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ON THE MOTION AND RADIATION OF CHARGED PARTICLES IN STRONG ELECTROMAGNETIC WAVES

I. MOTION IN PLANE AND SPHERICAL WAVES*

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ABSTRACT

In this first paper we treat the motion of test particles in strong electromagnetic waves considering both plane and spherical wavefronts, both with and without radiative reaction. Various limiting cases are discussed. Initially low-energy particles dropped into a strong plane wave are accelerated in the propagation direction up to an energy proportional to B^2 while oscillating with a "transverse energy" proportional to B. Radiative "losses" paradoxically lead to an increase rather than a decrease in the total energy of these particles. In spherical waves particles injected very close to the source become "phase locked" with the driving wave and are accelerated in the manner described in the authors' earlier work; particles injected further out follow orbits like classical particles in a repulsive inverse cube force-field except that, for certain classes of initial conditions, an essentially random energy redistribution (of limited range) is possible.

Charged particles of arbitrary initial energy will radiate strongly when injected into strong wave fields. The non-linear Compton ("NIC") radiation so produced depends on particle energy and field-energy density like, and has a spectrum like, synchrotron radiation. It is suggested that this mechanism is, in fact, responsible for the radiation in many astronomical "synchrotron" sources.

The physical processes elaborated here provide channels for converting rotational kinetic energy of condensed bodies into fast particles and the fields in which these particles can radiate. Further work will be needed to see if these processes are operating in any of the extragalactic "nonthermal" sources, but existing observations and calculations indicate that wave acceleration and NIC radiation probably are occurring in the Crab Nebula.

Further, we suggest that since wave acceleration of ions is likely to occur in the debris of young supernova remnants, pulsars may be able to produce the bulk of the galactic cosmic rays.

I. INTRODUCTION

A widely accepted class of theories[1] treats pulsars as rotating magnetized neutron stars, which are remnants of some common category of supernova explosions. The initial rotational energy is quite uncertain but estimated to be in the range $10^{50.0}$ to $10^{52.5}$ ergs. This energy is sufficiently large and the birth of pulsars sufficiently frequent (cf. Gunn and Ostriker 1970, "NP III") so that the average luminosity from pulsars in the galactic plane - $10^{-1.5}$ to $10^{+1.0}$ $L_{\odot}/pc^2$ - may be competitive with that from the burning of nuclear fuel in stars.

Since the observed pulsed luminosity is a trivial fraction of the total pulsar output, it remains important to study the final disposition of this significant energy resource.

In one subclass of the theories mentioned (cf. Pacini 1968, and Gunn and Ostriker 1969) the magnetic and rotational axes are assumed to be not aligned and energy is lost, in the first instance, by the emission of magnetic dipole radiation. If this picture is correct, then there are electromagnetic wave fields in the environs of pulsars of very high intensity, low frequency, and largely unexplored properties. Since these waves cannot propagate in the interstellar medium, their energy and momentum must ultimately be transformed into higher frequency radiation or particle motions in order to escape the pulsars' vicinity.

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[1] See Goldreich and Julian (1969), Ostriker and Gunn (1969, "NPI") and Michel (1969) for details and references to earlier work.

In this paper we will examine the simplest relevant problem, to wit, the interaction between charged test particles and a strong monochromatic wave field. Although we will consider neither the effect of particles on one another, nor the back effect of particles on the wave, both effects may be estimated after the fact.

The "strength" of a wave field can be measured in terms of a dimensionless Lorentz-invariant parameter, ν, defined in terms of the charge e and mass m of a test particle (an electron, unless otherwise specified), the frequency of the wave, Ω, and the maximum magnetic field in the wave, B_m:

$$\nu \equiv \frac{eB_m}{mc\Omega} \qquad (1)$$

Interstellar starlight has a strength parameter $\nu \simeq 10^{-14}$ and, even in the center of highly evolved stars ($T \mathrel{\bar{\sim}} 2 \times 10^8$ °K) $\nu < 10^{-2}$. In contrast, $\nu \simeq 10^{11}$ in the near wave zone of the Crab pulsar and may reach values of $\simeq 10^{13}$ in the early history of such objects.[2] Various equivalent physical interpretations of this parameter are possible. It represents the ratio of the formal cyclotron frequency $\omega_o \equiv eB_m/mc$ in the wave field to the wave frequency, and so may be thought of as a measure of whether the particle feels it is moving in a nearly static field ($\nu >> 1$) or oscillates in direct response to field variations ($\nu << 1$). Alternatively, we shall see that very large or very small ν corresponds to ultra-relativistic or non-relativistic motion of a particle initially at rest.

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[2]For a given pulsar and a given point in space $\nu \propto \Omega$, but at the wave radius (which is a function of Ω) $\nu \propto \Omega^2$.

A first problem, to find the motion of a "phase-locked" particle injected near the center of a strong spherical wave field, was treated in Paper I and will be recovered here as a special case. In this paper we will consider arbitrarily strong (or weak) plane and spherical waves, find the basic motion, the radiative losses, and the effects of the radiative reaction on the basic motion. In the limit $\nu \to 0$, the problem degenerates into the classical Compton case. In contrast, we will find that the bremsstrahlung of particles in strong waves ($\nu >> 1$) bears closer resemblance to synchrotron emission than inverse Compton radiation.

Although some remarks will be made here on the character of the radiated spectrum, a detailed discussion will be reserved for a subsequent paper.

In Section II we treat the motion in stationary and slowly-changing plane waves; in Section III the effect of radiative reaction on the motion in a plane wave; and in Section IV we discuss the radiation itself. Section V considers various aspects of the motion of charged particles in spherical waves. We reserve for Section VI a summary and a discussion of the applications of the previous mathematical results to some situations of astrophysical interest.

II. MOTION IN A PLANE WAVE

a) The Equations

The motion of a charged particle in a strong plane wave has been considered by several authors, notably as an exercise in Landau and Lifschitz Classical Theory of Fields (1951). We will review the problem briefly here in a slightly generalized form to bring out its fundamental features and to establish the units and notation we will use throughout.

Neglecting radiative reaction, the four equations of motion can be written

$$mc \frac{d\underset{\sim}{u}_\perp}{d\tau} = e \quad (\gamma - u_z) \underset{\sim}{E},$$

$$mc \frac{du_z}{d\tau} = e\, u_\perp \cdot \underset{\sim}{E}, \qquad (2)$$

and

$$mc \frac{d\gamma}{d\tau} = e\, u_\perp \cdot \underset{\sim}{E}.$$

Here the propagation direction is taken to be $+\hat{z}$ so that $\underset{\sim}{E}$, $\underset{\sim}{B}$, and $\underset{\sim}{u}_\perp$ are 2-vectors in the x-y plane. The quantities $(\underset{\sim}{u}_\perp, u_z, \gamma)$ are the dimensionless components of the four velocity $(c^{-1}dx/d\tau, c^{-1}dy/d\tau, c^{-1}dz/d\tau, dt/d\tau)$ and τ the proper time. We will further choose the x and y axes to be aligned with the principal polarization axes so that

$$\underset{\sim}{E} = \left[E_{xo}\cos \Omega(t-z/c),\ E_{yo}\sin \Omega(t-z/c)\right]$$
$$\underset{\sim}{B} = \left[-E_{yo}\sin \Omega(t-z/c),\ E_{xo}\cos \Omega(t-z/c)\right] \qquad (3)$$
$$B_m \equiv (E_{xo}^2 + E_{yo}^2)^{\frac{1}{2}},\ \lambda \equiv 2\pi c/\Omega$$

We introduce dimensionless variables

$$\eta \equiv \Omega\tau, \quad \zeta \equiv \Omega\, z/c, \quad \xi = \Omega t \tag{4}$$

$$\underset{\sim}{\nu} \equiv \underset{\sim}{E}e/mc\Omega, \quad \chi \equiv \Omega\,(t-z/c),$$

and, noting that $|\underset{\sim}{\nu}|$ is just the nonlinearity parameter (1) we rewrite (2)

$$\frac{d\underset{\sim}{u}_\perp}{d\eta} = (\gamma - u_\zeta)\ \underset{\sim}{\nu}(\chi), \tag{5a}$$

$$\frac{du_\zeta}{d\eta} = \frac{d\gamma}{d\eta} = \underset{\sim}{u}_\perp \cdot \underset{\sim}{\nu}(\chi). \tag{5b}$$

Note that $u_\zeta = d\zeta/d\eta = u_z$, and $u_\perp$ is unchanged. It is immediately verified that the four-force is orthogonal to the four-velocity, so that

$$\gamma^2 - u_\zeta^{\ 2} - u_\perp^{\ 2} = \text{const} \equiv 1 \tag{6}$$

Also, since from (5b)

$$\frac{d}{d\eta}\,(\gamma - u_\zeta) = 0, \tag{7}$$

$$\gamma - u_\zeta \equiv \frac{d\chi}{d\eta} \equiv \alpha, \text{ a constant} \tag{8}$$

quite independent of the form for $\underset{\sim}{\nu}$, provided only that the fields are those for a wave in the z-direction. Thus, we have the immediate solution from equations (5) - (8)

$$\chi = \alpha\eta + \chi_o \tag{9a}$$

$$\gamma = u_\zeta + \alpha \tag{9b}$$

$$u_\zeta = \frac{1 - \alpha^2}{2\alpha} + \frac{u_\perp^2}{2\alpha} \tag{9c}$$

$$\underset{\sim}{u}_\perp = \underset{\sim}{u}_{\perp o} + \int_o^\chi \underset{\sim}{\nu}\,(\chi')\,d\chi' \,. \tag{9d}$$

b) Plane-Polarized Wave

Consider first a plane-polarized wave, with $\underset{\sim}{E}$ in the x-direction. Then $\underset{\sim}{u}_\perp$ is parallel to the x-axis, and

$$u_\perp = u_{\perp o} + \nu_{ox} \sin \chi \tag{10}$$

where $\nu_{ox} = eE_{xo}/mc\Omega$. The motion can be decomposed into two parts. The particle oscillates in the x and ζ directions, and in addition, the phase-averaged (represented by $\langle\ \rangle_\chi$) position undergoes an arbitrary uniform translation. The latter will be called the motion of the "guiding center." In the frame in which the "guiding center" is at rest the particle moves in a figure-8 orbit:

$$u_\perp = \nu_{ox} \sin \chi, \quad \alpha = (1 + \tfrac{1}{2}\nu_{ox}^2)^{\frac{1}{2}}$$

$$u_\zeta = -\frac{\nu_{ox}^2}{4(1+\frac{1}{2}\nu_{ox}^2)^{\frac{1}{2}}} \cos 2\chi, \quad \langle\gamma\rangle_\chi = (1 + \tfrac{1}{2}\nu_{ox}^2)^{\frac{1}{2}} \tag{11}$$

Note that the motion is relativistic or non-relativistic as ν_{ox} is greater or less than one. In the weak or "linear" case ($\nu << 1$) there is negligible ζ motion; as ν becomes larger, the x velocity becomes relativistic before the wave reverses so magnetic forces begin to influence the motion. For very large ν the ratio of the x and ζ amplitudes is of order unity and is independent of the intensity of the wave; the orbit has changed from a line to a figure 8 of fixed shape.

It is instructive to consider next the motion of a particle injected at rest into the wave at arbitrary phase χ_o. In this case $\alpha = 1$ and

$$u_\perp = \nu_{ox} (\sin \chi - \sin \chi_o)$$

$$u_\zeta = \tfrac{1}{2} \nu_{ox}^2 (\sin \chi - \sin \chi_o)^2 \tag{12}$$

Although the particle does come to rest periodically, the phase-averaged velocity and energy are not zero:

$$\langle u_\perp\rangle_\chi = -\nu_{ox} \sin \chi_o \ , \ \langle u_\zeta\rangle_\chi = \tfrac{1}{4} \nu_{ox}^2 (1 + 2 \sin^2\chi_o),$$
$$\langle\gamma\rangle_\chi = 1 + \tfrac{1}{4} \nu_{ox}^2 (1 + 2 \sin^2\chi_o) \tag{13}$$

The direction of translational velocity in the plane of the wave depends on polarization and initial phase. In the strong-wave case the average four-velocity component in the direction of wave propagation is much greater than the transverse component; the former of order ν^2, the latter of order ν.

In this idealized case, as mentioned earlier, the particle returns to rest periodically; the frequency and fundamental length of this motion are simply expressed in terms of the wave frequency and wavelength:

$$\Omega_\nu = \Omega\left[1 + \tfrac{1}{4}\nu_{ox}^2 \,(1 + 2\sin^2\chi_o)\right]^{-1}$$

$$\lambda_\nu = \lambda\nu_{ox}^2 \,(1 + 2\sin^2\chi_o)/4 \tag{14}$$

The cycle, of course, coincides with the particles seeing one complete cycle of the wave, a change in χ of 2π. The maximum energy reached in every period is

$$\gamma_{max} = 1 + \tfrac{1}{2}\nu_{ox}^2 \,(1 + |\sin\chi_o|)^2 \tag{15}$$

Notice in (14) that as $\nu \rightarrow \infty$ the "period" becomes very long and the motion approaches the "phase-locked" condition described in NPI. However, the energies given by (13) and (15) are much greater than those found in NPI where $\gamma_{max} \propto \nu^{2/3}$. The origin of the difference is the assumption made here of an infinite train of plane waves. This is clearly not applicable when considering particles injected in the near wave zone of the Crab pulsar. There $\nu_o \simeq 10^{11}$ and $\lambda_\nu \approx 10^{30}$cm, only somewhat larger than the Hubble radius and very much larger than the Crab Nebula. However, over most of the nebula $\nu_o \simeq 10^1 - 10^3$ and the analysis given here would be valid for particles injected at rest, the criterion for local applicability at radius r being $r >> \lambda_\nu$.

c) Elliptically Polarized Waves

The results for circularly and elliptically polarized waves are easily obtained and are substantially the same. In the guiding center frame the amplitude of the z-motion -- $\frac{1}{4}(\nu_{xo}^2 - \nu_{yo}^2)\ (1 + \frac{1}{2}\nu_{xo}^2 + \frac{1}{2}\nu_{yo}^2)^{-\frac{1}{2}}$ - decreases as the wave departs from linear polarization and vanishes for circular polarization. Thus the orbits of particles in a circularly polarized wave field of arbitrary strength are similar to those in a uniform magnetic field; they are skewed circular helices. A particle starting at rest in an elliptically polarized wave acquires a drift velocity now in the y-direction as well, $\langle u_{\perp, y}\rangle_\chi = \nu_{yo} \cos \chi_o$. The energy and the z-drift velocity depend, as before, primarily on ν_o^2 and slightly on the initial phase - the phase dependence disappears, of course, for circular polarization.

d) Motion in a Slowly Changing Plane Wave Field

Let us now consider the motion of a charged particle in a plane wave within which the strength E and frequency Ω vary slowly with phase. Thus, we assume that the driving fields are given, as before, by equation (3) with the phase $\Omega(t - z/c)$ replaced by χ and E_{xo}, E_{yo}, and Ω considered to be weak functions of χ. Then equation (2) still represents the particle equations of motion and it is easily shown that the integral α is still strictly constant. If we further define

$$d\eta \equiv \Omega d\tau \tag{16}$$

then the phase χ is still given by equation (9a) and equations (5a) and (5b) are unchanged. Choosing as independent variable χ and specializing, for simplicity, to the case of linear polarization, we have

$$\frac{du_\perp}{d\chi} = \nu_o (\chi) \cos \chi \tag{17}$$

where $$\nu_o (\chi) \equiv \frac{eE_{xo} (\chi)}{mc\Omega (\chi)} . \tag{18}$$

Assuming now

$$\nu_o (\chi) = \nu_{oo} (1 + \beta\chi), \quad |\beta| << 1, \tag{19}$$

we can integrate (17) to give

$$\begin{aligned} u_\perp &= \nu_{oo} \left[(1 + \beta\chi) \sin \chi + \beta \cos \chi\right] + \text{const} \\ &= \nu_o (\chi) \sin \chi + \frac{d\nu_o}{d\chi} \cos \chi + \text{const} \\ &= \nu_o^* (\chi) \sin \left[\chi + \varepsilon (\chi)\right] + \langle u_\perp \rangle_\chi \end{aligned} \tag{20}$$

where

$$\nu_o^*(\chi) = \nu_o(\chi)\left[1 + \left|\frac{d\ln\nu_o}{d\chi}\right|^2 + 0\left(\frac{1}{\nu_o^2}\frac{d^2\nu_o}{d\chi^2}\right)\right] \tag{21}$$

and

$$\varepsilon(\chi) = \tan^{-1}\left[\frac{d\ln\nu_o}{d\chi} + 0\left(\frac{1}{\nu_o^2}\frac{d^2\nu_o}{d\chi^2}\right)\right] \tag{22}$$

The corresponding z-motion and energy are given in terms of (u, α) as before by equations (9b) and (9c). Thus, for example:

$$\langle\gamma\rangle_\chi = \left[(1+\alpha^2) + \langle u_\perp^2\rangle_\chi + \tfrac{1}{2}\nu_o^*(\chi)^2\right]/2\alpha, \tag{23}$$

The principal result of this exercise is that, apart from unimportant changes in phase $[$ i.e. $\varepsilon(\chi)]$ the energy and longitudinal velocity of a particle are determined by its initial, or its current, conditions depending on which environment has the stronger field. Particles remember the strongest field region (largest ν) they have encountered and, in general, do not lose substantial amounts of energy in going to weak field regions.

III. RADIATION REACTION IN A PLANE WAVE

a) The Equations

The equations of motion are now

$$mc\frac{du^\mu}{d\tau} = e F^{\nu\mu} u_\nu + \frac{2e^2}{3c^2}\left[\frac{du^\alpha}{d\tau}\frac{du_\alpha}{d\tau}u^\mu + \frac{d^2u^\mu}{d\tau^2}\right], \tag{24}$$

where $F^{\nu\mu}$ is the Maxwell field tensor and the second and third terms represent the reaction force on the particle due to the radiation it emits. We shall assume in this treatment that the radiative reaction forces are relatively very small so that the change in any quantity per cycle due to the radiative reaction is very small compared to the quantity itself. In this approximation, the terms in the bracket can be evaluated at the current value of the dynamical quantities without inclusion of radiative reaction. That is, we will linearize about a small parameter to be introduced shortly and later investigate the domain of validity for the resulting approximations. In the dimensionless variables introduced in equation (4), equation (24) can be written like equation (5) if we add to the right-hand sides a four-vector

$$\phi^{\mu} \equiv \frac{2e^2\Omega}{3mc^3}\left[\frac{du^{\alpha}}{d\eta}\ \frac{du_{\alpha}}{d\eta}\ u^{\mu} + \frac{d^2u^{\mu}}{d\eta^2}\right] \tag{25}$$

representing the radiative reaction force. Now, using the solution found in Section II, we have

$$u^{\mu}\ \frac{du^{\beta}}{d\eta}\ \frac{du_{\beta}}{d\eta} = -\alpha^2\ \underset{\sim}{\nu}\cdot\underset{\sim}{\nu}\ u^{\mu}$$

$$\frac{d^2u^{\mu}}{d\eta^2} = \alpha\left[\alpha\ \frac{d\nu_x}{d\chi},\ \alpha\ \frac{d\nu_y}{d\chi},\ \nu^2 + \underset{\sim}{u}_{\perp}\cdot\frac{d\underset{\sim}{\nu}}{d\chi},\ \nu^2 + \underset{\sim}{u}_{\perp}\cdot\frac{d\underset{\sim}{\nu}}{d\chi}\right] \tag{26}$$

Consistent with our assumption of a small radiative reaction, we will average its effect over one gyration cycle and obtain

$$\left\langle \frac{du^\beta}{d\eta}\frac{du_\beta}{d\eta} u_\zeta \right\rangle = -\tfrac{1}{2}\alpha^2\nu_o^{\,2}\ \langle u_\zeta\rangle + \alpha(\nu_{ox}^{\,2} - \nu_{o\eta}^{\,2})^2/16$$

$$\left\langle \frac{du^\beta}{d\eta}\frac{du_\beta}{d\eta} \gamma \right\rangle = -\tfrac{1}{2}\alpha^2\nu_o^{\,2}\ \langle\gamma\rangle + \alpha(\nu_{ox}^{\,2} - \nu_{o\eta}^{\,2})^2/16$$

$$\left\langle \frac{du\beta}{d\eta}\frac{du_\beta}{d\eta} \underset{\sim}{u}_\perp \right\rangle = -\tfrac{1}{2}\alpha^2\nu_o^{\,2}\langle \underset{\sim}{u}_\perp\rangle\ ,\quad \frac{d^2u^\mu}{d\eta^2} = 0 \qquad (27)$$

where, for notational simplicity, we have omitted the χ subscript on the phase averages. Equations (25) - (27) determine the phase-averaged radiation reaction force. The equations of motion now reduce to

$$\frac{d\underset{\sim}{u}_\perp}{d\chi} = \underset{\sim}{\nu} - \varepsilon\alpha\langle \underset{\sim}{u}_\perp\rangle,\quad \frac{d\alpha}{d\chi} = -\varepsilon\alpha^2$$

$$\gamma = u_\zeta + \alpha,\ u_\zeta = (1 - \alpha^2 + u_\perp^2)/2\alpha, \qquad (28)$$

where

$$\varepsilon \equiv \frac{e^2\Omega\nu_o^{\,2}}{3\ mc^3} = \frac{e^4B_m^2}{3m^3c^5\Omega} \qquad (29)$$

is a dimensionless parameter which is a measure of the radiative drag. Except in the immediate vicinity of pulsars ε will be very small. In the Crab Nebula where $\nu_o \simeq 10^1$ to 10^3, the radiation parameter $\varepsilon \simeq 10^{-19}$ to 10^{-15}. In deriving the radiative reaction force, we have treated α as a constant which it no longer is. Rather, we see from (28) that this (positive) quantity is a monotonically decreasing function of time. A satisfactory requirement for the validity of our linearization procedure is $|d\ln \alpha/d\chi| << 1$. From (28) we see that this is equivalent to the requirement $\epsilon\alpha<<1$ for weak radiative reaction. If one does <u>not</u> average over phase, it is still possible to treat the equation for $\underset{\sim}{u}_\perp$ with reasonable ease, and one finds that the only effect is an additional phase lag in the particle motion of order $\epsilon\alpha$. Returning to the case of weak radiative reaction in a plane wave, we find some rather bizarre long-term effects. The equation for α can be immediately integrated. We find

$$\alpha = \frac{\alpha_o}{(1 + \epsilon\alpha_o\chi)}, \tag{30}$$

where α_o is the initial value of α. The solution for $u_\perp$ is as given in Section II except now the average perpendicular velocity is not constant but decreases with time

$$\langle u_\perp \rangle = \frac{\langle u_\perp \rangle_o}{(1 + \epsilon\alpha_o\chi)} . \tag{31}$$

Now, since

$$\langle u_\perp^2 \rangle = \langle u_\perp \rangle^2 + \tfrac{1}{2} \nu_o^2 \tag{32}$$

we find, from (28), (30) -(32),

$$\langle \gamma \rangle = \frac{1}{4\alpha_o} (2 + \nu_o^2) (1 + \epsilon\alpha_o\chi) + \frac{1}{2\alpha_o} (\langle u_\perp^2 \rangle_o + \alpha_o^2)/ (1 + \epsilon\alpha_o\chi), \tag{33}$$

and a similar expression for u_ζ . Equations (31) through (33) are valid even if $\epsilon\alpha_o\chi>>1$ so long as $\epsilon\alpha<<1$.

We see that for any initial conditions, the radiative losses will ultimately lead to increases in the particle energy, and that for the conditions considered here, the energy will continue to increase without limit. This paradoxical result is perhaps most easily understood by considering motion in a circularly polarized wave.

Here the rate of change of γ is proportional to $\underset{\sim}{u}_\perp \cdot \underset{\sim}{\nu}$, simply the projection of the electric field along the motion. In the absence of radiation $\underset{\sim}{u}_\perp$ lags $\underset{\sim}{\nu}$ by exactly $\pi/2$; the field is always perpendicular to the velocity, and $d\gamma/d\eta$ vanishes. The effect of radiation drag, however, is to induce a phase lag in $u_\perp$, as remarked earlier, making $u_\perp \cdot \underset{\sim}{\nu}$ always positive around the orbit. Since, for a plane wave, $\underset{\sim}{u}_\perp \cdot \underset{\sim}{E} = |\underset{\sim}{u}_\perp \times \underset{\sim}{B}|$, a similar acceleration is induced in the ζ motion in the direction of propagation.

b) "Radiative Pumping"

We can easily find the dependence of γ on time in the asymptotic region where $\varepsilon\alpha_o\chi$ is large. Here

$$\langle\gamma\rangle \simeq \tfrac{1}{4}\,(2 + \nu_o^{\,2})\,\varepsilon\chi, \qquad (\chi \to \infty). \tag{34}$$

The dependence of time ($t = \xi/\Omega$) on phase is found by noting that, since $\gamma = \alpha\, d\xi/d\chi$,

$$d\xi = \frac{\gamma}{\alpha}\,d\chi \simeq \tfrac{1}{4}\,(2 + \nu_o^{\,2})\,\varepsilon^2\chi^2 d\chi,$$

$$\xi \simeq (2 + \nu_o^{\,2})\,\varepsilon^2\chi^3/12 + \text{const}, \qquad (\chi \to \infty) \tag{35}$$

Finally, (34),(35) and the definition of ξ give

$$\langle\gamma\rangle = \left[\frac{3}{16}\,(2 + \nu_o{}^2)^2 \quad \varepsilon\Omega\right]^{1/3} t^{1/3} \qquad (t \to \infty) \tag{36}$$

The energy increases very slowly with time. This mechanism of "radiative pumping" is probably not important in the Crab Nebula now. Taking $\nu_o = 10^2$, which might be typical in the inner portions of the nebula, ε is $\simeq 5 \times 10^{-17}$ and $\langle\gamma\rangle$ for a particle whose guiding center is at rest is of order ν_o; the time required for the energy to double, say, is of order 10^8 yr, much longer than the age (or expected lifetime) of the Crab. In addition, we shall see that particles trapped in spherical wave fields always suffer radiative losses larger than their gains. The fact that charged particles moving in the direction of a strong wave will gain, not lose, energy when they radiate remains true, however, and may prove to be of interest in other applications.

IV. RADIATION FROM PARTICLES MOVING IN A PLANE WAVE

a) The Equations

The nature of the radiation from particles moving in a strong wave field will be investigated in detail in a subsequent paper; let us now note a few of the salient features. The total energy radiated by a particle of charge e per unit time, integrated over all frequencies, is

$$P = -\frac{2e^2\Omega^2}{3c}\left[\frac{du^\mu}{d\eta}\frac{du^\mu}{d\eta} + \frac{1}{\gamma}\,\frac{d^2\gamma}{d\eta^2}\right], \tag{37}$$

where the bracketed terms in the standard formula have been expressed in our dimensionless variables. To find the energy radiated per cycle, we must average over time (ξ), not phase (χ). Then, since $d\xi = \gamma\, d\eta = \gamma\, d\chi/\alpha$, the time average of any quantity Ψ is

$$\overline{\Psi} = \frac{\langle \Psi\gamma \rangle_\chi}{\langle \gamma \rangle_\chi} , \tag{38}$$

and the time-averaged power output of a particle in a plane wave is, from equations (27), (37), and (38),

$$\overline{P} = \frac{e^2\Omega^2}{3c\langle\gamma\rangle_\chi} \left[\alpha^2 \nu_o^2 \langle\gamma\rangle_\chi - \frac{1}{8}\alpha(\nu_{ox}^2 - \nu_{oy}^2)^2 \right] . \tag{39}$$

This can be written in the more transparent form

$$\overline{P} = \varepsilon\, mc^2\alpha^2\Omega\, Q , \tag{40}$$

where ε is the radiation parameter [cf. eq. (29)] and Q is a factor of order unity:

$$Q = \frac{1 + \alpha^2 + \langle u_\perp{}^2\rangle_\chi + (\nu_{ox}{}^4 + 6\nu_{oy}{}^2\nu_{ox}{}^2 + \nu_{oy}{}^4)/4\nu_o{}^2}{1 + \alpha^2 + \langle u_\perp{}^2\rangle_\chi^2 + \frac{1}{2}\nu_o{}^2} . \tag{41}$$

The total range possible for Q is

$$\tfrac{1}{2} < Q < 2 ,$$

and (42)

$$Q \to 1 \quad \text{as} \quad \langle u_\perp\rangle/\nu_o \to \infty$$

in that commonly important limit. Equation (40) simply says that in units of its rest energy mc^2 a charged particle loses an amount $\sim\alpha^2\varepsilon$ per period of the wave which forces its motion.

b) Radiation from a Particle Injected at Rest

Let us look at this result [equations (39) and (40)] in two interesting limiting cases. First, consider a particle starting at rest. Noting that $\alpha = 1$ and Q is near unity under all circumstances we find from (29) and (40) that an electron dropped at rest into a strong field will radiate

$$\overline{P} \simeq \frac{e^4 B_m^2}{3m^2c^3} = 8 \times 10^{-16}\ B_m^2 \ \text{erg/sec} \tag{43}$$

independent of the frequency of the driving wave.

c) Highly Relativistic Particles

For the second case, consider $|u_\perp| >> \nu_o + 1$ corresponding to a highly relativistic particle moving obliquely to the wave front. The energy is nearly constant and deviations from rectilinear motion are slight. Here $\overline{\gamma} = \langle\gamma\rangle_\chi$ and

$$\overline{P} = \frac{e^2\ \Omega^2\ \nu^2\ \alpha^2}{3c}, \qquad (u_\perp >> \nu_o + 1) \tag{44}$$

Now, noting that $\frac{1}{2}\Omega^2\nu_o^{\ 2} = e^2E_o^2/m^2c^2 = 4\pi U/m^2c^2$, where E_o is the R.M.S. electric field and U the mean energy density in the field, we can write

$$\overline{P} = \frac{8\pi e^4}{3m^2c^3} U \gamma^2 (1 - v_z/c)^2 , \qquad (|u_\perp| >> \nu_0 + 1) \qquad (45)$$

which is identical to the standard expression for the power radiated via "inverse Compton" losses. For an isotropic distribution of relativistic particles (γ>>1), the power radiated per particle is exactly the same as the same particles radiating via synchrotron radiation in a uniform field of energy density U, to wit:

$$\overline{P} = \frac{32\pi e^4}{9m^2c^3} U \gamma^2 \qquad \left[\begin{array}{l}\gamma >> 1 \\ \text{isotropic velocity} \\ \text{distribution}\end{array}\right] \qquad (46)$$

d) <u>"NIC" Radiation</u>

The character of the radiation emitted by these processes [eqs. (40) and (46)] is, in general, different from both inverse Compton and synchrotron radiation. Consider ν_0>>1 (if ν_0<<1 the radiation calculated by eq. (46) <u>is</u> ordinary inverse Compton, of course,) and $u_\perp$>> ν_0 -- the conditions of (c) above. Let us investigate the angular deviation in the trajectory produced by the wave. The amplitude in $u_\perp$ is just ν_0, so the angular amplitude is ν_0/γ (an exact result in the limit considered). The beamwidth of the radiation from a relativistic particle is of order $1/\gamma$, so for ν_0> 1 the beam of the particle sweeps back and forth past the observer more or less in the manner of a gyrating <u>synchrotron</u> particle, and one expects the spectrum to resemble more nearly that of synchrotron radiation than that of inverse Compton. One can estimate the critical frequency simply as follows: The angular change in velocity is $2\nu_o/\gamma$ in the time it takes for the particle to travel from one wave node to the next, $\Delta t = 2\pi\Omega^{-1} (1 - v_z)^{-1}$. This corresponds to an average circular frequency

$$\omega_c = \frac{2\nu_o}{\gamma\ \Delta t} = \frac{\nu_o\ \Omega(1 - \nu_z)}{\pi\gamma} = \frac{e\ E_o\ (1 - v_z)}{\pi mc\gamma}\ , (|u_\perp| >> \nu_o >> 1) \qquad (47)$$

<u>independent</u> of the wave frequency. It depends only on the wave amplitude and the particle energy, and, in fact, does so in the same manner as the relativistic gyrofrequency in a static magnetic field depends on field strength and energy. Thus, the critical frequency is

$$\nu_{crit} = 2\pi\omega_c\gamma^3 = \frac{e\ E_o\gamma^2}{\pi mc} \qquad (u_\perp >> \nu_o >> 1) \qquad (48)$$

The highest frequencies emitted in ordinary inverse Compton radiation are of order $\nu_{IC} \sim 2\pi\gamma^2\Omega$ so the ratio of frequencies is

$$\frac{\nu_{crit}}{\nu_{IC}} = O\ (\nu_o) \qquad (|u_\perp| >> \nu_o >> 1) \qquad (49)$$

for relativistic particles traveling at large angles to the wave propagation direction. Thus, the behavior changes at $\nu_o \sim 1$ from the frequency-dependent, strength-independent classical inverse Compton process to the frequency-independent synchrotron-like process described here. We propose to call radiation in this regime nonlinear inverse Compton (NIC) radiation. The radiation rates are given by equations (40) or (46) and the peak frequency by equation (48). NIC radiation is clearly polarized if the radiation field is coherent (or even reasonably unidirectional and itself polarized.)

V. MOTION IN A SPHERICAL WAVE FIELD

a) The Equations of Motion

While the plane wave theory given in parts I and II is in many cases satisfactory locally, we have seen that extreme caution must be used in applying its predictions to the real world. A much better model for most cases of interest is a spherical radiation field. We will treat in some detail a dipole field, but there is no essential complication in treating fields with arbitrary angular dependence. We consider only the far-field region; there the propagation direction is accurately radial.

The equations of motion are

$$\frac{du^{\mu}}{d\tau} + \Gamma^{\mu}_{\nu\sigma}\, u^{\nu}u^{\sigma} = \frac{e}{m} F^{\nu\mu}u_{\nu} \; . \tag{50}$$

Now let

$$\zeta \equiv \Omega r/c, \; u^{0} \equiv \gamma, \; u^{\theta} \equiv \gamma v^{\theta}, \; u^{\phi} \equiv \gamma v^{\phi}$$

$$f^{\theta} = \frac{eE_{\theta}r}{mc^2}, \; f^{\phi} = \frac{eE_{\phi}r}{mc^2} \tag{51}$$

and (ξ, η, χ) retain the definitions given in equation (4) with r substituted for z. In these variables equation (50) becomes

$$\zeta \frac{du_\zeta}{d\eta} - u_\perp^2 = \underset{\sim}{f} \cdot \underset{\sim}{u}_\perp , \quad \zeta \frac{d\gamma}{d\eta} = \underset{\sim}{f} \cdot \underset{\sim}{u}_\perp , \quad \frac{\delta}{\delta\eta} (\zeta \underset{\sim}{u}_\perp) = (\gamma - u^\zeta) \underset{\sim}{f} \qquad (52)$$

where $\underset{\sim}{f} = (f_\theta , f_\phi)$, $\underset{\sim}{u}_\perp = (u^\theta, u^\phi)$, and $\frac{\delta}{\delta\eta}$ denotes the absolute derivative on the sphere (reducing to the ordinary derivative at the equator). Note also that u^θ, u^ϕ are physical components, not contravariant components, so that

$$\gamma^2 - u_\zeta^2 - u_\perp^2 = 1 \qquad (53)$$

as can be verified by integration of (52) (the notation differs slightly from that used in NPI.) In the dipole case, the field strength $\underset{\sim}{f}$ can be given in terms of a constant f_o representing the magnetic dipole luminosity L_{md}:

$$f_\theta = f_o \sin\chi, \quad f_\phi = f_o \cos\theta \cos\chi , \quad f^2 = f_\theta^2 + f_\phi^2$$

where (54)

$$f_o = \left[\frac{3e^2 L_{md}}{2m^2 c^5}\right]^{1/2} = \text{const.}$$

Note that f_o is just ν_o evaluated at the wave radius $\zeta = 1$ ($r = c/\Omega$) at the equator. If we again let $\alpha \equiv \gamma - u^\zeta = d\chi/d\eta$, then, using (53) we find

$$u_\zeta = (1 - \alpha^2 + u_\perp^2)/2\alpha \qquad (55)$$

as before. The other equations of motion can now be written

$$\frac{\delta}{\delta\chi} (\zeta \underset{\sim}{u}_\perp) = \underset{\sim}{f}, \quad \frac{d\alpha}{d\eta} = - \frac{u_\perp^2}{\zeta} \qquad (56)$$

b) Motion of the Guiding Center

First we discuss the orbits of relativistic particles injected into the far wave zone. Following the example of the plane wave treatment, we shall first look at the regime where the guiding center approximation is valid; i.e., the approximation in which the amplitude of the gyration in space is small compared to the scale length of the fields -- typically the radial distance from the center. Note that in a dipole field $\partial f_\theta / \partial \phi = \partial f_\phi / \partial \phi = 0$. It then easily follows (since excursions in θ do not produce first-order changes in f_θ, and likewise for θ) that, averaging over phase

$$\frac{\delta}{\delta \chi} (\zeta \langle \underset{\sim}{u}_\perp \rangle) = 0 . \qquad (57)$$

Thus, the guiding center angular momentum is conserved, and the orbit of the guiding center is planar, if the excursions introduced by the fields are small. The oscillatory part of $\underset{\sim}{u}_\perp$ is clearly just

$$\underset{\sim}{u}_\perp - \langle \underset{\sim}{u}_\perp \rangle = \frac{1}{\zeta} (- f_o \cos \chi, f_o \cos \theta \sin \chi), \quad (\Delta\zeta/\zeta<<1) \qquad (58)$$

where the parenthesized condition expresses the requirement that the excursions in ζ be small compared to ζ. Thus, if we denote the angular momentum per unit mass vector by $\underset{\sim}{h} \equiv \zeta \langle \underset{\sim}{u}_\perp \rangle$,

$$h_\theta^2 + h_\phi^2 = h^2 = \text{const.} \qquad (59)$$

and

$$\underset{\sim}{u}_\perp = \frac{1}{\zeta}\,(h_\theta - f_o \cos\chi,\ h_\phi + f_o \cos\theta \sin\chi),$$

(60)

$$\langle u_\perp^2 \rangle = \left[h^2 + \tfrac{1}{2} f_o^{\,2}\,(1+\cos^2\theta) \right] \zeta^{-2} = (\underset{\sim}{u}_\perp)^2 + \tfrac{1}{2} f_o^{\,2}\,(1+\cos^2\theta)\zeta^{-2}.$$

Note from (56) that α is a monotonically, and slowly (if $u_\perp$ is not too big) decreasing function of η. Thus, we can replace (56) with

$$\alpha \frac{d\alpha}{d\chi} = - \frac{\langle u_\perp^2 \rangle}{\zeta} . \qquad (61)$$

Let us pause for a moment to investigate the conditions under which our approximations are valid. We require first that the total excursion in θ or ϕ in one cycle be small. The RMS $u_\perp$ is of order $\zeta^{-1}\,(h^2 + f_o^{\,2})^{\frac{1}{2}}$ and

$$\Delta\theta,\ \Delta\phi \sim \zeta^{-1} u_\perp\, \alpha^{-1} \sim \frac{(h^2+f_o 2)^{\frac{1}{2}}}{\alpha\zeta^2} \ll 1. \qquad (62)$$

The excursion in ζ is of order

$$\frac{f_o^{\,2} + h^2}{\alpha^2 \quad \zeta^2} \ll \zeta\ ,\ \text{or}\ \frac{f_o 2 + h^2}{\alpha^2 \quad \zeta^3} \ll 1 \qquad (63)$$

for $\zeta^2 < fo^2 + h^2$.[3]

The condition that $\alpha^{-1} \frac{d\alpha}{d\chi}$ be small reduces to the same inequality as (63); we assume that we are in the far wave zone ($\zeta \gg 1$), so (62) is satisfied whenever (63) is, and the satisfaction of (63) is necessary and sufficient for the guiding center picture to be valid.

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[3] It is clear that the case $\zeta^2 > fo^2 + h^2$ is uninteresting, since in this case neither the field nor the angular momentum is significantly affecting the motion of the particle.

We can solve for the orbit simply in one interesting case which contains all the essential features of the problem. Let the guiding center travel initially in the equatorial plane. Thus we have no θ-variation. Let

$$\ell^2 \equiv h^2 + f_o^2/2 = \zeta^2 \langle u_\perp{}^2 \rangle \tag{64}$$

Then the problem is solved if we can find α and ζ, the radial coordinate of the guiding center; the differential equations are

$$\frac{d\alpha}{d\eta} = - \frac{\ell^2}{\zeta^3}$$

$$\frac{d\zeta}{d\eta} = \langle u_\zeta \rangle = \frac{1 - \alpha^2}{2\alpha} + \frac{\ell^2}{2\alpha\zeta^2} \tag{65}$$

Let us assume that the particle recedes to ∞ as $\eta \to \infty$, and let α_∞ be the limiting value of α. Then the system (65) can be integrated (eliminating η) to yield

$$\frac{\ell^2}{\zeta^2} \equiv \langle u_\perp^2 \rangle = \frac{1 + \alpha_\infty^2}{\alpha_\infty} \alpha - (1 + \alpha^2)$$

$$\langle u_\zeta \rangle = \left[\frac{1 + \alpha_\infty^2}{2\alpha_\infty} - \alpha \right], \langle \gamma \rangle = \frac{1 + \alpha_\infty^2}{2\alpha_\infty} = \text{const} \tag{66}$$

Thus, the mean energy of the particle is conserved.

Let us assume that $\langle \gamma \rangle >> 1$. Then the particle initially came from infinity with $u_\zeta \simeq -\langle \gamma \rangle$ and $\alpha \simeq 2\langle \gamma \rangle$. It reaches periastron ($\langle u_\zeta \rangle = 0$) when $\alpha = \langle \gamma \rangle$ and then recedes, with α approaching $(2\langle \gamma \rangle)^{-1}$ as the radius goes to infinity. The orbit and periastron distance are found by noting that

$$\langle u_\perp \rangle = \frac{\ell}{\zeta}\left[1 + \frac{f_o^{\,2}}{2h^2}\right]^{-\frac{1}{2}} , \tag{67}$$

and that, using (66) and the definition of α,

$$\langle u_\zeta \rangle^2 = \langle \gamma \rangle^2 - 1 - \frac{\ell^2}{\zeta^2} \tag{68}$$

From (67) and (68) we determine the orbit in the usual way and find

$$\zeta = \zeta_p \sec \left[(1 + f_o^{\,2}/2h^2)^{\frac{1}{2}}(\theta - \theta_o)\right] \tag{69}$$

where

$$\zeta_p \equiv \left[\frac{h^2 + f_o^{\,2}/2}{\langle \gamma \rangle^2 - 1}\right]^{\frac{1}{2}} \simeq \frac{\ell}{\langle \gamma \rangle} \tag{70}$$

is the periastron distance.

It can be seen from (68), in fact, that the orbit of the guiding center is the same as that for a classical particle moving in a central-force field having a repulsive potential $f_o^{\,2}/4\zeta^2$.

c) Validity of the Guiding Center Approximation

Let us check the goodness of our approximations. It is clear that, since α is of order $\langle \gamma \rangle$ for the entire first half of the orbit, that the criterion (63) is worst satisfied on the incoming branch at periastron. There $\alpha = \langle \gamma \rangle, \zeta = \zeta_p$ and $\ell^2/\alpha^2\zeta^3 = \gamma/\ell$; so the criterion is satisfied if $\gamma \ll \ell$, $\zeta_p \gg 1$. Thus, if the particle remains in the far wave zone, there is no trouble while it is incoming. For the outgoing branch we can write

$$\alpha \simeq \frac{1}{2\langle\gamma\rangle}\left[\frac{\ell^2}{\zeta^2}+1\right] \text{ and } \frac{\ell^2}{\alpha^2\zeta^3} \simeq \frac{4\langle\gamma\rangle^2\ \zeta\ell^2}{(\ell^2+\zeta^2)^2} \quad . \tag{71}$$

This latter quantity is less than unity (our criterion) only if $\langle\gamma\rangle \lesssim \ell^{\frac{1}{2}}$.

The phenomena that occur at higher energies, however, are relatively simple. It is easy to see, first, that if the criterion (63) is violated, the trouble occurs at a value of $\zeta < \ell$; physically, for $\zeta > \ell$, $\alpha \sim (2\langle\gamma\rangle)^{-1}$ and the particle essentially no longer interacts with the wave. Let us look at the radius as a function of phase, and let us assume that $\zeta_p >> 1$. Thus, the approximations run into trouble when $\zeta >> \zeta_p$, $\alpha << \langle\gamma\rangle$. If α is small compared to $\langle\gamma\rangle$, $\langle u_\zeta\rangle \simeq \langle\gamma\rangle$, and we can write

$$\frac{d\zeta}{d\chi} = \frac{1}{\alpha} \quad u^\zeta \simeq \frac{2\langle\gamma\rangle^2\zeta^2}{\ell^2} \tag{72}$$

recalling that $\alpha \simeq \ell^2 \ (2\langle\gamma\rangle\zeta^2)^{-1}$, since $\zeta << \ell$.

Thus, the residual phase from ζ to "infinity" (actually to $\zeta \sim \ell$, since it is there that our expression for α breaks down) is

$$\Delta\chi \simeq \frac{\ell^2}{2\langle\gamma\rangle^2\zeta} \tag{73}$$

This becomes of order unity at the same value of ζ for which the amplitude of the guiding center motion becomes large. Thus the trouble occurs because the particle becomes "phase-locked" -- it is traveling radially at so nearly the velocity of light that large changes in the radius occur for small changes in the phase.

It is easy to see that this causes real difficulty only if $f_o \gtrsim h$, for if $h >> f_o$ (and hence $\ell \sim h$), the perpendicular motion is always dominated by the "constant" (h/ζ) part, $d\alpha/d\eta$ is then independent of the oscillatory part and it <u>doesn't</u> <u>matter</u> whether the phase locks or not; $\gamma \simeq \langle\gamma\rangle$ and is nearly constant over the whole orbit (which is nearly rectilinear). If $f_o \gtrsim h$, however, one expects major modifications to the guiding center picture.

We can see how this goes easily for the case $\langle\gamma\rangle >> \ell^{\frac{1}{2}}$; i.e., for those particles for which the breakdown is in some sense "strong." First of all, we have an <u>exact</u> expression for $u_\perp$ [cf. Eqs. (59) - (60)] as a function of ζ and χ. Using Equation (52), and again assuming that $\alpha << \overline{\gamma}$, it is easy to show that the fractional change in γ from ζ to infinity on the exit orbit is bounded by

$$\left|\frac{\Delta\gamma}{\langle\gamma\rangle}\right| < \frac{\ell^2}{\langle\gamma\rangle^2\zeta} , \qquad (74)$$

which is small for $\zeta > \ell^2 \langle\overline{\gamma}\rangle^2$. Comparing (74) with (73) we see that γ changes little after phase locking occurs, so we may confine our attention to the region near $\zeta_{crit} \simeq \ell^2/\langle\gamma\rangle^2$, for which the value of α is about $\alpha_{crit} \simeq \langle\gamma\rangle^{3}/2\ell^2$. Now $\alpha^2_{crit} << \langle u_\perp{}^2\rangle$ at ζ_{crit}, so we can write the expression (55) for u_ζ as

$$\frac{d\zeta}{d\chi} \simeq \frac{u_\perp^2}{2\alpha^2}, \qquad (\zeta_{crit} \lesssim \zeta << \ell) \qquad (75)$$

which will be true except (possibly) when cos χ is near zero. On the other hand, Equation (56) tells us that $\alpha d\alpha/d\chi = - u_\perp^2/\zeta$ exactly. Thus, we can solve (75) to obtain

$$\alpha = \frac{\ell^2}{2\langle\gamma\rangle\zeta^2} \qquad (\zeta_{crit} \lesssim \zeta << \ell) \qquad (76)$$

which is the same as previously obtained under other circumstances [Eq. (71)]. Now if we insert (60) into (75) and integrate, we find

$$\frac{\ell^4}{2\langle\gamma\rangle^2}\left[\frac{1}{\zeta_r} - \frac{1}{\zeta}\right] = \ell^2\chi + \tfrac{1}{4} f_o^2 \sin 2\chi \qquad (77)$$

where ζ_r is some earlier value of ζ at which the phase was zero (modulo 2π).

It is clear that the phase at any value of ζ well before phase locking occurs is essentially random for an assembly of particles with random initial conditions, and so we obtain the somewhat surprising result that γ locks and becomes large at an essentially random phase χ_L. The value of γ is asymptotically

$$\gamma \sim \langle\gamma\rangle \frac{(h^2 + f_o^2 \cos^2\chi_L)}{\ell^2} \qquad (78)$$

and is thus distributed between $(f_o^2 + h^2)\,\ell^{-2}\langle\gamma\rangle$ and $h^2\ell^{-2}\langle\gamma\rangle$. The phase-locking phenomenon thus introduces a dispersion in energy which is large for orbits of small angular momentum.

Note that this dispersion represents a net energy input for particle distribution functions which increase toward lower energies, and thus, in a sense, the phase locking gives rise to an acceleration mechanism. The phenomenon is essentially unchanged for particle orbits out of the equatorial plane, and the redistribution in energy is of the same order.

d) Particles Injected at Rest

Before leaving this topic, let us investigate the behavior of particles dropped at rest into the wave. The initial value of α is, of course, unity. Let the phase at the initial instant be χ_o and the radius ζ_o. Then from (60)

$$h_\theta = f_o \cos \chi_o, \; h_\phi = -f_o \cos \theta_o \sin \chi_o \tag{79}$$

so that $h^2 \leqslant f_o^{\,2}$ and $\ell^2 \leqslant 3/2\, f_o^{\,2}$. The formal "average" energy of the guiding center motion is, from (66),

$$\langle\gamma\rangle = \frac{\ell^2}{2\zeta_o^{\,2}} + 1 = O\left(\frac{f_o^{\,2}}{\zeta^2}\right) . \tag{80}$$

This energy is in general, however, not reached because, as we saw in the last section, phase locking can invalidate the guiding center picture. There are no problems if $\zeta_o \gtrsim \ell^{3/4}$; phase locking does not occur, $\langle\gamma\rangle \leq O\ (\ell^{1/2})$ and the particle goes to infinity with $\gamma = \langle\gamma\rangle$. For particles with ζ_o in the range $f_o^{\,2/3} \leqslant \zeta_o \leqslant f_o^{\,3/4}$ phase locking occurs before they reach $\zeta = \ell$ and $\gamma \to O\ (\langle\gamma\rangle)$ at infinity, but randomized, as described in Section Vc. Let us now look at the particles starting nearer the center of the wave source, for which $\zeta_o < f_o^{2/3}$. For these the total phase change from rest is small, as was pointed out, using other techniques, in NPI. First, we calculate the total phase change occurring as the particle goes from ζ_o to some large ζ using equation (77) (rather than equation (73), which only applies if the guiding center approximation is valid)

$$\frac{2\zeta_o^4}{f_o^2}\left(\frac{1}{\zeta_o} - \frac{1}{\zeta}\right) = \frac{1}{3}\sin^2\chi_o\,(\chi - \chi_o)^3 , \quad \left[(\chi - \chi_o) << 1\right]. \tag{81}$$

Recalling that, in this case,

$$\alpha = (\zeta_o/\zeta)^2 , \quad u_\perp = f_o\,(\cos\chi_o - \cos\chi)\zeta^{-1}, \tag{82}$$

we obtain

$$\gamma \simeq \frac{u_\perp^2}{2\alpha} \cong \frac{f_o^2\sin^2\chi_o\,(\chi - \chi_o)^2}{2\zeta_o^2} = \left[\frac{3}{\sqrt{2}}\,f_o\,(1 - \zeta_o/\zeta)\,\sin\chi_o\right]^{2/3} , \tag{83}$$

in agreement with the result in NPI. The result is easily generalized to arbitrary initial angular coordinates and becomes

$$\gamma = \left[\frac{3}{\sqrt{2}}\,f_o\,(1 - \zeta_o/\zeta)\right]^{2/3}(\sin^2\chi_o + \cos^2\theta_o\cos^2\chi_o)^{1/3} , \quad \left(1<\zeta_o\leqslant f_o^{2/3}\right) \tag{84}$$

Note that γ goes to a value at infinity which is independent of ζ_o for this case.

We thus have the following behavior: For $1 < \zeta_o \leqslant f_o^{2/3}$ the final energy is independent of position ζ_o, depends weakly on initial phase χ_o and is of order $f_o^{2/3}$. As ζ_o is increased past $f_o^{2/3}$, the total phase change becomes rapid at first but later locks, the ultimate γ being approximately $\langle\gamma\rangle$, which for particles starting from rest takes the form [cf. Eq. (66)]

$$\langle\gamma\rangle = 1 + \tfrac{1}{4} f_o^2 \zeta_o^{-2}\left[1 + 2\cos^2\chi_o + \cos^2\theta_o\,(1 + 2\sin^2\chi_o)\right] \qquad (85)$$

but distributed between $\gamma_{min} \approx 0$ and $\gamma_{max} = 2 f_o^2 \zeta_o^{-2}$ for small changes in ζ_o. Finally, for very large ζ_o, γ is exactly $\langle\gamma\rangle$ and is given by Equation (85).

We note that the approximate results agree, as they must, in the interface regions where $\zeta_o \approx f_o^{2/3}$ or $\zeta_o \approx f_o^{3/4}$.

e) Motion Out of the Equatorial Plane

We begin our discussion of the guiding center motion of particles out of the equatorial plane by noting from Equation (65) and the definition of α that

$$\langle\gamma\rangle = \frac{1+\alpha^2}{2\alpha} + \frac{\ell^2}{2\alpha\zeta^2} \qquad (86)$$

for particles which are not phase-locked, where here $\ell^2 = f^2 + h^2$. Also, from equation (65) we have

$$\frac{d\langle\gamma\rangle}{d\eta} = \frac{1}{2\alpha\zeta^2}\,\frac{d\ell^2}{d\eta} = \frac{1}{4\alpha\zeta^2}\,\frac{df^2}{d\eta}\ . \qquad (87)$$

We found $\langle\gamma\rangle$ constant for motion in the equatorial plane because $|f|$ and consequently ℓ are constant in that plane. Although ℓ^2 is now variable, its range of variation is small;

$|f|$ changes at most by a factor of two from equator to pole for a dipole radiation pattern and the changes in f are diluted by any angular momentum since $\ell \equiv (h^2 + \frac{1}{2} f^2)^{1/2}$. We will, therefore, do the analysis under the assumption that the deviations of ℓ from some average value $\bar{\ell}$ are small. We assume that the orbits and α-dependence are described adequately by constant $\ell \approx \bar{\ell}$. In this case, we obtain from Equation (66) that

$$\frac{1}{2\alpha\zeta^2} = \frac{1}{\bar{\ell}^2}\left[\langle\gamma\rangle - \frac{1}{2}\left(\frac{1}{\alpha} + \alpha\right)\right]. \tag{88}$$

and it is easily shown that the $\frac{1}{\alpha}$ term has negligible effect when $\langle\gamma\rangle \gg 1$(essentially because α small implies that $u_\zeta \sim \gamma$, the motion is nearly radial, and ℓ is, therefore, nearly constant). Then, from this result and Equation (69) we find

$$\alpha \simeq \langle\gamma\rangle \left(1 - \sin \frac{\bar{\ell}\psi}{h}\right) \tag{89}$$

where ψ is the angle measured in the plane of the orbit from the periastron direction to the position of the particle (Fig. 1) with vertex at the origin; we assume for definiteness that the particle passes <u>above</u> the pole.

We then find

$$\Delta\alpha = \frac{\langle\gamma\rangle}{2} \int (1 + \sin \frac{\bar{\ell}\psi}{h}) \frac{d\ln\ell^2}{d\psi}\, d\psi \tag{90}$$

where $\Delta\gamma$ is the difference between final (exit) and initial (entrance) energies. If we integrate by parts and use the assumption that ℓ is nearly constant, we obtain

$$\Delta\gamma = \frac{\langle\gamma\rangle}{4\bar{\ell}^2}\left[f_f^{\,2} - f_i^{\,2} - \int \left(f^2 - \bar{f}^2\right) \cos\left(\frac{\ell\psi}{h}\right) d\left(\frac{\ell\psi}{h}\right)\right], \tag{91}$$

where $\bar{f}^2 = \frac{1}{2}(f_f^{\,2} + f_i^{\,2})$. But from Equation (54), $f^2 = f_o^{\,2}(1 + \cos^2\theta)$, and from geometrical considerations (See Figure 1),

$$\cos\theta = \cos\theta_o \sin\psi + \sin\theta_o \cos\beta \cos\psi \tag{92}$$

where θ_o is the polar angle to the tangent to the orbit at periastron (essentially the incident direction for small deflection) and β is the angle between the orbital plane and the plane of the tangent and the polar axis. If we specialize to nearly rectilinear orbits ($\bar{f}$ small compared to h), we can easily perform the integration in Equation (91) and obtain

$$\begin{aligned}\frac{\Delta\gamma}{\gamma} = &\frac{\pi}{32} f_o^{\,4}\,\bar{\ell}^{-4} \sin 2\theta_o (1 + \cos^2\theta_o)\cos\beta \\ &+ \frac{1}{3} f_o^{\,2}\,\bar{\ell}^{-2}\left(\cos^2\theta_o - \cos^2\beta\sin^2\theta_o\right)\end{aligned} \tag{93}$$

The first term is negligible for small deflections, since then $f_o \ll \bar{\ell}$; most of the change in γ comes from the effects of changing amplitude with θ near periastron passage. Note that the deflection is always in the sense of decreasing θ for passage above the pole. (The particle is repelled by the wave source.)

The sense of the change is most easily seen from Equation (87). All else being equal, the rate of change of γ is proportional to $1/\alpha$. Thus, at a given ζ and through a given change in ψ the exit segment of the orbit contributes more heavily than the corresponding entrance portion because α is smaller there. Thus, for θ_o near zero the particle exits in a field region in which f is decreasing, and the energy decreases; for θ_0 near $\pi/2$, the exit orbit is toward the pole where f is increasing; the energy thus increases. It is easily shown that the average $\Delta\gamma$ over all incident directions vanishes, so that this phenomenon provides a diffusion in energy of the same character as the phase locking.

f) Radiative Effects

We conclude the discussion with a brief look at radiative reaction. It is clear that the equations for $u_\perp$ and α in the spherical case become [cf. Equations (28), (65)]

$$\frac{d}{d\eta}\left(\zeta\langle u_\perp\rangle\right) = -\rho\alpha^2\langle u_\perp\rangle\,\zeta^{-1}$$
$$\frac{d\alpha}{d\eta} = -\frac{\ell^2}{\zeta^3} - \frac{\rho\alpha^3}{\zeta^2} \tag{94}$$

for non-phase-locked particles in the equatorial plane; here we define a new radiation parameter ρ:

$$\rho \equiv \frac{e^2\Omega f_o^{\,2}}{3\,mc^3}\,, \tag{95}$$

a constant. Note that ρ is simply the old parameter ε evaluated at $\zeta=1$ $(r = c/\Omega)$, so that ρ/ζ^2 is the "local" value of ε. From Equation (86), by direct differentiation, we find

$$\frac{d\overline{\gamma}}{d\eta} = \frac{\overline{u}_\zeta\rho\alpha^2}{\zeta^2} - \frac{\rho\alpha h^2}{\zeta^4} \tag{96}$$

Thus, the situation is analogous to the plane-wave case; if the motion is radial (h=0) and $\langle\overline{u}_\zeta\rangle$ is positive (outgoing branch), the particle gains energy by the same "pumping" mechanism as discussed before; on the incoming branch, the particle loses energy, and it is clear that losses far overbalance gains, since α is, on average, much smaller going out than coming in. To estimate radiative effects, let us assume they are small, h is nearly constant, and we can integrate Equation (96) along the unperturbed orbit. Omitting the remaining straightforward analytical steps, we present the conclusions that due to radiative effects the energy changes by an amount

$$(\Delta\gamma)_{rad} = - \frac{\pi\rho\langle\gamma\rangle^2}{\zeta_p} (1 + \tfrac{1}{4} h^2/\ell^2) \qquad (97)$$

This is, happily, in accord with one's intuitions; $\rho\zeta_p^{-2}$ is proportional to the square of maximum field encountered on the orbit. The particle feels this amplitude for a time of order ζ_p. Thus, the "power" is proportional to $\rho\gamma^2\zeta_p^{-2}$ and the total loss to $\rho\gamma^2\zeta_p^{-1}$.

The energy gained in the outgoing orbit due to radiative pumping can be shown to be

$$(\Delta\gamma)_{out} = \frac{\rho\langle\gamma\rangle^2}{\zeta_p} \left[5/3 - \pi/2 - h^2/\ell^2 \left(\pi/8 - 1/6\right)\right] , \qquad (98)$$

and unless h is quite small ($h \lesssim 0.6\ f_o$) losses dominate over gains on the exit orbit. For the complete orbit, gains on the exit balance only about three percent of the losses on the entrance orbit even in the most favorable case when h = 0.

The radiative changes in $\langle\gamma\rangle$ must, of course, be added to the changes in $\langle\gamma\rangle$ due to the independent effects due to the variation in ℓ and phase locking.

The orbit is greatly modified if the particle loses a significant fraction of its energy in one passage. The condition for this is

$$\langle\gamma\rangle \geqslant \min\left[\frac{\zeta_p}{\pi\rho}\ ,\ \left(\frac{\ell}{\pi\rho}\right)^{\frac{1}{2}}\right] \tag{99}$$

which for the Crab ($\rho \approx 5$) happens for $\gamma \gtrsim \frac{1}{4} f_o^{\ 1/2}$ in the extreme case $h = 0$. This implies that electrons from infinity, regardless of energy, cannot get closer to the star than the periastron distance corresponding to this critical energy, about $4\ f_o^{\ \frac{1}{2}}$, or in physical units about 10^{14} cm.

VI. SUMMARY AND DISCUSSION

a) Summary

Two important parameters appear in the theory developed here. The first, $\nu = \frac{eB}{mc\Omega}$ is a measure of the strength of an electromagnetic wave in terms of its ability to accelerate particles of a given e/m. In the usual model of wave particle interactions, ν is implicitly assumed to be very small as indeed it must be for any known particles in any plausible thermal radiation. In this usual weak case a particle initially at rest is not accelerated in the direction of the wave; it absorbs energy from the wave and oscillates non-relativistically about its original position. As a consequence it radiates at the driving frequency, Ω. This is ordinary electron scattering. If the particle was relativistic to begin with, then in the usually studied weak case, it oscillates about its initial rectilinear motion and as a consequence emits what is often called "inverse Compton" radiation at frequencies of order $\gamma^2\Omega$.

In the case of a strong wave ($\nu >> 1$), the particle motion has a fundamentally different character. In <u>any</u> chosen frame, the particle will become relativistic at some phase of its periodic motion. In the frame in which the average velocity vanishes (the"guiding center" velocity is zero) the orbit is a figure-8 described by equation (11); in this frame the average energy is of order ν. Particles accelerated from rest reach energies $\langle\gamma\rangle$ of order ν^2 and travel in one cycle a distance of order ν^2 times the wavelength of the radiation, [eq. (14)] in the wave propagation direction. It is clear that in this regime ($\nu >> 1$) familiar concepts like the plasma frequency have no immediate significance (see NPI).

Inclusion of the radiative reaction produces a slow runaway solution (whose character is fundamentally unlike the classical spurious runaway solutions) in which the energy of the particles grows as $t^{1/3}$ [eq. (36)], motion transverse to the wave decays as $t^{-1/3}$, and the radiated power decreases as $t^{-2/3}$. This process of "radiative pumping" is probably too slow to be of any interest in likely astrophysical situations.

The second important parameter, $\varepsilon = (e^4 B^2/3m^3c^5\Omega)$ measures the significance of radiative losses, the average power emitted by an isotropically distributed collection of particles with energy γ being about $\varepsilon\, \Omega\, \gamma^2 (mc^2)$ per particle [Eq. (40)]. The power output has the same dependence on particle energy and field density as both inverse Compton and synchrotron radiation. The radiation, which will be discussed in detail subsequently, becomes inverse Compton for $\nu << 1$, but in the more interesting strong-wave case ($\nu >> 1$), this non-linear inverse Compton ("NIC") radiation is qualitatively different from either of the usual high-energy processes. In terms of total radiated power and peak frequency as a function of B and γ NIC radiation is similar to synchrotron emission. For particles injected at rest into a strong wave the power radiated is simply $1.6 \times 10^{-15}\ B^2_{rms}$ erg sec^{-1} per particle. In more astrophysical units, this come to $5 \times 10^8\ B^2_{rms}$ solar luminosities per solar mass injected at rest into a strong electromagnetic field. This result, like all those presented in this paper, is dependent on the assumption of independent test particles and as such is meaningless if the luminosity of the radiating particles becomes comparable to the power in the underlying low-frequency radiation field; the particles act as catalysts, transforming very low-frequency waves to radio or higher frequency output.

The treatment of spherical waves introduces many new facets, some due to the geometry and others to the dependence of field amplitude on radius. The phase averaged energy $\langle\gamma\rangle$ is found to be a constant for a particle injected far from the radiating sources and the particle's guiding center moves as a classical particle in a repulsive r^{-3} force field. It follows that the angular momentum of the guiding center motion is conserved. An orbit is characterized by $\langle\gamma\rangle$ and an angular momentum parameter $\ell \equiv (1/2\ f_o^{\ 2} + h^2)^{1/2}$ where h is the normalized guiding center angular momentum and f_o measures the strength of the spherical wave field. If $\langle\gamma\rangle > \ell^{1/2}$ the particle on its exit branch finds itself traveling nearly radially at a velocity sufficiently near c that it "locks phase" with the driving wave. This causes particles with slightly different initial orbits to undergo an essentially random redistribution of energy about the value $\langle\gamma\rangle$. A dipolar (or higher multipolar) pattern in the driving radiation field produces much the same effect. The acceleration of particles from rest is found to involve two rather distinct regimes. For initial radii satisfying $r_o < r_c = c\ \Omega_o^{-1}\ f_o^{2/3}$ we find the particle is phase-locked throughout the part of its flight in which it interacts significantly with the wave and, on exit, $\gamma \approx f_o^{2/3}$ independent of $r_o (r_o < r_c)$, thus recovering the result of NPI. For particles injected at $r \geq r_c$ ($r_c \approx 3 \times 10^{15}$ cm for the Crab pulsar), at least one gyration is completed and $\gamma \approx f_o^{\ 2}(c/\Omega r_o)^2$

which is less than they would receive if injected close to the star and approximately the energy they would reach if the spherical wave at injection were treated as a local piece of a plane wave. "Radiative pumping" is never important in a spherical wave since the gains on the outgoing branch of an orbit are always greatly exceeded by the losses on the incoming branch.

b) Applications

Detailed application of these results will be made elsewhere but two of the most interesting possibilities will be mentioned here.

In the Introduction we noted that rotating magnetic neutron stars are good candidate objects for the pulsars. In NP III we argued briefly that collections of these objects in their associated nebulae would have some of the properties of the extragalactic point sources. Morrison (1969) and Fowler (1970) have suggested that single , very massive, rotating magnetic objects power the extragalactic sources and recently Bardeen (1970) and Wagoner (1969) have shown that as much as 40 percent of the rest mass energy can be liberated by rotating, highly-relativistic, slowly collapsing discs (this is about 10 times the amount of energy that can be liberated from non-rotating configurations).

It appears now that such rotating magnetic objects either singly like the Crab pulsar, like a concert of such objects, or like a "super-pulsar" can: a) provide an energy input to the surrounding medium via the emission of low-frequency electromagnetic waves; b) accelerate particles to very high energies in these waves; and c) provide the field (the waves themselves) in which the fast particles can produce synchrotron-like continuum radiation. These properties are essential prerequisites for the sources in a large class of astronomical objects. The results for the Crab Nebula are quantitatively reassuring; the amplitude of the waves at the edge of the Nebula is about 10^{-4} gauss, if the previously derived neutron star field (Gunn and Ostriker 1969) is correct. This is essentially the field value obtained from other arguments (cf. Scargle 1968); the synchrotron spectrum of the Crab can be understood on the basis of the mechanism and will be considered in detail in a later paper.

The other immediate application of the results concerns the origin of cosmic rays. Earlier we (Gunn and Ostriker 1969) suggested that the high-energy tail of the cosmic ray distribution may be ions accelerated from the wave zone ($r < r_c$) of pulsars via the mechanism which we have rederived in § V of this paper. Although pulsars are energetically capable of producing all the cosmic rays, they assuredly do not make them by this process, because, as we have pointed out elsewhere (Ostriker 1969) moderate energy (~ GeV) particles cannot be made in this way and the composition of particles originating on the surface of neutron stars, while entirely conjectural, is most unlikely to resemble the common distribution found in cosmic rays. We also pointed out in the earlier work that unless the estimates used for particle injection rates were entirely wrong [these were adapted from the work of Goldreich and Julian (1969)], the waves were not likely to be saturated with particles near the source. Thus they will reach the debris in the supernova remnant with the wave energy density diluted mainly by the geometrical inverse square factor. We have seen in this paper that such waves are still capable of accelerating ions in the nebula provided that $ZeB/Am_p\Omega$ is large there. This condition is satisfied early in the life a supernova remnant containing a pulsar. Using parameters appropriate to the Crab, we show elsewhere that ions would have been accelerated to relativistic energies for about the first 5 years after the explosion, during which time the pulsar loses $\sim 10^{48}$ ergs of kinetic energy. Thus cosmic rays can be produced in situ from the highly evolved nuclear material in the remnant. It is interesting to note in this connection that the helium-to-hydrogen ratio in cosmic rays is high (both as observed and as inferred at the "source") just as it is in the Crab Nebula.

Using the theory developed in this paper, we will return at a later date to examine the possibility that the bulk of the galactic cosmic rays are produced by wave acceleration in supernova remnants.

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Figure 1: Schematic of the guiding center orbit near periastron for arbitrary initial conditions, identifying the angles used in the text. The vector $\underset{\sim}{\tau}$ is the tangent to the orbit at periastron.

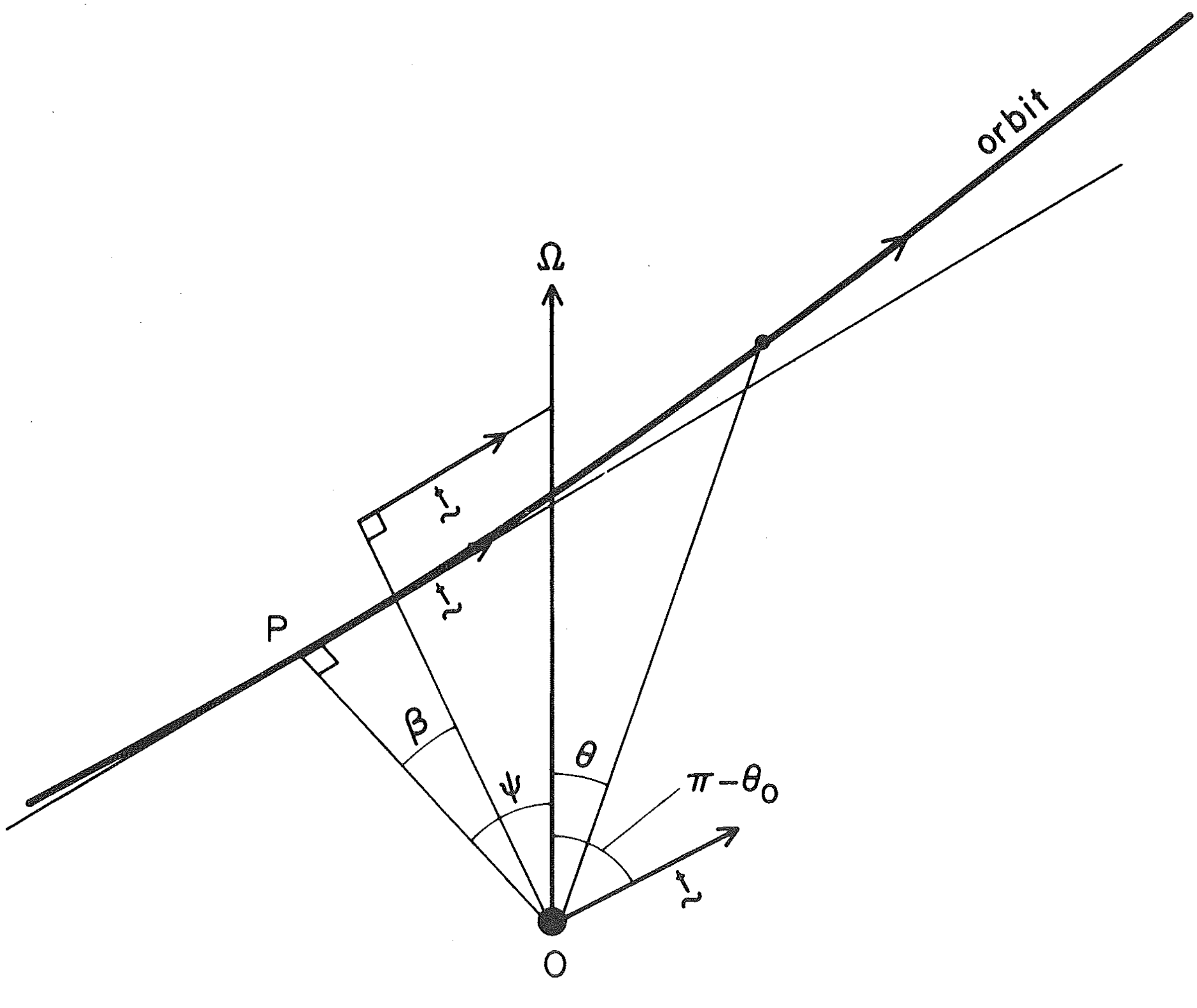

Fig. 1