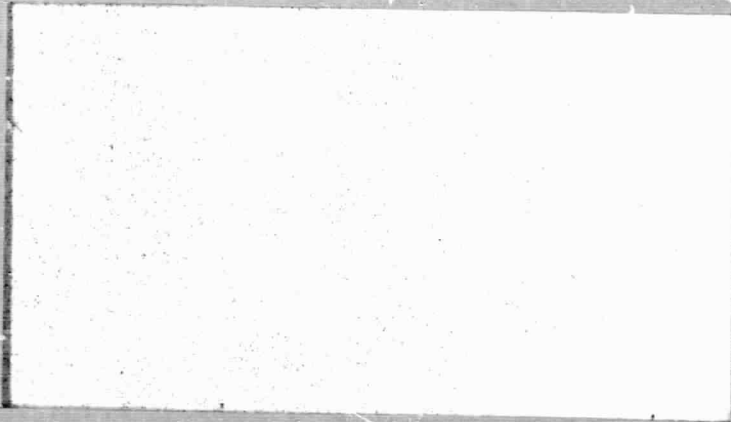


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A. R. A. P.

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A.R.A.P. WORKING PAPER NO. 71-4
A STUDY OF COMPRESSIBLE TURBULENT
BOUNDARY LAYERS USING THE
METHOD OF INVARIANT MODELING

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I. INTRODUCTION

The present report summarizes the current status of a program to analyze a compressible turbulent boundary layer. It is part of a continuing effort at A.R.A.P. to develop equations and investigate general turbulent shear flows by the method of invariant modeling. This method has been highly successful in the treatment of incompressible turbulent flows (Refs. 1, 2, 3). As this report will show, we are now on the verge of being able to make calculations for the fully compressible case.

This document is in the nature of a status report. No pretense is made of giving a fully detailed explanation of the modeling procedure or of the computer routines that were developed to program the model equation. A more detailed account will be forthcoming upon completion of this work.

II. MODEL EQUATIONS

The model equations that are appropriate for the study of compressible turbulent boundary layers are given in the appendix. Here we outline briefly the key steps and assumptions that are made to derive the model equations. The compressible Navier-Stokes equations are the fundamental equations from which the model equations are derived. In tensor notation, they are

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\rho u^j)_{,j} &= 0 \\ \rho \left(\frac{\partial u_i}{\partial t} + u^j u_{i,j} \right) &= - \frac{\partial p}{\partial x^i} + \tau_{i,j}^j \\ \frac{\partial \rho J}{\partial t} + (\rho J u^j)_{,j} &= \frac{\partial p}{\partial t} + (u^j \tau_{j,k}^k)_{,k} - q_{,j}^j \end{aligned} \quad (1)$$

$$p = \rho RT$$

State Equation

where

$$\begin{aligned} \tau_{ij} &= \mu \left(u_{i,j} + u_{j,i} - \frac{2}{3} \delta_{ij} u^k_{,k} \right) && \text{Viscous Stress} \\ q_j &= -k \frac{\partial T}{\partial x^j} && \text{Heat Flux} \\ J &= C_p T + \frac{u^2}{2} && \text{Total Enthalpy} \end{aligned} \quad (2)$$

To derive turbulent equations, we assume each variable to be the sum of a mean and fluctuating part, e.g.,

$$\rho = \bar{\rho} + \rho' \quad (3)$$

Substitute these expressions into the Navier-Stokes equations and compute statistical moments to arrive at differential equations for statistical quantities. We have derived equations for the following variables:

		No. of Equations
Mean Density	$\bar{\rho}$	1
Mean Velocity	\bar{u}_i	3
Mean Temperature	\bar{T}	1
Density Fluctuation	ρ'^2	1

Temperature Fluctuation	$\overline{T'^2}$	1
Density-Temperature Correlation	$\overline{\rho' T'}$	1
Density-Velocity Correlation	$\overline{\rho' u'_i}$	3
Temperature-Velocity Correlation	$\overline{T' u'_i}$	3
Velocity Correlation	$\overline{u'_i u'_j}$	$\frac{6}{20}$

Each of these equations (twenty in total) involves higher correlations, both third and fourth order. To close the equation, we must give the higher order correlations in terms of variables that we want to solve for. The method of invariant modeling permits us to accomplish this task in a rational way. This method has been described in detail and used successfully in previous work with the incompressible equations (Ref. 1,2). Here we give three typical model terms that illustrate how the two turbulent length scales enter into the formulation. The models are as follows:

$$\overline{u'_i u'_j u'_k} = -\Lambda \sqrt{K} \left[\left(\overline{u'_i u'_j} \right)_{,k} + \left(\overline{u'_i u'_k} \right)_{,j} + \left(\overline{u'_j u'_k} \right)_{,i} \right]$$

$$\overline{\rho' (u'_{i,k} + u'_{k,i})} = \frac{\bar{\rho} \sqrt{K}}{\Lambda} \left(g_{ik} \frac{K}{3} - \overline{u'_i u'_k} \right) \quad (4)$$

$$g^{mn} \overline{u'_{i,m} u'_{k,n}} = \frac{\overline{u'_i u'_k}}{\lambda^2}$$

where $K = \overline{u'_m u'_m}$ is the turbulent kinetic energy. The length parameters Λ and λ are related to the macroscale and microscale of the turbulence. In our work, they are assumed to be related by the formula

$$\lambda = \frac{\Lambda}{\sqrt{a + b \text{Re}_\Lambda}} \quad (5)$$

$$\text{Re}_\Lambda = \frac{\bar{\rho} \Lambda K}{\bar{\mu}}$$

where a and b are constants to be determined by comparison with experimental data. Equation (5) has been used by Glushko (Ref. 4) and Beckwith and Bushnell (Ref. 5) in their models of the turbulent kinetic energy equation.

One final assumption that is used to derive the turbulent equations in the appendix requires some discussion. It has been customary, in previous work, to assume that the fluctuating velocity field is incompressible, i.e.,

$$u_{,j}^{j'} = 0 \quad (6)$$

even though the mean flow field may be highly compressible. An alternate procedure is followed in the present work. Based on experimental and theoretical estimates, we assume that the temperature and density fluctuations are large compared to the pressure fluctuations, i.e.,

$$\frac{\sqrt{\rho'^2}}{\bar{\rho}} = 0 \left(\frac{\sqrt{T'^2}}{\bar{T}} \right) \gg \frac{\sqrt{p'^2}}{\bar{p}} \quad (7)$$

From the equation of state, we then obtain a simple proportionality between ρ' and T' , i.e.,

$$T' = -\frac{\bar{T}}{\bar{\rho}} \rho' \quad (8)$$

Finally, Eq. (8) is combined with the equations for density and temperature fluctuations to actually solve for the turbulent compressibility. The result is

$$\bar{p} u_{,j}^{j'} = \frac{\gamma - 1}{\gamma} (\phi' - H') \quad (9)$$

where

$$\begin{aligned} \phi' &= \bar{u}_{,k}^j \tau_{,j}^{k'} + u_{,k}^{j'-k} \bar{\tau}_{,j}^k \\ H' &= q_{,j}^{j'} \end{aligned} \quad (10)$$

This result is subsequently used to eliminate all equations for variables involving T' . The final number of equations is reduced from twenty to fifteen. These equations are summarized in the appendix.

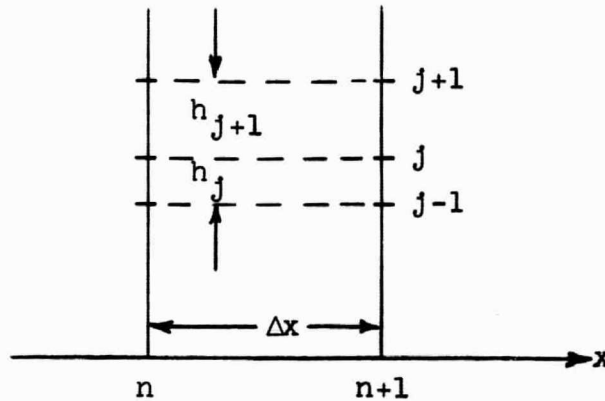
III. METHOD OF SOLUTION

The model equations given in the appendix are valid for fairly arbitrary turbulent compressible flow. To obtain equations for a two-dimensional turbulent boundary layer, we first eliminate the mean velocity component \bar{u}^3 or \bar{w} . Also, in all diffusion-type terms, we retain only second derivatives with respect to y , the coordinate normal to the wall. The final result is a set of fourteen nonlinear, coupled, parabolic, differential equations. They are to be solved subject to boundary conditions at the wall and in the free stream. Initial data must also be specified at a given x location on the wall.

Because of the large number of equations involved, it was decided that the derivation of the finite difference equations and the selection of the difference grid must be fully automated. The first step was necessary to eliminate needless errors in performing a very large amount of algebraic detail. The second step is needed to minimize machine computation time and to control the accuracy of the calculations.

To illustrate the method of solution and show how the two automatic routines fit into the scheme, we consider a single nonlinear diffusion equation

$$\frac{\partial u}{\partial x} = F(u) \frac{\partial^2 u}{\partial y^2} \quad (11)$$



Referring to the sketch above, we can write Eq. (11) as an implicit finite difference equation as follows:

$$\frac{u_j^{n+1} - u_j^n}{\Delta x} = \frac{1}{(h_{j+1} + h_j)} \left[F_{j+\frac{1}{2}}^n \frac{u_{j+1}^{n+1} - u_j^{n+1}}{h_{j+1}} - F_{j-\frac{1}{2}}^n \frac{u_j^{n+1} - u_{j-1}^{n+1}}{h_j} \right] \quad (12)$$

where

$$F_{j+\frac{1}{2}}^n = F(u_{j+\frac{1}{2}}^n)$$

$$u_{j+\frac{1}{2}}^n = \frac{u_{j+1}^n + u_j^n}{2}$$

$$u_{j-\frac{1}{2}}^n = \frac{u_j^n + u_{j-1}^n}{2}$$

Rearranging terms in (12), we obtain

$$-Au_{j-1}^{n+1} + Bu_j^{n+1} - Cu_{j+1}^{n+1} = D \quad (13)$$

where

$$A = \frac{\Delta x F_{j-\frac{1}{2}}^n}{(h_j + h_{j+1})h_j} \quad C = \frac{(\Delta x) F_{j+\frac{1}{2}}^n}{(h_j + h_{j+1})h_{j+1}} \quad (14)$$

$$B = \frac{\Delta x}{h_j + h_{j+1}} \left(\frac{F_{j-\frac{1}{2}}^n}{h_j} + \frac{F_{j+\frac{1}{2}}^n}{h_{j+1}} \right) \quad D = u_j^n$$

The last result is a linear difference equation for the unknown u at station $n+1$. Given data at station n , we can solve this equation by a familiar standard algorithm (Ref. 6) for u at $n+1$. The difference equations that result from the boundary layer equations in the appendix are exactly of the same form as Eq. (13). The only difference is that u is a vector with fourteen components. Thus, A , B , and C are 14×14 matrices and D is a 14 -component vector. Each element of A , B , C , and D requires a defining equation. In all there are 602 such equations that must be written out. It is precisely this number that dictates the need for an automatic routine to write the equations. Furthermore, an estimate of the time required to make a single step, Δx , is of the order of 4 seconds on a CDC 6600 computer or 15 minutes on an IBM 1130. This means that each step must be made efficiently. Thus, the routine to automatically adjust the grid spacing in the x and y directions was developed. Each of these routines is described in the following section.

IV. AUTOMATIC DIFFERENCING AND GRID SPACING

A. Automatic Differencing

The routine that automatically differences the tensorial equations can be reduced to the following basic steps:

1. Code tensor equations (input);
2. Expand into additive terms and set metric tensor;
3. Drop terms that are zero or of higher order and redefine coefficients;
4. Compute difference formula (output).

The actual computer program consists of 9000 cards! A detailed description would require a small book in itself. Here we can only give a general description of the four basic steps.

1. Code tensor equations (input)

The tensor equations are coded directly in a form that the computer program will accept. All tensor notations and operations are given special symbols, e.g.,

$$\begin{array}{ll}
 A_i \rightarrow A'I & g_{ij} \rightarrow @'IJ \\
 A^i \rightarrow A''I & \delta_j^i \rightarrow \&'I'J \\
 A_{,i} \rightarrow (A),I &
 \end{array} \quad (15)$$

The equation for the mean density, for example, is coded as follows:

$$U''J*(D),J=-D*(U''J),J-(DU''J),J \quad (16)$$

The asterisk denotes multiplication as in FORTRAN programming.

2. Expand into additive terms and set metric tensor

After the input is carefully checked, the program proceeds to expand tensor sums and parenthesized expressions into a string of additive terms. Then the zero elements of the metric tensor are eliminated and unit multipliers are dropped. The reduced terms are then printed and punched for the purpose of editing if it is deemed necessary. For the present problem, there are 1544 terms in total at this point in the operation. The equation for the mean density has 10 terms that are displayed below (cf. Eq. (16)):

$$\begin{array}{rcl}
 1 & & U''1*(D),1 \\
 2 & & +U''2*(D),2 \\
 3 & & +U''3*(D),3 \\
 4 & & = \\
 5 & & -D*(U''1),1 \\
 6 & & -D*(U''2),2 \\
 7 & & -D*(U''3),3 \\
 8 & & -(DU''1),1 \\
 9 & & -(DU''2),2 \\
 10 & & -(DU''3),3
 \end{array} \tag{17}$$

3. Drop terms that are zero or higher order and predefine coefficients

The next step is to hand-edit the terms for the particular problem at hand. For the two-dimensional boundary layer, we set $U'3 = 0$, $() ,3 = 0$ and drop all but second y derivatives in the diffusive terms. The number of terms then reduces to 226. For example, the mean density equation becomes

$$\begin{array}{rcl}
 1 & & U''1*(D),1 \\
 2 & & +U''2*(D),2 \\
 3 & & = \\
 4 & & -D*(U''1),1 \\
 5 & & -D*(U''2),2 \\
 6 & & -(DU''2),2
 \end{array} \tag{18}$$

Another part of the editing process is to gather all coefficients of differential expressions into a single term and give it a name for the purpose of precalculation. This is an economy measure to avoid multiple evaluation of the same expression in the final program. The mean density equation does not happen to have any such coefficients.

4. Compute difference formula (output)

The final step is to give the edited equations to the program for differencing. The output at this stage is an explicit FORTRAN formula on card for each nonzero element of the matrices A, B, C, and D discussed in Section III. We omit the actual formula for the mean density equation here to avoid explanation of the notation changes. Suffice it to say that we get the (1,1), (1,3), and (1,7) elements of the A, B, and C matrices and the first element of D. In total, there are approximately 70 elements in each of the A, B, C matrices that are nonzero. With these formulae in hand, it becomes routine to write the subroutine to define A, B, C, D in the solution algorithm.

B. Automatic Grid Spacing

The routine that adjusts the grid spacing is composed of two parts. The first part chooses the step size for integration along the wall (x-direction) and decides whether to back up or go forward at each step. The second part redistributes the y points normal to the wall. A brief description of the logic involved in each part follows.

After arriving at station n by integration, we have the dependent variables available at stations n and n-1. Thus we can compute the maximum percentage change of each variable in the last step. If that change exceeds a given limit, the program backs up and takes a smaller Δx increment. If the change is within its bound, the next step is taken. The new step size is computed from the maximum gradient of the most rapidly changing dependent variable.

Once the decision is made to go forward, the second part of the program examines the distribution of y points and adds and removes points as needed. The main criterion for adding or removing points is based on a local curvature parameter δ_j that is computed from three successive y points.

If $\delta_j < \epsilon_1$ (given limit), the center point is removed.

If $\epsilon_1 < \delta_j < \epsilon_2$ (given limit), no change is made.

If $\delta_j > \epsilon_2$, two points are added.

Further limitations are imposed on addition and removal of points to ensure that the ratio of two successive step sizes does not become too large.

The automatic grid spacer has been written and debugged on an existing program for incompressible flow. It has been found to work satisfactorily.

V. SUMMARY AND RECOMMENDATIONS

The accomplishments of the present program are summarized as follows:

1. The model equations appropriate for the investigation of compressible turbulent shear flow were derived.
2. The model equations were written in finite difference form by an automatic differencing program that was developed under this contract.
3. An automatic grid spacing program was developed to ensure the economic solution of the difference equations.

All main parts of the full computer program have been written. It remains to debug the final program and proceed to the solution of actual problems.

It is recommended that the following steps be carried out in the sequel to the present work:

1. Complete the final debugging of the turbulent boundary layer program. Perform check runs to compare with previous incompressible results.
2. Make complete transition runs for the following four cases:
 - a. $M = 0$ adiabatic wall
 - b. $M = 3$ adiabatic wall
 - c. $M = 3$ $T_{\text{wall}}/T_e^0 \cong 0.5$
 - d. $M = 3$ $T_{\text{wall}}/T_e^0 \cong 0.1$

Exact values of M and T_{wall}/T_e^0 should be chosen for comparison with available experimental data.

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APPENDIX

TURBULENT MODEL EQUATIONSMean Density ($\bar{\rho}$)

$$\bar{u}^j \bar{\rho}_{,j} = - \bar{\rho} \bar{u}^j_{,j} - \overline{(\rho' u^{j'})}_{,j}$$

Mean Velocity (\bar{u}_i)

$$\begin{aligned} \bar{\rho} \bar{u}^j \bar{u}_{i,j} &= - \overline{\rho' u^{j'} \bar{u}_{i,j}} - \overline{(\bar{\rho} u'_i u^{j'})}_{,j} - \overline{(\rho' u'_i \bar{u}^j)}_{,j} \\ &+ g^{jk} \left(\Lambda \sqrt{K} \left[\overline{(\rho' u'_i)_{,k}} + \overline{(\rho' u'_k)_{,i}} \right] \right)_{,j} \\ &+ \bar{\mu} \left[g^{j\ell} (\bar{u}_{i,\ell} + \bar{u}_{\ell,i}) - \frac{2}{3} \delta_i^j \bar{u}^{\ell}_{,\ell} \right]_{,j} \end{aligned}$$

Mean Temperature (\bar{T})

$$\begin{aligned} \bar{\rho} \bar{u}^j \bar{T}_{,j} &= \bar{T} \overline{(\rho' u^{j'})}_{,j} + \left(\frac{\bar{T}}{\bar{\rho}} \bar{u}^j \overline{\rho'^2} \right)_{,j} \\ &+ g^{jk} \left(\frac{\bar{T}}{\bar{\rho}} \Lambda \sqrt{K} \overline{\rho'^2}_{,k} \right)_{,j} - \frac{\bar{p}}{C_v} \bar{u}^j_{,j} \\ &+ \frac{\bar{\mu}}{C_v} \bar{u}^j_{,k} \left[g^{k\ell} (\bar{u}_{j,\ell} + \bar{u}_{\ell,j}) - \frac{2}{3} \delta_j^k \bar{u}^{\ell}_{,\ell} \right] + g^{jk} \left(\frac{\bar{k}}{C_v} \bar{T}_{,j} \right)_{,k} \end{aligned}$$

Density Fluctuation ($\overline{\rho'^2}$)

$$\begin{aligned}
 \bar{u}^j \overline{\rho'^2}_{,j} &= -2\overline{\rho'^2} \bar{u}^j_{,j} - 2\overline{\rho' u^j}_{,j} \bar{\rho} + g^{jk} (\Lambda \sqrt{K} \overline{\rho'^2}_{,k})_{,j} \\
 &+ \frac{2\bar{\rho}(\gamma - 1)}{\gamma \bar{\rho}} \left\{ \frac{\bar{\mu} \sqrt{\overline{\rho'^2}}}{\Lambda \sqrt{K}} g^{kl} \bar{u}^j_{,k} \left(\overline{u'_j u'_l} - g_{jl} \frac{K}{3} \right) \right. \\
 &+ \frac{\mu_s \bar{T}}{\bar{\rho}} \overline{\rho'^2} \bar{u}^j_{,k} \left[g^{kl} (\bar{u}_{j,l} + \bar{u}_{l,j}) - \frac{2}{3} \delta_j^k \bar{u}^l_{,l} \right] \\
 &+ \frac{\bar{\mu} \sqrt{\overline{\rho'^2}}}{2\Lambda \sqrt{K}} g^{jl} \left(\overline{u'_l u'_k} - g_{kl} \frac{K}{3} \right) \left[g^{km} (\bar{u}_{j,m} + \bar{u}_{m,j}) - \frac{2}{3} \delta_j^k \bar{u}^m_{,m} \right] \\
 &- \frac{2\bar{\rho}(\gamma - 1)}{\gamma \bar{\rho}} g^{kl} \left\{ \bar{k}_{,k} \left[\left(\frac{\bar{T}}{\bar{\rho}} \right)_{,l} \overline{\rho'^2} + \frac{\bar{T}}{\bar{\rho}} \frac{\overline{\rho'^2}_{,l}}{2} \right] \right. \\
 &+ \bar{k} \left[\left(\frac{\bar{T}}{\bar{\rho}} \right)_{,l,k} \overline{\rho'^2} + \left(\frac{\bar{T}}{\bar{\rho}} \right)_{,l} \frac{\overline{\rho'^2}_{,k}}{2} + \left(\frac{\bar{T}}{\bar{\rho}} \right)_{,k} \frac{\overline{\rho'^2}_{,l}}{2} + \frac{\bar{T}}{\bar{\rho}} \frac{\overline{\rho'^2}_{,l,k}}{2} \right. \\
 &- \left. \left. \frac{g_{kl}}{3} \frac{\overline{\rho'^2}}{\lambda^2} \right] + \frac{k_s \bar{T}}{\bar{\rho}} \overline{\rho'^2} \bar{T}_{,l,k} + \bar{T}_{,l} \left[\left(\frac{k_s \bar{T}}{\bar{\rho}} \right)_{,k} \overline{\rho'^2} + \frac{k_s \bar{T}}{\bar{\rho}} \frac{\overline{\rho'^2}_{,k}}{2} \right] \right\} \\
 &+ \frac{\bar{\mu}(\gamma - 1) \overline{\rho'^2}}{\gamma \bar{\rho} \Lambda \sqrt{K}} \left\{ 2g^{kl} \bar{u}^j_{,k} \left(\overline{u'_j u'_l} - g_{jl} \frac{K}{3} \right) \right. \\
 &+ \left. g^{jl} \left(\overline{u'_l u'_k} - g_{lk} \frac{K}{3} \right) \left[g^{km} (\bar{u}_{j,m} + \bar{u}_{m,j}) - \frac{2}{3} \delta_j^k \bar{u}^m_{,m} \right] \right\}
 \end{aligned}$$

Density-Velocity Correlation ($\overline{\rho' u'_i}$)

$$\begin{aligned}
 \overline{\rho u^j (\rho' u'_i)},_j &= - \overline{\rho'^2} \bar{u}^j \bar{u}_{1,j} + 2 \overline{\rho' u'_i (\rho' u'^j)},_j - \overline{\rho' u'_i} \bar{\rho} \bar{u}^j{}_j \\
 &\quad - \overline{\rho' u'^j} (\bar{\rho} \bar{u}_{1,j}) - \overline{u'_i u'^j} \bar{\rho} \bar{\rho},_j \\
 &\quad + \bar{\rho} g^{jk} \left(\Lambda \sqrt{K} \left[(\overline{\rho' u'_i}),_k + (\overline{\rho' u'_k}),_i \right] \right),_j \\
 &\quad + 2 \Lambda \sqrt{K} g^{jk} \left[(\overline{\rho' u'_i}),_k + (\overline{\rho' u'_k}),_i \right] \bar{\rho},_j + \Lambda \sqrt{K} g^{jk} \overline{\rho'^2} \bar{u}_{1,j} \\
 &\quad + 2 \Lambda \sqrt{K} \overline{\rho'^2} \bar{u}^j{}_j + \bar{u}^j (\Lambda \sqrt{K} \overline{\rho'^2}),_j \\
 &\quad + \left(\bar{\rho} \left(b \Lambda^2 \sqrt{\epsilon_{st} \epsilon^{st}} + \Lambda \sqrt{K} \right) (\overline{\rho' u'^k}),_k \right),_i \\
 &\quad - \bar{\rho} \left(a \sqrt{\epsilon_{st} \epsilon^{st}} + \frac{\sqrt{K}}{\Lambda} \right) (\overline{\rho' u'_i}) \\
 &\quad - \frac{\bar{\rho}^2 (\gamma - 1)}{\gamma \bar{\rho}} \left\{ \bar{\mu} g^{kl} \bar{u}^j{}_k (\overline{u'_j u'_l} - g_{jk} \frac{K}{3}),_i \right. \\
 &\quad - \frac{\mu_s \bar{T}}{\bar{\rho}} \bar{u}^j{}_k \overline{\rho' u'_i} \left[g^{kl} (\bar{u}_{j,l} + \bar{u}_{l,j}) - \frac{2}{3} \delta_j^k \bar{u}^l{}_l \right] \\
 &\quad \left. - \frac{\bar{\mu}}{2} g^{jl} (\overline{u'_l u'_k} - g_{lk} \frac{K}{3}),_i \left[g^{km} (\bar{u}_{j,m} + \bar{u}_{m,j}) - \frac{2}{3} \delta_j^k \bar{u}^m{}_m \right] \right\} \\
 &\quad + \frac{\bar{\rho}^2 (\gamma - 1)}{\gamma \bar{\rho}} g^{kl} \left\{ \bar{k},_k \left[\left(\frac{\bar{T}}{\bar{\rho}} \right),_l \overline{\rho' u'_i} + \frac{\bar{T}}{\bar{\rho}} (\overline{\rho' u'_i}),_l \right. \right. \\
 &\quad \left. \left. + \frac{\bar{T} \sqrt{\rho'^2}}{2 \bar{\rho} \Lambda \sqrt{K}} (\overline{u'_i u'_l} - g_{il} \frac{K}{3}) \right] + \frac{\bar{T}}{\bar{\rho}} \frac{\overline{\rho' u'_i} g_{lk}}{\lambda^2} \frac{K}{3} \right. \\
 &\quad \left. + \frac{k_s \bar{T}}{\bar{\rho}} \overline{\rho' u'_i} \bar{T},_{l,k} + \bar{T},_l \left[\left(\frac{k_s \bar{T}}{\bar{\rho}} \right),_k \overline{\rho' u'_i} + \frac{k_s \bar{T}}{\bar{\rho}} (\overline{\rho' u'_i}),_k \right. \right. \\
 &\quad \left. \left. + \frac{k_s \bar{T} \sqrt{\rho'^2}}{2 \bar{\rho} \Lambda \sqrt{K}} (\overline{u'_i u'_k} - g_{ik} \frac{K}{3}) \right] \right\} - \frac{\sqrt{\rho'^2}}{\Lambda \sqrt{K}} g^{jl} (\overline{u'_i u'_l} - g_{il} \frac{K}{3}) \bar{\mu},_j \\
 &\quad - \frac{\mu_s \bar{T}}{\bar{\rho}} \overline{\rho'^2} \left[g^{kl} (\bar{u}_{1,l} + \bar{u}_{l,1}) - \frac{2}{3} \delta_1^k \bar{u}^l{}_l \right],_k
 \end{aligned}$$

Velocity Correlation $(\overline{u'_i u'_k})$

$$\begin{aligned}
\bar{\rho} \bar{u}^j (\overline{u'_i u'_k})_{,j} &= - \overline{\rho' u'_k} \bar{u}^j \bar{u}_{1,j} - \overline{\rho' u'_i} \bar{u}^j \bar{u}_{k,j} - \overline{\rho u'_k u'^j} \bar{u}_{1,j} \\
&- \bar{\rho} \overline{u'_i u'^j} \bar{u}_{k,j} - \overline{u'_i u'_k} (\bar{\rho} \bar{u}^j)_{,j} + g^{j\ell} \left(\bar{\rho} \Lambda \sqrt{K} \left[(\overline{u'_i u'_\ell})_{,k} \right. \right. \\
&+ \left. \left. (\overline{u'_i u'_k})_{,\ell} + (\overline{u'_\ell u'_k})_{,i} \right] \right)_{,j} + \left(\Lambda \sqrt{K} \bar{u}^j \left[(\overline{\rho' u'_i})_{,k} \right. \right. \\
&+ \left. \left. (\overline{\rho' u'_k})_{,i} \right] \right)_{,j} + \Lambda \sqrt{K} g^{lj} \left[(\overline{\rho' u'_\ell})_{,k} + (\overline{\rho' u'_k})_{,\ell} \right] \bar{u}_{1,j} \\
&+ \Lambda \sqrt{K} g^{j\ell} \left[(\overline{\rho' u'_\ell})_{,i} + (\overline{\rho' u'_i})_{,\ell} \right] \bar{u}_{k,j} \\
&- \left(\bar{\rho} \left[b \Lambda^2 \sqrt{\epsilon_{st} \epsilon^{st}} + \Lambda \sqrt{K} \right] (\overline{u'_i u'^{\ell i}})_{,\ell} \right)_{,k} \\
&- \left(\bar{\rho} \left[b \Lambda^2 \sqrt{\epsilon_{st} \epsilon^{st}} + \Lambda \sqrt{K} \right] (\overline{u'_k u'^{\ell i}})_{,\ell} \right)_{,i} \\
&+ \bar{\rho} \left(a \sqrt{\epsilon_{st} \epsilon^{st}} + \frac{\sqrt{K}}{\Lambda} \right) \left(\overline{u'_i u'_k} - g_{ik} \frac{K}{3} \right) \\
&+ \bar{\mu} \left[g^{j\ell} (\overline{u'_i u'_k})_{,j,\ell} - 2 \frac{\overline{u'_i u'_k}}{\lambda^2} \right] + \bar{\mu}_{,j} \left[g^{j\ell} (\overline{u'_i u'_k})_{,\ell} \right. \\
&+ \left. (\overline{u'_i u'^j})_{,k} + (\overline{u'_k u'^j})_{,i} + g^{j\ell} (\overline{u'_i u'_k} - g_{ik} \frac{K}{3})_{,\ell} \right] \\
&- \frac{\mu_s \bar{T}}{\bar{\rho}} \overline{\rho' u'_k} \left[g^{j\ell} (\bar{u}_{1,\ell} + \bar{u}_{\ell,1}) - \frac{2}{3} \delta_1^j \bar{u}^\ell_{,\ell} \right]_{,j} \\
&- \frac{\mu_s \bar{T}}{\bar{\rho}} \overline{\rho' u'_i} \left[g^{j\ell} (\bar{u}_{k,\ell} + \bar{u}_{\ell,k}) - \frac{2}{3} \delta_k^j \bar{u}^\ell_{,\ell} \right]_{,j} \\
&- \left[\left(\frac{\mu_s \bar{T}}{\bar{\rho}} \overline{\rho' u'_k} \right)_{,j} + \frac{\mu_s \bar{T} \sqrt{\rho'^2}}{2 \bar{\rho} \Lambda \sqrt{K}} (\overline{u'_j u'_k} - g_{jk} \frac{K}{3}) \right] \\
&\cdot \left[g^{j\ell} (\bar{u}_{1,\ell} + \bar{u}_{\ell,1}) - \frac{2}{3} \delta_1^j \bar{u}^\ell_{,\ell} \right] - \left[\left(\frac{\mu_s \bar{T}}{\bar{\rho}} \overline{\rho' u'_i} \right)_{,j} \right. \\
&+ \left. \frac{\mu_s \bar{T} \sqrt{\rho'^2}}{2 \bar{\rho} \Lambda \sqrt{K}} (\overline{u'_j u'_i} - g_{ji} \frac{K}{3}) \right] \left[g^{j\ell} (\bar{u}_{k,\ell} + \bar{u}_{\ell,k}) - \frac{2}{3} \delta_k^j \bar{u}^\ell_{,\ell} \right]
\end{aligned}$$