TM-70-1022-20

TECHNICAL MEMORANDUM

A FINITE DIFFERENCE SOLUTION TO A MIXED BOUNDARY VALUE PROBLEM FOR LAPLACE'S EQUATION

Bellcomm

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W., WASHINGTON, D.C. 20024

COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- A Finite Difference Solution to a Mixed Boundary Value Problem for Laplace's Equation

TM- 70-1022-20

DATE- December 28, 1970

FILING CASE NO(S)- 320

AUTHOR(S)- V. Thuraisamy
S. C. Chu

FILING SUBJECT(S)
(ASSIGNED BY AUTHOR(S))- Finite Difference Techniques
Laplace's Equation

ABSTRACT

In this memorandum we present a finite difference scheme for the solution of a mixed boundary value problem for Laplace's equation. This problem arises in the analysis of longitudinal vibrations of tanks, partially filled with fluid. The discretization process presented permits both equal-size and variable mesh grids. The convergence property of the process is proved.

BA-145A (8-68)

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

SUBJECT: A Finite Difference Solution to a Mixed Boundary Value Problem for Laplace's Equation - Case 320

DATE: December 28, 1970

FROM: V. Thuraisamy
S. C. Chu

TM-70-1022-20

TECHNICAL MEMORANDUM

INTRODUCTION

A mixed boundary value problem for Laplace's equation arises in the study of the interaction dynamics of fluid and its flexible container. In a recent report Goldman [1] studies the longitudinal vibrations of partially filled ellipsoidal tanks. To describe the Goldman work briefly: The fluid is assumed non-viscous, irrotational, and incompressible. The perturbations of the tank and fluid are assumed small. There exists a velocity potential function ϕ defined everywhere in the fluid such that

$$\nabla^2\phi = 0 \tag{1}$$

Here ∇^2 is the three-dimensional Laplacian. It is assumed that ϕ vanishes on the liquid surface. Using axial symmetry, equation (1) is reduced to a two-dimensional form. A finite difference analogue is then employed to compute ϕ at a finite number of points in terms of $\frac{\partial\phi}{\partial n}$ (the normal derivative at the tank wall). From this the energy integral $\int_s \phi \frac{\partial\phi}{\partial n}$ is evaluated resulting in a mass matrix M. This M and the corresponding stiffness matrix are used to solve the eigenvalue problem for the frequencies and the mode shapes.

A finite difference scheme should have the theoretical property of convergence; that is, if $\hat{\phi}$ is a solution to the discrete problem, then $\hat{\phi}$ should approach the true solution ϕ as the mesh size approaches zero. The convergence property, thus, insures that the discretization process does indeed provide a valid mathematical approximation to the continuous problem.

In the report [1], the question of convergence was not specifically addressed. In this memorandum, we discuss a different finite difference scheme for which convergence is guaranteed. The proof of convergence is achieved by taking the well-known discrete Green's function approach. The crucial fact is that our finite difference scheme leads to a matrix of "positive type". This enables us to obtain a discrete maximum principle and use it to estimate various sums in the discrete Green's function,which are in turn used to estimate a bound for the error.

FINITE DIFFERENCE

Wherever possible we shall follow the same notation as in [1]. We take full advantage of axial symmetry. Thus for the hemispherical tank, one need work with only a quadrant of a circle, and similarly for other geometries. Accordingly, Figure 1 shows a portion of the vertical axial cross-section. O' is the center of the ellipse (circle) and the origin of coordinates O is taken to be the axial point on the surface of the fluid. OA represents the depth of the fluid and OB is the radius of the disc forming the free surface. z and λ are respectively the normalized (downward) vertical and horizontal coordinates in the OAB plane.

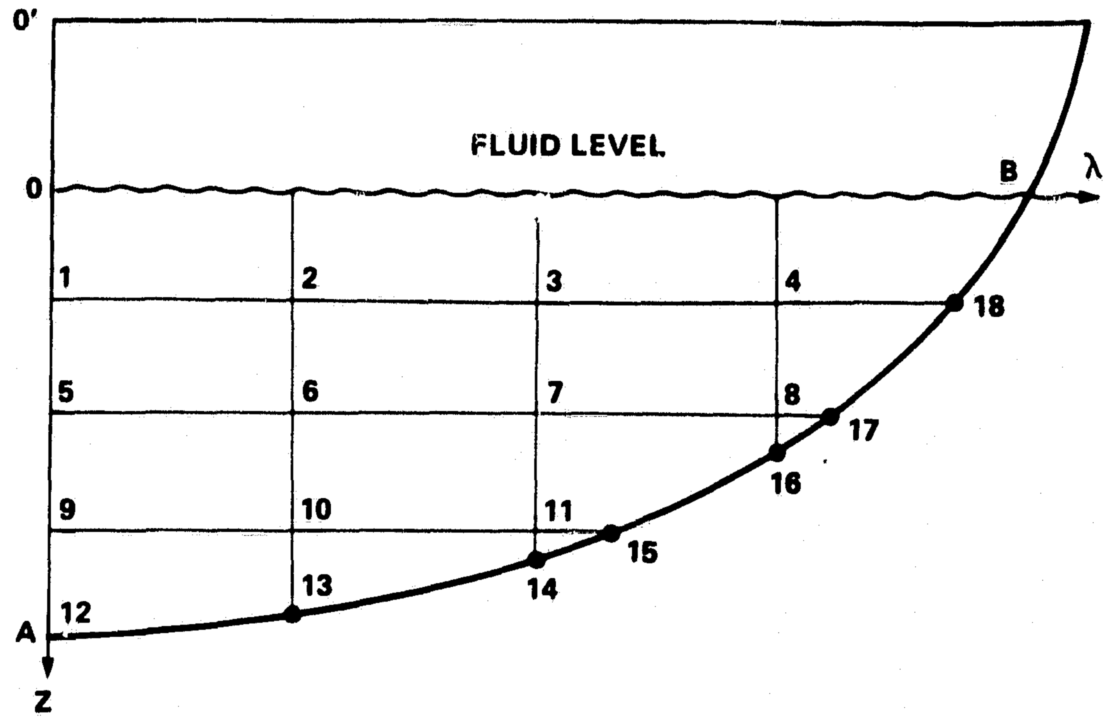

FIGURE 1

For the sake of definiteness we now assume that our grid is such that we have uniform mesh-widths d in the z-direction and h in the λ-direction. Later we comment on how to extend our results to non-uniform spacing. Without loss of generality we also assume that $d \leq h$. In practice a variable mesh would allow one to increase or decrease the number of boundary points in relation to the number of interior points. In Figure 1, we have seven boundary points and eleven interior points. Variable mesh spacing will also permit one to choose the boundary points before hand (e.g., at equal intervals along the arc) and to avoid the peripheral interior points (4, 8, 10, 11 in Figure 1) getting too close to the boundary or even abolishing the peripheral points. The price one pays for imposing a variable mesh grid is, of course, a complicated five point operator in the interior, involving increased computation at every mesh point. This is not an insurmountable problem, however.

Let R be the region comprising the fluid and $\bar{R}$ its closure. Referring now to the two-dimensional grid diagram such as Figure 1, we let D be the set of mesh points in R and $\dot{D}$ be the set on the boundary. Thus in Figure 1, the interior points are numbered 1 to 11 and the boundary points are numbered 12 to 18. Moreover, we occasionally have to distinguish between the "peripheral" points and the "regular" points of D. A peripheral point is one which has at least one of its four "neighbors" in $\dot{D}$. Thus in Figure 1, the points numbered 4, 8, 10, 11 are peripheral points. This distinction is irrelevant when working with non-uniform mesh. We have excluded the grid points on the fluid surface since $\phi = 0$ there.

Laplace's equation now takes the form (by axial symmetry)

$$\frac{1}{\lambda}\phi_{\lambda} + \phi_{\lambda\lambda} + \phi_{zz} = 0 \tag{2}$$

The boundary conditions are $\phi = 0$ on $z = 0$ and $\frac{\partial\phi}{\partial n} = V^{(1)}$ at the tank wall. Thus we assume that the outward normal derivative $\frac{\partial\phi}{\partial n}$ is prescribed on $\dot{D}$.

Let us first consider a regular point x of D. For any function u defined in $\bar{D}$ ($\bar{D} = D \cup \dot{D}$), we write

$$\hat{u}_{\lambda\lambda}(x) = \frac{1}{h^2}\{u(x-\tilde{h}) - 2u(x) + u(x+\tilde{h})\} \tag{3}$$

Here we use x as a vector representing the mesh point and $\tilde{h}$ is the vector (0,h) in (z,λ) coordinates. We also write $\tilde{d} = (d,0)$. If ϕ is assumed to be $C^3(\bar{R})$ (i.e., thrice continuously differentiable in $\bar{R}$), then it is clear that

$$\hat{\phi}_{\lambda\lambda}(x) = \phi_{\lambda\lambda}(x) + M_3 h \tag{4}$$

where $M_3 \leq \max_{i=0,1,2,3} |D^i\phi|$, $D^i\phi$ being any i^{th} derivative of ϕ in $\bar{R}$.

$$\hat{\phi}_{zz}(x) = \frac{1}{d^2}\{\phi(x+\tilde{d}) - 2\phi(x) + \phi(x-\tilde{d})\} \tag{5}$$

and for $\lambda \neq 0$,

$$\frac{1}{\lambda}\hat{\phi}_{\lambda}(x) = \frac{1}{2h}\{\phi(x+\tilde{h}) - \phi(x-\tilde{h})\} . \tag{6}$$

Since

$$\lim_{\lambda\to 0} \frac{\phi_\lambda}{\lambda} = \phi_{\lambda\lambda} , \quad (\text{for } \lambda = 0) \tag{7}$$

we use (3) (instead of (6)) in approximating $\frac{1}{\lambda}\phi_\lambda$ at the axis points. Also axial symmetry is used for these points in the obvious manner.

For a peripheral point, such as in Figure 2, we define

$$\hat{\phi}_{\lambda\lambda}(x) = \frac{2}{h^2(1+\alpha)}\{\frac{\phi(x+\alpha\tilde{h})}{\alpha} - (1+1/\alpha)\phi(x) + \phi(x-\tilde{h})\} \tag{8}$$

$$\hat{\phi}_{zz}(x) = \frac{2}{d^2(1+\beta)}\{\frac{\phi(x+\beta\tilde{d})}{\beta} - (1+1/\beta)\phi(x) + \phi(x-\tilde{d})\} \tag{9}$$

$$\frac{1}{\lambda}\hat{\phi}_\lambda(x) = \frac{1}{\lambda h}\frac{1}{\alpha+\alpha^2}\{\phi(x+\alpha\tilde{h}) - (1-\alpha^2)\phi(x) - \alpha^2\phi(x-\tilde{h})\} \tag{10}$$

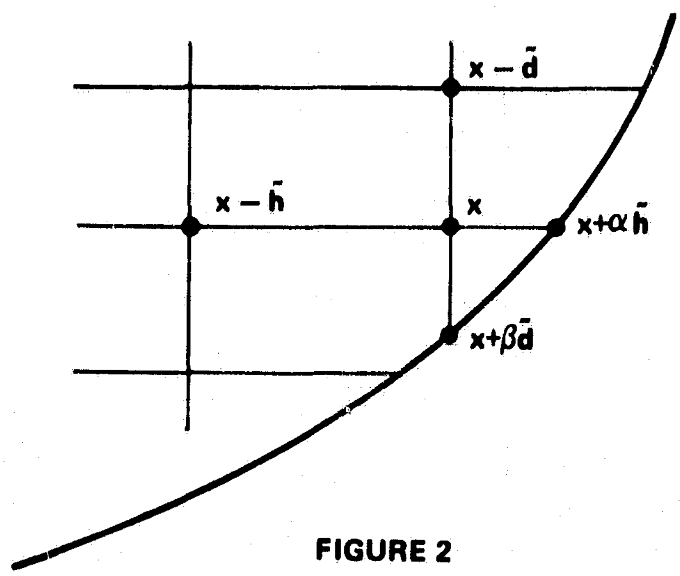

FIGURE 2

To justify these definitions, let us look at equation (8) in detail. Expanding $\phi(x+\alpha\tilde{h})$ and $\phi(x-\tilde{h})$ in terms of Taylor series about the point x, we have

$$\phi(x+\alpha\tilde{h}) = \phi(x) + \alpha\tilde{h}\phi_\lambda(x) + \frac{(\alpha\tilde{h})^2}{2}\phi_{\lambda\lambda}(x) + \frac{(\alpha\tilde{h})^3}{6}\phi_{\lambda\lambda\lambda}(x')$$

$$\phi(x-\tilde{h}) = \phi(x) - \tilde{h}\phi_\lambda(x) + \frac{\tilde{h}^2}{2}\phi_{\lambda\lambda}(x) - \frac{\tilde{h}^3}{6}\phi_{\lambda\lambda\lambda}(x'')$$

Adding α times the second to the first we obtain

$$\phi(x+\alpha\tilde{h}) + \alpha\phi(x-\tilde{h}) = (1+\alpha)\phi(x) + \frac{h^2}{2}(\alpha+\alpha^2)\phi_{\lambda\lambda}(x) + \varepsilon$$

where ε is a third order error term. Thus defining the quantity $\hat{\phi}_{\lambda\lambda}(x)$ by equation (8) implies

$$\hat{\phi}_{\lambda\lambda}(x) - \phi_{\lambda\lambda} = \frac{h}{3(1+\alpha)} \{\alpha^2\phi_{\lambda\lambda\lambda}(x') - \phi_{\lambda\lambda\lambda}(x'')\}$$

$$= O(h).$$

Similarly $\hat{\phi}_{zz}(x)$ approximates $\phi_{zz}(x)$ and $\frac{1}{\lambda}\hat{\phi}_{\lambda}(x)$ approximates $\frac{1}{\lambda}\phi_{\lambda}(x)$ with $O(h)$ error.

Equation (8) reduces to (3) when $\alpha = 1$. Similar remarks hold for (9) and (10) vis-a-vis (5) and (6). Thus all types of peripheral points (and indeed regular interior points) are covered by equations (8), (9), and (10). Now if we write

$$L\phi(x) \equiv \frac{1}{\lambda}\phi_{\lambda}(x) + \phi_{\lambda\lambda}(x) + \phi_{zz}(x) \tag{11}$$

$$\tilde{L}\phi(x) \equiv \frac{1}{\lambda}\hat{\phi}_{\lambda}(x) + \hat{\phi}_{\lambda\lambda}(x) + \hat{\phi}_{zz}(x), \; x\varepsilon D \tag{12}$$

then for $\phi\varepsilon C^3(\overline{R})$, the following inequality is true:

$$|L\phi - \tilde{L}\phi| < K_1 h. \tag{13}$$

Here and in the sequel, K_1, K_2, ... will be used to represent constants which are independent of d and h but may depend on M_3.

On the boundary we use a first order approximation to the normal derivative. A typical case is shown in Figure 3.

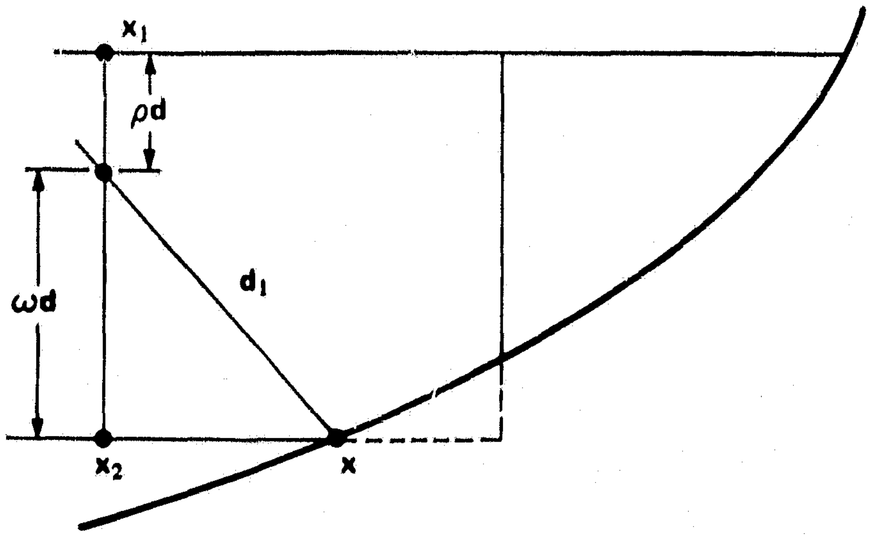

FIGURE 3

If $\delta_1\phi$ is defined by

$$\delta_1\phi = \frac{\phi(x) - \{\omega\phi(x_1) + \rho\phi(x_2)\}}{d_1} \tag{14}$$

where $\omega + \rho = 1$ and d_1 is as in Figure 3, then it is easily verified that

$$\left|\frac{\partial\phi(x)}{\partial n} - \delta_1\phi(x)\right| < K_2 h. \tag{15}$$

We note that equation (14) covers all types of boundary points. At the bottom point, e.g., we have

$$\delta_1\phi(x) = \frac{\phi(x) - \phi(x-\tilde{d})}{d} \text{ ,}$$

Thus the continuous problem

$$\begin{aligned} -L\phi &= 0 && \text{in the liquid} \\ \frac{\partial\phi}{\partial n} &= V^{(1)} && \text{at the tank wall} \\ \phi &= 0 && \text{on liquid surface,} \end{aligned} \tag{16}$$

is discretized by

$$\begin{aligned} -\tilde{L}\phi &= 0 && \text{in } D \\ \delta_1\phi &= V^{(1)} && \text{on } \dot{D}\text{ .} \end{aligned} \tag{17}$$

The information $\phi = 0$ on the fluid surface is incorporated in $\tilde{L}$ when writing equations for points of D which are at depth d below the free surface. We shall refer to the nxn coefficient matrix of the linear system (17) by A.

METHOD

The idea of the method is to solve (17) for ϕ on $\dot{D}$ in terms of $V^{(1)}$ and then evaluate the energy integral $\int_S \phi\frac{\partial\phi}{\partial n}$ as a discrete sum over the points of $\dot{D}$. The result would be a quadratic form $\sum_{i,j} m_{ij}\, V_i^{(1)}\, V_j^{(1)}$, where $M = \{m_{ij}\}$ may be assumed to be symmetric. This M would be the matrix used to compute the natural frequencies and mode shapes.

Let n_1, n_2 respectively be the number of interior and boundary points. Let us rewrite equation (17) in the form

$$A_\phi = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ V^{(1)} \end{pmatrix}$$

In equation (18) A_1 and A_2 are square matrices of order n_1 and n_2 respectively. ϕ_2 is the n_2-vector giving the n_2 boundary values of ϕ. The rest of the quantities then are self explanatory. If A_1^{-1}, A_2^{-1} exist, then

$$\phi_1 = -A_1^{-1} B_1 \phi_2$$

and

$$\phi_2 = A_2^{-1}(V^{(1)} - B_2\phi_1)$$

$$= A_2^{-1}(V^{(1)} + B_2A_1^{-1}B_1 \phi_2) .$$

Thus if

$$C_2 \equiv [I - A_2^{-1} B_2 A_1^{-1} B_1]$$

is invertible, then ϕ_2 is obtained from

$$\phi_2 = C_2^{-1} A_2^{-1} V^{(1)}. \qquad (19)$$

Once ϕ_2 is obtained, the approximation to the energy integral by evaluating the discrete sum at the same n_2 points of the boundary yields the mass matrix M.

We observe here that in practice, A_2 in (18) is in fact an $n_2 \times n_2$ diagonal matrix for reasonably smooth boundaries and is certainly the case for the simple boundaries such as the ellipsoid. This is easily seen from (14). (In Figure 1, $n_2 = 7$.)

CONVERGENCE

We shall now give a proof of convergence of the finite difference solution to the continuous solution. Specifically, we shall prove, with reference to equations (16) and (17), the following theorem.

Theorem

The error $\varepsilon(x) = \Phi(x) - \phi(x), \forall x \varepsilon \bar{D}$ is uniformly bounded and is given by

$$\max_{x \varepsilon \bar{D}} |\varepsilon(x)| \leq K_3 h \tag{20}$$

The proof will be developed in several lemmas below. It will be convenient for the reader to refer to Figure 1 and the corresponding matrix A presented in the Appendix when reading through the proof.

Lemma 1 If

$$-\tilde{L}\, U(x) \geq 0 \qquad , x \varepsilon D \tag{21}$$

$$\delta\, \bar{U}(x) \geq 0 \qquad , x \varepsilon \dot{D} \tag{22}$$

then

$$\bar{U}(x) \geq 0 \quad \text{in } \bar{D}. \tag{23}$$

Proof

Suppose there is a negative minimum at $x_0 \varepsilon D$. Then if x_0 is one of the points on the "top row" (such as 1, 2, 3, 4 of Figure 1) then (21) gives

$$a_0 \, U(x_0) - \sum_{i=1}^{3} a_i \, U(x_i) \geq 0 . \tag{24}$$

But by the definition of $\tilde{L}$, we have $a_i > 0$ for $i = 0, \ldots, 3$ and x_i $(i = 1, 2, 3)$ are the three neighbors of x_0. Also $\tilde{L}$ is such that

$$a_0 - \sum_{i=1}^{3} a_i > 0 .$$

Thus if we rewrite (24) as

$$(a_0 - \sum_{i=1}^{3} a_i) \, U(x_0) + \sum_{i=1}^{3} a_i (U(x_0) - U(x_i)) \geq 0 \tag{25}$$

then we have a contradiction.

If x_0 is any other type of point in D, then x_0 has four neighbors. Now

$$a_0 - \sum_{i=1}^{4} a_i = 0$$

and in place of (25) we now have

$$(a_0 - \sum_{i=1}^{4} a_i)\ U(x_0) + \sum_{i=1}^{4} a_i(\ U(x_0 - U(x_i)) \geq 0\ .$$

Again $a_i > 0$ and thus we have a contradiction. From the construction of the δ_1 operator in (14) we see that a negative minimum on $\dot{D}$ is not possible either. Hence we have the lemma.

Definition

For $y \varepsilon \bar{D}$ define $R(x,y)$ by

$$-\tilde{L}\ R(x,y) = h^{-1}d^{-1}\ \delta(x,y) \quad , \quad x \varepsilon D$$

$$\delta_1\ R(x,y) = \tilde{\tilde{h}}^{-1}\ \delta(x,y) \quad , \quad x \varepsilon \dot{D} \tag{26}$$

where

$$\delta(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

and $\tilde{\tilde{h}}$ is a quantity of order h depending on the choice of x. $R(x,y)$ is the so-called discrete Green's function. $R(x,y)$ exists because A is nonsingular.

Lemma 2

$$R(x,y) \geq 0 \quad \forall\ x,y\ \varepsilon\ \bar{D}\ .$$

Proof

Lemma 1 states that $AU \geq 0$ implies $U \geq 0$. But (26) implies

$$AR = Q \tag{27}$$

where R is an nxn matrix and Q is a diagonal matrix with diagonal entries 1, $1/\tilde{\tilde{h}}$ or 1/hd. Hence AR $\geqslant$ 0 and therefore R $\geqslant$ 0.

<u>Lemma 3</u>

For every grid function U(x), we have

$$U(x) = hd \sum_{y \varepsilon D} R(x,y)[-\tilde{L}\ U(y)] + \tilde{\tilde{h}} \sum_{y \varepsilon \dot{D}} R(x,y)[\delta_1\ U(y)] \tag{28}$$

<u>Proof</u>

Let the right hand side be W(x). For xεD,

$$-\tilde{L}W(x) = hd \sum_{y \varepsilon D} - \tilde{L}_x[R(x,y)][-\tilde{L}\ U(y)] + \tilde{\tilde{h}} \sum_{y \varepsilon \dot{D}} - \tilde{L}_x[R(x,y)][\delta_1\ U(y)]$$

implies

$$-\tilde{L}W(x) = -\tilde{L}\ U(x)$$

from the definition of $R(x,y)$. The subscript x of $\tilde{L}_x$ is to point out that the operation is with respect to the argument x.

Similarly if $x\varepsilon\dot{D}$, we have

$$\delta_1 W(x) = \delta_1 U(x) .$$

Thus we have

$$-\tilde{L}(W(x) - U(x)) = 0 \quad , \quad x\varepsilon D$$

$$\delta_1 (W(x) - U(x)) = 0 \quad , \quad x\varepsilon\dot{D}.$$

Lemma 1 applied to $W(x) - U(x)$ and then to $-[W(x) - U(x)]$ gives $W(x) \equiv U(x)$ and the lemma.

<u>Lemma 4</u>

There exists a function $\psi\varepsilon C^3(\bar{R})$ which is non-negative in $\bar{R}$;

$$-L\psi \geqslant 2 \quad \text{in} \quad R \tag{29}$$

and

$$\frac{\partial\psi}{\partial n} \geqslant 2 \quad \text{on tank wall} . \tag{30}$$

Proof

It is clear that if (29) and (30) are satisfied by a function ψ then by adding a sufficiently large constant one could always get a non-negative function with the same properties as ψ. A general existence proof for arbitrary regions is not available. However, we shall exhibit a ψ explicitly which is acceptable for hemispherical tanks and which can therefore be modified to ellipsoidal tanks.

We verify that, with respect to Figure 1, the function ψ defined by

$$\psi = C_3\{\lambda^2 - 3z^2 + C_1 z + C_2\} \tag{31}$$

for appropriate choice of constants C_1, C_2 and C_3 satisfies the conditions of the lemma. We have

$$\frac{1}{\lambda}\psi_\lambda = 2C_3$$

$$\psi_{\lambda\lambda} = 2C_3$$

$$\psi_{zz} = -6C_3 .$$

Hence

$$-L_\psi = 2C_3 \quad \text{for } (z,\lambda) \text{ in } R.$$

Also (Figure 4)

$$\left.\frac{\partial\psi}{\partial n}\right|_{\dot{R}} = C_3\{2\lambda \sin\theta - 6z\cos\theta + C_1\cos\theta\}\Big|_{\dot{R}} \tag{32}$$

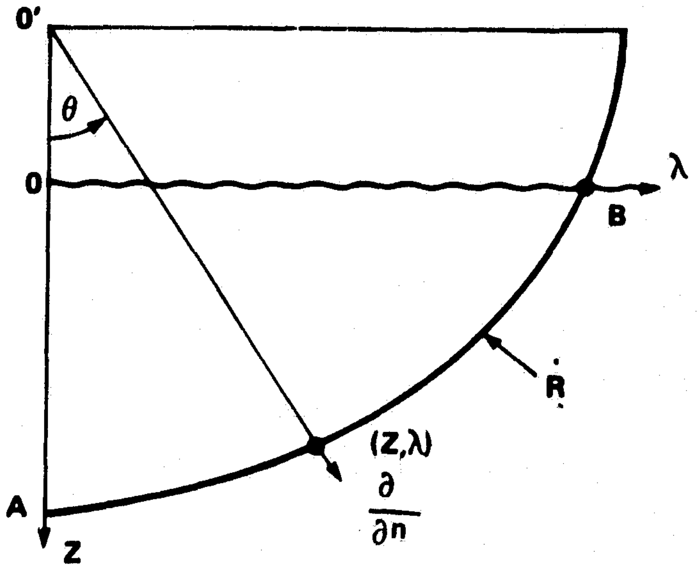

FIGURE 4

Equation (32) is easily verified if one expresses ψ in terms of polar conditions (r,θ) about 0' and uses the fact that $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ on the boundary $\dot{R}$.

The right hand side of (32) can be rewritten

$$C_3\{C_1 \cos\theta - 8r \cos^2\theta + 6|00'|\cos\theta + 2r\}|_{\dot{R}}.$$

Thus if C_1 is taken to be the number $\{8|0'A| - 6|00'|\}$ then

$$\left.\frac{\partial\psi}{\partial n}\right|_{\dot{R}} > 2|0'A|C_3 .$$

Hence taking C_1 to exceed eight times the radius of the hemisphere and C_3 to exceed the reciprocal of the radius guarantees ψ with $\frac{\partial\psi}{\partial n}$ greater than two. Also it is clear that taking C_2 to be any constant exceeding the maximum value of $3z^2$, for all z in R, is sufficient to guarantee a non-negative ψ. The adjustment required for the elliptic case is simple enough and shall not be discussed here.

Lemma 5

For sufficiently small h

$$-\tilde{L}\psi(x) \geq 1 \quad , \quad x\varepsilon D$$

$$\delta_1\psi(x) \geq 1 \quad , \quad x\varepsilon \dot{D} \; .$$

Proof

According to equations (13) and (15), there are constants K_1 and K_2 independent of h such that

$$\tilde{L}\psi(x) = L\psi(x) - K_1 h \quad \text{for all x in D}$$

$$\delta_1\psi(x) = \frac{\partial\psi}{\partial n}(x) - K_2 h \quad \text{for all x in } \dot{D} \; .$$

Thus when h is sufficiently small, K_1h and K_2h would be smaller than unity. An application of lemma 4 now yields the result.

Lemma 6

$$hd \sum_{y\varepsilon D} R(x,y) \leq K_4$$

$$\tilde{\tilde{h}} \sum_{y\varepsilon\dot{D}} R(x,y) \leq K_4$$

Proof

Applying Lemma 3 to the function $\psi(x)$, we have

$$\psi(x) = hd \sum_{y\varepsilon D} R(x,y) \; [-\tilde{L}\psi(y)]$$

$$+ \tilde{\tilde{h}} \sum_{y\varepsilon\dot{D}} R(x,y) \; [\delta_1\psi(y)].$$

Lemma 5 now yields

$$\psi(x) \geq hd \sum_{y\varepsilon D} R(x,y) + \tilde{\tilde{h}} \sum_{y\varepsilon\dot{D}} R(x,y).$$

We therefore have, for all $x\varepsilon\bar{D}$,

$$hd \sum_{y\varepsilon D} R(x,y) < \max_{x\varepsilon\bar{R}} \psi(x) = K_4$$

$$\tilde{\tilde{h}} \sum_{y\varepsilon\dot{D}} R(x,y) < \max_{x\varepsilon\bar{R}} \psi(x) = K_4$$

Note that in the case of a hemisphere, with unit radius we may take $K_4 = 5$.

Proof of Theorem:

Apply Lemma 3 to the function

$$\varepsilon(x) = \Phi(x) - \phi(x).$$

$$\varepsilon(x) = hd \sum_{y \in D} R(x,y)\,[-\tilde{L}\varepsilon(y)] + \tilde{\tilde{h}} \sum_{y \in \dot{D}} R(x,y)\ \delta_1 \varepsilon(y). \qquad (33)$$

By the triangle inequality

$$|\tilde{L}(\Phi-\phi)| \leq |\tilde{L}\Phi - L\phi| + |L\phi - \tilde{L}\phi|$$

and by definition

$$\tilde{L}\Phi - L\phi = 0.$$

Also

$$\delta_1(\Phi-\phi) = \delta_1\Phi - \frac{\partial\phi}{\partial n} + \frac{\partial\phi}{\partial n} - \delta_1\phi.$$

$$= \frac{\partial\phi}{\partial n} - \delta_1\phi\ .$$

These together with (13) and (15) fed into (33) yield

$$|\varepsilon(x)| \leq hd \sum_{y \in D} R(x,y)\ [K_1 h] + \tilde{\tilde{h}} \sum_{y \in \dot{D}} R(x,y)\,[K_2 h]\ .$$

Applying Lemma 6 to this inequality proves the theorem.

Conclusion

The reader may verify that the proofs all remain valid for a completely variable grid spacing and

$$|\varepsilon(x)| < Kh$$

is true provided h is now taken to be the maximum of all h_i, d_i, the variable grid widths.

We also emphasize that the proofs here hold for a large class of surfaces and are not limited to the hemisphere or the ellipsoid. The only requirement is that the surface is such that Lemma 4 holds.

With reference to equation (19), it should be noted that we have not proved that C_2 is invertible. We gave it there as a simple practical procedure. Our proofs, however, do not depend on this equation since we considered the whole matrix A. This matrix has several pleasant properties, which we had no need to mention explicitly. For example, A^{-1} exists with every element positive. A_1 of (18) is also of positive type [2].

V. Thuraisamy

S. C. Chu

1022-$\frac{\text{VT}}{\text{SCC}}$-mef

Attachments
References
Appendix A

REFERENCES

[1] Goldman, R. L., Longitudinal Vibration Analysis of Partially-Filled Ellipsoidal Tanks by Finite Differences, RIAS Tech Report No. TR 70-6C.

[2] Bramble, J. H. and Hubbard, B. E., New Monotone Type Approximation for Elliptic Problems, Math Comp. Vol. 18 (1964) pp. 349-367.

APPENDIX A

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18		Rt
1	$\frac{-2h^2}{d^2}-4$	4			h^2/d^2															0
2	1/2	$-2-\frac{2h^2}{d^2}$	3/2			h^2/d^2														0
3		3/4	$-2-\frac{2h^2}{d^2}$	5/4			h^2/d^2													0
4			$\frac{6-\alpha}{3(1+\alpha)}$	$\frac{-7-\alpha}{3\alpha}-\frac{2h^2}{d^2}$				h^2/d^2										$\frac{7}{3\alpha(1+\alpha)}$		0
5	h^2/d^2				$-\frac{2h^2}{d^2}-4$	4			h^2/d^2											0
6		h^2/d^2			1/2	$-2-\frac{2h^2}{d^2}$	3/2			h^2/d^2										0
7			h^2/d^2			3/4	$-2-\frac{2h^2}{d^2}$	5/4			h^2/d^2									0
8				$\frac{2}{1+\beta}\,\frac{h^2}{d^2}$			$\frac{6-c}{3(1+\alpha)}$	$-\frac{7-\alpha}{3\alpha}-\frac{2h^2}{\beta d^2}$								$\frac{2}{\beta(1+\beta)}\,\frac{h^2}{d^2}$	$\frac{7}{3\alpha(1+\alpha)}$			0
9					h^2/d^2				$-4-\frac{2h^2}{d^2}$	4		h^2/d^2								0
10						$\frac{2}{1+\beta}\,\frac{h^2}{d^2}$			1/2	$-2-\frac{2h^2}{\beta d^2}$	3/2		$\frac{2}{\beta(1+\beta)}\,\frac{h^2}{d^2}$							0
11							$\frac{2}{1+\beta}\,\frac{h^2}{d^2}$			$\frac{4-\alpha}{2(1+\alpha)}$	$-\frac{5-\alpha}{2\alpha}-\frac{2h^2}{\beta d^2}$			$\frac{2}{\beta(1+\beta)}\,\frac{h^2}{d^2}$	$\frac{5}{2\alpha(1+\alpha)}$					0
12									-1			1								dv_1
13									$-\lambda$	$-\mu$			1							$\bar{dv}_2$
14										$-\lambda$	$-\mu$			1						$\bar{dv}_3$
15							$-\bar{\lambda}$				$-\bar{\mu}$				1					$\bar{hv}_4$
16							$-\lambda$	$-\mu$								1				$\bar{dv}_5$
17				$-\bar{\lambda}$				$-\bar{\mu}$									1			$\bar{hv}_6$
18				$-\bar{\mu}$														1		$\bar{hv}_7$