NASA CONTRACTOR REPORT

NASA CR-1728

DIFFRACTION OF A PULSE BY A THREE-DIMENSIONAL CORNER

by Lu Ting and Fanny Kung

Prepared by
NEW YORK UNIVERSITY
New York, N. Y. 10453
for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1971

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-1728		
4. Title and Subtitle DIFFRACTION OF A PULSE BY A THREE-DIMENSIONAL CORNER		5. Report Date March 1971
		6. Performing Organization Code
7. Author(s) Lu Ting and Fanny Kung		8. Performing Organization Report No. NYU Report AA-70-06
9. Performing Organization Name and Address New York University University Heights New York, N.Y. 10453		10. Work Unit No.
		11. Contract or Grant No. NGL-33-016-119
12. Sponsoring Agency Name and Address National Aeronautics & Space Administration Washington, D. C. 20546		13. Type of Report and Period Covered CONTRACTOR REPORT
		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

For the diffraction of a pulse by a three-dimensional corner, the solution is conical in three variables $\zeta = r/(Ct)$, θ and φ. The three-dimensional effect is confined inside the unit sphere $\zeta = 1$, with the vertex of the corner as the center. The boundary data on the unit sphere is provided by the appropriate solutions for the diffraction of a pulse by a two-dimensional wedge. The solution exterior to the corner and inside the unit sphere is constructed by the separation of the variable ζ from θ and φ. The associated eigenvalue problem is subjected to the differential equation in potential theory with an irregular boundary in $\theta - \eta$ plane. A systematic procedure is presented such that the eigenvalue problem is reduced to that of a system of linear algebraic equations. Numerical results for the eigenvalues and functions are obtained and applied to construct the conical solution for the diffraction of a plane pulse. For the diffraction of a general incident wave by corners or edges, two theorems are presented so that the value at the vertex of the corner or along the edges can be determined without the construction of the three-dimensional non-conical diffraction solutions. Relevant numerical programs for the analysis are presented in the appendix.

17. Key Words (Selected by Author(s))	18. Distribution Statement
Sonic Boom Diffraction Plane pulse	Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	91	$3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

TABLE OF CONTENTS

LIST OF FIGURES

DIFFRACTION OF A PULSE BY A THREE-DIMENSIONAL CORNER

by Lu Ting and Fanny Kung
New York University, New York, N.Y.

SUMMARY

For the diffraction of a pulse by a three-dimensional corner, e.g., the corner of a cube, the solution is conical in three variables $\zeta = r/(Ct)$, θ and φ. The three-dimensional effect is confined inside the unit sphere $\zeta = 1$, or the sonic sphere $r = Ct$ with the vertex of the corner as the center. The boundary data on the unit sphere is provided by the appropriate solutions for the diffraction of a pulse by a two-dimensional wedge. The solution exterior to the corner and inside the unit sphere is constructed by the separation of the variable ζ from θ and φ. The associated eigenvalue problem is subjected to the same differential equation in potential theory for the spherical angle variables θ and φ, but with an irregular boundary in θ - φ plane. A systematic procedure is presented such that the eigenvalue problem is reduced to that of a system of linear algebraic equations. Numerical results for the eigenvalues and functions are obtained and are applied to construct the conical solution for the diffraction of a plane pulse. For the diffraction of a general incident wave by corners or edges, the solutions are no longer conical. Two theorems are presented so that the value at the vertex of the corner or along the edges can be determined without the construction of the three-dimensional non-conical diffraction solutions. Relevant numerical programs for the analysis are presented in the appendix.

1. Introduction

The problem of diffraction and reflection of acoustic waves or electromagnetic waves by wedges, corners and other two-dimensional or axially symmetric obstacles has received extensive investigations. A survey of these investigations can be found in Reference 1.

The present investigation is motivated by the study of the effect of sonic boom on structures. The pressure wave created by a supersonic airplane is three-dimensional in nature. However, the radius of curvature of the wave front is usually much larger than the length scale of a structure. Therefore, the incident waves can be approximated by progressing plane waves with the wave form usually in the shape of the letter N and are referred to as N-waves.

For two-dimensional structures in the shape of a rectangular block, the diffraction of a plane pulse by the first corner is given explicitly by the two-dimensional conical solution of Keller and Blank [2]. The solution for each subsequent diffraction by the next corners can be obtained by the use of Green's function for a wedge [1, pp. 108-115]. For right angled corners, the diffraction solutions can be obtained by the solution of an Abel type integral equation [3]. By means of the integral of Duhamel, the solution for the diffraction of a plane pulse by a two-dimensional structure was employed to construct the solution for the diffraction of an N-wave by the same structure [4].

For the three-dimensional problem of the diffraction of sonic boom by structures, the first step is the construction of the solution for the diffraction of a plane pulse by a corner of a structure. The formulation of the differential equation and the appropriate boundary conditions for the problem in three-dimensional conical variables is presented in the next section.

2. Formulation

For the acoustic disturbance pressure p, the governing differential equation is the simple wave equation,

$$p_{xx} + p_{yy} + p_{zz} - C^{-2} p_{tt} = 0 \qquad (1)$$

in the region outside a trihedron simulating the corner of a structure. As shown in Fig. 1, two edges, OA and OB, of the trihedron are in the x-y plane and are bisected by the negative x-axis with half angle α and the third edge OD, which is the negative z-axis with the vertex O as the origin. Let t=0 be the instant when the pulse front hits the vertex. The boundary condition on the three faces of the trihedron is

$$\partial p/\partial n = 0 \qquad (2)$$

The three-dimensional disturbance due to the vertex is confined inside the sonic sphere $r = Ct$ where $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$. Outside the sonic sphere, the pressure distribution is given either by the regular reflection of the plane pulse from the surfaces of the trihedron or by the diffraction of the pulse by the edges. When the pulse front is parallel to the edge, the diffracted wave is given by a two-dimensional unsteady conical solution. When the pulse front is not parallel to the edge, the diffracted wave is given by a steady three-dimensional conical solution. For both cases, solutions are given in two conical variables in [2] by means of Busemann's conical flow method.

Due to the absence of a time scale and a length scale, the disturbance pressure p nondimensionalized by the strength of the incident pulse should be a function of the three-dimensional conical variables, x/(Ct), y/(Ct) and z/(Ct) or in terms of the spherical coordinates by $\zeta = r/(Ct)$, θ and φ. The simple wave equation for $p(\zeta,\theta,\varphi)$ becomes,

$$\zeta^2 (1-\zeta^2) \frac{\partial^2 p}{\partial \zeta^2} + 2\zeta (1-\zeta^2) \frac{\partial p}{\partial \zeta} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial p}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 p}{\partial \varphi^2} = 0 \qquad (3)$$

inside the unit sphere, $\zeta = 1$, and exterior to the trihedron. The boundary conditions are:

$$\partial p/\partial\theta = 0 \qquad \text{on surface OAB, } \theta = \pi/2, \ -(\pi-\alpha) < \varphi < \pi-\alpha \qquad (4)$$

$$\partial p/\partial\varphi = 0 \qquad \text{on surface OAD, } \varphi = \pi-\alpha, \ \pi-\alpha, \ \pi/2 < \theta < \pi \qquad (5a)$$

$$\partial p/\partial\varphi = 0 \qquad \text{on surface OBD, } \varphi = -\pi+\alpha, \ \pi/2 < \theta < \pi \qquad (5b)$$

and $p = F(\theta,\varphi)$ on unit sphere $\zeta = 1$ outside of the trihedron.

The jump acorss the sonic sphere is inversely proportional to the square root of the area of the ray tube, $(dS)^{\frac{1}{2}}$. Since all the rays reaching the sphere come from the origin where $dS_o = 0$, the jump acorss the sonic sphere is zero. The pressure is continuous across the sphere $\zeta = 1$; and $F(\theta,\varphi)$ is defined by the solutions outside the sonic sphere in two conical variables given by [2].

To construct the solution by the method of separation of variables, the usual trial substitution $p(\zeta,\theta,\varphi) = Z(\zeta)\, G(\mu,\varphi)$ is introduced where $\mu = \cos\theta$ and eq. (3) becomes

$$\zeta^2 (1-\zeta^2)\, Z''(\zeta) + 2(1-\zeta^2)\, Z'(\zeta) - \lambda(\lambda+1)\, Z(\zeta) = 0 \tag{6}$$

for $1 > \zeta \geq 0$, and

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial G}{\partial\mu}\right] + \lambda(\lambda+1)\, G + \frac{1}{1-\mu^2}\frac{\partial^2 G}{\partial\varphi^2} = 0 \tag{7}$$

for the domain in $\mu - \varphi$ plane with $|\varphi| < \pi-\alpha$ when $0 \geq \mu \geq -1$, and with $|\varphi| \leq \pi$ when $1 \geq \mu > 0$. Since the pressure p and also G are single valued, G should be periodic in φ when $1 \geq \mu > 0$, i.e.,

$$G(\mu,\varphi+2\pi) = G(\mu,\varphi) \tag{8}$$

The range of φ is therefore restricted from $-\pi$ to π and the two ends are connected by the periodicity condition.

The replacement of the variable θ by μ and the use of the constant of separation $\lambda(\lambda+1)$ are motivated by the intention of representing $G(\mu,\theta)$ by the spherical harmonics.

The boundary conditions on the surfaces of the trihedron, eq. (4), become

$$\partial G/\partial\mu = 0 \qquad \text{along } \mu=0 \text{ with } \pi-\alpha < |\varphi| \leq \pi \tag{9a}$$

$$\partial G/\partial\varphi = 0 \qquad \text{along } \varphi=\pm(\pi-\alpha) \text{ with } -1 < \mu < 0 \tag{9b}$$

The condition that p is bounded in particular at the two poles, $\theta=0$ and $\theta=\pi$, yields the condition,

$$|G| < \infty \text{ at } \mu = \pm 1 \tag{9c}$$

The periodicity condition, eq. (8), supplies the condition along the remaining boundaries,

$$G(\mu,-\pi) = G(\mu,\pi) \text{ and}$$

$$G_\varphi(\mu,-\pi) = G_\varphi(\mu,\pi) \text{ for } 1 \geq \mu > 0 \tag{9d}$$

The differential equation, (7), and the boundary conditions, (9a-d), define an eigenvalue problem. The determination of the eigenvalues, λ's, and the associated eigenfunctions, $G_\lambda(\mu,\varphi)$, are described in the next section.

3. The Eigenvalue Problem

For the eigenvalue problem formulated in the preceding section, two edges of the trihedron, OA and OB, are assumed to be normal to the third edge. This restriction is imposed due to two consideration: 1) the surfaces of the corner can be defined by surfaces of constant θ or constant φ's and 2) the solutions outside the sonic sphere for the diffraction problem can be constructed as solutions of two-dimensional problems [2]. For the solution of the eigenvalue problem itself the second consideration is irrelevant. The eigenvalue problem in this section will, therefore, be formulated for a wider class of corners as shown in Fig. 2. The surface OAB is a conical surface with $\theta = \beta$ and the two surfaces OAD and OBD are planes with $\varphi = \pi-\alpha$ and $\varphi = -\pi + \alpha$ respectively. The boundary conditions for the solution of eq. (7) are

$$\partial G/\partial\mu = 0 \quad \text{along } \mu=\mu_o=\cos\beta \text{ with } \pi-\alpha < |\varphi| \leq \pi \tag{10a}$$

$$\partial G/\partial\varphi = 0 \quad \text{along } \varphi = \pm(\pi-\alpha) \text{ with } -1 \leq \mu < \mu_o \tag{10b}$$

$$|G| < \infty \text{ at } \mu = \pm 1 \tag{10c}$$

$$G(\mu,\pi) = G(\mu,-\pi),\ G_\varphi(\mu,\pi) = G_\varphi(\mu,-\pi) \text{ for } \mu_o < \mu \leq 1 \tag{10d}$$

For the special case of $\beta = \pi/2$, or $\mu_o = 0$, the boundary conditions of eqs. (10a - d) and the domain in the μ - φ plane (Fig. 2) reduce to those of eqs. (9a - d).

The eigenvalue problem is now defined by the differential eq. (7) and the boundary conditions of eqs. (10a - d). In order to reduce the problem to that for a set of algebraic equations, two representations of the eigenfunction $G_\lambda(\mu,\varphi)$ associated with the eigenvalue λ will be sought: one for the region, R^+, with $\mu > \mu_o$ and the other for R^- with $\mu < \mu_o$ (Fig. 2). These two solutions and their normal derivatives will be matched across the dividing line $\mu = \mu_o$ for $|\varphi| < \pi - \alpha$.

For the upper region R^+, the eigenfunction $G_\lambda^+(\mu,\varphi)$ which is periodic in φ on account of eq. (10d) or eq. (8) can be represented by the Fourier series in φ with period of 2π,

$$G_\lambda^+(\mu,\varphi) = \sum_{m=0,1,\ldots} A_m P_\lambda^{-m}(\mu) \cos m\varphi + \sum_{m=1,2,\ldots} B_m P_\lambda^{-m}(\mu) \sin m\varphi \tag{11}$$

For each m, eq. (7) yields the Legendre equation for $P_\lambda^{-m}(\mu)$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}P_\lambda^{-m}(\mu)\right] + \left[\lambda(\lambda+1) - \frac{m^2}{1-\mu^2}\right]P_\lambda^{-m}(\mu) = 0 \tag{12}$$

Since eq. (11) is defined for $1 \geq \mu \geq \mu_o > -1$, P_λ^{-m} should be finite at

$\mu = 1$. P_λ^{-m} is identified as the generalized Legendre function [5] and defined by

$$P_\lambda^{-m}(\mu) = \left(\frac{1-\mu}{1+\mu}\right)^{m/2} F\left\{-\lambda, \lambda+1, 1+m; (1-\mu)/2\right\} \tag{13}$$

where F denotes the Gaussian hypergeometric function. The factor $1/\Gamma(1+m)$ is omitted on the right side of eq. (13) since it is automatically abosrbed into coefficients A_m and B_m with a net saving of programming and computing time.

For the lower region R^-, i.e., $-1 \le \mu < \mu_o$ the eigenfunction G_λ^-, which fulfills the boundary conditions of eqs. (10b and c), can be expressed as a cosine series in $(\varphi - \pi + \alpha)$ with half period of $2(\pi - \alpha)$ i.e.,

$$G_\lambda^-(\mu,\varphi) = \sum_{n=0,1,2..} E_n P_\lambda^{-(n\pi/\Phi)}(-\mu) \cos\left[\frac{n\pi}{\Phi}\left(\varphi + \frac{\Phi}{2}\right)\right] \tag{14}$$

where $\Phi = 2(\pi - \alpha)$. Similarly for each n, eq. (7) yields the Legendre equation for the generalized Legendre function $P_\lambda^{-n\pi/\Phi}(-\mu)$. It is finite for $-1 \le \mu \le \mu_o < 1$.

Both expressions (12) and (14) fulfill the differential eq. (7) and the boundary conditions (10b-d) appropriate for regions R^+ and R^- respectively.

Across the dividing line of R^+ and R^-, where $\mu = \mu_o$ and $|\varphi| < \pi - \alpha$, the matching conditions are the continuity of the eigenfunction G_λ and its normal derivative $\partial G_\lambda/\partial\mu$. The continuity of the second derivatives are then assured since both G_λ^+ and G_λ^- fulfill the same differential eq. (7).

The matching conditions and the remaining boundary condition, eq. (10a), can be written as

$$G_\lambda^+(\mu_o^+,\varphi) = G_\lambda^-(\mu_o^-,\varphi) \qquad \text{for } |\varphi| \le \pi - \alpha \tag{15}$$

$$\partial G^+(\mu_o^+,\varphi)/\partial\mu = \begin{cases} \partial G_\lambda^-(\mu_o^-,\varphi)/\partial\mu & \text{for } |\varphi| < \pi-\alpha \\ 0 & \text{for } \pi-\alpha < |\varphi| \le \pi \end{cases} \tag{16}$$

Equations (15) and (16) can be reduced to a system of linear homogeneous algebraic equations for the unknown constants A_m, B_m and E_n by Fourier analysis or by fulfilling eqs. (15,16) at a number of φ's in the appropriate intervals. Before carrying out the reduction, it should be noted that the differential eq. (7) and the boundary conditions (10a-d) are symmetric with the plane $\varphi = 0$. The eigenfunction can, therefore, be expressed either as even functions or odd functions of φ. Indeed, in the expression (11) for

G_λ^+ the cosine terms represent the even solution and the sine terms represent the odd solution. In the expression (14) for G_λ^-, the terms with even and odd values of n are the even and odd solutions of φ respectively. They can be rewritten as

$$G_\lambda^-(\mu,\varphi) = \sum_{j=0,1...} C_j P_\lambda^{-\nu_j}(-\mu)\cos\nu_j\varphi + \sum_{j=1,2...} D_j P_\lambda^{-\bar{\nu}_j}(-\mu)\sin\bar{\nu}_j\varphi \quad (17)$$

where

$$\nu_j = 2j\pi/\Phi,\ \bar{\nu}_j = (2j-1)\pi/\Phi,\ C_j = E_{2j}(-1)^j \text{ and } D_j = E_{2j-1}(-1)^j .$$

It is now obvious that the cosine terms in eq. (17) are even and the sine terms are odd in φ. Since the odd solution and the even solution are orthogonal to each other, the conditions of eqs. (15) and (16) for G_λ^+ and G_λ^- should be fulfilled by their even terms and also by their odd terms respectively. The eigenvalues and the associated even eigenfunction can be determined separately from the eigenvalues and the associated odd eigenfunction. By this uncoupling, the rank of the characteristic determinant is reduced by a half, and a double root of the characteristic equation of the mixed solutions may now be split to one single root for the even and one for the odd solutions respectively. In actual numerical computations, it is much easier to locate the eigenvalue and construct the eigenfunction for a single root than for a double root of the characteristic equation.

The solution of the eigenvalue problem defined by eqs. (15) and (16) can be carried out by collocation method, i.e., to truncate the series of eqs. (11) and (17) and to impose eqs. (15) and (16) at a finite number of value of φ's in the appropriate intervals. In this method, it is necessary to uncouple the solutions first and then impose eqs. (15) and (16) to the even and to the odd solutions separately.

When the solution of the eigenvalue problem is to be carried out by Fourier analysis, the uncoupling of the even and odd solutions will come about automatically as follows:

Equation (16) holds for the interval $-\pi < \varphi < \pi$, it will be multiplied by $\cos n\varphi$ and will be integrated over the interval. Equation (16) then becomes the algebraic equations for the coefficients of the even terms,

$$A_n \dot{P}_\lambda^{-n}(\mu_o)\left[1+\delta_{on}\right] + \sum_{j=0,1,2} C_j \dot{P}_\lambda^{-\nu_j}(-\mu_o)\, I_{jn} = 0 \qquad \text{for } n=0,1... \quad (18)$$

where

$$I_{jn} = \frac{1}{\pi} \int_{-\Phi/2}^{\Phi/2} \cos \nu_j\varphi \cos n\varphi \, d\varphi$$

$$= \begin{cases} \Phi[1+\delta_{on}]/(2\pi) & , \nu_j = n \\ -2(-1)^j n \sin(n\Phi/2)/[\pi(\nu_j^2 - n^2)] & , \nu_j \neq n \end{cases} \quad (19)$$

and ($\dot{}$) means the first derivative with respect to its argument.

Similarly eq. (15), which holds for the interval $|\varphi| \leq \pi - \alpha/2 = \Phi/2$, will be multiplied by $\cos[2k\pi(\varphi+\Phi/2)/\Phi]$ or $\cos \nu_k\varphi$ with $\nu_k = 2k\pi/\Phi$ and will be integrated over the interval $-\Phi/2 < \varphi < \Phi/2$. The result is

$$\sum_{m=0,1,2...} A_m P_\lambda^{-m}(\mu_o) \; I_{km} - C_k P_\lambda^{-\nu_k}(-\mu_o) \; \Phi[1+\delta_{ok}]/(2\pi) = 0$$

$$\text{for } k=0,1,2.... \quad (20)$$

Since $\cos n\varphi$ and $\cos \nu_k\varphi$ are even functions of φ, eqs. (18) and (20) contain only the coefficients A_m and C_j associated with the even parts in G_λ^+ and G_λ^- respectively. Similarly the algebraic equations for the coefficients B_m and D_j associated with the odd parts are obtained from eqs. (15) and (16). Equation (16) will be multiplied by $\sin n\varphi$ and integrated over the interval $-\pi < \varphi < \pi$ and eq. (15) will be multiplied by $\sin \bar{\nu}_k\varphi$ and integrated over the interval $-\Phi/2 < \varphi < \Phi/2$ with $\bar{\nu}_k = (2k-1)\pi/\Phi$. The results are,

$$B_n \dot{P}_\lambda^{-n}(\mu_o) + \sum_{j=1,2,...} D_j \dot{P}_\lambda^{-\bar{\nu}_j}(-\mu_o) L_{jn} = 0$$

$$\text{for } n = 1,2,..... \quad (21)$$

and

$$\sum_{m=1,2...} B_m P_\lambda^{-m}(\mu_o) \; L_{km} - D_k P_\lambda^{-\bar{\nu}_j} \; (-\mu_o) \; \Phi/(2\pi) = 0$$

$$\text{for } k = 1,2,..... \quad (22)$$

where

$$L_{jn} = \frac{1}{\pi} \int_{-\Phi/2}^{\Phi/2} \sin \bar{\nu}_j\varphi \sin n\varphi \, d\varphi$$

$$= \begin{cases} \Phi/(2\pi) & , \nu_j = n \\ -2n \; (-1)^j \cos \frac{n\Phi}{2} \; / \; [\pi(\bar{\nu}_j^2 - n^2)] & , \bar{\nu}_j \neq n \end{cases} \quad (23)$$

Equations (18) and (20) define the eigenvalue problem for the even solution, while eqs. (21) and (22) for the odd solution. The numerical method for the solution of the problem will be described in detail in the next section.

4. Numerical Solutions of the Eigenvalue Problem

To obtain the numerical solutions, the infinite cosine series in eq. (11), the even part of G_λ^+, will be truncated with the maximum of m equal to M_c. Similarly the cosine series in eq. (17), the even part of G_λ^-, will be truncated with the maximum of j equal to J_c. Likewise, M_c and J_c will be the maxima of n and k, respectively, so that eqs. (18) and (20) yield $M_c + J_c + 2$ linear homogeneous equations for the $M_c + J_c + 2$ constants A_o, $A_1, \ldots A_{M_c}$ and C_o, $C_1 \ldots C_{J_c}$. These solutions are nontrivial if λ is a root of the characteristic equation,

$$\Delta_c(\lambda) = | a_{hi} | = 0 \tag{24}$$

where $a_{hi} = \dot{P}_\lambda^{-i+1}(\mu_o)\,[1+\delta_{1i}]\,\delta_{hi}$ for h, $i=1,2\ldots M_c+1$,

$a_{hi} = \dot{P}_\lambda^{-\nu_j}(-\mu_o)\, I_{ji}$ for $h=1,2\ldots M_c+1$, $i=M_c+2,\ldots\ldots M_c+J_c+2$, and $j=i-M_c-2$

$a_{hi} = P_\lambda^{-i+1}(\mu_o)\, I_{ki}$ for $h=M_c+2,\ldots,M_c+J_c+2$, $i=1,2,\ldots,M_c+1$, and $k=h-M_c-2$

and $a_{hi} = -\, P_\lambda^{-\nu_k}(-\mu_o)\,\Phi\,[1+\delta_{ok}]\,\delta_{hi}/(2\pi)$

for $h,i = M_c+2,\ldots,M_c+J_c+2$, with $k=i-M_c-2$.

For given values of M_c and J_c, the roots of $\Delta_c(\lambda^*) = 0$ in a given interval of λ can be located by numerical evaluation of $\Delta_c(\lambda)$ as a function of λ. When the derivative of $\Delta_c(\lambda)$ at a root is non-zero, the coefficients A_m, C_j are proportional to the cofactors of the determinant, i.e.,

$$A_m = N^{-1}\, A_{1,M+1} \qquad \text{for } M=0,1,2.\ldots,M_c$$
$$C_j = N^{-1}\, A_{1,M_c+1+j} \qquad \text{for } j=0,1,2\ldots,J_c \tag{25}$$

where A_{hi} is the cofactor of the element a_{hi} of the determinant with $\lambda = \lambda^*$. The constant N is defined by the normalization condition,

$$\int_{\mu_o}^{1} d\mu \int_{o}^{\pi} d\varphi \left[G^{+}_{\lambda*}(\mu,\theta)\right]^2 + \int_{-1}^{\mu_o} d\mu \int_{o}^{\Phi/2} d\varphi \left[G^{-}_{\lambda*}(\mu,\theta)\right]^2 = 1 \tag{26}$$

The equation for N is

$$N^2 = \frac{\pi}{2}\int_{\mu_o}^{1} d\mu \sum_{m=0}^{M_c} \left[P^{m}_{\lambda*}(\mu)\ A_{1,m+1}\right]^2 (1+\delta_{om})$$

$$+ \frac{\Phi}{4}\int_{-\mu_o}^{1} d\mu \sum_{j=0}^{J_c} \left[P^{-\nu j}_{\lambda*}\ (\mu)\ A_{1,M_c+1+j}\right]^2 (1+\delta_{oj}) \tag{27}$$

A numerical program is described in the appendix so that for a given pair of M_c and J_c, the determinant $\Delta_c(\lambda)$ can be evaluated as a function of λ. The search for the roots can be restricted to $\lambda > -\frac{1}{2}$ because of the symmetry of the function $P^{-\nu}_{\lambda}(\mu)$ about $\lambda = -\frac{1}{2}$, i.e., $P^{-\nu}_{\lambda}(\mu) = P^{-\nu}_{-\lambda-1}(\mu)$. For the given range of λ, say $-\frac{1}{2} \le \lambda \le 3$, the eigenvalues for the even solutions are located and the first derivative of $\Delta_c(\lambda)$ at each root $\lambda*$ are computed. In the examples considered in this paper, the derivative does not vanish at $\lambda*$, therefore, the numerical program in the appendix carries out the determinations of the coefficients A_m and C_j in the eigenfunctions by means of eqs. (25) and (27).

Similarly for the odd solutions, the series in eqs. (11) and (17) will be truncated with the maxima of m and j equal to M_s and J_s respectively. There will be M_s equations for eq. (21) and M_j equations for eq. (22). The roots of the characteristic equation yield the eigenvalues for the odd solutions. The ratios of the cofactors and the normalization condition of eq. (26) define the coefficients $B_1 .... B_{M_s}$ and $D_1 .... D_{J_s}$ in the eigenfunctions.

It should be pointed out here that the convergence of the eigenvalues and eigenfunctions of the truncated problem has been assumed and the equivalence between the matching conditions of eqs. (15) and (16) and matching of the Fourier coefficients is also assumed. Some confidence in these assumptions will be offered by the following numerical results.

For the special case of a two-dimensional corner, namely $\alpha = \pi/2$, $\beta = \pi/2$ in Fig. 2, the eigenvalue problem can be solved exactly by choosing the edge of the corner as the $\bar{z}$ axis (Fig. 3). In spherical coordinates $\bar{\theta}$ and $\bar{\varphi}$, the two sides are $\bar{\varphi} = 0$ and $\bar{\varphi} = 3\pi/2$. The eigenfunction can be written as

$$\bar{G}_{\lambda}\ (\bar{\theta},\bar{\varphi}) = P^{-2j/3}_{\lambda}\ (\bar{\mu})\ \cos\left[(2j/3)\bar{\varphi}\right] \tag{28}$$

where $\bar{\mu} = \cos\bar{\theta}$ and $j = 0,1,2,\ldots\ldots\ldots$ The eigenvalue λ is defined by the condition that $P_\lambda^{-2j/3}$ should be finite at $\bar{\theta} = \pi$ or $\bar{\mu} = -1$. Instead of eq. (13), an equivalent expression for $P_\lambda^{-\nu}(\bar{\mu})$ is [5].

$$P_\lambda^{-\nu}(\bar{\mu}) = 2^{-\nu}(1-\bar{\mu}^2)^{\nu/2} F(\nu-\lambda,\ \nu+\lambda+1,\ \nu+1;\ X) \tag{29}$$

where $\nu=2j/3$ and $X=(1-\bar{\mu})/2$

The power series representation for the hypergeometric function is divergent at $X=1$ or $\bar{\mu} = -1$ unless the series has only finite number of terms, say $k + 1$ terms. The condition for λ is

$$\nu-\lambda+k = 0 \quad \text{or} \quad \lambda_{jk} = k+2j/3 \tag{30}$$

For $j = 0,1,2,3\ldots\ldots$, the eigenvalues* are $\lambda_{o,k} = 0,1,2,3\ldots\ldots$, $\lambda_{1,k} = 2/3,\ 5/3,\ 8/3\ldots\ldots$, $\lambda_{2,k} = 4/3,\ 7/3,\ldots\ldots$, $\lambda_{3,k} = 2,3,\ldots\ldots$, $\lambda_{4,k} = 8/3,\ldots$

The numerical program in the appendix developed for a three-dimensional corner will now be tested by setting $\beta = \pi/2$ and $\alpha = \pi/2$. The eigenvalues between -0.5 and 3.0 for the even and the odd solutions respectively are shown in Table I for various combinations of M_c, J_c, M_s and J_s. The numerical results for each combination yield the same number of eigenvalues in the same interval of λ as the exact solution. For the combination of $M_c=8$, $J_c=4$, $M_s=9$, $J_s=5$, the eigenvalues given by the numerical program are in the agreement with the corresponding exact values within 0.2%. This is a confirmation of the procedures developed in this paper and the numerical program for the eigenvalue problem.

A few words should now be said about how to pick the numbers M_c and J_c (also M_s and J_s), i.e., how many terms in the series representations of G^+ and G^- should be employed.

Since the eigenvalues of the truncated problem are assumed to converge as M and J increase simultaneously, their limits should be independent of the differences between M and J where M and J stand for M_c, J_c, or M_s, J_s. This fact is also confirmed by all of our numerical results. Nevertheless, it is relevant to point out the implications of the difference between M and J, i.e., the difference between the number of terms in the series solution for G^+ and that for G^-. The conditions of eqs. (15) and (16) call for the matching of G^+ and G^- and of their normal derivatives G_μ^+ and G_μ^-

---

* Note that $G(\theta,\mu)$ = constant is an eigensolution, therefore, $\lambda=0$ is an eigenvalue for any combinations of β and α.

across $\mu = \mu_o$ with $|\varphi| < \pi-\alpha$ and the vanishing of G^+_μ at $\mu = \mu_o$ with $\pi-\alpha < |\varphi| \le \pi$. Intuitively, the two matching conditions would require the same number of terms in both series while the latter condition on G^+_μ is handled by the extra M-J terms in G^+.

The qualitative relationship between M and J can be established clearly if eqs. (15) and (16) are fulfilled by the collocation method, i.e., eq. (15) and (16) will be enforced at the grid points uniformly distributed in the intervals for φ. The grid sizes in these intervals are $2(\pi-\alpha)/J$ and $(2\alpha/(M-J)$ respectively. The condition for these two grid sizes to be equal or almost the same is

$$\pi/M \simeq (\pi-\alpha)/J \tag{31}$$

It means that for given J, the integer M should be so chosen that the ratio M/J will be the nearest rational number to $\pi/(\pi-\alpha)$.

For the special corner considered in the diffraction problem, it is the corner of a cube with $\alpha = \pi/4$, $\beta = \pi/2$. Some of the eigenvalues can be found by inspection, namely, all the even integers. The first few exact eigensolutions are grouped as even and odd solutions of φ and listed in Table II. In the truncation of the series for G^+ and G^-, it is, desirable that these exact eigensolutions with $\lambda \le M$ should be reproduced. This implies that the factor of φ in the last term of the truncated series of G^+ should be equal to or be the nearest one to that of G^-. For the even solutions, the condition is

$$M_c \cong (4/3)\ J_c \tag{32a}$$

and for the odd solution it is

$$M_s \cong (2/3)\ (2J_s-1) \tag{32b}$$

Condition (32a) is identical to eq. (31) from collocation method while condition (32b) will be equivalent to (31) if the grid points in the collocation method are shifted by a half of the grid size.

Listed in Table I are the numerical results for the eigenvalues corresponding to various combinations of M_c, J_c and M_s, J_s. It is clear that for the same total number of terms M + J, the combination of M and J which fulfill eqs. (32a and b) are in better agreement with the exact solution than those with M equal to J.

With the relationship between M and J established, the next step is to choose the integer J for the truncation of the series. To give a measure of the accuracy of the eigenvalue $\lambda_{(J)}$ and the corresponding eigenfunctions $G^+_{(J)}$ and $G^-_{(J)}$, the following norms are introduced

$$I_1 = |\lambda_{(J_1)} - \lambda_{(J_2)}| \tag{33a}$$

$$I_2 = \int_0^{\Phi/2} \left\{G^+_{(J_1)} + G^-_{(J_1)} - \left[G^+_{(J_2)} + G^-_{(J_2)}\right]\right\}^2_{\mu=\mu_o} d\varphi \Bigg/ \int_0^{\Phi/2} \left\{G^+_{(J_1)} + G^-_{(J_1)}\right\}^2_{\mu=\mu_o} d\varphi \tag{33b}$$

$$I_3 = 4\int_0^{\Phi/2} \left[G^+ - G^-\right]^2_{\mu=\mu_o} d\varphi \Bigg/ \int_0^{\Phi/2} \left[G^+ + G^-\right]^2_{\mu=\mu_o} d\varphi \tag{33c}$$

$$I_4 = 4\left\{\int_0^{\Phi/2} \left[\frac{\partial G^+}{\partial\mu} - \frac{\partial G^-}{\partial\mu}\right]^2_{\mu=\mu_o} d\varphi + \int_{\Phi/2}^{\pi} \left[\frac{\partial G^+}{\partial\mu}\right]^2_{\mu=\mu_o} d\varphi\right\} \Bigg/ \int_0^{\Phi/2} \left[\frac{\partial G^+}{\partial\mu} + \frac{\partial G^-}{\partial\mu}\right]^2_{\mu=\mu_o} d\varphi \tag{33d}$$

I_1 and I_2 measure the convergence of the eigenvalues and the eigenfunctions as the number of the term increases. I_3 and I_4 measure the degree of accuracy of the approximations to the matching conditions and the boundary condition at $\mu = \mu_o$, i.e., eqs. (15) and (16).

It should be pointed out here that although the eigenfunctions are continuous, its derivatives may not exist along the edges. The singularity along the edge $\theta = \pi$ is built in by the representation for $G^-(\mu,\varphi)$. The singularity along the edges, $\theta = \beta$ and $\varphi = \pi-\alpha$ or $\varphi = -\pi+\alpha$, can be ascertained by investigating the behavior of the solutions of the differential equations for $G(\mu,\varphi)$, eq. (7). In the neighborhood of an edge, i.e., $|\tilde{\mu}| \ll 1$ and $|\tilde{\varphi}| \ll 1$ with $\tilde{\mu} = \mu_o-\mu$, $\tilde{\varphi}=\pi-\alpha-\varphi$ eq. (7) can be approximated by the Laplace equation

$$(1-\mu_o^2)^2 \, \partial^2 G/\partial\tilde{\mu}^2 + \partial^2 G/\partial\tilde{\varphi}^2 = 0$$

in the first three quadrants of the $\tilde{\mu}$ - $\tilde{\varphi}$ plane subjected to the boundary condition that the normal derivative of G should vanish along the positive $\tilde{\mu}$-axis and along the negative $\tilde{\varphi}$-axis. The potential solution G near the origin $\tilde{\mu} = \tilde{\varphi} = 0$ should behave as the real part of $[\tilde{\mu}(1-\mu_o^2)^{-1} + i\tilde{\varphi}]^{2/3}$ [6]. The same result can be obtained by observing the three-dimensional corner directly: the surfaces of $\varphi = \pi-\alpha$ and $\theta = \beta$ intersect at right angle and in the neighborhood of the edge away from the vertex, the solution behaves as that of a two-dimensional convex right corner.

Along the interface of G^+ and G^-, i.e., $\mu = \mu_o$ and $\tilde{\varphi} > 0$, the normal derivative of G behaves as the real part of $(\tilde{\varphi}i)^{-1/3}$ i.e.,

$$G_\mu(\mu = \mu_o, \ \varphi = \pi-\alpha-0^+) \sim (\pi-\alpha-\varphi)^{-1/3} .$$

G_μ is singular near the edge but its product with cosine or sine functions of φ is integrable and itself are square integrable. Therefore, eqs. (18) and (21) are meaningful and the norm I_4 exists.

Table III shows the eigenvalues between -1/2 and 3 for the corner of a cube, i.e., $2\alpha = \pi/2$ and $\beta = \pi/2$. The series representations are truncated with $M_c = 8$ and $J_c = 6$ for the even solutions and with $M_s = 10$ and $J_s = 8$ for the odd solutions. The coefficients in the series representations of the eigenfunctions are also listed. The norms, I_3 and I_4, are computed for each eigenvalue. It is found that all the I_3's are less than 1% and all the I_4's are less than 10%. The matching of the normal derivative of G^+ and G^- as measured by I_4 is less satisfactory as expected due to the singularity at the edge and due to the fact that the exact value of $\partial G/\partial\mu$, i.e. zero, has been used in the denominator for I_4 in the interval $\Phi/2 < \varphi < \pi$.

Calculations have been performed for various values of J_c and J_s. It is found that when $J_{1c} = 6$ and $J_{2c} = 9$ and also $J_{1s} = 8$ and $J_{2s} = 11$, all the I_1's and I_2's corresponding to the eigenvalues between -1/2 and 3 are less than .1%.

Table IV compares the eigenvalues for the corner of a cube with those for a circular cone with the same solid angle, i.e., with half angle, $\Theta = \arccos(3/4)$.

For the circular cone, the eigenfunctions are $P_\lambda^{-n}(\mu)\exp(in\varphi)$. The eigenvalues λ's are defined by the condition of $dP_\lambda^{-n}(\mu)/d\mu=0$ at $\mu=-\cos\Theta$. For each integer $n \geq 1$, the eigenvalue λ has a multiplicity of 2 and only for $n = 0$ the eigenvalue λ has a multiplicity of unity.

Table IV shows that the eigenvalues for the three-dimensional corner can be approximated by those of a circular cone with the same solid angle. It also shows that when an eigenvalue for the circular cone has a multiplicity of two, i.e., for $n\geq 1$, the corresponding values are found for the corner with even and odd eigensolutions and when the multiplicity is unity, i.e., $n=0$, the corresponding value for the corner has only an even eigensolution. It should be noted that when a corner degenerates to a plane, it is then identical with the equivalent circular cone of solid angle 2π.

The results in Table IV not only serves as a indirect confirmation of the numerical program but also suggests the use of an equivalent circular cone for a quick approximate determination of the eigenvalues of a three-dimensional corner.

5. Construction of the Conical Solution

In 2 the general formulation of the conical problem, the method of separation of variables was introduced. The disturbance pressure $p(\zeta,\theta,\varphi)$ can be written as

$$p(\zeta,\theta,\varphi) = \sum_{\lambda} K_{\lambda} Z_{\lambda}(\zeta) G_{\lambda}(\mu,\varphi) \tag{34}$$

The eigenvalues λ's and the eigenfunction G_{λ} are determined in 4. The function $Z_{\lambda}(\zeta)$ is a solution of the differential eq. (6), i.e.,

$$\zeta^2 (1-\zeta^2) Z''(\zeta) + 2(1-\zeta^2)\zeta Z'(\zeta) - \lambda(\lambda+1) Z(\zeta) = 0 \tag{35}$$

The boundary conditions are $Z(0) < \infty$ and $Z(1) = 1$. The condition on the unit sphere $\zeta=1$ permits the determination of the coefficients K_{λ} from the boundary data on the unit sphere, eq. (5), and the eigenfunction $G_{\lambda}(\mu,\varphi)$ independent of $Z_{\lambda}(\zeta)$. The equation for the coefficient K_{λ} is

$$K_{\lambda} = 1/2 \int_{0}^{\mu_o} d\mu \int_{-\pi}^{\pi} d\varphi \, F(\mu,\varphi) \, G_{\lambda}^{+}(\mu,\varphi)$$

$$+ 1/2 \int_{\mu_o}^{\pi} d\mu \int_{-\pi+\alpha}^{\pi-\alpha} d\varphi \, F(\mu,\varphi) \, G_{\lambda}^{-}(\mu,\varphi) \tag{36}$$

The factor 1/2 is due to the normalization condition of eq. (12b) for G_{λ} in which the integration with respect to φ is carried over only half the interval. In the numerical determination of K_{λ} the double integral can be reduced to a line integral by splitting the boundary data to even and odd solutions and then to a cosine and a sine series in φ respectively. For the even solution the results are:

$$K_{\lambda} = \int_{\mu_o}^{1} \sum_{m=0,1}^{M_c} A_m P_{\lambda}^{-m}(\mu) \, f_m^{+}(\mu) \, d\mu$$

$$+ \int_{-1}^{\mu_o} \sum_{j=0,1}^{J_c} C_j \, P_{\lambda}^{-\nu_j}(-\mu) \, f_j^{-}(\mu) \, d\mu \tag{37}$$

where

$$f_m^+(\mu) = \frac{1}{2}\int_0^{\pi} [F(\mu,\varphi) + F(\mu,-\varphi)] \cos m\varphi \, d\varphi ,$$

$$f_j^-(\mu) = \frac{1}{2}\int_0^{\Phi/2} [F(\mu,\varphi) + F(\mu,-\varphi)] \cos \nu_j\varphi \, d\varphi ,$$

and $\nu_j = 2j\pi/\Phi$.

Corresponding to the even boundary data, eq. (37) holds for all the eigenvalues of the even solutions and A_m and C_j are the coefficients defined by eq. (25) for each λ.

Similarly the coefficients K_λ for odd eigensolutions are obtained as follows:

$$K_\lambda = \int_{\mu_o}^{1} \sum_{m=1,2}^{M} {}^{s} B_m P_\lambda^{-m}(\mu) \, f_m^+(\mu) \, d\mu$$

$$+ \int_{-1}^{\mu_o} \sum_{j=1,2}^{J} {}^{s} D_j P_\lambda^{-\bar{\nu}_j}(-\mu) \, f_j^-(\mu) \, d\mu \qquad (38)$$

where

$$f_m^+ = \int_0^{\pi} \frac{1}{2} [F(\mu,\varphi) - F(\mu,-\varphi)] \sin m\varphi \, d\varphi$$

$$f_j^- = \int_0^{\Phi/2} \frac{1}{2} [F(\mu,\varphi) - F(\mu,-\varphi)] \sin \bar{\nu}_j\varphi \, d\varphi$$

and $\bar{\nu}_j = (2j-1)\pi/\Phi$.

For the determination of the pressure distribution inside the unit sphere, it is necessary to construct the solution $Z_\lambda(\zeta)$ of the differential eq. (35), subjected to the boundary conditions at $\zeta=0$ and $\zeta=1$. The solution is obtained numerically by the "shooting method". For Z_λ to be finite at $\zeta=0$ the solution can be represented by the power series,

$$Z(\zeta) = a_o\zeta^\lambda \left[1 + \frac{\lambda(\lambda+1)}{4\lambda+6}\zeta^2 + \ldots\right] \qquad (39)$$

By setting $a_o=1$, the differential equation can be integrated numerically

from a small ζ_o (say 0.001) with initial data $\tilde{Z}_\lambda(\zeta_o)=\zeta_o^\lambda\left[1+\lambda(\lambda+1)\zeta_o^2/(4\lambda+6)\right]$ and $\tilde{Z}'_\lambda(\zeta_o) = \lambda\zeta_o^{\lambda-1}\left[1+(\lambda+1)(\lambda+2)\zeta_o^2/(4\lambda+6)\right]$.
The numerical integration for $\tilde{Z}_\lambda(\zeta)$ is continued until ζ is close to 1 (say 0.99). The value of $\tilde{Z}_\lambda(1)$ is obtained by patching the numerical solution with the series solution of ζ-1, i.e.,

$$\tilde{Z}_\lambda(\zeta) = \tilde{Z}_\lambda(1)\left\{1 - \frac{\lambda(\lambda+1)}{4}(1-\zeta) + \frac{\lambda^2(\lambda+1)^2}{16}(1-\zeta)^2 + a_1(1-\zeta)^3\log(1-\zeta) + \dots\right\} \tag{40}$$

Since the problem is linear, the correct solution is

$$Z_\lambda(\zeta) = \tilde{Z}_\lambda(\zeta)/\tilde{Z}_\lambda(1)$$

The appropriate value for a_o or $\lim_{\zeta\to 0} Z\zeta^{-\lambda}$ is $1/\tilde{Z}_\lambda(1)$. The numerical program in the appendix computes a_o automatically and generates the function $Z_\lambda(\zeta)$. Thus for a given boundary data $F(\mu,\varphi)$ on the unit sphere, the conical solution of eq. (3) is defined.

6. A Numerical Example

Figure 4 shows the incidence of a plane pulse normal to a face say OAD of the corner of a cube. In order to demonstrate the use of both the even and the odd eigensolutions, one of the two edges parallel to the plane pulse is chosen as the negative z-axis as shown in the figure. After the incidence, $t > 0$, part of the plane pulse is reflected by the face OAD of the cube with double the intensity. The remaining part advancing over the corner is undisturbed. The diffraction by either one of the edges OA and OD, is given by a two dimensional conical solution. It is confined in a circular cylindrical surface with radius Ct and the edge as the axis provided it is outside the domain of influence of the vertex.

Figure 5 shows the incident plane pulse, the reflected pulse and the sonic circle in a cross-section normal to the edge OA at a distance greater than Ct from the vertex. In terms of polar coordinates ρ and τ, the boundary condition on the sonic circle $\rho = Ct$ for the pressure variation from that behind the incident pulse is

$$p_c = 1 \qquad \pi < \tau < 3\pi/2$$

$$p_c = 0 \qquad 0 < \tau < \pi$$

The strength of the incident pulse is chosen as unity. For the two-dimensional conical solution, the Busemann conical transformation is introduced [2],

$$\rho^* = \bar{\rho} \Big/ \left\{1+[1-\bar{\rho}^2]^{\frac{1}{2}}\right\}, \quad \bar{\rho} = \rho/(Ct) \text{ and } \tau^* = \tau \tag{41}$$

In the new variables, the disturbance pressure becomes a potential solution and the region inside the circle $\rho^* < 1$ with $0 < \tau^* < 3\pi/2$ is mapped to a half circle with $\tilde{\rho}\exp(i\tilde{\tau}) = [\rho^*\exp(i\tau^*)]^{2/3}$. The solution in terms of $\tilde{\rho},\tilde{\tau}$ is [2]

$$p_c(\rho,\tau) = \frac{1}{\pi}\arctan\frac{(1-\tilde{\rho}^2)\sqrt{3/2}}{(1+\tilde{\rho}^2)/2+2\tilde{\rho}\cos\tilde{\tau}} \tag{42}$$

where the arctangent lies in the first and the second quadrants. The circular cylinder with axis OA and radius Ct intersects the sonic sphere about the vertex along a greater circle and covers the spherical surface on the right side of plane OBD. Similarly, the diffraction by the other edge, OD, created another two-dimensional conical solution given by eq. (42) with ρ the distance from the edge OD and τ the angle measured from face OBD. Again, the circular cylinder with axis OD and radius Ct intersects the sonic sphere about the vertex along another great circle and covers the part of the spherical surface below plane OAB, i.e., $\mu < 0$.

The surface of the sonic sphere can be divided into four regions:

1) not covered by either one of these circular cylinders

2) covered only by the cylinder with axis OA

3) covered only by the cylinder with axis OD and

4) covered by both cylinders.

With the plane $\varphi=0$ defined as the plane bisecting the exterior angle of the two vertical surface OAD and OBD, the four regions and the boundary data are defined as

i) Region I $\quad 1 > \mu > 0;\ -3\pi/4 < \varphi < \pi/4$

$$F(\mu,\varphi) = 0 \tag{43a}$$

ii) Region II $\quad 1 > \mu > 0;\ \pi/4 < \varphi \le \pi$ and $-\pi \le \varphi < -3\pi/4$ (or $\pi \le \varphi < 5\pi/4$)

$$F(\mu,\varphi) = p_c(\bar{\rho}_1, \tau_1) \tag{43b}$$

iii) Region III $\quad 0 > \mu > -1;\ -3\pi/4 < \varphi < \pi/4$

$$F(\mu,\varphi) = p_c(\bar{\rho}_2, \tau_2) \tag{43c}$$

iv) Region IV $\quad 0 > \mu > -1;\ \pi/4 < \varphi < 3\pi/4$

$$F(\mu,\varphi) = p_c(\bar{\rho}_1, \tau_1) + p_c(\bar{\rho}_2, \tau_2) - 1 \tag{43d}$$

where

$$\bar{\rho}_1 = [\mu^2 + (1-\mu^2)\sin^2(\varphi - 3\pi/4)]^{\frac{1}{2}}$$

$$\tau_1 = \begin{cases} \sin^{-1}|\mu/\bar{\rho}_1| & \text{for } \sin(\varphi-3\pi/4) \ge 0,\ \mu \ge 0 \\ \pi - \sin^{-1}|\mu/\bar{\rho}_1| & \text{for } \sin(\varphi-3\pi/4) < 0,\ \mu \ge 0 \\ \pi + \sin^{-1}|\mu/\bar{\rho}_1| & \text{for } \mu < 0 \end{cases}$$

and

$$\bar{\rho}_2 = (1-\mu^2)^{\frac{1}{2}}$$

$$\tau_2 = \varphi + 3\pi/4$$

With the boundary data $F(\mu,\varphi)$ defined by eq. (43) the coefficients K_λ in eq. (34) for the pressure variations inside the sonic circle are computed for all the eigenvalues between $-\frac{1}{2}$ and 3. The pressure distribution on the three faces of the corner are shown in Figs. 6 & 7. The pressure distributions for $r/(Ct) < 1$ are obtained from the present analysis. They match with the corresponding two-dimensional values across the sonic sphere with a discontinuity in slope. They confirm, with less than 1% variation, the symmetry of the solution with respect to the plane bisecting the faces OAB and OAD and the value of 8/7 at the vertex as predicted by Theorem I in 8. When the calculation is performed with only eigenvalues less or equal to 2, the results differ from those in Figs. 6,7 within 6%.

7. General Incident Wave

The preceding concial solutions for an incident plane pulse can be employed to construct diffraction solutions for an incident wave of more general type by superposition of plane pulses. In particular, when the incident wave is a plane wave of the type

$$p_i = \psi(\eta) \tag{44}$$

with

$$\eta = Ct - (n_1 x + n_2 y + n_3 z) \tag{45}$$

where the wave form ψ, is an arbitrary function of its phase η and n_1, n_2 and n_3 are the direction cosines of the normal to the plane wave.

The incident wave due to the sonic boom can be locally represented as a plane wave since the length of a structure is usually much smaller than the radius of curvature of the wave front. The wave form ψ is in general a sequence of a weak shock wave and an expansive wave or a compression wave as shown in Fig. 8.

If the wave form is a Heaviside function, the diffraction due to the three-dimensional corner is given by the preceding conical solution and will be designated as $p^*(r/(Ct),\theta,\varphi)$. The solution of the diffracted wave corresponding to a general wave form of a sonic boom is

$$p(r,\theta,\varphi,t) = \sum_i \left\{ (\Delta p)_i p^* \left(\frac{r}{Ct-\eta_i}, \theta, \varphi\right) + \int_{\eta^i}^{\eta^{i+1}} p^* \left(\frac{r}{Ct-\eta}, \theta, \varphi\right) \frac{d\psi}{d\eta} d\eta \right\} \tag{46}$$

where η_i is the phase of the i-th shock wave with strength $(\Delta p)_i = \psi(\eta_i + 0) - \psi(\eta_i - 0)$. Note that η increases in the direction opposite to the normal of the plane wave.

Another method for the construction of nonconical diffraction solutions should be pointed out here. When the boundary data on the sonic sphere $\zeta = r/(Ct) = 1$ around the vertex of a three-dimensional corner, which is now a function of t in addition to the spherical angles θ and φ, can be written as a power series with respect to t, i.e.,

$$F(t,\theta,\varphi) = \sum_\gamma t^\gamma F_\gamma(\theta,\varphi) \text{ for } t > 0 \tag{47}$$

where the summation is carried over a sequence of positive numbers, $\gamma \geq 0$. The solution inside the sonic sphere can likewise be written as

$$p(t,r,\theta,\varphi) = \sum_\gamma t^\gamma p^{(\gamma)}(\zeta,\theta,\varphi) \tag{48}$$

The wave equation yields the governing equation for $p^{(\gamma)}(\zeta,\theta,\varphi)$

$$\zeta^2(1-\zeta^2)\frac{\partial^2 p^{(\gamma)}}{\partial\zeta^2} + 2\zeta[1+(\gamma-1)\zeta^2]\frac{\partial p^{(\gamma)}}{\partial\zeta} - \gamma(\gamma-1)\zeta^2 p^{(\gamma)}$$

$$+ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial p^{(\gamma)}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 p^{(\gamma)}}{\partial\varphi^2} = 0 \tag{49}$$

$p^{(\gamma)}$ is subjected to the boundary condition on the sphere $\zeta=1$,

$$p^{(\gamma)}(1,\theta,\varphi) = F_\gamma(\theta,\varphi)$$

and the boundary conditions on the surfaces of the corner remain the same as eqs. (3,4).

Since the differential operators with respect to θ and φ are independent of γ, the corresponding eigenvalue problem in θ-φ plane after the separation of the variable ζ is the same as that for $\gamma=0$, i.e., the conical problem. The solution $p^{(\gamma)}$ can therefore be written as,

$$p^{(\gamma)}(\zeta,\theta,\varphi) = \sum_\lambda K_\lambda^{(\gamma)}\, Z_\lambda^{(\gamma)}(\zeta)\, G_\lambda(\mu,\varphi) \tag{50}$$

where $\mu = \cos\theta$. The eigenvalues λ's and the eigenfunctions $G_\lambda(\mu,\varphi)$ are identical with those obtained for $\gamma = 0$ in 3,4. With the boundary condition $Z_\lambda^{(\gamma)}(1) = 0$ imposed on $Z_\lambda^{(\gamma)}$, the constants $K_\lambda^{(\gamma)}$ are also related to the boundary data $F_\gamma(\mu,\theta)$ by the same set of equations, eqs. (37) and (38), and can be determined by the same numerical program for $\gamma = 0$. The appearance of γ occurs only in the differential equation for $Z_\lambda^{(\gamma)}(\zeta)$, which is

$$\zeta^2(1-\zeta^2)\frac{d^2 Z}{d\zeta^2} + 2\,[1+(\gamma-1)\zeta^2]\frac{dZ}{d\zeta} - [\gamma(\gamma-1)\zeta^2 + \lambda(\lambda+1)]\,Z = 0 \tag{51}$$

The boundary conditions are the same, i.e., $Z(1) = 1$ and $Z(0)$ is finite. $Z_\lambda^{(\gamma)}(\zeta)$ can be determined by the same procedure described in 5 for $Z_\lambda^{(0)}(\zeta)$.

Now it is clear that the extension of the proceeding conical solutions to the diffraction of a general incident wave by a three-dimensional corner can be carried out provided that the boundary data $F(t,\theta,\varphi)$ on the sonic sphere can be found. This suggests that the next step is to solve the diffraction problem of a three-dimensional wave by a two-dimensional corner.

For a two dimensional corner, the three dimensional diffracted waves can be constructed by superposition of solutions for incident plane waves with different wave forms and directions of incidence or by the method of separation of variables in cylindrical coordinates and time. These schemes in general are tedious for the actual construction of the solution. By making use of the special geometry of the 90° convex corner, an integral equation can be set up and solved by an iteration scheme similar to the procedure for a two dimensional incident wave [3]. This method of approach is presently under investigation.

In the next section, two theorems are presented so that the value along the edge or the vertex of a corner can be related directly to an incident wave without the construction of the diffraction solution. The incident wave can be completely arbitrary and hence can also be a diffracted wave from an adjacent corner.

8. "Mean-Value" Theorems for Diffraction by a Conical Surface and Applications

In a previous article [7], a mean value theorem for the diffraction of an incident wave by a cone was derived. It states that the value at the vertex of the cone is equal to $4\pi/(4\pi-\Omega)$ times the value of the incident wave at the vertex as if the value is averaged locally over the exterior solid angle of the cone, $4\pi-\Omega$ instead of over the whole space, 4π. This relationship was obtained under the restriction that the incident wave does not hit any portion of the conical surface prior to its encounter with the vertex of the cone. In this section, the derivation of the theorem is partially repeated with modifications so that the theorem is valid under a weaker restriction, that is, the incident wave may hit the conical surface within a finite time interval T ahead of the epoch of its incidence on the vertex. In case such a finite time interval T cannot be found, a second theorem is introduced. Extension of the results to solutions of inhomogeneous wave equations is also presented in this section.

Let t=0 be the instant when an acoustic incident wave $\varphi^{(i)}$ hits the vertex of the cone located at the origin. Let T be the finite time interval such that for $t \leq -T$ the incident wave does not hit the conical surface. Let D(t) denote the domain outside of which the incident wave vanishes at the instant t, e.g., $\varphi^{(i)}(x,y,z,-T)=0$ for (x,y,z) not in D(-T). Domain $D(t\leq -T)$ does not intersect the cone G with solid angle Ω, as shown in Fig. 9. In the absence of the cone, the incident wave $\varphi^{(i)}$ at the origin can be related to the initial data at an instant $t=t_o<-T$ by the Poisson Formula [8],

$$\varphi^{(i)}(o,o,o,t) = \frac{1}{4\pi C}\left[\frac{\partial}{\partial t}\iint_S \frac{f}{r}\,dS + \iint_S \frac{g}{r}\,dS\right]$$

where r is distance from the origin,

$$f(x,y,z) = \varphi^{(i)}(x,y,z,-t_o),$$

$$g(x,y,z) = \varphi^{(i)}_t(x,y,z,-t_o)$$

and S is the sphere with radius $R=C(t+t_o) > C(t+T)$ and with its center at the origin. The Poisson formula can be identified as a special form of the Kirchhoff's formula [8],

$$\varphi^{(i)}(o,o,o,t) = -\frac{1}{4\pi}\iint_S \left\{\left[\varphi^{(i)}\right]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{\partial\varphi^{(i)}}{\partial n}\right] - \frac{1}{Cr}\frac{\partial r}{\partial n}\left[\frac{\partial\varphi^{(i)}}{\partial t}\right]\right\}dS \qquad (52)$$

where $\partial/\partial n$ means differentiation along the outward normal to the surface S and $[\varphi]$ is the retarded value of φ. On the sphere S with $R=a(t+t_o)$, it follows $[\varphi^{(i)}(x,y,z,t)] = \varphi^{(i)}(x,y,z,-t_o)$. Similarly, the retarded values of

$\varphi_n^{(i)}$ and $\varphi_t^{(i)}$ are the corresponding values at $t = -t_o$. All of them vanish outside the domain $D(-t_o)$. Let S_c denote the part of the sphere outside of the cone G, and S^* denote the part inside the domain $D(-t_o)$. Since $D(-t_o)$ with $-t_o < -T$ does not intersect G, S_c contains S^*. The integrand is non-zero only inside S^*, therefore, the domain of the surface integration can be reduced from S to S^* or to S_c.

To obtain the resultant wave φ at the vertex of the cone, the passages in the derivation of the Kirchhoff formula [7,9] from the Green's theorem will be repeated. The Green's theorem is now applied to the volume bounded by the two concentric spheres S^* and S_σ with radii r=R and r=σ respectively and the conical surface ∂G yields

$$\iint\limits_{S^*+S_\sigma+\partial G^*} \left\{\left(\frac{[\varphi]}{r^2} + \frac{1}{rC}\left[\frac{\partial\varphi}{\partial t}\right]\right)\frac{\partial r}{\partial n} + \frac{1}{r}\left[\frac{\partial\varphi}{\partial n}\right]\right\} dS = 0 \qquad (53)$$

where ∂G^* is the part of ∂G lying between S^* and S_σ.

On the surface of the cone ∂G, $\partial r/\partial n=0$. With the boundary condition of $\partial\varphi/\partial n=0$, the integral over ∂G^* vanishes.

As $\sigma \to 0$, the integral over S_σ approaches $-(4\pi-\Omega)\,\varphi(o,o,o,t)$. Equation (53) reduces to

$$(4\pi-\Omega)\,\varphi(o,o,o,t) = -\iint\limits_{S^*} \left\{[\varphi]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{\partial\varphi}{\partial n}\right] - \frac{1}{Cr}\frac{\partial r}{\partial n}\left[\frac{\partial\varphi}{\partial t}\right]\right\} dS \qquad (54)$$

On the spherical surface S^*, $[\varphi] = \varphi(x,y,z,-t_o) = \varphi^{(i)}(x,y,z,-t_o) = [\varphi^{(i)}]$ and similar relationship holds for the t- and n- derivatives. Eqs. (52) and (54) yield:

$$\varphi(o,o,o,t) = \frac{4\pi}{4\pi-\Omega}\,\varphi^{(i)}(o,o,o,t) \text{ for } t > -T \qquad (55)$$

This result can be stated as follows:

Theorem I

The resultant value at the vertex of a cone with solid angle Ω is equal to $4\pi/(4\pi-\Omega)$ times the incident wave at the vertex, if a finite T can be found such that the incident wave hits the conical surface within the time interval T prior to its encounter with the vertex of the cone.

In a practical problem, the conical surface which forms a part of the surface of an obstacle, has a finite length, say L. In applying the theorem, it is essential that the part of the obstacle inside the sphere S with radius C(t + T) is conical, i.e.,

$$C(t + T) < L \qquad (56)$$

It defines an upper bound for T. On the other hand, for a given T with $CT < L$, the inequality (56) defines an upper bound for t for which eq. (55) holds.

When an incident wave is diffracted first by a part of the surface of the obstacle other than the conical surface of finite length L, condition (56) cannot be fulfilled. Even for a cone of infinite length, the incident pulse may be in contact with a part of the conical surface all the time, consequently, the finite time interval T assumed in Theorem I does not exist. For both examples, the value at the vertex of the cone cannot be related directly to the incident wave by Theorem I. A theorem relating the value at the vertex of one cone to that of another cone will be presented.

Figure 10 shows the relative orientations of two cones G_1 and G_2, with solid angles Ω_1 and Ω_2. Their vertices coincide at the origin and their boundaries ∂G_1 and ∂G_2 have a common region Γ, i.e.,

$$\Gamma = \partial G_1 \cap \partial G_2 \tag{57}$$

Let $\varphi_1(x,y,z,t)$ denote the resultant of an incident wave $\varphi^{(i)}$ and its reflected and diffracted waves by cone G_1 alone with $\partial\varphi_1/\partial n=0$ on the surface of the cone ∂G_1. $D_1(t)$ denotes the domain outside of which $\varphi_1=0$ at the instant t. At $t=0$, the origin lies on the boundary of $D_1(t)$ and $D_1(t)$ does not contain the origin for $t < 0$.

It will be assumed that there exists a finite time interval T such that the domain $D(t \le -T)$ is in partial contact with the cone G_1 only over the surface Γ, i.e.,

$$\left(\partial D(t) \cap \partial G_1\right) \subset \left(\partial G_1 \cap \partial G_2\right) = \Gamma \qquad \text{for } t \le -T \tag{58}$$

The next step is to examine the solution φ_2 which is the resultant of the same incident wave $\varphi^{(i)}$ and its reflected and diffracted waves by cone G_2 alone. The initial data for φ_2 are prescribed by $\varphi_1(x,y,z,-t_o)$ and $\varphi_{1,t}(x,y,z,-t_o)$ for $-t_o < -T$. If cone G_2 does not intersect the domain $D_1(t)$ for $t < -T$, it is evident from conditions (58) that

$$\varphi_2(x,y,z,t) = \varphi_1(x,y,z,t)$$

and

$$D_2(t) \equiv D_1(t) \qquad \text{for } -T \ge t \ge -t_o \tag{59}$$

The derivation of eq. (54) from the Green's theorem will be repeated for the volume bounded by two concentric spheres S and S_σ and by the conic surface ∂G_1 or ∂G_2 respectively. The two spheres, S and S_σ are centered at the origin and their radii are $C(t+t_o)$ and σ. The following result equivalent to eq. (54) is obtained:

$$(4\pi-\Omega_j)\,\varphi_j(o,o,o,t) = -\iint_{S_j^*} \left\{[\varphi_j]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{\partial\varphi_j}{\partial n}\right] - \frac{1}{rC}\frac{\partial r}{\partial n}\left[\frac{\partial\varphi_j}{\partial t}\right]\right\} dS$$

$$\text{for } j = 1,2 \quad t \geq -T \tag{60}$$

On the sphere S with radius $C(t+t_o)$, it is evident $[\varphi_j] = \varphi_j(x,y,z,-t_o)$. Equation (59) shows that $[\varphi_i]$, $[\partial\varphi_j/\partial n]$, $[\partial\varphi_j/\partial t)$ and $D_j(-t_o)$ are invariant with respect to j. S_j^* is defined as the part of the spherical surface S lying in $D_j(-t_o)$ and is also independent of j. With the right side of Eq. (60) invariant with respect to j, it yields

$$(4\pi-\Omega_1)\,\varphi_1(o,o,o,t) = (4\pi-\Omega_2)\,\varphi_2(o,o,o,t) \text{ for } t \geq -T \tag{61}$$

So far the problem was formulated as propagation and diffraction of waves. It can also be formulated as an initial and boundary value problem and restated as follows:

<u>Theorem II</u>

For two conical surfaces ∂G_1 and ∂G_2 with the same location for the vertices and with the same initial data $\varphi=f(x,y,z)$ and $\varphi_t=g(x,y,z)$ at the instant $t=-t_o<0$, the resultant disturbance at the vertex for each cone alone is inversely proportional to the exterior solid angle of the cone provided that the support D for f and g does not intersect either one of the cones, and that the part of the boundary ∂D in common with one of the conical surface is also with the other, i.e.,

$$(\partial D \cap \partial G_1) = (\partial D \cap \partial G_2) \subset (\partial G_1 \cap \partial G_2) \tag{62}$$

It is obvious that when the wave equation has an inhomogeneous term $h(x,y,z,t)$ with support $E(t)$ similar conclusions can be obtained:
<u>Corollary</u> Theorem I and II will also be valid for solutions of the inhomogeneous wave equation $\Delta\varphi - C^{-2}\varphi_{tt} = h(x,y,z,t)$ provided that the support $E(t)$ of h does not intersect the cone or both cones respectively.

An interesting application for Theorem I and the Corollary will be the initial and boundary value problem for an inhomogeneous wave equation in the interior of a conical surface ∂G_c with solid angle Ω_c (or the exterior of a cone G with solid angle $\Omega=4\pi - \Omega_c > 2\pi$). The mathematical problem is:

D.E. $\Delta\varphi - C^{-2}\varphi_{tt} = h(x,y,z,t)$ inside the cone G_c for $t > 0$,

B.C. $\varphi_n = 0$ on ∂G_c,

I.C. $\varphi=f(x,y,z)$, $\varphi_t = g(x,y,z)$ at $t=0$.

From the definition of the problem both the support E of h and the support

D of f and g lie in G_c and, therefore, do not intersect G. Theorem I and the corollary state that:

The value at the vertex, $\varphi(o,o,o,t)$ is related to the solution, $\varphi_{3D}(o,o,o,t)$, of the initial value problem for the inhomogeneous wave equation in three-dimensional space with the removal of the conical surface by an amplification factor equal to the local enlargement in solid angle, i.e., $4\pi/\Omega_c$. This statement is expressed by the following equation,

$$\varphi(o,o,o,t) = \frac{4\pi}{\Omega_c}\left\{\varphi_{3D}(o,o,o,t)\right\}$$

$$= \frac{4\pi}{\Omega_c}\left\{\frac{1}{4\pi C}\left[\frac{\partial}{\partial t}\iint_{S^*_{Ct}} \frac{f}{r}\, dS + \iint_{S^*_{Ct}} \frac{g}{r}\, dS\right]\right.$$

$$\left. + \frac{1}{4\pi}\int_o^{Ct} dr \iint_{S^*_r} \frac{[h]}{r}\, dS\right\} \tag{63}$$

where S^*_{Ct} is the part of the spherical surface inside the support D of f and g and S^*_r is the part of the spherical surface inside the support E of h at the retarded instant t-r/C. The spheres are centered at the origin and the subscript denotes the radius.

It is of interest to note that the value at the vertex of the cone is independent of the geometry of the cone G_c nor does it depend on the distribution of f, g, and h and their supports D and E with respect to the spherical angles θ and φ so long as the integrals of f, g and h over the spherical surface are invariant as functions of r and t.

Another example is the diffraction problem being analyzed in this paper. As a matter of fact, the investigations in this section are motivated by the desire to determine the values along an edge or a corner of a rectangular block or structure due to the arrival of a conical solution originated from an adjacent edge or corner without the construction of the non-conical solution.

Figure 11 shows an incident plane pulse of strength ϵ impinging on a block with normal parallel to an edge OB. After the instant t = 0, the pulse is reflected by the face OAD and diffracted around the edges OA, OD and the corner O. Let OA, OB and OD to be designated as the x-, y-, z-, axis respectively. From Theorem I, the value at the corner O of solid angle $\pi/2$ is

$$p_1(o,o,o,t) = \epsilon[4\pi/(4\pi-\pi/2)] = (8/7)\,\epsilon \text{ for } W/C > t > 0. \tag{64}$$

This is in agreement with the numerical result in 6 and is valid until the arrival at t = W/C of the diffracted wave from the adjacent corners.

For the convenience of description, it is assumed that the height OD is equal to the width W(=OA) and is related to the length L(=OB) by the inequality

$$2L > W > L \tag{65}$$

For a point (x_o, o, o) on the edge of OA with solid angle π, Theorem I gives

$$p_1 (x_o, o, o, t) = \epsilon[4\pi/(4\pi-\pi)] = (4/3)\,\epsilon \quad \text{for } x_o/C > t > 0 \tag{66}$$

Due to symmetry, x_o can be assumed to be less than W/2 and the upper limit for t is the epoch for the arrival of the diffracted wave from the vertex O.

For the disturbance pressure elsewhere such as on the surfaces of the block it is necessary to construct the diffracted waves. Behind the incident front the disturbance pressure is $p_1 = \epsilon$ outside the envelopes of sonic spheres along the edges OA and OD. Inside the sonic sphere around the vertex O the pressure is given by the three-dimensional conical solution constructed in 7, i.e., $p_1 = \epsilon + \epsilon p_{3c}(\bar{x},\bar{y},\bar{z})$ with $\bar{x},\bar{y},\bar{z}$ equal to $x/(Ct)$, $y/(Ct)$, $z/(Ct)$ respectively. Outside the sonic sphere but inside one of the cylindrical envelopes, the solutions are conical in two dimensions i.e., $p_1 = \epsilon + \epsilon p_{2c}(\bar{x},\bar{y})$ and $p_1 = \epsilon + \epsilon p_{2c}(\bar{z},\bar{y})$ for edges OA and OD respectively. Of course there are also diffracted waves from vertices A, D, E, and edges AD and DE.

When $t > L/C$, the wave front advances over the edges BA', BD' and the vertex B. If the edges BA', BD' were absent and the planes OBA and OBD were extended beyond BA' and BD', point B can be considered as the vertex of a cone, G_1, with solid angle $\Omega_1 = \pi$ and the solution p_1 at B is valid for $t > L/C$ (Fig. 12). In the presence of the edges BA' and BD', point B can be considered as the vertex of a cone G_2, with solid angle $\Omega_2 = \pi/2$. Cone G_1 and cone G_2 have faces OBAB' and ODAD' in common and time interval T can be equated to zero. Theorem II yields the resultant value at B for cone G_2, i.e., including the diffraction of p_1 by corner B,

$$p_2 (o, L, o, t) = [(4\pi-\pi)/(4\pi-\pi/2)]\; p_1 (o, L, o, t)$$

$$= (6/7)\,\epsilon[1+p_{3c}(o, L/Ct, o)] \quad \text{for } L/C > t-L/C > 0 \tag{67}$$

The upper limit of $t = 2L/C$ is due to the finite length L of both cone G_1 and cone G_2.

In case of $(W^2 + L^2)^{\frac{1}{2}} < 2L$, the conical solutions originated at the vertices A and D arrive at corner B before $t = 2L/C$, additional terms should be added for $t > (W^2 + L^2)^{\frac{1}{2}}/C$. They can again be defined by Theorem II. The incoming wave from corner A is $\epsilon p_{3c}(\bar{x}', \bar{y}, \bar{z})$ with $\bar{x}' = (W-x)/(Ct)$.

Point B can be considered as the vertex of cone G_2 defined as before and as the vertex of cone G_1 with the extension of face OABA' over the edges OB and BA'. The solid angle for G_1 is 2π and the ratio of solid angles exterior to G_1 and G_2 is $(4\pi-2\pi)/(4\pi-\pi/2) = 4/7$. Theorem II yields the extra term (4/7) $\epsilon p_{3c}(W/(Ct), L/(Ct), 0)$. Similarly a term (4/7) $\epsilon p_{3c}(0, L/(Ct), W/(Ct))$ should be added to Eq. (67) to take into account the incoming wave from vertex D for $t > (W^2 + L^2)^{\frac{1}{2}}/C$.

The disturbance pressure at a point $P(x_o, L, 0)$ on the edge BA' can also be related to the diffracted waves from edge OA and from vertices O and A by means of Theorem II. Point p can be considered as the vertex of cone G_2 with solid angle $\Omega_2 = \pi$ (Fig. 11). When the face OABA' is extended over the edge BA', point P becomes the vertex of cone G_1 with solid angle $\Omega_1 = 2\pi$ (Fig. 12). The resultant pressure at point P due to diffraction is equal to 2/3 of the value of the incoming wave alone. The factor 2/3 is the ratio of exterior solid angles of G_1 and G_2. This result is valid until the arrival of the non-conical diffracted wave from vertex B' or A'.

It has been demonstrated that the theorems are useful to extend the knowledge of conical solutions to the adjacent edges and vertices for limited time intervals. To obtain the pressure distribution on the surfaces of the rectangular block or structure, it is still necessary to construct three-dimensional non-conical solutions for diffractions by edges and corners.

Conclusion

In this paper the following results are obtained:

1) the conical solution for the diffraction of a plane acoustic pulse by a three-dimensional corner of a cube is obtained by separation of variables.

2) the solutions of the eigenvalue problem in spherical angles for the conical problem remain the same for the generalized conical solutions described in 7 and also for potential or unsteady heat conduction problems, so long as the boundary conditions in θ,φ plane are the same.

3) the technique for the solution of the eigenvalue problem is applicable to a more general shape of boundary on the unit sphere formed by two great circles of given longitudes and a horizontal circle of given latitude. The boundary conditions can be generalized to the type $ap+b\ \partial p/\partial n = 0$ with both a and b as non-negative constants.

4) the numerical results suggest that the eigenvalues for corners can be approximated, one by one, by the eigenvalues for circular cones of the same solid angle.

5) "mean-value" theorems in 8 are derived for solutions of wave equations so that the resultant wave at the vertex of a cone can be related to the incident wave or the value at the vertex of a different cone. These theorems are useful to extend the knowledge of the conical solutions to the adjacent corners or edges.

6) applications of the Theorems in 8 yield interesting and simple results: for an initial value problem of an inhomogeneous wave equation inside a conical surface with vanishing normal derivative on the surface, the value at the vertex of the cone can be related directly to the same initial value problem in three-dimensional space. The value at the vertex depends only on the value of the solid angle of the cone but not on the geometry of the cone. It depends on the radial distribution of the initial data and the inhomogeneous term but not on their distribution with respect to the spherical angles.

Acknowledgment

The authors wish to acknowledge Professor Antonio Ferri, Director of the Aerospace Laboratory and Professor J.B. Keller, Head of the Department of Mathematics, New York University, Bronx, New York, for their valuable discussions.

REFERENCES

1) Friedlander, F.G., "Sound Pulse," Cambridge University Press, New York, New York, 1958.

2) Keller, J.B. and Blank, A., "Diffraction and Reflection of Pulses by Wedges and Corners," Communication on Pure and Applied Mechanics, Vol. 4, No. 1, pp. 75-94, June 1951.

3) Ting, L, "Diffraction of Disturbances Around a Convex Right Corner with Applications in Acoustics and Wing-body Interference," Journal of Aerospace Sciences, Vol. 24, No. 11, pp. 821-831, November 1957.

4) Ting, L., and Pan, Y.S., "Incident of N-Waves on Structures," NASA Second Conference on Sonic Boom Research, NASA SP-180, pp. 89-98, May 1968.

5) Hobson, E.W., "The Theory of Spherical and Ellipsoidal Harmonics," Chelsea Publishing Company, New York, New York, pp. 178-190, 1955.

6) Churchill, R.V., "Complex Variables and Applications, 2nd Ed. McGraw-Hill Company, New York, New York, pp. 218-229, 1960.

7) Ting, L., "On the Diffraction of an Arbitrary Pulse by a Wedge in a Cone," Quarterly of Applied Mathematics, XVIII, No. 1, pp. 89-92, April 1960.

8) Tychonow, A.N. and Samarski, A.A., "Partial Differential Equations of Mathematical Physics, Vol. II, Holden-Day, Inc., San Francisco, California, pp. 381-392, 1967.

9) Baker, B.B. and Copson, E.T., "Mathematical Theory of Huygen's Principle," 2nd Ed., Oxford University Press, New York, New York, pp. 36-38, 1950.

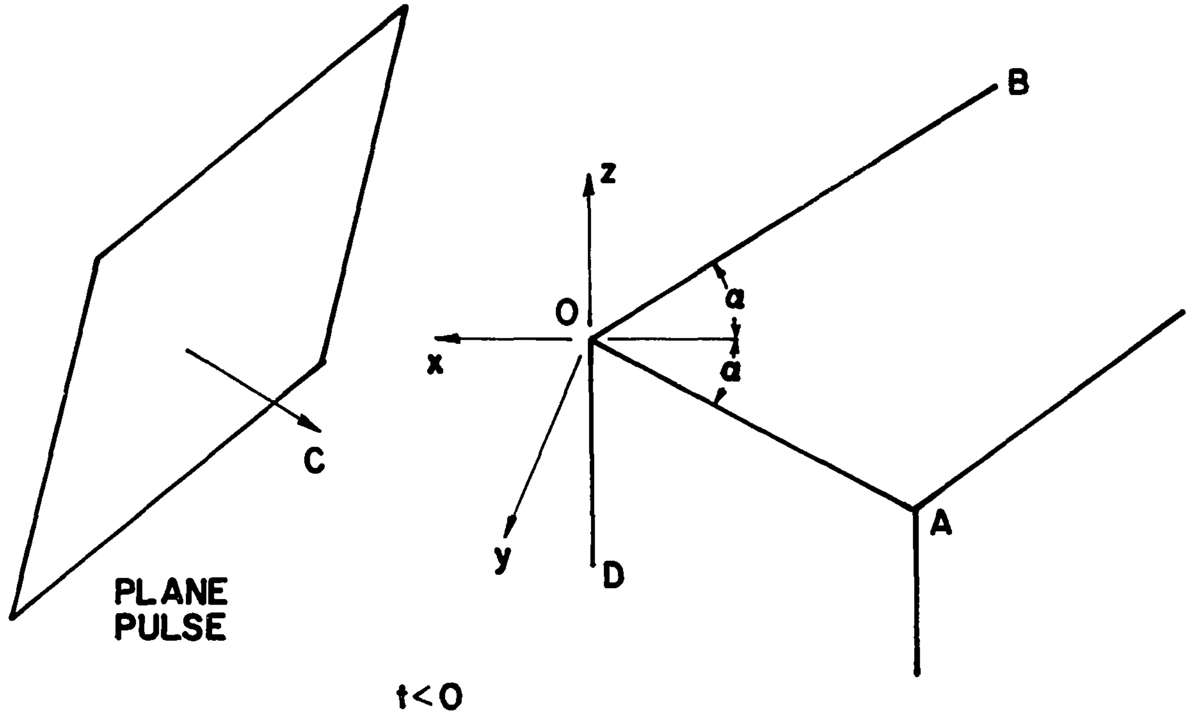
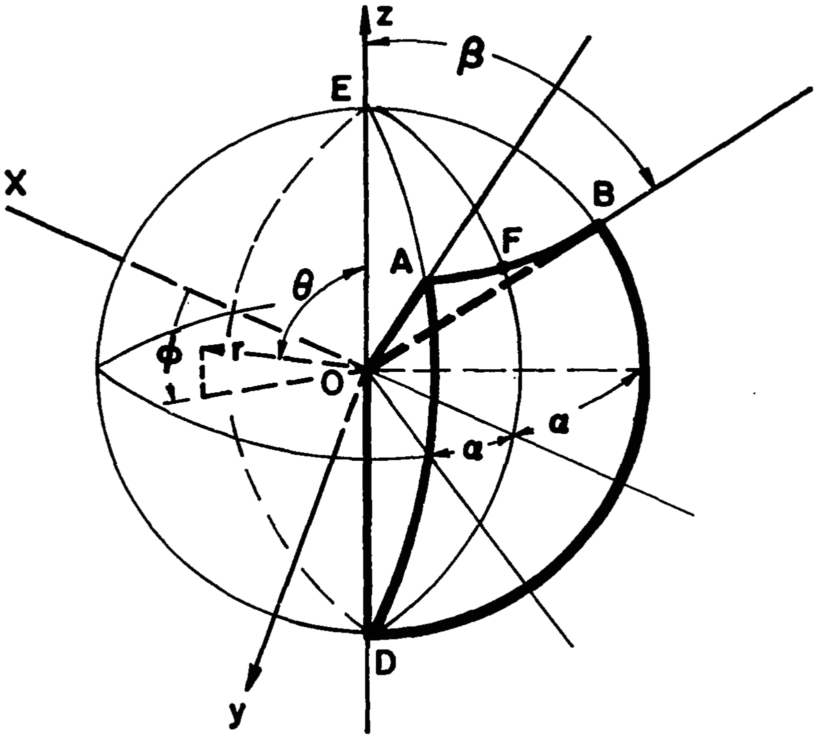


Fig. 1 Incidence of a plane pulse on a corner



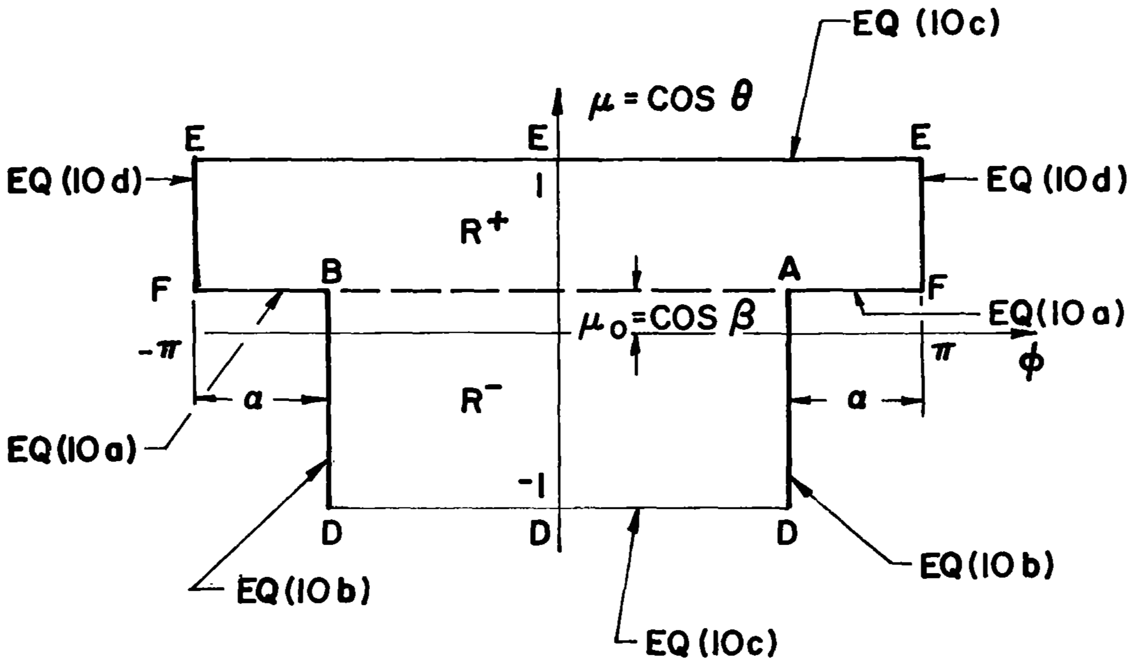


Fig. 2 Eigenvalue problem for the corner

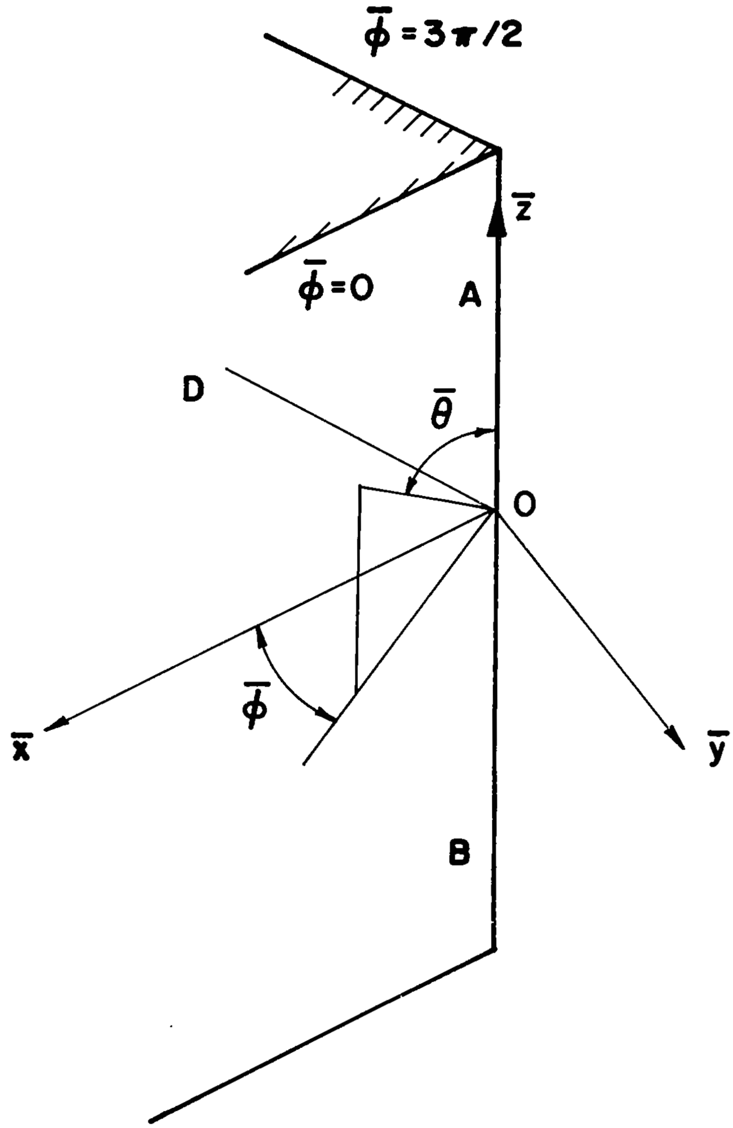


Fig. 3 A two-dimensional corner

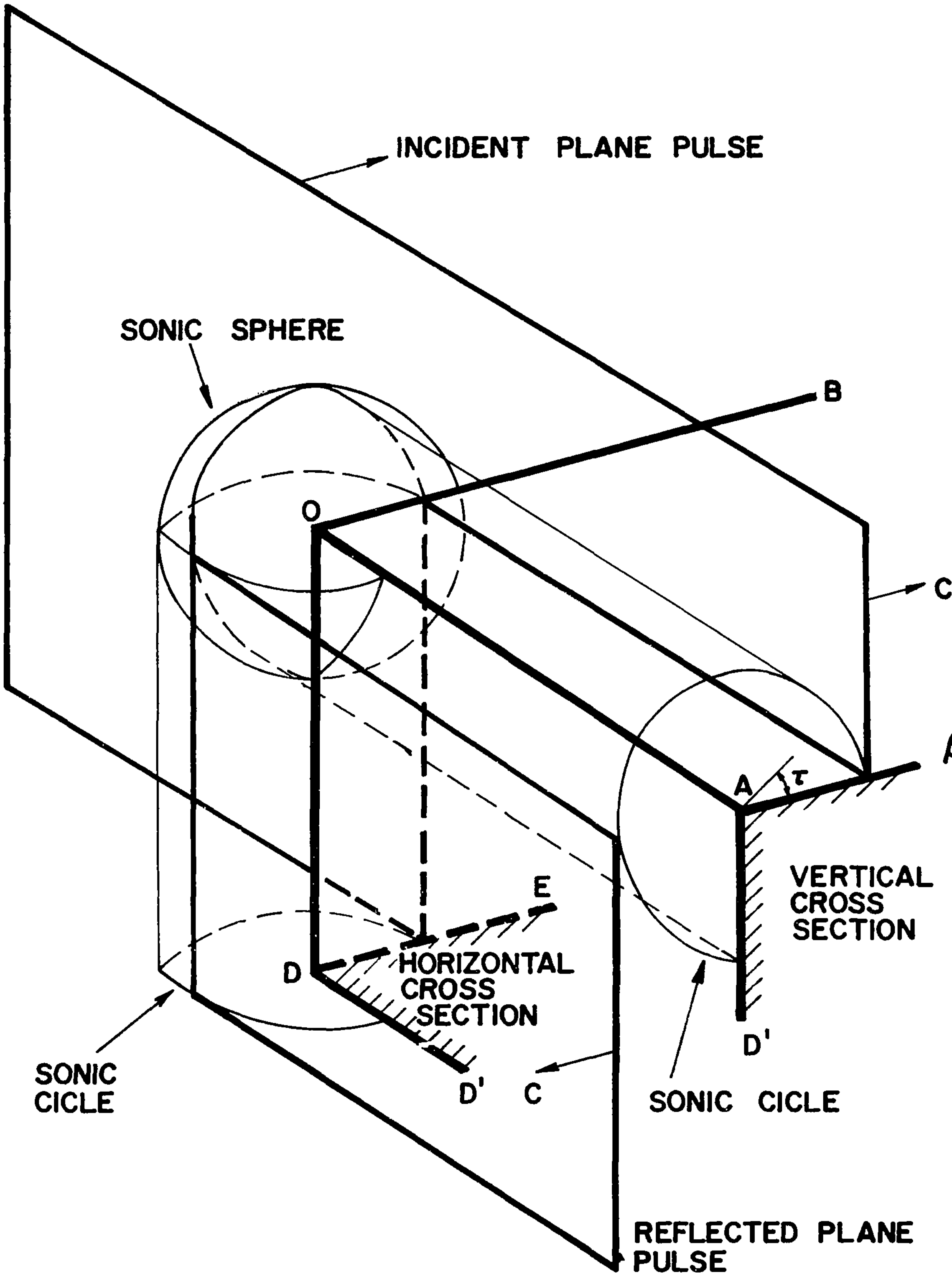
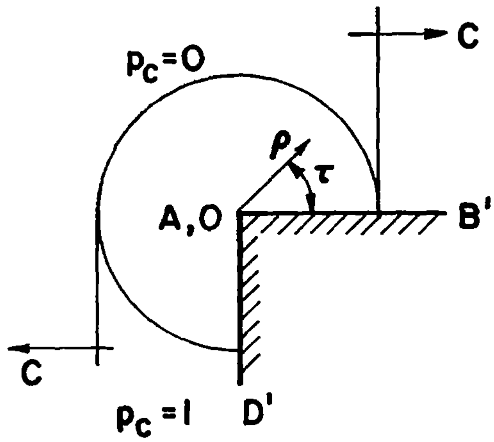
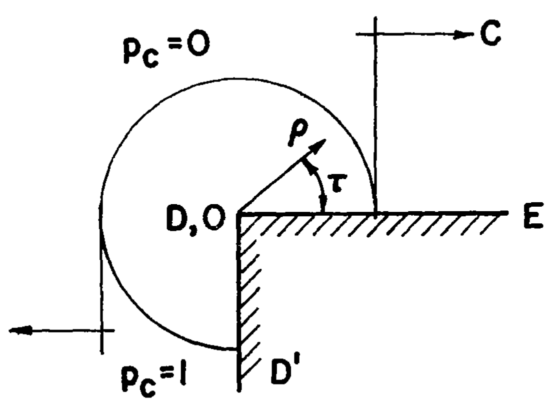


Fig. 4 Plane pulse incidence head on to face OAD, the reflected plane pulse and the diffracted fronts around edges OA, OD, and vertex O



SECTION NORMAL TO EDGE OA



SECTION NORMAL TO EDGE OD

Fig. 5 Two-dimensional conical solutions around edges OA and OD

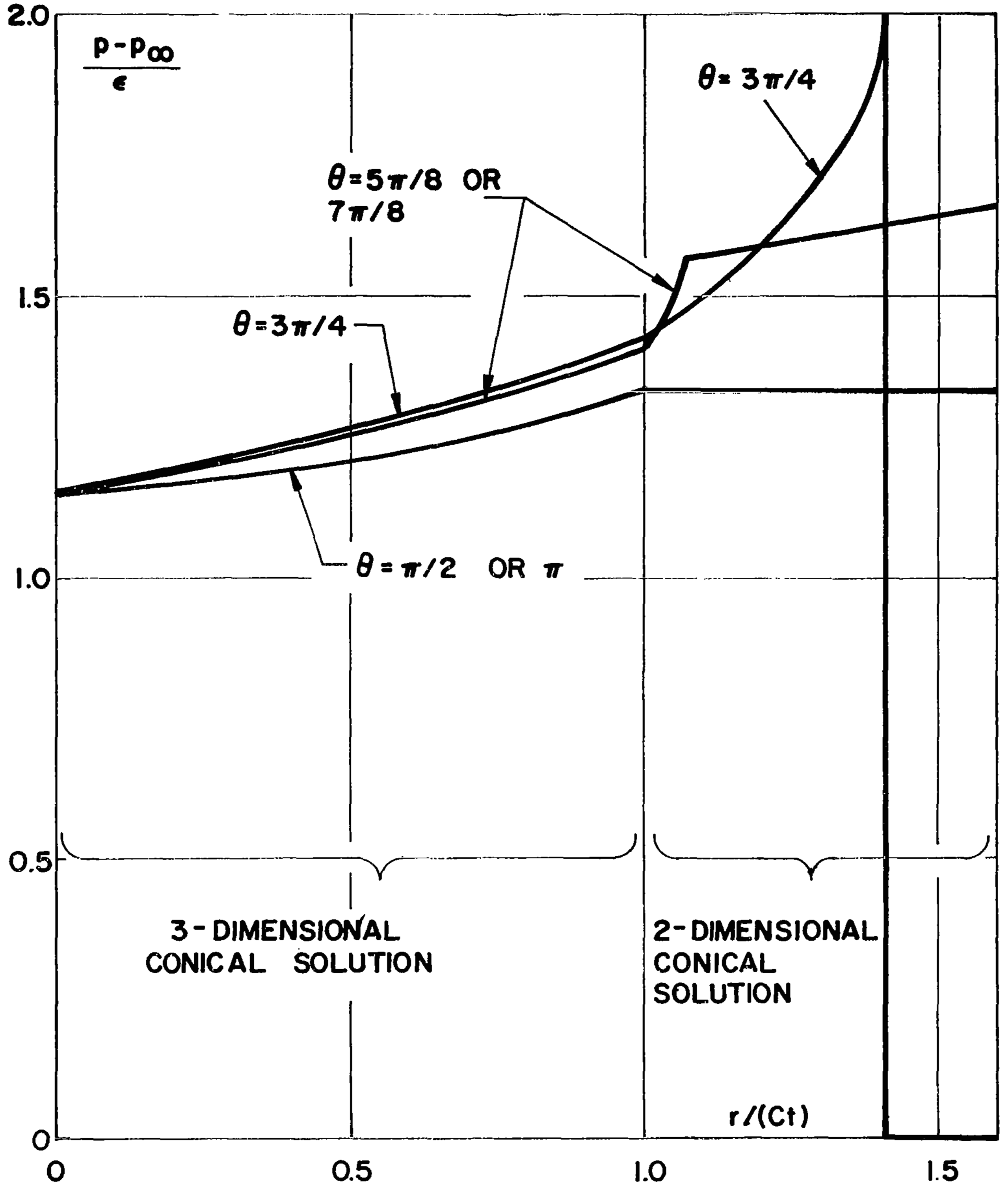


Fig. 6 Over pressure along constant θ lines face OAD($\varphi=3\pi/4$)

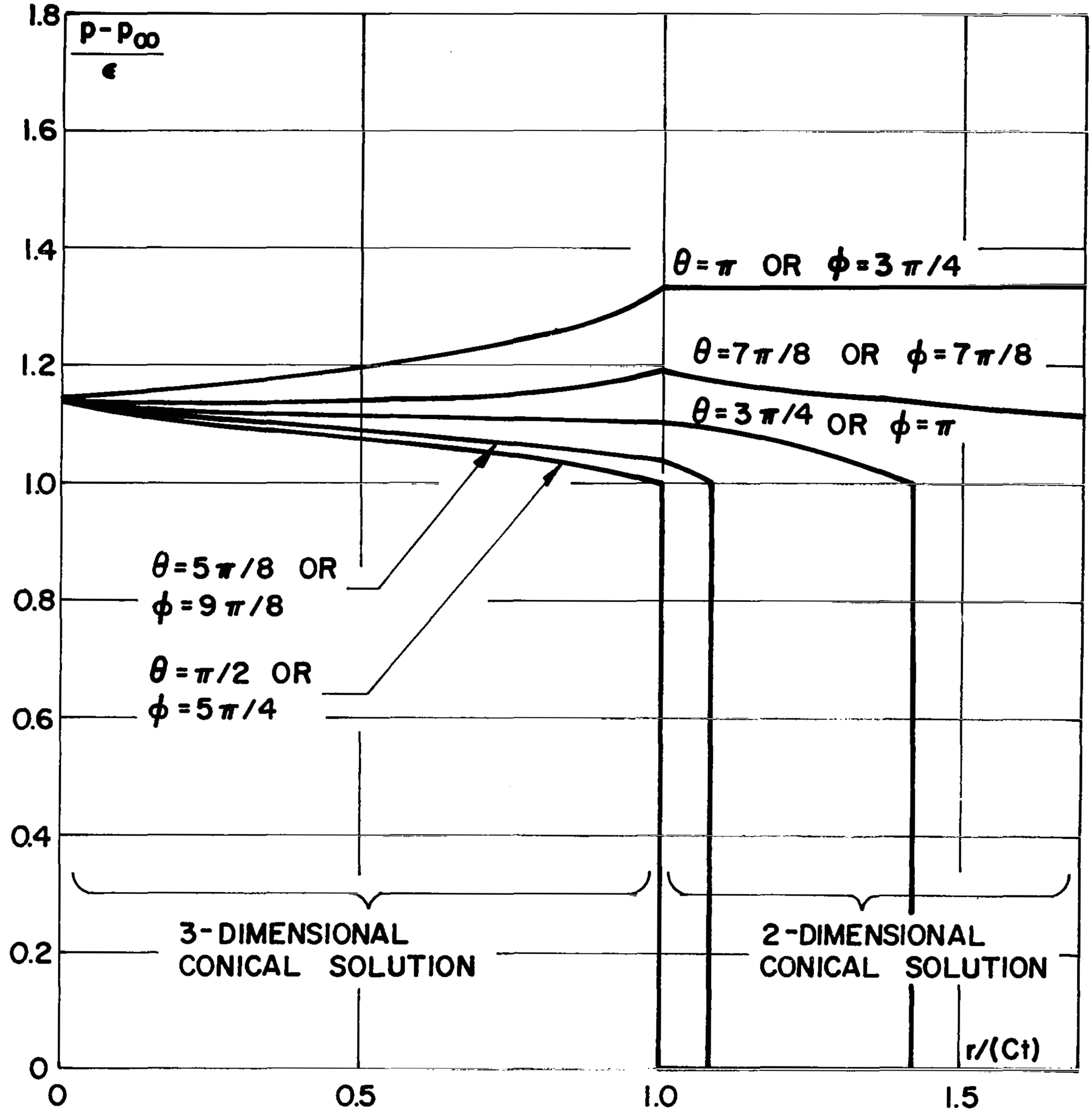


Fig. 7 Over pressure along constant φ lines on face OAB (θ=π/2) and along constant θ lines on face OBD (φ=3π/4)

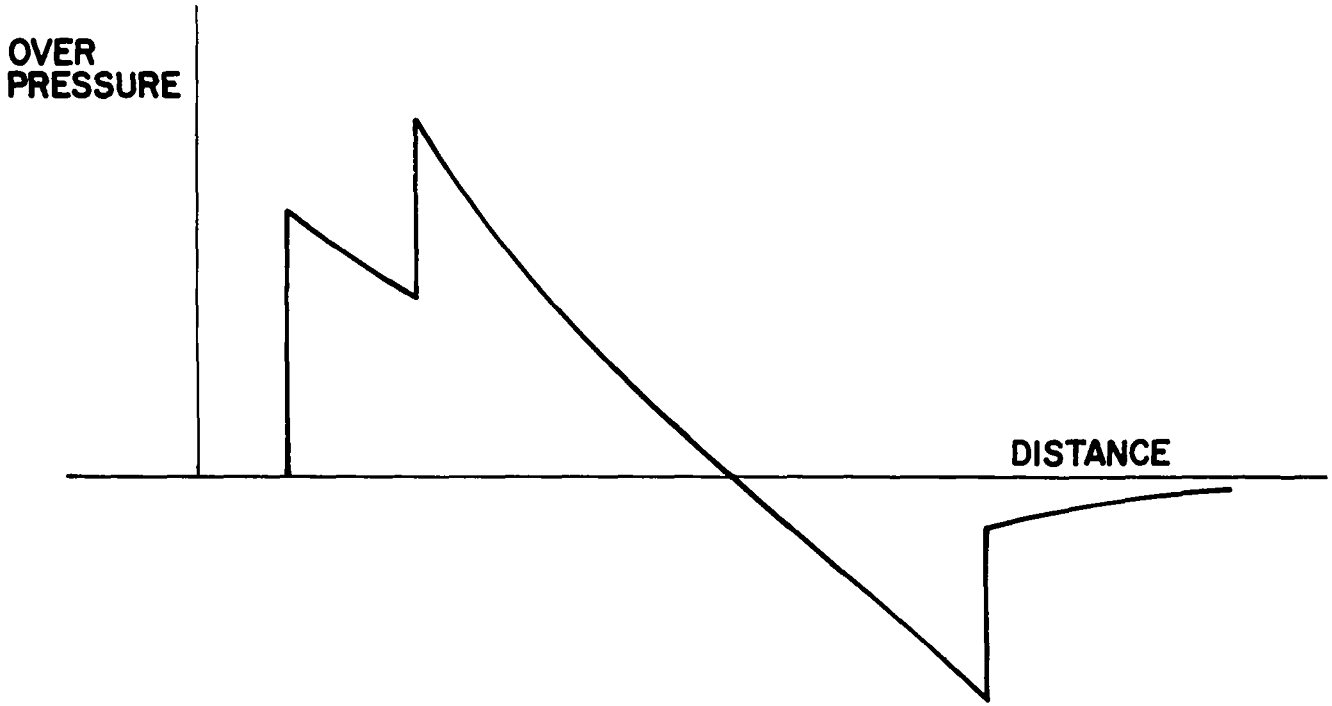


Fig. 8 Typical sonic boom signature

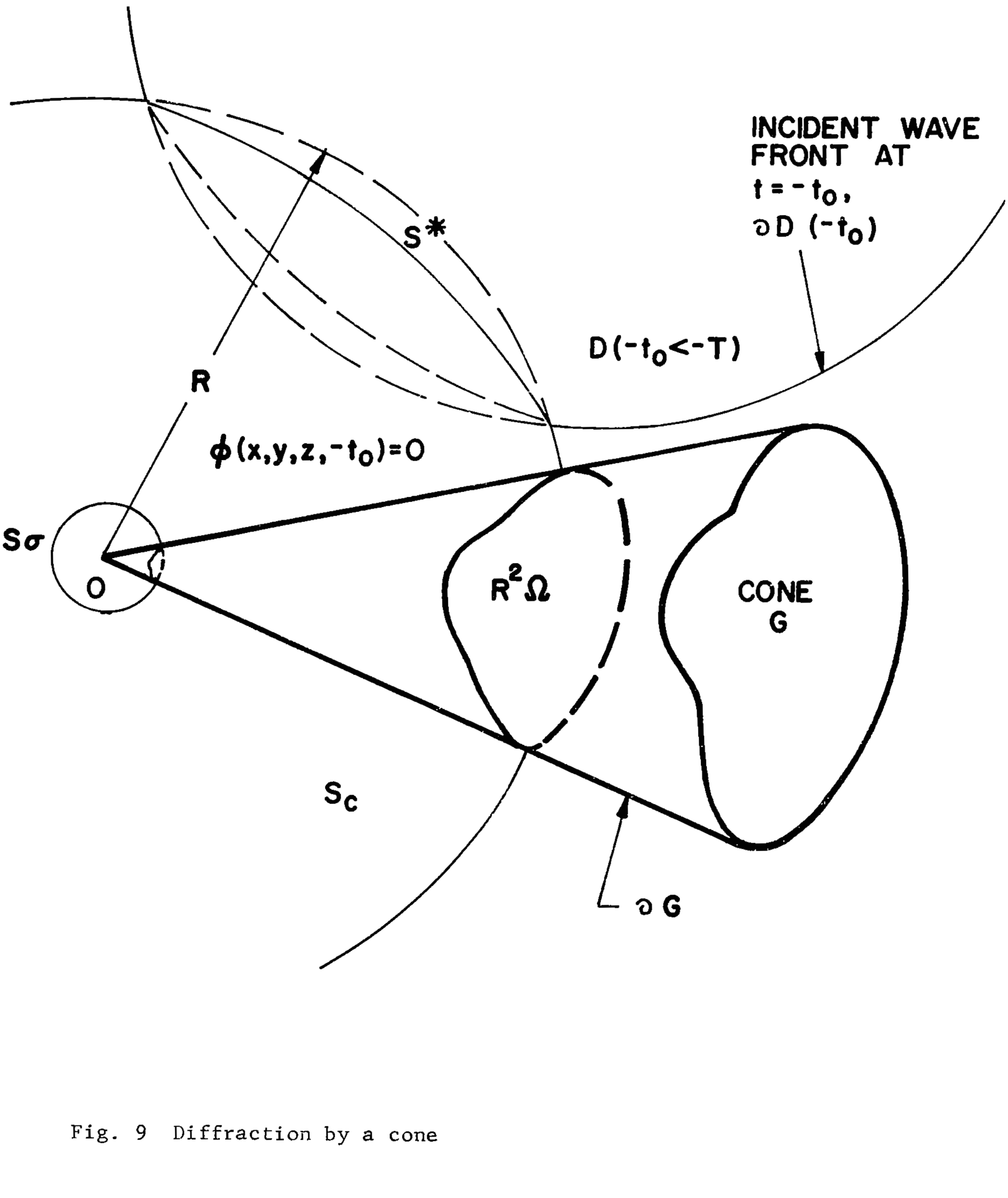


Fig. 9 Diffraction by a cone

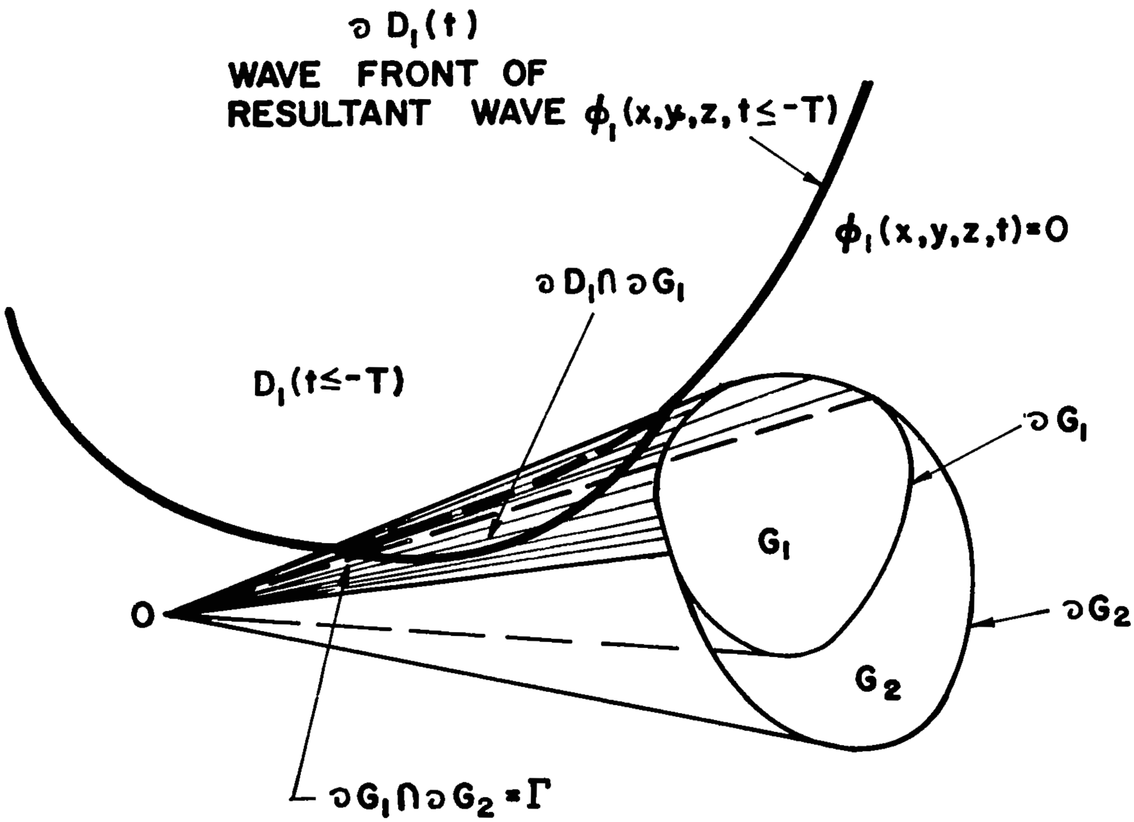


Fig. 10 Comparison of two cones

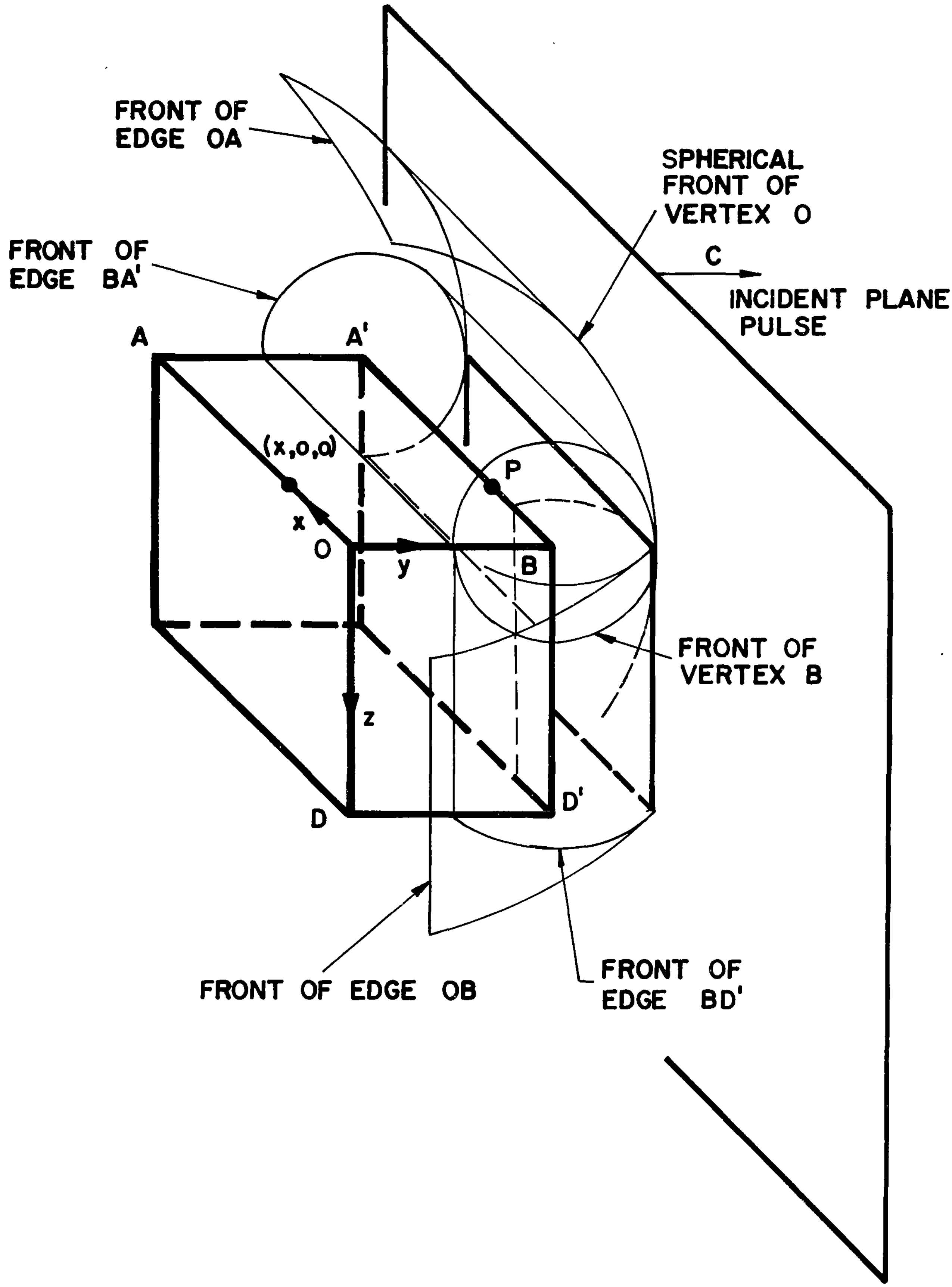


Fig. 11 Incident of a plane pulse on a block at instant $t > L/C$ ($OB=L$, $OA=OD=W$; also shown are the diffracted wave fronts around edges OA,OD,BA',BD' and around vertices O and B).

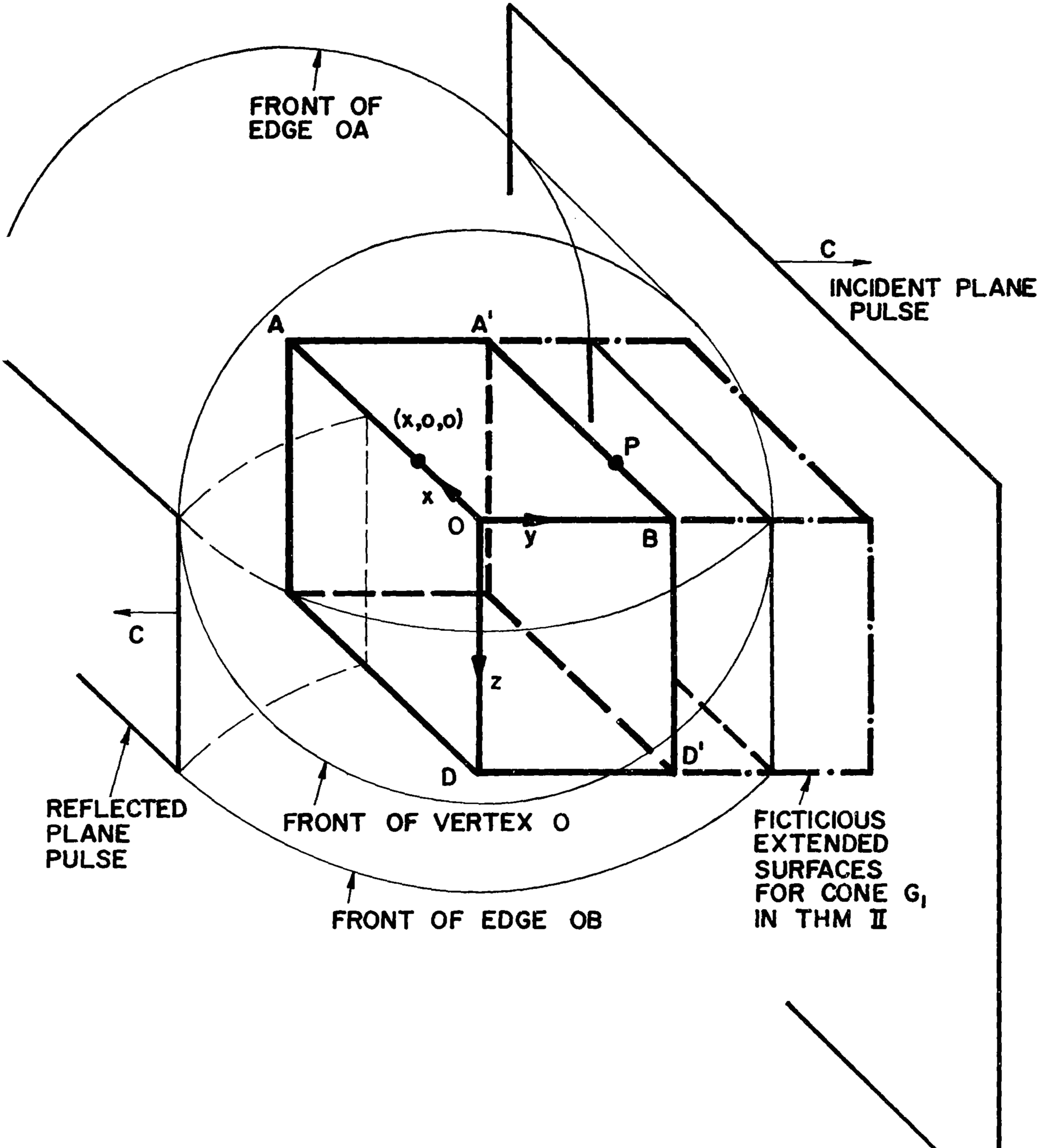


Fig. 12 Incident of a plane pulse on a block with extended fictitious surfaces at instant t >L/C (also shown are the diffracted wave fronts around edges OA,OD and vertex O)

APPENDIX

NUMERICAL PROGRAMS

Four complete sets of listing programs for CDC-6600 are given in the appendix. They are called first program (A), first program (B), second program and third program. Their input and output formats and the operating instructions are presented as follows:

First Program (A): determination of eigenvalues and eigenfunctions for odd function.

First Program (B): determination of eigenvalues and eigenfunctions for even function.

Input Definition

IIPP . a control constant
To calculate determinant and search for eigenvalues; let IIPP equal to any negative or zero integer.
To calculate determinant and coefficients of eigenfunction; let IIPP equal to any positive integer.

NMAX : for odd function NMAX=M_s; for even function NMAX=M_c

LMAX : for odd function LMAX=J_s; for even function LMAX=J_c

XLAM : λ
If IIPP $\leq$ 1; XLAM is only a searching eigenvalue. More accurate value for the roots of $\Delta(\lambda)=0$ can be obtained by interpolation. Be careful not to overlook multiple root across which $\Delta(\lambda)$ may not change sign.
If IIPP > 1; XLAM is an eigenvalue as located by the preceding method, and ready to calculate the coefficients of eigenfunctions.

Input

Card 1; IIPP, NMAX, LMAX (Format 315)

Card 2; XLAM (Format F15.0)

Output and Definition

NMAX, LMAX : input data previously described

IIMAX : number of linear homogeneous equations, for the odd solutions; IIMAX=LMAX+NMAX; for the even solution: IIMAX=LMAX+NMAX+2

ALPHA : 2α

Output and Definition

PPHI : $\Phi=2(\pi-\alpha)$

XNUI : $\nu_o = \pi/\Phi$

XMU : $\mu_o=\cos(\beta)$ where β is the latitude of upper surface

XLAM : previously described

DET : the value of determinant for a given λ; $\Delta c(\lambda)=|a_{ij}|$

BMIN(J), DMIN(L) : coefficients in the series representation of eigenfunctions,

for even function — First M_c+1 output are Am's; Next J_c+1 output are Cj's

for odd function — First M_s are Bm's; Next J_s are Dj's

These Am's, Cj's, Bm's, Dj's are defined by eq. (25).

Second Program: Determination of the coefficient K_λ of the eigenfunction expansion from the initial data.

Input Definition

II : control constant
II=1; to calculate K_λ for odd function
II equal to any integer other than one; to calculate K_λ for even function

NMAX, LMAX, XLAM : previously described

BMIN(J), DMIN(L) : these are the coefficients calculated from program one (A) for odd function or program one (B) for even function

Input

II=1, NMAX, LMAX	(3I5)
Set 1 { XLAM }	(F15.0)
Set 1 { BMIN(J) }	(5F15.0)
Set 1 { DMIN(L) }	(5F15.0)
Set { }	
.	
.	
.	
END FILE	

Input

II=2, NMAX, LMAX

{ }

{ }

.
.
.

{ }

END FILE

Note that the bracket indicates a set of data composed of a given λ and its corresponding coeffcients BMIN(J) and DMIN(L). Note also that these sets are then listed into a group of odd functions and a group of even functions separated by an end file card.

This use of the end file card facilitates the orderly calculations of all odd functions first and then all even functions.

Output and Definition

XLAM NMAX, LMAX, IIMAX, ALPHA, PPHI, XNUI, XMU BMIN(J) DMIN(L)	These are input data which are listed for identification and verification
K(Lamda)	K_λ the coefficient of the eigenfunction expansion from the initial data

Remark: the subroutine FSF and subroutine CON2D prepare the boundary data on the sonic sphere for the problem described in 6. For a different problem only these two subroutines should be changed.

Third Program: determination of the pressure distribution on the surfaces.

Input same as the second program, except the first card of each set of input data contains both λ and its corresponding K_λ which was calculated from the second program.

Output and Definition

XLAM : λ

AAO : defined in eq. (39)

K(Lamda) : K_λ

PHI, XMU	:	φ where θ and φ are spherical coordinates $\mu = \cos\theta$ for points on the surfaces
P_1	:	pressure on surface OAB on figure (4) where $\theta = \beta$, $\mu = \cos\beta$; $\varphi_N = \pi - \alpha + \frac{2\alpha}{4}(N-1)$, $N = 1,2,\ldots 5$
P_2	:	pressure on surface OAD where $\varphi = \frac{3\pi}{4}$; $\theta_N = \frac{\pi}{2} + \frac{\pi}{8}(N-1)$ where $N = 1,2,3,4,5$
P_3	:	pressure on surface OBD where $\varphi = \frac{-3\pi}{4}$; $\theta_N = \frac{\pi}{2} + \frac{\pi}{8}\cdot(N-1)$, $N = 1,2,\ldots 5$

Note: P_1, P_2, P_3 are defined in eq. (34)

Output Format

Odd function

NMAX, LMAX, IIMAX,
ALPHA, PPHI, XNUI, XMU

Set 1 { XLAM, AAO, K(Lamda)
BMIN(J)
DMIN(J) }

Set 2 { }

.
.
.

Even function

NMAX, LMAX, IIMAX,
ALPHA, PPHI, XNUI, XMU

Sect. 1 { }

Sect. 2 { }

.
.
.

At the last page of the output, a table of PHI, P_1, XMU, P_2, P_3 is tabulated in five columns with Format (2E16.5, 15X, 3E16.5).

Remark: In this program, there are 10 $Z(\zeta)$'s where ζ's are 0.111, 0.211, to 0.9989. In case of a different increment of ζ, the subroutine T102 must then use a different value of XTEST.

```
C    FIRST PROGRAM (A)
C   ODD FUNCTION TO CALCULATE DETERMINANT AND COEFFICIENTS
      DIMENSION AEL(50,50),B(50,50)
      DIMENSION ADET(50,50),EEEE(50,50)
      DIMENSION SUMA(101),SUMB(101),BMIN(30),DMIN(30),SBMIN(30),
     1SDMIN(30)
      LLMAX=50
      READ(5,5) IIPP,NMAX,LMAX
    5 FORMAT(3I5)
      IIMAX=LMAX+NMAX
      ITTEM=IIMAX
      PI=3.1415926
      ALPHA=PI/2.
      PPHI=2.*(PI-ALPHA/2.)
      XNU1=PI/PPHI
      INTERV=30
      LEP=LMAX
      JEP=NMAX
      U0=0.
      XMU=0.
      XMUFIX=XMU
      WRITE(6,502) NMAX,LMAX,IIMAX,ALPHA,PPHI,XNU1,XMU
  502 FORMAT(     *   NMAX=*,I5,5X,*LMAX=*,I5,5X,*IIMAX=*,I5/5X,            *
     1ALPHA=*,E12.5,5X,*PPHI=*,E12.5,5X,*XNU1=*,E12.5,5X,*XMU=*,E12.5)
  400 READ(5,31) XLAM
   31 FORMAT(F14.0)
      IF(ENDFILE 5) 5559,5556
 5556 CONTINUE
C    CALCULATION OF ELEMENTS OF DETERMINANT
      DO 303 J=1,IIMAX
      DO 300 K=1,IIMAX
      IF(J-NMAX   ) 10,10,200
   10 IF(K-NMAX   ) 11,11,30
   11 CONTINUE
      EEL=1.
   14 IF(J-K) 16,15,16
   15 DELTA=1.
      GO TO 17
   16 DELTA=0.
   17 SS=K
      CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      AEL(J,K)=DP*EEL*DELTA
      GO TO 300
   30 SS=(2*(K-NMAX)-1)*XNU1
      CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      LH=K-NMAX
      LI=J
      CALL XLLLL(LH,LI,ALPHA,XNU1,XII)
      AEL(J,K)=DP*XII
      GO TO 300
  200 IF(K-NMAX   ) 212,212,230
  212 LH=J-NMAX
      LI=K
      CALL XLLLL(LH,LI,ALPHA,XNU1,XII)
      SS=K
      CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      AEL(J,K)=PP*XII
      GO TO 300
```

```
  230 IF(J-K) 233,232,233
  232 DELTA=1.
      GO TO 234
  233 DELTA=0.
  234 CONTINUE
      EEL=1.
  237 SS=(2 *(K-NMAX)-1)*XNU1
      CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      AEL(J,K)=-(1.-ALPHA/(2.*PI))*EEL*PP*DELTA
  300 CONTINUE
  303 CONTINUE
      DO 307 I=1,ITTEM
      DO 306 J=1,ITTEM
      B(I,J)=0.
      EEEE(I,J)=AEL(I,J)
  306 CONTINUE
  307 CONTINUE
      CALL LEQ(AEL,B,IIMAX,0,50,50,DET)
      WRITE(6,32) XLAM ,DET
   32 FORMAT(*       XLAM=*,E15.8,5X,*DET=*,E15.8)
      IF (IIPP) 400,400,1002
 1002 CONTINUE
C   CALCULATION OF COFACTORS
      DO 903 I=1,ITTEM
      DO 902 J=1,ITTEM
      ADET(I,J)=EEEE(I,J)
  902 CONTINUE
  903 CONTINUE
      ICORD=1
      DO 511 J=1,JEP
      JCORD=J
      CALL COFACT(ADET,ITTEM,ICORD,JCORD,DETCOE)
      BMIN(J)=-(-1.)**JCORD*DETCOF
  511 CONTINUE
      DO 512 L=1,LEP
      JCORD=L+JEP
      CALL COFACT(ADET,ITTEM,ICORD,JCORD,DETCOE)
      DMIN(L)=-(-1.)**JCORD*DETCOE
  512 CONTINUE
      DELU=(1.-UO)/INTERV
      ISTEP=INTERV+1
      DO 611 I=1,ISTEP
      XMM=(I-1)*DELU
      SUM10=0.
      DO 601 J=1,JEP
      SS=J
      CALL PPDD(SS,XLAM,XMM, LLMAX,PP,DP)
      SUM10=SUM10+(BMIN(J)*PP)**2
  601 CONTINUE
      SUMA(I)=SUM10
  611 CONTINUE
      DELV=(1.+U0)/INTERV
      DO 711 I=1,ISTEP
      XMM=(I-1)*DELV
      SUM11=0.
      DO 603 L=1,LEP
      SS=(2*L-1)*XNU1
      CALL PPDD(SS,XLAM, XMM,LLMAX,PP,DP)
```

```
      SUM11=SUM11+(DMIN(L)*PP)**2
  603 CONTINUE
      SUMB(I)=SUM11
  604 FORMAT(15X,I5,3E20.8)
  711 CONTINUE
      TOSUM1=SUMA(1)
      TOSUM2=SUMB(1)
      DO 622 I=2,INTERV
      COEF=3.+(-1.)**I
      TOSUM1=TOSUM1+COEF*SUMA(I)
      TOSUM2=TOSUM2+COEF*SUMB(I)
  622 CONTINUE
      TOSUM1=(TOSUM1+SUMA(ISTEP))*DELU/3.
      TOSUM2=(TOSUM2+SUMB(ISTEP))*DELV/3.
      ODDN=TOSUM1*PI/2.+PPHI*TOSUM2/4.
  641 FORMAT(9X,3E20.8)
      DO 633 I=1,JEP
      SBMIN(I)=BMIN(I)/SQRT(ODDN)
      BMIN(I)=SBMIN(I)
  633 CONTINUE
C     CALCULATION OF NORMALIZED COFACTORS --COEFFICIENT OF EIGEN FUNCTION
      WRITE(6,6)
    6 FORMAT(//* COEFFICIENTS OF  ODD FUNCTION*)
  304 FORMAT(5E20.8/(5X,5E20.8))
      WRITE(6,304) ( BMIN(I),I=1,JEP)
      DO 634 L=1,LEP
      SDMIN(L)=DMIN(L)/SQRT(ODDN)
      DMIN(L)=SDMIN(L)
  634 CONTINUE
      WRITE(6,304) ( DMIN(L),L=1,LEP)
      GO TO 400
 5559 STOP
      END
```

```
      SUBROUTINE XLLLL(LH,LI,ALPHA,XNU1,XII)
      XH=LH
      XI=LI
      EPSIL=1.E-09
      PI=3.1415926
      PPHI=PI/XNU1
      VC=(2.*XH-1.)*XNU1
      TEST=ABS(VC-XI)-(1.E-08)
      IF(TEST)20,20,10
   10 VAL=XI/(2.*XNU1)
      VAA=VAL+0.5
      J2V0=VAA+EPSIL
      IF(ABS(J2V0-VAA)-2.*EPSIL)14,11,11
   11 JZ=LH+1
      XII=2.*(-1.)**JZ*COS(VAL*PI)*XI/(PI*(VC**2-XI**2))
      GO TO 30
   14 XII=0
      GO TO 30
   20 IF(LI)21,21,23
   21 EEL=2.
      GO TO 24
   23 EEL=1.
   24 XII=PPHI/(2.*PI)*EEL
   30 RETURN
      END
```

```
      SUBROUTINE PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      XNU=SS
      DD=1.
      TEM1=0.
      TEM2=0.
      ZX=(1.-XMU)/2.
      IF(ABS(ZX)-0.000001) 35,35,31
   31 CONTINUE
      DO 40 L=1,LLMAX
      ZL=L
      TEM1=TEM1+DD*ZX**(L-1)
      TEM2=TEM2+(ZL-1.)*DD*ZX**(L-2)
      DD=DD*(ZL-XLAM-1.)*(ZL+XLAM)/(ZL**2+ZL*XNU)
   40 CONTINUE
      ARTFL=((1.-XMU)/(1.+XMU))**(XNU/2.)
      PP=ARTFL*TEM1
      DP=-XNU*PP/(1.-XMU**2)-0.5*ARTFL*TEM2
      GO TO 41
   35 PP=0.
      DP=0.
   41 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE LEQ(A,B,NEQS,NSOLNS,IA,IB,DET)
CLEQ  LINEAR EQUATIONS SOLUTIONS  FORTRAN II VERSION                    LEQF0020
C     SOLVE A SYSTEM OF LINEAR EQUATIONS OF THE FORM AX=B BY A MODIFIED LEQF0030
C     GAUSS ELIMINATION SCHEME                                          LEQF0040
C                                                                       LEQF0050
C     NEQS = NUMBER OF EQUATIONS AND UNKNOWNS                           LEQF0060
C     NSOLNS = NUMBER OF VECTOR SOLUTIONS DESIRED                       LEQF0070
C     IA = NUMBER OF ROWS OF A AS DEFINED BY DIMENSION STATEMENT ENTRY  LEQF0080
C     IB = NUMBER OF ROWS OF B AS DEFINED BY DIMENSION STATEMENT ENTRY  LEQF0090
C     ADET = DETERMINANT OF A, AFTER EXIT FROM LEQ                      LEQF0100
C                                                                       LEQF0110
      DIMENSION A(IA,IA),B(IB,IB)                                       LEQF0120
      NSIZ = NEQS                                                       LEQF0130
      NBSIZ = NSOLNS                                                    LEQF0140
C     NORMALIZE EACH ROW BY ITS LARGEST ELEMENT. FORM PARTIAL DETERNT   LEQF0150
      DET=1.0                                                           LEQF0160
      DO 1 I=1,NSIZ                                                     LEQF0170
      BIG=A(I,1)                                                        LEQF0180
      IF(NSIZ-1)50,50,51                                                LEQF0190
   51 DO 2 J=2,NSIZ                                                     LEQF0200
      IF(ABSF(BIG)-ABSF(A(I,J))) 3,2,2                                  LEQF0210
 3    BIG=A(I,J)                                                        LEQF0220
 2    CONTINUE                                                          LEQF0230
      BG=1.0/BIG                                                        LEQF0240
      DO 4 J=1,NSIZ                                                     LEQF0250
4     A(I,J)=A(I,J)*BG                                                  LEQF0260
      DO 41 J=1,NBSIZ                                                   LEQF0270
41    B(I,J)=B(I,J)*BG                                                  LEQF0280
      DET=DET*BIG                                                       LEQF0290
 1    CONTINUE                                                          LEQF0300
C     START SYSTEM REDUCTION                                            LEQF0310
      NUMSYS=NSIZ-1                                                     LEQF0320
      DO 14 I=1,NUMSYS                                                  LEQF0330
C     SCAN FIRST COLUMN OF CURRENT SYSTEM FOR LARGEST ELEMENT           LEQF0340
C     CALL THE ROW CONTAINING THIS ELEMENT, ROW NBGRW                   LEQF0350
      NN=I+1                                                            LEQF0360
      BIG=A(I,I)                                                        LEQF0370
      NBGRW=I                                                           LEQF0380
      DO 5 J=NN,NSIZ                                                    LEQF0390
      IF(ABSF(BIG)-ABSF(A(J,I))) 6,5,5                                  LEQF0400
 6    BIG=A(J,I)                                                        LEQF0410
      NBGRW=J                                                           LEQF0420
 5    CONTINUE                                                          LEQF0430
      BG=1.0/BIG                                                        LEQF0440
C     SWAP ROW I WITH ROW NBGRW UNLESS I=NBGRW                          LEQF0450
      IF(NBGRW-I) 7,10,7                                                LEQF0460
C     SWAP A-MATRIX ROWS                                                LEQF0470
 7    DO 8 J=I,NSIZ                                                     LEQF0480
      TEMP=A(NBGPW,J)                                                   LEQF0490
      A(NBGRW,J)=A(I,J)                                                 LEQF0500
 8    A(I,J)=TEMP                                                       LEQF0510
      DET = -DET                                                        LEQF0520
C     SWAP B-MATRIX ROWS                                                LEQF0530
      DO 9 J=1,NBSIZ                                                    LEQF0540
      TEMP=B(NBGRW,J)                                                   LEQF0550
      B(NBGRW,J)=B(I,J)                                                 LEQF0560
 9    B(I,J)=TEMP                                                       LEQF0570
C     ELIMINATE UNKNOWNS FROM FIRST COLUMN OF CURRENT SYSTEM            LEQF0580
```

```
 10    DO 13 K=NN,NSIZ
C      COMPUTE PIVOTAL MULTIPLIER
       PMULT=-A(K,I)*BG
C      APPLY PMULT TO ALL COLUMNS OF THE CURRENT A-MATRIX ROW
       DO 11 J=NN,NSIZ
 11    A(K,J)=PMULT*A(I,J)+A(K,J)
C      APPLY PMULT TO ALL COLUMNS OF MATRIX B
       DO 12 L=1,NBSIZ
 12    B(K,L)=PMULT*B(I,L)+B(K,L)
 13    CONTINUE
 14    CONTINUE
C      DO BACK SUBSTITUTION
C      WITH B-MATRIX COLUMN = NCOLB
   50 DO 15 NCOLB=1,NBSIZ
C      DO FOR ROW = NROW
       DO 19 I=1,NSIZ
       NROW=NSIZ+1-I
       TEMP=0.0
C      NUMBER OF PREVIOUSLY COMPUTED UNKNOWNS = NXS
       NXS=NSIZ-NROW
C      ARE WE DOING THE BOTTOM ROW
       IF(NXS) 16,17,16
C      NO
 16    DO 18 K=1,NXS
       KK=NSIZ+1-K
 18    TEMP=TEMP+B(KK,NCOLB)*A(NROW,KK)
 17    B(NROW,NCOLB)=(B(NROW,NCOLB)-TEMP)/A(NROW,NROW)
C      HAVE WE FINISHED ALL ROWS FOR B-MATRIX COLUMN = NCOLB
 19    CONTINUE
C      YES
C      HAVE WE JUST FINISHED WITH B-MATRIX COLUMN NCOLB=NSIZ
 15    CONTINUE
C      YES
C      NOW FINISH COMPUTING THE DETERMINANT
       DO 20 I=1,NSIZ
 20    DET=DET*A(I,I)
C      WE ARE ALL DONE NOW
C      WHEW...
       RETURN
       END
```

```
      SUBROUTINE COFACT(ADET,IIMAX,ICORD,JCORD,DETCOE)
      DIMENSION ADET(50,50),BDET(50,50),CDET(50,50)
      M=1
      MM=1
   10 N=1
      MN=1
      IF(M-ICORD) 20,12,20
   12 MM=M+1
   20 IF(N-JCORD) 23,22,23
   22 MN=N+1
   23 BDET(M,N)=ADET(MM,MN)
      N=N+1
      MN=MN+1
      IF(N-IIMAX+1) 20,20,31
   31 M=M+1
      MM=MM+1
      IF(M-IIMAX+1) 10,10,35
35    ITERM=IIMAX-1
      DO 37 I=1,ITERM
      DO 36 J=1,ITERM
      CDET(I,J)=0.
   36 CONTINUE
   37 CONTINUE
      CALL LEQ(BDET,CDET,ITERM,0,50,50,DETCOE)
      RETURN
      END
```

```
C    FIRST PROGRAM(B)
C    EVEN FUNCTION TO CALCULATE DETERMINANT AND COEFICIENTS
       DIMENSION AEL(50,50),B(50,50)
       DIMENSION ADET (50,50),EEEE(50,50)
       DIMENSION SUMA(101),SUMB(101),BMIN(30),DMIN(30),SBMIN(30),
      1SDMIN(30)
       LLMAX=50
       READ(5,5) IIPP,NMAX,LMAX
     5 FORMAT(3I5)
       IIMAX=LMAX+NMAX+2
       ITTEM=IIMAX
       PI=3.1415926
       ALPHA=PI/2.
       PPHI=2.*(PI-ALPHA/2.)
       XNU1=PI/PPHI
       INTERV=30
       LEP=LMAX+1
       JEP=NMAX+1
       U0=0.
       XMU=0.
       XMUFIX=XMU
       WRITE(6,502) NMAX,LMAX,IIMAX,ALPHA,PPHI,XNU1,XMU
   502 FORMAT(     *   NMAX=*,I5,5X,*LMAX=*,I5,5X,*IIMAX=*,I5/5X,            *
      1ALPHA=*,E12.5,5X,*PPHI=*,E12.5,5X,*XNU1=*,E12.5,5X,*XMU=*,E12.5)
   400 READ(5,31) XLAM
    31 FORMAT(F14.0)
       IF(ENDFILE 5) 5559,5556
  5556 CONTINUE
C    CALCULATION OF ELEMENTS OF DETERMINANT
       DO 303 J=1,ITTEM
       DO 300 K=1,ITTEM
       IF(J-NMAX-1) 10,10,200
    10 IF(K-NMAX-1) 11,11,30
    11 IF(J-1) 12,12,13
    12 EEL=2.
       GO TO 14
    13 EEL=1.
    14 IF(J-K) 16,15,16
    15 DELTA=1.
       GO TO 17
    16 DELTA=0.
    17 SS=K-1
       CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
       AEL(J,K)=DP*EEL*DELTA
       GO TO 300
    30 SS=2*(K-NMAX-2) *XNU1
       CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
       LH=K-NMAX-2
       LI=J-1
       CALL XLLLL(LH,LI,ALPHA,XNU1,XII)
       AEL(J,K)=DP*XII
       GO TO 300
   200 IF(K-NMAX-1) 212,212,230
   212 LH=J-NMAX-2
       LI=K-1
       CALL XLLLL(LH,LI,ALPHA,XNU1,XII)
       SS=K-1
       CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
```

```
      AEL(J,K)=PP*XII
      GO TO 300
  230 IF(J-K) 233,232,233
  232 DELTA=1.
      GO TO 234
  233 DELTA=0.
  234 IF(J-NMAX-2) 236,235,236
  235 EEL=2.
      GO TO 237
  236 EEL=1.
  237 SS=2 *(K-NMAX-2)*XNU1
      CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      AEL(J,K)=-(1.-ALPHA/(2.*PI))*EEL*PP*DELTA
  300 CONTINUE
  303 CONTINUE
      DO 307 I=1,ITTEM
      DO 306 J=1,ITTEM
      B(I,J)=0.
      EEEE(I,J)=AEL(I,J)
  306 CONTINUE
  307 CONTINUE
      CALL LEQ(AEL,B,ITTEM,0,50,50,DET)
      WRITE(6,32) XLAM ,DET
   32 FORMAT(*      XLAM=*,E15.8,5X,*DET=*,E15.8)
      IF(IIPP) 400,400,1002
 1002 CONTINUE
C    CALCULATION OF COFACTORS
      DO 903 I=1,ITTEM
      DO 902 J=1,ITTEM
      ADET(I,J)=EEEE(I,J)
  902 CONTINUE
  903 CONTINUE
      ICORD=1
      DO 511 J=1,JEP
      JCORD=J
      CALL COFACT(ADET,ITTEM,ICORD,JCORD,DETCOE)
      BMIN(J)=-(-1.)**JCORD*DETCOE
  526 CONTINUE
  511 CONTINUE
      DO 512 L=1,LEP
      JCORD=L+JEP
      CALL COFACT(ADET,ITTEM,ICORD,JCORD,DETCOE)
      DMIN(L)=-(-1.)**JCORD*DETCOE
  536 CONTINUE
  512 CONTINUE
      DELU=(1.-U0)/INTERV
      ISTEP=INTERV+1
      DO 611 I=1,ISTEP
      XMM=(I-1)*DELU
      SS=0.
      CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)
      SUM10=2.*(BMIN(1)*PP)**2
      DO 601 J=2,JEP
      SS=J-1
      CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)
      SUM10=SUM10+(BMIN(J)*PP)**2
  601 CONTINUE
      SUMA(I)=SUM10
```

```
  611 CONTINUE
      DELV=(1.+U0)/INTERV
      DO 711 I=1,ISTEP
      XMM=(I-1)*DELV
      SS=0.
      CALL PPDD(SS,XLAM,XMM,LLMAX,PP,DP)
      SUM11=2.*(DMIN(1)*PP)**2
      DO 603 L=2,LEP
      SS=2.*(L-1)*XNU1
      CALL PPDD(SS,XLAM, XMM,LLMAX,PP,DP)
      SUM11=SUM11+(DMIN(L)*PP)**2
  603 CONTINUE
      SUMB(I)=SUM11
  711 CONTINUE
      TOSUM1=SUMA(1)
      TOSUM2=SUMB(1)
      DO 622 I=2,INTERV
      COFF=3.+(-1.)**I
      TOSUM1=TOSUM1+COFF*SUMA(I)
      TOSUM2=TOSUM2+COFF*SUMB(I)
  622 CONTINUE
      TOSUM1=(TOSUM1+SUMA(ISTEP))*DELU/3.
      TOSUM2=(TOSUM2+SUMB(ISTEP))*DELV/3.
      ODDN=TOSUM1*PI/2.+PPHI*TOSUM2/4.
C   CALCULATION OF NORMALIZED COFACTORS --COEFFICIENT OF EIGEN FUNCTION
      DO 633 I=1,JEP
      SBMIN(I)=BMIN(I)/SQRT(ODDN)
      BMIN(I)=SBMIN(I)
  633 CONTINUE
      WRITE(6,6)
    6 FORMAT( /* COEFFICIENTS OF EVEN FUNCTION*)
  304 FORMAT(5E20.8/(5X,5E20.8))
      WRITE(6,304)  (BMIN(I),I=1,JEP)
      DO 634 L=1,LEP
      SDMIN(L)=DMIN(L)/SQRT(ODDN)
      DMIN(L)=SDMIN(L)
  634 CONTINUE
      WRITE(6,304) (DMIN(L),L=1,LEP)
      WRITE(6,401)
  401 FORMAT(////)
      GO TO 400
 5559 STOP
      END
```

```
      SUBROUTINE XLLLL(LH,LI,ALPHA,XNU1,XII)
C
C
C   THIS SUBROUTINE IS FOR '''BETA-'''
C
C
      XH=LH
      XI=LI
      EPSIL=1.E-09
      PI=3.1415926
      PPHI=PI/XNU1
      VC=2.*XH*XNU1
      TEST=ABS(VC-XI)-(1.E-08)
      IF(TEST)20,20,10
   10 VAL=XI/(2.*XNU1)
      J2VO=VAL+EPSIL
      IF(ABS(J2VO-VAL)-2.*EPSIL)14,11,11
   11 JZ=LH+1
      XII=2.*(-1.)**JZ*SIN(VAL*PI)*XI/(PI*(VC**2-XI**2))
      GO TO 30
   14 XII=0
      GO TO 30
   20 IF(LI)21,21,23
   21 EEL=2.
      GO TO 24
   23 EEL=1.
   24 XII=PPHI/(2.*PI)*EEL
   30 RETURN
      END
```

```
      SUBROUTINE PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      XNU=SS
      DD=1.
      TEM1=0.
      TEM2=0.
      ZX=(1.-XMU)/2.
      IF(ABS(ZX)-0.000001) 35,35,31
   31 CONTINUE
      DO 40 L=1,LLMAX
      ZL=L
      TEM1=TEM1+DD*ZX**(L-1)
      TEM2=TEM2+(ZL-1.)*DD*ZX**(L-2)
      DD=DD*(ZL-XLAM     -1.)*(ZL+XLAM     )/(ZL**2+ZL*XNU)
   40 CONTINUE
       ARTFL=((1.-XMU)/(1.+XMU))**(XNU/2.)
      PP=ARTFL*TEM1
      DP=-XNU*PP/(1.-XMU**2)-0.5*ARTFL*TEM2
      GO TO 41
   35 PP=0.
      DP=0.
   41 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE LEQ(A,B,NEQS,NSOLNS,IA,IB,DET)
CLEQ  LINEAR EQUATIONS SOLUTIONS  FORTRAN II VERSION                    LEQF0020
C     SOLVE A SYSTEM OF LINEAR EQUATIONS OF THE FORM AX=B BY A MODIFIED LEQF0030
C     GAUSS ELIMINATION SCHEME                                          LEQF0040
C                                                                       LEQF0050
C     NEQS = NUMBER OF EQUATIONS AND UNKNOWNS                           LEQF0060
C     NSOLNS = NUMBER OF VECTOR SOLUTIONS DESIRED                       LEQF0070
C     IA = NUMBER OF ROWS OF A AS DEFINED BY DIMENSION STATEMENT ENTRY  LEQF0080
C     IB = NUMBER OF ROWS OF B AS DEFINED BY DIMENSION STATEMENT ENTRY  LEQF0090
C     ADET = DETERMINANT OF A, AFTER EXIT FROM LEQ                      LEQF0100
C                                                                       LEQF0110
      DIMENSION A(IA,IA),B(IB,IB)                                       LEQF0120
      NSIZ = NEQS                                                       LEQF0130
      NBSIZ = NSOLNS                                                    LEQF0140
C     NORMALIZE EACH ROW BY ITS LARGEST ELEMENT. FORM PARTIAL DETERNT   LEQF0150
      DET=1.0                                                           LEQF0160
      DO 1 I=1,NSIZ                                                     LEQF0170
      BIG=A(I,I)                                                        LEQF0180
      IF(NSIZ-1)50,50,51                                                LEQF0190
   51 DO 2 J=2,NSIZ                                                     LEQF0200
      IF(ABSF(BIG)-ABSF(A(I,J))) 3,2,2                                  LEQF0210
 3    BIG=A(I,J)                                                        LEQF0220
 2    CONTINUE                                                          LEQF0230
      BG=1.0/BIG                                                        LEQF0240
      DO 4 J=1,NSIZ                                                     LEQF0250
4     A(I,J)=A(I,J)*BG                                                  LEQF0260
      DO 41 J=1,NBSIZ                                                   LEQF0270
41    B(I,J)=B(I,J)*BG                                                  LEQF0280
      DET=DET*BIG                                                       LEQF0290
 1    CONTINUE                                                          LEQF0300
C     START SYSTEM REDUCTION                                            LEQF0310
      NUMSYS=NSIZ-1                                                     LEQF0320
      DO 14 I=1,NUMSYS                                                  LEQF0330
C     SCAN FIRST COLUMN OF CURRENT SYSTEM FOR LARGEST ELEMENT           LEQF0340
C     CALL THE ROW CONTAINING THIS ELEMENT, ROW NBGRW                    LEQF0350
      NN=I+1                                                            LEQF0360
      BIG=A(I,I)                                                        LEQF0370
      NBGRW=I                                                           LEQF0380
      DO 5 J=NN,NSIZ                                                    LEQF0390
      IF(ABSF(BIG)-ABSF(A(J,I))) 6,5,5                                  LEQF0400
 6    BIG=A(J,I)                                                        LEQF0410
      NBGRW=J                                                           LEQF0420
 5    CONTINUE                                                          LEQF0430
      BG=1.0/BIG                                                        LEQF0440
C     SWAP ROW I WITH ROW NBGRW UNLESS I=NBGRW                          LEQF0450
      IF(NBGRW-I) 7,10,7                                                LEQF0460
C     SWAP A-MATRIX ROWS                                                LEQF0470
 7    DO 8 J=I,NSIZ                                                     LEQF0480
      TEMP=A(NBGRW,J)                                                   LEQF0490
      A(NBGRW,J)=A(I,J)                                                 LEQF0500
 8    A(I,J)=TEMP                                                       LEQF0510
      DET = -DET                                                        LEQF0520
C     SWAP B-MATRIX ROWS                                                LEQF0530
      DO 9 J=1,NBSIZ                                                    LEQF0540
      TEMP=B(NBGRW,J)                                                   LEQF0550
      B(NBGRW,J)=B(I,J)                                                 LEQF0560
 9    B(I,J)=TEMP                                                       LEQF0570
C     ELIMINATE UNKNOWNS FROM FIRST COLUMN OF CURRENT SYSTEM            LEQF0580
```

```
 10     DO 13 K=NN,NSIZ                                              LEQF0590
C       COMPUTE PIVOTAL MULTIPLIER                                   LEQF0600
C       APPLY PMULT TO ALL COLUMNS OF THE CURRENT A-MATRIX ROW       LEQF0620
        PMULT=-A(K,I)*BG                                             LEQF0610
        DO 11 J=NN,NSIZ                                              LEQF0630
C       APPLY PMULT TO ALL COLUMNS OF MATRIX B                       LEQF0650
 11     A(K,J)=PMULT*A(I,J)+A(K,J)                                   LEQF0640
        DO 12 L=1,NBSIZ                                              LEQF0660
 12     B(K,L)=PMULT*B(I,L)+B(K,L)                                   LEQF0670
 13     CONTINUE                                                     LEQF0680
 14     CONTINUE                                                     LEQF0690
C       DO BACK SUBSTITUTION                                         LEQF0700
C       WITH B-MATRIX COLUMN = NCOLB                                 LEQF0710
   50   DO 15 NCOLB=1,NBSIZ                                          LEQF0720
C       DO FOR ROW = NROW                                            LEQF0730
        DO 19 I=1,NSIZ                                               LEQF0740
        NROW=NSIZ+1-I                                                LEQF0750
        TEMP=0.0                                                     LEQF0760
C       NUMBER OF PREVIOUSLY COMPUTED UNKNOWNS = NXS                 LEQF0770
        NXS=NSIZ-NROW                                                LEQF0780
C       ARE WE DOING THE BOTTOM ROW                                  LEQF0790
        IF(NXS) 16,17,16                                             LEQF0800
C       NO                                                           LEQF0810
 16     DO 18 K=1,NXS                                                LEQF0820
        KK=NSIZ+1-K                                                  LEQF0830
 18     TEMP=TEMP+B(KK,NCOLB)*A(NROW,KK)                             LEQF0840
 17     B(NROW,NCOLB)=(B(NROW,NCOLB)-TEMP)/A(NROW,NROW)              LEQF0850
C       HAVE WE FINISHED ALL ROWS FOR B-MATRIX COLUMN = NCOLB        LEQF0860
 19     CONTINUE                                                     LEQF0870
C       YES                                                          LEQF0880
C       HAVE WE JUST FINISHED WITH B-MATRIX COLUMN NCOLB=NSIZ        LEQF0890
 15     CONTINUE                                                     LEQF0900
C       YES                                                          LEQF0910
C       NOW FINISH COMPUTING THE DETERMINANT                         LEQF0920
        DO 20 I=1,NSIZ                                               LEQF0930
C       WE ARE ALL DONE NOW                                          LEQF0950
C       WHEW...                                                      LEQF0960
 20     DET=DET*A(I,I)                                               LEQF0940
        RETURN                                                       LEQF0970
        END                                                          LEQF0980
```

```
      SUBROUTINE COFACT(ADET,IIMAX,ICORD,JCORD,DETCOE)
      DIMENSION ADET(50,50),BDET(50,50),CDET(50,50)
      M=1
      MM=1
   10 N=1
      MN=1
      IF(M-ICORD) 20,12,20
   12 MM=M+1
   20 IF(N-JCORD) 23,22,23
   22 MN=N+1
   23 BDET(M,N)=ADET(MM,MN)
      N=N+1
      MN=MN+1
      IF(N-IIMAX+1) 20,20,31
   31 M=M+1
      MM=MM+1
      IF(M-IIMAX+1) 10,10,35
   35 ITERM=IIMAX-1
      DO 37 I=1,ITERM
      DO 36 J=1,ITERM
      CDET(I,J)=0.
   36 CONTINUE
   37 CONTINUE
      CALL LEQ(BDET,CDET,ITERM,0,50,50,DETCOE)
      RETURN
      END
```

```
C    SECOND PROGRAM
C    DETERMINATION OF THE COEFFICIENTS OF THE EIGENFUNCTION EXPANSION FROM
C    THE INITIAL DATA
      DIMENSION F(70),FUP(70,70),FBT(70,70),SFL(70,70)
      DIMENSION SUMJ(70),SUML(70),BMIN(70),DMIN(70)
      LLMAX=50
      IDIM=70
      MAXU1=60
      MAXTOP=60
      MAXU2=60
      MAXBOT=60
 1000 READ(5,1001) II,NMAX,LMAX
 1001 FORMAT(3I5)
      IF(ENDFILE 5) 9999,1002
 1002 CONTINUE
      IF(II .EQ. 1) 101,103
C
C
C   ODD FUNCTION
  101 IIMAX=LMAX+NMAX
      WRITE(6,1003)
 1003 FORMAT(1H1*   ODD FUNCTION*)
      NCALL=NMAX
      LCALL=LMAX
      GO TO 104
C
C    EVEN FUNCTION
  103 IIMAX=LMAX+NMAX+2
      WRITE(6,1004)
 1004 FORMAT(1H1,*   EVEN FUNCTION*)
      NCALL=NMAX+1
      LCALL=LMAX+1
  104 CONTINUE
       PI=3.1415926
      ALPHA=PI/2.
      XMU=0.
      PPHI=2.*(PI-ALPHA/2.)
      XNU1=PI/PPHI
      HALPHI=PPHI/2.
      EPS    =0.000001
      WRITE(6,502) NMAX,LMAX,IIMAX,ALPHA,PPHI,XNU1,XMU
  502 FORMAT(     *   NMAX=*,I5,5X,*LMAX=*,I5,5X,*IIMAX=*,I5/5X,          *
     1ALPHA=*,E12.5,5X,*PPHI=*,E12.5,5X,*XNU1=*,E12.5,5X,*XMU=*,E12.5)
  400 READ(5,100)XLAM
      IF(ENDFILE 5) 1000,1111
 1111 CONTINUE
  100 FORMAT(5F15.0)
      READ(5,100)   (BMIN(I),I=1,NCALL)
      READ(5,100)   (DMIN(L),L=1,LCALL)
      U0=0.
      MAXPUS=MAXTOP+1
      INDEX=1
      WRITE(6,32) XLAM
   32 FORMAT(     *   XLAM=*,E15.8)
      WRITE(6,503)
  503 FORMAT(//*   COEFICIENTS OF EIGENFUNCTION*)
      WRITE(6,304) (BMIN(I),I=1,NCALL)
      WRITE(6,304) (DMIN(L),L=1,LCALL)
```

```
  304 FORMAT(5E20.8/(5X,5E20.8))
      DELU1=(1.-U0-2.*EPS   )/MAXU1
      MAXUST=MAXU1+1
      UK=U0+EPS
      DO 5 I=1,MAXUST
      TOTARG=0.
      CALL FFF(UK,INDEX,MAXTOP,F,II,PPHI,IDIM)
      DO 4 J=1,MAXPUS
      FUP(I,J)=F(J)
    4 CONTINUE
      IROW=I
      DO 13 J=1,NCALL
      IF(II .EQ. 1) 111,113
C
C   ODD FUNCTION
  111 CONTINUE
      XVAL=J
      CALL SININT(FUP,XVAL,IROW,MAXTOP,PI,0,IDIM ,TRIINT)
      GO TO 114
C
C   EVEN FUNCTION
  113 CONTINUE
      XVAL=J-1
      CALL COSINT(FUP,XVAL,IROW,MAXTOP,PI,0,IDIM ,TRIINT)
  114 CONTINUE
      SS=XVAL
      CALL PPDD(SS,XLAM,UK,LLMAX,PP,DP)
      TOTARG=TOTARG+BMIN(J)*PP*TRIINT
   13 CONTINUE
      SUMJ(I)=TOTARG
      UK=UK+DELU1
    5 CONTINUE
      TOTEM1=SUMJ(1)
      DO 19 I=2,MAXU1
      COEF=3.+(-1.)**I
      TOTEM1=TOTEM1+COEF*SUMJ(I)
   19 CONTINUE
      TOTEM1=(TOTEM1+SUMJ(MAXUST))*DELU1/3.
      MAXPUS=MAXPOT+1
      INDEX=2
      DELU2=(1.+U0-2.*EPS   )/MAXU2
      MAXUSM=MAXU2+1
      UK=-1.+EPS
      DO 9 I=1,MAXUSM
      TOTARG=0.
      CALL FFF(UK,INDEX,MAXBOT,F,II,PPHI,IDIM)
      DO 8 J=1,MAXPUS
      FBT(I,J)=F(J)
    8 CONTINUE
      IROW=I
      DO 23 L=1,LCALL
      IF(II .EQ. 1) 121,123
C
C  ODD FUNCTION
  121 CONTINUE
      XVAL=(2*L-1)*XNU1
      CALL SININT(FBT,XVAL,IROW,MAXBOT,HALPHI,0,IDIM ,TRIINT)
      GO TO 124
```

```
C
C   EVEN FUNCTION
  123 CONTINUE
      XVAL=2*(L-1)*XNU1
      CALL COSINT(FBT,XVAL,IROW,MAXBOT,HALPHI,0,IDIM ,TRIINT)
  124 CONTINUE
      SS=XVAL
       UF=-UK
      CALL FPDD(SS,XLAM,UF,LLMAX,PP,DP)
      TOTARG=TOTARG+DMIN(L)*PP*TRIINT
   23 CONTINUE
      SUML(I)=TOTARG
      UK=UK+DELU2
    9 CONTINUE
      TOTEM2=SUML(1)
      DO 29 I=2,MAXU2
      COEF=3.+(-1.)**I
      TOTEM2=TOTEM2+COEF*SUML(I)
   29 CONTINUE
      TOTEM2=(TOTEM2+SUML(MAXUSM))*DELU2/3.
      EEEM=TOTEM1+TOTEM2
      WRITE(6,11)EEEM
   11 FORMAT( /20X,*K(LAMDA)=*,E15.8)
      WRITE(6,401)
  401 FORMAT(////)
      GO TO 400
 9999 STOP
      END
```

```
      SUBROUTINE SININT(FP,XVAL,IROW,MAXST,UPLIM,BOTLIM,IDIM,TRIINT)
C    FILON'S METHOD FOR THE NUMERICAL EVALUATION OF TRIGONAMETRICAL
C    INTEGRALS--(INTEGRAND=FP(P)*SIN(X*P))
      DIMENSION FP(IDIM,IDIM)
      HH=(UPLIM-BOTLIM)/MAXST
      S2S=0.5*FP(IROW,1)*SIN(XVAL*BOTLIM)
      DO 14 J=3,MAXST,2
      P=BOTLIM+(J-1)*HH
      S2S=S2S+FP(IROW,J)*SIN(XVAL*P)
   14 CONTINUE
      J=MAXST+1
      S2S=S2S+0.5*FP(IROW,MAXST+1)*SIN(XVAL*UPLIM)
      S2SM=0.
      DO 16 J=2,MAXST,2
      P=BOTLIM+(J-1)*HH
      S2SM=S2SM+FP(IROW,J)*SIN(XVAL*P)
   16 CONTINUE
      THE=XVAL*HH
      IF(THE-0.2) 25,21,21
   21 ALPHA=(THE**2+THE*SIN(THE)*COS(THE)-2.*SIN(THE)**2)/THE**3
      BETA=2.*(THE*(1.+COS(THE)**2)-2.*SIN(THE)*COS(THE))/THE**3
      GARM=4.*(SIN(THE)-THE*COS(THE))/THE**3
      GO TO 31
   25 ALPHA=2.*THE**3/45.-2.*THE**5/315.+2.*THE**7/4725.
      BETA=2./3.+2.*THE**2/15.-4.*THE**4/105.+2.*THE**6/567.
      GARM=4./3.-2.*THE**2/15.+THE**4/210.-THE**6/11340.
   31 FA=FP(IROW,1)
      FB=FP(IROW,MAXST+1)
      TRIINT=HH*(-ALPHA*(FB*COS(XVAL*UPLIM)-FA*COS(XVAL*BOTLIM))+BETA*
     1S2S+GARM*S2SM)
      RETURN
      END
```

```
      SUBROUTINE COSINT(FP,XVAL,IROW,MAXST,UPLIM,BOTLIM,IDIM,TRIINT)
C     FILON'S METHOD FOR THE NUMERICAL EVALUATION OF TRIGONAMETRICAL
C     INTEGRALS--(INTEGRAND=FP(P)*COS(X*P))
      DIMENSION FP(IDIM,IDIM)
      HH=(UPLIM-BOTLIM)/MAXST
      S2S=0.5*FP(IROW,1)*COS(XVAL*BOTLIM)
      DO 14 J=3,MAXST,2
      P=BOTLIM+(J-1)*HH
      S2S=S2S+FP(IROW,J)*COS(XVAL*P)
   14 CONTINUE
      J=MAXST+1
      S2S=S2S+0.5*FP(IROW,MAXST+1)*COS(XVAL*UPLIM)
      S2SM=0.
      DO 16 J=2,MAXST,2
      P=BOTLIM+(J-1)*HH
      S2SM=S2SM+FP(IROW,J)*COS(XVAL*P)
   16 CONTINUE
      THE=XVAL*HH
      IF(THE-0.2) 25,21,21
   21 ALPHA=(THE**2+THE*SIN(THE)*COS(THE)-2.*SIN(THE)**2)/THE**3
      BETA=2.*(THE*(1.+COS(THE)**2)-2.*SIN(THE)*COS(THE))/THE**3
      GARM=4.*(SIN(THE)-THE*COS(THE))/THE**3
      GO TO 31
   25 ALPHA=2.*THE**3/45.-2.*THE**5/315.+2.*THE**7/4725.
      BETA=2./3.+2.*THE**2/15.-4.*THE**4/105.+2.*THE**6/567.
      GARM=4./3.-2.*THE**2/15.+THE**4/210.-THE**6/11340.
   31 FA=FP(IROW,1)
      FB=FP(IROW,MAXST+1)
      TRIINT=HH*( ALPHA*(FB*SIN(XVAL*UPLIM)-FA*SIN(XVAL*BOTLIM))+BETA*
     1S2S+GARM*S2SM)
      RETURN
      END
```

```
      SUBROUTINE FFF(UU,INDEX,MAX,F,II,PPHI,IDIM)
      DIMENSION F(IDIM)
      PI=3.1415926
      IF(INDEX .EQ. 1) 4,6
    4 DELPSI=PI/FLOAT(MAX)
      GO TO 7
    6 DELPSI=PPHI/FLOAT(2*MAX)
    7 CONTINUE
      PSI=0.
      MAXPUS=MAX+1
      DO 19 N=1,MAXPUS
      BPSI=PSI-PPHI/2.
      CALL FSF(BPSI,UU,FS ,PPHI)
      FA=FS
      BPSI=2.*PI-PSI-PPHI/2.
      CALL FSF(BPSI,UU,FS ,PPHI)
      FB=FS
      IF(II .EQ. 1) 14,16
   14 F(N)=(FA-FB)/2.
      GO TO 17
   16 F(N)=(FA+FB)/2.
   17 CONTINUE
      PSI=PSI+DELPSI
   19 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE FSF(BPSI,UU,FS ,PPHI)
      PI=3.1415926
      PI23=2.*PI/3.
      PI43=4.*PI/3.
      CONS23=2./3.
      HALPI=PI/2.
      HALPHI=PPHI/2.
      TEST1=2.*PI-HALPHI
      TEST2=2.*PI-PPHI
      IF(UU) 11, 1,1
    1 F0=1.
      GO TO 31
   11 IF(BPSI .LE. TEST1 .AND. BPSI .GE. HALPI .OR. BPSI .LT. -HALPI
     1 .AND. BPSI .GE.(-HALPHI-0.0000001)) 1,14
   12 F0=0.
      GO TO 31
   14 IF(BPSI .LE. 0 .AND. BPSI .GT. -HALPI) 12,15
   15 IF(BPSI .LT. HALPI .AND. BPSI .GT. 0) 16,18
   16 WRITE(6,17) UU,BPSI
   17 FORMAT(*  PRORAM STOP IN CASE 1  U=*,E12.5,5X,*BPSI=*,E12.5)
      STOP
   18 IF(ABS(BPSI+HALPI) -0.00001) 19,19,20
   19 F0=0.5
      GO TO 31
   20 WRITE(6,21) UU,BPSI
   21 FORMAT(*  PRORAM STOP IN CASE 2  U=*,E12.5,5X,*BPSI=*,E12.5)
      STOP
   31 ABSPSI=ABS(BPSI)
      IF(ABSPSI .GT. HALPI) 32,33
   32 F1=0.
      GO TO 51
   33 RHO=SQRT(UU**2+(1.-UU**2)*SIN(BPSI)**2)
      TAU0=ASIN(ABS(UU)/RHO)
      IF(UU .GE. 0) 34,37
   34 IF(BPSI .GE. 0) 35,36
   35 TAU=TAU0
      GO TO 42
   36 TAU=PI-TAU0
      GO TO 42
   37 IF(BPSI .LE. 0) 38,39
   38 TAU=PI+TAU0
      GO TO 42
   39 WRITE(6,40) UU,BPSI
   40 FORMAT(*  PRORAM STOP IN CASE 3  U=*,E12.5,5X,*BPSI=*,E12.5)
      STOP
   42 CALL CON2D(RHO,TAU,PI23,PI43,CONS23,FIF)
      IF(ABSPSI-HALPI) 45,43,43
   43 F1=FIF/2.
      GO TO 51
   45 F1=FIF
   51 IF(UU ) 56,54,53
   53 F2=0.
      GO TO 71
   54 IF(BPSI .LT. HALPI .AND.   BPSI .GT. 0) 55,56
   55 F2=0.
      GO TO 71
   56 RHO=SQRT(1.-UU**2)
      IF(BPSI .LE. TEST1 .AND. BPSI .GE. HALPI) 57,58
```

```
57 TAU=BPSI-HALPI
   GO TO 62
58 IF(BPSI .LE. 0 .AND. BPSI .GE. -HALPHI) 59,60
59 TAU=BPSI+3.*PI/2.
   GO TO 62
60 WRITE(6,61) UU,BPSI
61 FORMAT(*  PRORAM STOP IN CASE 4  U=*,E12.5,5X,*BPSI=*,E12.5)
   STOP
62 CALL CON2D(RHO,TAU,PI23,PI43,CONS23,FIF)
   IF(UU ) 66,64,55
64 F2=0.5*FIF
   GO TO 71
66 F2=FIF
71 FS =F0+F1+F2
   RETURN
   END
```

```
      SUBROUTINE CON2D(RHO,TAU,W1,W2,SS,PC)
      PI=3.1415926
      EPSIL=0.000001
      CON1=1.-RHO**2
      IF(CON1+EPSIL) 10,10,14
   10 WRITE(6,11)
   11 FORMAT(*  1-RHO**2+EPSIL .LE. ZERO, CHECK THE PROGRAM *)
      STOP
   14 IF(CON1 )15,15,17
   15 GG=1.
      GO TO 21
   17 GG=(RHO/(1.+SQRT(CON1)))**SS
   21 BTAU=SS*TAU
      DD=(1.-GG**2)*SIN(0.5*(W2-W1))
      CC=(1.+GG**2)*COS(0.5*(W2-W1)) -2.*GG*COS(BTAU-(W2+W1)*0.5)
      SQCD=SQRT(CC**2+DD**2)
      IF(SQCD-EPSIL) 23,25,25
   23 PC=0.5
      GO TO 30
   25 IF(CC) 28,26,26
   26 PC=ASIN(DD/SQCD)/PI
      GO TO 30
   28 PC=ASIN(DD/SQCD)/PI*(-1.)+1.
   30 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      XNU=SS
      DD=1.
      TEM1=0.
      TEM2=0.
      ZX=(1.-XMU)/2.
      IF(ABS(ZX)-0.000001) 35,35,31
   31 CONTINUE
      DO 40 L=1,LLMAX
      ZL=L
      TEM1=TEM1+DD*ZX**(L-1)
      TEM2=TEM2+(ZL-1.)*DD*ZX**(L-2)
      DD=DD*(ZL-XLAM     -1.)*(ZL+XLAM     )/(ZL**2+ZL*XNU)
   40 CONTINUE
       ARTFL=((1.-XMU)/(1.+XMU))**(XNU/2.)
      PP=ARTFL*TEM1
      DP=-XNU*PP/(1.-XMU**2)-0.5*ARTFL*TEM2
      GO TO 41
   35 PP=0.
      DP=0.
   41 CONTINUE
      RETURN
      END
```

```
C    THIRD PROGRAM
C    DETERMINATION OF THE PRESURE DISTRIBUTION ON SURFACES
      DIMENSION EDIM(20),P1(20,20),P2(20,20),P3(20,20),G1(20,20),
     1G2(20,20),G3(20,20) ,BMIN(20),DMIN(20)
      LLMAX=50
       PI=3.1415926
      ALPHA=PI/2.
      PPHI=2.*(PI-ALPHA/2.)
      XNU1=PI/PPHI
      HALPHI=PPHI/2.
      DO 106 M=1,10
      DO 105 N=1,5
      P1(M,N)=0.
      P2(M,N)=0.
      P3(M,N)=0.
  105 CONTINUE
  106 CONTINUE
 1000 READ(5,1001) II,NMAX,LMAX
 1001 FORMAT(3I5)
      IF(ENDFILE 5) 9999,1002
 1002 CONTINUE
      IF(II .EQ. 1) 101,103
C
C  ODD FUNCTION
  101 IIMAX=LMAX+NMAX
      WRITE(6,2)
    2 FORMAT(1H1,//*   ODD FUNCTION*)
      NCALL=NMAX
      LCALL=LMAX
      GO TO 104
C
C    EVEN FUNCTION
  103 IIMAX=LMAX+NMAX+2
      WRITE(6,3)
    3 FORMAT(1H1,//*   EVEN FUNCTION *)
      NCALL=NMAX+1
      LCALL=LMAX+1
  104 CONTINUE
      XMU=0.
      WRITE(6,502) NMAX,LMAX,IIMAX,ALPHA,PPHI,XNU1,XMU
  502 FORMAT(     *  NMAX=*,I5,5X,*LMAX=*,I5,5X,*IIMAX=*,I5/5X,          *
     1ALPHA=*,E12.5,5X,*PPHI=*,E12.5,5X,*XNU1=*,E12.5,5X,*XMU=*,E12.5)
  110 READ(5,100) XLAM,EEEM
  100 FORMAT(5F15.0)
      IF(ENDFILE 5) 1000,1111
 1111 CONTINUE
      READ(5,100)  (BMIN(I),I=1,NCALL)
      READ(5,100)  (DMIN(L),L=1,LCALL)
      XMU=0.
      CALL T102(XLAM,1.,EDIM,BBBBBB)
      AAAAAA=BBBBBB
      WRITE(6,32) XLAM,AAAAAA,EEEM
   32 FORMAT(    *    XLAM=*,F14.9,5X,*AA0=*,F14.9,5X,*K(LAMDA)=*,F14.9)
      WRITE(6,503)
  503 FORMAT(//  *  COEFFICIENTS OF EIGEN-FUNCTIONS*)
      WRITE(6,304) (BMIN(I),I=1,NCALL)
      WRITE(6,304) (DMIN(L),L=1,LCALL)
  304 FORMAT(5E20.8/(5X,5E20.8))
```

```
      WRITE(6,401)
  401 FORMAT(////)
      DELPHI=(PI-HALPHI)/2.
      IF(II .EQ. 1) 200,203
C
C    ODD FUNCTION
  200 DO 202 I=1,5
      PHI=PI+(I-3)*DELPHI
      PPUS=0.
      DO 201 J=1,NCALL
      SS=J
      CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      SINFU=SIN(J*PHI)
      PPUS=PPUS+BMIN(J)*PP*SINFU
  201 CONTINUE
      G1(I)=PPUS
  202 CONTINUE
      GO TO 209
C
C     EVEN FUNCTION
  203 CONTINUE
  204 DO 207 I=1,5
      PHI=PI+(I-3)*DELPHI
      PPUS=0.
      DO 206 J=1,NCALL
      SS=J-1
      CALL PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      COSFU=COS(SS*PHI)
      PPUS=PPUS+BMIN(J)*PP*COSFU
  206 CONTINUE
      G1(I)=PPUS
  207 CONTINUE
  209 CALL T102(XLAM,AAAAAA,EDIM,BBBBBB)
      DO 228 M=1,10
      DO 227 N=1,5
      P1(M,N)=P1(M,N)+EDIM(M)*G1(N)*EEEM
  227 CONTINUE
  228 CONTINUE
      MMP=1
      PHI=HALPHI
      DELTH=PI/8. -0.0005
  300 IF(II .EQ. 1) 301,400
C
C    ODD FUNCTION
  301 DO 309 I=1,5
      THE=PI/2.+(I-1)*DELTH
      XMU=COS(THE)
      PMIN=0.
      DO 302 L=1,LCALL
      SS=(2*L-1)*XNU1
      CALL PPDD(SS,XLAM,-XMU,LLMAX,PP,DP)
      SINFU=SIN(SS*PHI)
      PMIN=PMIN+DMIN(L)*PP*SINFU
  302 CONTINUE
  303 G2(I)=PMIN
  305 G3(I)=PMIN*(-1.)
  309 CONTINUE
      GO TO 500
```

```
C
C    EVEN FUNCTION
  400 DO 409 I=1,5
      THE=PI/2.+(I-1)*DELTH
      XMU=COS(THE)
      PMIN=0.
      DO 402 L=1,LCALL
      SS=2*(L-1)*XNU1
      CALL PPDD(SS,XLAM,-XMU,LLMAX,PP,DP)
      COSFU=COS(SS*PHI)
      PMIN=PMIN+DMIN(L)*PP*COSFU
  402 CONTINUE
  403 G2(I)=PMIN
  404 G3(I)=PMIN
  409 CONTINUE
  500 CONTINUE
  501 DO 505 M=1,10
      DO 504 N=1,5
      P2(M,N)=P2(M,N)+EDIM(M)*G2(N)*EEEM
      P3(M,N)=P3(M,N)+EDIM(M)*G3(N)*EEEM
  504 CONTINUE
  505 CONTINUE
      GO TO 110
 9999 CONTINUE
      WRITE(6,9998)
 9998 FORMAT(//8X,*PHI*,14X,*P1*,28X,*XMU*,14X,*P2*,14X,*P3*)
  601 DO 605 N=1,5
      PHI=PI+(N-3)*DELPHI
      THE=PI/2.+(N-1)*DELTH
      XMU=COS(THE)
      DO 604 M=1,10
      WRITE(6,607)      PHI,P1(M,N),XMU,     P2(M,N),P3(M,N)
  607 FORMAT(2E16.5,15X,3E16.5)
  604 CONTINUE
  605 CONTINUE
      STOP
      END
```

```
      SUBROUTINE T102(XLAM,AAAAAA,EDIM,BBBBBB)
      DIMENSION EDIM(20)
      COMMON XXXXL
      XXXXL=XLAM
      XMAX=0.9989
      X0=0.001
      X0XL=F3(XLAM,X0)
      X0XLP=F3(XLAM+1.,X0)
      X0XLM=F3(XLAM-1.,X0)
      X0XLM2=F3(XLAM+2.,X0)
   10 F0=(X0XL+XLAM*(XLAM+1.)*X0XLM2/(4.*XLAM+6.))*AAAAAA
      G0=(XLAM*X0XLM+(XLAM+1.)*(XLAM+2.)*XLAM*X0XLP/(4.*XLAM+6.))*AAAAAA
      DELX=0.001
      X=X0
      F=F0
      G=G0
      I=1
      XTEST=0.101
   20 XK1=F1(X,F,G)*DELX
      XL1=F2(X,F,G)*DELX
      XK2=F1(X+DELX/2., F+XK1/2., G+XL1/2.)*DELX
      XL2=F2(X+DELX/2., F+XK1/2., G+XL1/2.)*DELX
      XK3=F1(X+DELX/2., F+XK2/2., G+XL2/2.)*DELX
      XL3=F2(X+DELX/2., F+XK2/2., G+XL2/2.)*DELX
      XK4=F1(X+DELX, F+XK3, G+XL3)*DELX
      XL4=F2(X+DELX, F+XK3, G+XL3)*DELX
      DELF=1./6.*(XK1+2.*XK2+2.*XK3+XK4)
      DELG=1./6.*(XL1+2.*XL2+2.*XL3+XL4)
      X=X+DELX
      F=F+DELF
      G=G+DELG
      IF(X .GE. XTEST) 201,203
  201 EDIM(I)=F
      XTEST=XTEST+0.1
      I=I+1
  203 CONTINUE
      IF(X-0.01) 20,20,31
   31 IF(X-0.99) 32,32,33
   32 DELX=0.01
      GO TO 20
   33 IF(X-0.9989) 34,40,40
   34 DELX=0.001
      GO TO 20
   40 ZETA=1.-X
      C1=-XLAM*(XLAM+1.)/4.
      C2=C1**2
      C3=((XLAM*(XLAM+1.)-2.)*C1**2+2.*C1)/6.
      GEND=F/(1.+C1*ZETA+C2*ZETA**2+C3*ZETA**3*ALOG(ZETA))
   41 BBBBBB=1./GEND
      EDIM(I)=GEND
      RETURN
      END
```

```
FUNCTION F1(A,B,C)
COMMON  XLAM
F1=C
RETURN
END
```

```
FUNCTION F2(A,B,C)
COMMON  XLAM
F2=(-2.*A*(A**2-1.)*C-XLAM*(XLAM+1.)*B)/( A**2*(A**2-1.))
RETURN
END
```

```
FUNCTION F3(D,E)
F3=EXP(D*ALOG(E))
RETURN
END
```

```
      SUBROUTINE PPDD(SS,XLAM,XMU,LLMAX,PP,DP)
      XNU=SS
      DD=1.
      TEM1=0.
      TEM2=0.
      ZX=(1.-XMU)/2.
      IF(ABS(ZX)-0.000001) 35,35,31
   31 CONTINUE
      DO 40 L=1,LLMAX
      ZL=L
      TEM1=TEM1+DD*ZX**(L-1)
      TEM2=TEM2+(ZL-1.)*DD*ZX**(L-2)
      DD=DD*(ZL-XLAM     -1.)*(ZL+XLAM     )/(ZL**2+ZL*XNU)
   40 CONTINUE
       ARTFL=((1.-XMU)/(1.+XMU))**(XNU/2.)
      PP=ARTFL*TEM1
      DP=-XNU*PP/(1.-XMU**2)-0.5*ARTFL*TEM2
      GO TO 41
   35 PP=0.
      DP=0.
   41 CONTINUE
      RETURN
      END
```

TABLE I

Comparison of numerical results with the exact eigenvalues for a two dimensional corner.

Exact Solution	Numerical Solution			
	$M \neq J$		$M = J$	
	even	odd	even	odd
	$M_c=8 \quad J_c=4$	$M_s=9 \quad J_s=5$	$M_c=6 \quad J_c=6$	$M_s=7 \quad J_s=7$
0.0	0.0		0.0	
2/3	.667394		.669715	
1.0		1.0		1.0
1+1/3	1.333342		1.333518	
1+2/3		1.668060		1.672477
2.0*	2.0*		2.0*	
2+1/3		2.333356		2.333793
2+2/3*	{2.666667, 2.669734}		{2.666669, 2.679591}	
3.0*		3.0*		3.0*

* Double root

TABLE II

Some exact eigensolutions for the three dimensional corner with $\beta = \pi/2$, $\alpha = \pi/4$.

Eigenvalue	Eigensolutions	
λ	Even solutions	Odd solutions
0	$P_o(\mu)$	2
2	$P_2(\mu)$	$P_2^2(\mu)\sin 2\varphi$
4	$P_4(\mu), P_4^4(\mu)\cos 4\varphi$	$P_4^2(\mu)\sin 2\varphi$

TABLE III

Tabulations of the eigenvalues and the coefficients in the series representations of the eigenfunctions ($2\alpha = \pi/2$ $\beta_j = \pi/2$).

	even	
	Coef. Am	Coef. Cj
	.426487	.426487
	0.0	0.0
λ = 0.0	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	
	0.0	0.0
	0.0	0.0
	0.0	

	odd	
	Coef. Bm	Coef. Dj
	-1.135600	-1.444600
	.073230	-.097647
λ=.839948	.004226	.031486
	-.021135	-.016324
	.017872	.010371
	-.007779	-.007452
	-.001831	.005917
	.007199	-.005446
	.007525	
	.004085	

	.496980	-.662630
	-.85376	-.394590
λ=.840341	.19722	.070879
	-.086956	-.032794
	.024596	.020205
	.009953	-.014725
	-.022866	.012976
	.020424	
	-.009732	

	2.676300	-2.692700
λ =	-.141180	.188240
1.807053	.224060	.027981
	-.131070	-.019274
	.059485	.013975
	-.010481	-.011068
	-.016294	.009643
	.024045	-.010008
	-.018772	
	.007506	

	even	
	Coef. Am	Coef. Cj
	.379050	-.505400
	1.656200	1.188100
λ=1.206357	.143140	-.038965
	-.010898	.011436
	-.008577	-.005016
	.008833	.002579
	-.004721	-.001293
	.000934	
	.000970	

	odd	
	Coef. Bm	Coef. Dj
	0.0	0.0
	3.303556	3.303556
λ=2.0	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	
	0.0	

	-.172240	.229650
	-1.263800	2.980100
λ=1.805246	2.193400	.529750
	-.359600	-.135050
	.101280	.065806
	.009582	-.041720
	-.050099	.032932
	.048334	
	-.024699	

	-4.072100	-2.750300
λ =	3.765900	5.021200
2.814159	1.818200	.698380
	-.358120	-.103210
	.135310	.043530
	-.032647	-.025606
	-.016271	.018233
	.031678	-.015960
	-.026615	
	.011970	

	even	
	Coef. Am	Coef. Cj
	.953654	.953654
	0.0	0.0
λ=2.0	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	0.0
	0.0	
	0.0	

	odd	
	Coef. Bm	Coef. Dj
	2.584400	3.142500
λ =	-4.047200	5.396300
2.868927	3.291700	1.576600
	-.657200	-.249830
	.283730	.108690
	-.081520	-.064265
	-.027367	.045417
	.067622	-038976
	-.060812	
	.029232	

	-.208840	.278460
	2.725500	-3.976900
λ=2.446688	4.206800	1.747500
	.358770	-.103810
	.077859	.032182
	-.065595	-.014013
	.030583	.006305
	-.005977	
	-.004729	

	even	
	Coef. Am	Coef. Cj
λ=2.813928	.530570	-.707430
	-2.247100	-.855920
	1.003000	5.289000
	4.135700	.342700
	-.257020	-.116920
	-.039995	.064339
	.097445	-.047471
	-.079136	
	.035604	

λ=2.958670	-.507640	.676780
	2.214800	4.076500
	-3.983000	5.162800
	3.733900	.323730
	-.242800	-.065636
	.101760	.024029
	-.035758	-.010174
	.003411	
	.007631	

TABLE IV

Comparison of the eigenvalues for a three dimensional corner with those for a circular cone of the same solid angle $\Theta = \cos^{-1}(3/4)$.

Three Dimensional Corner	
even	odd
.840341	.839948
1.206357	
1.805246	1.807053
2.0	2.0
2.446688	
2.813928	2.814159
2.958670	2.868927
3.205219	3.205133

Circular Cone	
$\Theta = \cos^{-1}(3/4)$	n
.863382	1
1.210120	0
1.893798	2
1.961940	1
2.477387	0
2.865298	2
2.936459	3
3.190234	1

NASA-Langley, 1971 — 12 CR-1728