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by

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CENTER FOR RADAR ASTRONOMY

RADIOSCIENCE LABORATORY

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Abstract

The problem of polarization measurement is considered under the assumption that observations are carried out using unknown antennas and a variety of calibration signals. The explicit effects of antenna parameters on observations of the coherency matrix of the received waves is determined. A simple method for determining percentage polarization is discussed. The results have been applied to the reduction of bistatic-radar observations of the moon.

Introduction

Studies of electromagnetic polarization are common exercises in optics, astronomy and geophysics. Usually one wishes to determine the polarization of an arriving electromagnetic wave, either from a natural source such as the sun, to obtain information regarding the source mechanisms, or from an experimental source such as a radio transmitter or laser, to obtain information regarding the media through which the signals propagate. Several descriptors for polarization are well known (Stokes, 1852; Poincaré, 1892; Born and Wolf, 1959) and enjoy widespread usage. Elementary descriptions of monochromatic waves and antennas may be found in textbooks (Kraus, 1966). Radio and radar astronomy polarization measurement techniques have tended to follow closely the commonly used optical intensity measurement techniques. The power received on pairs of calibrated orthogonal antennas is used to specify directly the Stokes parameters of the received wave. Conversion to any of the other systems might be carried out then for convenience.

Analyses of the measurement problem in the radio case have been somewhat restricted. Certainly, the response of an arbitrary antenna to a narrowband signal of any general polarization is well understood (Cohen, 1958; Ko, 1950), as is the range of polarization measurements required to completely specify the scattering properties of a medium in scattering experiments (Hagfors, 1968). But the measurement problem itself seems not to have been considered very generally. Typically, it has been assumed that the measurements are carried out using perfect antennas of a specified polarization, but that the receiving channels associated with those antennas introduce some corruption in phase and amplitude. Many authors have extended this analysis to include cross-coupling effects of the antennas themselves. For perhaps the majority of measurement problems these approaches are more than adequate. However, due to measurement or construction difficulties, particularly at the longer wavelengths, it often happens that the polarization of the receiving antennas is known only approximately, or to less than the desired precision. The purpose of this paper is to characterize the polarization measurement problem in such a way as will exhibit the errors involved in a general way. Then we describe certain simple calibration techniques which can permit the use of almost arbitrary pairs of antennas for polarization measurements. The solution to the general problem is not completed here. For example, the detailed error statistics are not calculated. It is hoped that the results and ideas here will lead to further work. For convenience, we first briefly review one way in which polarization may be specified. A matrix formulation is used in the development to facilitate the ease with which the results may be adapted to digital data reduction schemes and computations.

Specification of Polarization

Elliptical

Consider a monochromatic, electromagnetic plane wave. Let $\bar{e}_1$ and $\bar{e}_2$ be two real, orthogonal unit vectors lying in a plane perpendicular to the direction of propagation. Let the complex electromagnetic field strengths along the $\bar{e}_1$ and $\bar{e}_2$ directions be denoted E_1 and E_2, respectively. Then the total electric field is

$$\begin{aligned} \bar{E} &= E_1\,\bar{e}_1 + E_2\,\bar{e}_2 = E_1\left(\bar{e}_1 + \frac{E_2}{E_1}\,\bar{e}_2\right) \\ &= E_1(\bar{e}_1 + p\,\bar{e}_2) \qquad p = E_2/E_1 \end{aligned} \tag{1}$$

The quantity p is the complex polarization of the wave. If we let the vector $\bar{e}_1 \times \bar{e}_2$ lie along the positive Poynting vector and assume an $e^{j\omega t}$ time variation, then the locations in the complex p plane $p = +j$ and $p = -j$ represent left and right circular polarization, respectively. All linear polarizations lie on the real axis, while left elliptical polarization occupies the upper half plane and right elliptical polarization the lower. The relationship of p to other representations has been given elsewhere (Beckmann, 1968).

If

$$\bar{E}' = E'(\bar{e}_1 + p' \, \bar{e}_2) \qquad 2$$

Then $\bar{E}$ and $\bar{E}'$ are orthogonal provided

$$\bar{E} \cdot \bar{E}'^* = 0 \Rightarrow pp'^* = -1, \quad p = \frac{-1}{p'^*}$$

where $*$ indicates the complex conjugate. In terms of the representative polarization ellipses, orthogonal polarizations have equal eccentricities, opposite senses of rotation and mutually perpendicular major axes. Any polarized wave may be decomposed into a linear combination of two such orthogonal components.

In real, isotropic, homogeneous media the physical significance of orthogonality lies in the response of an instrument to $\bar{E}$ and $\bar{E}'$. If the antenna has the response c_i to a unit field aligned with vector $\bar{e}_i$, then the voltage at the antenna terminals in response to fields $\bar{E}$ is

$$V_{out} = E_1 \, c_1 + E_2 \, c_2 = c_1 \, E_1 (1 + p \frac{c_2}{c_1}) = c_1 \, E_1 (1 + p \cdot p_a^*) \qquad 3$$

where the complex polarization of the antenna is $p_a = (c_2/c_1)^*$.

The condition $p \cdot p_a^* = -1$ guarantees that the voltage at the antenna terminals will be zero. For a constant incident flux f

$$f = |E_1|^2 + |E_2|^2 = |E_1|^2 (1 + |p|^2) \qquad 4$$

the output power is proportional to

$$\frac{1 + |p_a^* p|^2 + 2\,\mathrm{Re}(p_a^* p)}{1 + |p|^2} \qquad 5$$

which is zero if $p_a^* \cdot p = -1$, and is maximum if $p = p_a^*$. It can be shown that this requirement is equivalent to the specification that the antenna transmit waves of the same polarization as p, albeit traveling in the opposite direction.

Partially Polarized Waves

Waves of natural origin can seldom be characterized by a simple ellipse. Consider a wave made up of two stochastic signals $E_1(t)$ and $E_2(t)$, oriented along $\bar{e}_1$ and $\bar{e}_2$, such that

$$\bar{E}(t) = E_1(t)\,\bar{e}_1 + E_2(t)\,\bar{e}_2 \qquad 6$$

Throughout the remainder of this report, we shall use E to denote stochastic signals of this more general type, but the functional dependencies will not be explicitly shown. Clearly such usage includes the monochromatic wave as a special case. It is then appropriate to characterize the wave by its various correlation products. In particular, the coherency matrix (Born and Wolf, 1959) for this wave is

$$\underline{J} = \begin{pmatrix} \langle E_1 E_1^* \rangle & \langle E_1 E_2^* \rangle \\ \langle E_1^* E_2 \rangle & \langle E_2 E_2^* \rangle \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \qquad 7$$

where $< >$ denotes statistical averaging (either time or ensemble) appropriate to the analysis being undertaken and the sub-bar denotes a matrix. In experimental situations, the $< >$ would denote the averaging time for the observations.

Such waves may be decomposed into polarized and unpolarized components on the basis of the statistical behavior of E_1 and E_2. The unpolarized component (superscript u) consists of independently varying orthogonal components containing equal power. That is

$$\langle E_1^u \, E_2^{u*} \rangle = 0 \ ,$$

and

$$\langle |E_1^u|^2 \rangle = \langle |E_2^u|^2 \rangle = \tfrac{1}{2} \langle |E^u|^2 \rangle \qquad 8$$

In contrast, the polarized part (superscript p) consists of perfectly correlated orthogonal components. Thus, the polarized wave may be written as

$$\bar{E}^p = E_1^p \, \bar{e}_1 + E_2^p \, \bar{e}_2 = E^p (\bar{e}_1 + p \, \bar{e}_2) \ , \qquad 9$$

where p has the same meaning as before. Such a wave might undergo considerable fluctuations in amplitude or phase since E^p is no longer required to be monochromatic. Regardless of such fluctuations it is distinguished by a polarization ellipse fixed in shape and orientation that may vary in size with the magnitude of E^p.

An analog may be drawn between these concepts and those of a narrowband modulation. In a plane normal to the direction of propagation the complex vector $\bar{e}_1 + p\,\bar{e}_2$ represents a rotating vector defined by the coordinate system $\bar{e}_1$ and $\bar{e}_2$ and p. The quantity E^p represents the deviations from this average motion. Unlike a uniformly rotating phasor, $\bar{e}_1 + p\,\bar{e}_2$ has a periodically varying angular rate and magnitude. It follows that there always exists an antenna, viz., $p_a = -\frac{1}{p^*}$, with zero response to such a wave.

The unpolarized part describes a random, irregular figure in a plane normal to the direction of propagation. There is no single p such that the simple vector $\bar{e}_1 + p\,\bar{e}_2$ can describe the wave. In all representations of the unpolarized part (i.e., for all p) the power carried by the wave is the same, and is equal to one-half the total unpolarized power. A general representation is

$$\bar{E}^u = E(\bar{e}_1 + p\,\bar{e}_2) + E'(\bar{e}_1 - \frac{1}{p^*}\,\bar{e}_2)\,, \qquad 10$$

where

$$\langle E \cdot E'^* \rangle = 0\,, \quad \langle |E|^2 \rangle = \langle |E'|^2 \rangle = \frac{\langle |E^u|^2 \rangle}{2}\,,$$

for any p. The specific form of E and E' is not fixed and will vary with p. Thus antennas of all polarizations will have the identical response, except for differences in gain, to a given unpolarized wave.

Properties of Coherence Matrix

It can be shown (Born and Wolf, 1959) that for statistically independent sources, each with coherency matrix $\underline{J}^i$, that the coherency matrix of the sum is the sum of the individual coherency matrices, or

$$\underline{J} = \Sigma \underline{J}^i \tag{11}$$

The total power density in the wave is given by the trace of the coherency matrix $\mathrm{Tr}\,\underline{J} = J_{11} + J_{22}$. A wave which is neither totally polarized nor totally unpolarized is said to be partially polarized. Such a wave may be decomposed into polarized and unpolarized parts (Ko, 1950).

$$\underline{J} = \frac{1}{2}(1 - \gamma) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathrm{Tr}\,\underline{J} + \gamma \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \cdot \mathrm{Tr}\,\underline{J} \tag{12}$$

where the factor

$$\gamma = \left\{ 1 - \frac{4\,\mathrm{Det}\,\underline{J}}{(\mathrm{Tr}\,\underline{J})^2} \right\}^{\frac{1}{2}} \tag{13}$$

is the normalized fraction of the total power in the polarized part, $(1 - \gamma)$ is the fraction in the unpolarized part. The elements q are defined by

$$q_{12} = \frac{J_{12}}{\gamma\,\mathrm{Tr}\,\underline{J}}, \quad q_{21} = \frac{J_{21}}{\gamma\,\mathrm{Tr}\,\underline{J}}, \quad q_{ii} = \frac{1}{\gamma}\left(\frac{J_{ii}}{\mathrm{Tr}\,\underline{J}} - \frac{1}{2}(1 - \gamma)\right) \tag{14}$$

It is sometimes convenient to deal with the normalized quantities $\rho_{ij} = \frac{J_{ij}}{Tr\ \underline{J}}$, for which the results are similar.

Antenna Transformations

It is our purpose here to provide the basis for an analysis with as few restrictive assumptions as possible. For generality, therefore, assume that we are interested in receiving signals of the form

$$\left.\begin{aligned} \bar{E}_1(t) &= E_1(t)(\bar{e}_1 + p_1\ \bar{e}_2) = E_1\ \hat{e}_1 \\ \bar{E}_2(t) &= E_2(t)(\bar{e}_1 + p_2\ \bar{e}_2) = E_2\ \hat{e}_2 \end{aligned}\right\} \qquad p_1 \cdot p_2^* = -1 \qquad 15$$

That is, the arriving signals are expressed in terms of independently modulated orthogonal polarizations. Further, assume that the received signals, E_1' and E_2' at the terminals of two non-identical antennas are of the form

$$\begin{pmatrix} E_1' \\ E_2' \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \qquad 16$$

where the c's are complex and completely arbitrary. The matrix elements may be thought of as the transmission coefficients of a four port network consisting of pairs of antenna elements and terminals. Only the linearity of the network is assumed. Physically, the c's may represent attenuation, gain, and cross-coupling. They may be the result of an intentional design, as in the coupling of linear antenna elements to form circular polarization, or result from construction errors. But, overall, the $\underline{c}$ matrix and the original $\hat{e}$'s define two new antennas created by a superposition of signals from the original elements. In a perfect, non-transforming system, matched to the polarizations $\hat{e}_1$ and $\hat{e}_2$, $c_{11} = c_{22} = 1$, while $c_{12} = c_{21} = 0$. Usually it is assumed that $c_{11} \simeq c_{22}$, $c_{12} = c_{21}^*$, and that $\left|\frac{c_{12}}{c_{11}}\right|$ is small. Such restrictions are not implied here. In an imperfect system the primed quantities may be thought of as new (transformed) polarizations.

$$E_1' = c_{11}\,\hat{e}_1 + c_{12}\,\hat{e}_2 = (c_{11} + c_{12})\,\bar{e}_1 + (p_1\,c_{11} + p_2\,c_{12})\,\bar{e}_2$$
$$E_2' = c_{21}\,\hat{e}_1 + c_{22}\,\hat{e}_2 = (c_{21} + c_{22})\,\bar{e}_1 + (p_1\,c_{21} + p_2\,c_{22})\,\bar{e}_2 \qquad 17$$

By our previously defined convention (c.f., eqn. 3 and f.f.), the polarizations of the primed antennas are

$$p_{a1}' = \left[(p_1\,c_{11} + p_2\,c_{12})/(c_{11} + c_{12})\right]^*$$

and

$$p_{a2}' = \left[(p_1\,c_{21} + p_2\,c_{22})/(c_{21} + c_{22})\right]^*$$

For orthogonality of the transformed polarizations we require:

$$(c_{11} + c_{12})(c_{21}^* + c_{22}^*) + (p_1\,c_{11} + p_2\,c_{12})(p_1^*\,c_{21}^* + p_2^*\,c_{22}^*) = 0$$

which reduces to

$$c_{11}\,c_{21}^*(1 + |p_1|^2) + c_{12}\,c_{22}^*(1 + |\frac{1}{p_1}|^2) = 0\ . \qquad 18$$

If the input vectors have the same length,

$$(c_{11}\,c_{21}^{*} + c_{12}\,c_{22}^{*})(1 + |p|^{2}) = 0 \tag{19}$$

This latter expression is a necessary (but not sufficient) condition for the c matrix to represent a unitary transformation. In addition, the output power must be proportional to the input (i.e., length must be preserved) so

$$\alpha|c_{11}|^{2} + \beta|c_{12}|^{2} + \alpha|c_{21}|^{2} + \beta|c_{22}|^{2} = (\alpha + \beta)K \tag{20}$$

for all α, β. It follows that the requirements for unitary transformation are

$$c_{11}\,c_{21}^{*} = -\,c_{12}\,c_{22}^{*}$$

and 21

$$|c_{11}|^{2} + |c_{21}|^{2} = |c_{22}|^{2} + |c_{12}|^{2}$$

Clearly these are special conditions which will usually not be fulfilled.

The effect of the c̲ transform on the J matrix is easily shown to be

$$\underline{J'} = \underline{M} \cdot \underline{J}$$

$$\begin{pmatrix} J'_{11} \\ J'_{12} \\ J'_{21} \\ J'_{22} \end{pmatrix} = \begin{pmatrix} |c_{11}|^{2} & c_{11}c_{12}^{*} & c_{11}^{*}c_{12} & |c_{12}|^{2} \\ c_{11}c_{21}^{*} & c_{11}c_{22}^{*} & c_{12}c_{21}^{*} & c_{22}^{*}c_{12} \\ c_{11}^{*}c_{21} & c_{21}c_{12}^{*} & c_{22}c_{11}^{*} & c_{22}c_{12}^{*} \\ |c_{21}|^{2} & c_{22}^{*}c_{21} & c_{22}c_{21}^{*} & |c_{22}|^{2} \end{pmatrix} \cdot \begin{pmatrix} J_{11} \\ J_{12} \\ J_{21} \\ J_{22} \end{pmatrix} \tag{22}$$

where J' is the coherency matrix of the wave associated with $\underline{J}$ observed at the antenna terminals. The apparent power is

$$\mathrm{Tr}\ \underline{J}' = J'_{11} + J'_{22}$$

$$\mathrm{Tr}\underline{J}' = \left(|c_{11}|^2 + |c_{21}|^2\right) J_{11} + \left(|c_{22}|^2 + |c_{12}|^2\right) J_{22} \qquad 23$$

$$+ 2\ \mathrm{Re}\left[\left(c_{11}\ c^*_{12} + c_{21}\ c^*_{22}\right) J_{12}\right] \ , \text{ where } \ J_{12} = J^*_{21} \ .$$

The determinant of $\underline{J}'$ is

$$\mathrm{Det}\ J' = \left\{|c_{11}|^2\ |c_{22}|^2 + |c_{12}|^2\ |c_{21}|^2 \right. \qquad 24$$

$$\left. - 2\ \mathrm{Re}(c_{11}\ c^*_{12}\ c^*_{21}\ c_{22})\right\} \mathrm{Det}\ J$$

The apparent degree of polarization is

$$\gamma' = \left\{1 - \frac{4\ \mathrm{Det}\ \underline{J}'}{(\mathrm{Tr}\ \underline{J}')^2}\right\}^{\frac{1}{2}}$$

The relations just derived have considerable utility. With the advent of high speed digital computers, the numerical manipulation of radio observations has become increasingly common. In the case of polarimetry data, particularly at low signal-to-noise ratios, a large number of calculations may be necessary to obtain the coherency matrix (or an equivalent representation) from sampled data. However, once the coherency matrix is available in numerical form, only matrix multiplications are required to transform the antenna system. Such transformations provide the basis for the calibration and, if necessary, correction of observations. Observations may be carried out with the best available systems or instruments and the data reduced. Then, with little additional effort, provided only that suitable calibration signals may be obtained, the observations may be corrected for instrument errors through manipulations such as those represented by (22).

Manipulations of the coherency matrix

For an unpolarized input

$$J'_{11} = (|c_{11}|^2 + |c_{12}|^2) J_o \qquad 25$$

$$J'_{12} = (c_{11}\, c^*_{21} + c^*_{22}\, c_{12}) J_o$$

$$J'_{21} = J'^*_{12}$$

$$J'_{22} = (|c_{21}|^2 + |c_{22}|^2) J_o$$

The apparent polarization

$$\gamma' = \left\{ 1 - \frac{4 \text{ Det } \underline{J'}}{(\text{Tr } \underline{J'})^2} \right\}^{\frac{1}{2}} \qquad 26$$

$$= \left\{ 1 - \frac{4(|c_{11}|^2\, |c_{22}|^2 + |c_{12}|^2\, |c_{21}|^2 - 2 \text{ Re}(c_{11}\, c_{22}(c_{12}\, c_{21})^*))}{(|c_{11}|^2 + |c_{12}|^2 + |c_{21}|^2 + |c_{22}|^2)^2} \right\}^{\frac{1}{2}}$$

$$\gamma' \geq 0$$

The signal will appear unpolarized if $\gamma' = 0$ which requires

$$J'_{12} = 0 \quad \text{and} \quad J_{11} = J_{22}$$

or

$$c_{11}\, c_{21}^{*} = - c_{22}^{*}\, c_{12} \tag{27}$$

and

$$|c_{11}|^2 + |c_{12}|^2 = |c_{21}|^2 + |c_{22}|^2$$

This requirement that an unpolarized signal appear unpolarized is sufficient to guarantee that the c matrix represent a unitary transformation. Note that

$$c_{11}\, c_{21}^{*} = - c_{22}^{*}\, c_{12} \Rightarrow \left|\frac{c_{12}}{c_{11}}\right|^2 = \left|\frac{c_{21}}{c_{22}}\right|^2 \tag{28}$$

$$\therefore |c_{11}|^2 + |c_{12}|^2 = |c_{11}|^2\left(1 + \left|\frac{c_{21}}{c_{22}}\right|^2\right) = |c_{22}|^2\left(1 + \left|\frac{c_{12}}{c_{11}}\right|^2\right)$$

$$\Rightarrow |c_{11}|^2 = |c_{22}|^2$$

and

$$|c_{21}|^2 = |c_{12}|^2$$

$$\therefore |c_{11}|^2 + |c_{21}|^2 = |c_{11}|^2 + |c_{12}|^2$$

Furthermore, deviations from the unitary transform always produce an apparent increase in the percentage polarization of an originally unpolarized wave. It can be shown also that a polarized wave appears to be completely polarized for all <u>c</u> matrices. The situation is more complicated for the partially polarized case.

Given sufficient information the response to partially polarized waves may be predicted. Since $\underline{J} = \Sigma \underline{J}_i$, for any coherency matrix, $\underline{J}' = \Sigma \underline{J}_i'$ where $\underline{J}_i' = \underline{M} \cdot \underline{J}_i$, the polarized and unpolarized parts may be considered separately. The unpolarized portion of the original $\underline{J}$ is converted into apparent polarized and unpolarized parts, which may interfere constructively or destructively with the transformed polarized parts of the original wave. Thus for an unknown $\underline{M}$ transform the fractional polarization

$$\gamma' = \left\{1 - \frac{4 \operatorname{Det} \underline{J}'}{(\operatorname{Tr} \underline{J}')^2}\right\}^{\frac{1}{2}} = \left\{1 - \frac{4 (\operatorname{Det} \underline{M}) \operatorname{Det} (\Sigma \underline{J}_i)}{(\Sigma \operatorname{Tr} \underline{M} \cdot \underline{J}_i)^2}\right\}^{\frac{1}{2}} \qquad 29$$

behaves in an unpredictable way.

Consider a general signal described by $\underline{J}' = \underline{M} \cdot \underline{J}$. In terms of the normalized matrices $\underline{\rho}$ and $\underline{\rho}'$ we have

$$\underline{\rho}' = \frac{1}{2}(1 - \gamma')\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \gamma' \begin{pmatrix}q_{11}' & q_{12}'\\ q_{21}' & q_{22}'\end{pmatrix} \qquad 30$$

$$= \underbrace{\frac{1}{2}(1 - \gamma')\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{\mathrm{I}} + \underbrace{\gamma' \begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}}_{\mathrm{II}} + \underbrace{\gamma' \begin{pmatrix}q_{11}' - q_{11} & q_{12}' - q_{12}\\ q_{21}' - q_{21} & q_{22}' - q_{22}\end{pmatrix}}_{\mathrm{III}}. \qquad 31$$

Here

$$q_{ii}' = \frac{1}{\gamma'} \left[\rho_{ii}' - \frac{1}{2}(1 - \gamma')\right] \qquad i = 1, 2 \qquad 32$$

$$q_{ij}' = \frac{1}{\gamma'} \rho_{ij}', \quad i \neq j \qquad j = 1, 2$$

The parts I, II and III may be associated with specific portions of the wave.

I = apparent unpolarized part

II = apparent polarized part due to polarized input

III = apparent polarized part due to deviation of $\underline{c}$ matrix from unitary transformation; i.e., the unpolarized input transformed by the c matrix

As a result of the summation property discussed above, the power associated with the matrices I and II will always be in a constant ratio. Thus, assuming that a suitable calibration can be obtained (e.g., by applying two uncorrelated signals to the antenna terminal) III can always be determined. But the ratio of polarized to unpolarized power may or may not be available from this technique. We had

$$\mathrm{Tr}(\underline{J}') = (|c_{11}|^2 + |c_{21}|^2)\, J_{11} + (|c_{22}|^2 + |c_{12}|^2)\, J_{22} + 2\,\mathrm{Re}\left[(c_{11}\, c_{12}^* + c_{21}\, c_{22}^*)\, J_{12}\right] \quad 33$$

The ratio of the part due to the unpolarized input to the total input is

$$\frac{\mathrm{Tr}(\underline{J}'^{u})}{\mathrm{Tr}(\underline{J}')} = \frac{(|c_{11}|^2 + |c_{21}|^2)J_{11}^{u} + (|c_{22}|^2 + |c_{12}|^2)J_{22}^{u}}{\left\{\begin{array}{c}(|c_{11}|^2 + |c_{21}|^2)J_{11}^{p} + (|c_{22}|^2 + |c_{12}|^2)J_{22}^{p} \\ 2\,\mathrm{Re}\left[(c_{11}\, c_{12}^* + c_{21}\, c_{22}^*)\, J_{12}\right] + \mathrm{Tr}(J'^{u})\end{array}\right\}} \quad 34$$

$$= \frac{1}{1 + \dfrac{\mathrm{Tr}(\underline{J}'^{p})}{\mathrm{Tr}(\underline{J}'^{u})}} = 1 - \gamma'$$

With suitable choice of input signals the matrix $\underline{M}$ can be determined.

a) $J_{11} = 1,\ J_{12} = J_{21} = J_{22} = 0$ 35

b) $J_{22} = 1,\ J_{12} = J_{21} = J_{11} = 0$

c) $J_{11} = J_{22} = J_{12} = J_{21} = \frac{1}{2}$

is sufficient and permits straightforward determination of the c's.

A quantity of considerable interest is the percentage polarization. A single unpolarized calibration signal, $J_{11} = J_{22} = \frac{1}{2}$, $J_{12} = J_{21} = 0$ is sufficient to permit its determination. The procedure produces an orthonormal set of polarizations, albiet of unknown description. Consider the $\underline{J'}$ matrix resulting from the input specified above. We wish to generate a second matrix $\underline{J}'' = \underline{M}'\,\underline{J}'$, such that $J''_{12} = J''_{21} = 0$, and $J''_{11} = J''_{22}$. For convenience we require the polarization of E'_1 to be preserved, so $c'_{12} = 0$. Then

$$\begin{pmatrix} E''_1 \\ E''_2 \end{pmatrix} = \begin{pmatrix} c'_{11} & 0 \\ c'_{21} & c'_{22} \end{pmatrix} \begin{pmatrix} E'_1 \\ E'_2 \end{pmatrix} \qquad 36$$

and it follows that

$$J''_1 = |c'_{11}|^2\ J'_{11} \qquad 37$$

$$J''_{12} = c'_{11}\ c'^*_{21}\ J'_{11} + c'_{11}\ c'^*_{22}\ J'_{12} = c'_{11}(c'^*_{21}\ J'_{11} + c'^*_{22}\ J'_{12})$$

$$J''_{21} = J''^*_{12}$$

$$J''_{22} = |c'_{22}|^2\ J'_{22} + |c'_{21}|^2\ J'_{11} + 2\,\mathrm{Re}(c'_{22}\ c'^*_{21}\ J'_{21})$$

For an orthogonal pair of transformed polarizations we require

$$J''_{12} = 0 \Rightarrow c'_{21}/c'_{22} = - \frac{J'^{*}_{12}}{J'_{11}} = - \frac{J'_{21}}{J'_{11}} \; , \; c'_{21} = - \frac{J'^{*}_{12}}{J'_{11}} c'_{22} \qquad 38$$

We may chose $c'_{22} = 1$. Only c'_{11} remains to be determined. In order that the apparent polarizations be zero, $J''_{11} = J''_{22}$.

$$|c'_{11}|^2 J'_{11} = J'_{22} - \frac{1}{J'_{11}} |J'_{21}|^2 \qquad 39$$

$$|c'_{11}|^2 = \frac{J'_{11} J'_{22} - |J'_{21}|^2}{J'^{2}_{11}} = \frac{\text{Det } \underline{J'}}{J'^{2}_{11}}$$

$$|c'_{11}| = \frac{(\text{Det } \underline{J'})^{1/2}}{J'_{11}} = c'_{11}$$

since the phase of c'_{11} is unimportant. Then

$$J''_{11} = \frac{\text{Det } J'}{J'_{11}} \qquad 40$$

$$J''_{22} = J'_{22} + \left|\frac{J'_{21}}{J'_{11}}\right|^2 J'_{11} + 2 \,\text{Re}\left(- \frac{J'^{*}_{21}}{J'_{11}} J'_{21}\right) = J'_{22} - \frac{1}{J'_{11}} |J'_{21}|^2$$

$$J''_{12} = c'_{11}\left(J'_{12} - \frac{J'^{*}_{21}}{J'_{11}} \cdot J'_{11}\right) = 0$$

is the desired transformation.

This procedure for finding the elements of $\underline{M}'$ required to orthonormalize the antenna output must be applied carefully. We have $\underline{E}'' = \underline{C}' \cdot \underline{C} \cdot E$ where $\underline{E}$ is the incident field vector and $\underline{M}$ is the unknown transform introduced by the antenna. An $\underline{M}'$ may be calculated for any input signal, but it will be the proper correction only if an unpolarized (i.e., uncorrelated signals of equal power) input is used.

Under certain conditions a transformation $\underline{M}'$ can be found using partially polarized source. A two step process is required. The polarization of the transformed antennas are not arbitrary but must match that of the polarized part of the source. Then a procedure similar to that just described for the unpolarized source may be carried out. We again require that the percentage polarization after the transformation equal that prior to the transformation. However, it is also necessary that one of the transformed polarizations E_1' say, correspond to the polarization of the source. There seems to be no clear-cut way to achieve this condition other than a calibration scheme or *a priori* knowledge of the antenna. If this condition is achieved, then we might proceed as follows.

Two coherency matrices represent equal percentage polarization if, for both

$$J_{11} J_{22} - |J_{12}|^2 = k(J_{11}^2 + J_{22}^2 + 2J_{11} J_{22}) \quad \text{where} \quad k = \left(\frac{1 - \gamma^2}{4}\right) \tag{41}$$

Again consider transforms of $\underline{J}'$

$$J_{11}'' = |c_{11}|^2 J_{11}' \tag{42}$$

$$J_{22}'' = |c_{22}|^2 J_{22}' + |c_{21}|^2 J_{11}' + 2\,\mathrm{Re}(c_{22} c_{21}^* J_{12}'^*)$$

$$J_{12}'' = c_{11}(c_{21}^* J_{11}' + c_{22}^* J_{12}')$$

Requiring $J''_{12} = 0$, restricts J''_{11} or J''_{22} to the transformed polarizations of the polarized part of the incident wave. As before this leads to $\frac{-J'^{*}_{12}}{J_{11}} = \frac{c_{21}}{c_{22}}$. Take $c_{22} = 1$. Now J''_{22} is independent of c_{11}, while $J''_{12} = 0$. The requirement for equal percentage polarizations becomes $k({J''_{11}}^2 + {J''_{22}}^2 + (2 - \frac{1}{k})J''_{11}\, J''_{22}) = 0$ which may be solved to yield

$$|c_{11}|^2 = \frac{1 \pm \gamma}{1 \mp \gamma} \cdot \frac{\text{Det } \underline{\underline{J'}}}{{J'_{11}}^2} \qquad 43$$

Note that

$$|c_{11}|^2 = \frac{\text{Det } \underline{\underline{J'}}}{{J'_{11}}^2} \quad \text{if} \quad \gamma = 0 ,$$

as before.

The plus and minus solutions correspond to an adjustment of the relative gains of the two orthogonal channels so that J''_{11} is greater than or less than J''_{22}, by the same fractional amount relative to their sum. By our assumption that the polarized part is matched to E_1, the sign ambiguity is resolved with the positive sign. It can be shown for reasonably constructed systems $(J'_{11} \simeq J'_{22})$, that the error introduced in $-J'_{21}/J'_{11}$ is of the order of the percent polarization. If we do not require that J'_{11} correspond to the polarized part of the incident wave, a set of c's satisfying all the requirements just given may still be found. The results, however, will be in error because the correction matrix will remove the portion of the incident polarized power orthogonal to that in J'_{11}, thus causing the assumptions in the calculation of $|c_{11}|^2$ to be violated. In other words, for a signal not matched to J'_{11}, J'_{12} will not be zero, as assumed. Since the correct J'_{12} is unknown, c_{11}, c_{12}, and c_{22} cannot be determined.

Polarization Calibration Schemes

While a large number of calibration techniques are known for polarimeters, the manipulations above suggest several straightforward methods. An example apropos of low signal-to-noise ratio measurements is given below. Several steps are required.

a) i) With the antenna terminals terminated in matched loads of known temperature, T, inject a known $\underline{J}$ matrix. The resultant $\underline{J}'$ matrix observed at the receiver terminals is

$$\underline{J}'_1 = \underline{J}_{\substack{\text{noise at}\\ \text{temp. } T}} + \underline{J}_{\text{preamplifier}} + \underline{J}_{\text{injected}} \qquad 44$$

Change the magnitude of the injected signal by a factor of k and repeat to find J'_2. Then

$$(1 - k)(\underline{J}_{\text{noise}} + \underline{J}_{\text{preamp}}) = \underline{J}'_2 - k\underline{J}'_1 \qquad 45$$

ii) Alternately $\underline{J}_{\text{injected}}$ may be inserted at the antenna terminals. Then

$$(1 - k)(\underline{J}_{\text{preamp}}) = \underline{J}'_2 - k\underline{J}'_1 \qquad 46$$

$\underline{J}_{\text{preamp}}$ may now be subtracted from all future measurements.

iii) If this calibration is carried out while the antenna is directed toward the sky, the quantity determined is $\underline{J}_{\text{sky}} + \underline{J}_{\text{preamp}}$.

b) Inject a signal into a single input port. Measure ${}^{(1)}\underline{J}'$. Inject the same signal into second input port, measure ${}^{(2)}\underline{J}'$. Correct

$$ {}^{(1)}\underline{J}' \text{ for } \underline{J}_{preamp} \text{, as in a)} \tag{47} $$

Then

$$ {}^{(1)}J'_{11} = |c_{11}|^2 J_{11} , \quad {}^{(2)}J'_{11} = |c_{12}|^2 J_{22} , \quad J_{22} = J_{11} $$

$$ {}^{(1)}J'_{12} = c_{11} c^*_{21} J_{11} , \quad {}^{(2)}J'_{12} = c^*_{22} c_{12} J_{22} $$

$$ {}^{(1)}J'_{22} = |c_{21}|^2 J_{11} , \quad {}^{(2)}J'_{22} = |c_{22}|^2 J_{22} $$

Thus

$$ |c_{11}| = \frac{{}^{(1)}J'_{11}}{J_{11}} , \quad |c_{21}| = \frac{{}^{(1)}J'_{22}}{J_{11}} , \quad |c_{22}| = \frac{{}^{(2)}J'_{22}}{J_{22}} , \tag{48} $$

$$ |c_{12}| = \frac{{}^{(2)}J'_{11}}{J_{22}} $$

If we take $c_{11} = |c_{11}|$ as a zero phase reference, then

$$ c_{21} = \frac{{}^{(1)}J'^*_{12}}{c_{11} J_{11}} = \frac{{}^{(1)}J'_{21}}{c_{11} J_{11}} \tag{49} $$

Inject equal coherent signals simultaneously so that $J_{11} = J_{12} = J_{21} = J_{22}$. At the output we observe

$$ {}^{(3)}J'_{12} = {}^{(1)}J'_{12} + {}^{(2)}J'_{12} + (c_{11} c^*_{22} + c_{12} c^*_{21}) J_{12} \tag{50} $$

$$ c_{11} \left(\frac{{}^{(3)}J'_{12} - {}^{(1)}J'_{12} - {}^{(2)}J'_{12}}{J_{12}} \right) = c_{11}(c_{11} c^*_{22} + c_{12} c^*_{21}) $$

$$= |c_{11}|^2 c_{22}^* + c_{11} c_{12} c_{21}^* = |c_{11}|^2 c_{22}^* + \frac{{}^{(1)}J'_{12}}{J_{11}} c_{12}$$

$$\therefore c_{22}^* = c_{11} \frac{{}^{(3)}J'_{12} - {}^{(1)}J'_{12} - {}^{(2)}J'_{12}}{|c_{11}|^2 J_{12}} - \frac{{}^{(1)}J'_{12}}{|c_{11}|^2 J_{11}} c_{12}$$

But thus far only the magnitude of c_{12} is known. The real and imaginary parts may be separated using

$${}^{(3)}J'_{11} = {}^{(1)}J'_{11} + {}^{(2)}J'_{11} + c_{11}(c_{12}^* + c_{12})J_{12} \qquad 51$$

$$\frac{{}^{(3)}J'_{11} - {}^{(1)}J'_{11} - {}^{(2)}J'_{11}}{c_{11} J_{12}} = 2 \operatorname{Re}(c_{12}) \qquad 52$$

$$\operatorname{Im} c_{12} = \pm \left\{ |c_{12}|^2 - (\operatorname{Re} c_{12})^2 \right\}^{\frac{1}{2}}$$

Only one value of c_{12} i.e., $\operatorname{Re} c_{12} \pm \operatorname{Im} c_{12}$ will give the correct magnitude for c_{22}. This permits the sign ambiguity to be resolved.

c) As an alternative to the above, measure c_{11}, c_{12}, c_{21}, c_{22} directly (with a vector voltmeter).

d) Estimate of accuracy

For orthogonal, or almost orthogonal antennas, absolute bounds on the error in the estimation of the percentage polarization may be derived in terms of the actual percentage polarization at the input. Let this percentage be γ, with ϵ the apparent percentage polarization of unpolarized input, and γ' the fraction of the polarized input power associated with a particular (unspecific) polarization.

Then

$$\left|\frac{\Delta\gamma}{\gamma}\right| = \frac{2\epsilon\left(\frac{1}{\gamma} - 1\right)\left(\gamma' + \frac{1}{2}\left(\frac{1}{\gamma} - 1\right)\right)}{1 - \epsilon + 2\epsilon\, \gamma' + \frac{1}{\gamma} - 1} \qquad 53$$

For a given ϵ and γ , γ' must be varied between 0 and 1 to find the true max. For small ϵ , this occurs for $\gamma' \simeq 1$.

Conclusions and Remarks

The techniques described in this report have been used to obtain the percentage polarization of bistatic-radar echoes from the moon. The receiving system (Tyler, 1968) consists of two coherent, narrow band channels, and (nominally) right and left circular polarized feeds in the Stanford Research Institute 150-foot dish antenna. Initial data analyses were based on a spectral decomposition of signals in each channel using sampled data and fast Fourier transform techniques (Tyler, 1969). For subsequent analyses, the data were separated into polarized and unpolarized parts, using the method in (40) to calibrate the antenna-receiver combination. At the time the data were taken, an attempt was made to maintain equal signal levels in the two receiving channels. While the polarization of the antenna feed structure was known, the effects of the dish surface and tripod could not be measured. Thus, the nominal polarization of the system was known, but the details could not be determined.

Observations were carried out at a wavelength of 2.2 m. In this region of the radio spectrum radio background is approximately 5-10% polarized and has nominal temperature of 500°K. Matched preamplifiers were used, with temperatures of $\sim 250^{\circ}$K. The J' matrix was computed from data as a function of frequency, using the spectral components previously determined. Then, since each $\underline{J}'$ spectrum contained a sample of the cosmic noise, an $\underline{M}'$ matrix, equivalent to (40), could be determined. The resulting $\underline{J}'' = \underline{M} \cdot \underline{J}'$ was used to compute the fractional percentage polarization, γ , again as a function of frequency.

The total power is $\mathrm{Tr}\underline{J}''$, the polarized part is $\gamma \cdot \mathrm{Tr}\underline{J}''$, and the unpolarized part is $(1 - \gamma) \cdot \mathrm{Tr}\underline{J}''$. As expected, the principal corrections required corresponded to gain adjustments, i.e., $|c_{11}| \sim 0.8 \pm 0.2$ while $|c_{21}| < 0.1$, typically.

In future experiments it would be highly desirable to use known calibration signals, such as those suggested in the preceding section of this report. With such signals, it would be a simple matter to apply corrections for systems errors or changes in the data processing.

Considerably more analysis of the statistical behavior of the $\underline{M}$ transformations needs to be carried out before these techniques can be completely evaluated.

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