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Abstract

The variation with time of the bond stress in a case bonded solid propellant rocket assembly is studied under conditions of plane stress, when the external temperature changes abruptly. Maxwell, Voigt, and four element models are assumed for the propellant, and effects arising from changes in viscoelastic properties, thermal expansion coefficients, and heat transfer characteristics of the assembly are studied. It is found that the bond stress equations are separable into a stress factor identical to the solution of the equivalent elastic problem, and a time dependent factor, which is a function of the viscoelastic constants. Quantitative aspects of the solution are presented in dimensionless form.

Introduction

It has been shown[1,2]* that when a case bonded rocket assembly is subjected to ambient temperature variations, thermal stresses develop at the bond surface due to differential thermal expansion of the solid propellant and its casing. Under certain conditions, these stresses become high enough to cause failure of the bond. Their magnitude may be determined from the pro-

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* Superscript numbers in parenthesis refer to the references listed in Bibliography.

pellant material properties, the casing material, the geometrical configuration, and the rapidity of the ambient temperature change.

Initially the thermal bond stress was explored[1,2] under the assumption that the propellant has only elastic characteristics. Subsequently,[3] the analysis was extended to include some viscoelastic characteristics, which represent a more apt description of the propellant material behavior. The extension was based on the assumption that the viscoelastic propellant material is of the Maxwell or Voigt type. The present report proposes to extend this analysis to a somewhat more physically realistic material representation of the propellant.

Statement of Problem

In order to define the problem so that the resulting computations yield meaningful physical results which do not become excessively unwieldy, the range of the present analysis will be restricted by several assumptions.

Attention will be confined to a propellant grain consisting of an annular disk, as shown in Fig. 1. The disk has an internal radius a and an external radius b, surrounded by a casing whose internal and external radii are b and c respectively. The disk is thin compared to its radial dimensions, so that the assembly may be analyzed as a problem of plane stress.

The casing material may be either metallic or plastic; in either event it is assumed to be an isotropic elastic material. The solid propellant is a viscoelastic material. Following the usage of Alfrey and others[4,5] its properties may be described in terms of one dimensional behavior which characterizes the stress-strain relationship in two independent modes, i.e., compression and shear. For simplicity, the propellant will be assumed elastic rather than viscoelastic in compression, and in the limit, taken to be incompressible.

The bond stress will be studied by comparing results obtained from three shear models; the two element Maxwell and Voigt models, and a more general four element model. The one dimensional analogues of the material properties in pure shear are shown in Fig. 2. The Maxwell model is shown in Fig. 2a, while the Voigt model is shown in Fig. 2b. The characteristics of each model are defined by two parameters, an elastic and a viscous constant. The four element model, shown in Fig. 2c, consists of a Voigt and Maxwell model in series. It approximates the behavior of some types of non-cross-linked polymers fairly well. The existence of four independent constants allows greater freedom in correlating data which describe actual materials with mathematical equations which describe the behavior of the model.

In discussing results arising from variation in the material properties, it will be convenient to speak of a Maxwell-like or Voigt-like change, by noting that in a limiting sense, the four element model may be transformed into either of the two element models. It is useful in doing this, to evaluate the properties of the four element model quantitatively in terms of an elastic parameter E_1, and three moduli which have the dimensions of reciprocal time, namely

$$K_1 = \frac{E_1}{\eta_1}, \quad K_2 = \frac{E_1}{\eta_2}, \quad K_3 = \frac{E_2}{\eta_2} \tag{1}$$

The Maxwell material is characterized by an initial elastic deformation and unrestricted viscous flow under stress. Its behavior may be described in terms of the four element model as the limiting case, either as $\eta_2 \to \infty$, for which $K_1 >> K_2$ and $K_1 >> K_3$, or when $E_2 \to \infty$ and $K_3 \to \infty$. The Voigt model has an infinite resistance to suddenly applied stress, and under constant stress tends toward an elastic-like behavior. This characteristic occurs in the four element model when both $E_1 \to \infty$ and $\eta_1 \to \infty$.

The ratio E_1/η_1 should not be left ambiguous by the preceding limiting process. Any ambiguity may be avoided by observing that in the Voigt-like material, the relaxation effects associated with the Maxwell element have a relaxation time $1/K_1$ which is much greater than the corresponding retardation time $1/K_3$. A Voigt-like material thus behaves in a manner similar to the four element material for which $K_2 >> K_3 >> K_1$; the two element model is formed when $K_2 \to \infty$, and $K_1 \to 0$.

Another important factor in evaluating the bond stress is the temperature distribution in a rocket assembly subsequent to an ambient temperature change. When the ambient temperature changes abruptly after the assembly has been in thermal equilibrium with its surroundings, the assembly is said to be subjected to a thermal shock. To describe the thermal characteristics of the assembly, it is convenient to introduce a dimensionless parameter called the Biot number

$$N_{Bi} = \frac{Hb}{k} \tag{2}$$

where b is the external radius of the propellant, k is its thermal conductivity, and

$$\frac{1}{H} = \frac{1}{h_o} + \frac{1}{h_c} + \frac{1}{h_i} \tag{3}$$

The surface heat transfer coefficient h_o is evaluated at the external surface of the assembly, while the surface heat transfer coefficient h_i is evaluated at the propellant-casing interface. The coefficient h_c is equal to $k_c/b\ell n(c/b)$ where k_c is the thermal conductivity of the casing material.

When a rocket assembly is subjected to a thermal shock, the ambient temperature change T_A, measured from an arbitrary equilibrium temperature, varies with time t in accordance with the relation

$$T_A(t) = 0 \quad t < 0 \quad ; \quad T_A(t) = T_o \quad t \geq 0 \tag{4}$$

If the casing is thermally thin, its temperature T_c is immediately affected by a change in ambient temperature so that

$$T_c(t) = 0 \quad t < 0 \quad ; \quad T_c(t) = T_o \quad t \geq 0 \tag{5}$$

The temperature within the propellant T_p, on the other hand, may be shown to vary independently of the radius, to a degree of approximation determined by the Biot number[6],

$$T_p(t) = 0 \quad t < 0$$
$$T_p(t) = T_o\,(1 - e^{-\beta t}) \quad t \geq 0 \tag{6}$$

where β is a temperature time constant equal to twice the product of the Biot number and the Fourier number. It is a function of geometric and thermal conditions.

Use of the preceding temperature equations restrict the range of applicability of the subsequent stress calculations to a Biot number consistent with the assumption of uniform temperature throughout the propellant.

General Method of Solution

E.H. Lee[7] has shown that there exists a correspondence principle whereby viscoelastic stresses are related to the stresses of an equivalent elastic problem. The correspondence is applicable to quasi-static problems of small strain in which the surface forces or surface displacements, whichever are prescribed, are identical in the elastic and in the viscoelastic problems.

To transform an elastic solution to a viscoelastic solution, it is observed that the viscoelastic linear operators P, Q and P', Q' are related to the elastic constants E and μ, Young's modulus and Poisson's ratio respectively, by the relations

$$\frac{1}{E} = \frac{2}{3}\left(\frac{P}{Q}\right) + \frac{1}{3}\left(\frac{P'}{Q'}\right)$$
$$\frac{1+\mu}{E} = \frac{P}{Q} \tag{7}$$

and to G and K , the shear modulus and bulk modulus respectively, by the relations

$$\frac{Q}{P} = 2G \quad , \quad \frac{Q'}{P'} = 3K \tag{8}$$

The viscoelastic linear operators appear in the stress-strain relation of a viscoelastic material

$$\begin{aligned} P\, s_{ij} &= Q\, e_{ij} \quad ; \quad i,j = r,\ \theta,\ z \\ P'\sigma_{kk} &= Q'\epsilon_{kk} \end{aligned} \tag{9}$$

where the linear differential operators

$$\begin{aligned} P &= \sum_{r=0}^{n} p_r \frac{\partial^r}{\partial t^r} \qquad & Q &= \sum_{r=0}^{n} q_r \frac{\partial^r}{\partial t^r} \\ P' &= \sum_{r=0}^{n} p_r' \frac{\partial^r}{\partial t^r} \qquad & Q' &= \sum_{r=0}^{n} q_r' \frac{\partial^r}{\partial t^r} \end{aligned} \tag{10}$$

s_{ij} and e_{ij} are the stress and strain deviators respectively, and where σ_{kk} and ϵ_{kk} are three times the mean normal stress and strain, that is

$$\begin{aligned} \sigma_{kk} &= \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} \\ \epsilon_{kk} &= \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} \end{aligned} \tag{11}$$

in a cylindrical coordinate system in which r , θ and z are the radial, circumferential and axial directions respectively. The coefficients p_r and p_r' are constants which depend upon the material properties. The stress and strain deviators are related to the stress tensor σ_{ij} and the strain tensor ϵ_{ij} by the relations

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$
$$e_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_{kk} \delta_{ij} \qquad (12)$$

where δ_{ij} is the Kronecker delta, equal to unity when $i = j$ and equal to zero when $i \neq j$.

The strains are small in the problem under consideration because the viscoelastic propellant is surrounded by an elastic casing. The boundary requirements are met by virtue of the fact that the radial displacement of the propellant's outer surface is restricted by the casing both in the elastic and in the viscoelastic problems; the internal surface of the propellant is traction free. Thus the elastic solution for the stress in a case-bonded propellant disk is identical to the Laplace transform of the stress in the viscoelastic problem, when time t is the transform parameter. By making use of Equations (7) or (8) and (10), where the time derivatives $\partial^n/\partial t^n$ are replaced by powers of the inverse transform parameter p^n, the transformed expression for stress may be obtained directly from an elastic solution by replacing the material constants with the aforementioned equivalents. The viscoelastic solution for stress can then be obtained by taking an inverse Laplace transform of the resulting expression.

Using the foregoing concept, Shaffer and Levitsky[3] have shown that the Laplace transform of the time dependent viscoelastic radial bond stress is given by the expression

$$\bar{\sigma}(p) = \frac{b^2 - a^2}{2b^2} \left[\frac{\alpha_c \bar{T}_c(p) - \alpha_p \bar{T}_p(p)}{\frac{1}{2}\left(\frac{1}{3} + \frac{a^2}{b^2}\right) \frac{P}{Q} + \frac{1}{3} \frac{P'}{Q'} + J} \right] \tag{13}$$

where the subscripts c and p refer to the casing and the propellant respectively; and where

$$J = \left(\frac{b^2 - a^2}{c^2 - b^2}\right) \frac{1}{E_c} + \frac{1 + \mu_c}{2\, E_c} \left(1 - \frac{a^2}{b^2}\right) \tag{14}$$

a function of the dimensions and elastic constants of the casing,

μ_c = Poisson's ratio for the casing material

E_c = Young's modulus for the casing material

a,b,c = dimensions of the assembly as shown in Fig. 1

α_c = thermal expansion coefficient of the casing

α_p = thermal expansion coefficient of the propellant.

The variable p is the transform parameter identified by the operator relation $p = \partial/\partial t$, $\bar{T}_c(p)$ and $\bar{T}_p(p)$ are the Laplace transforms with respect to time of the mean temperature of the casing and propellant respectively, i.e.,

$$\bar{T}_c(p) = \int_0^\infty T_c(t)\, e^{-pt}\, dt \tag{15}$$

$$\bar{T}_p(p) = \int_0^\infty T_p(t)\, e^{-pt}\, dt \tag{16}$$

where

$$T_p(t) \left(1 - \frac{a^2}{b^2}\right) = \frac{2}{b^2} \int_a^b T(\rho, t)\, d\rho \tag{17}$$

$$T_c(t)\left(1 - \frac{c^2}{b^2}\right) = \frac{2}{b^2}\int_c^b T(\rho, t)\, d\rho \tag{18}$$

$T(\rho, t)$ is the space-time temperature distribution within the assembly expressed as a function of the dummy radial variable ρ. When the temperature throughout the casing or propellant is uniform radially, Equations (15) and (16) show

$$\overline{T}_c(p) = \int_0^\infty T_o H(t)\, e^{-pt} dt = \frac{T_o}{p} \tag{19}$$

where $H(t)$ is the Heaviside unit function defined by

$$\begin{aligned} H(t) &= 0 \quad t < 0 \\ H(t) &= 1 \quad t \geq 0 \end{aligned} \tag{20}$$

and

$$\overline{T}_p(p) = \int_0^\infty T_o(1 - e^{-\beta t})\, e^{-pt} dt = \frac{\beta T_o}{p\,(p + \beta)} \tag{21}$$

Use is made here of the restricted temperature variations expressed by Equations (5) and (6).

For a material which is elastic in compression, it is convenient to write the ratio of linear operators,

$$\frac{P'}{Q'} = \frac{1}{E_B} \tag{22}$$

where, in view of Equations (8) and (22) E_B is a viscoelastic constant related to the elastic bulk modulus K by the relation

$$E_B = 3K \tag{23}$$

Substituting Equations (19), (21), and (23) into Equation (13) shows that bond stress in the assembly may be written

$$\overline{\sigma}(p) = T_o \left(\frac{b^2 - a^2}{2b^2}\right)\left[\frac{\alpha_c}{p} - \frac{\alpha_p \beta}{p(p+\beta)}\right]\left[\frac{1}{\frac{1}{2}\left(\frac{1}{3} + \frac{a^2}{b^2}\right)\frac{\overline{P}}{\overline{Q}} + \frac{1}{3E_B} + J}\right] \quad (24)$$

It is now convenient to define a new parameter,

$$L = \frac{2\left(\frac{1}{3E_B} + J\right)}{\frac{1}{3} + \frac{a^2}{b^2}} \quad (25)$$

which is independent of and thus not affected by any subsequent variations in the viscoelastic shear properties of the propellant. With this new parameter the bond stress expression may be written in the form

$$\overline{\sigma}(p) = \frac{3T_o(b^2 - a^2)}{b^2 + 3a^2}\left[\frac{\alpha_c}{p} - \frac{\alpha_p \beta}{p(p+\beta)}\right]\left(\frac{1}{\frac{\overline{P}}{\overline{Q}} + L}\right) \quad (26)$$

The time dependent bond stress $\sigma(t)$ is then, in principle determined once the material properties of the propellant in shear are specified by the operator fraction P/Q .

<u>Solutions for Viscoelastic Materials</u>

Let us now specialize the bond stress equation derived in the previous section for a Maxwell material, for a Voigt material, and for the general four element material.

For a Maxwell material in shear, the stress strain law may be written

$$\left(\frac{1}{E_1} + \frac{1}{\eta_1 \frac{\partial}{\partial t}}\right) s_{ij} = e_{ij} \quad (27)$$

where E_1 is an elastic parameter and η_1 a viscous constant.

Comparison with Equation (9) indicates that for a Maxwell material the polynomial fraction $\overline{P}/\overline{Q}$ may be written in terms of the operator p as

$$\frac{\overline{P}}{\overline{Q}} = \frac{1}{E_1} + \frac{1}{\eta_1 p} \tag{28}$$

Substitution of Equation (28) into Equation (26) shows that

$$\overline{\sigma}(p) = \frac{3T_o\ (b^2 - a^2)}{b^2 + 3a^2}\left[\frac{\alpha_c}{p} - \frac{\alpha_p \beta}{p(p + \beta)}\right] \frac{1}{\frac{1}{E_1} + \frac{1}{\eta_1 p} + L} \tag{29}$$

Rearranging terms of the last equation gives

$$\overline{\sigma}(p) = \frac{\alpha_c T_o\ (1 - a^2/b^2)}{\left(\frac{1}{3} + \frac{a^2}{b^2}\right)\frac{1}{E_1} + \frac{2}{3E_B} + 2J}\left[\frac{(1 - \alpha_R)\ \beta + p}{(p + \beta)(p + \delta)}\right] \tag{30}$$

where the new parameter δ is defined by the relation

$$\left[\left(\frac{1}{3} + \frac{a^2}{b^2}\right)\frac{1}{E_1} + \frac{2}{3E_B} + 2J\right]\delta = \frac{1}{\eta_1}\left(\frac{1}{3} + \frac{a^2}{b^2}\right) \tag{31}$$

and

$$\alpha_R = \alpha_p/\alpha_c \tag{32}$$

The inverse Laplace transform to Equation (30) may be obtained by partial fractions, yielding

$$\sigma(t) = \frac{\alpha_c T_o\ (1 - a^2/b^2)}{\left(\frac{1}{3} + \frac{a^2}{b^2}\right)\frac{1}{E_1} + \frac{2}{3E_B} + 2J}\left\{\frac{\left[\delta - (1-\alpha_R)\beta\right] e^{-\delta t} - \beta\ \alpha_R e^{-\beta t}}{\delta - \beta}\right\} \tag{33}$$

which is the expression for the time dependent bond stress in a case bonded rocket assembly, with a Maxwell type propellant, under temperature conditions of Equations (5) and (6).

Similarly, since the stress-strain law for the shear properties of a Voigt material is

$$\left(E_2 + \eta_2 \frac{\partial}{\partial t}\right) e_{ij} = s_{ij} \tag{34}$$

the operator expression for $\overline{P}/\overline{Q}$ may be written

$$\frac{\overline{P}}{\overline{Q}} = \frac{1}{E_2 + \eta_2 p} \tag{35}$$

so that Equation (26) becomes

$$\overline{\sigma}(p) = \frac{3T_o(b^2 - a^2)}{b^2 + 3a^2} \; \frac{\left[\frac{\alpha_c}{p} - \frac{\alpha_p \beta}{p(p + \beta)}\right]}{\frac{1}{E_2 + \eta_2 p} + L} \tag{36}$$

Subsequent calculations are made more convenient by rearranging Equation (36) so that it reads

$$\overline{\sigma}(p) = \frac{\alpha_c T_o(1-a^2/b^2)}{\frac{2}{3E_B} + 2J} \left\{\frac{p^2 + \{(1-\alpha_R)\beta + E_2/\eta_2\}\, p + (1-\alpha_R)\beta E_2/\eta_2}{p(p + \beta)(p + \delta)}\right\} \tag{37}$$

where

$$\delta = \frac{\frac{1}{3} + \frac{a^2}{b^2} + 2E_2 \left(\frac{1}{3E_B} + J\right)}{2\eta_2 \left(\frac{1}{3E_B} + J\right)} \tag{38}$$

The inverse Laplace transform to Equation (37) is taken again by partial fractions, and gives

$$\sigma(t) = \frac{\alpha_c T_o(1 - a^2/b^2)}{2\delta\left(\frac{1}{3E_B} + J\right)} \left[(1 - \alpha_R)\, \frac{E_2}{\eta_2} + \Gamma_1 + \Gamma_2\right] \tag{39a}$$

where

$$\Gamma_1 = \frac{1}{(\beta - \delta)} \left\{ (1-\alpha_R) \frac{E_2}{\eta_2} - \left((1-\alpha_R)\ \beta + \frac{E_2}{\eta_2} \right) + \beta \right\} \delta\ e^{-\beta t} \quad (39b)$$

$$\Gamma_2 = \frac{1}{(\delta - \beta)} \left\{ (1-\alpha_R)\ \beta \frac{E_2}{\eta_2} - \left((1-\alpha_R)\ \beta + \frac{E_2}{\eta_2} \right) \delta + \delta^2 \right\} e^{-\delta t} \quad (39c)$$

For future comparison with the corresponding bond stress equation for a Maxwell propellant, the preceding equation is rewritten as

$$\sigma(t) = \frac{\alpha_c T_o\ (1 - a^2/b^2)}{\left(\frac{1}{3} + \frac{\alpha^2}{b^2}\right)\frac{1}{E_2} + \frac{2}{3E_B} + 2J} \left[(1 - \alpha_R) + \Gamma_3 \right] \quad (40a)$$

where

$$\Gamma_3 = \frac{1}{\beta-\delta} \left\{ \delta\alpha_R \left(\frac{\beta\eta_2}{E_2} - 1 \right) e^{-\beta t} + \left((\alpha_R - 1)\beta + \delta \right) \left(1 - \frac{\eta_2\delta}{E_2} \right) e^{-\delta t} \right\}$$

(40b)

The stress-strain law for the four element model of the form illustrated by Fig. 2c is

$$e_{ij} = \left[\frac{1}{E_1} + \frac{1}{E_2 + \eta_2 \frac{\partial}{\partial t}} + \frac{1}{\eta_1 \frac{\partial}{\partial t}} \right] s_{ij} \quad (41)$$

It may be written in the form of Equation (9), namely

$$\left[\left(\frac{\partial}{\partial t}\right)^2 + \frac{E_2}{\eta_2} \frac{\partial}{\partial t} \right] e_{ij} = \left[\frac{1}{E_1} \left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{1}{\eta_1} + \frac{1}{\eta_2} + \frac{E_2}{E_1\eta_1} \right) \frac{\partial}{\partial t} + \frac{E_2}{\eta_1\eta_2} \right] s_{ij}$$

(42)

Making the formal substitution $p = \partial/\partial t$, Equation (42) may be rewritten

$$\frac{P}{Q} = \frac{Ap^2 + Bp + c}{Dp + p^2} \tag{43}$$

where the parameters A,B,C and D are given by

$$A = \frac{1}{E_1} \qquad B = \frac{E_2}{E_1\eta_1} + \frac{1}{\eta_1} + \frac{1}{\eta_2}$$

$$C = \frac{E_2}{\eta_1\eta_2} \qquad D = \frac{E_2}{\eta_2} \tag{44}$$

Substituting Equation (43) into Equation (26) shows that the transformed bond stress takes the form

$$\bar{\sigma}(p) = \frac{3T_o(b^2-a^2)}{b^2 + 3a^2}\left[\frac{(\alpha_c-\alpha_p)\ \beta + \alpha_c}{(p + \beta)}\right]\left[\frac{p + D}{Ap^2 + B_p + C + Lp^2 + LDp}\right] \tag{45}$$

When the quadratic expression in the denominator is factored in terms of its roots,

$$\bar{\sigma}(p) = \frac{3T_o(b^2-a^2)\alpha_c}{(b^2 + 3a)(A+L)}\left[\frac{p^2 + [(B+D) - \alpha_R\beta]p + (1 - \alpha_R)\ \rho D}{(p - R^+)\ (p - R^-)\ (p + \beta)}\right] \tag{46}$$

where the roots R^+ and R^- are given, for the moment, as

$$R^{\pm} = \frac{1}{2}\left[- \frac{B + LD}{A + L} \pm \sqrt{\left(\frac{B + LD}{A + L}\right)^2 - \frac{4C}{A + L}}\right] \tag{47}$$

In this form, it is simple to take the inverse Laplace transform of the bond stress by partial fractions, resulting in the time dependent bond stress expression

$$\sigma(t) = \frac{3T_o(a^2-b^2)\alpha_c}{(b^2+ 3a^2)(A+L)}\left\{\left[\frac{\beta^2-(\beta+D-\alpha_R\beta)\ \beta+(1-\alpha_R)\beta D}{(\beta + R^+)(\beta + R^-)}\right]e^{-\beta t}+\lambda_1+\lambda_2\right\} \tag{48}$$

where

$$\lambda_1 = \frac{R^{-2} + (\beta + D - \alpha_R\beta)\, R^- + (1 - \alpha_R)\beta D}{(R^- - R^+)(R^- + \beta)}\, e^{R^- t} \tag{49a}$$

$$\lambda_2 = \frac{R^{+2} + (\beta + D - \alpha_R\beta)\, R^+ + (1 - \alpha_R)\, \beta D}{(R^+ - R^-)(R^+ + \beta)}\, e^{R^+ t} \tag{49b}$$

With the notation $R^{\pm} = R \pm \epsilon$, where

$$R = -\frac{E_2}{2\eta_2} - \frac{E_1}{2(1 + E_1 L)}\left[\frac{1}{\eta_1} + \frac{1}{\eta_2}\right] \tag{50}$$

$$\epsilon = +\frac{1}{2}\sqrt{\left(\frac{E_2}{\eta_2}\right)^2 + \frac{2E_1E_2}{\eta_2(1+E_1L)}\left(\frac{1}{\eta_2} - \frac{1}{\eta_1}\right) + \frac{E_1^{\,2}}{(1+E_1L)^2}\left(\frac{1}{\eta_2} + \frac{1}{\eta_1}\right)^2} \tag{51}$$

the bond stress may be more simply written in the form

$$\sigma(t) = \left[\frac{\alpha_c T_o\ (1 - a^2/b^2)}{\left(\frac{1}{3} + \frac{a^2}{b^2}\right)\frac{1}{E_2} + \frac{2}{3E_B} + 2J}\right] F(t) \tag{52}$$

where

$$\frac{2\epsilon}{e^{Rt}} F(t) = \left(1 - \frac{\alpha_R\beta}{R + \epsilon + \beta}\right)\left(R + \epsilon + D\right) e^{+\epsilon t} - \left(1 - \frac{\alpha_R\beta}{E - \epsilon + \beta}\right)\cdot$$

$$\left(R - \epsilon + D\right) e^{-\epsilon t} + \left[\frac{\alpha_R\beta\ (\beta - D)}{(\beta + R)^2 - \epsilon^2}\right] e^{-\beta t} \tag{53}$$

Equations (50) - (53), taken together, give the variation with respect to time of the bond stress with a four element propellant model in a case bonded assembly.

Discussion of Results

The expressions developed for the bond stress variation in the three cases of the Maxwell type, Voigt, and four element propellant models have been put into a general form

$$\sigma(t) = S_o \, F(t) \tag{54}$$

A comparison among the three pertinent Equations (33), (40), and (52) shows that S_o is the same in each case,

$$S_o = \frac{\alpha_c \, E \, T_o \, (1 - a^2/b^2)}{\frac{1}{3} + \frac{a^2}{b^2} + \frac{2E}{3E_B} + 2JE} \tag{55}$$

subject to proper interpretation of the elastic parameter E. That is, under suddenly applied constant loading for the Maxwell and four element models, E represents an elastic modulus associated with an initial deformation of the propellant; for the Voigt model, it represents the elastic behavior at infinite time. The common consideration is that viscous effects are inoperative in the coefficient S_o. Further study then shows S_o to be identical with the elastic stress produced when the temperature of the casing changes by an amount T_o, while the propellant is still at the initial temperature. The conditions under which the magnitude of S_o may be decreased for a fixed temperature change are essentially similar to those discussed for the elastic case in prior papers[1,2] Of course, in the viscoelastic problem, variations of the time dependent factor $F(t)$ may be more significant than S_o in determining the maximum bond stress to which an assembly may be subjected.

The complex manner in which the equations for $F(t)$ depend upon the various physical parameters precludes much quantitative analytical discussion of their significance. Instead, a number of computations will be performed in order to indicate the general characteristics of the time dependent factor. Several conclusions may be drawn on this basis.

The numerical work will be restricted to a steel encased solid propellant rocket whose physical dimensions, $a/b = .5$ and $c - b = .01b$ are representative of an assembly. The material properties typical of a steel casing are

$$E_c = 30 \times 10^6 \text{ psi} \qquad \alpha_c = 7.3 \times 10^{-6} \text{ in/in/}^\circ\text{F}$$

$$\mu_c = 0.3 \qquad G = 11.53 \times 10^6 \text{ psi}$$

where μ_c is Poisson's ratio, and G is the shear modulus, obtained from E_c and μ_c. It is further assumed that the thermal expansion coefficient of the propellant is similar to vulcanized rubber, namely $\alpha_p = 2\text{x}10^{-4}$ in/in/$^\circ$F. Since the behavior of the propellant in compression has been assumed elastic, there is no loss in present calculation if, as a limiting case, the propellant is made incompressible, so that E_B may be taken infinitely large.

In the case of the four element model, the viscoelastic properties of the propellant in shear are governed by the choice of four independent parameters, two elastic constants E_1 and E_2, and two viscous parameters, η_1 and η_2. To avoid a direct numerical choice of all four quantities, and thereby render the results somewhat more general, it will be convenient to present the numerical work in terms of the ratios K_1, K_2, and K_3, previously defined. In addition, by studying the dimensionless stress ratio $\sigma(t)/S_o$ rather than the bond stress itself, the magnitude of the remaining elastic parameter may be left arbitrary. No numerical values need be assigned to the K's, providing that the reciprocal of one, say K_1, is chosen as the unit of the time scale against which the bond stress variation will be measured, for the other time-like parameters K_2, K_3 and β, may be measured with respect to K_1. It is then necessary only to specify the ratios K_2/K_1 and K_3/K_1 for the propellant.

Let us first consider the influence of the rate of temperature change on the thermal bond stress. Since there is little information available on the magnitude of viscoelastic material constants, it will be necessary to choose an arbitrary set of material properties, say $K_3 = K_2 = K_1$. The time dependent factor in the bond stress may be evaluated from Equations (50), (51), and (53) for different values of the temperature rate parameter β. The results of this calculation are presented in Fig. 3, which shows variation of the stress ratio $\sigma(t)/S_o$ with respect to the dimensionless time $t \cdot K_1$, plotted on semilog coordinates.

For all rates of temperature change there is a characteristic change of sign in the bond stress as the curve β = (a constant) crosses the dimensionless time axis. This arises, as a subsequent calculation will show, from the relative magnitudes of the thermal expansion coefficients, as well as the rate at which the temperature of the propellant changes. Increasing values of β , corresponding to higher rates of temperature variation, cause an increasingly severe change from the initial stress and a more rapidly occurring peak. Alternately, slow variations in the temperature of the propellant result in a lower peak stress which occurs somewhat later in dimensionless time. Hence small values of β are desirable as a means of minimizing stress.

In view of the difference in properties which may occur among real propellants, it is of interest to study the effects of different combinations of the four element parameters on the bond stress variation. Specific studies cannot be conducted because data is lacking. Instead, some of the coefficients will be permitted to vary but in accordance with a definite sequence. One such sequence is to consider those changes in the relative values of K_1 , K_2 and K_3 which lead, in the limit, to a two element Maxwell type propellant model. It can be seen from Fig. 2c that such a pattern may be obtained by keeping E_1 and η_1 constant, but varying the viscosity of the element η_2 . The effect is equivalent to keeping K_1 constant while varying K_2 and K_3 . If, arbitrarily, K_2 is set equal to K_3 , then in either limit, as K_1 becomes very much larger than $K_2 = K_3$, or as K_1 becomes very much smaller than $K_2 = K_3$, the four element material tends toward a Maxwell-like limit. The corresponding stress-time curves for various values of the ratio K_2/K_1 are shown in Fig. 4 for $\beta = 2K_1$.

In looking for a sequence of changes for K_1 , K_2 and K_3 which would alter a four element model to a Voigt-like model, several factors must be taken into consideration. In a Voigt material the final stress is determined by the elastic parameter E_2 and the final stress should remain constant in the limiting process. Consequently, K_3 should be kept constant and tK_3 used as the dimensionless time scale. The parameters K_1 and K_2 may then be varied as discussed earlier to evaluate the effect of increasing stiffness in the propellant material. The results of some numerical computations with $\beta = 2K_3$ are shown in Fig. 5. The time-ratio scale has been distorted in order to present details near the origin but still accommodate a relatively large range. It is seen that high values of the initial elastic parameter E_1 lead to an increasing initial stress as well as a large peak in the subsequent stress reversal, although the latter will be governed by the Maxwell-like compliance K_1 , as well.

Methods for determining the parameters E_1 , E_2 , η_1 and η_2 based either upon vibrating reed tests, or creep and relaxation data have been discussed in the literature [8]. Appropriate values of the parameters for specific propellants may be determined experimentally and substituted into the equations of the present paper to evaluate the affect of changes of propellant chemistry.

The change in sign of the bond stress may be interpreted physically. As the initial dimensions of the casing change, followed by an expansion or contraction of the propellant, due to the relative thermal expansion coefficient, there develops an over-compensation of the dimensional change relative to the casing. For example, if the casing contracts initially causing a compressive bond stress, it is subsequently followed by a contraction of the propellant which could be large enough so that the propellant would draw away from the casing if it were unbonded. With bonding, however, contact is maintained and tensile stresses develop. Their magnitude depend upon the thermal expansion coefficients, the relative rapidity of the temperature rise, and the viscoelastic constants.

The influence of the ratio of thermal expansion coefficients may be readily determined and the results of some computations are shown in Fig. 6. As the ratio $\alpha_R = \alpha_p/\alpha_c$ decreases, the magnitude of the stress reversal also decreases. It seems that sufficiently low values of the ratio α_R also cause the stress peak to appear at later times.

Concluding Remarks

Equations have been obtained for the thermal bond stress in a case bonded cylindrical propellant grain for radially independent propellant temperature profiles. Various viscoelastic propellant materials have been considered, such as those represented by a Maxwell model, Voigt model, and a four element model which may be thought of as a series combination of the aforementioned. The resulting bond stress equation is separable into the product of two factors, S_o and $f(t)$. The former, S_o, is identical with the elastic bond stress produced by the temperature distribution at zero time for the Maxwell and four-element models, and at infinite time for the Voigt model. In making use of this identification, the elastic propellant constants appearing in S_o must be evaluated at the corresponding time limits.

The time dependent factor $f(t)$ is a function of all the propellant parameters. It has a significantly large range of variation, which may produce peaks of stress several times the initial value.

In attempting to minimize the bond stress, the following points appear to be of importance. The elastic behavior of the propellant, either under initial loading or at long times directly affects S_o . This, together with other factors pointed out in previous elastic analyses indicate a direction for decreasing the magnitude of S_o . In particular, it would appear that the initial elastic modulus should be kept as small as practical.

The stress-time pattern is directly influenced by the rate of temperature change and the ratio of the thermal expansion coefficient of the propellant to that of the casing. Both quantities should be kept small to minimize the stress peak arising from the change of sign in $f(t)$.

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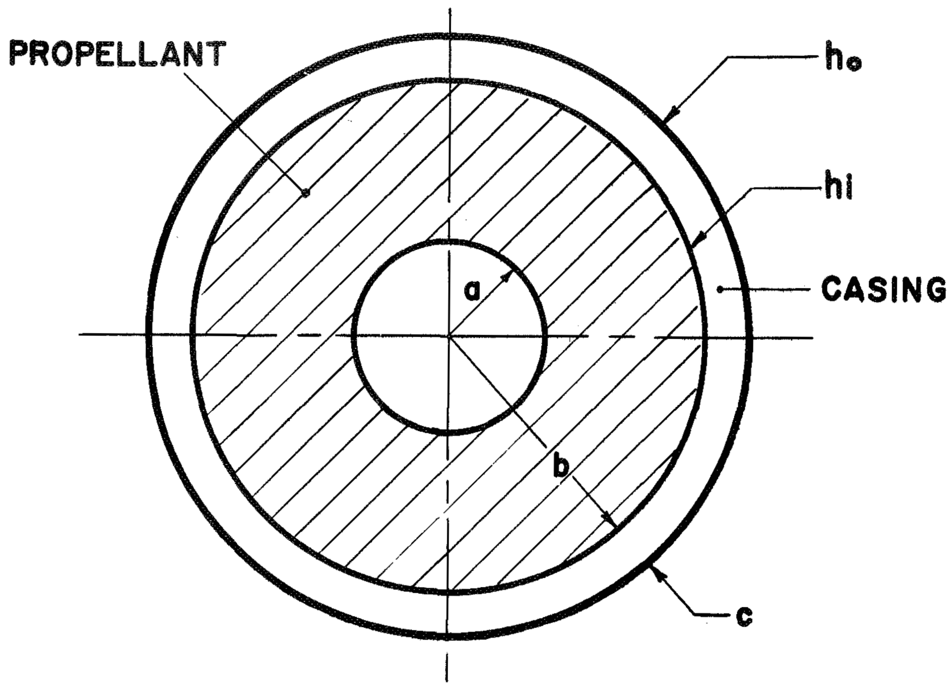

FIGURE 1

CROSS SECTION OF A

SOLID PROPELLANT ROCKET

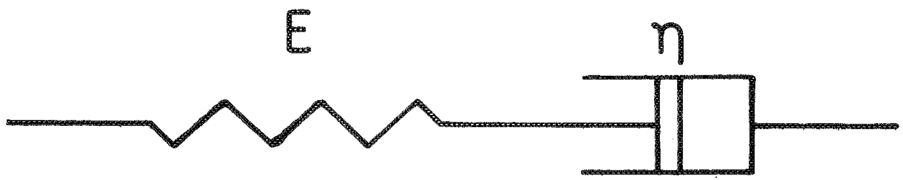

FIGURE 2A

TWO ELEMENT MAXWELL MODEL

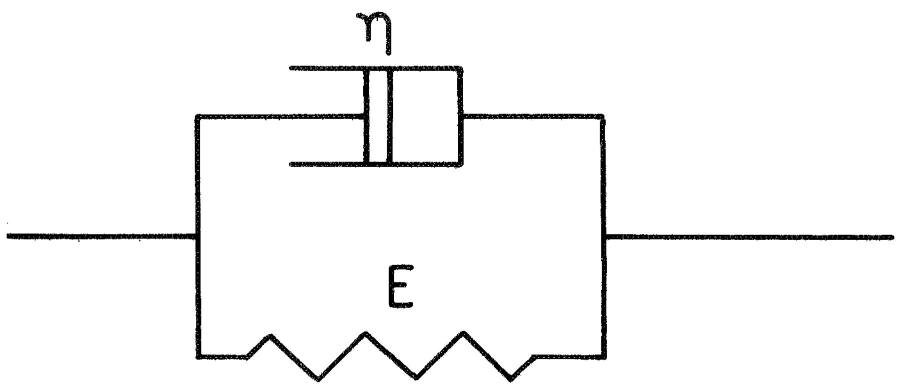

FIGURE 2B

TWO ELEMENT VOIGT MODEL

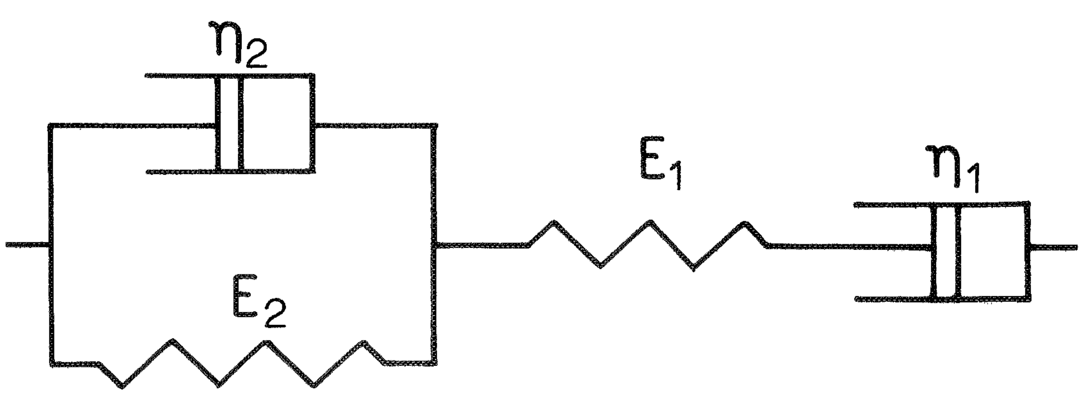

FIGURE 2C

FOUR ELEMENT MODEL

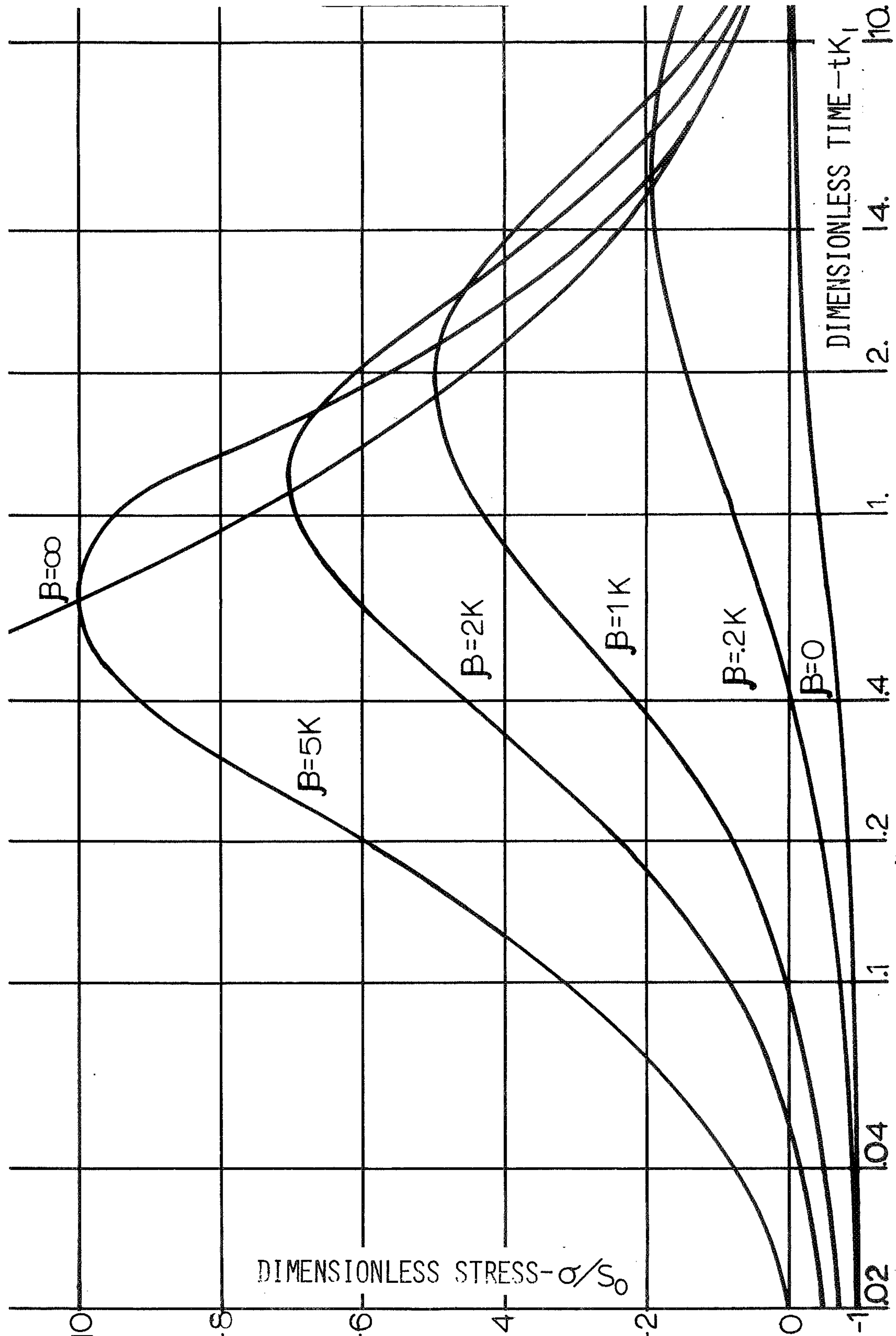

Figure 3

Bond Stress History for Several Rates of Temperature Change in the Propellant. Propellant Model $K_1 = K_2 = K_3$.

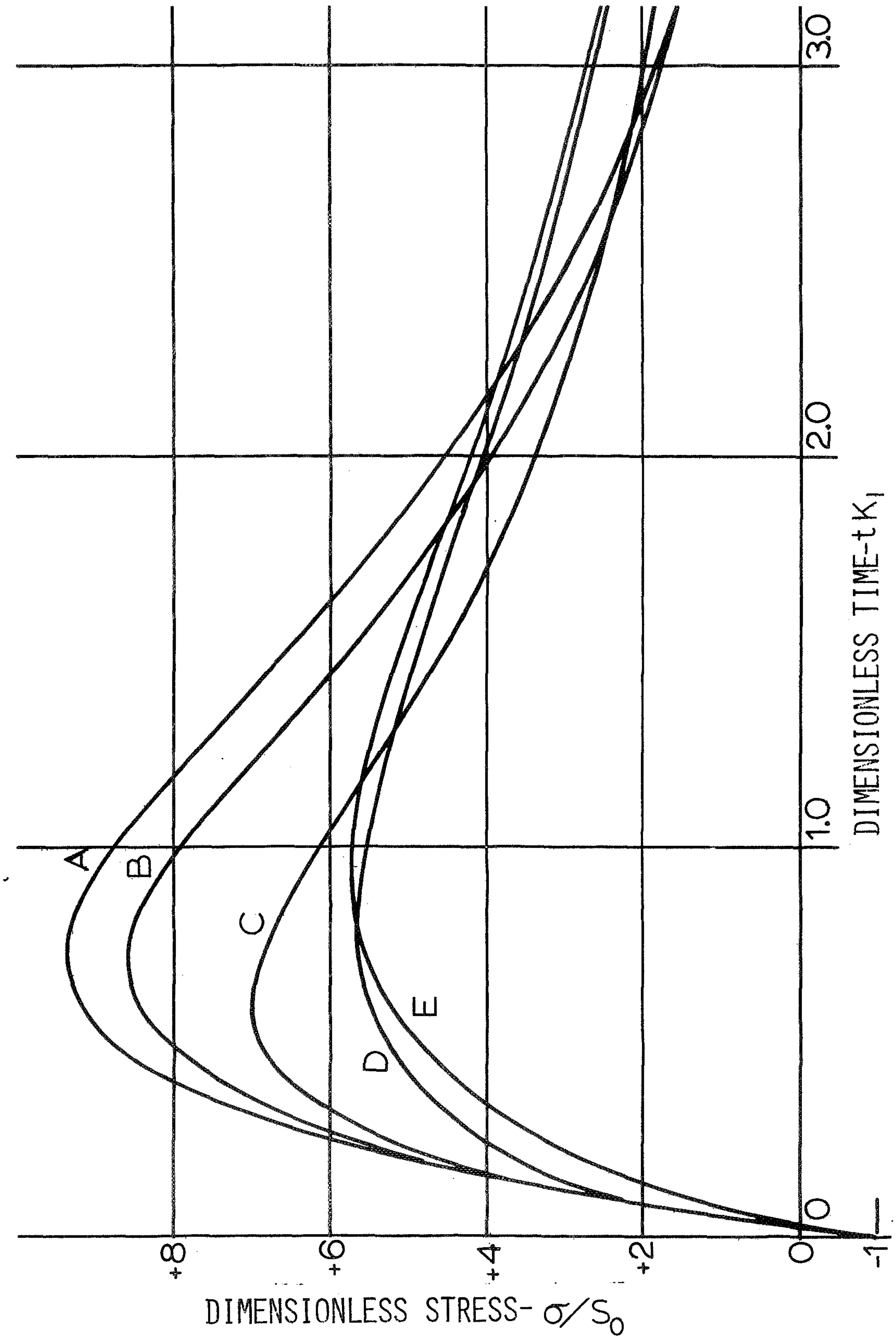

Figure 4

Bond Stress History for Maxwell-like Variations of Propellant Properties. Temperature Time Constant $\beta = 2K_1$

A - Pure Maxwell

B - $K_2 = K_3 = .2K_1$

C - $K_2 = K_3 = K_1$

D - $K_2 = K_3 = 5K_1$

E - $K_2 = K_3 = 20K_1$

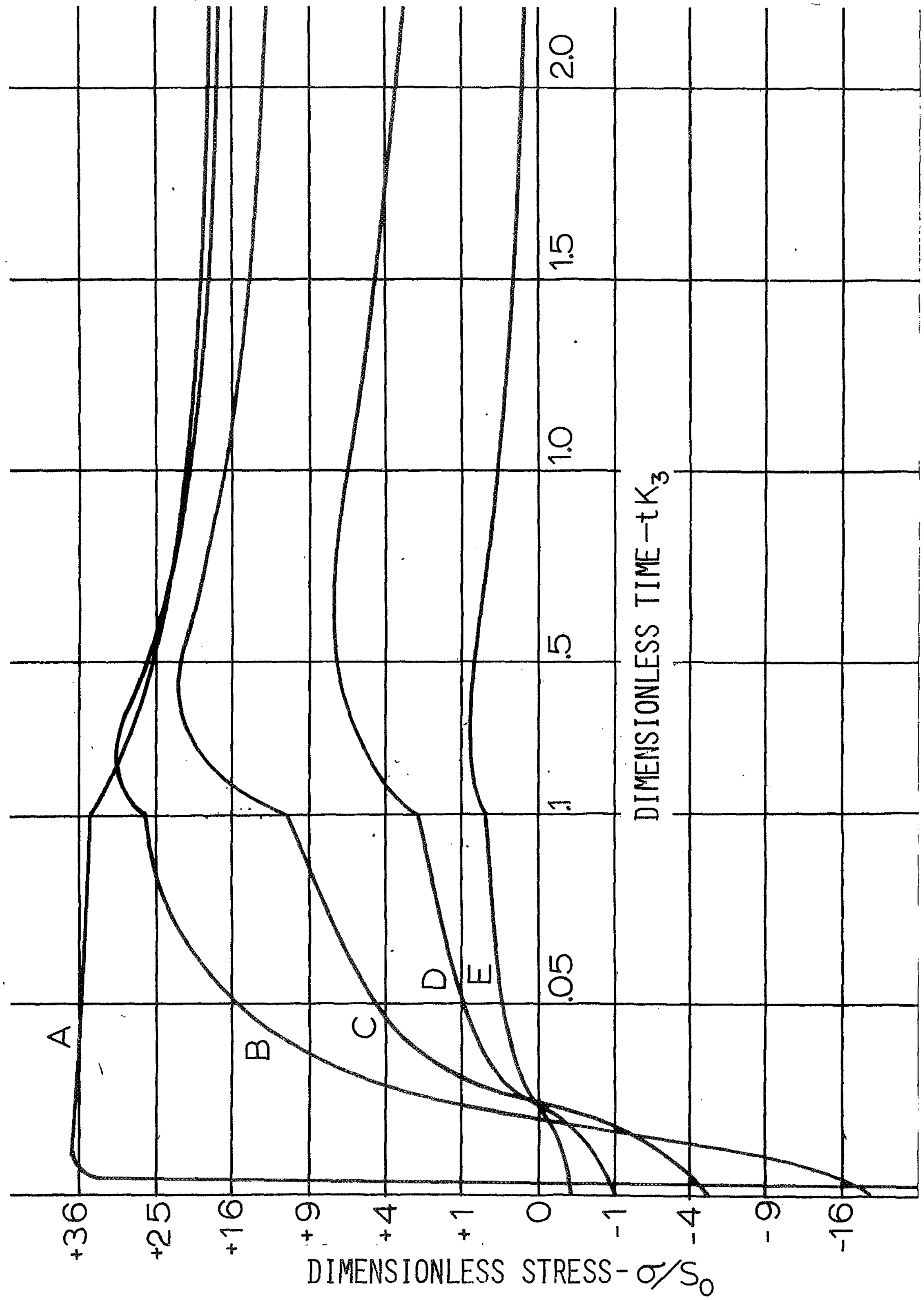

Figure 5

Bond Stress History for Voigt-like Variations of Propellant Properties. Temperature Time Constant $\beta = 2K_3$

A - Pure Voigt
B - $K_2 = 20K_3$: $K_1 = .05K_3$
C - $K_2 = 5K_3$: $K_1 = .2K_3$
D - $K_1 = K_2 = K_3$
E - $K_2 = .2K_3$: $K_1 = 5K_3$

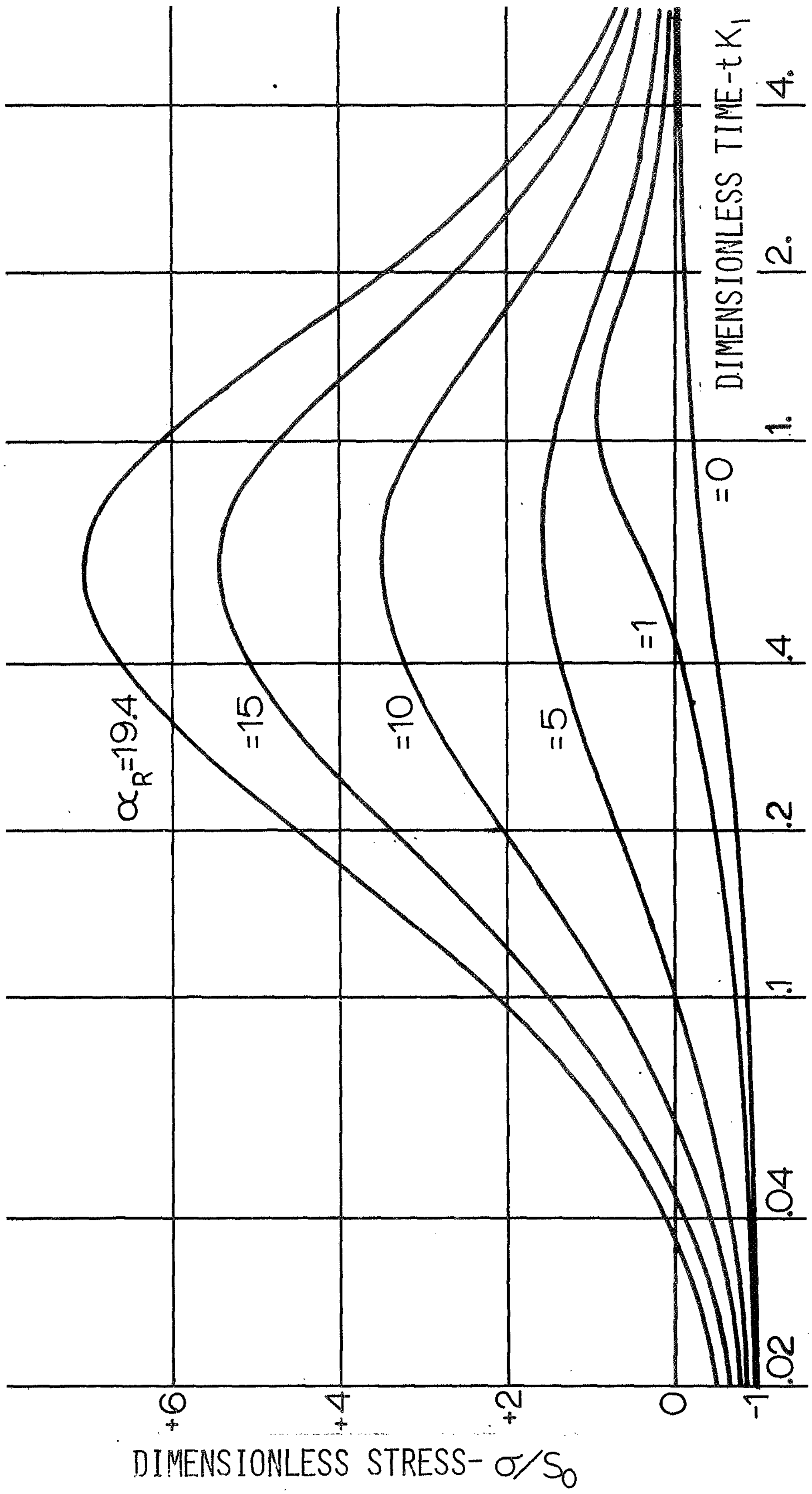

Figure 6

Bond Stress History for Several Values of the Ratio of Thermal Expansion Coefficients α_R . Temperature Time Constant $\beta = 2K_1$. Propellant Model $K_1 = K_2 = K_3$