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BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D. C. 20024

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SUBJECT: Fitting a Mass Distribution to a Potential - Case 310

DATE: February 1, 1971

FROM: S. L. Levie, Jr.

ABSTRACT

For a fixed mass distribution, McLaughlin presented an integral criterion for determining the set of mass values which will render the gravitational potential of the distribution a "best match" to a given potential over a fixed region of space. The criterion is here extended to include arbitrary bodies in the mass distribution and then is reformulated as a classical least squares matrix problem. The new formulation permits convenient calculation of the mass values, allowing one to determine an optimum mass distribution iteratively. Analytical evaluation of the weight matrix in the new formulation is performed for a case of general interest.

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MEMORANDUM FOR FILE

1.0 INTRODUCTION

This paper presents an elaboration of McLaughlin's method [1] for calculating a unique set of mass values, assuming that a mass distribution, a region of space, and a gravitational potential to be matched by the distribution in the region have been specified. The motivation for the work was a desire to find a simple mass distribution which would reproduce the L1 potential.* The distribution obtained is reported in [2].

In its generalized form, McLaughlin's method consists of selecting an arbitrary number of bodies whose net gravitational potential $\tilde{v}$ could be computed if the bodies' masses were known. The masses are selected in such a way that $\tilde{v}$ as closely as possible reproduces the given potential v throughout a region R of space, in the sense that the criterion integral

$$f = \int_R (v-\tilde{v})^2 \, d\tau \tag{1}$$

is minimized. Since potentials are linear in the mass, $\tilde{v}$ can be written as

$$\tilde{v} = \sum_{k=1}^{n} m_k \tilde{v}_k, \tag{2}$$

where n is the total number of bodies and $m_k \tilde{v}_k$ is the potential of the k^{th} body. This shows that f is quadratic in the masses, from which it follows that the set of masses minimizing f is unique. Presented this way, McLaughlin's method

*The L1 potential is the lunar gravitational potential currently used in the Apollo program. It consists of a particular superposition of six solid spherical harmonies.

displays its formal equivalence to the Ritz method (in the calculus of variations) for determining an approximate representation of the solution to the variational problem of minimizing the functional f [3].

A particular virtue of McLaughlin's integral criterion is that it allows the matching of a potential function with respect to a finite region of space. For example, the moon's potential -- or, more accurately, the first few terms of a solid spherical harmonic expansion of its potential -- has been determined from observations of orbiting spacecraft. Since these vehicles have been restricted to a thin shell of the space surrounding the moon, it seems proper to require that the potential $\tilde{v}$ generated by the mass distribution match v only inside the thin shell. Otherwise, v would have to be extended into a region in which it has not been observed.

The reason for wanting some criterion for determining only the masses is that including size, shape, and location parameters can lead to ambiguities in the process of fitting a mass distribution to a potential. Therefore the fitting should be done with these additional parameters held fixed. A benefit of this approach is that it allows the investigator to shape the candidate mass distribution into conformity with his prejudices concerning the actual physical distribution. The mass values, for whose selection there is no a priori basis, will then follow automatically from minimization of the criterion function. Since the final value of the criterion depends on the assumed fixed parameters of the distribution, the closeness of the criterion to its absolute minimum serves as a critique of the candidate distribution and the investigator's prejudices about the actual distribution.

The next section presents a reformulation of McLaughlin's integral criterion as a classical least squares matrix problem. It is based on the assumption that both potentials v and $\tilde{v}$ are expandable in the same complete set of functions, which is just the format of present studies of planetary potentials. In the new formulation, the integral in (1) appears only in the weight matrix, in a form admitting analytic precomputation. This and the matrix format adapt (1) to computer applications.

Section 3 presents a way of constraining the sum of the masses in the integral criterion, and Section 4 gives a simple example to illustrate the workings of the criterion. The weight matrix is the subject of Section 5, where its formulation is demonstrated for the case that the complete set of functions is the solid spherical harmonics. Some conclusions are given in Section 6.

2.0 MATRIX FORMULATION

Suppose the elements of a set of functions which is complete over a region A of space are ordered and arranged to form the infinite vector*

$$O^T = (O_0, O_1, O_2, \ldots.), \tag{3}$$

where O_i is the i^{th} function in the set. Assuming that v can be expanded in terms of these functions, there exists a vector of numbers

$$S^T = (S_0, S_1, S_2, \ldots.) \tag{4}$$

such that

$$v = \sum_{i=0}^{\infty} O_i S_i = O^T S. \tag{5}$$

If the set of functions is orthogonal, the coefficients S_j may be computed from

$$S_j = [\int_A O_j \, v \, d\tau] \, / \, [\int_A O_j^2 \, d\tau]. \tag{6}$$

Referring to (2), suppose that for each k the function $\tilde{v}_k$ has an expansion in terms of the same complete set used in expanding v. Then for each k there exists an infinite set of numbers, the i^{th} of which is denoted J_{ik}, such that

$$\tilde{v}_k = \sum_{i=0}^{\infty} O_i J_{ik}. \tag{7}$$

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*"T" denotes the transpose of a vector. Thus (3) implies that O is a column vector.

The numbers J_{ik} can be computed by analogy with (6), if the expansion functions are orthogonal. Writing the n mass values appearing in (2) as the vector

$$M^T = (m_1, m_2, \ldots, m_n), \tag{8}$$

we see that (2) has the equivalent matrix form

$$\tilde{v} = \sum_{k=1}^{n} \sum_{i=0}^{\infty} O_i \, J_{ik} \, m_k = O^T JM. \tag{9}$$

J represents the $(\infty x n)$ matrix whose k^{th} column contains the expansion coefficients of $\tilde{v}_k$.

Looking now at the argument of the integral in (1), and using (5)and (9), it is seen that the scalar

$$v-\tilde{v} = O^T S-O^T JM = O^T (S-JM) \tag{10}$$

must be squared. Exploiting the scalar property, the square is

$$(v-\tilde{v})^2 = (S-JM)^T \, OO^T (S-JM). \tag{11}$$

Note that OO^T is the infinite square matrix formed by taking the outer product of the vector O with itself. Substituting (11) and (1) finally gives the matrix reformulation of McLaughlin's integral criterion. It is

$$f = (S-JM)^T \, W(S-JM), \tag{12}$$

where W is the weight matrix*

$$W = \int_R OO^T \, d\tau. \tag{13}$$

---

*The region R has been implicitly assumed to be a subspace of A.

For ordinary applications, the functions θ_i will be solid spherical harmonics, for which the integrals in (13) can be performed analytically. The weight matrix is discussed in detail in Section 5.

The form of (12) shows that the minimization of f is a classical least squares problem, the solution of which is easily shown to be

$$M = (J^T WJ)^{-1} J^T WS. \tag{14}$$

This is the fundamental result of this paper. It is shown in Appendix I that the inverse of $J^T WJ$ exists if each mass of the distribution has a distinct set of parameters and if R is chosen so that W has full rank.

The practical application of (14) rests on the assumption that the series in (5) and (7) are uniformly convergent. This allows them to be truncated at some term which represents an acceptable level of error in the expansions. Thus upper limits of infinity are to be replaced by some "satisfactorally small" integer N in the sums implicit in (14). This converts J to an ($N x n$) matrix, W to an (N+1 x N+1) matrix and S to an (N+1)-dimensional vector, rendering (14) a finite matrix equation.

A further word must be said about determining the elements of S and J. Although (6) can be used to obtain the expansion of a potential in terms of a complete, orthogonal set of functions, a simpler, precomputed alternative is available for problems of interest. These problems -- such as finding mass distributions which will reproduce the moon's or the earth's potential -- are conveniently posed so that the potential to be matched is given as a set of coefficients for a sum of solid spherical harmonics. Thus, with the θ_i chosen as solid spherical harmonics, the elements of S are provided in the statement of the problem. The columns of J, each of which will be the solid spherical harmonic expansion coefficients for a constituent unit mass, are available from potential theory [4]. In general these coefficients must be computed as certain integrals of an assumed density function, but for simple bodies, such as point masses and oblate spheroids, the coefficients are available in closed form [5].

3.0 CONSTRAINED PROBLEM

For the potential of a proposed mass distribution to be correct at large distances from the center, the total mass must be required to equal the mass m of the body whose potential is being matched by the distribution. This constraint may be written as

$$\sum_{i=1}^{n} m_i = m. \tag{15}$$

With this constraint operating, the minimization problem treated in the last two sections must be modified. This can be done in a simple way, using the Lagrange undetermined multiplier technique.

To begin, noting that (12) is quadratic in the masses, let us square (15) to get the quadratic matrix expression

$$M^T N M = m^2, \tag{16}$$

where N is an $(n x n)$ matrix with ones everywhere and M is defined in (8). This constraint will be used instead of (15) in the minimization problem.

In the Lagrange method, the problem of minimizing (12) subject to (16) is replaced by the auxiliary problem of minimizing

$$f' = f + \lambda(M^T NM - m^2), \tag{17}$$

where λ is a constant to be determined so that the constraint is satisfied. It is easily shown that the solution of this new problem is

$$M = [\lambda N + J^T WJ]^{-1} J^T WS. \tag{18}$$

Comparing (14) and (18), the constraint's presence is seen to modify $J^T WJ$ by adding λ to every element.

Formally, λ may be determined by substituting (18) into (16), getting

$$S^T WJ[\lambda N + J^T WJ]^{-1} N[\lambda N + J^T WJ]^{-1} J^T WS = m^2, \tag{19}$$

which is quadratic in λ. The correct root is the one which causes (15) to be satisfied. (The extra root was introduced by squaring (15).) Practically, however, it is simpler and numerically more significant to iterate (18) as a function of λ until (15) is satisfied.

In forming the inverse of $\lambda N + J^T WJ$, one should be alert for peculiarities, as the following example warns. Suppose

$$J^T WJ = \begin{pmatrix} a & a & a \\ a & b & c \\ a & c & b \end{pmatrix}, \tag{20}$$

which is the result of selecting a point mass at the origin, two other point masses symmetrically located on the z-axis, and W as the identity matrix. Then the inverses of $J^T WJ$ and $\lambda N + J^T WJ$ are identical except for the (1, 1) elements. Consequently, satisfaction of the constraint is achieved in this example by adjusting only the value of the point mass at the origin.

4.0 SIMPLE THEORETICAL EXAMPLE

To illustrate the workings of McLaughlin's integral criterion for matching potentials, a simple, two-dimensional example admitting an *a priori* solution will be presented. In the example, the known potential will be assumed to be

$$v = Gm_1/r, \tag{21}$$

where G is the gravitational constant, m_1 a known mass, and r the distance from the origin of an x-y coordinate system to the point at which v is to be computed. This is recognized as the potential due to a point mass m_1 located at the origin.

Suppose it is assumed that a mass distribution capable of producing a good match to (21) consists of a point mass m_2

located at a fixed point d on the positive x-axis. Then the criterion integral (12) may be used to compute the best value for m_2, after a region R is specified in which the match is to be optimized. Let the region be a small rectangle on the negative x-axis at a distance b from the origin. The coordinate system, the masses m_1 and m_2, and the region R are shown in Figure 1. Also shown are the equipotential surfaces of v and the equipotential surfaces generated by the "matching mass" m_2.

By inspecting Figure 1 it may be seen that m_2 will generate an optimum match in R provided its potential $\tilde{v}$ equals v in the center of R. This requires

$$m_2 = m_1(b+d)/b, \tag{22}$$

a value greater than m_1. This value of m_2 will produce excellent agreement between v and $\tilde{v}$ throughout the region R, where the equipotentials generated by m_1 and m_2 are nearly parallel.

If the constraint (15) is applied to the problem, the solution is trivial:

$$m_2 = m_1. \tag{23}$$

Notice that a penalty -- a poorer matching of the potentials v and $\tilde{v}$ in R -- has been paid for satisfying the criterion.

The quality of a match, measured by the criterion integral f, depends on the interaction of R and the assumed mass distribution. This may be seen for the unconstrained problem by moving the rectangle in Figure 1 to a point on the y-axis a distance d from the origin. For this location, no choice of m_2 can produce a match as good as the one just computed in (22), for in the new region the equipotentials are more nearly perpendicular than parallel. Conversely, for a fixed region R, the quality of the fit can vary with changes in the assumed mass distribution. This allows one to iterate the distribution to improve the fit, a useful feature mentioned in the introduction.

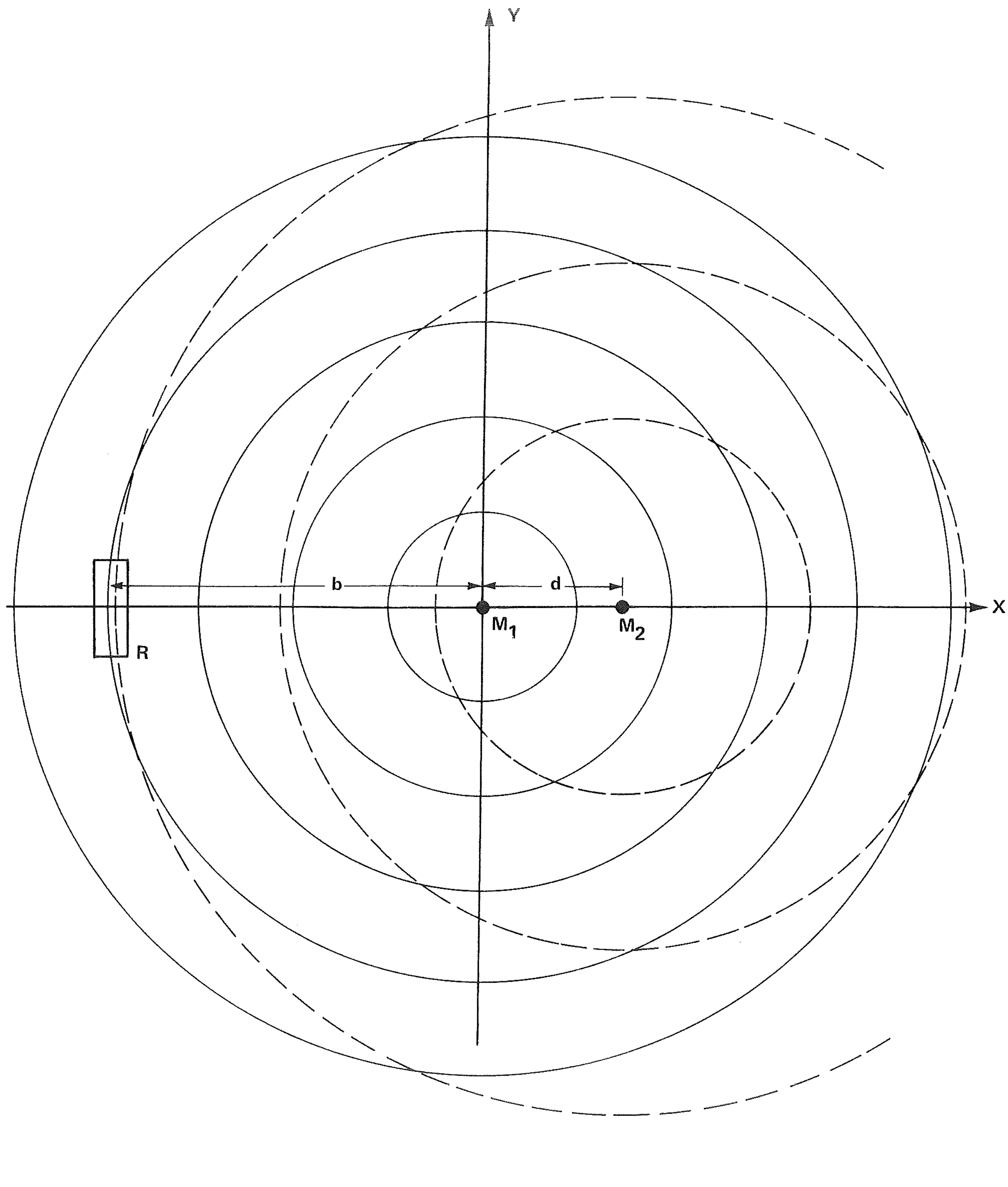

FIGURE 1 - SOLID CIRCLES ARE THE EQUIPOTENTIALS OF THE GIVEN POTENTIAL, GENERATED BY M_1. BROKEN CIRCLES ARE THE EQUIPOTENTIALS DUE TO THE "MATCHING MASS" M_2. THE REGION R IN WHICH THE MATCH IS TO BE OPTIMIZED IS SHOWN ON THE X-AXIS.

5.0 THE WEIGHT MATRIX

5.1 General Case

The weight matrix W was defined in (13) as a square matrix whose elements are the integrals of products of the functions over which the potentials v and $\tilde{v}$ were expanded. This matrix isolates the integral in McLaughlin's criterion function f from the parameters of v and $\tilde{v}$. An advantage accrues from this isolation because the integrals in W often can be performed analytically, in advance of the minimization of f. The integration will be demonstrated in this section for the case in which solid spherical harmonics are the functions over which the potentials are expanded. The integration region will be a torus-shaped shell of finite thickness. This is a useful region when one is dealing with spacecraft-sampled potentials. The infinitely thick shell is discussed in Section 5.2

By solid spherical harmonics is meant the infinite set of functions whose general element is

$$\frac{1}{r}\left(\frac{c}{r}\right)^{\ell} P_{\ell}^{m}(\cos)\begin{pmatrix}\cos m\phi \\ \\ \sin m\phi\end{pmatrix} \tag{24}$$

where the indices ℓ and m run from zero to infinity subject to $m \leq \ell$. The parentheses indicate that the set is actually doubly infinite, containing both sines and cosines. (r, θ, ϕ) are the spherical polar coordinates of a point in space, and c is an arbitrary constant. $P_{\ell}^{m}(\cos\theta)$ is the unnormalized associated Legendre polynomial, definable for non-negative values of ℓ and m by [6]

$$P_{\ell}^{m}(x) = (1-x^2)^{\frac{m}{2}} \sum_{t=0}^{k_t} T_{\ell m t}\, x^{\ell-m-2t} \qquad (|x|\leq 1,\ m\leq\ell) \tag{25}$$

in which

$$T_{\ell m t} = \frac{(-)^{t}(2\ell-2t)!}{2^{\ell} t!\,(\ell-t)!\,(\ell-m-2t)!} \tag{26}$$

and k_t is the greatest integer in $(\ell-m)/2$, written as

$$k_t = \left[\frac{\ell - m}{2}\right] . \tag{27}$$

Notice that $P_0^{\,0}(x) = 1$.

The solid spherical harmonics may be organized into an infinite vector, as required by (3), by forming an "inner loop" on the index m, a "middle loop" on the index ℓ, and an "outer loop" on the trigonometric functions. The resulting vector $\mathcal{O}$ is shown in Figure 2. Notice that $\mathcal{O}$ has been written with two parts:

$$\mathcal{O} = \begin{bmatrix} \mathcal{O}_I \\ \\ \mathcal{O}_{II} \end{bmatrix} \tag{28}$$

where $\mathcal{O}_I$ contains all the cosines and $\mathcal{O}_{II}$ all the sines. Both $\mathcal{O}_I$ and $\mathcal{O}_{II}$ must be truncated if the series expansions of the potentials are truncated for practical work. Using (28) in (13), the weight matrix becomes

$$W = \begin{pmatrix} A & B \\ \\ B^T & C \end{pmatrix} , \tag{29}$$

where

$$\left.\begin{aligned} A &= \int_R \mathcal{O}_I \mathcal{O}_I^T \, d\tau \\ B &= \int_R \mathcal{O}_I \mathcal{O}_{II}^T \, d\tau \\ C &= \int_R \mathcal{O}_{II} \mathcal{O}_{II}^T \, d\tau \end{aligned}\right\} \tag{30}$$

$$
O = \begin{bmatrix}
1/r \\
(c/r^2)\ P_1^{\,0}(\cos\theta) \\
(c/r^2)\ P_1^{\,1}(\cos\theta)\ \cos\phi \\
(c^2/r^3)\ P_2^{\,0}(\cos\theta) \\
(c^2/r^3)\ P_2^{\,1}(\cos\theta)\ \cos\phi \\
(c^2/r^3)\ P_2^{\,2}(\cos\theta)\ \cos 2\phi \\
(c^3/r^4)\ P_3^{\,0}(\cos\theta) \\
(c^3/r^4)\ P_3^{\,1}(\cos\theta)\ \cos\phi \\
(c^3/r^4)\ P_3^{\,2}(\cos\theta)\ \cos 2\phi \\
(c^3/r^4)\ P_3^{\,3}(\cos\theta)\ \cos 3\phi \\
\vdots \\
(c/r^2)\ P_1^{\,1}(\cos\theta)\ \sin\phi \\
(c^2/r^3)\ P_2^{\,1}(\cos\theta)\ \sin\phi \\
(c^2/r^3)\ P_2^{\,2}(\cos\theta)\ \sin 2\phi \\
(c^3/r^4)\ P_3^{\,1}(\cos\theta)\ \sin\phi \\
(c^3/r^4)\ P_3^{\,2}(\cos\theta)\ \sin 2\phi \\
(c^3/r^4)\ P_3^{\,3}(\cos\theta)\ \sin 3\phi \\
\vdots
\end{bmatrix}
$$

FIGURE 2 - THE SOLID SPHERICAL HARMONICS ARRANGED AS A VECTOR IN FUNCTION SPACE.

in which R is the integration region and $d\tau$ is the volume element $r^2 dr\, d(\cos\theta)\, d\phi$.

Let the region R be chosen so that

$$\left.\begin{array}{ll} c + \Delta_1 \leq r \leq c + \Delta_2 & (\Delta_2 > \Delta_1 \geq 0) \\ -a \leq \cos\theta \leq a & (|a| \leq 1) \\ 0 \leq \phi < 2\pi & \end{array}\right\} . \qquad (31)$$

Thus R is a torus-shaped shell of thickness $\Delta_2 - \Delta_1$, whose symmetry axis coincides with the z-axis and whose symmetry plane coincides with the x-y plane. Note that if a=1, R becomes a spherical shell. If $\Delta_2 = \infty$, the shell is infinitely thick. This case is treated in Section 5.2.

The r, θ, and ϕ integrals in (30) are independent of each other and may be performed separately. Since the ϕ integrals in sub-matrix B have the character

$$\int_0^{2\pi} \cos i\phi \, \sin j\phi \, d\phi ,$$

every element of B is zero. A general element of A is

$$\int_0^{2\pi} \int_{-a}^{a} \int_{c+\Delta_1}^{c+\Delta_2} \frac{1}{r^2} \left(\frac{c}{r}\right)^{\ell+n} P_\ell^{\,i}(x)\, P_n^{\,j}(x) \cos i\phi \cos j\phi \, r^2 dr\, dx\, d\phi$$

$$= \pi c\, \delta_{ij} \int_{-a}^{a} P_\ell^{\,i}(x)\, P_n^{\,j}(x)\, dx \left\{ \begin{array}{ll} \dfrac{(1+\Delta_2/c)^{1-\ell-n}}{1-\ell-n} - \dfrac{(1+\Delta_1/c)^{1-\ell-n}}{1-\ell-n} & (\ell+n \neq 1) \\ \ln \dfrac{1+\Delta_2/c}{1+\Delta_1/c} & (\ell+n=1) \end{array} \right\} \qquad (32)$$

where δ_{ij} is the Kronecker delta. The general element of C differs only in having the cosines in (32) replaced by sines, so the integrals are the same. The remaining integral in (32), an integral of arbitrary pairs of associated Legendre polynomials over an arbitrary interval, is given in Appendix II. This completes the demonstration that W can be computed analytically for at least one interesting region and one interesting set of expansion functions.

5.2 Infinitely Thick Shell

If in (32) we let Δ_2 tend to infinity and consider the element of the weight matrix characterized by $\ell=i=n=j=0$, we find that the radial integral diverges. Referring to (12), this means that f may be infinite when R is infinitely thick.

To explore the consequences of this, suppose that the given potential is generated by a body of mass m. Then (12) can be expanded into an infinite series all of whose terms are finite except the first, which contains the divergent integral. Emphasizing this term, the series may be represented as

$$f = 2\pi ac \left(m - \sum_{i=1}^{n} m_i\right)^2 \lim_{u\to\infty} \int_{c+\Delta_1}^{u} dr + \ldots. \tag{33}$$

As u tends to infinity, the first term dominates the series. Thus, if in the minimization of (33) a non-infinite limiting value is attainable, it will be achieved by satisfying

$$\sum_{i=1}^{n} m_i = m, \tag{34}$$

for which the first term of (33) is zero. This happens to be the constraint discussed in Section 3.0. For large but finite values of u, (34) will be approximately satisfied.

Minimization of f is seen to force a condition which nullifies the divergent integral which appears when the region of integration is infinitely thick. Although the divergence has been discussed in connection with expansion in solid spherical harmonics, it is actually a property inherent in any potential function which goes to zero like 1/r.

5.3 Unity Weight Matrix

It is natural to inquire about the special case in which the weight matrix is set equal to the unit matrix in (12) and (14). It is easy to see from (12) that this case is equivalent to reformulating the matching criterion so that the expansion coefficients of v and $\tilde{v}$ are to be matched directly, in the ordinary least squares sense. This obviously can be extended to a non-unity diagonal weight matrix, which would emphasize the matching of particular coefficients.

There is an interesting special case in which the minimization of f is independent of the weight matrix which is selected. This occurs when the mass distribution generating the matching potential fortuitously is chosen so that the global solution f = 0 becomes available. This solution implies that v equals $\tilde{v}$ for all finite R, which is equivalent to having identical expansion coefficients for v and $\tilde{v}$ for all weight matrices. This, of course, is the solution one usually would hope to approach by iterating the candidate mass distribution, since it is a consequence of hitting upon the unknown distribution which generated v.

6.0 CONCLUSIONS

This paper has presented a method for calculating the optimum mass values for an arbitrary set of bodies whose net gravitational potential is required to approximate a given potential. As presented, the method is an elaboration of an integral criterion suggested by McLaughlin [1]. It is intimately related to the Ritz method in the calculus of variations. The elaboration consists of generalizing McLaughlin's conception and then obtaining a classical least squares matrix reformulation of it.

There are two principal advantages to the new formulation. The first is that it isolates the integral from the parameters of the assumed mass distribution, allowing the integral to be precomputed analytically, as demonstrated for the solid spherical harmonics. The second is that the matrix formulation permits the convenient utilization of a computer for determining mass distributions which arbitrarily closely reproduce a given potential, in the sense of minimizing McLaughlin's integral. In the limit, a zero minimum of the integral signals a perfect reproduction of the field.

Due to the fact that for any given potential field there are many mass distributions which generate the field precisely [7], nothing can be said regarding the physical reality of a solution obtained with the integral criterion unless information in addition to the given field is provided. Treatment of this subject depends on the specific character of the additional information and is not handled herein.

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2014-SLL-cjz S. L. Levie, Jr.

Attachments
Appendices I and II

APPENDIX I

INVERTIBILITY OF J^TWJ

In the expression J^TWJ, J is an (N×n) matrix and W is an (N×N) weight matrix which depends on an integration region R. Theoretically, N may be infinite. The number of bodies in the assumed or "matching" distribution is n, and each column of J contains the expansion coefficients for the potential of one constituent body, the mass of which is set to unity. It will be assumed that the density functions of the constituent bodies are selected so that the columns of J are linearly independent, or that J has full rank n. Although the assumption is violated by two or more radially symmetric bodies centered at the origin, for example, it is satisfied by all natural distributions, such as a collection of distinct point masses. With this assumption in force, it will be shown that J^TWJ is invertible if R is favorably selected.

From matrix theory, J^TWJ is invertible if and only if it is positive definite, i.e., if and only if

$$x^TJ^TWJ\,x > 0 \qquad \text{(I-1)}$$

for all vectors x. Referring to its definition (13), W is symmetric, so there exists an (N×N) matrix C such that

$$W = C^TC. \qquad \text{(I-2)}$$

Thus (I-1) may be rewritten as

$$(CJx)^T\,(CJx) > 0. \qquad \text{(I-3)}$$

Thus J^TWJ fails to be invertible when $CJx = 0$, which will be the case when C does not have full rank, (since J has full rank by assumption), that is, when

$$\mathrm{rank}(C) < N. \qquad \text{(I-4)}$$

But, due to (I-2), this means that

$$\mathrm{rank}(W) < N \qquad \text{(I-5)}$$

is the equivalent condition for $J^T WJ$ not to be invertible. This is to say that the weight matrix W must have full rank for $J^T WJ$ to be invertible.

To interpret the significance of this, the defining equation (13) should be more closely examined. Because W is the integral over R of an outer product of a vector of functions of a complete set, it certainly has full rank before the integration. Thus, the full rank property of W depends only on the favorable selection of the integration region R. The invertibility of $J^T WJ$ therefore hinges on the favorable selection of R.

APPENDIX II

SOLUTION OF AN INTEGRAL

This appendix is concerned with the evaluation of the integral

$$I_a(\ell,i,n,j) \equiv \int_{-a}^{a} P_\ell^{\,i}(x)\; P_n^{\,j}(x)\; dx \qquad \text{(II-1)}$$

where the P's denote the unnormalized associated Legendre polynomial defined in (25), (26), and (27). This may be regarded as a generalization of the "orthogonality integral" connected with these functions, which is $I_1(\ell,i,n,i)$ in this notation.

If the definitions of $P_\ell^{\,i}(x)$ and $P_n^{\,j}(x)$ are substituted into (II-1), there results

$$I_a(\ell,i,n,j) = \sum_{t=0}^{k_t}\sum_{\nu=0}^{k_\nu} T_{\ell i t}T_{nj\nu}\int_{-a}^{a}(1-x^2)^{\frac{i+j}{2}}\; x^{-2(\nu+t)+(\ell+n)-(i+j)}\, dx, \qquad \text{(II-2)}$$

where

and

$$\left.\begin{aligned} k_t &= \left[\frac{\ell-i}{2}\right] \\ k_\nu &= \left[\frac{n-j}{2}\right] \end{aligned}\right\}. \qquad \text{(II-3)}$$

Since the first factor in the integral is always an even function of x, the integral will be zero whenever the second factor is odd. Hence the immediate result

$$I_a(\ell,i,n,j) = 0 \qquad [\text{if } (\ell+n)-(i+j) = \text{odd}]. \qquad \text{(II-4)}$$

Conversely, the integral will be non-zero when the second factor is even. There are two cases for which this may happen, and for each the integral in (II-2) can be performed in a straightforward way. The results are

$$I_a(\ell,i,n,j) = 2\sum_{t=0}^{k_t}\sum_{v=0}^{k_v}\sum_{q=0}^{\frac{i+j}{2}} (-)^q \, T_{\ell i t}\, T_{n j v} \binom{\frac{i+j}{2}}{q} \frac{a^{1+2p}}{1+2p}\,, \qquad \text{(II-5)}$$

[if $(i+j)$ = even and $(\ell+n)$ = even],

and

$$I_a(\ell,i,n,j) = 2\sum_{t=0}^{k_t}\sum_{v=0}^{k_v}\sum_{q=0}^{\frac{i+j-1}{2}} (-)^q \, T_{\ell i t}\, T_{n j v} \binom{\frac{j+\ell-1}{2}}{q}$$

$$\times \left\{ \int_0^{\arcsin(a)} (\sin\theta)^{2p}\, d\theta - \int_0^{\arcsin(a)} (\sin\theta)^{2p+2}\, d\theta \right\} \qquad \text{(II-6)}$$

[if $(i+j)$ = odd and $(\ell+n)$ = odd],

where

$$2p \equiv 2q - 2(v+t) + (\ell+n) - (i+j). \qquad \text{(II-7)}$$

The integrals remaining in (II-6) are given in [8] for integer p as

$$\int_0^{\psi} \sin^{2p}\theta \, d\theta = \frac{1}{2^{2p}} \binom{2p}{p} \psi$$

$$+ \frac{(-)^p}{2^{2p-1}} \sum_{r=0}^{p-1} (-)^r \binom{2p}{r} \frac{\sin(2p-2r)\psi}{(2p-2r)} \quad . \tag{II-8}$$

Due to its importance, the special case a=1 will be displayed. The case is trivial for (II-4) and (II-5), but not for (II-6), which reduces to

$$I_1(\ell,i,n,j) = \pi \sum_{t=0}^{k_t} \sum_{v=0}^{k_v} \sum_{q=0}^{\frac{i+j-1}{2}} (-)^q \, T_{\ell i t} \, T_{n j v} \binom{\frac{j+\ell-1}{2}}{q} \tag{II-9}$$

$$\times \frac{1}{(2p+1) \; 2^{2p+2}} \binom{2p+2}{p+1}$$

[if $(i+j)$ = odd and $(\ell+n)$ = odd].

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