BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D. C. 20024

B70 12075

SUBJECT: An Analytical Solution to Swing-by Trajectories
Case 310

DATE: December 28, 1970

FROM: K. M. Carlson

ABSTRACT

An analytical solution is derived to the problem of trajectories which encounter the Moon or a planet and must satisfy conditions before and after the encounter. Such trajectories include free-return and fly-by trajectories. The analysis is based on an analytical solution to patched-conic trajectories developed previously by the author. The conditions to be met are pre-encounter and post-encounter inclination and angular momentum. The trajectory energy referenced to the encountered body and the time of the encounter must also be known.

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D. C. 20024

B70 12075

SUBJECT: An Analytical Solution to Swing-by Trajectories Case 310

DATE: December 28, 1970

FROM: K. M. Carlson

MEMORANDUM FOR FILE

INTRODUCTION

Much analytical work has been performed during the last decade on various forms of swing-by trajectories, particularly Earth-Moon free return trajectories and planetary fly-by trajectories. Such trajectories are normally calculated using a patched-conic analysis and iterating to match the required boundary conditions. While this approach works quite well, and high speed computers overcome the labor required by the large number of calculations involved, the degree of insight into the problem is necessarily limited by such an approach. For this reason, an analytical solution to the problem has been derived which satisfies pre-encounter and post-encounter boundary conditions.

The analytical solution to swing-by trajectories is based on a new formulation of the patched-conic technique developed in Reference 1. This formulation reduces to an analytical form the problem of solving patched-conic trajectories which satisfy boundary conditions both inside and outside the sphere of influence. This is combined with the geometry of the trajectory turn at the minor (or target) body to produce the equations required for swing-by trajectories.

ANALYSIS

The Geometry of the Turn at the Minor Body

As the spacecraft trajectory passes through the minor body's sphere of influence, it is turned by the gravity field of the minor body. The geometry of the turn is shown in Figure 1. Two relationships for the turn angle, 2f, can be written,

$$2f = 2\tau + \gamma_a + \gamma_b \qquad (1)$$

and

$$2f = 2\psi + \phi_a + \phi_b \quad (2)$$

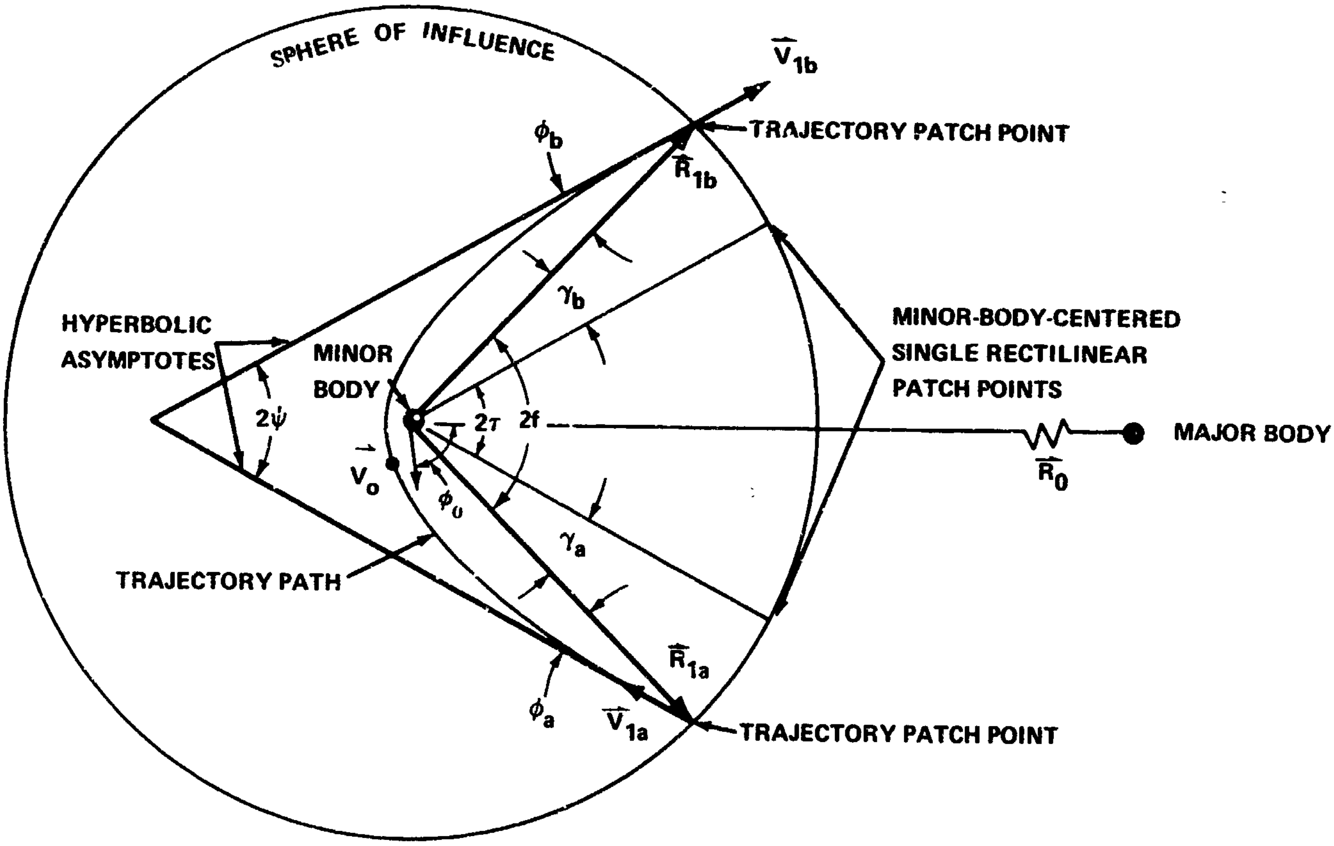


FIGURE 1 - THE GEOMETRY OF THE MINOR BODY TURN OF A SWING-BY TRAJECTORY

The angles γ_a and γ_b are the displacements of the trajectory patch points from the associated minor-body-centered single-rectilinear patch point;* ϕ_a and ϕ_b are the angles between the

---

*Definitions of all special terminology used may be found in Appendix A of Reference 1.

trajectory patch point radius vector and the hyperbolic asymptote of the trajectory. Since the pericenter radius is typically much smaller than the sphere of influence radius, the trajectory may be taken as traveling along the asymptote, and ϕ then becomes the flight path angle of the trajectory. The relationship between γ and ϕ is given by (Reference 1)

$$\sin\gamma \approx \frac{V_1 \sin \lambda_{DRT}}{V_o \sin \phi_o} \sin \phi . \tag{3}$$

The a and b subscripts have been deleted from the equation since it is valid for either case. The angle λ_{DRT} is the longitude of the dual rectilinear patch point and is found from

$$(R_1 V_o \cos\phi_o + R_o V_1 k) \sin\lambda_{DRT} + R_1 V_o \sin\phi_o \cos\lambda_{DRT} - R_o V_o \sin\phi_o = 0 \tag{4}$$

and

$$k_a = -1 \qquad k_b = +1 \tag{5}$$

For planetary configurations encountered in the solar system, R_1 is considerably smaller than R_o,* i.e.,

$$R_1 = \varepsilon R_o \tag{6}$$

---

*The worst case in the solar system is the Earth-Moon system, where R_1/R_o = .168.

Combining Eqns. (4) and (6) and solving for sin λ_{DRT} yields

$$\sin \lambda_{DRT} = \frac{V_o}{V_1}\left\{\frac{\sin\phi_o\left[\varepsilon\frac{V_o}{V_1}\cos\phi_o + k\pm\varepsilon\sqrt{\left(\frac{V_o}{V_1}\right)^2\left(\varepsilon^2 - \sin^2\phi_o\right) + 2k\varepsilon\frac{V_o}{V_1}\cos\phi_o + 1}\right]}{\varepsilon^2\left(\frac{V_o}{V_1}\right)^2 + 2k\varepsilon\frac{V_o}{V_1}\cos\phi_o + 1}\right\} \quad (7)$$

Now, if the eccentricity of the minor planet orbit is less than .15, then ϕ_o is always between 81° and 90°. Thus, $\varepsilon\cos\phi_o$, as well as ε^2, may be dropped from Eqn. (7) as second order terms. Equation (7) then becomes

$$\sin\lambda_{DRT} = \frac{V_o}{V_1}\sin\phi_o\left[k \pm \varepsilon\sqrt{1 - \left(\frac{V_o}{V_1}\right)^2\sin\phi_o}\right] \quad (8)$$

Combining Eqns. (3) and (8),

$$\sin\gamma = \left[k \pm \varepsilon\sqrt{1 - \frac{V_o}{V_1}\sin^2\phi_o}\right]\sin\phi \quad (9)$$

For a small pericenter radius, $\sin\phi$ is a small number, so that $\varepsilon\sin\phi$ is a second order term. Thus,

$$\sin\gamma = k\sin\phi \quad (10)$$

as a first order approximation. Equations (1), (2) and (10) combined show that

$$\psi = \tau \quad (11)$$

That is, the angle between the asymptotes of the hyperbola is equal to the angle between the minor-body-centered single-rectilinear patch points associated with the trajectory, to the degree that Eqn. (10) is true. With regard to the accuracy of the approximation, note that ϕ is a small angle, normally in the range of 5° to 10°, so that a second order error would be expected to be about 1°. The asymptote angle, ψ, is a large angle, typically 45° or more. Thus, the error in τ implied in Eqn. (11) is roughly one part in 45.

Equation (11) provides the wedge required to reduce the swing-by problem to analytical form. The angle ψ is related to the trajectory energy and angular momentum by

$$\tan\psi = \frac{h_1}{\sqrt{\mu_1 a_1}} \qquad (12)$$

and τ is related to the minor-body-centered single-rectilinear patch points, as will be shown in the next section. The coordinates of the minor-body-centered single-rectilinear patch point are calculable from the equations of Reference 1.

The Geometry of the Encounter

To find τ from the coordinates of the minor-body-centered single-rectilinear trajectory, consider the geometry of the minor-body-centered trajectory segment in three dimensional space. Figure 2 shows the geometrical detail for one form of a swing-by encounter. The geometry is slightly different for trajectories which cross the minor body's orbital plane twice while inside the sphere of influence instead of once, as shown in Figure 2. Also, some significant geometrical differences occur as the approach or departure trajectory portions lie interior or exterior to the minor body orbital path. To distinguish between the various possibilities, descriptive trajectory names such as "interior-exterior, double-crossing trajectories" will be used, where "interior-exterior" refers to the geometry of the approach and departure trajectory segments, respectively, and "double-crossing" refers to the geometry within the sphere of influence. Thus, Figure 2 represents an interior-interior, single-crossing trajectory; that is, the spacecraft approaches the minor body sphere of influence from the direction of the major body, crosses the orbital plane of the minor body only once while inside the sphere of influence, and exits the sphere of influence traveling towards the major body.

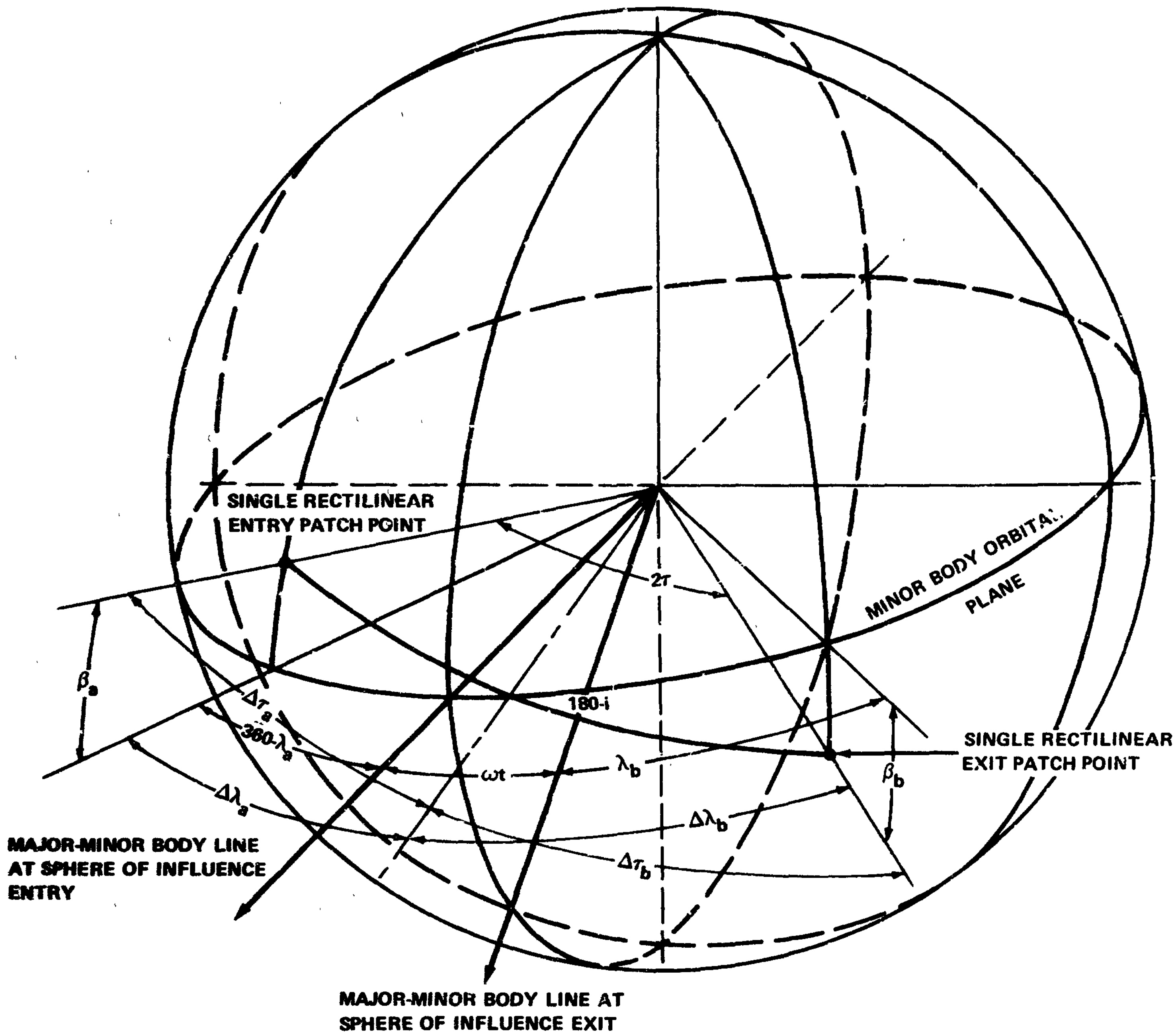


FIGURE 2 - THE GEOMETRY OF AN INTERIOR-INTERIOR, SINGLE-CROSSING TRAJECTORY

In Figure 2, β_a, λ_a and β_b, λ_b are the coordinates of the minor-body-centered single-rectilinear patch points, and are calculated according to the equations of Reference 1. They are referenced to the line joining the major and minor bodies at the time the spacecraft pierces the sphere of influence; thus, β_a, λ_a is referenced to the major-minor body line at the time the spacecraft enters the sphere of influence, and β_b, λ_b is referenced to the major-minor body line at the time the spacecraft exits the sphere of influence. These two major-minor body lines are separated by the angular distance the minor body travels during the time the spacecraft is within the sphere of influence. This is shown as ωt_s. From Figure 2, one may then write

$$\Delta\lambda_a + \Delta\lambda_b = 360 - \lambda_a + \lambda_b + \omega t_s \quad (13)$$

From the relationships of spherical trigonometry

$$\sin\Delta\lambda_a = \frac{\tan \beta_a}{\tan (180 - i_1)}$$

or

$$\sin\Delta\lambda_a = - \frac{\tan \beta_a}{\tan i_1} \quad (14)$$

and, similarly,

$$\sin\Delta\lambda_b = \frac{\tan \beta_b}{\tan i_1} \quad (15)$$

The inclination has been taken as retrograde since an interior-interior trajectory cannot be formed with a posigrade inclination. Combining Eqns. (13) and (15),

$$\frac{\tan\beta_b}{\tan i_1} = \sin[(\omega t_s + \lambda_b - \lambda_a) - \Delta\lambda_a] \qquad (16)$$

Then, using Eqn. (14) and the trigonometric identity for the difference of two angles,

$$\tan\Delta\lambda_a = \frac{\sin(\omega t_s + \lambda_b - \lambda_a)}{\cos(\omega t_s + \lambda_b - \lambda_a) - (\tan\beta_b/\tan\beta_a)} \qquad (17)$$

Equations (15) and (17) give $\Delta\lambda_a$ and $\Delta\lambda_b$ in terms of the minor-body centered, single-rectilinear patch point coordinates. The angle τ is related to $\Delta\lambda_a$ and $\Delta\lambda_b$ by

$$\tau = \frac{1}{2}(\Delta\tau_a + \Delta\tau_b) \qquad (18)$$

where

$$\cos\Delta\tau = \cos\beta\cos\Delta\lambda \qquad (19)$$

It was shown in the previous section that τ is well approximated by ψ, and the relationship (Eqn. (12)) between ψ and angular momentum was stated. Using these relationships, Eqn. (18) becomes

$$h_1 = \sqrt{\mu_1 a_1}\tan\frac{1}{2}(\Delta\tau_a + \Delta\tau_b) \qquad (20)$$

From Eqn. (14)

$$\tan i_1 = -\frac{\tan\beta_a}{\sin\Delta\lambda_a}, \quad 90° < |i_1| < 180° \qquad (21)$$

Calculation of the Sphere of Influence Stay Time

Equations (20) and (21) provide the required bridge between the approach and departure trajectories. The only missing link is the time the spacecraft stays within the sphere of influence. To approximate the sphere of influence stay time, consider the equation for the stay time

$$t_s = 2\sqrt{\frac{a_1^3}{\mu_1}} \left(\frac{R_p + a_1}{a_1} \sinh F - F \right) \qquad (22)$$

where

$$\cosh F = \frac{a_1 + R_1}{a_1 + R_p} \qquad (23)$$

Now, a_1 and R_p are normally much smaller than R_1. Then, from Eqn. (23), cosh F is a large number and we may make the approximation

$$\sinh F = \cosh F = \frac{1}{2} e^F \qquad (24)$$

Equation (22) then becomes

$$t_s = 2\sqrt{\frac{a_1^3}{\mu_1}} \left\{ \frac{a_1 + R_1}{a_1} - \ln 2 \left(\frac{a_1 + R_1}{a_1 + R_p} \right) \right\} \qquad (25)$$

The behavior of the first and second terms of Eqn. (25) is shown in Figure 3 for the range of values of R_p and a_1 normally encountered in Earth-Moon trajectories. The second term is much smaller than the first and is relatively invariant. Since R_p and a_1 are of the same order, approximating the second term with $\ln \frac{a_1 + R_1}{a_1}$ suggests itself. A plot of this expression is shown as a dotted line in Figure 3. For large values of a_1, the approximation is excellent, and for small values of a_1 the second term is very small compared to the first; hence, the approximating term is quite acceptable. Therefore, the stay time equation may be written as

$$t_s = 2\sqrt{\frac{a_1^3}{\mu_1}}\left\{\frac{a_1 + R_1}{a_1} - \ln\left(\frac{a_1 + R_1}{a_1}\right)\right\} \tag{26}$$

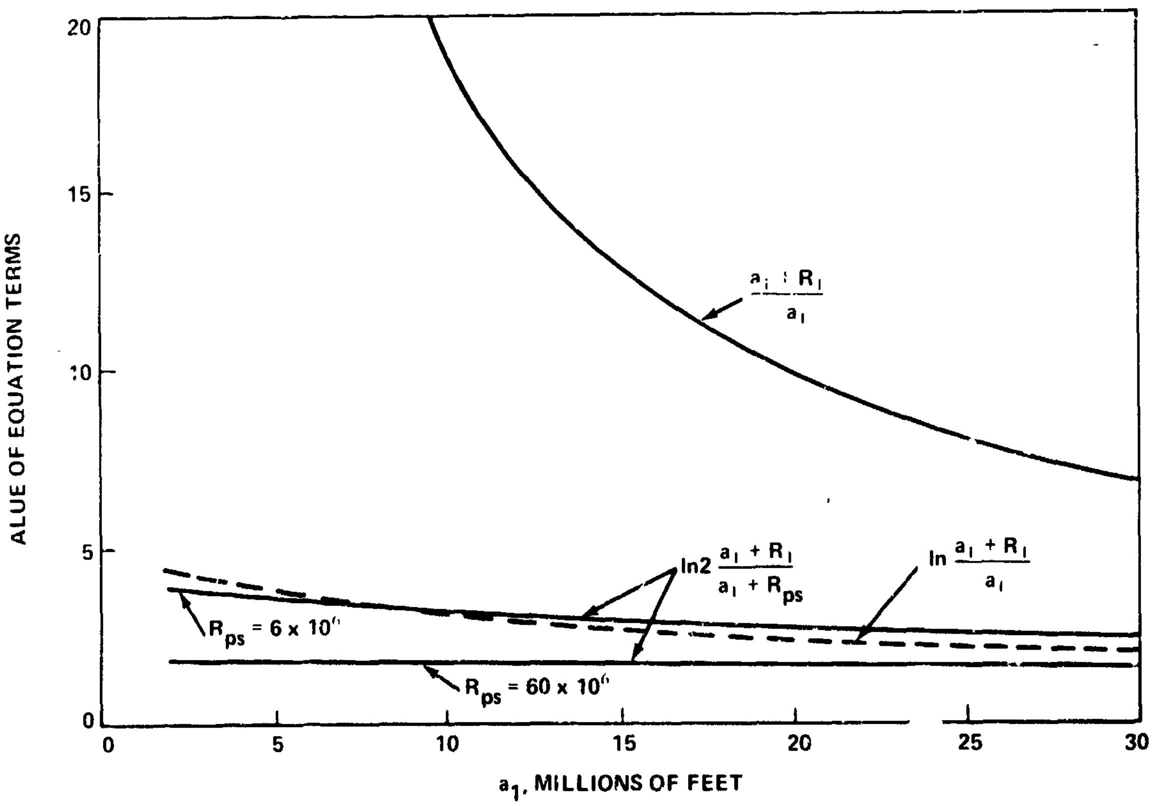


FIGURE 3 - VALUES OF THE TERMS IN EQN. (26) FOR LUNAR TRAJECTORIES

A plot of Eqn. (26) is given in Figure 4 along with the true stay times for a set of lunar trajectories. While the correlation shown is not as good as might be hoped, it is adequate for our purposes.

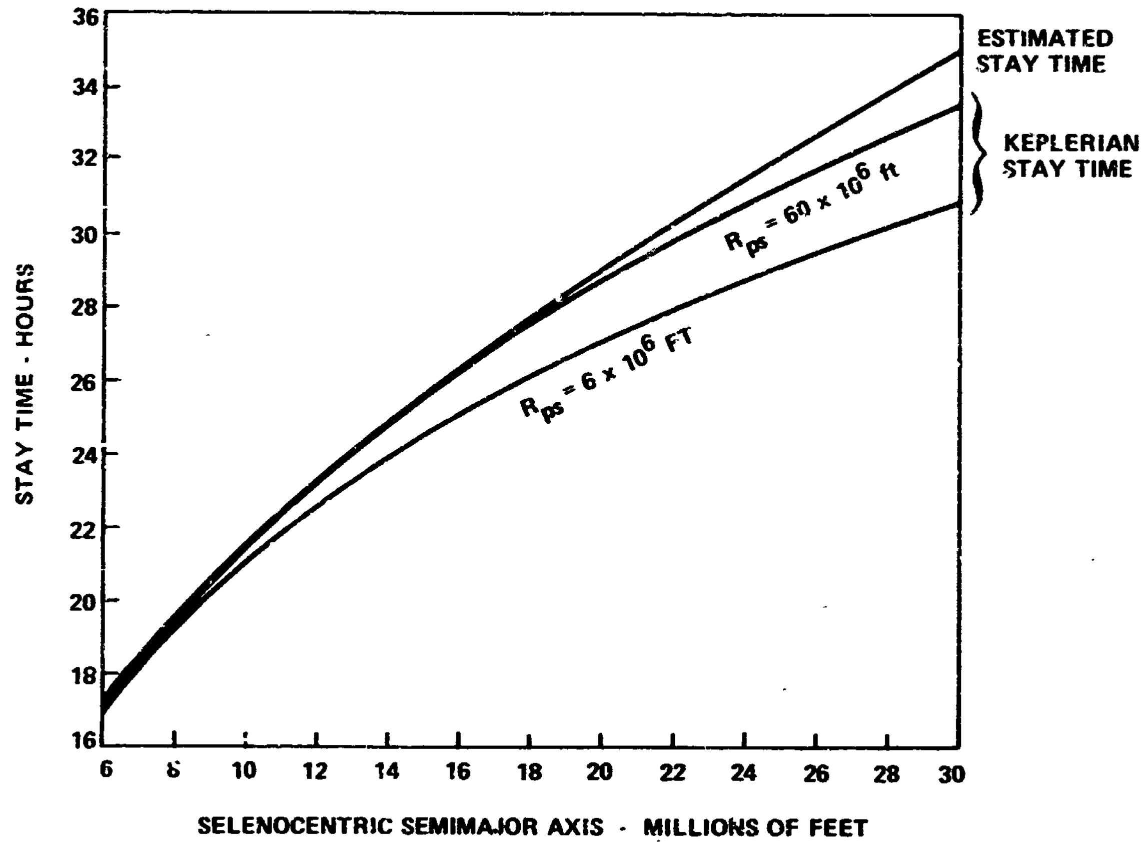


FIGURE 4 - COMPARISON OF ESTIMATED AND KEPLERIAN SPHERE OF INFLUENCE STAY TIMES FOR LUNAR TRAJECTORIES

Correction of the Sphere of Influence Stay Time

Once the stay time has been estimated by Eqn. (26), values for h_1 and i_1 may be obtained. Once h_1 is obtained, t_s may be recalculated using

$$t_s = 2\sqrt{\frac{a_1^3}{\mu_1}}\left(\sqrt{1 + \frac{h_1^2}{\mu_1 a_1}}\sinh F - F\right)$$

where

$$\cosh F = \frac{a_1 + R_1}{\sqrt{1 + \frac{h_1^2}{\mu_1 a_1}}} \qquad (27)$$

Equation (27) is a slightly modified version of Eqn. (25) and is an exact expression, although the value of t_s obtained from the equation is approximate since the value used for h_1 is still approximate. If the new t_s is significantly different from the old value, h_1 and i_1 may be recalculated using the new t_s. It does not appear to be necessary to recalculate λ_b or to perform the recalculation of h_1 and i_1 more than once to obtain good accuracy.

The General Solution to Swing-by Trajectories

The set of equations composed of Eqns. (15), (17), (19), (20) and (26), when combined with the equations of Reference 1, constitutes the solution to interior-interior single-crossing swing-by trajectories. For the other trajectory cases, modifications are required in Eqns. (15), (17), (20), and (21) as follows:

i) For all cases, Eqns. (15) and (17) may be written as

$$\sin \Delta\lambda_b = \left|\frac{\tan \beta_b}{\tan i_1}\right|$$

$$\tan \Delta\lambda_a = \left|\frac{\sin(\omega_o t_s + \lambda_b - \lambda_a)}{\cos(\omega_o t_s + \lambda_b - \lambda_a) - \dfrac{\tan \beta_b}{\tan \beta_a}}\right|$$

ii) For double-crossing trajectories, Eqn. (20) becomes

$$h_1 = \sqrt{\mu_1 a_1} \cot \frac{1}{2} (\Delta\tau_a + \Delta\tau_b)$$

iii) For all trajectories except interior-interior, Eqn. (21) uses the principal value of $\tan i_1$.

The equations for swing-by trajectories which have just been derived are restated in the Appendix in the form of a computational algorithm. It should be noted that two alternative expressions for combining loci were presented for the equation sets in Reference 1. Of these two, the expression chosen for use here retains tighter control of the longitude of the node.

Example Free Return Trajectory

Several approximations have been made in the foregoing derivation. An example problem is now considered to show that these have not degraded the results significantly. Specifically, consider a free return trajectory in the Earth-Moon system with the following characteristics:

Injection Radius = 2.15333×10^7 ft (100 n. mi. alt.)

Injection Inclination = 30°

Return Vacuum Perigee = 2.10333×10^7 ft (100,000 ft alt.)

Return Inclination = 15°

Radius of Periselene = 6.0670×10^6 ft (60 n. mi. alt.)

The trajectory is to first enter the Moon's sphere of influence when the Moon is at apogee. By iterative patched-conic analysis, the trajectory is found to have the following properties:

Approach Angular Momentum = 7.7377×10^{11} ft^2/sec

Approach Semi-Major Axis = 8.6855×10^8 ft

Departure Angular Momentum = 7.6483×10^{11} ft^2/sec

Departure Semi-Major Axis = 8.6658×10^8 ft

Selenocentric Inclination = 174.878°

Selenocentric Angular Momentum = 4.9551×10^{10} ft^2/sec

Entry Patch Point

Latitude = - 4.931°

Longitude = -47.897°

Exit Patch Point

Latitude = 2.792°

Longitude = 46.570°

Entry and exit patch points are referred to the Earth-Moon lines at the time of entry and exit, respectively. These lines are found to be separated by 12.720°.

Using the above input information, the trajectory is calculated using the formulations derived here and in Reference 1. The angular separation between the entry and exit Earth-Moon lines is given as 13.471° using Eqn. (26) and the angular velocity of the Moon at apogee. From Eqns. (20) and (21):

Selenocentric Angular Momentum = 5.0519×10^{10} ft^2/sec

Selenocentric Inclination = 174.848°

So, for this case, angular momentum is found to be within 2% by Eqn. (20) and the inclination is found almost exactly by Eqn. (21). The angular momentum error causes the radius of periselene to be shifted to 6.2752×10^6 feet for an error of 3%. Using Eqn. (27), the angular separation between the Earth-Moon line at entry and exit is found to be

$$\omega_o t_s = 12.732°$$

Recalculating the angular momentum and inclination:

Selenocentric Angular Momentum = 4.9617×10^{10} ft^2/sec

Selenocentric Inclination = 174.828°

The angular momentum error is reduced to less than 0.1%. The corresponding periselene radius is 6.0811×10^6 feet for an error of roughly 0.2%.

Using the angular momentum and inclination just obtained, and the general equation set of Reference 1, the entry and exit patch points are found to be:

Entry Patch Point

Latitude = - 4.977°

Longitude = -47.904°

Exit Patch Point

Latitude = 2.826°

Longitude = 46.576

Using the patch points just obtained, the sphere of influence radius, and the selenocentric angular momentum and inclination obtained from Eqns. (20) and (21), a state vector for the trajectory may be calculated. From this state vector the approach and departure geocentric trajectory parameters may be obtained; they are:

Approach Trajectory

Angular Momentum = 7.7571×10^{11} ft^2/sec

Semi-Major Axis = 8.6861×10^8 ft

Inclination = 30.222°

Departure Trajectory

Angular Momentum = 7.6554×10^{11} ft^2/sec

Semi-Major Axis = 8.6661×10^8 ft

Inclination = 15.182°

The agreement with the input values is quite good. Using these numbers, the approach and departure perigee radii are calculated as

Approach Perigee = 2.1647×10^7 ft

Departure Perigee = 2.1073×10^7 ft

which are within 1% of the desired values.

SUMMARY

The problem of swing-by trajectories has been reduced to an analytical form. The resultant equations show swing-by trajectories to be dependent on six parameters, or their equivalents: the angular momentum and inclination of the major-body-centered trajectories both before and after the encounter with the minor body, the minor-body-centered energy, and the time of the encounter. Thus, swing-by trajectories are dynamically shaped by angular momentum and energy as are Keplerian two body trajectories. The only independent geometrical parameters are the two inclinations. The other geometrical trajectory properties of node line location and pericenter location are implicit with the geometry of the swing-by trajectory.

Two assumptions are made within the derivation, and the accuracy of these assumptions will affect the results. These are:

i) The eccentricity of the minor body's orbit is less than 0.15.

ii) The minor-body-centered trajectory flight path angle is small at the sphere of influence.

These assumptions are usually met by real trajectory problems and the results are acceptably accurate, as was shown for an Earth-Moon free return trajectory.

The analytical solution developed here has the disadvantage that swing-by trajectory problems are most commonly stated in terms of pericenter radii and travel times, parameters which do not readily convert to angular momentum and energy. It is generally true, however, that a concise analytical statement of the principles underlying a problem facilitates its analysis. An immediate benefit gained from this work is the identification of the important variables. A useful extension of the analysis should lie in the development of derivatives of these equations to produce useful tools for targeting swing-by trajectories. In general, the formulation derived here should expedite analysis of problems ranging from Earth-Moon free return trajectories to interplanetary grand tour missions.

2011-KMC-vh

K. M. Carlson
K. M. Carlson

Attachment
Appendix

BELLCOMM, INC.

REFERENCE

1. Carlson, K. M., "An Analytical Solution to Patched-Conic Trajectories Satisfying Initial and Final Boundary Conditions," Technical Memorandum TM-70-2011-1, Bellcomm, Inc., Washington, D. C., November 30, 1970.

APPENDIX

A COMPUTATIONAL ALGORITHM FOR SWING-BY TRAJECTORIES

The required input information for computing swing-by trajectories is:

h_a = the approach angular momentum

i_a = the approach inclination

a_1 = the minor-body-centered semi-major axis

h_b = the departure angular momentum

i_b = the departure inclination

f_a = true anomaly of the minor body at the time of entry of the sphere of influence.

The inclinations are referred to the minor body orbital plane. In addition, one must know whether the trajectory is to be single-crossing or double-crossing, in order to set

$c = 1 \sim$ single crossing

$c = 2 \sim$ double crossing

Also, the constant k must be set according to

$k_a = +1 \sim$ interior approach

$k_a = -1 \sim$ exterior approach

$k_b = +1 \sim$ interior departure

$k_b = -1 \sim$ exterior departure

The gravitational constants for the major and minor bodies must be known and are denoted:

μ_1 = minor body gravitational constant

μ_2 = major body gravitational constant

In addition, the minor body orbital semi-major axis and eccentricity are:

a_o = semi-major axis of the minor body orbit

e_o = eccentricity of the minor body orbit

The distance between the minor body and the major body at the time of entry of the sphere of influence is

$$R_o = \frac{a_o^2(1 - e_o^2)}{1 + e_o \cos f_a} \tag{A-1}$$

and the minor body orbital velocity is

$$V_o^2 = \mu_2\left(\frac{2}{R_o} - \frac{1}{a_o}\right) \tag{A-2}$$

The minor body flight path angle is

$$\sin \phi_o = \left[\frac{a_o^2(1 - e_o^2)}{R_o(2a_o - R_o)}\right]^{1/2} \tag{A-3}$$

The sphere of influence radius is

$$R_1 = \left(\frac{\mu_1}{\mu_2}\right)^{2/5} R_o \tag{A-4}$$

At this point one is ready to begin the calculations. The stay time within the sphere of influence is estimated from

$$t_s = 2\sqrt{\frac{a_1^3}{\mu_1}}\left[\frac{a_1 + R_1}{a_1} - \ln\left(\frac{a_1 + R_1}{a_1}\right)\right] \qquad \text{(A-5)}$$

The angular velocity of the minor body at sphere of influence entry is

$$\omega_o = \frac{V_o \sin \phi_o}{R_o} \qquad \text{(A-6)}$$

The minor-body-centered single-rectilinear patch point latitude is:

$$\sin \beta_{a,b} = - \frac{h_{a,b} \sin i_{a,b} \sin \Omega_{a,b}}{R_1 V_o \sin\phi_o}$$

where (A-7)

$$\tan \Omega_{a,b} = - \frac{R_1 V_o \sin\phi_o}{R_1 V_o \cos\phi_o + k_{a,b} R_o V_1}$$

The a, b subscript means the equation is valid for either entry or exit. The equation is read with the "a" subscript for entry and with the "b" subscript for exit.

The minor-body-centered single-rectilinear patch point longitude is:

$$\sin \lambda_{a,b} = \frac{-AC \pm B\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2}$$

where (A-8)

$$A = R_o V_1 k_{a,b} + R_1 V_o \cos \phi_o$$

$$B = R_1 V_o \sin \phi_o$$

$$C = - \frac{R_o V_o \sin \phi_o - h_{a,b} \cos i_{a,b}}{\cos \beta_{a,b}}$$

Now,

$$\tan \Delta\lambda_a = \left| \frac{\sin(\omega_o t_s + \lambda_b - \lambda_a)}{\cos(\omega_o t_s + \lambda_b - \lambda_a) - \dfrac{\tan \beta_b}{\tan \beta_a}} \right| \tag{A-9}$$

Then, the minor-body-centered inclination is

$$\tan i_1 = - \frac{\tan \beta_a}{\sin \Delta\lambda_a} \tag{A-10}$$

If both k_a and k_b are positive, then

$$i_1 = 180 - i_1 \tag{A-11}$$

Now

$$\sin \Delta\lambda_b = \frac{\tan \beta_b}{\tan i_1} \tag{A-12}$$

and hence

$$\Delta\tau_{a,b} = \cos \beta_{a,b} \cos \Delta\lambda_{a,b} \tag{A-13}$$

The minor body centered angular momentum is then

$$\left. \begin{aligned} h_1 &= \sqrt{\mu_1 a_1} \tan \tfrac{1}{2}(\Delta\tau_a + \Delta\tau_b), \ c = 1 \\ &\text{or} \\ h_1 &= \sqrt{\mu_1 a_1} \cot \tfrac{1}{2}(\Delta\tau_a + \Delta\tau_b), \ c = 2 \end{aligned} \right\} \tag{A-14}$$

The stay time is now recalculated using

$$t_s = 2\sqrt{\frac{a_1^{\ 3}}{\mu_1}}\left(\sqrt{1 + \frac{h_1^{\ 2}}{\mu_1 a_1}}\ \sinh F - F\right)$$

where

$$\cosh F = \frac{a_1 + R_1}{\sqrt{1 + \frac{h_1^{\ 2}}{\mu_1 a_1}}} \qquad \text{(A-15)}$$

Using the new value of t_s, return to Eqn. (A-9) and proceed as before to obtain new values for h_1 and i_1. This loop may be cycled as often as desired to improve the t_s estimate; however, it does not appear that more than one cycle is normally justified.

The values of h_1 and i_1 having been determined, the patch point coordinates for entry and exit are obtained using the method of Reference 1.

Converting to the notation of Reference 1,

$h_2 = h_{a,b}$

$i_2 = i_{a,b}$ for single crossing trajectories

$i_2 = i_a, -i_b$ for double crossing trajectories

$\beta_{SRT2} = \beta_{a,b}$

$\lambda_{SRT2} = \lambda_{a,b}$

Now, calculate the following quantities for both the entry and exit conditions

$$\sin \gamma_{NRT} = \frac{-AC \pm B\sqrt{A^2 + B^2 - C^2}}{A^2 + B^2} \qquad \text{(A-16)}$$

with

$$A = R_1V_o[(\sin\phi_o)\ (\cos\lambda_{SRT2}\sin\beta_{SRT2}\sin i_1' - \sin\lambda_{SRT2}\cos i_1')$$
$$+ (\cos\phi_o)\ (\sin\lambda_{SRT2}\sin\beta_{SRT2}\sin i_1' + \cos\lambda_{SRT2}\cos i_1')]$$

$$+ R_oV_1\ [(\sin\phi_1)\ (-\sin\lambda_{SRT2}\cos\beta_{SRT2})$$
$$+ (\cos\phi_1)\ (-\sin\lambda_{SRT2}\sin\beta_{SRT2}\sin i_1' - \cos\lambda_{SRT2}\cos i_1')]$$

$$B = R_1V_o[-\sin\phi_o\cos\lambda_{SRT2}\cos\beta_{SRT2} - \cos\phi_o\sin\lambda_{SRT2}\cos\beta_{SRT2}]$$

$$+ R_oV_1[(\sin\phi_1)\ (-\sin\lambda_{SRT2}\sin\beta_{SRT2}\sin i_1' - \cos\lambda_{SRT2}\cos i_1')$$
$$+ \cos\phi_1\sin\lambda_{SRT2}\cos\beta_{SRT2}]$$

$$C = h_1\cos i_1' - h_2\cos i_2 + R_oV_o\sin\phi_o$$

where

$$\cos i_1' = \frac{\cos i_1}{\cos\beta_{SRT2}},$$

and

$$h_1 = R_1V_1\ \sin\phi_1\ .$$

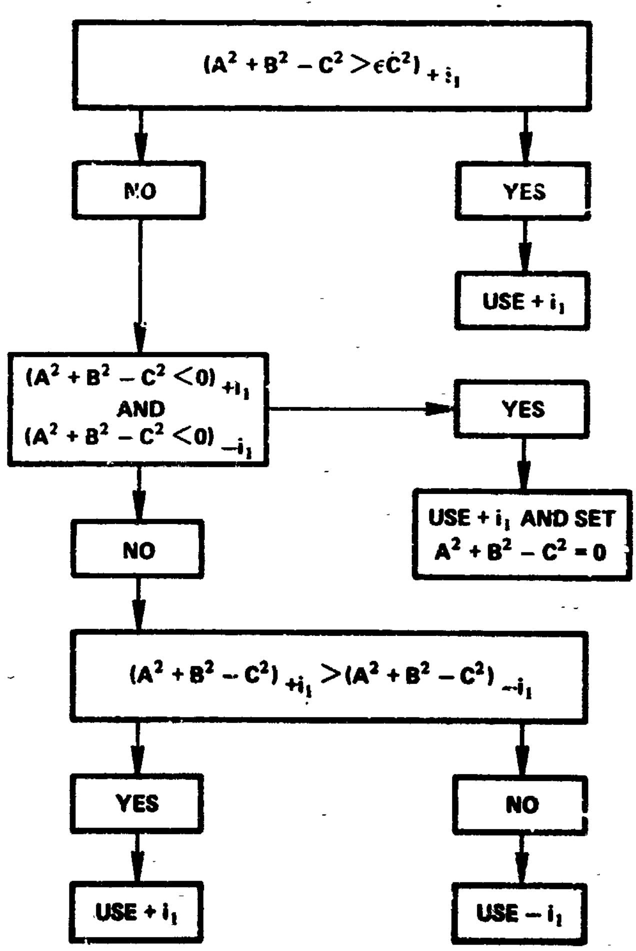
The quantities i_1, i_1' and i_2 carry algebraic signs, according to the rules:

Nearest minor-body-centered node	
Ascending	Descending
i_1 and i_1' are negative	i_1 and i_1' are positive

and

Nearest major-body-centered node	
Ascending	Descending
i_2 is positive	i_2 is negative

Having chosen the appropriate sign for i_1 and i_1' by the above rule, this sign must be modified (for both i_1 and i_1') for use in the γ_{NRT} equation according to the following algorithm:



Note that the sign of i_1 is changed in the γ_{NRT} equation only. The sign on the radical in the γ_{NRT} equation is chosen to obtain the smallest positive value of γ_{NRT} for trajectories from the minor body to the major body and to obtain the smallest negative value of γ_{NRT} for trajectories from the major body to the minor body.

The Non-Rectilinear Patch Point:

The patch point latitude is given by

$$\sin\beta_{NRT} = \sin i_1 \sin\left[\gamma_{NRT} + \sin^{-1}\left(\frac{\sin\beta_{SRT2}}{\sin i_1}\right)\right] \quad \text{(A-17)}$$

and the longitude by

$$\lambda_{NRT} = \lambda_{SRT2} + \sin^{-1}\left(\frac{\tan\beta_{SRT2}}{\tan i_1}\right) - \sin^{-1}\left(\frac{\tan\beta_{NRT}}{\tan i_1}\right) \quad \text{(A-18)}$$

The State Vectors at Sphere of Influence Penetration:

A) The Minor-Body-Centered State Vector

The longitude of the nearest node line is

$$\Omega_1 = \lambda_{NRT} + \sin^{-1}\left(\frac{\tan\beta_{NRT}}{\tan i_1}\right) \quad \text{(A-19)}$$

The argument of latitude of the patch point with respect to Ω_1 is

$$\omega_1 = \sin^{-1}\left(\frac{\sin\beta_{NRT}}{\sin i_1}\right) \quad \text{(A-20)}$$

The position vector is then

$$\left.\begin{aligned} x_1 &= R_1(\cos\Omega_1\cos\omega_1 + \sin\Omega_1\cos i_1\sin\omega_1) \\ y_1 &= R_1(\sin\Omega_1\cos\omega_1 - \cos\Omega_1\cos i_1\sin\omega_1) \\ z_1 &= R_1\sin i_1\sin\omega_1 \end{aligned}\right\} \quad \text{(A-21)}$$

The velocity vector is

$$\left.\begin{aligned} \dot{x}_1 &= V_1[\sin\phi_1(\cos\Omega_1\sin\omega_1 - \sin\Omega_1\cos i_1\cos\omega_1) \\ &\qquad - \cos\phi_1(\cos\Omega_1\cos\omega_1 + \sin\Omega_1\cos i_1\sin\omega_1)] \\ \dot{y}_1 &= V_1[\sin\phi_1(\sin\Omega_1\sin\omega_1 + \cos\Omega_1\cos i_1\cos\omega_1) \\ &\qquad - \cos\phi_1(\sin\Omega_1\cos\omega_1 - \cos\Omega_1\cos i_1\sin\omega_1)] \\ \dot{z}_1 &= V_1[-\sin\phi_1\sin i_1\cos\omega_1 - \cos\phi_1\sin i_1\sin\omega_1] \end{aligned}\right\} \quad \text{(A-22)}$$

B) The Major-Body-Centered State Vector

$$\left.\begin{aligned} x_2 &= R_o - x_1 \\ y_2 &= -y_1 \\ z_2 &= z_1 \end{aligned}\right\} \quad \text{(A-23)}$$

$$\left.\begin{aligned} \dot{x}_2 &= -\dot{x}_1 - V_o\cos\phi_o \\ \dot{y}_2 &= -\dot{y}_1 + V_o\sin\phi_o \\ \dot{z}_2 &= \dot{z}_1 \end{aligned}\right\} \quad \text{(A-24)}$$

The solution is now complete.