only 2 dB.

For the general bandlimited signals we shall proceed in the following way. Noise at the receiver is of two kinds:

i) noise due to imperfect recovery of $y_1(t)$ because the zero-crossings have been displaced by the channel noise and filtering,

ii) noise due to a wrong decision on the sign of the samples of $x(t)$. But $n_1(t)$ and $n_2(t)$ are independent since they are Gaussian, zero-mean and occupy different frequency bands therefore the two noise contributions at the receiver output are themselves independent and the total noise power is merely the sum. The first contribution is given by (IV-1-20).

$$a_1^{-1} = \pi^2 Q^2\left(\sqrt{a_{c1}}\right) \qquad \text{(IV-2-5)}$$

where a_{c1} is the signal-to-noise ratio for the channel which transmits the random square wave. The probability of error for the second channel is**

$$P_e = Q\left(\sqrt{a_{c2}}\right) \qquad \text{(IV-2-6)}$$

---

* which still rely on the hypothesis made in IV-1.

** since $x(t)$ is zero-median.

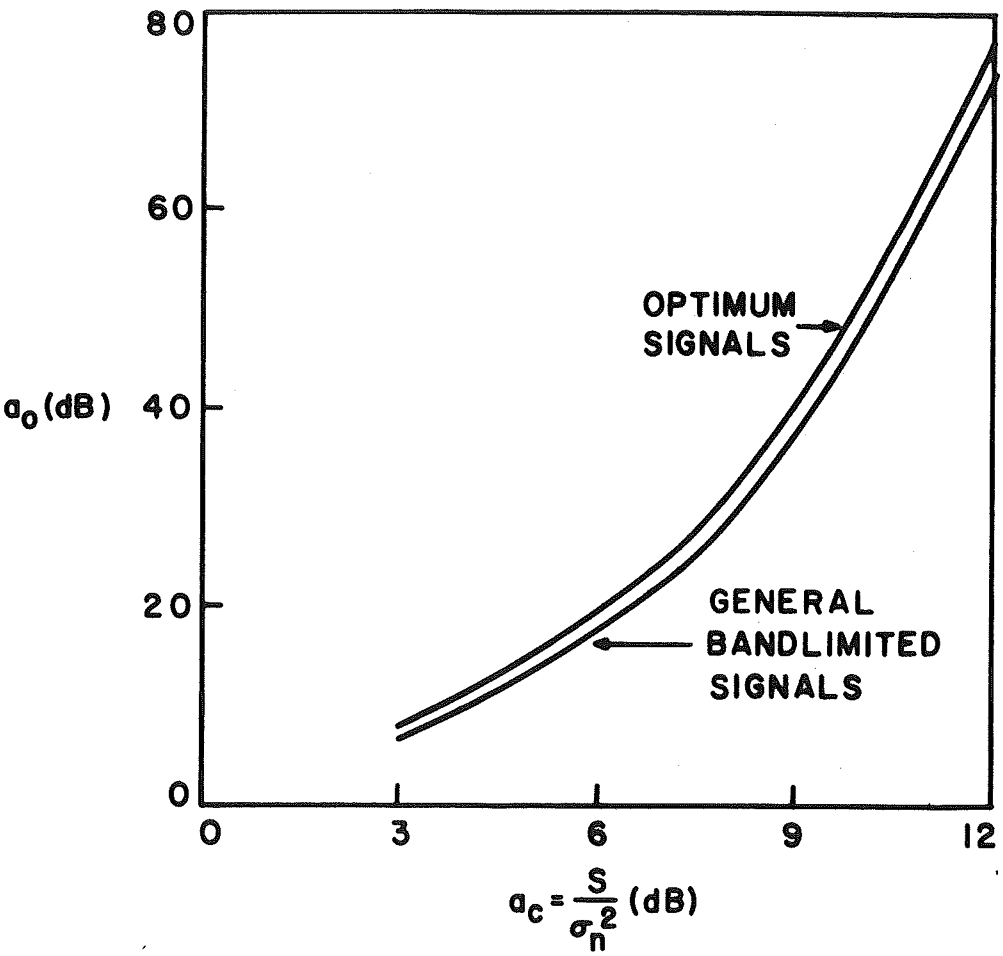

Fig. IV-6

Signal-to-noise ratio at the output of the communication system. (Asymptotic behavior: $\frac{B}{W} \rightarrow \infty$).

where a_{c2} is the signal-to-noise ratio for the channel which transmits the samples sign. If a wrong decision is made at k/2W the resulting additional error at the receiver output is about

$$2x\,(k/2W)\,\psi_k(t) \qquad \text{(IV-2-7)}$$

The corresponding noise power is

$$\lim_{T \to \infty} (2T)^{-1} \int_{-T}^{+T} \Big[\, 2 \sum_k x(k/2W)\, \psi_k(t) \,\Big]^2 dt \qquad \text{(IV-2-8)}$$

where the summation is extended over the samples for which a wrong decision is made. Therefore the second noise contribution is merely about $4P_e\sigma_x^2$ and

$$a_2^{-1} \approx 4P_e \approx \frac{4}{\sqrt{2\pi a_{c2}}} \exp\left(-\frac{a_{c2}}{2}\right) \qquad \text{(IV-2-9)}$$

Finally

$$a^{-1} \approx \frac{4}{\sqrt{2\pi a_{c2}}} \exp\left(-\frac{a_{c2}}{2}\right) + \frac{\pi}{2}\, a_{c1}^{-1} \exp\left(-\, a_{c1}\right) \qquad \text{(IV-2-10)}$$

Maximization of $\underline{a}$ by choice of an optimum sharing of the available power between the two channels leads to a transcendental equation; on the other hand, the optimum sharing is function of the noise level. Thus, it is more convenient to consider the following sub-optimum system

$$a_{c1} = a_{c2}/2. \qquad \text{(IV-2-11)}$$

When A_1 and A_2 have been set at the appropriate value (IV-2-11) is independent of the channel conditions. On the other hand, the main factor in (IV-2-10) is the exponential: by chosing the same exponent the two terms remain of the same order of magnitude. Then

$$a^{-1} \approx \left(\frac{2}{\sqrt{\pi a_{c1}}} + \frac{\pi}{2a_{c1}}\right) \exp\left(-a_{c1}\right) \qquad \text{(IV-2-12)}$$

But

$$a_{c1} = A_1^2/\sigma_{n1}^2 \qquad \text{(IV-2-13)}$$

$$a_{c2} = A_2^2/\sigma_{n2}^2 \tag{IV-2-14}$$

$$a_c = (A_1^2 + A_2^2)/(\sigma_{n1}^2 + \sigma_{n2}^2) \tag{IV-2-15}$$

$$= a_{c1}(1 + \frac{\sigma_{n2}^2}{\sigma_{n1}^2 + \sigma_{n2}^2}$$

and since

$$\sigma_{n2}^2 << \sigma_{n1}^2 \tag{IV-2-16}$$

$$a_c \approx a_{c1} \tag{IV-2-17}$$

Finally

$$a_{o,x}^{-1} \approx a_x^{-1} \approx (2\sqrt{\frac{a_c}{\pi}} + \frac{\pi}{2})\, a_c^{-1} \exp(-a_c) \tag{IV-2-18}$$

(IV-2-18) is plotted on fig. IV-6. The output signal-to-noise ratio is essentially an exponential function of the channel signal-to-noise ratio. This property is related to the transmission of the information by a random square wave and the Gaussian nature of the channel noise.

3. Effect of the Channel $H_c(\omega) = \frac{\sin \omega T}{\omega T}$

For optimum signals and high channel signal-to-noise ratio we have shown in section IV-1 that

$$a_{y_1} \approx (1 - \frac{1}{\pi})\, \frac{4}{\pi^2} \left[1 - R_{y_2 y_4}(k/2W,\ k/2W)\right]^{-2} \tag{IV-3-1}$$

$R_{y_2 y_4}$ is given by (III-2-79) where d(t) (given by (III-2-61) has to be evaluated at k/2W:

$$d(k/2W) = \sqrt{\frac{1}{3}(1 - \frac{2}{\pi})} = .348 \tag{IV-3-2}$$

Therefore

$$R_{y_2 y_4}(k/2W,\ k/2W) \approx 1 - \sqrt{\frac{2}{\pi}}\, \frac{\sigma_{n_1}}{A_1} \left[(1 - \frac{.348}{b}) \exp(-\frac{A_1^2}{2\sigma_{n_1}^2}) + \frac{.348}{b}\right]$$

(IV-3-3)

and

$$a_{y_1} = (1 - \frac{1}{\pi}) \frac{2}{\pi} (\frac{A_1^2}{\sigma_{n_1}^2}) \left[(1 - \frac{.348}{b}) \exp (- \frac{A_1^2}{2\sigma_{nl}^2}) + \frac{.348}{b} \right]^{-2} \tag{IV-3-4}$$

or since by (III-2-70)

$$a_c = \frac{S}{N_o W} = \sigma_{nl}^{-2} (b - .511) A_1^2 \tag{IV-3-5}$$

$$a_{y_1} = (1 - \frac{1}{\pi}) \frac{2}{\pi} \frac{a_c}{b-.511} \left[(1 - \frac{.348}{b}) \exp (- \frac{a_c}{2(b-.511)}) + \frac{.348}{b} \right]^{-2} \tag{IV-3-6}$$

$a_{o, y_1} = a_{y_1} - 1$ vs. a_c is plotted on fig. IV-7 for b = 4 and 9.

For the general bandlimited signals (IV-1-20) gives a first contribution (noise in the recovery of the optimum signal transmitted)

$$a_1^{-1} = \frac{\pi^2}{4} \left[1 - R_{y_2 y_4} (k/2W, k/2W) \right]^2 \tag{IV-3-7}$$

$$= \frac{\pi}{2} (\frac{\sigma_{n_1}^2}{A_1^2}) \left[(1 - \frac{.348}{b}) \exp (- \frac{A_1^2}{2\sigma_{n_1}^2}) + \frac{.348}{b} \right]^2 \tag{IV-3-8}$$

The second contribution due to errors on the sign of the samples of x(t) is given by (IV-2-9)

$$a_2^{-1} = \frac{4}{\sqrt{2\pi}} \frac{\sigma_{n_2}}{A_2} \exp (- \frac{A_2^2}{2\sigma_{n_2}^2}) \tag{IV-3-9}$$

And therefore, the output signal-to-noise ratio for general bandlimited signals is given by

$$a_x^{-1} = \frac{\pi}{2} (\frac{\sigma_{n_1}}{A_1})^2 \left[\frac{\beta - 1.35}{\beta - 1} \exp (- \frac{A_1^2}{2\sigma_{n_1}^2}) + \frac{.348}{\beta - 1} \right]^2$$

$$+ \frac{4}{\sqrt{2\pi}} \frac{\sigma_{n_2}}{A_2} \exp (- \frac{A_2^2}{2\sigma_{n_2}^2}) \tag{IV-3-10}$$

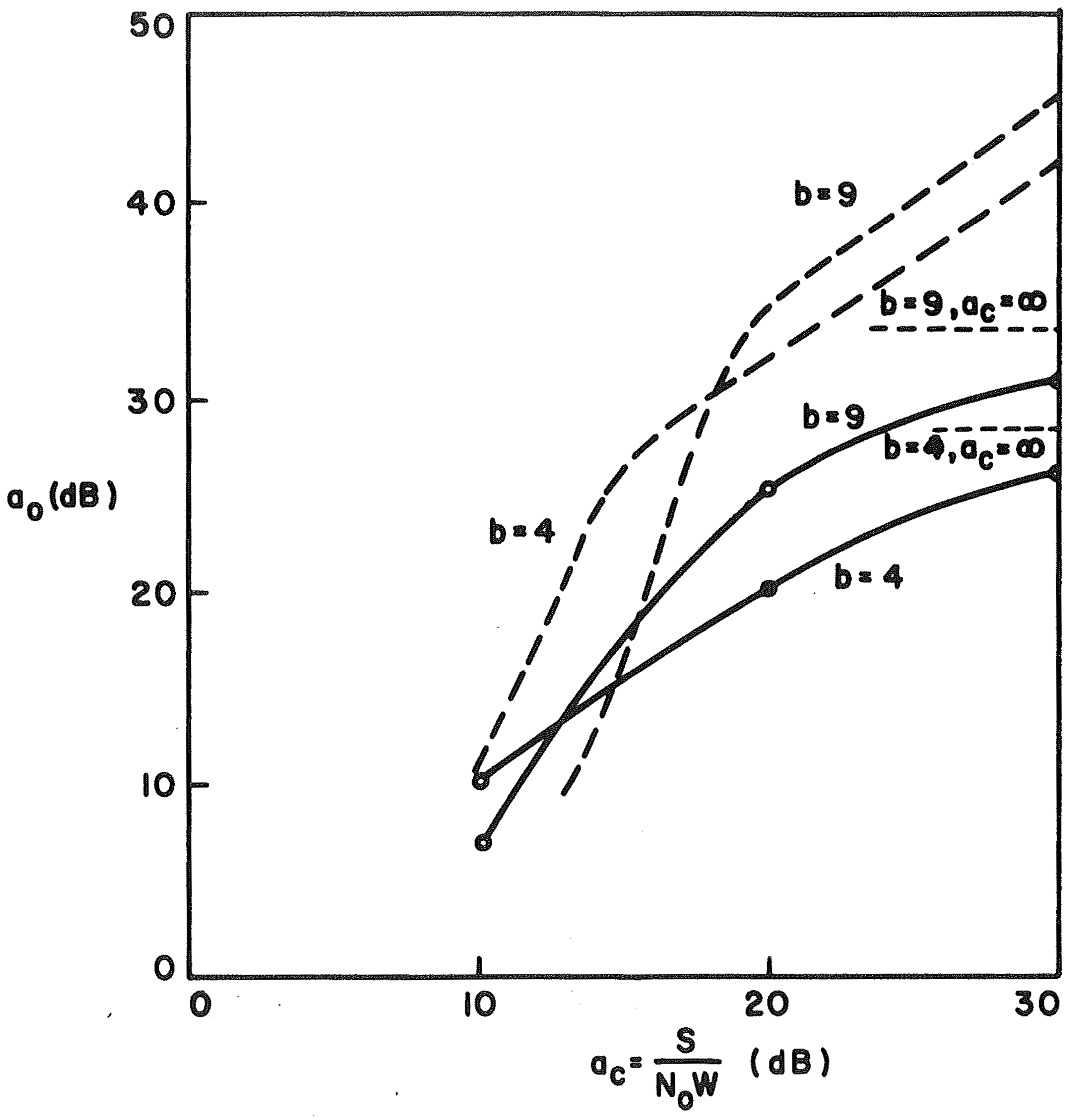

Fig. IV-7

Signal-to-noise ratio at the output of the

Communication System. Optimum signal.

Dotted curves : theoretical results. $H_c(\omega) = \frac{\sin \omega T}{\omega T}$.

Plain curves : experimental results. $H_c(\omega) = \pi p_{2\pi B}(\omega)$.

$b = B/W$.

where β is the bandwidth expansion factor for the complete system (β = b + 1).

Now, to optimize a by appropriate signal power sharing we define the parameter k:

$$\frac{A_1^2}{\sigma_{n_1}^2} = k \frac{A_2^2}{\sigma_{n_2}^2} \qquad \text{(IV-3-11)}$$

Then

$$S = A_1^2 \left(1 - \frac{.511}{b}\right) + A_2^2 = A_2^2 \left[(\beta - 1.51)\, k + 1\right] \qquad \text{(IV-3-12)}$$

and

$$a_c = \frac{S}{N_o W} = \frac{A_2^2}{\sigma_{n_2}^2} \left[(\beta - 1.51)\, k + 1\right] \qquad \text{(IV-3-13)}$$

(IV-3-10) becomes

$$a_x^{-1} = \frac{\pi}{2} \frac{1+k\,(\beta - 1.51)}{k\, a_c} \left[\frac{\beta - 1.35}{\beta - 1} \exp\left(- \frac{k\, a_c}{2\left[1 + k\,(\beta - 1.51)\right]}\right) + \frac{.348}{\beta - 1}\right]^2 + \frac{4}{\sqrt{2\pi}} \sqrt{\frac{1+k\,(\beta - 1.51)}{a_c}} \exp\left(- \frac{a_c}{2\left[1+k(\beta - 1.51)\right]}\right)$$

(IV-3-14)

or with

$$m = \frac{a_c}{1 + k\,(\beta - 1.51)} \qquad \text{(IV-3-15)}$$

$$a_x^{-1} = \frac{\pi}{2km} \left[\frac{\beta - 1.35}{\beta - 1} \exp\left(- \frac{km}{2}\right) + \frac{.348}{\beta - 1}\right]^2 + \frac{4}{\sqrt{2\pi m}} \exp\left(- \frac{m}{2}\right)$$

(IV-3-16)

$$k = \frac{a_c - m}{m(\beta - 1.51)} \qquad \text{(IV-3-17)}$$

To make things easier we shall consider two cases. When a_c is big enough we have

$$a_x^{-1} \approx \frac{\pi}{2km} \left(\frac{.348}{\beta -1}\right)^2 + \frac{4}{\sqrt{2\pi m}} \exp\left(-\frac{m}{2}\right) \qquad \text{(IV-3-18)}$$

and the optimum value of m is given by

$$\frac{\pi}{4} \sqrt{2\pi} \, (\beta - 1.51) \left(\frac{.348}{\beta-1}\right)^2 \frac{m^{\frac{3}{2}}}{m+1} \exp\left(\frac{m}{2}\right) = (a_c - m)^2 \qquad \text{(IV-3-19)}$$

For lower values of a_c where the exponential is the main term in the brackets we choose k = 1/2 which makes the two exponentials in (IV-3-16) identical as in section IV-2. Then

$$a_x^{-1} = \frac{\pi}{2} \frac{\beta - .49}{a_c} \left[\frac{\beta - 1.35}{\beta-1} \exp\left(-\frac{a_c}{2(\beta-.49)}\right) + \frac{.348}{\beta-1}\right]^2$$

$$+ \frac{2}{\sqrt{\pi}} \sqrt{\frac{\beta-.49}{a_c}} \exp\left(-\frac{a_c}{\beta - .49}\right) \qquad \text{(IV-3-20)}$$

$a_{o,x} = a_x - 1$ is plotted for β = 5 and 10 on fig. IV-8.

For purposes of comparison we have plotted on the same figure the output signal-to-noise ratio given by an FM system for the same bandwidth expansion. We assume that the signals to be transmitted in our system (clipped version of the optimum signal and signal carrying the samples sign) are frequency multiplexed, and the composite signal transmitted by SSB. Then the curves plotted previously give the output signal-to-noise ratio as we have seen in I-5 and the bandwidth required by the system is B+W = βW. The performance of the FM system are given by (3-8-25a) of (17) (p. 152) for unmodulated carrier and rectangular channel bandwidth. In our notations

$$a_{o,x} = \frac{\frac{3}{8} (\beta - 2)^2 a_c}{1 + 2\sqrt{3}\, \beta \, a_c \, Q\left(\sqrt{\frac{2a_c}{\beta}}\right)} \qquad \text{(IV-3-21)}$$

Figure IV-8 shows that the system based on the transmission of the zero-crossings of the optimum process associated to the signal x(t) gives better performances than the classical FM system in the linear region (+ 8dB for β = 5, + 3dB for β = 10). This is paid, however, by a shift of the threshold

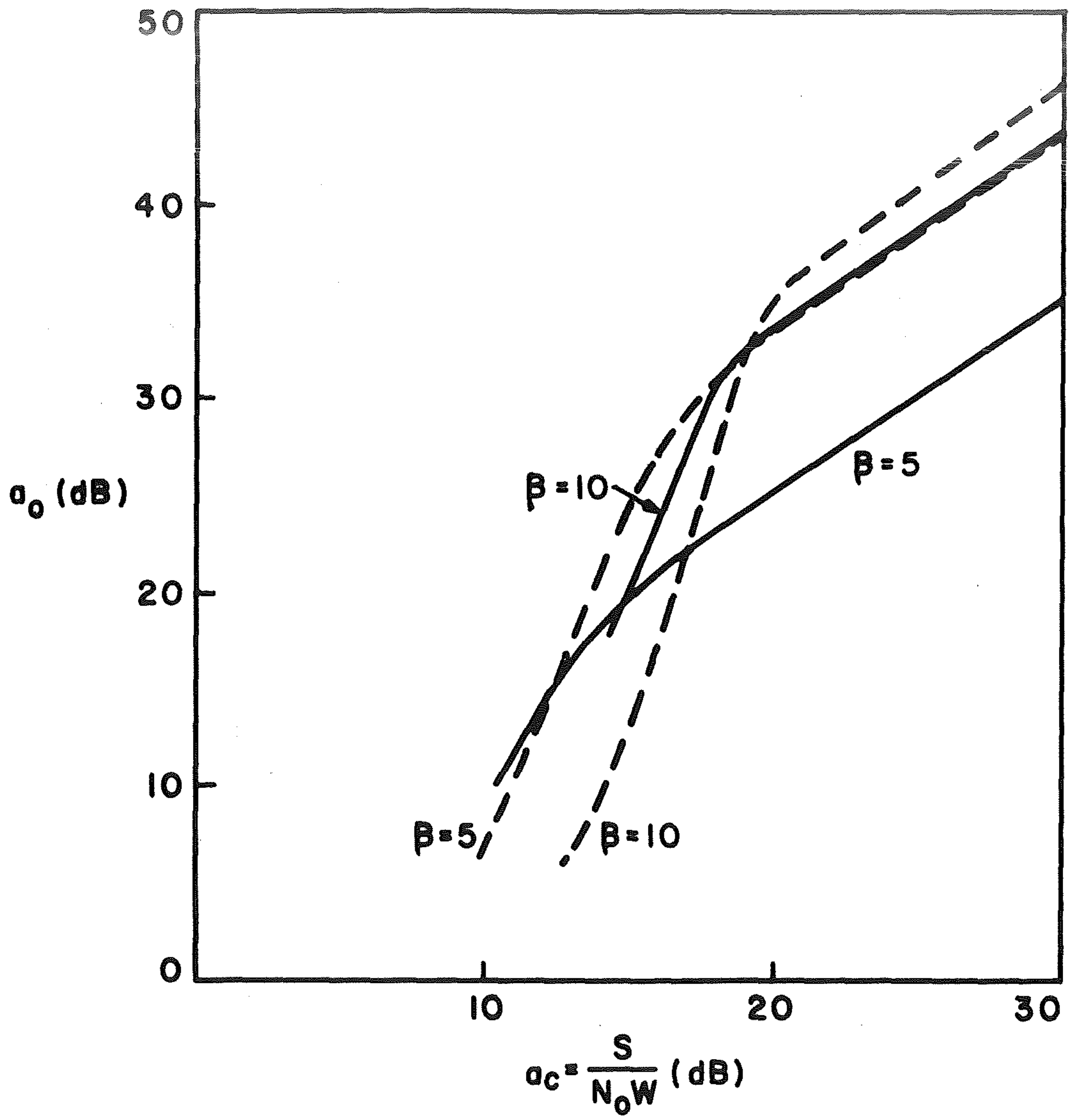

Fig. IV-8

Signal-to-noise ratio at the output of the Communication system. Non-optimum signals.
Dotted curves : theoretical results. $H_c(\omega) = \frac{\sin \omega T}{\omega T}$.
Plain curves : FM performances for unmodulated carrier and rectangular channel bandwidth. β = total channel bandwidth/information bandwidth.

toward higher channel signal-to-noise ratio, and by a more rapid decrease of the performances below threshold.

4. Experimental results for the channel $H_c(\omega) = \pi \; p_{2\pi B}(\omega)$.

The listing of the program used to simulate the communication system of fig. IV-2 is given in appendix C. Actually only part of the system has been investigated, namely the transmission of the optimum signal itself. Without loss of generality, we set $2W = 1$ in the following; sampling occurs at the integers which is convenient for programming.

4a. Generation of y_1 (t) and its zero-crossings.

The first problem we encounter is the computation of the optimum signal itself. By standard IBM subroutines (GAUSS and RANDU) we can generate independent Gaussian random variables x_k with specified mean and variance. Thus we can easily generate a realization of a bandlimited Gaussian zero-mean random process with power spectrum flat in $(-\Omega, +\Omega)$ by means of the sampling formula(+)

$$x(t) = \sum_{k=-\infty}^{+\infty} x_k \psi_k (t) \qquad \text{(IV-4-1)}$$

and similarly the optimum signal

$$y_1(t) = \sum_{k=-\infty}^{+\infty} (-1)^k |x_k| \psi_k(t) \qquad \text{(IV-4-2)}$$

(the random variable Z of (I-2-7) can be dropped since we consider one realization only). Actually (IV-4-1 and 2) must be truncated:

$$x(t) \approx \sum_{k=[t]-N_4+1}^{[t]+N_4} x_k \psi_k(t) \qquad \text{(IV-4-3)}$$

$$y_1(t) \approx \sum_{k=[t]-N_4+1}^{[t]+N_4} (-1)^k |x_k| \psi_k(t) \qquad \text{(IV-4-4)}$$

$2N_4$ is the number of samples around t which are taken into account and [t] is the largest integer smaller than or equal to t. In appendix D the signal-to-mean-square-error ratio due to truncation (a_t) is derived. The result is plotted in fig. IV-9. Thus a first limitation appears: with a reasonable amount of computation, namely 20 terms in the sampling formula, we cannot find the signal itself with an accuracy better than 20 dB (to reach 25 dB we need about 70 terms). However this does not mean that we shall be unable to find output signal-to-noise ratio higher than 20 dB if we set

(+). See (10) section 10-6. Other techniques are available which from independent random variables gives a process with any power spectrum (S. Stein and J. E. Storer, "Generating a Gaussian Sample," IEEE Trans., IT-2, n°2, pp. 87-90; June, 1956). Filtering of white noise leads to the same precision as this technique but is less tractable; the other technique proposed in this paper would require manipulation of large matrices (150 x 150).

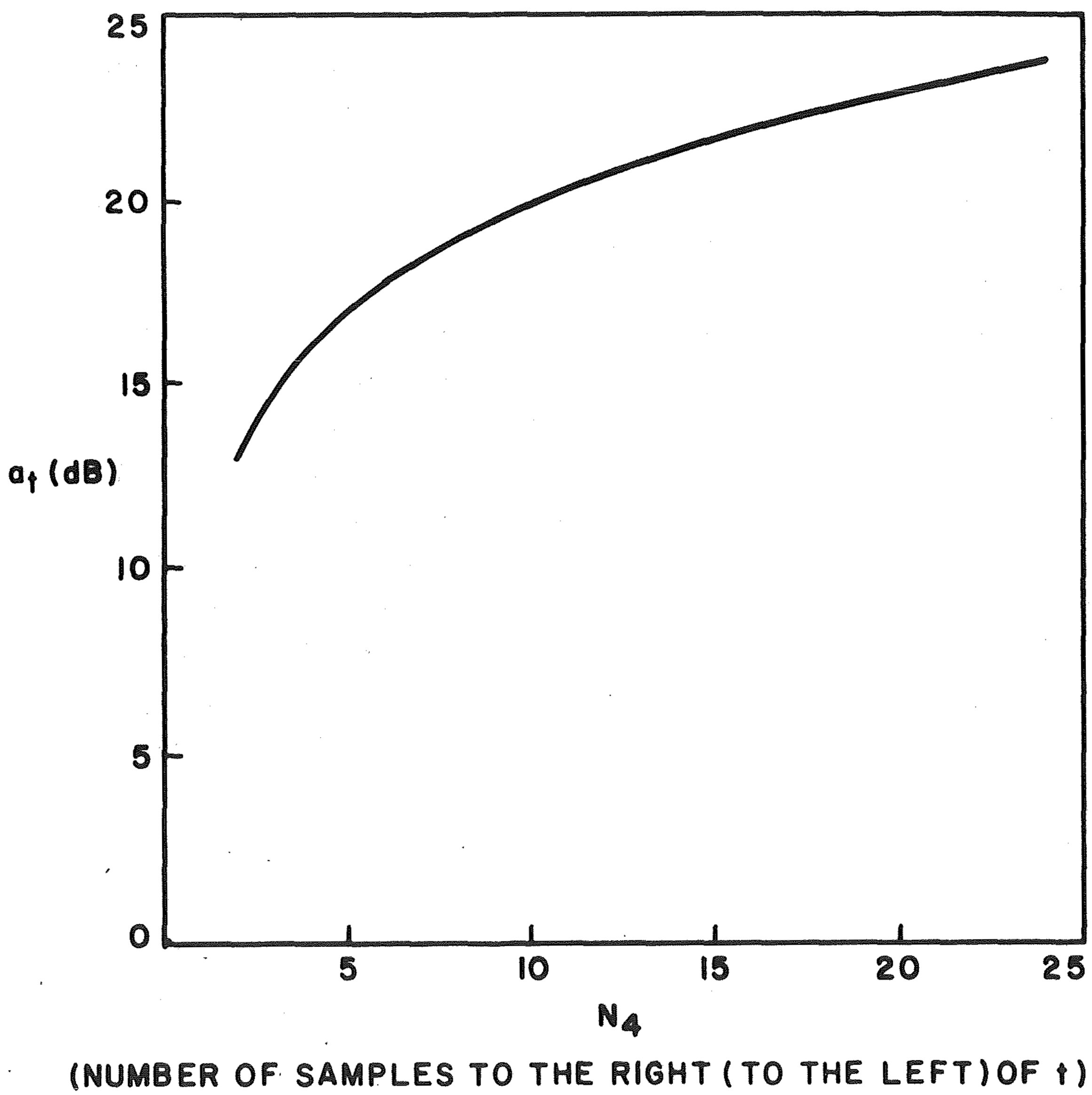

Fig. IV-9

Signal-to-mean-square-error ratio for the truncation of the sampling formula (case of a zero-mean Gaussian process with independent samples.).

N_4=10 in the simulation: (IV-4-3 and 4) define a new signal and its associated optimum process, and the zero-crossings used by the computer algorithm are now those of (IV-4-4). Thus the approximation will tend to this last expression.

The zero-crossings of $y_1(t)$ as defined by (IV-4-4) are determined by the subroutine ZERO. This step naturally introduces a quantization effect: the error on the position of each zero-crossing is less or equal to 2^{-L} where L is a parameter to be specified. L = 8 was finally chosen after investigation of the behavior of the signal-to-mean-squared-error ratio vs L (a higher value does not improve the final result). This corresponds to a rate of about 9 bits/sample (1 bit for the sample sign)(+).

We are now in a position to investigate a fundamental limit of the simulation: from the zeros just found and the algorithm described in II-5 we can find the estimate of $y_1(t)$ and plot a curve of the signal-to-mean-square-error ratio achieved vs the number of zeros in the interval: we get fig. IV-10 for N_4=10, L=8 and double precision used in the algorithm itself. a first increases with the number of zero-crossings (as expected) then drops above 30. In single precision the same occurs above 10 zero-crossings. The problem is obviously one of truncation error: each series coefficient is the sum of a number of complex exponentials (all terms are therefore ≤ 1 in magnitude) increasing very quickly with the number of zero-crossings. In the following we shall therefore set the length of the interval of time at 30.

4b. Distribution of the zero-crossings of an optimum signal.

A problem which is often considered in the literature is the distribution of the interval between successive zero-crossings. Even in the case of Gaussian processes the theoretical problem is a difficult one to analyze. For optimum processes we can obviously write

$$F_\tau(\tau) = 1 \quad \text{for} \quad \tau \geq W^{-1} \qquad \text{(IV-4-5)}$$

since each zero belongs to an interval of length $(2W)^{-1}$. On fig. IV-11 we have plotted an experimental distribution of τ and also the distribution of the zero-crossing in the Nyquist interval (the left end being taken as origin). The density associated with the latter is symmetrical about the mid-point of the interval and this has been used in deriving the experimental distribution.

(+) For the exact rate we should take the distribution of the zero-crossings into account. Some information on this point of view will be given in the next section.

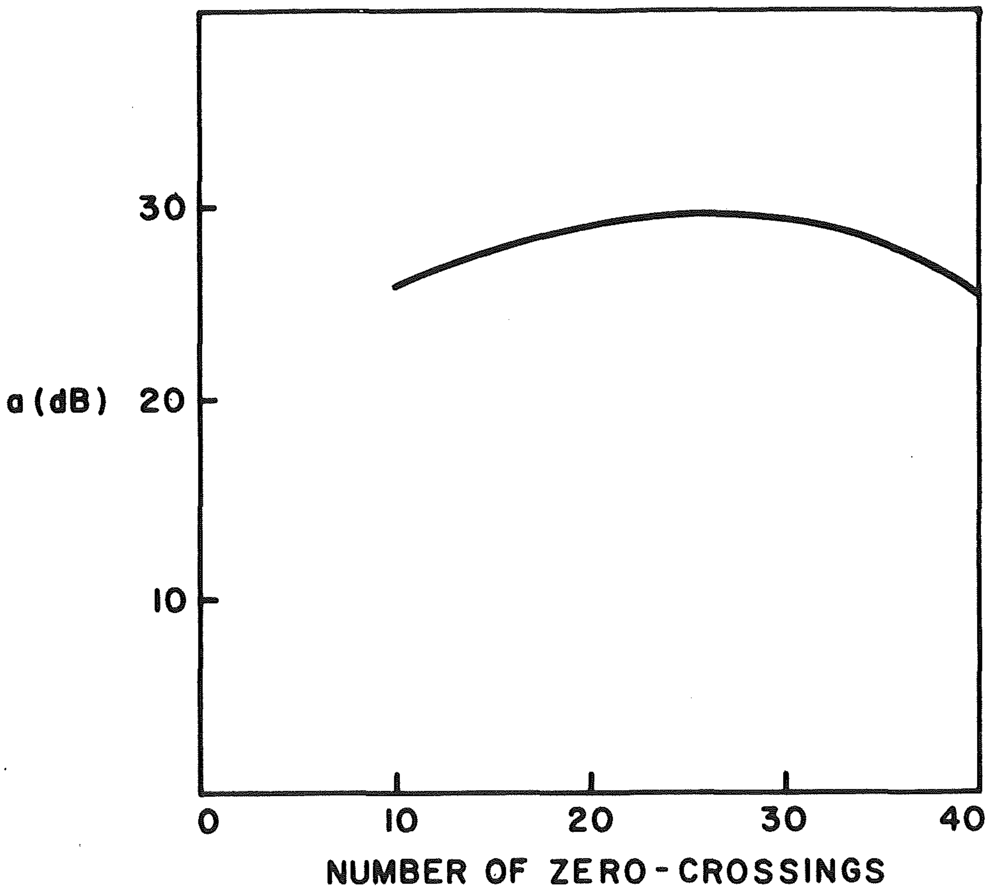

Fig. IV-10

Signal-to-mean-square-error ratio
vs. number of zero-crossing
(N_4 = 10, L = 8, double precision)

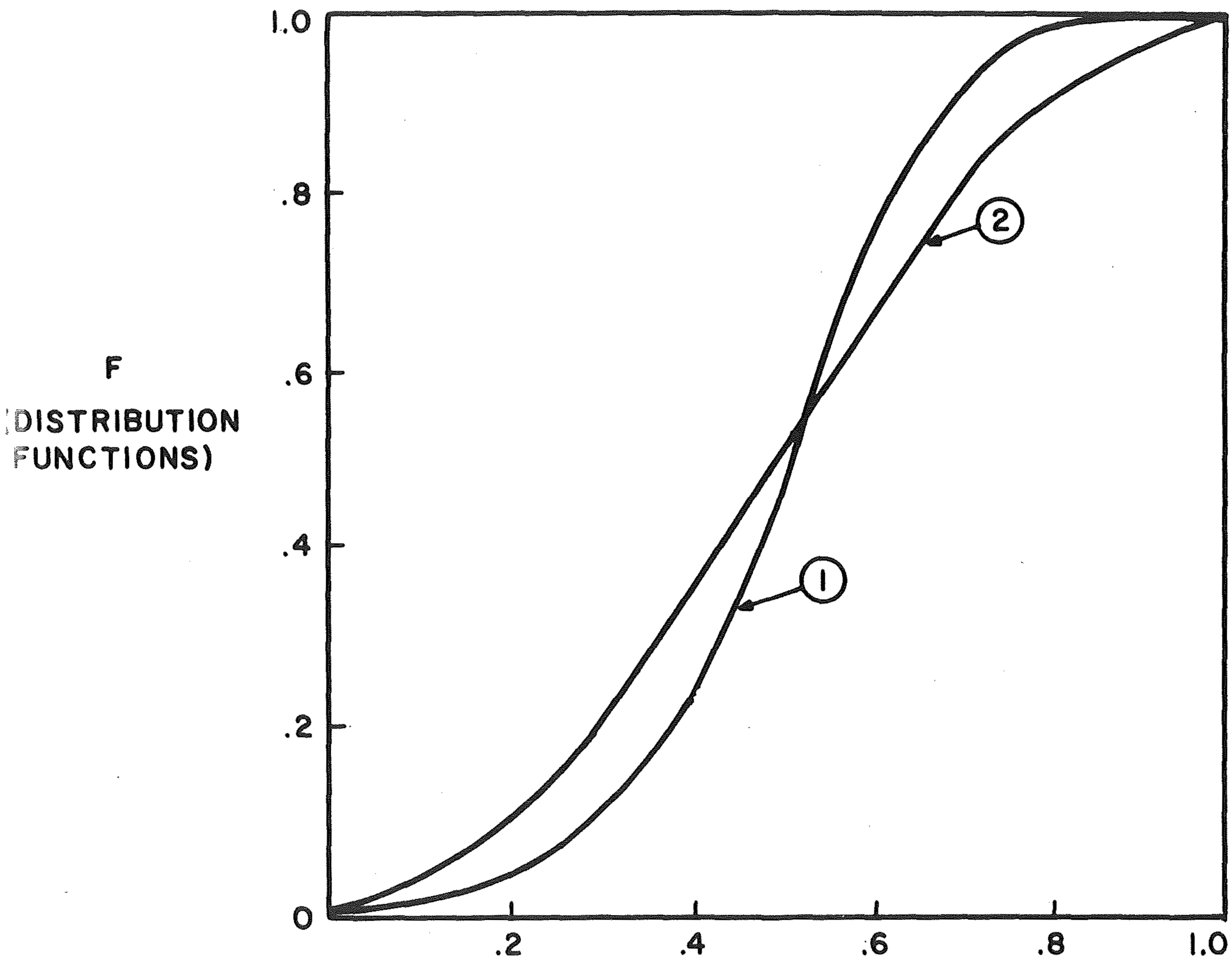

Fig. IV-11

Distribution of the zero-crossings of the optimum signal derived from a Gaussian process with independent samples.

1. Interval between successive zeros (in percentage of W^{-1})
2. Position of the zero in the Nyquist interval (in percentage of $(2W)^{-1}$)

The probability density of a zero-crossings interval close to 0 or W^{-1} is very small. For instance

$$F_{\tau}(.1/W) = .01 \qquad \text{(IV-4-6)}$$

a result which has already been used in III-2. As expected the probability density is maximum near $\tau = (2W)^{-1}$. Similar conclusions hold for the density of the zero-crossing position.

4c. <u>Computation of the channel output.</u>

The effect of the channel on the random square wave which carries the zero-crossings of the optimum signal is obtained by the subroutine CHNNL1. We have seen in 4a that we can easily generate a realization of a Gaussian zero-mean process with a flat power spectrum. For this reason we assume here the ideal channel $\pi p_{2\pi B}(\omega)$. Another reason is that the channel output is perfectly bandlimited and therefore defined (in the mean-square-sense) by its samples. Thus CHNNL1 computes at the sampling times $k/2B$ the channel output due to the random square wave and adds to it a Gaussian zero-mean random variable of specified variance given by the subroutine GAUSS.

In III-3 we have seen that the response of the channel $\pi p_{2\pi B}(\omega)$ to a random square wave of unit amplitude is

$$y_3(t) = \sum_{k=-\infty}^{+\infty} (-1)^{k+1} \left\{ \mathrm{Si}\,[\pi b(z_{k+1}-t)] - \mathrm{Si}[\pi b(z_k-t)] \right\} + n(t) \qquad \text{(IV-4-7)}$$

(we have dropped the random variable Z; $b=2B/2W=2B$ is the bandwidth expansion factor for optimum signals). Again we have to truncate an infinite series. Taking account of N_5 zero-crossings on the left (on the right) of the Nyquist interval containing t we get

$$y_3(t) = \sum_{k=[t]-N_5}^{[t]+N_5-1} (-1)^{k+1} \left\{ \mathrm{Si}\,[\pi b(z_{k+1}-t)] - \mathrm{Si}[\pi b\,(z_k-t)] \right\} + n(t) \qquad \text{(IV-4-8)}$$

We chose N_5 in such a way that the terms left are negligible; for instance

$$\pi b N_5 \approx 150$$

To find the variance of the noise for a given channel signal-to-noise ratio we need a first run of the program (without noise) to get the signal power S at the channel output (function of B). The specified channel signal-to-noise ratio $a_c = S/N_oW$ is achieved with a noise variance

$$\sigma_n^2 = bS/a_c \qquad \text{(IV-4-10)}$$

4d. Zero-crossings at the receiver input.

Clipping at the receiver input is implemented by RECCLP. Actually this subroutine also performs a decision task: it picks in each Nyquist interval the received zero-crossing close (hopefully) to the zero-crossing of the random square wave. This decision is necessary since in the algorithm the number of zeros is not allowed to exceed the length of the interval of time. The decision is based on the following property: when the received signal is noise free we can write(+)

$$2(z_k-k)-1 = \mathrm{sgn}\,[y_3(k)] \int_k^{k+1} y_4(t)dt \qquad \text{(IV-4-11)}$$

or

$$z_k = k + \tfrac{1}{2}\left[1 + \mathrm{sgn}\,[\,y_3(k)]\int_k^{k+1} y_4(t)dt\right] \qquad \text{(IV-4-12)}$$

We use this formula to get an approximate position of the zero-crossing in the noisy case. From there we go to the left and the right until we hit the actual first zero-crossing. For numerical purpose we replace the integral by a sum. Thus we compute the sign of the receiver input at N_6 equally distant points in the Nyquist interval, then

$$z_k \approx k + \tfrac{1}{2}\left[1 + N_6^{-1}\ \mathrm{sgn}\,[y_3(k)] \sum_{i=o}^{N_6} \mathrm{sgn}\ y_3(k+\frac{i}{N_6})\right] \qquad \text{(IV-4-13)}$$

and we compare $\mathrm{sgn}\,[y_3(z_k)]$ with $\mathrm{sgn}\,[y_3(k+\frac{i}{N_6})]$ as indicated above. Thus at high channel signal-to-noise ratio the actual received zero-crossings are determined with an error less than N_6^{-1}. N_6 has been set at 250 which corresponds to the same quantization as in the transmitter ($2^8 = 256$).

The end of the main program computes the approximation $\hat{y}_1(t)$ of the optimum signal from the set of these zero-crossings by (II-5-10 and 11). It also computes the output signal-to-noise ratio for the entire interval and also for the 10 Nyquist intervals in the center (the 10 first and last being considered as a guard time).

(+) Experiment shows that as soon as $B > W$ the number of zero-crossings of the channel output and the optimum signal is the same.

The results obtained for the central part of the interval (and the parameters: $N_4 = 10$, $N_5 b \approx 50$, $N_6 = 250$, $L = 8$, length of the interval = 30) are plotted on fig. IV - 7 for b = 4 and 9. At low channel signal-to-noise ratio we get better performances than the theoretical ones found in IV-3. This comes from replacing the ideal clipper by a device which eliminates all but one of the received zero-crossing in each Nyquist interval. Thus our knowledge of the structure of the optimum signal has merely been used. At high channel signal-to-noise ratio (a_c = 30 dB) we lose 14 (b = 9) or 16 dB(b = 4) by truncation error in the algorithm. Thus from here a special attention should be given to the design of other computer techniques.

Summary and Conclusions.

We have started with the fact that in some well defined cases (speech and signals with real zeros only) bandlimited signals are determined by their zero-crossings.

In chapter II we have investigated the recovery of a signal from a set of related zero-crossings. Among the solutions available mapping into what we have called an optimum signal is the only one acceptable. The scheme suggested for estimating the signal from the zero-crossings of the associated optimum process is the algorithm (II-5-4) followed by inverse mapping.

Random square waves as clipped versions of the optimum process are the simplest way to send the zero-crossings. Thus in chapter III a detailed analysis of the transmission of random square waves has been carried out. We have investigated the performances of an elementary receiver to recover a square wave from the channel output, namely the ideal clipper. We have found that as the channel bandwidth increases the normalized cross-correlation function between the two random square waves behaves as

$$r_{y_2y_4}(t,t) \rightarrow 1 - 2Q(\sqrt{a_c}) \qquad \text{(III-1-18)}$$

$$a_c = \frac{A_1^2}{\sigma_{n_1}^2}$$

where A_1 is the amplitude of the random square wave and $\sigma_{n_1}^2$ the channel noise power. As a consequence a_o behaves exponentially

$$a_o \rightarrow \left[4Q(\sqrt{a_c}) \right]^{-1} \approx \frac{\sqrt{2\pi a_c}}{4} \exp(a_c/2) \qquad \text{(III-1-26)}$$

We were able to analyze the effect of finite channel bandwidth for $H_c(\omega) = \sin(\omega T)/\omega T$. An additional term proportional to $\sigma_n b^{-1}$ appears where b is the bandwidth expansion factor for optimum signals (see III-2-79 and 81). This result is based on the approximation of some properties of the optimum process by their expressions for a Gaussian process.

In chapter IV the communication system of fig. IV-2 is analyzed. First we derive a relationship valid for the range of output signal-to-noise ratio of interest:

$$r_{y_1\hat{y}_1}(k/2W, k/2W) = \sin\left[\frac{\pi}{2} r_{y_2 y_4}(k/2W, k/2W)\right] \qquad \text{(IV-1-8)}$$

From this relation we get the performances for the transmission of optimum signals:

$$a_{y_1} = (1 - \frac{1}{\pi}) \frac{1}{\pi^2} \left[1 - R_{y_2 y_4}(k/2W, k/2W)\right]^{-2} \qquad \text{(IV-1-15)}$$

and for $b \rightarrow \infty$

$$a_{y_1} \rightarrow .434\, a_c \exp(a_c) \qquad \text{(IV-2-2)}$$

When we take account of the finite channel bandwidth we get (IV-3-6 and 10) for optimum signals and general signals respectively. To find to what extent finite bandwidth impairs the exponential behavior we have plotted (IV-2-2) and (IV-3-6) as functions of a_c=signal power/total channel noise power (fig. IV-12). Very large bandwidth expansion are required to get close to the limit in the range of interest (say up to 40 dB). Also the system is inefficient as far as exchange between bandwidth and signal-to-noise ratio is concerned.

When we compare the performances for general signals to those of a classical system with bandwidth expansion we can see from fig. IV-8 that the system gives better results than FM in the linear region (3 dB and 8 dB for a total bandwidth expansion factor $\beta = 10$ and 5 respectively). A threshold appears at a channel signal-to-noise ratio slightly higher than for FM; this is due to the sharp increase of the exponential terms when $a_c \rightarrow 0$ (equation IV-3-14). The linear term introduced by finite bandwidth is related to the appearance of extra zeros in the 2T interval of time about the zero-crossing in each Nyquist interval; the exponential terms and the threshold associated with them are related to extra zeros occurring outside the 2T interval at low channel signal-to-noise ratio.

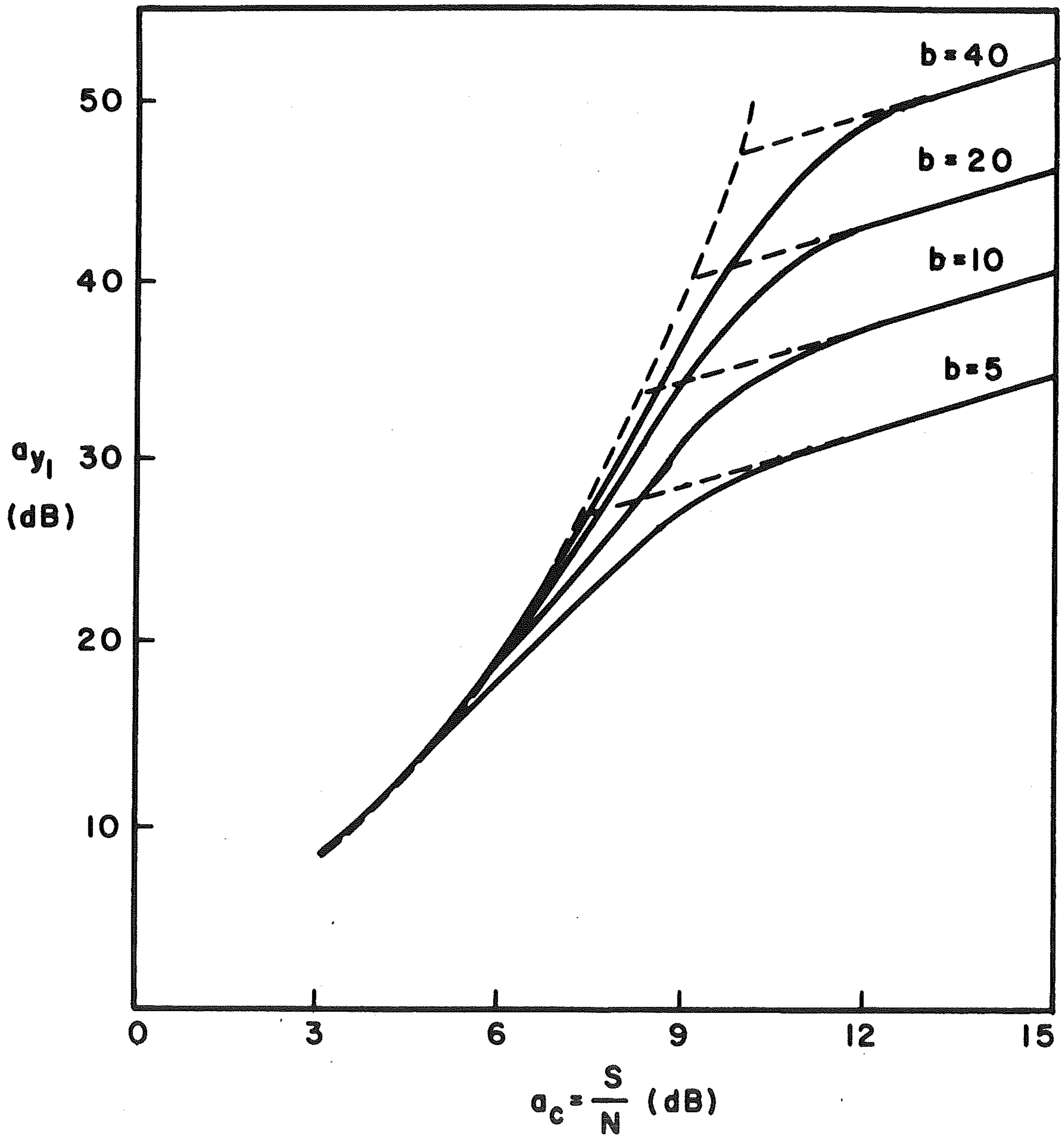

Fig. IV-12 Performances of the Communication System for Optimum Signals and the Channel
$H_c(\omega) = \frac{\sin \omega T}{\omega T}$

The experimental data are for a rectangular channel bandwidth. While keeping in mind that the theoretical results are for the channel with rectangular impulse response we think that the most interesting result is the improvement of the performances by the zero-crossings detector used in the simulation: the device takes advantage of the fact that only one zero-crossings should appear in each Nyquist interval. We have shown in IV-4a that truncation errors places a limit on the performances of the computer algorithm. Also the channel investigated introduces a loss of performances when compared to the channel $p_T(t)$ because here bandlimiting itself displaces the zero-crossings.

At this point we like to suggest a few directions of research. In fig. IV-2 we have the simplest scheme. Thus the clipper could be replaced by a nonlinear memoryless device with a smoother characteristic. In general, the channel should be followed by a filter to minimize the zero-crossings displacement. Actually the characteristic of the nonlinear device and the filter should be matched to the characteristic of the channel in order to get the minimum zero-crossings displacement. The mapping itself could be preceded by a nonlinear memoryless device to control the distribution of the zero-crossings of the optimum process. No attempt has been made to optimize the zero-crossings detector, and in the theoretical analysis a clipper has been assumed. The algorithm for estimation of the optimum process from the received zero-crossings is also an open question; we have considered a series approximation; Voelcker's real zero interpolator is another possible choice. Finally advantage could be taken of the statistics of the signal sign (a first improvement being to send a pulse only if the sign has changed).

APPENDIX A.

A theorem due to Polyá[8].

A function which has an absolute integrable Fourier transform vanishing outside an interval $(-\Omega, +\Omega)$ has one zero in each interval

$$\left(\frac{k-1}{2W}+\alpha,\ \frac{k}{2W}+\alpha\right) \qquad (|k| = 0, 1, \ldots) \tag{A1}$$

$$0 \le \alpha < 1/2W \ ; \ \Omega = 2\pi W$$

if and only if its values at $\frac{k}{2W}+\alpha$ alternate in sign.

Proof.

i) When the function y(t) has one zero in each interval the values at $\frac{k}{2W}+\alpha$ must alternate in sign since y(t) keeps the same sign between successive zeros.

ii) When the samples alternate in sign there is at least one zero-crossing in each interval. Therefore all that we have to show is the uniqueness of this zero. Without loss of generality we may assume that $\alpha = 0$. Since $|Y(\omega)|$ is integrable

$$y_a(\theta) = \frac{1}{2\pi}\int_{-\Omega}^{+\Omega} y(\omega)\, e^{j\omega\theta} d\omega \qquad \text{where } \theta = t + j\tau \tag{A2}$$

is an entire function. On the other hand $\sin(\Omega\theta)$ has all its zeros on the real axis since

$$|\sin(\Omega\theta)| = \frac{|e^{\Omega(jt-\tau)} - e^{-\Omega(jt-\tau)}|}{2} \ge \frac{|e^{-\Omega\tau} - e^{+\Omega\tau}|}{2} \tag{A3}$$

Therefore the poles of $y_a(\theta)/\sin(\Omega\theta)$ are on the real axis and at $\pm k/2W$ $(k = 0, 1, 2, \ldots)$. Let C_n be the circle of radius $(n+\frac{1}{2})/2W$ centered about the origin in the θ-plane.

$$\frac{1}{2\pi j}\int_{C_n} \frac{y_a(u)}{\sin(\Omega u)}\,\frac{du}{u-\theta} = \frac{y_a(\theta)}{\sin(\Omega\theta)} - \frac{1}{\Omega}\sum_{k=-n}^{+n} \frac{y_a\left(\frac{k}{2W}\right)}{\left(\theta - \frac{k}{2W}\right)\cos(k\pi)} \tag{A4}$$

Let us denote by $f_n(\theta)$ the right-hand side of this equality which is an analytic function. Since $y_a(\theta)/\sin(\Omega\theta)$ is bounded on C_n

$$|f_n(\theta)| \le M \frac{(n+\frac{1}{2})\frac{1}{2W}}{(n+\frac{1}{2})\frac{1}{2W} - |\theta|} \quad \text{for some } M \tag{A5}$$

and

$$f(\theta) = \lim_{n\to\infty} f_n(\theta) \le M \tag{A6}$$

Therefore the analytic function $f(\theta)$ is actually a constant K and we can write

$$\frac{y_a(\theta)}{\sin(\Omega\theta)} = K + \frac{1}{\Omega}\sum_{k=-n}^{+n} \frac{y_a(\frac{k}{2W})}{(\theta - \frac{k}{2W}) \cos(k\pi)} \tag{A7}$$

The right-hand side becomes $\pm\infty$ at the points $t = k/2W$ of the real axis and has only one zero between these points. Therefore since $\sin(\Omega t)$ is bounded $y(t)$ vanishes only once between $k/2W$ and $(k+1)/2W$ for all integers k.

APPENDIX B. Memoryless nonlinear transformation of a broad class of processes.

(II - 1 - 4) has been found previously by J. J. Bussgang[20] under more general conditions. Consider two processes x(t) and y(t) jointly Gaussian; assume they are zero-mean for the sake of simplicity. If one of them, say y(t), is subject to a memoryless nonlinear transformation T then

$$E\left\{x(t_1)T\left[y(t_2)\right]\right\} = K\, E\left\{x(t_1)\, y(t_2)\right\} \tag{B-1}$$

where K is a constant depending on T. The proof given by Bussgang merely requires T instantaneous. Here we shall again illustrate the Fourier transform technique for the description of a nonlinear device, while deriving his result. Thus we write

$$T(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \zeta(\omega)\, e^{j\omega y}\, d\omega \tag{B-2}$$

Then

$$E\left\{x(t_1)T\left[y(t_2)\right]\right\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \zeta(\omega_2)\, E\left\{x(t_1)\, e^{j\omega_2 y(t_2)}\right\} d\omega_2$$

$$= \frac{1}{2\pi j}\int_{-\infty}^{+\infty} \zeta(\omega_2)\left[\frac{\partial\Phi_{xy}(\omega_1,\omega_2;t_1,t_2)}{\partial\omega_1}\right]_{\omega_1=0} d\omega_2 \tag{B-3}$$

and since

$$\Phi_{xy}(\omega_1,\omega_2;t_1,t_2) = \exp\left[-\frac{\sigma_x^2(t_1)\,\omega_1^2 + 2R_{xy}(t_1,t_2)\omega_1\omega_2 + \sigma_y^2(t_2)\,\omega_2^2}{2}\right] \tag{B-4}$$

$$E\left\{x(t_1)T\left[y(t_2)\right]\right\} = -\frac{R_{xy}(t_1,t_2)}{2\pi j}\int_{-\infty}^{+\infty} \omega_2\, \zeta(\omega_2)\exp\left(-\frac{\sigma_y^2(t_2)\omega_2^2}{2}\right) d\omega_2 \tag{B-5}$$

Therefore in (B-1)

$$K = \frac{j}{2\pi}\int_{-\infty}^{+\infty} \omega\,\zeta(\omega)\exp\left(-\frac{\sigma_y^2(t_2)\,\omega^2}{2}\right) d\omega \tag{B-6}$$

and by Parseval's formula

$$K = \frac{1}{\sqrt{2\pi}\,\sigma_y}\int_{-\infty}^{+\infty} \frac{dT}{dy}\exp\left(-\frac{y^2}{2\sigma_y^2(t_2)}\right) dy \tag{B-7}$$

After integration by parts

$$K = \frac{1}{\sqrt{2\pi}\,\sigma_y^3(t_2)} \int_{-\infty}^{+\infty} yT(y) \exp\left(-\frac{y^2}{2\sigma_y^2(t_2)}\right) dy \tag{B-8}$$

which is identical to result (19) in Bussgang's report. Now if we let $y_1(t) = x(t) = y(t)$ and $y_2(t) = T[y_1(t)] = \text{sgn}\, y_1(t)$ we get

$$K = \sigma_{y_1}^{-1}(t_2)\sqrt{\frac{2}{\pi}} \tag{B-9}$$

and

$$R_{y_1 y_2}(t_1, t_2) = \frac{R_{y_1}(t_1, t_2)}{\sigma_{y_1}(t_2)} \sqrt{\frac{2}{\pi}} \tag{B-10}$$

i.e. (II-1-4) for a non-stationary process.

The result extends to the class of processes such that the joint characteristic function has a diagonal expansion in a series of orthonormal functions(19) Consider the two sets of functions solutions of the integral equations (+)

$$\int_{-\infty}^{+\infty} f_x(x)\, P_m(x) P_n(x) dx = \delta_{mn} \tag{B-11}$$

$$\int_{-\infty}^{+\infty} f_y(y) Q_m(y) Q_n(y) dy = \delta_{mn} \tag{B-12}$$

If the sets are complete we can write the joint density function $f_{xy}(x, y; t_1, t_2)$ as

$$f_{xy}(x, y) = f_x(x) f_y(y) \sum_{m,n=0}^{\infty} C_{mn} P_m(x) Q_n(y) \tag{B-13}$$

We restrict our attention to the class of processes such that the joint density can be written in the form

$$f_{xy}(x, y) = f_x(x) f_y(y) \sum_{n=0}^{\infty} C_n P_n(x) Q_n(y) \tag{B-14}$$

It follows from (B-11) and (B-12) that

$$C_n = \iint_{-\infty}^{+\infty} f_{xy}(x, y) P_n(x) Q_n(y) \tag{B-15}$$

(+) f_x and the P's are functions of t_1; similarly f_y and the Q's are functions of t_2. This dependence is not explicitly shown here.

Furthermore we consider the class of memoryless transformations which can be expanded in the Q's

$$T(y) = \sum_{m=0}^{\infty} T_m Q_m(y) \tag{B-16}$$

where

$$T_m = \int_{-\infty}^{+\infty} T(y) f_y(y) Q_m(y) \tag{B-17}$$

Then

$$E\left\{x(t_1) T\left[y(t_2)\right]\right\} = \sum_{m,\, n=0}^{\infty} T_m C_n \int_{-\infty}^{+\infty} x f_x(x) P_n(x) \int_{-\infty}^{+\infty} f_y(y) Q_m(y) Q_n(y)$$

$$= \sum_{n=0}^{\infty} T_n C_n \int_{-\infty}^{+\infty} x f_x(x) P_n(x) \tag{B-18}$$

For the sake of simplicity we shall assume zero mean processes then $\frac{x}{\sigma_x}$ is one of the orthonormal functions with respect to $f_x(x)$ and therefore

$$E\left\{x(t_1) T\left[y(t_2)\right]\right\} = T_1 C_1 \sigma_x(t_1) \tag{B-19}$$

with

$$C_1 = \iint_{-\infty}^{+\infty} f_{xy}(x, y; t_1, t_2) \frac{x(t_1)}{\sigma_x(t_1)} \frac{y(t_2)}{\sigma_y(t_2)} \, dx \, dy$$

$$= \frac{R_{xy}(t_1, t_2)}{\sigma_x(t_1)\, \sigma_y(t_2)} \tag{B-20}$$

and

$$T_1 = \int_{-\infty}^{+\infty} T(y) f_y(y) \frac{y}{\sigma_y(t_2)} \, dy \tag{B-21}$$

Finally form (B-19) and (B-20)

$$E\left\{x(t_1) T\left[y(t_2)\right]\right\} = T_1 \frac{R_{xy}(t_1, t_2)}{\sigma_y(t_2)} \tag{B-22}$$

Again for $y_1(t) = x(t) = y(t)$ and $T(y_1) = \text{sgn}\,(y_1)$ we get

$$T_1 = \sqrt{\frac{2}{\pi}}\, \sigma_{y_1}^{-1} \int_0^{\infty} y_1 \exp\left(-\frac{y_1^2}{2\sigma_{y_1}^2}\right) dy_1 = \sqrt{\frac{2}{\pi}} \tag{B-23}$$

and

$$R_{y_1 y_2}(t_1, t_2) = \frac{R_{y_1}(t_1, t_2)}{\sigma_{y_1}(t_2)} \sqrt{\frac{2}{\pi}} \tag{B-24}$$

Finally the most general class for which the result is valid is the class of separable processes.[18] We call two processes x(t) and y(t) jointly separable when(+)

$$E\{x(t_1) | y(t_2)\} = y(t_2) r_{xy}(t_1, t_2) \frac{\sigma_x(t_1)}{\sigma_y(t_2)} \tag{B-25}$$

Then

$$E\{x(t_1) T[y(t_2)]\} = E\Big\{ E\{x(t_1) T[y(t_2)] \; | y(t_2)\}\Big\}$$

$$= E\Big\{ T[y(t_2)] \; E\{x(t_1) | \; y(t_2)\}\Big\}$$

$$= \frac{\sigma_x(t_1)}{\sigma_y(t_2)} E\{y(t_2) T[y(t_2)]\} \; r_{xy}(t_1, t_2) \tag{B-26}$$

Gaussian processes belongs to the class. Thus again for a zero-mean Gaussian process and the ideal clipper

$$E\{y_1(t_2) T[y_1(t_2)]\} = E\{|y_1(t_2)|\} = \sigma_{y_1}(t_1) \sqrt{\frac{2}{\pi}} \tag{B-27}$$

and (B-24) follows. Processes with diagonal expansion of the joint characteristic function also belong to the class since from (B-14) we get

$$E\{x(t_1) | y(t_2)\} = \sum_{n=0}^{\infty} C_n Q_n(y) \int_{-\infty}^{+\infty} x f_x(x) P_n(x) \, dx$$

$$= \sigma_x(t_1) C_1 Q_1[y(t_2)] \tag{B-28}$$

and by (B-20)

$$E\{x(t_1) | y(t_2)\} = y(t_2) r_{xy}(t_1, t_2) \frac{\sigma_x(t_1)}{\sigma_y(t_2)} \tag{B-29}$$

(+) In the report just mentioned only one process is considered, and called separable if $E\{x(t+\tau) | \; x(t)\} = x(t) \, r_x(\tau)$. As it is shown here extension to two nonstationary processes is straight forward.

APPENDIX C - Computer Program

```
      DOUBLE PRECISION DARG, Q1, Q2, C1, C2, DCOS, DSIN, DBLE
      INTEGER VECTOR
      DIMENSION VECTOR (251)
      DIMENSION Q1( 30), Q2(30),C1(30), C2(30)
      DIMENSION SY1(64), Z(45), SY3(307), RECZC(30)
      DIMENSION Y1(151), Y1HAT(151)
      DIMENSION CROSS(151), VAR1(151),  VAR2(151)
      DIMENSION A(151), B(151), D(151), R(151)
      DIMENSION CROSSC(51), VAR1C(51), VAR2C(51)
      DIMENSION   AC(51), BC(51), DC(51), RC(51)
      PI = 3.1415927
      IX = 6433645
      N1 = 64
      N2 = 5
C     N4= HALF THE NUMBER OF SAMPLES TAKEN INTO ACCOUNT IN THE
C     SAMPLING FORMULA TO FIND THE ZERO-CROSSINGS (TRANSMITTED
C     AND RECEIVED) AND ALSO TO COMPUTE THE SIGNAL (OPTIMUM OR
C     NOT) IN GENERAL
      N4 = 10
C     N5=NUMBER OF ZEROS OF Y1(T) TAKEN INTO ACCOUNT ON THE LEFT
C     AND THE RIGHT OF  T  FOR THE CALCULATION OF THE CHANNEL OUT-
C     PUT
      N5 = 5
C     PRECISION OF THE RECEIVED ZERO-CROSSINGS IS 1/N6
      N6 = 250
C     IBEF=BANDWIDTH EXPANSION FACTOR FOR OPTIMUM SIGNAL
      IBEF = 9
      L = 8
      N6A = N6 + 1
      BN6 = 1.0/N6
      N5A = 2 *N5
      N5B = N4 + N5
      N5C = (N1-2*N5B)*IBEF+1
      N4A = 2*N4
      N4B = N4/IBEF+1
      N4C = N4B+N5B
      N4D = N4B+N5
      N1A = N1-2*N4C
      N1B = N1A*N2+1
      N1C = N1B-1
      N1D = N1B-5
      N1E = N1-2*N4+1
      PI2 = PI*IBEF
C
C     GENERATION OF N1 INDEPENDENT NUMBERS WITH DISTRIBUTION
C               N(0, 1)
      G = 1.0
      DO 100 I = 1, N1
      CALL GAUSS(IX, 1., 0., V)
      G = -G
  100 SY1(I) = G*ABS(V)
C
C     DETERMINATION OF THE ZERO-CROSSINGS OF Y1(T)
      DO 105 I = 1, NIE
      J = 1+N4-1
      CALL ZERO(J, L, T, SY1, N1, N4, N4A, PI)
  105 Z(I) = 1
C
```

```
C       POSITION OF THE ZERO-CROSSINGS
        WRITE (6,410)
 410    FORMAT ('ZERO-CROSSINGS OF THE OPTIMUM SIGNAL')
        WRITE (6,305) (Z(I), I=1, N1E)
 305    FORMAT (10F8.3)
C
        CALL CHNNL1(N5A, N5B, N5C, IBEF, Z, N1E, PI2, SY3)
C
        CALL RECCLP(N1A, SY3, N5C, N4C, N6, N6A, VECTOR, IBEF, N4, N4A,
       1 PI2, PI, BN6, RECZC, N5B)
        WRITE (6,420)
 420    FORMAT ('RECEIVED ZERO-CROSSINGS')
        WRITE (6,430) (RECZC(I), I=1, N1A)
 430    FORMAT (10F8.3)
C
C       COMPUTATION OF THE SERIES COEFFICIENTS
        W=2*PI/N1A
        DO 505 I=1, N1A
        DARG=DBLE(W*RECZC(I))
        Q1(I)=DCOS(DARG)
        Q2(I)=DSIN(DARG)
 505    CONTINUE
        DO 510 I=1, N1A
        C1(I)=0.0D0
        C2(I)=0.0D0
 510    CONTINUE
        DO 515 I=1, N1A
        J=N1C
 520    C1(J+1)=C1(J+1)+C1(J)*Q1(I)-C2(J)*Q2(I)
        C2(J+1)=C2(J+1)+C2(J)*Q1(I)+C1(J)*Q2(I)
        J=J-1
        IF(J.GE.1) GO TO 520
        C1(1)=C1(1)+Q1(I)
        C2(1)=C2(1)+Q2(I)
 515    CONTINUE
        G=1.0
        DO 525 I=1, N1A
        G=-G
        C1(I)=G*C1(I)
        C2(I)=G*C2(I)
 525    CONTINUE
C
C       CALCULATION OF Y1(T) AND Y1HAT(T)
        DO 110 I=1, N1B
        T=N4C+(I-1.0)/N2
        CALL SIG(T, SY1, N1, N4, N4A, PI, F)
        Y1(I)=F
        ARG=PI*T
        Y1HAT(I)=COS(ARG)
        DO 550 J=1, N1A
        ARG=(N1A/2-J)*W*T
 550    Y1HAT(I)=Y1HAT(I)+ SNGL (C1(J))*COS(ARG)-SNGL(C2(J))*SIN(ARG)
 110    CONTINUE
C
C       CALCULATION OF THE SMSER
        DO 120 I=1, N1B
        CROSS(I)=Y1(I)*Y1HAT(I)
        VAR1(I)=Y1(I)*Y1(I)
```

```
120 VAR2(I) = Y1HAT(I)*Y1HAT(I)
    CALL QSF(.2, CROSS,A,N1B)
    CALL QSF(.2, VAR1, B, N1B)
    CALL QSF(.2, VAR2, D, N1B)
    DO 125 I = 1, N1D
    J = I+ 5
    R(I) =(A(J)*A(J))/(B(J)*D(J))
125 R(I) = -10.0*ALOG10(1.0-R(I))
    WRITE (6, 460)
460 FORMAT ('SMSER')
    WRITE (6, 320) (R(J), J = 1, N1D)
320 FORMAT(10F8.2)
C
C   CALCULATION OF THE SMSER FOR THE CENTRAL PART OF THE
C            INTERVAL
    DO 140 I = 1, 51
    CROSSC(I) = CROSS(50+I)
    VAR1C(I)= VAR 1 (50 + I)
140 VAR2C(I) = VAR2(50+I)
    CALL QSF(.2, CROSSC, AC, 51)
    CALL QSF (.2, VAR1C, BC, 51)
    CALL QSF(.2, VAR2C, DC, 51)
    DO 150 I = 1, 46
    J = I+ 5
    RC(I) =(AC(J)*AC(J))/(BC(J)*DC(J))
150 RC(I) =-10.0*ALOG10(1.0-RC(I))
    WRITE (6, 170)
170 FORMAT ('SMSER FOR THE CENTRAL PART OF THE INTERVAL')
    WRITE (6, 180) (RC(J), J =1,46)
180 FORMAT(10F8.2)
C
C
C   CROSSCORRELATION OF THE INPUT AND OUTPUT RANDOM SQUARE
C            WAVES
    CORRSW=0.
    DO 200 I = 1, N1A
    CORRSW =ABS(Z(I+N4D)-RECZC(I)) + CORRSW
200 CONTINUE
    CORRSW =1.0-2.0*CORRSW/N1A
    WRITE (6, 210)
210 FORMAT('CORRELATION OF THE INPUT AND THE OUTPUT RANDOM
   1         SQUARE   WAVES')
    WRITE (6, 220) CORRSW
220FORMAT(F12.4)
    STOP
    END
```

```
      SUBROUTINE CAUSS(IX, S, AM, V)
      A =0.0
      DO 50 I = 1, 12
      CALL RANDU( IX, IY, Y)
      IX = IY
50    A =A+Y
      V =(A-6.0)*S+AM
      RETURN
      END

      SUBROUTINE RANDU(IX, IY, YFL)
      IY =IX*65539
      IF(IY)5, 6, 6
5     IY=IY+2147483647+1
6     YFL =IY .
      YFL =YFL*.4656613E-9
      RETURN
      END

      SUBROUTINE ZERO(J, L, T, SY1, N1, N4, N4A, PI)
      DIMENSION SY1(N1)
      T =J
      CALL SIG(T, SY1, N1, N4, N4A, PI, F)
      IF (F) 10, 50, 20
10    K1=-1
      GO TO 25
20    K1=1
25    DT1=.5
      DO 30 I=1, L
55    T=T+DT1
      CALL SIG(T, SY1, N1, N4, N4A, PI, F)
      IF (F) 35, 50, 40
35    K2=-1
      GO TO 45
40    K2=1
45    IF(IABS(K1-K2)) 50, 55, 60
60    K1=-K1
30    DT1=-DT1/2.0
50    RETURN
      END
```

```
      SUBROUTINE SIG(T, SY1, N1, N4, N4A, PI, F)
C
C     COMPUTES Y1(T) BY MEANS OF THE SAMPLING FORMULA
C
      DIMENSION SY1(N1)
C
C     CHECK FOR  T  INTEGER
C
      IT=T
      IF (T-IT) 200, 205, 200
  205 F =SY1(IT)
      RETURN
  200 F = 0.
      K1 =IT-N4
      DO 210 J =1, N4A
      K = K1+J
      X= PI*(T-K)
  210 F=F+SY1(K)*SIN(X)/X
      RETURN
      END

      SUBROUTINE RECSG1(T, IBEF, N5B, SY3, N5C, N4, N4A, PI2, PI, F)
      DIMENSION SY3(N5C)
      T1 = T-N5B+1.0/IBEF
      I =T1*IBEF
      F =0.
      K1 =I-N4
      DO 210 J =1, N4A
      K =K1+J
      X =PI2*T1-PI*K
      IF (ABS(X).LE.1.0E-03) GO TO 200
      G =SIN(X)/X
      GO TO 210
  200 G = 1.0
  210 F =F+SY3(K)*G
      RETURN
      END
```

```
      SUBROUTINE CHNNL1(N5A, N5B, N5C, IBEF, Z, N1E, PI2, SY3)
C     THIS SUBROUTINE PERFORMS THE FOLLOWING
C
C     1) IDEAL LOW-PASS FILTERING OF THE OPTIMUM RANDOM SQUARE
C             WAVE (OF UNIT AMPLITUDE) TO THE BANDWIDTH
C             IBEF*W. IBEF IS THE BANDWIDTH EXPANSION
C             FACTOR FOR OPTIMUM SIGNALS (FOR NON OPTIMUM
C             SIGNAL THE BANDWIDTH EXPANSION FACTOR IS ACTUALLY
C             IBEF+1).
C
C     2) ADDITION OF A ZERO-MEAN, GAUSSIAN NOISE WITH SINGLE-
C             SIDED POWER DENSITY NZERO (CHANNEL SNR = S/(NZERO*
C             IBEF*W). NOTE THAT S IS NOT EQUAL TO 1 SINCE THE
C             UNIT AMPLITUDE RANDOM SQUARE WAVE IS SUBJECT TO
C             FILTERING.)
C
C     THE SUBROUTINE COMPUTES THE SAMPLES (IBEF*2*W SAMPLES/SEC)
C             OF THE CHANNEL OUTPUT.
      DIMENSION SY3(N5C), Z(N1E)
      PI = 3.1415927
      IX = 267331
      SN = .98
      CNR = 100
      SIGMA = PI*SQRT(SN*IBEF/CNR)
      DO 110 I = 1, N5C
      T = N5B + (I-1.0)/IBEF
      SY3(I) = 0.
      DO 115 J = 1, N5A
      K = T - N5B+J
      ARG =(Z(K)-T)*PI2
      CALL SICI(U, ARG)
      ARG = (Z(K+1)-T)*P12
      CALL SICI(W, ARG)
  115 SY3(I) =SY3(I)+((-1)**K)*(W-U)
      CALL GAUSS(IX, SIGMA, 0., V)
      SY3(I) =SY3(I)+V
  110 CONTINUE
      CNR = 10*ALOG10(CNR)
      WRITE (6, 125)
  125 FORMAT ('CHANNEL SIGNAL TO NOISE RATIO S/NO*W  IN DB')
      WRITE (6, 120) CNR
  120 FORMAT (F8.2)
      RETURN
      END
```

```
      SUBROUTINE RECCLP(N1A, SY3, N5C, N4C, N6, N6A, VECTOR, IBEF,
     1 N4, N4A, PI2, PI, BN6, RECZC, N5B)
C
C     THIS SUBROUTINE GIVES THE ZERO-CROSSINGS OF THE CHANNEL
C     OUTPUT.
C
      INTEGER VECTOR
      DIMENSION RECZC(N1A), SY3(N5C), VECTOR(N6A)
      AN6 = N6
      SIGPOW = 0.
      DO 105 I = 1, N1A
      T =N4C+I-1
      T1=T
      CALL RECSG1(T, IBEF, N5B, SY3, N5C, N4, N4A, PI2, PI,F)
      POWER = .5*F*F
  300 IF (F) 10, 15, 20
   10 K1 =-1
      GO TO 25
   20 K1=1
   25 SUM=K1/2.0
      VECTOR(1)=K1
      DO 100 J=1, N6
      T=T1+J/AN6
      CALL RECSG1(T, IBEF, N5B, SY3, N5C, N4, N4A, PI2, PI, F)
      POWER=POWER+F*F
  310 IF (F) 30, 15, 35
   30 K2=-1
      GO TO 110
   35 K2=1
  110 VECTOR (J+1)=K2
  100 SUM= SUM+K2
      SUM= SUM-K2/2.0
      POWER=POWER-.5*F*F
      T=T1+.5*(1.0+K1*SUM*BN6)
      CALL RECSG1(T, IBEF, N5B, SY3, N5C, N4, N4A, PI2, PI, F)
  320 IF (F) 40, 15, 45
   40 K=-1
      GO TO 50
   45 K=1
   50 IT1=(T-T1)*N6+1
      IT2=IT1+1
   80 IF (IABS(VECTOR(IT1)-K)) 55, 60, 65
   65 T=(IT1-1.0)/N6+T1
      GO TO 15
   60 CONTINUE
      IF (IABS(VECTOR(IT2)-K)) 55, 70, 75
   75 T=(IT2-1.0)/N6+T1
      GO TO 15
   70 IT1=IT1-1
      IT2=IT2+1
      GO TO 80
   15 RECZC(I)=T
      SIGPOW=SIGPOW+POWER
  105 CONTINUE
      SIGPOW=SIGPOW/(N6*N1A*PI*PI)
      WRITE(6.200)
  200 FORMAT ('SIGNAL POWER AT THE CHANNEL OUTPUT')
      WRITE (6, 210) SIGPOW
  210 FORMAT (F12.4)
   55 RETURN
      END
```

```
      SUBROUTINE QSF(H, Y, Z, NDIM)
C
C
      DIMENSION Y(1), Z(1)
C
      HT=.3333333*H
      IF(NDIM-5)7, 8, 1
C
C     NDIM IS GREATER THAN 5.  PREPARATIONS OF INTEGRATION LOOP
    1 SUM1=Y(2)+Y(2)
      SUM1=SUM1+SUM1
      SUM1=HT*(Y(1)+SUM1+Y(3))
      AUX1=Y(4)+Y(4)
      AUX1=AUX1+AUX1
      AUX1=SUM1+HT*(Y(3)+AUX1+Y(5))
      AUX2=HT*(Y(1)+3.875*(Y(2)+Y(5))+2.625*(Y(3)+Y(4))+Y(6))
      SUM2=Y(5)+Y(5)
      SUM2=SUM2+SUM2
      SUM2=AUX2-HT*(Y(4)+SUM2+Y(6))
      Z(1)=0.
      AUX=Y(3)+Y(3)
      AUX=AUX+AUX
      Z(2)=SUM2-HT*(Y(2)+AUX+Y(4))
      Z(3)=SUM1
      Z(4)=SUM2
      IF(NDIM-6)5, 5, 2
C
C       INTEGRATION LOOP
    2   DO 4 I = 7, NDIM, 2
      SUM1=AUX1
      SUM2=AUX2
      AUX1=Y(I-1) + Y(I-1)
      AUX1=AUX1+AUX1
      AUX1=SUM1+HT*(Y(I-2)+AUX1+Y(I))
      Z(I-2)=SUM1
      IF (I-NDIM) 3, 6, 6
    3 AUX2=Y(I)+Y(I)
      AUX2=AUX2+AUX2
      AUX2=SUM2+HT*(Y(I-1)+AUX2+Y(I+1))
    4 Z (I-1)=SUM2
    5 Z(NDIM-1)=AUX1
      Z(NDIM)=AUX2
      RETURN
    6 Z(NDIM-1)=SUM2
      Z(NDIM)=AUX1
      RETURN
C     END OF INTEGRATION LOOP
C
    7 IF(NDIM-3)12, 11, 8
C
C     NDIM IS EQUAL TO 4 OR 5
    8 SUM2=1.125*HT*(Y(1) + Y(2) + Y(2) + Y(2) + Y(3) + Y(3) + Y(3) + Y(4))
      SUM1=Y(2) +Y(2)
      SUM1=SUM1 +SUM1
      SUM1=HT* (Y(1) + SUM1 + Y(3))
      Z(1)=0.
      AUX1=Y(3) + Y(3)
      AUX1=AUX1 +AUX1
```

```
      Z(2)=SUM2-HT*(Y(2) + AUX1 + Y(4))
      IF(NDIM-5) 10, 9, 9
 9    AUX1=Y(4) + Y(4)
      AUX1=AUX1 + AUX1
      Z(5)=SUM1 + HT*(Y(3) + AUX1 + Y(5))
 10   Z(3)=SUM1
      Z(4)=SUM2
      RETURN
C
C     NDIM IS EQUAL TO 3
 11   SUM1=HT*(1.25*Y(1) + Y(2) + Y(2) -.25*Y(3))
      SUM2=Y(2) + Y(2)
      SUM2=SUM2 + SUM2
      Z(3)=HT*(Y(1) + SUM2 + Y (3))
      Z(1)=0.
      Z(2)=SUM1
 12   RETURN
      END

      SUBROUTINE SICI(SI, X)
C
C     TEST ARGUMENT RANGE
C
      Z=ABS(X)
      IF(Z-4.) 10, 10, 50
C
C     Z IS NOT GREATER THAN 4
C
 10   Y=Z*Z
     0SI=-1.5707963+X*((((((.97942154E-11*Y-.22232633E-8)*Y+.30561233E-6
     1)*Y-.28341460E-4)*Y+.16666582E-2)*Y-.55555547E-1)*Y+1.)
 40   RETURN
C
C     Z IS GREATER THAN 4.
C
 50   SI=SIN(Z)
      Y=COS(Z)
      Z=4./Z
     0U=((((((((.40480690E-2*Z-.022791426)*Z +.055150700)*Z-.072616418)*Z
     1+.049877159)*Z-.33325186E-2)*Z-.023146168)*Z-.11349579E-4)*Z
     2+062500111)*Z+.25839886E-9
     0V=((((((((-.0051086993*Z + .028191786)*Z-.065372834)*Z + .079020335)*
     1Z-.044004155)*Z-.0079455563)*Z +.026012930)*Z-.37640003E-3)*Z
     2-.031224178)*Z-.66464406E-6)*Z +.25000000
      SI=-Z*(SI*U +Y*V)
C
C     TEST FOR NEGATIVE ARGUMENT
C
      IF(X) 60, 40, 40
C
C     X IS LESS THAN -4.
C
 60   SI=-3.1415927-SI
      RETURN
      END
```

APPENDIX D.

Truncation of the sampling formula.

If we take account of N_4 samples to the left and N_4 samples to the right of the time t the error on x(t) is

$$e(t) = \sum_{\substack{k < [t] - N_4 + 1 \\ \text{and } k > [t] + N_4}} x_k \psi_k(t) \tag{D-1}$$

Since the samples are independent and zero-mean

$$E\, e^2(t) = \sum_{\substack{k, \ell < [t] - N_4 + 1 \\ \text{and } k, \ell > [t] + N_4}} E\{x_k x_l\} \psi_k(t)\, \psi_l(t)$$

$$= \sigma_x^2 \sum_{\substack{k < [t] - N_4 + 1 \\ \text{and } k > [t] + N_4}} \psi_k^2(t) \tag{D-2}$$

This quantity is a periodic function of time (with period equal to the Nyquist interval). We are interested in its time average

$$\int_i^{(i+1)} E\{e^2(t)\}\, dt = \sigma_x^2 \sum_{\substack{k < i - N_4 + 1 \\ \text{and } k > i + N_4}} \int_i^{(i+1)} \psi_k^2(t)\, dt \tag{D-3}$$

Therefore the signal-to-mean-square-error ratio a_t for truncation of the sampling formula is given by

$$a_t^{-1} = \frac{1}{\pi} \sum_{\substack{k < i - N_4 + 1 \\ \text{and } k > i + N_4}} \int_{(i-k)\pi}^{(i+1-k)\pi} \left[\frac{\sin(u)}{u}\right]^2 du$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{-N_4\pi} \left[\frac{\sin(u)}{u}\right]^2 du + \int_{N_4\pi}^{\infty} \left[\frac{\sin(u)}{u}\right]^2 du \right]$$

$$= 1 - \frac{1}{\pi} \int_{-N_4\pi}^{N_4\pi} \left[\frac{\sin(u)}{u}\right]^2 du \tag{D-4}$$

D-4 is plotted on fig. IV-6. The corresponding expression for $y_1(t)$ is more complicated and is not reported here.

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