NASA CONTRACTOR REPORT

NASA CR-1736

THE INFLUENCE OF CHANGING END CONDITIONS ON THE RESONANT RESPONSE OF BEAMS AND PLATES

by D. M. Egle

Prepared by
UNIVERSITY OF OKLAHOMA
Norman, Okla.
for Langley Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1971

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-1736		
4. Title and Subtitle THE INFLUENCE OF CHANGING END CONDITIONS ON THE RESONANT RESPONSE OF BEAMS AND PLATES		5. Report Date March 1971 6. Performing Organization Code
7. Author(s) D. M. Egle		8. Performing Organization Report No.
9. Performing Organization Name and Address School of Aerospace and Mechanical Engineering The University of Oklahoma Norman, Oklahoma		10. Work Unit No. 11. Contract or Grant No. NGR 37-003-041
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Grant Report 14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

In the dynamic testing of structures, it is sometimes necessary to conduct response tests of a substructure while it is separated from the parent structure. In this report, the effect of changing end conditions on the maximum resonant response of a free-elastically supported beam excited by a spatially uniform load, and a plate, pinned on three edges and elastically supported on the fourth, excited by a concentrated force is investigated. It is demonstrated, in both cases, that it is possible to estimate the range of maximum resonant response for a wide range of elastic edge restraints by employing a single term principal mode approximation.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Vibration, Beams, Plates, Natural frequencies, End conditions, Forced response	Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	65	$3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

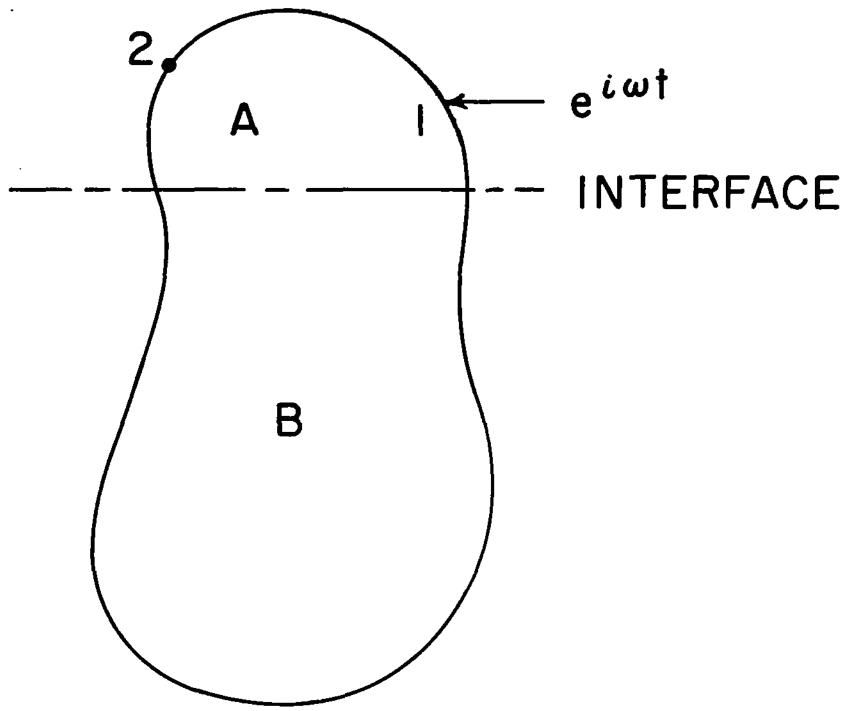
A	cross sectional area of beam
a	plate width
b	plate length
E	elastic modulus of beam or plate
E^*	complex elastic modulus, $E(1 + i\zeta)$
F_m	generalized force for the m^{th} natural mode of the beam (eqn. 2.23)
h	plate thickness
I	second moment of beam cross sectional area
K^2	a factor proportional to the generalized mass of the plate (eqn. 4.3)
k	beam wave number, $(\rho A \omega^2/E^*I)^{1/4}$
L	beam length
M_m	generalized mass for the m^{th} natural mode of the beam (eqn. 2.24)
P	magnitude of uniformly distributed load applied to beam
P_o	magnitude of concentrated load applied to the plate
p,q	plate dimensionless wave numbers (see eqn. 3.11)
Q	beam load parameter, (P/E^*I)
$u(\ )$	unit step function
W	deflection amplitude of beam, eqn. 2.3 or plate, eqn. 3.17
W'	dimensionless displacement of beam (see eqn. 2.20) or plate (see eqn. 3.16)
w	deflection of beam or plate
x	beam, plate coordinate
x_k^*	x coordinate of the k^{th} plate antinode counting from $x = 0$
x^*, y^*	antinode coordinates

y	plate coordinate
Z_{11}, Z_{12}, Z_{21}, Z_{22}	nondimensional stiffnesses of support structure (see eqn. 2.5b or 3.3)
α	support structure damping factor (see eqn. 2.19)
γ	plate aspect ratio parameter (see eqn. 3.11)
$\delta(\ )$	Dirac delta function
ζ	structural damping factor of beam or plate
ν	Poisson's ratio
ρ	density of beam or plate
φ	natural mode of the beam or plate
ω'	plate dimensionless frequency (see eqn. 3.11)

1. INTRODUCTION

In the design of a large structure, whether it be intended for use as aircraft, spacecraft, sea-going vehicle, land craft, or a stationary structure, an important step, if it is to be subjected to a dynamic load environment, is the determination of the response, which may be a displacement, acceleration, or strees, of the structure to the dynamic loads. From a knowledge of the dynamic response at various locations in the structure, the vibration or stress specifications to which the payloads of the structure itself are subjected to may be determined. Uncertainty in these specifications may well lead to overdesign and increased weight and cost or to underdesign and increased probability of failure.

The ascertainment of this dynamic response often involves the experimental determination of the characteristics of the structure subjected to a concentrated or distributed load which varies sinusoidally in time. Suppose the structure may be conveniently divided into two substructures, as indicated in figure 1.1. If the excitation, which is taken to be a concentrated force for the purpose of illustration, is being applied to substructure A at point 1 and if the response is being measured at point 2, also on substructure A, then it is reasonable to expect for excitation frequencies sufficiently high and for points 1 and 2 sufficiently distant from the interface between substructures A and B, the response at point 2 will be independent of substructure B. That is, the dynamic characteristics of substructure A, for some frequencies and points of excitation and response, will not be changed if substructure B is detached from A. The determination of the conditions for which the dynamic response of a substructure is independent of its end conditions could substantially reduce the complexity



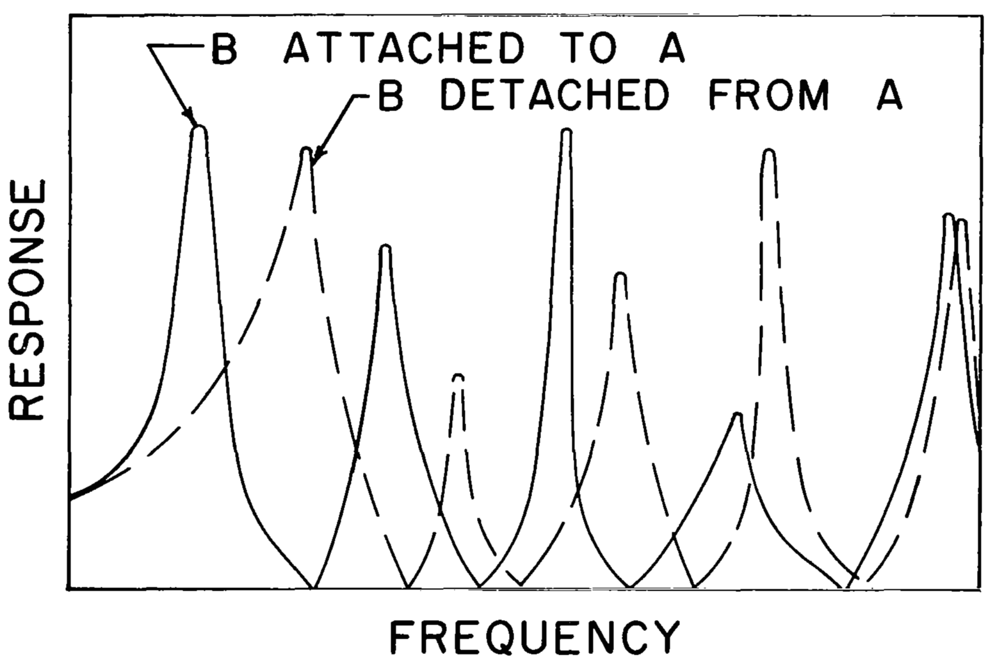


Figure 1.1 Dynamic Response of a Structure.

and hence the cost of the experimental evaluation of the dynamic characteristics of the substructure, especially if the substructure is only a small part of the total structure.

The effects of changing end conditions of a structure on the dynamic response quite obviously depends on what is meant by "dynamic response". The response-frequency plot shown in figure 1.1 illustrates the change in the displacement measured at point 2 for an excitation at point 1 if the substructure B is detached from A. The change in the response at a given frequency may be attributed to a combination of three factors. First, a change in end conditions will change the natural frequencies. If the original response is at a natural frequency of the combined structure, then even a slight change in natural frequency may result in a large decrease in response because the response of A is not evaluated at a resonance. Secondly, the normal mode shapes of the structure will change if the end conditions are varied. If either points 1 or 2 are near a node of the original structure or the modified one, the response may change substantially with only a small change in the mode. Thirdly, there is another factor which is not as easily identified as the first two but which is related to the generalized mass and the generalized force of the modes of the structures. It is this third factor with which this report is concerned.

If it is desirable to evaluate the dynamic response under the most unfavorable circumstances, then the first two factors may be eliminated by requiring that "dynamic response" refer to the resonant response at an antinode. Under this limited definition, the change in "dynamic response" does not refer to the change in response at point 2 due to an excitation of frequency ω at point 1 but rather the change in maximum response due to an excitation whose frequency and whose position maximizes the response. It is in this limited sense that the problem of determining the effect of end conditions on the dynamic response shall be approached.

There is a substantial body of published literature on the forced vibration of structures. References [1-20] are a representative but not complete list of such studies. Most of this work has dealt with the response of various types of structures subjected to different types of loading but have only incorporated simple boundary conditions (such as simply-supported, clamped, or free). However Gayman [17] studied the effect of interface flexibility on the response of A for excitation applied at the interface. White [18] presented frequency characteristics for beams with elastic and inertial end restraints and studied the response of a simply-supported beam with symmetric rotational constraints excited by a harmonic travelling pressure wave. White also discusses the natural frequencies of rectangular plates with elastic and inertial edge restraints. Hwang et al. [19,20] studied the design of simulated boundary conditions to achieve similarity in the dynamic response of shell structure-mounted components.

The results of a previous investigation [21] of the effects of boundary conditions on the forced vibration of a plate may be summarized as follows:

(1) The most important restraint in an elastic end structure is the restraint of linear deflections. The influence of rotational restraints on the displacements in the plate is confined to a region near the boundary even for the lowest mode of vibration.

(2) The point of application of the load may have a substantial effect on the response. This is primarily due to the second

factor (the change in the normal modes of the structure) mentioned above.

(3) The linear restraint of the elastic end structure strongly influences the response of the lowest mode of vibration but its influence is confined nearer the boundary as the frequency of the mode increases.

The present report is concerned with the effect of a linear restraint on the displacement response of a structure excited by a loading with a simple harmonic time variation. Two particular problems are considered: (1) that of a uniform beam, one end free and the other elastically supported, excited by a spatially uniform loading; and (2) that of a uniform plate, simply supported on three edges and elastically restrained on the fourth, excited by a concentrated force. It is demonstrated, in both cases, that it is possible to estimate the range of maximum resonant response as defined above for a wide range of elastic edge restraints by a relatively simple procedure.

2. DYNAMIC RESPONSE OF A UNIFORM FREE-ELASTICALLY SUPPORTED BEAM EXCITED BY A UNIFORM SIMPLE HARMONIC LOAD

The object of this analysis is to study the effects of the elastic properties of a boundary structure on the resonant response of a beam. The problem goemetry is shown in Fig. 2.1.

The energy dissipation in the beam will be accounted for by using the "complex" elastic modulus concept. Thus Young's modulus for the beam is taken in the form

$$E^* = E\ (1 + i\zeta) \tag{2.1}$$

where E - elastic modulus

ζ - structural damping constant

i - $\sqrt{-1}$

The equation of a beam loaded by a uniformly distributed simple harmonic load is

$$E^* I \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = P\ e^{i\omega t} \tag{2.2}$$

where I, ρ, A are the moment of inertia of the cross-section, density, and cross sectional area of the beam; P is the load per unit length and ω the frequency of the applied load. Assuming a solution in the form

$$w(x,t) = W(x)\ e^{i\omega t} \tag{2.3}$$

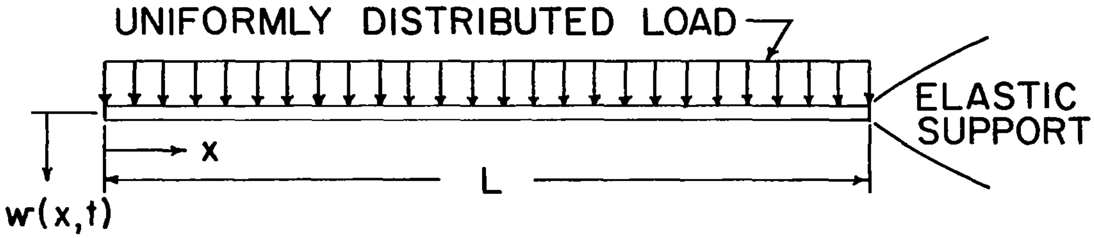
will transform eqn. 2.2 into

$$\frac{d^4 W}{dx^4} - (k)^4\ W = Q \tag{2.4}$$

where $k = (\rho A \omega^2 / E^* I)$

End Conditions

The end conditions are taken to be



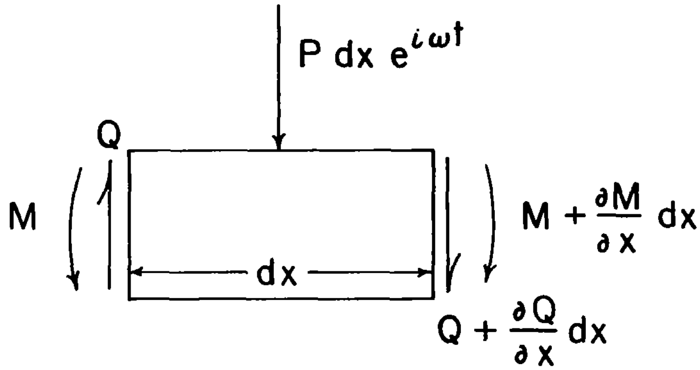


Figure 2.1 Notation and Geometry of a Free-Elastically Supported Beam Excited by a Spatially Uniform Load.

$$W_{,xx} = W_{,xxx} = 0 \qquad \text{for } x = 0 \tag{2.5a}$$

$$\left.\begin{aligned} W_{,xx} &= -\frac{Z_{11}}{L} W_{,x} - \frac{Z_{12}}{L^2} W \\ W_{,xxx} &= \frac{Z_{21}}{L^2} W_{,x} + \frac{Z_{22}}{L^3} W \end{aligned}\right\} \quad \text{for } x = L \tag{2.5b}$$

The quantities Z_{11}, Z_{12}, Z_{21}, Z_{22} are nondimensional stiffnesses of the support structure and may be defined as follows:

$$Z_{11} = \frac{L}{EI} \text{ (Moment/unit angular deflection)}_{x=b}$$

$$Z_{12} = \frac{L^2}{EI} \text{ (Moment/unit deflection)}_{x=b}$$

$$Z_{21} = \frac{L^2}{EI} \text{ (Vertical Force/unit angular deflection)}_{x=b}$$

$$Z_{22} = \frac{L^3}{EI} \text{ (Vertical Force/unit deflection)}_{x=b}$$

Thus Z_{11} and Z_{22} are rotational and linear stiffnesses, respectively, and Z_{12}, Z_{21} are stiffnesses coupling linear and rotatory displacements. It is assumed that these stiffnesses are independent of frequency.

If it is further assumed that the supporting structure is elastic and obeys Betti's law, then it may be shown that $Z_{12} = Z_{21}$. To demonstrate this first invert eqns. 2.5b.

$$\begin{Bmatrix} W_{,x} \\ W \end{Bmatrix}_L = \frac{1}{(Z_{21}\, Z_{12} - Z_{11}\, Z_{22})} \begin{bmatrix} Z_{22}L & Z_{12}L^2 \\ -Z_{21}L^2 & -Z_{11}L^3 \end{bmatrix} \begin{Bmatrix} W_{,xx} \\ W_{,xxx} \end{Bmatrix}_L \tag{2.6}$$

Since the moment and shear force at the end of the beam are

$$\begin{aligned} M(L) &= EIW_{,xx}(L) \\ V(L) &= -EIW_{,xxx}(L) \end{aligned} \tag{2.7}$$

eqn. 2.6 may be written

$$\begin{Bmatrix} W_{,x} \\ W \end{Bmatrix}_L = \frac{1}{EI(Z_{12}\,Z_{21} - Z_{11}\,Z_{22})} \begin{bmatrix} Z_{22}L & -Z_{12}L^2 \\ -Z_{21}L^2 & Z_{11}L^3 \end{bmatrix} \begin{Bmatrix} M \\ V \end{Bmatrix}_L \qquad (2.8)$$

Betti's Law requires the deflection, W_1, due to a moment, M_1, (but zero shear force) applied to the boundary structure and the slope, $(W_{,x})_2$, due to a shear force, V_2, (but zero moment) applied to the boundary structure be related by

$$M_1 (W_{,x})_2 = V_2 W_1 \qquad (2.9)$$

Eqns. 2.8 may be employed to calculate W_1 and $(W_{,x})_2$, thus

$$W_1 = \frac{-Z_{21}\,L^2\,M}{EI(Z_{12}\,Z_{21} - Z_{11}\,Z_{22})}$$

$$(W_{,x})_2 = \frac{-Z_{12}\,L^2\,V}{EI(Z_{12}\,Z_{21} - Z_{11}\,Z_{22})}$$

Equation 2.9 is satisfied only if $Z_{12} = Z_{21}$, as long as $Z_{12}\,Z_{21} - Z_{11}\,Z_{22} \neq 0$, which if not true would imply that the transformation from forces to deflection (i.e. eqns. 2.8) is singular and this is disregarded on physical grounds.

The range of the coeffieients Z_{11}, Z_{12}, Z_{21} may be further restricted if it is assumed that the boundary structure is passive, i.e. if a force or moment is applied to the boundary structure energy flows into the structure. If this is the case, the work done by those forces must be positive. The work done by the boundary forces acting on the boundary structure is

$$\psi = -\tfrac{1}{2} \begin{Bmatrix} M \\ V \end{Bmatrix}_L^T \begin{Bmatrix} W_{,x} \\ W \end{Bmatrix}_L \qquad (2.10)$$

and combining eqns. 2.5b and 2.7

$$\begin{Bmatrix} M \\ V \end{Bmatrix}_L = -EI \begin{bmatrix} Z_{11}/L & Z_{12}/L^2 \\ Z_{12}/L^2 & Z_{22}/L^3 \end{bmatrix} \begin{Bmatrix} W_{,x} \\ W \end{Bmatrix}_L \tag{2.11}$$

Substituting eqn. 2.11 into 2.10 yields

$$\psi = + \frac{EI}{2} \begin{Bmatrix} W_{,x} \\ W \end{Bmatrix}^T \begin{bmatrix} Z_{11}/L & Z_{12}/L^2 \\ Z_{12}/L^2 & Z_{22}/L^3 \end{bmatrix} \begin{Bmatrix} W_{,x} \\ W \end{Bmatrix} \tag{2.12}$$

A necessary condition for ψ to be positive is that the determinant of the matrix in eqn 2.12 be positive, thus

$$Z_{11}\, Z_{22} > Z_{12}^{\;2} \tag{2.13}$$

It should be noted that eqn. 2.13 pertains to elastic structures exhibiting no dynamic effects, i.e. Z_{11}, Z_{12}, Z_{22} are independent of frequency.

Beam Response

The general solution of eqn. 2.4 is

$$W(x) = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx - Q\,k^{-4} \tag{2.14}$$

The constants C_1 - C_4 may be evaluated by requiring eqn. 2.14 to satisfy the boundary conditions, eqns. 2.5. Equations 2.5a require that

$$C_2 = C_4$$

$$C_1 = C_3$$

Thus eqn. 2.14 may be reduced to

$$W(x) = C_1\,(\sin kx + \sinh kx) + C_2\,(\cos kx + \cosh kx) - Qk^{-4} \tag{2.15}$$

Substituting eqn. 2.15 in eqns. 2.5b results in

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad (2.16)$$

where

$D_{11} = S-s + (Z_{11}/kL)\ (C+c) + (Z_{12}/k^2L^2)\ (S+s)$

$D_{12} = C-c + (Z_{11}/kL)\ (S-s) + (Z_{12}/k^2L^2)\ (C+c)$

$D_{21} = C-c - (Z_{21}/k^2L^2)\ (C+c) - (Z_{22}/k^3L^3)\ (S+s)$

$D_{22} = S+s - (Z_{21}/k^2L^2)\ (S-s) - (Z_{22}/k^3L^3)\ (C+c)$

$F_1 = + (Z_{12}/k^2L^2)\ Qk^{-4}$

$F_2 = - (Z_{22}/k^2L^2)\ Qk^{-4}$

$s = \sin kL$

$c = \cos kL$

$S = \sinh kL$

$C = \cosh kL$

The solutions to eqn. 2.16 are

$$C_1 = (F_1\ D_{22} - F_2\ D_{12})/D$$
$$C_2 = (F_2\ D_{11} - F_1\ D_{22})/D \qquad (2.17)$$

where $D = D_{11}\ D_{22} - D_{12}\ D_{21}$ (2.18)

If $\zeta = 0$, D is the frequency determinant from which the natural frequencies may be determined.

The resonant response of the beam may be calculated by first computing the values of kL which satisfy

$$D = 0$$

for $\zeta = 0$. Then eqn. 2.17, 2.18 and 2.15 may be evaluated using a nonzero value for the damping factor of the beam. Note that if the Z_{ij}'s are taken to be real quantities it is implicity assumed that the damping factor of the boundary structure is equal to ζ. If the boundary damping factor is not equal to ζ but instead α, the Z_{ij}'s must be modified according to

$$Z_{ij} = Z'_{ij} \left(\frac{1 + i\alpha}{1 + i\zeta}\right) \tag{2.19}$$

where Z'_{ij} is the boundary stiffness for zero damping.

Figure 2.2 shows the variation of kL (which is proportional to the square root of the natural frequency) for the m^{th} lowest resonance over a wide range of values of Z_{22}. For this plot Z_{11}, Z_{12} and Z_{21} are zero; thus the end condition may be represented by a massless linear spring as shown at the top of the figure.

Figures 2.3, 2.4 and 2.5 show the resonant response of the beam as a function of position along the length of the beam. The damping factors for the beam and the supports were both taken to be 0.01. The response at the lowest and second lowest resonances (figures 2.3 and 2.4) for Z_{22} larger than 10^4 is essentially the same as that for $Z_{22} = 10^4$. The dimensionless response plotted in these three figures is

$$W' = \frac{\rho A \omega^2 \zeta}{P} |W| \tag{2.20}$$

Figure 2.6 shows the peak dimensionless displacements plotted as a function of Z_{22} for the lowest four resonances. Note that for the lowest resonance and $Z_{22} < 10$, the maximum displacement occurs at $x/L = 1$ and is not a relative maximum in the sense that the slope is zero. For the second, third, and fourth resonances, the maximum at $x/L = 1$, if it existed, was ignored and only the relative maxima were plotted. Also for the higher resonances, there often is more than one relative maximum, but as is shown in several cases in figure 2.6, the maxima are nearly equal.

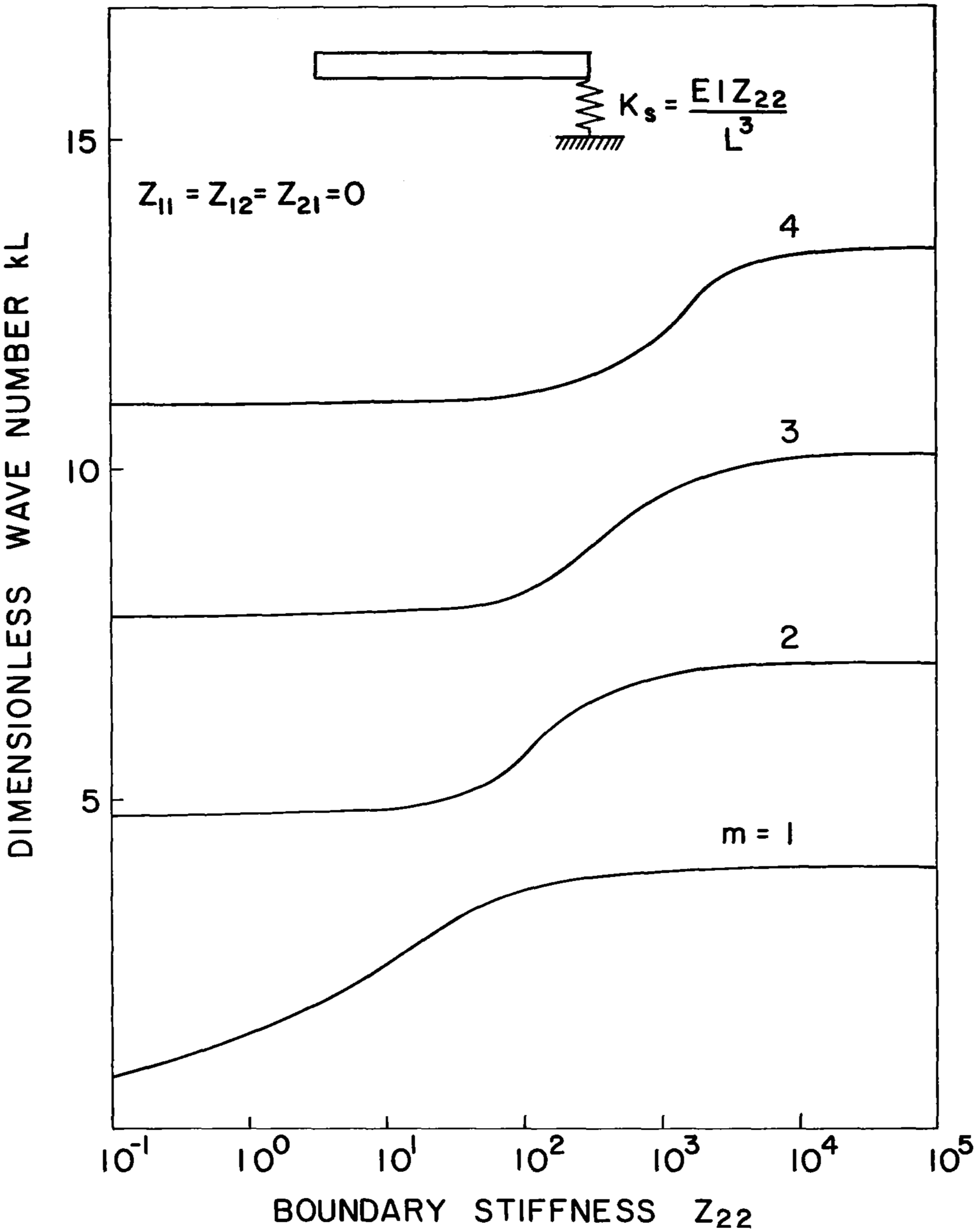


Figure 2.2 Dimensionless Wave Number as a Function of Boundary Stiffness for a Free-Elastically Supported Beam.

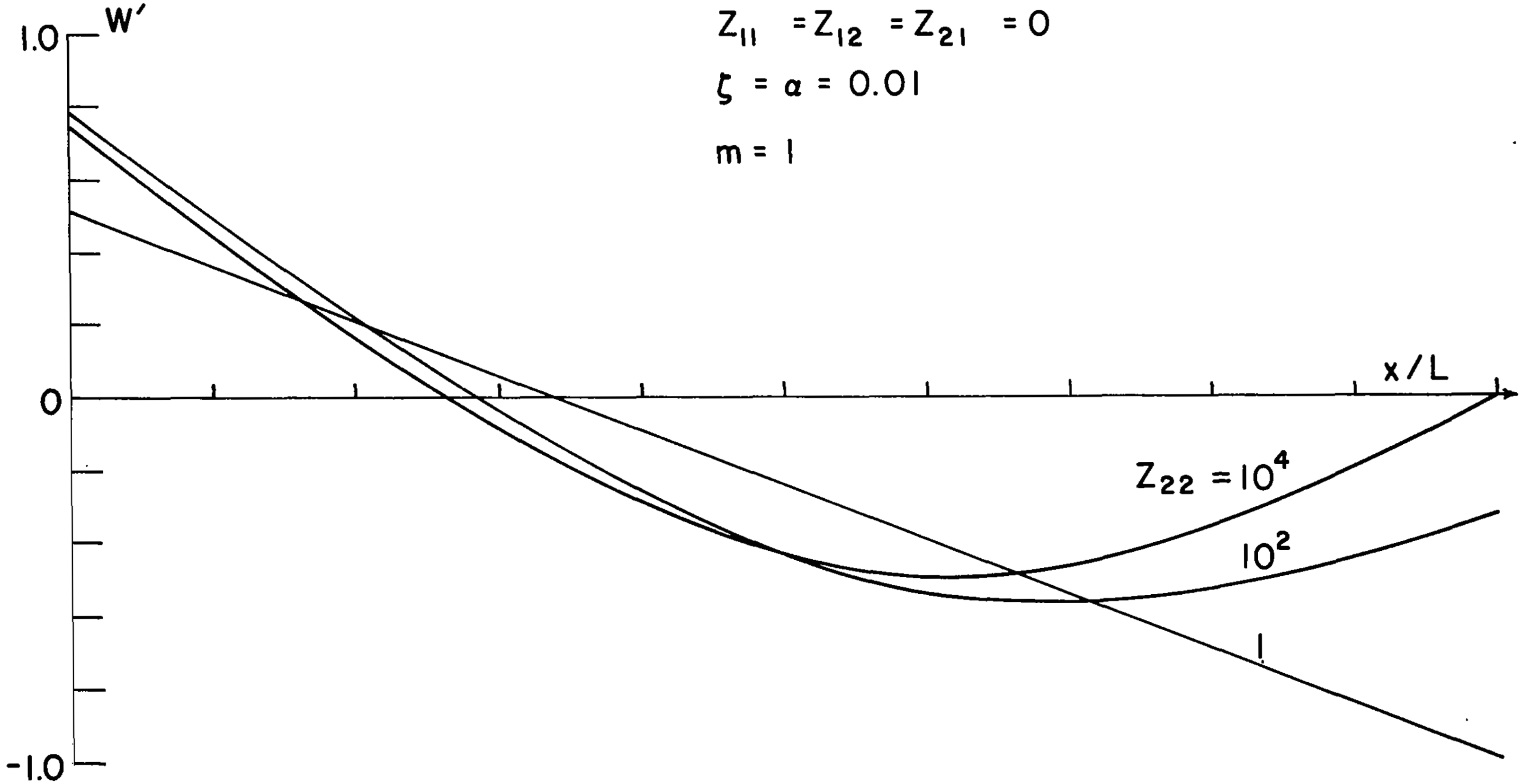


Figure 2.3 Resonant Response of a Free-Elastically Supported Beam-Fundamental Mode.

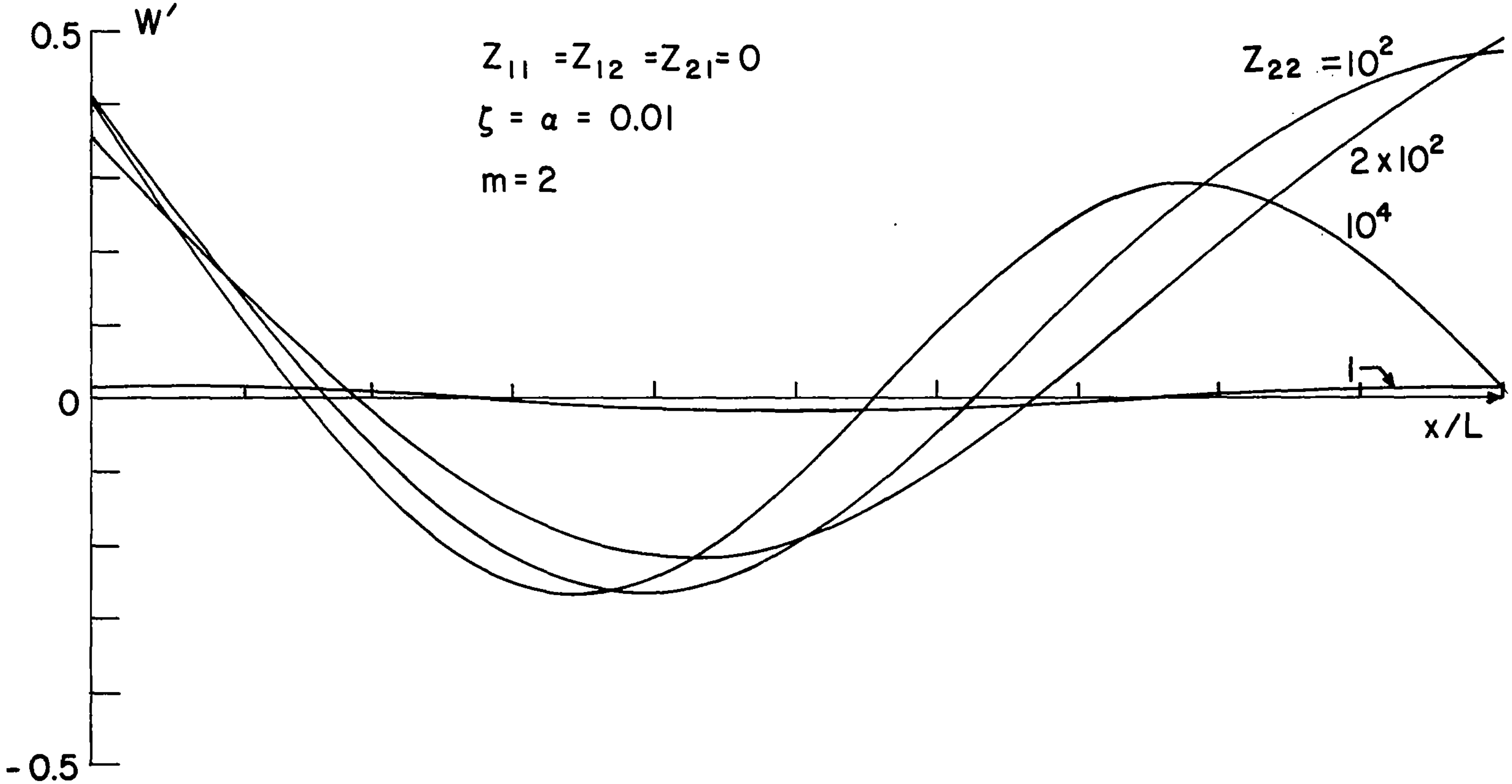


Figure 2.4 Resonant Response of a Free-Elastically Supported Beam - Second Mode.

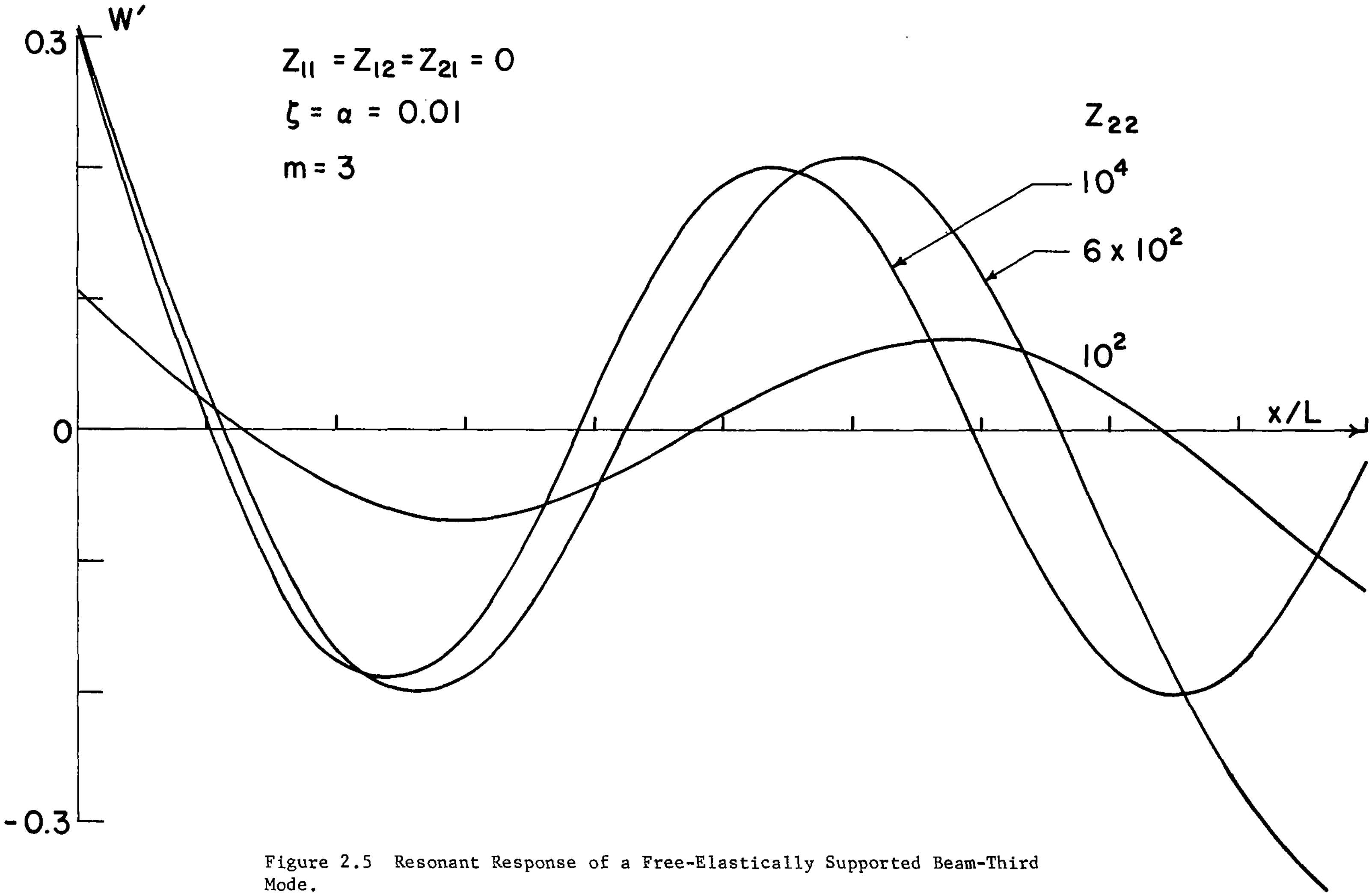


Figure 2.5 Resonant Response of a Free-Elastically Supported Beam-Third Mode.

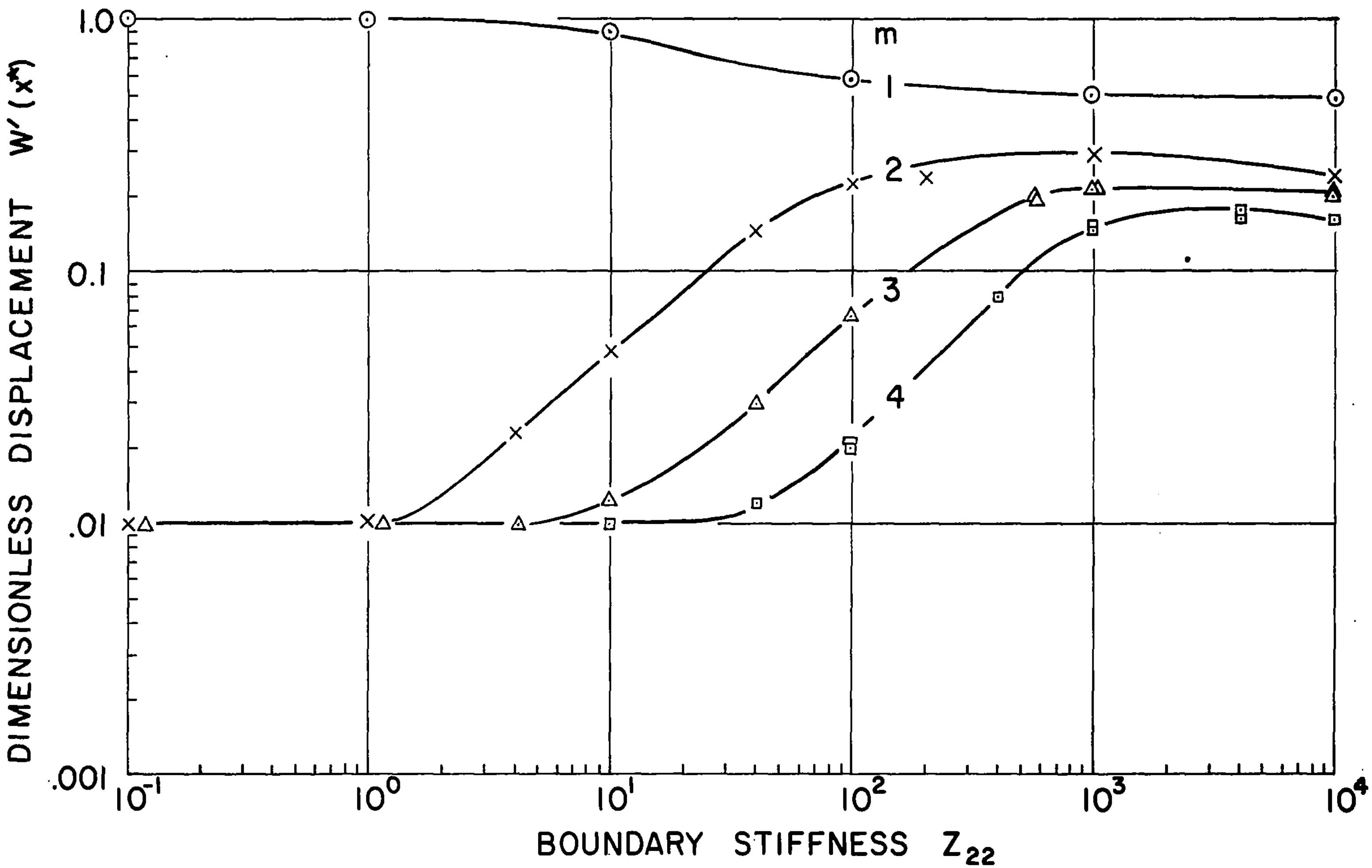


Figure 2.6 Maximum Resonant Response as a Function of Boundary Stiffness for a Free-Elastically Supported Beam.

Note that the peak displacements for the three higher modes (m = 2,3,4) shown in figure 2.6 approach a common value asymptotically as Z_{22} approaches zero. It may be shown by examining the normal mode solution discussed below that this asymptotic value is equal to the structural damping factor ζ. The mode shape for this case is a displacement which is practically uniform along the beam and out of phase with the loading.

Single Term Normal Mode Solution

This same problem may be solved in an approximate manner by using a normal mode approach. The results obtained lend some insight into interpreting the response plots shown previously.

If the response w(x,t) is expanded in a normal mode series, i.e.

$$w(x,t) = \sum_{j=1}^{\infty} \varphi_j(x)\, q_j(t) \tag{2.21}$$

where $\varphi_j(x)$ is the jth mode of the free vibration problem and satisfies

$$EI \frac{d^4\varphi}{dx^4} = \rho A \omega_j^2 \varphi$$

where ω_j is the j^{th} natural frequency, then it can be shown that the solution to eqn. 2.2 is

$$w(x,t) = \sum_{j=1}^{\infty} \frac{P F_j \varphi_j(x) e^{i\omega t}}{\rho A M_j (\omega_j^2 - \omega^2 + i\zeta\omega_j^2)} \tag{2.22}$$

where

$$F_j = 1/L \int_o^L \varphi_j \, dx \tag{2.23}$$

and

$$M_j = 1/L \int_0^L \varphi_j^2 \, dx \qquad (2.24)$$

If eqn. 2.22 is evaluated at the m^{th} resonance ($\omega = \omega_m$) and if F_m is not small compared to the rest of F_j's then the response may be approximated by

$$w \sim \frac{PF_m \varphi_m(x) \, e^{i\omega_m t}}{\rho A M_m \zeta \omega_m^2}$$

or, in terms of the dimensionless displacement W' (see eqn. 2.20)

$$W' \sim \frac{F_m \varphi_m(x)}{M_m} \qquad (2.25)$$

The normal modes may be determined from eqns. 2.15-2.18 by setting $Q = 0$. Thus

$$\varphi_m(x) = \sin k_m x + \sinh k_m x + R \, (\cos k_m x + \cosh k_m x) \qquad (2.26)$$

where $R = - D_{11}^m / D_{12}^m = - D_{21}^m / D_{22}^m$

and D_{11}^m, D_{12}^m, D_{21}^m, D_{22}^m are equal to D_{11}, D_{12}, D_{21}, D_{22} (following eqn. 2.16) evaluated at the m^{th} natural frequency.

It was shown in reference [21] that the stiffnesses Z_{11}, Z_{12}, Z_{21} have little effect on the frequencies or resonant response of a plate. Assuming that this is also true for the beam under consideration, Z_{11}, Z_{12}, Z_{21} will be set equal to zero for the remainder of this section. The quantities R, M_m, and F_m may then be shown to equal

$$R = - (S-s)/(C-c) \qquad (2.27)$$

$$M_m = (1/k_m L) \left\{ sC - cS + \tfrac{1}{2} (SC-sc) + R(s+S)^2 + R^2 \left[k_m L + (cS + sC) + \tfrac{1}{2} (SC + sc) \right] \right\} \qquad (2.28)$$

$$F_m = (1/k_m L) \left\{ C-c + R(s-S) \right\} \qquad (2.29)$$

where $s = \sin(k_m L)$ $c = \cos(k_m L)$

$S = \sinh(k_m L)$ $C = \cosh(k_m L)$

The variations of F_m, M_m and the ratio F_m/M_m with boundary stiffness, Z_{22}, are shown in figures 2.7, 2.8, and 2.9 for the four lowest resonances of the beam. Note that the variation in F_m accounts for the major part of the change in $W'(x^*)$ shown in figure 2.6. The variations in M_m are much smaller especially for the higher modes, $m \geq 2$.

Since the peak resonant response is of interest, it is necessary to determine the maxima of $\varphi_m(x)$. Let x_ℓ^* be the location of the ℓth anti-node of $\varphi_m(x)$ counting from $x = 0$; then x_ℓ^* is the root of

$$\cos(k_m x_\ell^*) + \cosh(k_m x_\ell^*) = R\,(\sin(k_m x_\ell^*) - \sinh(k_m x_\ell^*)) \tag{2.30}$$

Combining eqns. 2.26 and 2.30, the peak values of φ_m are seen to be

$$\varphi_m(x_\ell^*) = \sin(k_m x_\ell^*) + \sinh(k_m x_\ell^*) - R^2\,(\sinh(k_m x_\ell^*) - \sin(k_m x_\ell^*)) \tag{2.31}$$

Note that from figure 2.2

$$(m-\tfrac{1}{2})\pi \leq k_m L \leq (m+\tfrac{1}{4})\,\pi\,;\; m > 1 \tag{2.32}$$

For large values of kL (say $kL > 3\pi/2$), it may be shown that

$$R \sim -1 + K(-1)^m\, e^{-k_m L} \tag{2.33}$$

where $-2\sqrt{2} \leq K \leq 0$

Substituting eqn. 2.33 into 2.31, gives the following approximate peak values of φ_m

$$\varphi_m(x_\ell^*) \sim 2(1-K(-1)^m\, e^{-k_m L})\,\sin(k_m x_\ell^*) + 2K\,(-1)^m\, e^{-k_m L}\,\sinh(k_m x_\ell^*) \tag{2.34}$$

The roots of eqn. 2.30 may be written in the form

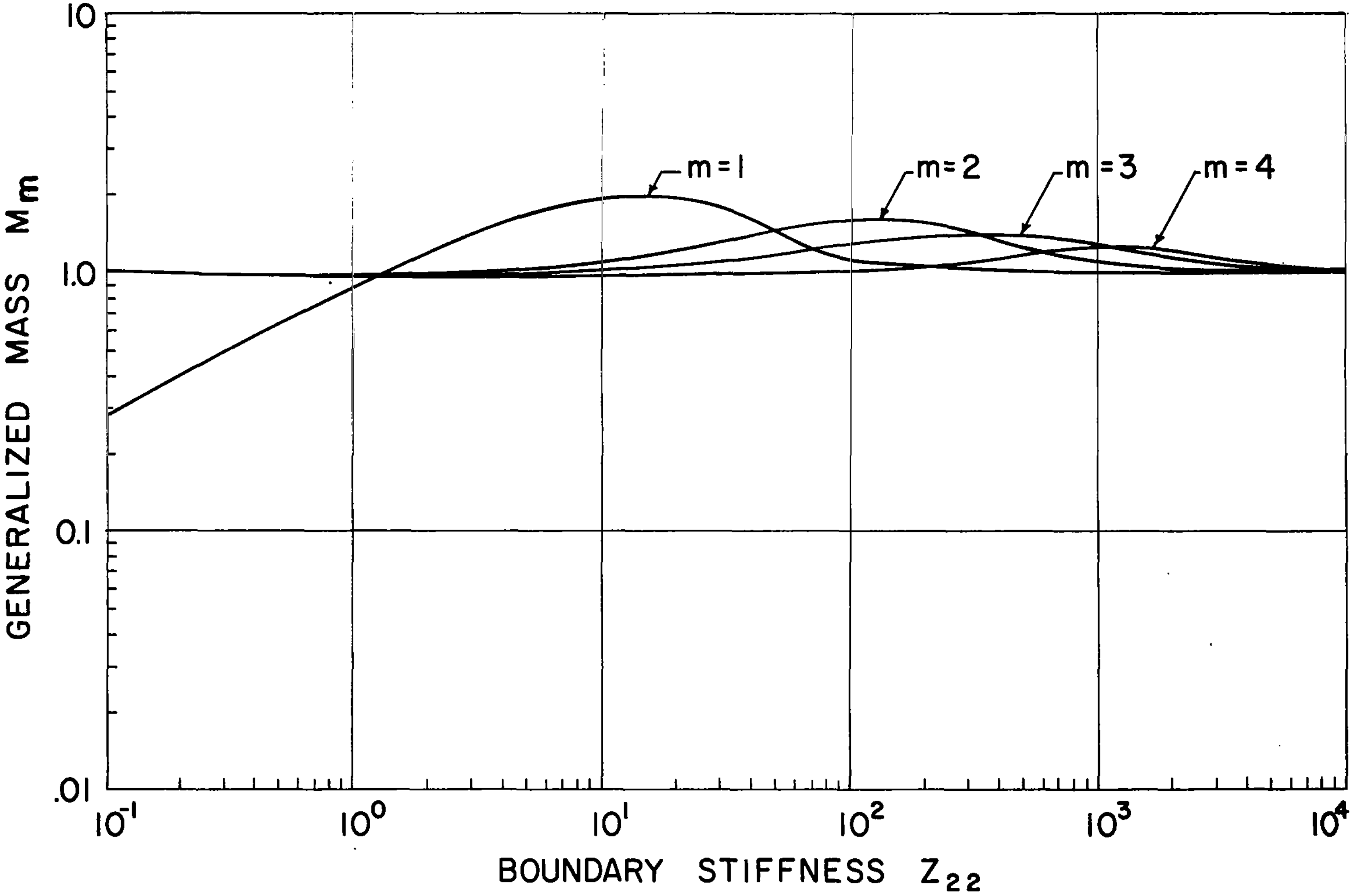


Figure 2.7 Generalized Mass for a Free-Elastically Supported Beam.

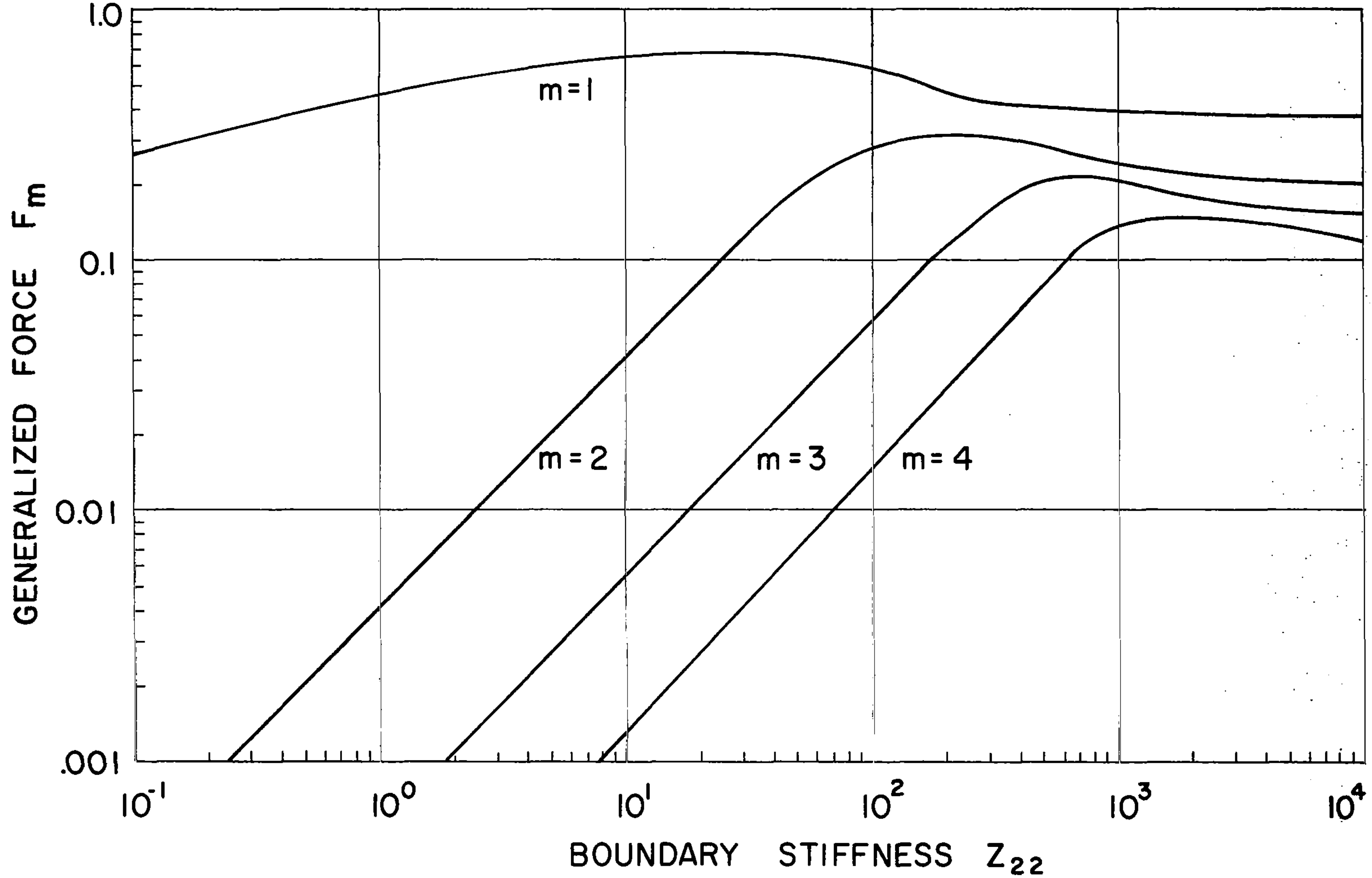


Figure 2.8 Generalized Force for a Free-Elastically Supported Beam.

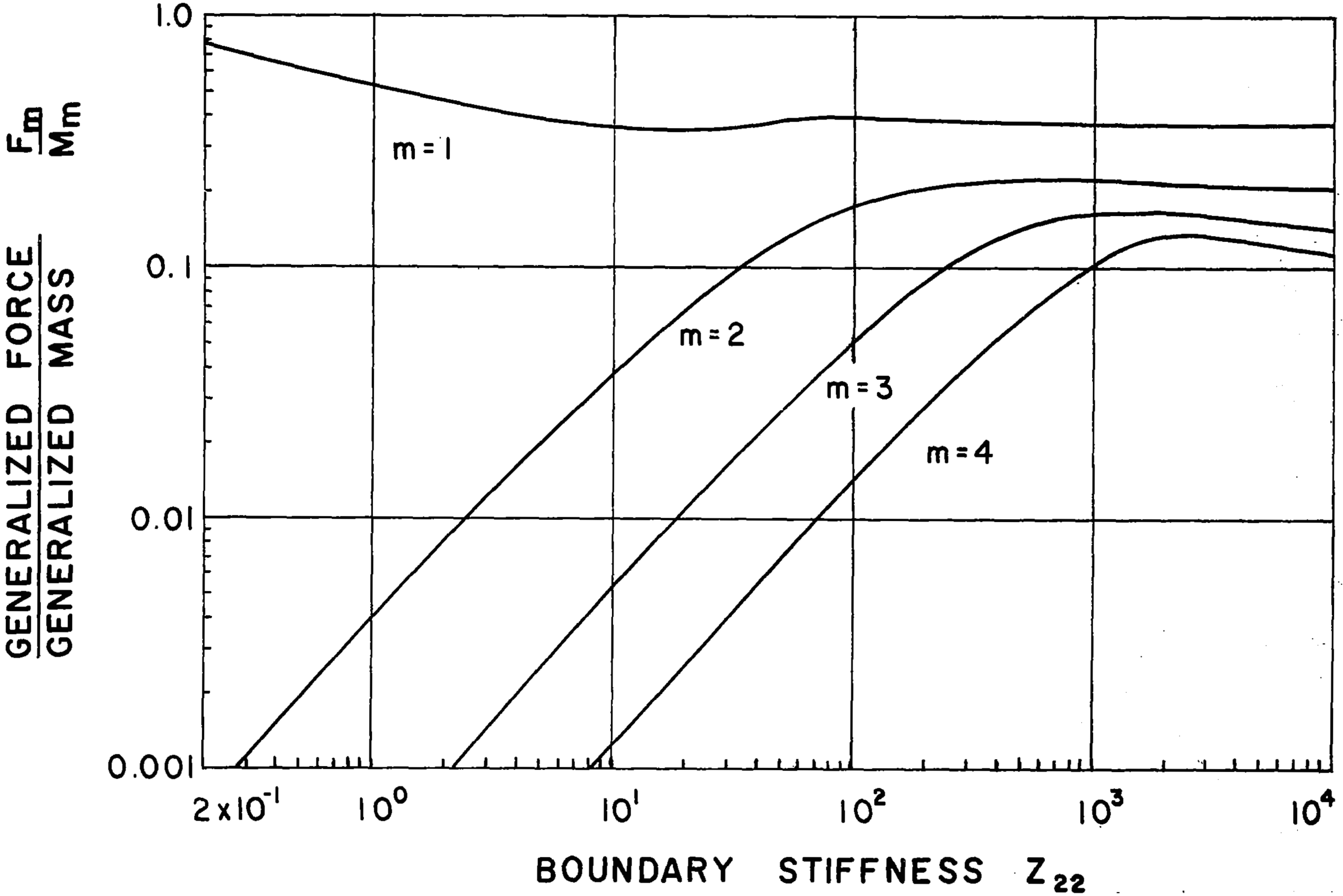


Figure 2.9 Ratio of Generalized Force to Generalized Mass for a Free Elastically Supported Beam.

$$k_m x_\ell^* = (\ell - \tfrac{1}{4})\pi + \Delta; \quad \ell = 1,2, \ldots m-1 \tag{2.35}$$

The limitation $\ell < m$ is necessary because $x_m^* > L$ for some values of Z_{22}. The quantity Δ is small and may be shown, by combining eqs. 2.30, 2.33, 2.35 to be approximated by

$$\Delta = - (H_1 + H_2)/(\sqrt{2}\ (-1)^\ell - H_1 + H_2) \tag{2.36}$$

where

$$H_1 = \tfrac{1}{2}\sqrt{2}\ K(-1)^{m+\ell}\ e^{-k_m L} + e^{\pi(\frac{1}{4} - \ell)}$$

$$H_2 = \tfrac{1}{2}K(-1)^m\ e^{(-k_m L + (\ell - \frac{1}{4})\pi)}$$

That Δ is small may be established from eqn. 2.36. The extreme values of Δ occur when K is at one of its limits and may be shown to be, for $\ell=1$, m=2 (the lowest mode to which eqn. 2.35 is applicable)

$$-0.013 \le \Delta \le 0.063$$

The effect of Δ on $\varphi(x_\ell^*)$ may be approximated by combining eqns. 2.34, 2.35 and 2.36 and neglecting terms of order exp $(-2(\ell-\frac{1}{4})\pi)$. Thus

$$\varphi(x_\ell^*) \sim -\sqrt{2}\ (-1)^\ell\ (1+\Delta) - 2e^{-(\ell-\frac{1}{4})\pi}\ (1-\Delta) \tag{2.37}$$

The effect of the boundary stiffness, Z_{22}, on the peak resonant displacements, $W'(x_\ell^*)$, may be estimated using eqns. 2.25, 2.37 and figure 2.9. For example, suppose the boundary condition at x = L changes from a simple support to an elastic support with finite Z_{22}. What will be the effect on the resonant response for m = 2?

The response for the simply supported end may be evaluated by noting that for $Z_{22} = 10^4$, $F_2/M_2 = 0.20$ from figure 2.9. Since $-2\sqrt{2} \le K \le 0$, the range of Δ (using 2.36) is $-0.013 \le \Delta \le 0.065$. Employing eqn. 2.37, $1.20 \le \varphi(x_1^*) \le 1.32$, and from eqn. 2.25, $0.24 \le W'(x_1^*) \le 0.264$ for the simply supported end. The actual value for $W'(x_1^*)$ is 0.267 as shown in figure 2.4.

The maximum response of the beam for arbitrary Z_{22} is proportional to the ratio F_2/M_2 which reaches a maximum of 0.23 (see figure 2.9) for $Z_{22} = 6 \times 10^2$. Since the maximum value of $\varphi(x_1*) = 1.32$, the maximum resonant response $W'(x_1*) = 0.304$. The minimum resonant response is zero since $F_2/M_2 \rightarrow 0$ as $Z_{22} \rightarrow 0$. In comparison, the exact values for the maximum and minimum resonant responses are 0.29 and 0.01 as shown in figure 2.6.

A comparison of the approximate limits of the peak resonant response calculated as shown above to the exact limits determined from the solution presented in the previous section and plotted in figure 2.6 is shown in table 2.1. The lower approximate limits are actually inaccurate because they reflect the fact that F_m is zero. The assumption that F_m is not small compared to the remainder of the F_j's (see paragraph following eqn. 2.24) is invalidated. However, agreement between the actual and approximate upper limits is satisfactory.

m	ℓ	Exact Limits (Figure 2.6)	Approximate Limits
2	1	$0.01 \leq W'(x_\ell^*) \leq 0.29$	$0 \leq W'(x_\ell^*) \leq 0.304$
3	2	$0.01 \leq W'(x_\ell^*) \leq 0.22$	$0 \leq W'(x_\ell^*) \leq 0.234$
4	3	$0.01 \leq W'(x_\ell^*) \leq 0.18$	$0 \leq W'(x_\ell^*) \leq 0.207$

Table 2.1 Comparison of Exact Limits to Approximate Limits of the Resonant Response of a Free-Elastically Supported Beam.

3. DYNAMIC RESPONSE OF A RECTANGULAR PLATE WITH AN ELASTICALLY SUPPORTED EDGE.

The equation of motion for a rectangular thin uniform isotropic and homogeneous plate acted on by a simple harmonic point load is

$$D^* \nabla^4 w + \rho h \ddot{w} = P_o \, \delta (x - x_o) \, \delta (y - y_o) \, e^{i\omega t} \tag{3.1}$$

where $D^* = D (1 + i \zeta)$ is a complex elastic modulus, which accounts for energy dissipation in the plate, ζ is the structural damping factor assumed to be independent of frequency.

The boundary conditions to be imposed on eqn. 3.1 are

$$\begin{aligned} &w (0,y,t) = w (x,o,t) = w (x,a,t) = 0 \\ &w,_{xx}(o,y,t) = w,_{yy}(x,o,t) = w,_{yy}(x,a,t) = 0 \end{aligned} \tag{3.2}$$

and, on the fourth edge, x = b

$$\begin{aligned} &-b^3 \left[w,_{xx} + \nu \, w,_{yy} \right] = b^2 Z_{11} \, w,_x + Z_{12} \, b \, w \\ &b^3 \left[w,_{xxx} + (2-\nu) \, w,_{xyy} \right] = Z_{12} b \, w,_x + Z_{22} \, w \end{aligned} \tag{3.3}$$

Equations 3.3 are an analytical representation of an elastic edge restraint along x = b. The edge restraint is not all inclusive in that the moment per unit length and the force per unit length at position y are assumed to depend only on the displacement and slope at that same position; whereas, in general, the moments and forces at y would be a function of the displacements and slopes along the entire edge, $0 \le y \le a$. Physically, eqns. 3.3 represent a set of closely spaced and equal, but independent, point elastic supports having a linear and a rotary restraint and cross coupling between linear and rotary motion.

The parameters, Z_{11}, Z_{12}, Z_{22} are nondimensional stiffnesses per unit length of the boundary structure and may be defined physically as follows:

Z_{11} = (b/D*)(moment per unit length per unit slope acting on the boundary structure needed to produce a unit slope and zero deflection at the boundary)

Z_{12} = (b^2/D*)(moment per unit length per unit deflection acting on the boundary structure needed to produce a unit deflection and zero slope at the boundary)

Z_{22} = (b^3/D*)(force per unit length per unit deflection acting on the boundary structure needed to produce a unit deflection and zero slope at the boundary)

Thus Z_{11} and Z_{22} are rotational and linear stiffnesses per unit length, respectively, and Z_{12} is a stiffness coupling linear and rotary displacements. It is assumed that these stiffnesses are independent of frequency.

If the boundary structure is further assumed to be passive (no energy sources) and stable then its potential energy function must be positive definite. Following a procedure similar to that in section 2 of this report or in reference [21], it may be shown that for the potential energy of the boundary structure to be positive definite, the following restrictions on the Z_{ij}'s are required

$$Z_{22} > 0 \quad ; \quad Z_{11} Z_{22} - Z_{12}^2 > 0$$

In the following discussion, the boundary structure will be taken to be positive definite with the single exception of the case of a completely free edge ($Z_{11} = Z_{12} = Z_{22} = 0$).

Assume a solution to eqn. 3.1 in the form

$$w\ (x,y,t) = \sum_{n=1}^{\infty} W_n(x)\ \sin\ (n\pi y/a)\ e^{i\omega t} \tag{3.4}$$

which, when substituted into eqn. 3.1 yields

$$\sum_{n=1}^{\infty} \left\{ \frac{d^4 W_n}{dx^4} - 2\left(\frac{n\pi}{a}\right)^2 \frac{d^2 W_n}{dx^2} + \left[\left(\frac{n\pi}{a}\right)^4 - \frac{\rho\omega^2}{D^*} \right] W_n \right\} \sin\ (n\pi y/a)$$

$$= (P_o/D^*)\ \delta(x-x_o)\ \delta(y-y_o) \tag{3.5}$$

Equation 3.5 may be uncoupled into the following set of differential equations by multiplying by sin $(m\pi y/a)$ dy and integrating from o to a.

$$\frac{d^4W_n}{dx^4} - 2\left(\frac{n\pi}{a}\right)^2 \frac{d^2W_n}{dx^2} + \left[\left(\frac{n\pi}{a}\right)^4 - \frac{\rho\omega^2}{D^*}\right] W_n$$
$$= (2P_o/aD^*) \sin (n\pi y_o/a)\, \delta(x - x_o) \qquad (3.6)$$

Equation 3.4 satisfies the two boundary conditions requiring the edges $y = o,a$ to be simply supported. The two remaining boundary conditions are

$$W_n\ (0) = \frac{d^2W_n}{dx^2}\ (0) = 0 \qquad (3.7)$$

$$-b^3\left[\frac{d^2W_n}{dx^2} - \nu\left(\frac{n\pi}{a}\right)^2 W_n\right] = b^2 Z_{11}\frac{dW_n}{dx} + Z_{12}\, b\, W_n\Bigg|_{x=b} \qquad (3.8)$$

$$b^3\left[\frac{d^2W_n}{dx^2} - (2-\nu)\left(\frac{n\pi}{a}\right)^2 \frac{dW_n}{dx}\right] = b\, Z_{12}\frac{dW_n}{dx} + Z_{22}W_n\Bigg|_{x=b} \qquad (3.9)$$

The solution to eqn. 3.6 is straightforward and may be found by several methods. The solution which satisfies eqn. 3.7 is

$$W_n(x) = A_n \sinh (qx/b) + B_n \sin (px/b)$$
$$+ C_n\Big[(1/q) \sinh (q(x-x_o)/b)$$
$$- (1/p) \sin (p(x-x_o)/b)\Big]\ u\ (x-x_o) \qquad (3.10)$$

where

$$p^2 = \omega' - \gamma^2$$
$$q^2 = \omega' + \gamma^2$$
$$\gamma = n\pi b/a$$
$$\omega' = \omega b^2 (\rho h/D^*)^{\frac{1}{2}} \qquad (3.11)$$

$$C_n = (b^3 P_o / aD^* \omega') \sin (n\pi y_o/a)$$

$$u(x-x_o) = \begin{cases} 0 , & x < x_o \\ 1 , & x > x_o \end{cases} \qquad (3.11) \text{ Cont'd}$$

The constants A_n, B_n are chosen to satisfy the remaining end conditions, eqn. 3.8, 3.9. Substituting eqn. 3.10 into 3.8 and 3.9 results in

$$\begin{aligned} D_{11} A_n + D_{12} B_n &= F_1 \\ D_{21} A_n + D_{12} B_n &= F_2 \end{aligned} \qquad (3.12)$$

where

$$D_{11} = -(q^2 - \nu \gamma^2 + Z_{12}) S - qZ_{11} C$$

$$D_{12} = (p^2 + \nu \gamma^2 - Z_{12}) s - pZ_{11} c$$

$$D_{21} = q(p^2 + \nu \gamma^2 - Z_{12}) C - Z_{22} S$$

$$D_{22} = - p (q^2 - \nu \gamma^2 + Z_{12}) c - Z_{22} s$$

$$F_1 = \Big\{ (1/q)(q^2 - \nu \gamma^2 + Z_{12}) S' + Z_{11} C' + (1/p) (p^2 + \nu \gamma^2 - Z_{12}) s' - Z_{11} c' \Big\} C_n$$

$$F_2 = \Big\{ -(p^2 + \nu \gamma^2 - Z_{12}) C' + (1/q) Z_{22} S' - (q^2 - \nu \gamma^2 + Z_{12}) c' - (1/p) Z_{22} s' \Big\} C_n$$

$s = \sin (p)$ $c = \cos (p)$

$S = \sinh (q)$ $C = \cosh (q)$

$s' = \sin p (1-x_o/b)$ $c' = \cos p (1-x_o/b)$

$S' = \sinh q (1-x_o/b)$ $C' = \cosh q (1-x_o/b)$

The solutions to eqns. 3.12 are

$$A_n = (C_n/q \det(D)) \Big\{ -2\omega' (Z_{22} s S' + pq Z_{11} c C')$$

$$+ q \Big[(\omega')^2 - \big[(1-\nu) \gamma^2 + Z_{12} \big]^2 + Z_{11} Z_{22} \Big] (s c' - s' c)$$

$$- p \Big[(q^2 - \nu \gamma^2 + Z_{12})^2 - Z_{11} Z_{22} \Big] c S'$$

$$+ q \Big[(p^2 + \nu \gamma^2 - Z_{12})^2 - Z_{11} Z_{12} \Big] s C' \Big\} \qquad (3.13)$$

$$B_n = (C_n/p \det(D)) \Big\{ 2 \omega' (Z_{22} s' S + pq Z_{11} c' C)$$

$$+ p \Big[(\omega')^2 - \big[(1-\nu) \gamma^2 + Z_{12} \big]^2 + Z_{11} Z_{22} \Big] (SC' - S' C)$$

$$+ p \Big[(q^2 - \nu \gamma^2 + Z_{12})^2 - Z_{11} Z_{22} \Big] c' S$$

$$- q \Big[(p^2 + \nu \gamma^2 - Z_{12})^2 - Z_{11} Z_{22} \Big] s' C \Big\} \qquad (3.14)$$

$$\det(D) = 2 \omega' \Big[Z_{22} s S + pq Z_{11} c C \Big]$$

$$+ p \Big[(q^2 - \nu \gamma^2 + Z_{12})^2 - Z_{11} Z_{22} \Big] c S$$

$$- q \Big[(p^2 + \nu \gamma^2 - Z_{12})^2 - Z_{11} Z_{22} \Big] s C \qquad (3.15)$$

To calculate the (m^{th}, n^{th}) resonant response, the dimensionless frequency, ω', or, equivalently, the dimensionless wavenumber, p, is determined as the m^{th} lowest root (for fixed n) of

$$\det(D') = 0$$

where det (D') is given by eqn. 3.15 with $\zeta = 0$. A plot of the resonant values of p, calculated in this manner, is shown in figure 3.1 for $Z_{11} = Z_{12} = 0$, a wide range of Z_{22} and several values of m and n b/a. Note

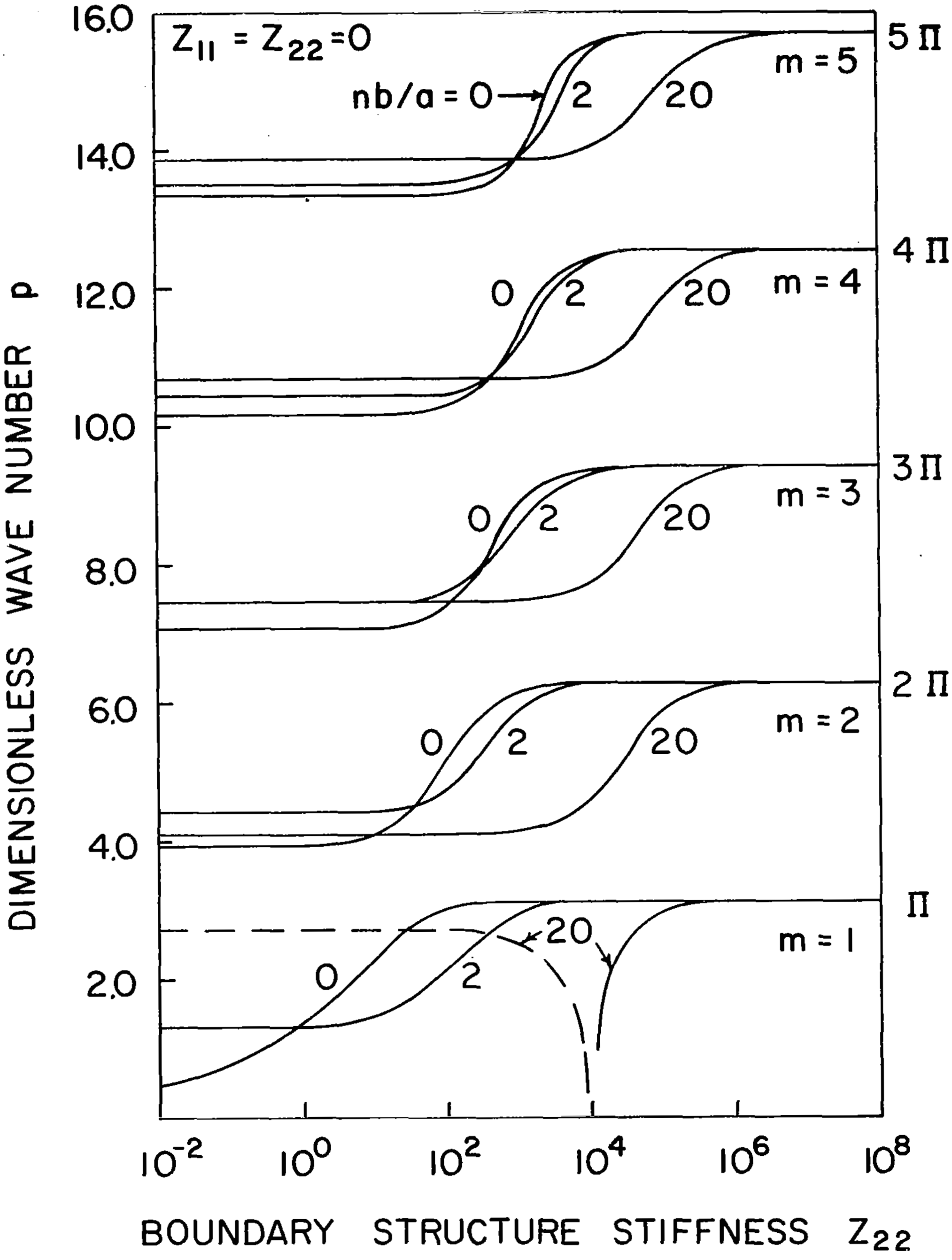


Figure 3.1 Dimensionless Wave Number for a Plate with One Edge Elastically Supported.

that for $m = 1$, $n\,b/a = 20$, p is imaginary for $Z_{22} < 10^4$ and $|p|$ is shown as the dashed line. Although not shown because of difficulty in numerical calculation, the values of p for large $n\,b/a$ ($\gamma \gg p$) may be determined analytically to approach the following limits, for $m \geq 2$,

$$p \sim (m-1)\pi \quad \text{for} \quad Z_{22} \ll \tfrac{1}{2}\sqrt{2}\,\nu^2\gamma^3$$

$$p \sim m\pi \quad \text{for} \quad Z_{22} \gg \tfrac{1}{2}\sqrt{2}\,\nu^2\gamma^3$$

The natural frequency, ω, of the plate corresponding to any value of p may be calculated with the first and the fourth of eqns. 3.11, with $\zeta = 0$.

Note that the boundary structure stiffnesses, Z_{ij}, are nondimensional quantities involving properties of both the boundary and the plate. For example, if k_{22} represents the linear stiffness per unit length of the boundary structure (force per unit length applied to the boundary/deflection at the boundary), then

$$Z_{22} = b^3 k_{22}/D^*$$

Now if Z_{22} is assumed to be a real quantity, k_{22} must be proportional to $1 + i\zeta$, which is equivalent to assuming a damping factor for the boundary structure equal to that of the plate. On the other hand, if the boundary structure is not to dissipate energy then Z_{22} must be proportional to $1/(1 + i\zeta)$.

A typical example of the effect of energy dissipation in the boundary structure is shown in figure 3.2. The magnitude of this effect is dependent not only on the damping factor of the boundary structure and the mode of vibration of the plate but also on the value of Z_{22}. If Z_{22} takes on either of its extreme values $(0, \infty)$, there is no effect from energy dissipation in the boundary. The numerical results presented

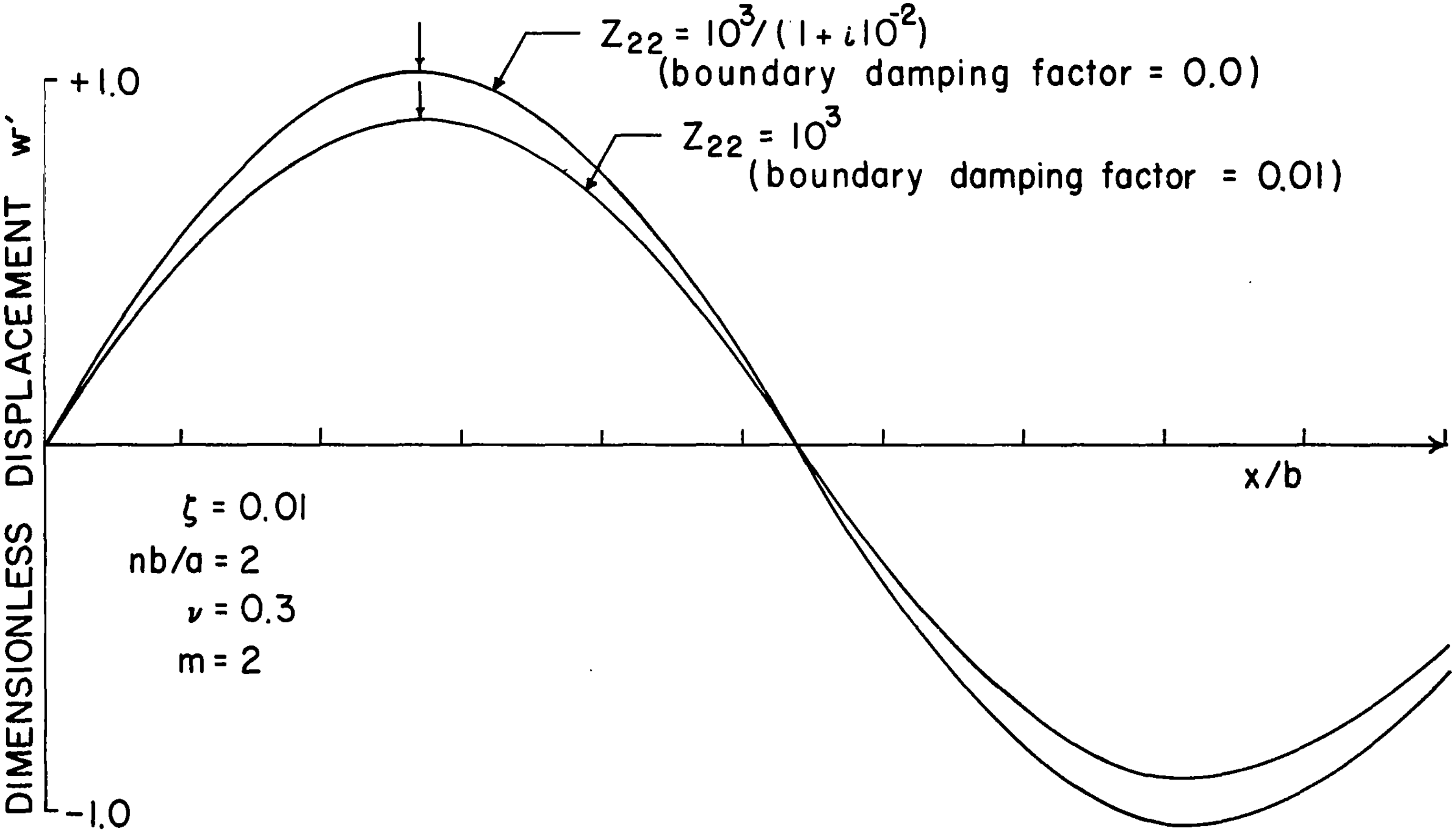


Figure 3.2 Effect of Boundary Damping on the Resonant Response of a Plate with one Edge Elastically Supported.

below are all based on a boundary damping factor equal to the plate damping factor.

The resonant responses of a plate for several values of Z_{22} and two different modes of vibration are shown in figures 3.3 and 3.4. The dimensionless displacement, W', plotted is defined as

$$W' = \frac{\rho hab\, \omega^2 \zeta}{4 P_o} |W| \tag{3.16}$$

where

$$w(x,y,t) = W(x,y)\, e^{i\omega t} \tag{3.17}$$

and w(x,y,t) is calculated with eqns. 3.4, 3.10, 3.13-3.15. The frequency ω in eqn. 3.16 is the resonant frequency for the given values of m, n b/a, ν, and Z_{22}; hence, each of the responses shown in figures 3.3 and 3.4 occur at different frequencies but each one represents the peak response for a particular boundary condition (Z_{22}), resonance (m,n) and plate configuration (ν, b/a). The displacement shown on the figures are those along the line y = a/2n and, hence, are a maximum with respect to the y coordinate. In each case, the point load is applied at an antinode of the motion, the y coordinate of which is a/2n and x coordinate is indicated by the arrow on each plot.

Several sets of numerical calculations were done to check the convergence of the series, eqn. 3.4. In the calculations, the plate damping factor, ζ, was set equal to 0.01 and m ranged from 1-5. The convergence of the series was found to improve with increasing b/a. For b/a = 0, the series does not converge; for b/a = 0.1, ten terms in the series gives convergence results to within 1%; and, for b/a > 1, one term in the series is sufficient. The responses shown in figures 3.3 and 3.4 required only a single term in the series for convergence.

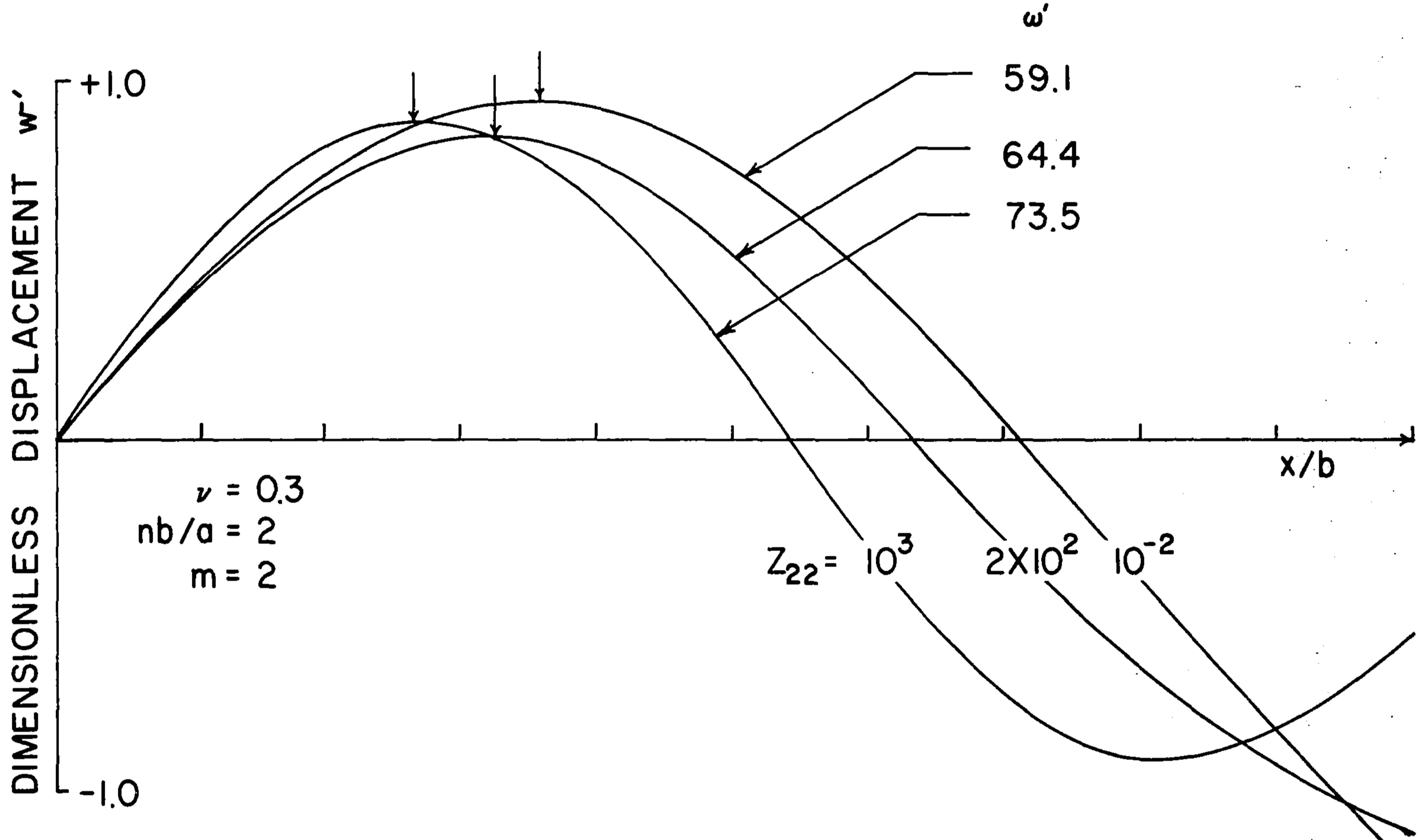


Figure 3.3 Resonant Response of a Plate with One Edge Elastically Supported-Second Longitudinal Mode.

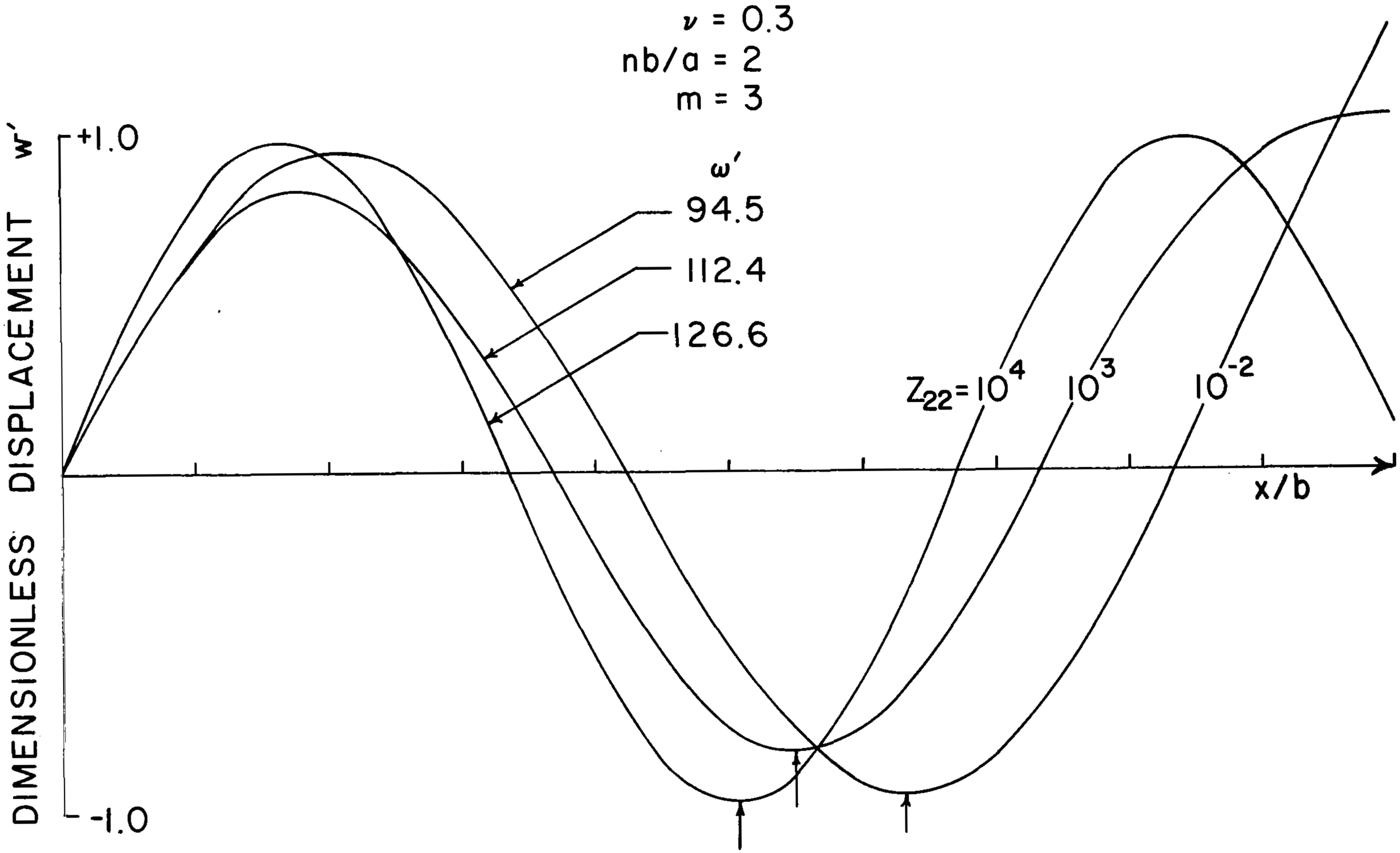


Figure 3.4 Resonant Response of a Plate with One Edge Elastically Supported-Third Longitudinal Mode.

The response of the plate for $Z_{22} = \infty$, although not shown in figure 3.3 and 3.4 is

$$W' = \sin (m \pi x/b)$$

The natural frequencies of these two cases are $\omega' = 78.9$, 128.6, respectively. It should also be remarked that the responses for $Z_{22} = 10^{-2}$ in these two figures are essentially the same as those for $Z_{22} = 0$.

Table 3.1 shows a comparison of the peak resonant response, which is W' evaluated at an antinode, x_k^*, for the cases plotted in figures 3.3 and 3.4. Note that the change in the peak resonant response as the end condition changes from pinned to free is relatively slight compared to the change in resonant frequency or Z_{22}. The largest change in peak resonant response occurs as the boundary condition varies from pinned to a partially supported end. Perhaps the most interesting point to be made from table 3.1 is that the dimensionless displacement, $W'(x_k^*)$, at an antinode is relatively insensitive to the vibration mode, m, the frequency, or the boundary stiffness Z_{22}. It will be shown in the following section that this is true in general if the natural frequency for the resonant mode under consideration is sufficiently different from the remainder of the plate's frequencies.

m = 2

Z_{22}	ω'	$W'(x_1^*)$
10^{-2}	59.1	0.953
$2x10^{2}$	64.4	0.832
10^{3}	73.5	0.953
∞	78.9	1.000

m = 3

Z_{22}	ω'	$W'(x_1^*)$	$W'(x_2^*)$
10^{-2}	94.5	0.945	0.940
10^{3}	112.4	0.824	0.830
10^{4}	126.6	0.978	0.978
∞	128.6	1.000	1.000

Table 3.1 Peak resonant response and frequencies of a plate with an elastically supported edge. ($n\, b/a = 2$, $\nu = 0.3$)

4. NORMAL MODE APPROXIMATION TO THE DYNAMIC RESPONSE OF A PLATE WITH AN ELASTICALLY SUPPORTED EDGE.

A normal mode solution for this problem was developed previously and presented in reference [21]. An estimate of the plate response at resonance may be had by employing only a single term of the normal mode series expansion. The term retained is, of course, the largest term in the series. Employing the results of reference [21] for $\omega = \omega_{mn}$

$$|W| \sim \frac{P_o}{\rho h} \frac{\varphi(x,y)\, \varphi\,(x_o,y_o)}{\zeta\omega^2_{mn}\, K^2} \tag{4.1}$$

where $\varphi(x,y) = \psi(x) \sin\,(n\pi y/a)$

$$\psi(x) = \sin\,(px/b) - C \sinh\,(qx/b) \tag{4.2}$$

$$K^2 = \int_o^a \int_o^b [\varphi(x,y)]^2\, dx\, dy \tag{4.3}$$

W = resonant response at m^{th}, n^{th} natural frequency

The quantity C may be determined using eqn. 16 in reference [21]. This expression is cumbersome but may be simplified considerably if we further assume $Z_{11} = Z_{12} = 0$. As the calculations presented in reference [21] demonstrated, Z_{11}, Z_{12} have a minor effect on the resonant response. Using eqn. 16 in reference [21] and taking $Z_{11} = Z_{12} = 0$

$$C = -\frac{p^2 + \nu\gamma^2}{q^2 - \nu\gamma^2} \frac{\sin p}{\sinh q} \tag{4.4}$$

It was shown in reference [21] that the normalizing factor K^2, which is proportional to the generalized mass of the (m^{th}, n^{th}) mode, may be expressed as

$$K^2 = \frac{ab}{4} \left\{1 - C^2 + \tfrac{1}{2}\left[\frac{C^2}{q} \sinh(2q) - \frac{1}{p}\sin(2p)\right]\right.$$

$$\left. - \frac{4C}{p^2 + q^2}\left[q \sin p \cosh q - p \cos p \sinh q\right]\right\} \qquad (4.5)$$

If the change in the end condition is known, the change in the response of the plate may be evaluated by first determining the two values of p (one for each of the boundary conditions) and then computing the two values of W using equations (4.1 - 4.5). Such a procedure is not especially laborious but there are circumstances in which it is desirable to be able to estimate the largest possible change in response for any change in a boundary condition. To do this, the "response" is taken to be the response at an antinode of the motion, say (x^*, y^*), and the point load is assumed to be applied at an antinode also. Under these conditions the dimensionless displacement W',defined in eqn. 3.16, is

$$W' \sim \frac{\psi(x_\ell^*)\ \psi(x_k^*)}{(4K^2/ab)} \qquad (4.6)$$

Note that K,C, and q are dependent on p which in turn is dependent on the boundary condition, Z_{22}. It is possible, however, to establish limits on the range of p, as follows.

The determination of the natural frequencies of the plate with arbitrary, but positive Z_{22} may be formulated as a variational problem. That is

$$\delta\left[V(\varphi) / T(\varphi)\right] = 0$$

where $V(\varphi)$ is the potential energy stored in the plate and the elastic boundary structure for the configuration $\varphi(x,y)$ and

$$T(\varphi) = \rho \int_o^a \int_o^b \varphi^2 \, dx \, dy$$

The function $\varphi(x,y)$ is any admissable function (one which satisfies the geometric constraints: $\varphi(o,y) = \varphi(x,o) = \varphi(x,a) = 0$) that renders $[V(\varphi)/T(\varphi)]$ stationary. There are an infinite set of functions which accomplish this, call them φ_j, $j = 1, 2, \ldots$. The eigenvalues are given by

$$\omega_j^2 = V(\varphi_j)/T(\varphi_j)$$

If the one constraint, $\varphi(b,y) = 0$, is added to this problem and if the new eigenvalues are labeled Ω_j^2, then, according to Rayleigh's theorem (see reference [22], pp. 70-71.

$$\omega_j^2 \leq \Omega_j^2 \leq \omega_{j+1}^2$$

or, alternatively,

$$\Omega_j^2 \leq \omega_{j+1}^2 \leq \Omega_{j+1}^2$$

Thus the $(j+1)^{th}$ frequency for arbitrary Z_{22} lies between the j^{th} and the $(j+1)^{th}$ frequencies of a plate simply-supported on four edges. Taking the wave number in the y direction to be fixed (n is constant), then

$$\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2 \leq \left(\frac{p}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2 \leq \left(\frac{(m+1)\pi}{b}\right)^2 + \left(\frac{n\,\pi}{a}\right)^2$$

or

$$m\pi \leq p_{m+1} \leq (m+1)\pi \qquad (4.7)$$

Knowing the range of p for the $(m+1)^{th}$ resonance, the largest and smallest values of K may be determined as follows. If eqn. 4.4 is substituted into 4.5 and terms of order exp (-2q) are ignored, K^2 may be put in the form

$$K^2 = \frac{ab}{4}\left\{1 + \frac{C_1}{q}\sin^2 p - \frac{C_2}{p}\sin p \cos p\right\} \qquad (4.8)$$

where

$$C_1 = \frac{p^2 + \nu\gamma^2}{p^2 + (2-\nu)\gamma^2}\left[\frac{2(p^2 + 2\gamma^2)}{p^2 + \gamma^2} + \frac{p^2 + \nu\gamma^2}{p^2 + (2-\nu)\gamma^2}\right]$$

$$C_2 = 1 + \frac{2p^2(p^2 + \nu\gamma^2)}{(p^2 + \gamma^2)(p^2 + (2-\nu)\gamma^2)} \qquad (4.9)$$

The maximum and minimum values of K^2 in the range $(m-1)\pi \leq p_m \leq m\pi$ are shown in figure 4.1 for a wide range of γ (or nb/a as in the figure).

To evaluate the change in $\psi(x_k^*)$ as Z_{22} ranges from $Z_{22} = 0$ to $Z_{22} = \infty$, note that the position of the antinodes are the roots x_k^* of the equation

$$\cos(px_k^*/b) + \frac{p^2 + \nu\gamma^2}{q^2 - \nu\gamma^2}\,\frac{q\sin p}{p\sinh q}\cosh(qx_k^*/b) = 0 \qquad (4.10)$$

The second term in eqn. 4.10 is small because of the sinh q in the denominator. Thus the roots of eqn. 4.10 will be nearly equal to the roots of the first term. Let Δ denote the difference, that is,

$$p\,x_k^*/b = (k - \tfrac{1}{2})\pi + \Delta\ ; \quad k = 1, 2, \ldots m-1 \qquad (4.11)$$

The restriction $k \leq m-1$ for the m^{th} resonance is necessary because for small values of Z_{22}, $x_m^* > b$, which is not physically permissable.

The quantity Δ is directly to the change in $\psi(x_k^*)$. Combining eqns. 4.2 and 4.4

$$\psi(x_k^*) \sim \sin(px_k^*/b) + \frac{p^2 + \nu\gamma^2}{q^2 - \nu\gamma^2}\,\frac{\sin p}{\sinh q}\sinh(qx_k^*/b)$$

and, combining this with eqn. 4.10

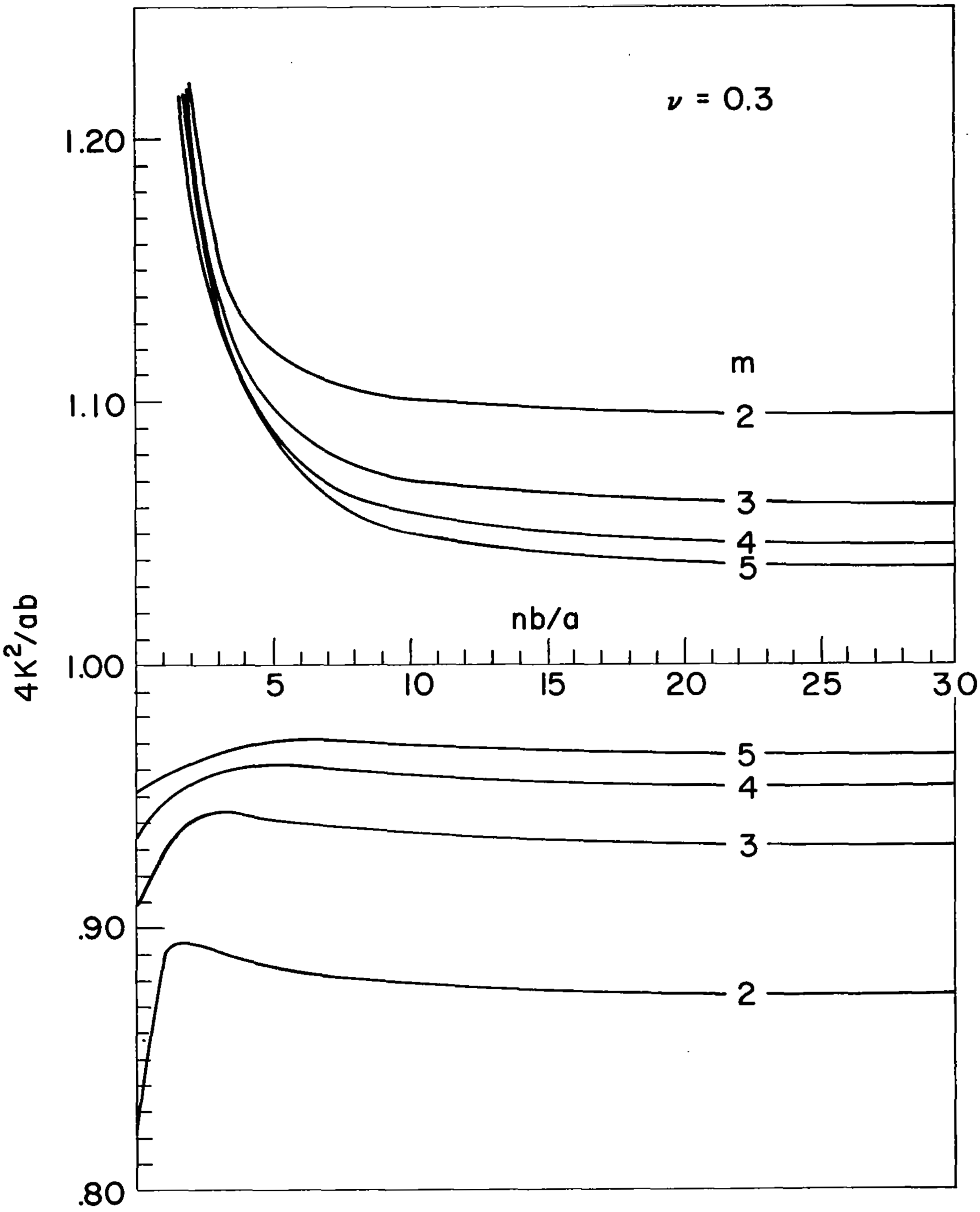


Figure 4.1 Maximum and Minimum Values of the Generalized Mass Parameter of a Plate with One Edge Elastically Supported.

$$\psi(x_k^*) \sim \sin(px_k^*/b) - \frac{p}{q} \tanh(qx_k^*/b) \cos(px_k^*/b) \tag{4.12}$$

Assuming that (qx_k^*/b) is large and substituting eqn. 4.11 into 4.12 results in

$$\psi(x_k^*) \sim -(-1)^k (\cos\Delta + (p/q)\sin\Delta)$$

or, for small Δ

$$\psi(x_k^*) \sim -(-1)^k (1 + (p/q)\Delta) \tag{4.13}$$

The range of $(p\Delta/q)$ may be estimated as follows. Substitute eqn. 4.11 in 4.10

$$(-1)^k \sin\Delta + \frac{q\,(p^2 + \nu\gamma^2)}{p\,(q^2 - \nu\gamma^2)} \frac{\sin p}{\sinh q} \cosh\left[\frac{q}{p}\{(k-\tfrac{1}{2})\pi + \Delta\}\right] = 0 \tag{4.14}$$

Assuming both Δ and $(q\Delta/p)$ are small, expanding both functions of Δ in series and retaining only the first order terms in Δ, and employing the assumption made previously that $(qx_k^*/b) \sim (q/p)(k-\tfrac{1}{2})\pi$ is large, eqn. 4.14 may be solved for Δ. Thus

$$\Delta' \sim \frac{-Z}{(-1)^k + (q/p)^2 Z} \tag{4.15}$$

where

$\Delta' = (p\Delta/q)$ and

$$Z = \frac{p^2 x \nu\gamma^2}{q^2 - \nu\gamma^2} \sin p \exp\left[\frac{q}{p}(k-\tfrac{1}{2})\pi - q\right]$$

Recall that for the m^{th} resonance $p = m\pi$ if $Z_{22} = \infty$ and that as Z_{22} decreases to zero, $(m-1)\pi \le p \le m\pi$. Thus, for the m^{th} resonance, Δ' either increases from zero to a small positive value (if $\sin p < 0$) or decreases from zero to a small negative value (if $\sin p > 0$). The maximum range of Δ' as Z_{22} spans the range $(0, \infty)$ is $(0, \bar{\Delta})$ where

$$\bar{\Delta} = \frac{(-1)^m \; |Z|_{max}}{(-1)^k - (-1)^m \; (q/p)^2 \; |Z|_{max}} \tag{4.16}$$

An upper bound on $|Z|$ may be shown to be

$$|Z| \leq \frac{p_2^2 + \nu \gamma^2}{q_2^2 - \nu \gamma^2} \exp\left[\frac{q_1}{p_1} (k-\tfrac{1}{2})\pi - q_1\right] \tag{4.17}$$

where $p_1 = (m-1)\pi \qquad q_1 = [(m-1)^2 \pi^2 + 2\gamma^2]^{\frac{1}{2}}$

$p_2 = m\pi \qquad q_2 = [m^2 \pi^2 + 2\gamma^2]^{\frac{1}{2}}$

Equations 4.13, 4.16 and 4.17 may be used to estimate the change in $\psi(x_k^*)$ as Z_{22} ranges from 0 to ∞ given values of m, k, ν and γ. The maximum and minimum values of the generalized mass parameter, $(4k^2/ab)$, may be determined from figure 4.1. The range of W' may then be calculated using eqn. 4.6.

For example, let $\gamma = 2\pi$, $\nu = 0.3$, m = 2, k = 1, then

$p_1 = \pi$

$p_2 = 2\pi$

$q_1 = 3\pi$

$q_2 = 2\sqrt{3}\,\pi$

Using eqns. 4.17 and 4.16, $|Z| \leq 0.004$ and $\bar{\Delta} = -0.004$.

Thus

$$0.996 \leq |\psi(x_1^*)| \leq 1.0 \tag{4.18}$$

From figure 4.1

$$0.895 \leq 4k^2/ab \leq 1.22 \tag{4.19}$$

Combining eqns. 4.6, 4.18 and 4.19, the range of W' may be determined

$$0.81 \leq W'(x_1^*) \leq 1.12$$

A comparison of the exact values of $W'(x_i^*)$ as first presented in table 3.1 to the approximate limits calculated with eqns. 4.13, 4.16, 4.17 is shown in table 4.1. The lower limits show good agreement with the exact values but the upper limits are conservative. The reason for this is that the minimum values of $4K^2/ab$ which determine the upper limit on W' occur for $(m-1)\pi \leq p \leq (m-\frac{1}{2})\pi$ or the lower half of the possible range of p, but the actual values of p are concentrated in the upper half of the range, at least for the configurations shown in figure 3.1. This is not generally true for, as was noted in the discussion of figure 3.1, p approaches the lower limit, $(m-1)\pi$, for large nb/a and small Z_{22}. Thus it is expected for those configurations in which nb/a is large and Z_{22} is small that the upper limit will be nearer to the exact value.

m = 2

Z_{22}	$W'(x_1^*)$	Limits
10^{-2}	0.953	
$2x10^2$	0.832	$0.81 \le W'(x_1^*) \le 1.12$
10^3	0.953	

m = 3

Z_{22}	$W'(x_1^*)$	Limits
10^{-2}	0.945	
10^3	0.824	$0.82 \le W'(x_1^*) \le 1.06$
10^4	0.978	

m = 3

Z_{22}	$W'(x_2^*)$	Limits
10^{-2}	0.940	
10^3	0.830	$0.79 \le W'(x_2^*) \le 1.06$
10^4	0.978	

Table 4.1 Comparison of Exact Values of Peak Resonant Response ($nb/a = 2$, $\nu = 0.3$) to Approximate Limits.

5. CONCLUSION

The following conclusions are drawn from the analyses presented above:

(1) A fairly accurate estimation of the range of the maximum resonant response for a change in an elastic edge restraint of a structure may be had by using the predominant term of the principal mode expansion of the response. This leads to an expression for the displacements in terms of the generalized force, generalized mass and principal modes of free vibration of the structure (e.g. eqns. 2.25 or 4.6). If the variations of these three factors for a change in end condition can be calculated or bracketed, then the change in the response due to the change in end condition may be determined.

(2) Application of the procedure just described to estimate the maximum resonant response of a free-elastically supported uniform beam excited by a spatially uniform load showed that the major change in response is due to the variation in the generalized force which vanishes as the elastic restraint vanishes. Estimates of the peak resonant response as the end condition varied from a pinned end to a free end showed good agreement with an exact solution.

(3) Application of (1) to estimate the maximum resonant response of a plate, pinned on three edges and elastically supported on the fourth, excited by a concentrated force lead to the introduction of a dimensionless displacement, eqn. 4.6, whose magnitude is relatively independent of the vibration mode (for modes higher than the fundamental) and the end restraint. Estimates of the range in maximum resonant response as the edge restraint varied from free to pinned showed good agreement with the values obtained by an exact solution.

REFERENCES

1. Reismann, H., "Forced Motion of Elastic Plates", J. Appl. Mech., Vol. 35, Trans. ASME, 510-515, 1968.

2. Jones, J.P., "Forced Motion of a Thin Bonded Plate", J. Acous. Soc. of Amer., Vol. 37, 1008-1015, 1965.

3. Thomas, D. A., "Mechanical Impedence for Thin Plate", J. Acous. Soc. of Amer., Vol. 32, 1302-1304, 1960.

4. Kalnins, A., "On Fundamental Solutions and Green's Functions in the Theory of Elastic Plates", J. Appl. Mech., Vol. 33, Trans. ASME, 31-38, 1966.

5. Weiner, R.S., "Forced Axisymmetric Motions of Circular Elastic Plates", J. Appl. Mech., Vol. 32, Trans. ASME, 893-898, 1965.

6. Williams, C.J.H. and Tobias, S.A., "Forced Undamped Nonlinear Vibrations of Imperfect Circular Disks", J. Mech. Engr. Sci., Vol. 5, 325-335, 1963.

7. McLeod, A. J. and Bishop, R.E.D., "The Forced Vibration of Circular Flat Plate", Mech. Engr. Sci. (Mono. No. 1), 1965.

8. Likhoded, A.I., "Steady-State Vibrations of Plates with Free Edges", PMM: Jour. Appl. Math. Mech., Vol. 29, No. 5, 1084-1089, 1965.

9. Bieniek, M.P. and Freudenthal, A.M., "Forced Vibrations of Cylindrical Sandwich Shells", J. Aero. Sci., Vol. 29, No. 2, 180-184, 1962.

10. Pilkey, W.D., "Mechanically and/or Thermally Generated Dynamic Response of Thick Spherical and Cylindrical Shells with Variable Materials", J. Sound and Vibr., Vol. 6, No. 1, 105-109, 1967.

11. Sheng, J., "The Response of a Thin Cylindrical Shell to Transient Loading", AIAA J., Vol. 3, No. 4, 701-709, 1965.

12. Payton, R.G., "Dynamic Membrane Stresses in a Circular Elastic Shell", Trans. ASME, Vol. 83E, J. Appl. Mech., Vol. 28, 417-420, 1961.

13. Bushnell, D., "Dynamic Response of Two-Layered Cylindrical Shells to Time-Dependent Loads", AIAA J., Vol. 3, No. 9, 1698-1703, 1965.

14. Haywood, J.H., "Response of an Elastic Cylindrical Shell to a Pressure Pulse", Quart. J. Mech. Appl. Mech., Vol. 22, part 2, 129-141, 1958.

15. Heckl, M., "Vibrations of Point-Driven Cylindrical Shells", Jour. Acous. Soc. Amer., Vol. 34, 1553-1557, 1962.

16. Crocker, M.J., "Theoretical and Experimental Response of Panels to Traveling Sonic Boom and Blast Waves", Wyle Laboratories-Research Staff Report WR 66-2, March 1966.

17. Gayman, W.H., "A Note on Boundary-Condition Simulation in the Dynamic Testing of Spacecraft Structures", Calif. Inst. Tech., JPL TR 32-938, April 15, 1966.

18. White, R.W., "Vibration Characteristics of Beams and Plates Mounted on Elastic and Inertial Supports", Wyle Laboratories-Huntsville Facility Research Staff Report WR 64-2, Aug. 11, 1964.

19. Hwang, C., Pi, W.S. and Bhatia, N.M., "Analytical and Experimental Determination of Localized Structure to be used in Laboratory Vibration Testing of Shell Structure-Mounted Components, Saturn V", Final Report NOR 67-127, NASA Contract NAS8-20025, Northrop Corporation, Hawthorne, Calif., July 1967.

20. Hwang, C. and Pi, W.S. "Analytical and Experimental Determination of Localized Structure to be used in Laboratory Vibration Testing of Shell Structure-Mounted Components, Saturn V", Progress Report NOR 67-19, NASA Contract NAS8-20025, Northrop Corporation, Hawthorne, Calif., Jan. 1967.

21. Henry, F.D. and Egle, D.M., "The Effect of an Elastic Edge Restraint on the Forced Vibration of a Rectangular Plate", presented at the Fifth Southeastern Conference on Theoretical and Applied Mechanics, Rayleight-Durham, North Carolina, April 16-17, 1970.

22. Gould, S.H., "Variational Methods for Eigenvalue Problems", University of Toronto Press, Second Edition, 1966.

APPENDIX

Computer Programs

In this appendix the Fortran IV digital computer programs used to implement the analyses presented above are listed.

1) Resonant response of a free-elastically supported beam excited by a uniform simple harmonic load

After the input data is read, the resonant wave number (kL) is determined by finding the appropriate root of the frequency determinant, eqn. 2.18. Using this value of (kL) and eqns. 2.17, 2.18 and 2.15, the dimensionless displacements (eqn. 2.20) are calculated at twenty one equally-spaced positions along the beam.

<u>Input Format</u>

One card - Z_{11}, Z_{12}, Z_{21}, Z_{22}, ζ, α, m (6F10.0, I2)

where Z_{11}, Z_{12}, Z_{21}, Z_{22}, are the support stiffnesses defined in eqn. 2.5b and following

ζ is the beam structural damping factor

∂ is the support structural damping factor

m is an integer defining the order of the mode

```
C      RESONANT RESPONSE OF A FREE-ELASTICALLY SUPPORTED BEAM
C      EXCITED BY A UNIFORM SIMPLE HARMONIC LOAD--OU131
       COMPLEX Z11,Z12,Z21,Z22,SH,CH,S,C,D11,D12,D21,D22,D,
      1C1,C2,F1,F2,F,W,KX,KC,Z
       REAL K,NEW,KI
       COMMON Z11P,Z12P,Z21P,Z22P
     1 WRITE(6,800)
   800 FORMAT('1  RESONANT RESPONSE OF A FREE-ELASTICALLY SUPPORTED'/
      1'     BEAM EXCITED BY A UNIFORM SIMPLE HARMONIC LOAD'/
      2'    EXACT SOLUTION--OU131            '//)
       READ(5,801) Z11P,Z12P,Z21P,Z22P,GAMMA,ALPHA,NRES
   801 FORMAT(6F10.0,I2)
C
C      DETERMINE NATURAL FREQUENCY
C
       N=0
       OLD =DETMT(0.001)
       K=0.1
     3    NEW=DETMT(K)
       IF(OLD*NEW)5,8,4
     4   K=K+0.1
       OLD=NEW
       GO TO 3
     5   N=N+1
       IF(N-NRES)4,6,1
     6   DELK=0.05
    23  K=K+DELK*OLD*NEW/ABS(OLD*NEW)
       NEW=DETMT(K)
       IF(OLD*NEW)21,8,24
    21  IF(DELK-0.00001)8,8,22
    22  DELK=DELK/2.0
       GO TO 23
    24   OLD=NEW
       GO TO 21
     8 WRITE(6,103)NRES,K
   103  FORMAT(//'KL FOR RESONANCE',I2,'  =',E14.6//)
```

```
C
C     EVALUATE RESONANT RESPONSE
C
      Z=CMPLX(1.0,GAMMA)/CMPLX(1.0,ALPHA)
      Z11=Z11P/Z
      Z12=Z12P/Z
      Z21=Z21P/Z
      Z22=Z22P/Z
      Z=CMPLX(1.0,GAMMA)
      Z=CSQRT(Z)
      KC=K/CSQRT(Z)
      SH=0.5*(CEXP(KC)-CEXP(-KC))
      CH=0.5*(CEXP(KC)+CEXP(-KC))
      S=CSIN(KC)
      C=CCOS(KC)
      D11=SH-S+Z11*(C+CH)/KC   +Z12*(S+SH)/KC**2
      D12=CH-C+Z11*(SH-S)/KC   +Z12*(CH+C)/KC**2
      D21=CH-C-Z21*(C+CH)/KC**2  -Z22*(S+SH)/KC**3
      D22=S+SH-Z21*(SH-S)/KC**2  -Z22*(C+CH)/KC**3
      F1=Z12/(KC)**2
      F2=-Z22/(KC)**3
      F=1.0/KC**4
      D=D11*D22-D12*D21
      C1=(F1*D22-F2*D12)/D
      C2=(F2*D11-F1*D21)/D
      WRITE(6,802)
  802 FORMAT('          MAG(W)       ARG(W)'/)
      DO 30 I=1,21
      X=0.05*FLOAT(I-1)
      KX=KC*X
      W    =C1*CSIN(KX)+C2*CCOS(KX)+0.5*(C2+C1)*CEXP(KX)
     1+0.5*(C2-C1)*CEXP(-KX)-1.0
      WR=REAL(W)
      WI=AIMAG(W)
      WMAG=SQRT(WR*WR+WI*WI)
      WARG=57.2957795*ATAN2(WI,WR)
```

```
C
C     EVALUATE RESONANT RESPONSE
C
      Z=CMPLX(1.0,GAMMA)/CMPLX(1.0,ALPHA)
      Z11=Z11P/Z
      Z12=Z12P/Z
      Z21=Z21P/Z
      Z22=Z22P/Z
      Z=CMPLX(1.0,GAMMA)
      Z=CSQRT(Z)
      KC=K/CSQRT(Z)
      SH=0.5*(CEXP(KC)-CEXP(-KC))
      CH=0.5*(CEXP(KC)+CEXP(-KC))
      S=CSIN(KC)
      C=CCOS(KC)
      D11=SH-S+Z11*(C+CH)/KC   +Z12*(S+SH)/KC**2
      D12=CH-C+Z11*(SH-S)/KC   +Z12*(CH+C)/KC**2
      D21=CH-C-Z21*(C+CH)/KC**2   -Z22*(S+SH)/KC**3
      D22=S+SH-Z21*(SH-S)/KC**2   -Z22*(C+CH)/KC**3
      F1=Z12/(KC)**2
      F2=-Z22/(KC)**3
      F=1.0/KC**4
      D=D11*D22-D12*D21
      C1=(F1*D22-F2*D12)/D
      C2=(F2*D11-F1*D21)/D
      WRITE(6,802)
  802 FORMAT('          MAG(W)      ARG(W)'/)
      DO 30  I=1,21
      X=0.05*FLOAT(I-1)
      KX=KC*X
      W    =C1*CSIN(KX)+C2*CCOS(KX)+0.5*(C2+C1)*CEXP(KX)
     1+0.5*(C2-C1)*CEXP(-KX)-1.0
      WR=REAL(W)
      WI=AIMAG(W)
      WMAG=SQRT(WR*WR+WI*WI)
      WARG=57.2957795*ATAN2(WI,WR)
```

```
      WRITE(6,810) X,WMAG,WARG
  810 FORMAT(F5.2,E12.4,F9.2)
   30 CONTINUE
      GO TO 1
      END
      FUNCTION DETMT(K)
      REAL K
      COMMON Z11,Z12,Z21,Z22
      C=COS(K)
      S=SIN(K)
      SH=0.5*(EXP(K)-EXP(-K))
      CH=0.5*(EXP(K)+EXP(-K))
      DETMT=2.0*S*CH*(Z11+Z22/K**2)/K +2.0*C*SH*(Z11-Z22/K**2
     1)/K+2.0*(1.0+C*CH)*(Z12*Z21-Z11*Z22)/K**4 +2.0*S*SH*
     1(Z12+Z21)/K**2 +2.0*(C*CH-1.0)
      RETURN
      END
```

2) Resonant response of a rectangular plate with one edge elastically supported

After reading and reproducing the input data, the natural frequency and dimensionless wave number for the mode being considered is determined by finding the appropriate root of the frequency determinant, eqn. 3.15, by a simple iterative technique. The dimensionless displacement as defined by eqns. 3.13, 3.14, 3.10, 3.4, and 3.16 is calculated at a number of equally spaced positions parallel to the x-axis

Input Format

One card - Z_{11}, Z_{12}, Z_{22}, ζ, ν, b/a, x_o, y_o, y, n, m, NMAX, JMAX (3F10.0, 6F5.0, 4I2

where Z_{11}, Z_{12}, Z_{22} are support stiffnesses defined in eqn. 3.3 and following

ζ is the plate structural damping factor

ν is the plate Poisson's ratio

b/a is the plate aspect ratio

x_o, y_o are the coordinate of the applied load

y is the y coordinate of the displacements being calculated

n, m are integers defining the vibration mode

NMAX is the number of terms in the series, eqn. 3.4

JMAX is the number of equally spaced points at which the displacement is to be calculated.

```
C      RESONANT RESPONSE OF A RECTANGULAR PLATE WITH ONE EDGE
C      ELASTICALLY SUPPORTED EXCITED BY A CONCENTRATED HARMONIC LOAD
C      OU154
       COMPLEX GAMMA,WP,P,Q,PS,QS,SH,CH,SHP,CHP,S,C,SP,CP,
      1W,PP,QQ,A,B,DETN,I,WW,Z11,Z12,Z22
       REAL NU,NEW,IW
       DIMENSION A(50),B(50),W(50),PP(50),QQ(50)
       COMMON Z11P,Z12P,Z22P,BOA,NU,N
       I=(0.0,1.0)
       PI=3.1415927
   10 READ(5,701) Z11P,Z12P,Z22P,ZETA,NU,BOA,X0,Y0,Y,N,NRES,NMAX,JMAX
  701 FORMAT(3F10.0,6F5.0,4I2)
       WRITE(6,700)
  700 FORMAT('1 RESONANT RESPONSE OF A RECTANGULAR PLATE WITH ONE'/
      1 '   EDGE ELASTICALLY SUPPORTED--OU154'//)
       WRITE(6,803)X0,Y0,Y,ZETA
       GAMMA=CMPLX(1.0,ZETA)
       Z11=Z11P
       Z12=Z12P
       Z22=Z22P
  803 FORMAT(' LOAD POINT OF APPLICATION, X/B =',F5.3,'  Y/A =',
      1F5.3/' LINE OF CALCULATION,Y/A =',F5.3/' DAMPING FACTOR =',
      1F8.4)
   11 WRITE(6,104)Z11P,Z12P,Z22P,NU,BOA,N,NRES,NMAX
  104 FORMAT(' Z11 =',E12.4,'  Z12 =',E12.4,'   Z22 =',E12.4,
      1/' POISSONS RATIO =',F6.3,'  B/A =',F6.3,'  N =',I2,
      1' M =',I2,' NMAX=',I2)
C
C      CALCULATE NATURAL FREQUENCY
C
       NN=0
       OLD=DETMT(0.01)
       WPP=5.0
    3 NEW=DETMT(WPP)
       IF(OLD*NEW)5,8,4
    4 WPP=WPP+5.0
```

```
      OLD=NEW
      GO TO 3
    5 NN=NN+1
      IF(NN-NRES)4,6,10
    6 DELWPP=2.5
   23 WPP=WPP+DELWPP*OLD*NEW/ABS(OLD*NEW)
      NEW=DETMT(WPP)
      IF(OLD*NEW)21,8,24
   21 IF(DELWPP-0.00001*WPP)8,8,22
   22 DELWPP=DELWPP/2.0
      GO TO 23
   24 OLD=NEW
      GO TO 21
    8 WRITE(6,103)N,NRES,WPP
  103 FORMAT(' RESONANT OMEGA PRIME FOR N =',I3,' M =',I3,
     1'  IS',E14.6//)
      PPP=SQRT(WPP-(N*PI*BOA)**2)
      WRITE(6,105)PPP
  105 FORMAT(' RESONANT P =',E14.6//)
C
C     EVALUATE RESONANT RESPONSE
C
      WP=WPP/CSQRT(GAMMA)
      DO 20 N=1,NMAX
      PS=(N*PI*BOA)**2+WP
      QS =-(N*PI*BOA)**2+WP
      P=CSQRT(PS)
      Q=CSQRT(QS)
      PP(N)=P
      QQ(N)=Q
      SH=0.5*(1.0-CEXP(-2.0*P))
      CH=0.5*(1.0+CEXP(-2.0*P))
      SHP=0.5*(CEXP(-P*X0)-CEXP(P*(X0-2.0)))
      CHP=0.5*(CEXP(-P*X0)+CEXP(P*(X0-2.0)))
      S=-I*0.5*(CEXP(I*Q-P)-CEXP(-I*Q-P))
      C=   0.5*(CEXP(I*Q-P)+CEXP(-I*Q-P))
```

```
      SP=-I*0.5*(CEXP(I*Q*(1.0-X0)-P)-CEXP(-I*Q*(1.0-X0)-P))
      CP=    0.5*(CEXP(I*Q*(1.0-X0)-P)+CEXP(-I*Q*(1.0-X0)-P))
      R=(1.0-NU)*(N*PI*BOA)**2
      DETN=Q*((R+WP+Z12)**2-Z11*Z22)*SH*C-P*((R-WP+Z12)**2-Z11*
     1Z22)*S*CH+2.0*WP*(Z22*S*SH+P*Q*Z11*C*CH)
      A(N)=(-2.0*WP*(Z22*SHP*S+P*Q*Z11*CHP*C)+P*(WP*WP-(R+Z12)**2
     1+Z11*Z22)*(S*CP-SP*C)-Q* ((R+WP+Z12)**2-Z11*Z22)*SHP*C
     1+P*((R-WP+Z12)**2-Z11*Z22)*CHP*S)/(DETN*P)
      B(N)=(2.0*WP*(Z22*SH*SP+P*Q*Z11*CH*CP)+Q*(WP*WP-(R+Z12)**2
     1+Z11*Z22)*(SH*CHP-SHP*CH)+Q*((R+WP+Z12)**2-Z11*Z22)*SH*CP
     1-P*((-R+WP-Z12)**2-Z11*Z22)*CH*SP)/(DETN*Q)
   20 CONTINUE
      WRITE(6,802)
  802 FORMAT('  X/B      MAG(W)       ARG(W)'/)
      DO 40 J=2,JMAX
      WW=(0.0,0.0)
      X=(FLOAT(J)-1.0)/(FLOAT(JMAX)-1.0)
      DO 30 N=1,NMAX
      W(N)=A(N)*0.5*(CEXP(PP(N)*X)-CEXP(-PP(N)*X))+B(N)*CSIN
     1(QQ(N)*X)
      IF(X-X0)26,26,25
   25 W(N)=W(N)+(0.5/PP(N))*(CEXP(PP(N)*(X-X0))-CEXP(PP(N)*
     1(X0-X)))-(1.0/QQ(N))*CSIN(QQ(N)*(X-X0))
   26 W(N)=W(N)*SIN(N*PI*Y)*SIN(N*PI*Y0)*WP*ZETA/(4.0*GAMMA)
   30 WW=WW+W(N)
      RW=REAL(WW)
      IW=AIMAG(WW)
      WMAG=SQRT(RW*RW+IW*IW)
      WARG=57.2957795*ATAN2(IW,RW)
      WRITE(6,810)X,WMAG,WARG
  810 FORMAT(F6.3,E12.4,F9.2)
   40 CONTINUE
  439 CONTINUE
      GO TO 10
      END
      FUNCTION DETMT(WP)
```

```
      REAL NU
      COMMON Z11,Z12,Z22,BOA,NU,N
      PI=3.1415927
      PS=N*N*PI*PI*BOA*BOA+WP
      P=SQRT(PS)
      QS=-N*N*PI*PI*BOA*BOA+WP
      IF(QS)1,2,2
    1 Q=SQRT(-QS)
      S=0.5*(EXP(Q   )-EXP(-Q   ))
      C=0.5*(EXP(Q   )+EXP(-Q   ))
      GO TO 3
    2 Q=SQRT(QS)
      S=SIN(Q)
      C=COS(Q)
    3 SH=0.5*(1.0-EXP(-2.0*P))
      CH=0.5*(1.0+EXP(-2.0*P))
      R=(1.0-NU)*(N*PI*BOA)**2
      DETMT=Q*((R+WP+Z12)**2-Z11*Z22)*SH*C-P*((R-WP+Z12)**2-Z11*Z22)
     1*S*CH+2.0*WP*(Z22*S*SH+Q*Z11*C*CH)
      RETURN
      END
```

NASA-Langley, 1971 — 32 CR-1736