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STABILITY OF THE SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH GENERAL BOUNDARY CONDITIONS

by

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FOREWORD

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ABSTRACT

STABILITY OF THE SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH GENERAL BOUNDARY CONDITIONS

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The object of this dissertation is to establish sufficient conditions to ensure the existence, uniqueness, stability and asymptotic stability of the solution to the following initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x\varepsilon\Omega, t\geq 0$$

with general boundary conditions

$$B_j(x,D)u(x,t) = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \quad (0\leq j\leq m-1)$$

and initial condition

$$u(x,o) = u_o(x)$$

where $A(x,D)$ is a strongly elliptic partial differential operator in $\Omega\subset R^n$, $n \geq 1$, and $\{B_j\}_{j=0}^{m-1}$ satisfies very general boundary conditions which include the Dirichlet boundary conditions as a subclass, and f is, in general, a nonlinear function defined on the appropriate function space. By setting $u(t) = u(\cdot,t)$ the above equation with the boundary conditions and initial condition are reduced to an abstract (nonlinear) operator differential equation

$$\frac{du(t)}{dt} + Au(t) = f(u) \qquad (t\geq 0)$$

$$u(o) = u_o$$

where A is an (unbounded) linear operator with domain and range both contained in the same real Hilbert space H, and f is a nonlinear function mapping all of H into H. With the proper definition of the base Hilbert space, H, A becomes an extension of the operator A(x,D). With additional

assumptions on A and the nonlinear function f(u) the existence, uniqueness, stability or asymptotic stability of the initial-boundary value problem is ensured from the results obtained for the abstract operator equation.

The investigation of stability criteria is extended to the study of the following initial-boundary value problem

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u)$$

with general boundary conditions. By setting $u_1 = u, u_2 = \frac{\partial u}{\partial t}$ this equation is reduced to a system of equations of the form

$$\frac{\partial \underline{u}}{\partial t} + \begin{bmatrix} 0 & -1 \\ A(x,D) & a \end{bmatrix} \underline{u} = \underline{f}(\underline{u})$$

where

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \underline{f}(\underline{u}) = \begin{bmatrix} 0 \\ f(x,u_1,u_2) \end{bmatrix}.$$

By a suitable choice of function space, we obtain the abstract operator equation of the form

$$\frac{d\underline{u}(t)}{dt} + A\underline{u}(t) = \underline{f}(\underline{u}) \qquad (t \geq 0)$$

$$u(o) = u_o$$

where A is an abstract linear operator extension of A(x,D) mapping some function space into itself. With certain restrictions on the system $(A(x,D),\{B_j\},\Omega)$, and on the nonlinear function f(u), stability criteria is established for the general boundary value problem from the results obtained for the abstract operational differential equation. The linear problem, $f(u) \equiv 0$, and the nonlinear problem are considered for $\Omega \subset R^n$, where the cases $n \geq 2$ and $n = 1$ are considered separately, since the boundary conditions differ for the two cases. Applications are given which show how the theory can be applied to a large class of physical and engineering problems.

1.0. INTRODUCTION

A.M. Lyapunov in [19]* developed his so-called "second method" or "direct method" which years later was used in answering the question of stability of differential equations from the given form of the equations together with the boundary conditions without explicit knowledge of the solutions. In the study of ordinary differential equations, the main idea of Lyapunov's direct method is the construction of a "Lyapunov functional," v(u) with u in some finite dimensional space and having the properties that v(u) is positive and the derivative of v(u) along solutions of the given equation is negative. Since many physical problems must be described by partial differential equations it was natural to extend Lyapunov's direct method to study the case for partial differential equations by the construction of a Lyapunov functional in infinite dimensional spaces, and by the use of function spaces on which a topology was defined. Zubov in [36] considered equations of the form

$$\frac{\partial u(x,t)}{\partial t} = f\ (x,u,\frac{\partial u}{\partial x})$$

and established a stability theory for the special case of f linear in $\frac{\partial u}{\partial x}$. In more recent years a growing number of results have been discovered as can be seen in a survey of the literature by Wang [34]. However, each one was only concerned with specific partial differential operators and considered only small classes of problems. No rigorous mathematical approach covering a large class of systems was used and often the existence of the solution was assumed. There were some authors who studied the stability problem for operational differential equations, for example, Taam in [31]

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*Numbers in brackets designate references at the end of this dissertation.

studied the stability properties of the equation

$$\frac{du(t)}{dt} + Au(t) = f(t,u,\lambda)$$

where A was either a bounded linear operator or the infinitesimal generator of a semi-group and sufficient conditions were given for the existence and asymptotic stability of a periodic solution.

1.1. Recent Results for the Dirichlet Problem

One of the problems of extending the Lyapunov stability theory from ordinary differential equations to partial differential equations is that the existence of the solution must first be established, since the derivative of the Lyapunov functional is taken along solutions of the given equation. Buis in [7] was the first to rigorously use the results for operational differential equations to solve the stability problem for a large class of initial - boundary value problems by considering a linear partial differential equation

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = 0 \qquad (1-1)$$

with Dirichlet boundary conditions and a given initial function, where A(x,D) is a linear partial differential operator and u(x,t) is in some prescribed function space. An operator differential equation is formed as follows, with the abstract equation

$$\frac{du(t)}{dt} + Au(t) = 0 \qquad (t \geq 0) \qquad (1-2)$$

where u(t) is a vector valued function with values in the real Hilbert space

H and A is a linear unbounded operator with domain and range in H, and A can be considered as the extension of the partial differential operator A(x,D) in (1-1) in the sense that for any u in the domain of A, Au is the function defined by

$$(Au)(x) = A(x,D)u(x)$$

where the domain, D(A), and base Hilbert space, H, are complete function spaces and the domain of A is characterized by the Dirichlet boundary conditions. This shows that the operator equation (1-2) can be considered as an abstract extension of (1-1). The stability problem of (1-2) is then studied. By using semi-group theory, the solution of (1-2) can be represented by a semi-group in the sense that if a solution of (1-2) with initial value at $t = t_o$ of $u_o \varepsilon D(A)$ is denoted by $u(t;u_o,t_o)$, then under suitable conditions the operator A is the infinitesimal generator of a semi-group $\{T_t | t \geq 0\}$ of bounded linear operators such that the solution of (1-2) exists and is given by

$$u(t;u_o,t_o) = T_t u_o \qquad (t \geq 0)$$

Thus, the stability properties of the solution of (1-2) are related to the properties of the semi-group generated by A. Buis established sufficient conditions for A to generate a semi-group (of class C_o) so that a solution of (1-2) exists and is stable. Then from the semi-group properties and the definition of D(A), this gave sufficient conditions for the solutions of (1-1) to exist, satisfy the initial condition, verify the Dirichlet boundary conditions and also be asymptotically stable or stable. Buis developed his stability criteria for the operator equation solving only the class of linear partial differential equations satisfying the Dirichlet boundary conditions.

Pao in [23] examined the non-linear operational differential equation or evolution equation

$$\frac{du(t)}{dt} + Au(t) = 0 \qquad (t \geq 0) \qquad (1\text{-}3)$$

where A, is in general an unbounded, non-linear operator with domain and range both contained in the same real (or complex) Hilbert space H. In [23], necessary and sufficient conditions were given so that A would generate a non-linear semi-group $\{T_t | t \geq 0\}$ of bounded operators in a Hilbert space which ensures the existence and uniqueness of the solution of (1-3), while the properties of the semi-group establish the stability of the solution to (1-3). As a subclass of the evolution equation (1-3), Pao in [23] also considered the operator equation of the form:

$$\frac{du(t)}{dt} + Au(t) = f(u) \qquad (t \geq 0) \qquad (1\text{-}4)$$

in which A is a linear, unbounded operator with domain and range both contained in a real Hilbert space H, and f is a non-linear function from H into H. A has the property of being the infinitesimal generator of a linear semi-group of bounded operators in a Hilbert space and conditions were given on f to ensure the existence, uniqueness, asymptotic stability or stability of the solution of (1-4). Pao only applied these results to the relatively small class of partial differential equations which satisfy Dirichlet boundary conditions, since it can be related to the evolution equation (1-3) or (1-4) readily. Because of the difficulty of defining the abstract operator A, its domain D(A), and the base Hilbert space H so it would satisfy the conditions imposed by Pao in [23] and also relate the equation (1-4) to the non-linear partial differential equation (1-5) with general boundary conditions work has not been done to solve the more general boundary value problem

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u). \qquad (1\text{-}5)$$

1.2. Recent Developments in General Boundary Value Problems

In studying the stability problem for more general initial-boundary value problems for partial differential equations one needs to ensure the existence and uniqueness as well as the stability of the solution. Schechter in [28] and [29], Lions and Magenes [18] considered the partial differential equation with more general boundary conditions in which they were interested in establishing criteria for the existence of the solution to the general boundary value problem. Necessary and sufficient conditions were established by placing certain restrictions on the system $(A(x,D),\{B_j\},\Omega)$ to ensure the existence and uniqueness of the solution to the elliptic partial differential equation

$$\begin{aligned} A(x,D)u &= f && \text{in } \Omega && (1\text{-}6)\\ B_j(x,D)u &= 0 && \text{on } \partial\Omega && (0\leq j\leq m-1) \end{aligned}$$

where $A(x,D)$ is an elliptic partial differential operator, $B_j(x,D)$ are m - linear partial differential operators, and Ω is a subset of the Euclidean n - space R^n, $n \geq 2$, with boundary, $\partial\Omega$. Certain restrictions are placed on $\{B_j(x,D)\}_{j=0}^{m-1}$, that of being a 'normal set' and satisfying the 'complementary condition', to assure that the boundary value problem is well-posed. It should be noted that these conditions include the Dirichlet conditions as a subclass. Also $A(x,D)$ is a 'properly elliptic' partial differential operator, and Ω is sufficiently smooth. With these conditions on $(A(x,D),\{B_j\},\Omega)$ the solution to (1-6) is found to exist and be unique. The functions, u, in (1-6) are in some prescribed function space satisfying certain differentiability conditions in Ω and near the boundary. Agmon, Douglis and Nirenberg [2], Agmon [1], Browder [6] and Friedman [11] studied the problem of differentiability of the solution to the general boundary-value problem (1-6) in Ω and near the boundary, $\partial\Omega$. The function space in which the solutions are found are complete Hilbert spaces and are characterized by the boundary conditions given. A discussion of this characterization is found in Grebb [12] and Friedman [11]. In solving the general boundary value problem (1-6), these authors did not consider the stability problem.

1.3. Area for Extension

In [23], utilizing the theory developed for operator differential equations, sufficient conditions were given to ensure the existence, uniqueness, asymptotic stability and stability for the solution of the nonlinear partial differential equation

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad (1\text{-}7)$$

with Dirichlet boundary conditions, where $f(u) \equiv 0$ gives us the linear equation. Since many physical and engineering problems are in the form of (1-7) with more general boundary conditions, for instance, the mixed problem, it is necessary to consider the stability problem for the partial differential equation (1-7) with general boundary conditions. Also since many physical problems are in the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad (1\text{-}8)$$

with general boundary conditions, such as the wave equation and the bending plate problem, this stability problem must also be considered. In order to study the stability problem one must ensure the existence and uniqueness of the solution as well as the stability of the solution. The work done in [23] on operator differential equations and the work in [18], [28], and [29] on the existence and uniqueness of solutions to more general boundary value problems, allows us to consider the stability problem for a large class of partial differential equations, i.e., (1-7) and (1-8), with general boundary conditions.

2.0. STATEMENT OF THE PROBLEM

Many physical or engineering problems can be placed in the following form

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x \varepsilon\ \Omega,\ t \geq 0 \qquad (2-1)$$

or in the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x \varepsilon\ \Omega,\ t \geq 0 \qquad (2-2)$$

where $u(x,t)$ is a function defined on $\Omega\ x[0,\infty)$, and where Ω is a bounded domain in the n - dimensional Euclidean space R^n, and $A(x,D)$ is a linear formal partial differential operator whose coefficients are infinitely differentiable functions defined on Ω and are time independent. $f(u)$ is, in general, a non-linear function defined on a function space, such that if $f(u) \equiv 0$ then (2-1) and (2-2) are linear and if $f(u) \neq 0$ we have a non-linear, or semi-linear, partial differential equation. To specify the solution of (2-1) or (2-2) a system of boundary conditions is given by

$$B_j(x',D)u(x',t) = 0 \qquad x' \varepsilon\ \partial\Omega,\ t \geq 0 \quad (0 \leq j \leq m-1) \quad (2-3)$$

where the $B_j(x,D)$ are m - linear partial differential operators defined on the boundary, $\partial\Omega$, and are independent of time. The coefficients are infinitely differentiable functions defined on $\partial\Omega$. Also, an initial condition is given by

$$u(x,0) = u_o(x) \qquad (2-4)$$

where $u_o(x)$ is a given space dependent function. Since $A(x,D)$ is a linear differential operator then (2-1) and (2-3) or (2-2) and (2-3) can be reduced to the following form

$$\frac{du(t)}{dt} + Au(t) = f(u) \tag{2-5}$$

where u(t) is a vector valued function defined on $[0,\infty)$ to a suitable Hilbert space H, and A, in general, is an unbounded linear operator whose domain and range are both contained in H, and f is a non-linear operator defined on H into H. In the case (2-2) and (2-3), the Hilbert space H is a product of Hilbert spaces, $H = H_1 \times H_2$, with u(t) a 2 - dimensional vector valued function, A a 2 x 2 matrix whose elements are linear partial differential operators and f(u) a 2 - dimensional vector in $H_1 \times H_2$. In all cases, the operator equation (2-5) can be considered as an abstract extension of the initial-boundary value problem (2-1) and (2-3), or (2-2) and (2-3), examples of which are the heat equation and the wave equation with mixed boundary conditions. The object of this investigation is to establish sufficient conditions on the system $(A(x,D),\{B_j\},\Omega)$ to ensure the existence, uniqueness and asymptotic stability or stability of the solution of (2-1), (2-3), and (2-4) or of (2-2), (2-3) and (2-4). This is done by defining the appropriate abstract operator in a base Hilbert space, H, and defining the correct domain, D(A), characterized by the boundary conditions (2-3) and forming the correct abstract operator equation. Utilizing the stability criteria in [23] which solve the stability problem for (2-5), the behavior of the corresponding partial differential equations can be deduced. Sections 2.1 and 2.2 introduce the types of partial differential equations to be studied and section 2.3 summarizes the results obtained in this investigation.

2.1. The Partial Differential Equation: $\frac{\partial u}{\partial t} + A(x,D)u = f(u)$

Pao in [23] considered the abstract operator equation of the form

$$\frac{du(t)}{dt} + Au(t) = f(u) \qquad (t \geq 0) \qquad (2\text{-}6)$$

where u(t) is a vector-valued function in a Hilbert space H, A is an unbounded linear operator mapping part of H into H, and f is a nonlinear function on H into H. Sufficient conditions were established to ensure the existence, uniqueness and stability of a solution of (2-6). These results were applied to the Dirichlet boundary value problem in which (2-6) is an abstract extension of the partial differential equation (2-1) with Dirichlet boundary conditions, where (2-6) is formed by defining the appropriate domain, D(A), characterized by the Dirichlet boundary conditions, and the appropriate base Hilbert space, H. The solving of the stability problem for (2-6) guarantees the existence and uniqueness of the solution of (2-1) which satisfies the Dirichlet boundary conditions. Furthermore, the stability of the solution is guaranteed. Since for boundary conditions more general than the Dirichlet problem the definition of the exact domain of the operator, the base Hilbert space and the abstract operator equation which satisfies the conditions found in [23] and which would relate the abstract operator equation to the general boundary value problem becomes much more difficult, more general boundary value problems have not been discussed.

To overcome this problem certain restrictions must be placed on the partial differential operator A(x,D), the boundary operators $B_j(x,D)$ and Ω. With these restrictions placed on the system $(A(x,D),\{B_j\},\Omega)$, as discussed in Schechter [28] and [29], the proper abstract operator, exact domain, base Hilbert space and abstract operator equation are defined so that the existence, uniqueness and stability is guaranteed for the newly formed operational equation. These results are related to the partial differential equation with general boundary conditions and from the proper definition of the domain of A and the properties of the solution of (2-6), the existence, uniqueness and stability of the general boundary value problem is ensured.

In this investigation the first case studied is the linear partial differential equation of the form

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = 0 \qquad x \in \Omega,\ t \geq 0 \qquad (2\text{-}7)$$

with general boundary conditions

$$B_j(x',D)u(x',t) = 0 \qquad x' \in \partial\Omega,\ t \geq 0 \quad (0 \leq j \leq m-1)$$

and initial condition

$$u(x,0) = u_o(x)$$

where $A(x,D)$ is a linear formal partial differential operator, $u(x,t)$ is a function defined on the subset Ω of the Euclidean n - space R^n, and $B_j(x,D)$ are m - linear partial differential operators defined on the boundary, $\partial\Omega$, and are independent of time. Since the general boundary conditions for the case $n \geq 2$ differ from the boundary conditions for the case $n = 1$, the two cases are studied separately. Be defining the appropriate abstract operator equation (2-6) with the correct base Hilbert space, abstract operator and domain, $D(A)$, and utilizing the results in [23] a stability theory is developed in which the existence, uniqueness as well as the stability of the solution to the general boundary value problem given in (2-7) is ensured. The cases $n \geq 2$ and $n = 1$ are solved in a similar manner.

The next step is to consider the nonlinear problem

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x \in \Omega,\ t \geq 0 \qquad (2\text{-}8)$$

where $A(x,D)$, the general boundary conditions, and initial function are defined as in the linear case, and $f(u)$ is a nonlinear function defined on the appropriate function space. By utilizing the results for the linear case the existence, uniqueness and stability of the solution to (2-8) is ensured by additional assumptions on $f(u)$. As in the linear problem, the cases $n \geq 2$ and $n = 1$ are treated separately. It is then shown that the Dirichlet problem studied by Buis [7] is just a special case of the theory developed for the general boundary value problem since the restrictions imposed on $B_j(x,D)$ include the Dirichlet conditions as a subclass. Some applications of the general boundary value problems of (2-7) and (2-8) are considered which shows how the stability theory can be applied to physical problems.

2.2. The Partial Differential Equation:

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + A(x,D)u = f(u)$$

The next problem studied is the linear partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = 0 \qquad x\epsilon\Omega,\ t\geq 0 \tag{2-9}$$

with constant $a \geq 0$, and with general boundary conditions

$$B_j(x',D)u(x',t) = 0 \qquad x'\epsilon\partial\Omega,\ t\geq 0 \quad (0\leq j\leq m-1)$$

and initial function

$$u(x,0) = u_o(x)$$

where $A(x,D)$ is a linear formal partial differential operator. Here Ω, the function u(x,t), the linear boundary operators $\{B_j(x,D)\}_{j=0}^{m-1}$ are defined as in the cases (2-7) and (2-8). By defining the appropriate base Hilbert space as a product space, $H_1 \times H_2$, and abstract operator A, a 2 x 2 matrix with linear partial differential operator elements, with the correct domain, D(A), an operator equation is formed and a stability criterion is deduced which ensures the existence, uniqueness and stability of the solution to (2-9).

The next case considered is the nonlinear general boundary value problem of the following form

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x\varepsilon\Omega,\ t\geq 0 \qquad (2\text{-}10)$$

with general boundary conditions. A(x,D), $B_j(x,D)$ and the initial value function are defined as in the linear case and f(u) is a nonlinear function defined on a function space. Stability results can be found by imposing additional conditions on f(u) and the existence, uniqueness and stability of the solution of (2-10) is guaranteed. The Dirichlet problem for an equation of the form (2-10) as worked out by Pao [24] is shown to be a special case of the problem (2-10) with general boundary conditions. Specific examples are worked out to show how the theory can be applied to various physical problems.

2.3. Summary of Results and Contributions to the Problem

The object of this study is to establish a stability theory so that the solution to a given partial differential equation with general boundary conditions not only exists and is unique but is also asymptotically stable or stable. The contribution of this dissertation is the establishment of criteria for the existence, uniqueness and stability of the solutions to a large class of partial differential equations with general boundary conditions and a given initial value function. This contribution can be stated in two separate stages in Chapters 6 and 7, and the results in these chapters are summarized as follows:

(i) In Chapter 6, the object is to find sufficient conditions on the system $(A(x,D),\{B_j\},\Omega)$ to ensure the existence, uniqueness, asymptotic stability and stability of the solution to the initial-general boundary value problem for the nonlinear partial differential equation

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x\varepsilon\Omega,\ t\geq 0$$

$$B_j(x',D)u(x',t) = 0 \qquad x'\varepsilon\partial\Omega,\ t\geq 0 \quad (0\leq j\leq m-1)$$

$$u(x,0) = u_o(x)$$

where $f(u) \equiv 0$ gives the linear equation. First, for the linear case, sufficient conditions are found to guarantee the existence and uniqueness of the solution as well as the stability of the solution. These results are given in theorems 6.2.1 and 6.3.1. Secondly, stability criteria are established for the nonlinear case which extend the results of the linear case by placing additional conditions on the nonlinear function f(u) and guarantees the existence, uniqueness and stability of the solution to the nonlinear problem. These results are given in theorems 6.41.2 and 6.42.1. In section 6.5, the main idea is to show that if the general boundary conditions are restricted to Dirichlet boundary conditions, then the existence, uniqueness and stability of the solution is ensured if A(x,D) satisfies an integral inequality. This shows that the Dirichlet problem is a special case of the theory developed in Chapter 6. These results are found in theorems 6.5.1 and 6.5.2. Finally, specific examples are considered which show that a large class of initial-boundary value problems fits into the theory developed in Chapter 6. These examples are seen to satisfy the conditions placed on $(A(x,D),\{B_j\},\Omega)$.

(ii) In Chapter 7, the main object is to find sufficient conditions on the system $(A(x,D),\{B_j\},\Omega)$ to ensure the existence, uniqueness, asymptotic stability and stability of the solution to the following initial-boundary value problem,

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x\varepsilon\Omega,\ t\geq 0$$

$$B_j(x',D)u(x',t) = 0 \qquad x'\varepsilon\partial\Omega,\ t\geq 0 \qquad (0\leq j\leq m-1)$$

$$u(x,0) = u_o(x)$$

where $f(u) \equiv 0$ gives the linear case. First the linear problem is considered and by forming the correct base Hilbert space, a product space, and an abstract operator A, a 2 x 2 matrix with linear operator elements and an abstract operator equation, sufficient conditions are found on the system $(A(x,D),\{B_j\},\Omega)$ to ensure the existence, uniqueness and stability of the solution to the linear equation. This result is found in theorem 7.1.1. Secondly, the nonlinear problem is considered and stability criteria are formed by placing additional assumptions on the nonlinear function f(u). This result is in theorem 7.2.1. In section 7.31, a Dirichlet problem for a partial differential equation is considered and shown to be a special case of the theory worked out in Chapter 7. Examples are considered to show that the theory developed in Chapter 7 can be applied to a large class of physical problems. These are found in examples 7.32.1 and 7.32.2.

3.0. FUNCTIONAL ANALYSIS

Fundamental to the study of the stability theory for Partial Differential Equations is a knowledge of Functional Analysis. In this chapter we will give a brief discussion on the basic definitions and properties needed in the remainder of this work. There are many references which will give a more complete discussion of the subject, among these are in [8], [9], [10] and [35].

3.1. Normed Spaces

A set X is a linear space over a field K if for any two elements $x,y \in X$, the sum $x + y$ is defined as an element of X, and similarly for any $\lambda \in K$, the scalar product λx is defined and is an element of X. The operations satisfy the following conditions: for any $x,y \in X$, and any $\lambda,\mu \in K$

(i) $(x + y) + z = x + (y + z)$;

(ii) $x + y = y + x$;

(iii) there exists an element $\underline{0}$ in X such that for any $x \in X$, $0 \cdot x = \underline{0}$;

(iv) $(\lambda + \mu)\, x = \lambda x + \mu x$;

(v) $\lambda\,(x + y) = \lambda x + \lambda y$;

(vi) $(\lambda\mu)\, x = \lambda(\mu x)$;

(vii) $1 \cdot x = x$.

Let X be a linear space over the field of real or complex numbers. The elements $x_1, x_2, \ldots, x_n$ of X are said to be linearly independent when the following relation is satisfied

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$$

if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Otherwise, the elements $x_1, x_2, \ldots, x_n$ are linearly dependent.

A linear space X is called a normed linear space if every element εX has associated with it a real number, denoted $||\ x\ ||$, called the norm of x, satisfying the following conditions: for any $x, y \varepsilon X$, and any $\lambda \varepsilon K$,

(i) $||\ x\ || \geq 0$, $||\ x\ || = 0$ if and only if $x = 0$;

(ii) $||\lambda\ x\ || = |\lambda|\ ||\ x\ ||$;

(iii) $||\ x + y\ || \leq ||\ x\ || + ||y||$.

The normed linear space is then denoted by $(X, ||\cdot||)$ or simply by X. A sequence $\{x_n\}$ in a normed linear space X is said to be a Cauchy sequence if for any $\varepsilon > 0$, there exists an integer, $N(\varepsilon)$, such that for any $n, m \geq N(\varepsilon)$, $||x_m - x_n|| < \varepsilon$. If every Cauchy sequence in X converges to an element $x \varepsilon X$, then X is called a complete normed linear space or a Banach space. This convergence is strong convergence and will be denoted by $\lim_{n\to\infty} x_n = x$, or more simply by

$$x_n \underset{X}{\to} x \quad \text{as } n \to \infty .$$

X is a real or complex Banach space depending on whether K is a real or complex field. A set $\{x\}$ in a normed linear space X is said to be bounded if there exists an $M \geq 0$ such that for any element of the set, we have $||\ x\ || \leq M$.

A complex linear space X is called a complex inner product space (or a pre-Hilbert space) if there is defined on X x X a complex valued function, denoted by (x,y), called the inner product of x and y, satisfying the following properties: for any $x, y, z \ \varepsilon\ X$ and any $\lambda, \mu \varepsilon K$

(i) $(\lambda\ x + \mu y, z) = \lambda(x,z) + \mu(y,z)$;

(ii) $(x,y) = \overline{(y,x)}$; (the bar denotes the complex conjugate)

(iii) $(x,x) \geq 0$ and $(x,x) = 0$ if and only if $x = 0$.

A real linear space is called a real inner product space if the properties (i),(ii)',(iii) are satisfied, where (ii)' replaces (ii) above

(ii)' $(x,y) = (y,x)$.

An inner product space becomes a normed linear space by defining $||x|| = (x,x)^{1/2}$ and the norm is said to be induced by the inner product $(.,.)$. Every normed linear space is not an inner product space. However, in a normed linear space X (complex or real), if the norm $||.||$ satisfies the parallelogram law: for any $x,y \in X$

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

than an inner product can be defined so that X is an inner product space. If an inner product space H (complex or real) is complete with respect to the norm induced by the inner product $(.,.)$, it is called a Hilbert space and denoted by $(H,(.,.))$ or more simply by H. H is a real or complex Hilbert space according to whether K is real or complex. A Hilbert space satisfies the following two important properties;

(i) The inner product is sesquilinear if H is a complex Hilbert space and is bilinear if H is a real Hilbert space. By sesquilinearity we mean: for any $x,y,z \in H$, and any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in K$

$$(\alpha_1 x + \alpha_2 y, z) = \alpha_1(x,z) + \alpha_2(y,z)$$

$$(x, \beta_1 y + \beta_2 z) = \overline{\beta}_1(x,y) + \overline{\beta}_2(x,z),$$

If $\overline{\beta}_1$ and $\overline{\beta}_2$ are replaced by β_1 and β_2 respectively then the inner product is said to be bilinear.

(ii) The inner product is continuous, in the sense that if

$$x_n \underset{H}{\rightarrow} x, \text{ and } y_n \underset{H}{\rightarrow} y \qquad \text{as } n \rightarrow \infty$$

then we have

$$(x_n, y_n) \underset{K}{\rightarrow} (x,y) \qquad \text{as } n \rightarrow \infty .$$

Let us consider the example $L^p(\Omega)$, $(1 \leq p < \infty)$. The set of all real-valued (or complex-valued) measurable functions $f(x)$ defined a.e. (almost everywhere) on Ω, where Ω is an open subset of R^n, such that $|f(x)|^p$ is Lebesque integrable over Ω constitutes a normed linear space $L^p(\Omega)$. It is a linear space from the definition of sum and scalar product: Let $f, g \varepsilon L^p(\Omega)$, $\lambda \varepsilon K$,

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x)$$

and the norm is defined by

$$|| x ||_{L^p(\Omega)} = [\int_\Omega |f(x)|^p dx]^{1/p} \qquad (dx = dx_1 \ldots dx_n).$$

$L^p(\Omega)$ is a Banach space whose elements are equivalence classes of p^{th} power integrable functions, where two functions f and g are said to be equivalent if $f(x) = g(x)$ a.e. on Ω. In particular, if $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product and norm denoted by

$$(f,g)_o = \int_\Omega f(x) \overline{g(x)} dx \qquad ; \qquad || f ||_o = (f,f)_o^{1/2}$$

<u>Definition 3.1.1.</u> Let $X_1 = (X, ||\cdot||_1)$ and $X_2 = (X, ||\cdot||_2)$, where X is a linear space. The two norms are said to be equivalent if there exists real numbers α, β with $0 < \alpha \leq \beta < \infty$ such that for any $x \varepsilon X$

$$\alpha || x ||_2 \leq || x ||_1 \leq \beta || x ||_2 .$$

<u>Definition 3.1.2.</u> If $H_1 = (H,(.,.)_1)$ and $H_2 = (H,(.,.)_2)$ are two inner product spaces then the inner products are equivalent if the norms they induce are <u>equivalent</u> norms.

Let X be a normed linear space. An element $x \in X$ is said to be a limit point of a set $D \subset X$ if there exists a sequence of distinct elements $\{x_n\} \subset D$ such that

$$x_n \underset{X}{\to} x \quad \text{as } n \to \infty.$$

The closure of a set D in X, denoted by $\overline{D}$, is the set comprising D and all of the limit points of D in X. A set D is said to be closed if $D = \overline{D}$ and is dense in X if $\overline{D} = X$. If X is complete then the closure of a set D in X is a complete normed linear space, and we will say $\overline{D}$ is the completion of D with respect to the norm on X.

3.2. Linear and Nonlinear Operators

Let X and Y be linear spaces over the same field of scalars K. Let A be an operator (or function) which maps part of X into Y. The domain of A, denoted by D(A), is the set of all $x \in X$ such that there exists a $y \in Y$ where $Ax = y$. The range of A, denoted by R(A), is the set $\{Ax | x \in D(A)\}$. The null space (or Kernel) of A, $Ker(A) = \{x \in X | Ax = 0\}$. A_2 is called an extension of A_1 if $D(A_1) \subset D(A_2)$ and $A_1x = A_2x$, for all $x \in D(A_1)$. If $D(A_1) = D(A_2)$ and $A_1x = A_2x$, for all $x \in D(A_1)$, then $A_1 = A_2$. If the operator A is one-to-one, then A is said to have an inverse and is denoted by A^{-1} and defined by $D(A^{-1}) = R(A)$, and

$$A^{-1}(y) = x \qquad \text{where } y \in D(A^{-1}) \text{ and } Ax = y.$$

An operator A with domain, $D(A)$, a linear subspace of X, and range, $R(A)$, in Y is called linear if for any $x,y \varepsilon D(A)$ and any $\alpha,\beta \varepsilon K$

$$A(\alpha x + \beta y) = \alpha\, A(x) + \beta\, A(y)$$

and is called nonlinear if it is not linear.

If X and Y are normed linear spaces and T is a linear operator with $D(T) \subset X$ and $R(T) \subset Y$, the following statements are equivalent:

(i) T is continuous on $D(T)$;

(ii) T is bounded on $D(T)$, i.e., there exists a real number $M > 0$, such that for any $x \,\varepsilon\, D(T)$

$$||T\, x\, ||_Y \leq M\, ||\, x\, ||_X.$$

If T is bounded, the norm of T is defined by

$$||T|| = \sup\{||T\, x\, ||_Y \mid ||\, x\, ||_X \leq 1,\ x \varepsilon D(T)\}.$$

With this norm, the space of all bounded linear operators with domain X and range in Y, denoted by $L(X,Y)$, is a normed linear space if we define addition of operators and scalar multiplication in the usual way. If in addition Y is a Banach space so is $L(X,Y)$. T^{-1} exists and is continuous if and only if there exists an $m > 0$, such that for any $x \varepsilon D(T)$ $||T\, x\, ||_Y \geq m||\, x\, ||_X$.

Let X, Y be normed linear spaces on the same scalar field. Then the product space X x Y is a normed linear space defined as the set of all ordered pairs $\{x,y\}$, such that $x \varepsilon X$, $y \varepsilon Y$ with addition and scalar multiplication defined by

$$\{x_1,y_1\} + \{x_2,y_2\} = \{x_1 + x_2, y_1 + y_2\},\ \alpha\, \{x,y\} = \{\alpha x, \alpha y\}$$

and with the norm given by

$$|| \{x,y\} || = (|| x ||_X^2 + || y ||_Y^2)^{1/2}.$$

If X and Y are Banach spaces then so is X x Y. If T is a linear operator with $D(T) \subset X$ and $R(T) \subset Y$, the graph of T, G(T), is the set of all ordered pairs $\{x,Tx\}$, such that $x \in D(T)$. Since T is linear, G(T) is a linear subspace of X x Y. A linear operator T is said to be closed in X if the graph, G(T), of T is closed in X x Y. An equivalent definition is the following: A linear operator T is closed if and only if $x_n \in D(T)$, $x_n \underset{X}{\to} x$, $Tx_n \underset{Y}{\to} y$ (as $n \to \infty$) imply $x \in D(T)$ and $Tx = y$. If T is closed then the inverse, T^{-1}, if it exists, is closed. A bounded linear operator need not be closed and a closed operator need not be bounded. However, we have the following well known theorem, the Banach Closed Graph Theorem.

<u>Theorem 3.2.1.</u> A closed linear operator T defined on a Banach space X into a Banach space Y is continuous.

A linear operator T is said to be closable if there exists a linear extension of T which is closed in X. If T is closable, there is a closed operator $\overline{T}$ with $G(\overline{T}) = \overline{G(T)}$. $\overline{T}$ is called the closure of T and is the smallest closed extension of T, in the sense that any closed extension of T is also an extension of $\overline{T}$. A linear operator T is closable if and only if $x_n \in D(T)$, $x_n \underset{X}{\to} 0$, and $Tx_n \underset{Y}{\to} y$ (as $n \to \infty$) imply that $y = 0$. In this case, the closure $\overline{T}$ of T can be defined as follows: $x \in D(\overline{T})$ if and only if there exists a sequence $\{x_n\} \subset D(T)$ such that $x_n \underset{X}{\to} x$ and $\lim_{n\to\infty} Tx_n = y$ exists; and we define $\overline{T}x = y$. It can be proved that y is unique and $\overline{T}$ is closed.

<u>Definition 3.2.1.</u> Let $H = (H,(.,.))$ be a Hilbert space and T be an operator with dense domain in H and range $R(T) \subset H$. The adjoint operator of

S, denoted by S^*, is defined as follows:

$$D(S^*) = \{y \varepsilon H \mid \text{there exists a } y^* \varepsilon H, \text{ such that, for any } x \varepsilon D(S),$$
$$(Sx,y) = (x,y^*)\}$$

$$S^*(y) = y^* \qquad y \varepsilon D(S^*).$$

S^* exists if and only if D(S) is dense in H, and S^* is a closed linear operator. S is symmetric if $S \subset S^*$, that is, S^* is an extension of S, and is self-adjoint if $S = S^*$. Hence, a self-adjoint operator is closed.

Definition 3.2.2. Let X and Y be normed linear spaces. Suppose T is a linear operator with domain in X and range in Y. T is completely continuous (or compact) if for every bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ contains a subsequence converging to some limit in Y.

Definition 3.2.3. Let $H_1 = (H,(.,.)_1)$ be a Hilbert space, and T a linear operator with domain and range in H_1. T is said to be dissipative with respect to the inner product of H_1 if for every $x \varepsilon D(T)$

$$\text{Re}(Tx,x)_1 \leq 0.$$

T is said to be strictly dissipative with respect to the inner product on H if there exists a $\beta > 0$, such that for every $x \varepsilon D(T)$

$$\text{Re}(Tx,x)_1 \leq -\beta || x ||_1^2 .$$

(The supremum of all β satisfying the inequality is called the dissipativity constant.)

Let X,Y be normed linear spaces on the same scalar field of real or complex numbers and let L(X,Y) be the class of all bounded linear operators on X to Y. If Y is the real or complex number field topologized in the usual way, $L(X,Y)$ is called the conjugate space (or dual space) of X and is denoted by X^*. An element of X^* is called a functional. Thus X^* is the set of all continuous linear functionals on X. The pairing between any elements x of X and f of X^* is denoted by $f(x)$ or by $\langle f,x\rangle$. If we define the norm of $f \varepsilon X^*$ by

$$||f||_{X^*} = \sup|f(x)| \qquad \text{for } ||\ x\ ||_X \leq 1$$

then X^* is a Banach space.

If X is a Hilbert space, X^* can be identified with X as can be seen from the Riesz representation theorem (the identification is with H as an abstract set).

Theorem 3.2.2. For any linear functional f on a Hilbert space $H = (H,(.,.))$, there exists an element $y_f \ \varepsilon\ H$, uniquely determined by the functional f, such that

$$f(x) = (x,y_f) \qquad \text{for every } x \ \varepsilon\ H.$$

Moreover, $||f|| = ||y_f||$.

Corollary 3.2.1. Let H be a Hilbert space. Then the totality of all bounded linear functionals H^* on H constitutes also a Hilbert space, and there is a norm preserving, one-to-one correspondence $f \leftrightarrow y_f$ between H^* and H.

We have introduced the concept of equivalent inner product which is useful in the development of the stability theory in Chapter 4. The following theorem which was formulated by P. Lax and A. N. Milgram plays an important role in the construction of an equivalent inner product. A proof can be found in [35].

Theorem 3.2.3. (Lax-Milgram). Let H be a Hilbert space. Let $V(x,y)$

be a complex-valued functional defined on the product space H x H which satisfies the conditions:

(i) Sesquilinearity, i.e.,

$$V(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 V(x_1, y) + \alpha_2 V(x_2, y) \quad \text{and}$$

$$V(x, \beta_1 y_1 + \beta_2 y_2) = \overline{\beta}_1 \, V(x, y_1) + \overline{\beta}_2 \, V(x, y_2);$$

(ii) Boundedness, i.e., there exists a positive constant γ such that

$$|V(x,y)| \leq \gamma \, || \, x \, ||_H \cdot ||y||_H \, ;$$

(iii) Positivity, i.e., there exists a positive constant δ such that

$$V(x,x) \geq \delta \, || \, x \, ||_H^2 \, .$$

Then there exists a uniquely determined bounded linear operator S with a bounded linear inverse S^{-1} such that

$$V(x,y) = (x, Sy)_H \qquad \text{whenever } x,y \in H$$

and $||S|| \leq \gamma$, $||S^{-1}|| \leq \delta^{-1}$.

Definition 3.2.4. A sequence $\{x_n\}$ in a normed linear space X is said to converge weakly to an element $x \in X$ if $\lim_{n\to\infty} f(x_n) = f(x)$, for every $f \in X^*$. This is denoted by $w\text{-}\lim_{n\to\infty} x_n = x$. It can be shown in this case that x is uniquely determined. It should be noted that if x_n converges strongly to x then x_n converges weakly to x. However, the converse is not true. We know that if H is a Hilbert space, and the sequence $\{x_n\}$ of H converges weakly to $x \in H$ and $\lim_{n\to\infty} ||x_n||_H = || \, x \, ||_H$, then $\{x_n\}$ converges strongly to x.

We will now define vector valued functions, weak continuity and weak differentiability.

Definition 3.2.5. Let $u(t)$ be a vector valued function on $[0,\infty)$ to X. $u(t)$ is said to be weakly continuous in t if $\langle f,u(t)\rangle$ is continuous for each $f\epsilon X^*$. $u(t)$ is said to be weakly differentiable in t if $\langle f,u(t)\rangle$ is differentiable for each $f\epsilon X^*$. If the derivative of $\langle f,u(t)\rangle$ has the form $\langle f,v(t)\rangle$, for each $f\epsilon X^*$, $v(t)$ is the weak derivative of $u(t)$ and we write $\frac{du(t)}{dt} = v(t)$ weakly.

3.3. Spectral Theory and Semi-Groups

Let T be a linear operator with domain $D(T)$ and range $R(T)$ both contained in a normed linear space X. The set of complex numbers λ for which the linear operator $(\lambda I-T)$ has an inverse and the properties of this inverse, if it exists, are called the spectral theory for the operator T.

Definition 3.3.1. The complex number λ_o is in the resolvent set, $\rho(T)$, of T if $R(\lambda_o I-T)$ is dense in X and $\lambda_o I-T$ has a continuous inverse, $(\lambda_o I-T)^{-1}$. The inverse $(\lambda_o I-T)^{-1}$ is denoted by $R(\lambda_o;T)$ and is called the resolvent of T at λ_o. The spectrum of $T,\sigma(T)$, is the set of all complex numbers λ not in $\rho(T)$.

Theorem 3.3.1. Let X be a Banach space and T a closed linear operator with $D(T)$ and $R(T)$ both in X. Then for any $\lambda\epsilon\rho(T)$; the resolvent $R(\lambda;T)$ is an everywhere defined continuous linear operator. The resolvent set, $\rho(T)$, of T is an open set of the complex plane.

This theorem tells us that for any $\lambda\epsilon\rho(T)$, $R(\lambda I-T) = D(R(\lambda;T)) = X$, and the spectrum, $\sigma(T)$, of T is a closed set of the complex plane. For a more detailed discussion of spectral theory see [14] and [35].

In the study of stability theory for operational differential equations found in Pao [23], much use was made of the Semi-group theory of Yosida and Hille-Phillips in [35] and [14], respectively. The basic definitions and properties will be defined here. Look in the above books for a more detailed duscussion.

Definition 3.3.2. For each $t \in [0,\infty)$, let $T_t \in L(X,X)$. The family $\{T_t | t \geq 0\} \subset L(X,X)$ is called a strongly continuous semi-group of class C_o (or a semi-group of class C_o) if the following conditions hold:

(i) $T_s T_t = T_{s+t}$ for $s,t \geq 0$;

(ii) $T_o = I$ (I is the identity operator);

(iii) $\lim_{t \to t_o} T_t x = T_{t_o} x$ for any $t_o \geq 0$ and any $x \in X$.

If $\{T_t | t \geq 0\}$ is a semi-group, its norm satisfies: there exists an $M \geq 1$ and a $\beta < \infty$, such that for any $t \geq 0$

$$||T_t|| \leq Me^{\beta t}.$$

If β can be taken to be zero, then $\{T_t | t \geq 0\}$ is said to be an equibounded semi-group of class C_o; if in addition $M = 1$, it is called a contraction semi-group of class C_o. If β can be taken as $\beta < 0$, $\{T_t | t \geq 0\}$ is a negative semi-group of class C_o and if, in addition, $M = 1$, it is called a negative contraction semi-group of class C_o.

Definition 3.3.3. The infinitesimal generator, A, of the semi-group $\{T_t | t \geq 0\}$ is defined by

$$Ax = \lim_{h \downarrow o} \frac{T_h x - x}{h}$$

for all $x \in X$ such that the limit exists.

The infinitesimal generator, A, of a semi-group of class C_o has the following properties;

(i) A is a closed linear operator with domain, D(A), dense in X and the zero vector $\underline{0} \varepsilon D(A)$.

(ii) If $x \varepsilon D(A)$, then for any $t \geq 0$, $T_t x \varepsilon D(A)$ and

$$\frac{d}{dt}(T_t x) = AT_t x = T_t Ax$$

where $\frac{d}{dt}(T_t x)$ is defined

$$\frac{d}{dt}(T_t x) = \lim_{h \downarrow 0} \left[\frac{T_{t+h}x - T_t X}{h} \right]$$

for $x \varepsilon X$, if the limit exists.

(iii) If $||T_t|| \leq Me^{\beta t}$, then all λ with $Re(\lambda) > \beta$ is in the resolvent set, $\rho(A)$, of A.

The following result known as the Hille-Yosida theorem gives necessary and sufficient conditions for a closed linear operator to be the infinitesimal generator of a semi-group.

<u>Theorem 3.3.2.</u> Let A be a closed linear operator with domain, D(A), dense in X and range, R(A), in X. Then A is the infinitesimal generator of a semi-group $\{T_t | t \geq 0\}$ satisfying the condition

$$||T_t|| \leq Me^{\beta t} \qquad \text{with } M \geq 1 \text{ and } \beta < \infty$$

if and only if there exists real numbers M and β as above such that for every integer $n > \beta$, $n \varepsilon \rho(A)$ and

$$||R(n;A)^m|| \equiv ||(nI-A)^{-m}|| \leq M \frac{1}{(n-\beta)} m \quad (m = 1,2,\ldots).$$

We have already introduced the concept of dissipative operator which gives a more aesthetic and useful result in the study of the stability theory of the abstract operator equation

$$\frac{du(t)}{dt} + Au(t) = 0.$$

The result is due to Phillips [27]

Theorem 3.3.3. Let A be a linear operator with domain, D(A), dense in H and range, R(A), in H. Then -A is the infinitesimal generator of a contraction semi-group of class C_o in H if and only if -A is dissipative with respect to the inner product on H and R(I+A) = H; and -A is the infinitesimal generator of a negative contraction semi-group of class C_o in H if and only if -A is strictly dissipative with respect to the inner product on H and R((1-β)I+A) = H, where β is the constant in definition 3.2.3.

3.4. Distributions and Function Spaces

In the study of stability theory for Partial Differential Equations, we need to examine the function spaces which define the domain of an infinitesimal generator, A, of a contraction semi-group satisfying the operator equation

$$\frac{du(t)}{dt} + Au(t) = 0.$$

To do this, we must first introduce the concept of Distributions and Fourier Transformations. Next, we will discuss the Sobolev spaces $H^m(\Omega)$, and the boundary spaces $H^s(\partial\Omega)$. This will lead to the 'trace theorems' or what we mean by the restrictions of functions on the boundary, $\partial\Omega$. Finally, we will discuss the function space $H_B^{2m}(\Omega)$ which is needed for the definition of the domain of the operator A.

3.41. Distributions

In this section we will define and give some basic properties of distributions. For a more complete discussion see Horvath [15], Treves [32] and Edwards [10]. First, we will use the following notations: $x = (x_1, x_2, \ldots, x_n) \epsilon R^n$, the Euclidean n-space, Ω is an open subset of R^n. $\overline{\Omega}$ denotes the closure of Ω in R^n, and $dx = dx_1 \ldots dx_n$, the usual Lebesque measure.

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \epsilon R^n$ with non-negative integer components and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

<u>Definition 3.41.1.</u> A real-valued function, $q(x)$, defined on a linear space X is called a semi-norm on X, if the following conditions are satisfied:

(i) $q(x + y) \leq q(x) + q(y)$;

(ii) $q(\alpha x) = |\alpha| \; q(x)$.

We can see directly that $q(x) \geq 0$ and $q(0) = 0$.

Let $f(x)$ be a complex-valued (or real-valued) function defined on Ω. The support of f, denoted by supp(f), is the smallest closed set containing the set $\{x\epsilon\Omega \mid f(x) \neq 0\}$ (or equivalently, the smallest closed subset of Ω outside of which f vanishes identically).

<u>Definition 3.41.2.</u> By $C^m(\Omega)$, $0 \leq m \leq \infty$, we mean the set of all complex-valued (or real-valued) functions defined in Ω which have continuous partial derivatives of order up to and including m (of order $< \infty$ if $m = \infty$). $C_o^m(\Omega)$ is the set of all functions of $C^m(\Omega)$ with compact support, that is

those functions of $C^m(\Omega)$ whose supports are compact subsets of Ω (a subset of R^n is compact if and only if it is closed and bounded). If $0 \leq m \leq \infty$, the set $C^m(\Omega)$ or $C_o^m(\Omega)$ is a linear space defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (\alpha f)(x) = \alpha f(x).$$

$C^k(\bar{\Omega})$, $0 \leq k \leq \infty$, is the set of all functions (complex or real-valued), such that for any $x\varepsilon\Omega$, and for any $\alpha, |\alpha| \leq k(|\alpha| < k$ if $k = \infty)$ $D^\alpha f(x)$ exists and $D^\alpha f$ has a continuous extension to $\bar{\Omega}$.

We will now define a topology on $C_o^\infty(\Omega)$. Using the notation of Treves, we let K be any compact subset of Ω. Then $C_o^\infty(K)$ is defined by

$$C_o^\infty(K) = \{\phi\varepsilon C_o^\infty(\Omega) \,|\, \mathrm{supp}\phi \subset K\}.$$

We define on $C_o^\infty(K)$ a family of semi-norms

$$q_p(\phi) = \sup_{|\alpha| \leq p, x\varepsilon K} |D^\alpha\phi(x)|.$$

This makes $C_o^\infty(K)$ a Frechet space (metrizable and complete). Then, if we let K_n be an increasing sequence of compact sets such that $K_n \subset \Omega$, and $\bigcup_n K_n = \Omega$, this defines the inductive limit topology on $C_o^\infty(\Omega)$, where a set O is open in this topology if and only if $O \cap C_o^\infty(K)$ is an open set, for all K, $1 \leq K < \infty$. Topologized in this way, $C_o^\infty(\Omega)$ is a locally convex linear topological space. The convergence $\lim_{n\to\infty} \phi_n = \phi$ in $C_o^\infty(\Omega)$ means that the following two conditions are satisfied:

(i) There exists a compact subset $K \subset \Omega$ such that $\mathrm{supp}(\phi_n) \subset K$ $(n = 1,2,\ldots)$.

(ii) For any differential operator D^α, the sequence $D^\alpha\phi_n(x)$ converges to $D^\alpha\phi(x)$ uniformly on K.

Definition 3.41.3. A linear functional defined and continuous on

$C_o^\infty(\Omega)$ is called a distribution or a generalized function in Ω. We denote by $D'(\Omega)$ the set of all distributions in Ω. $D'(\Omega)$ is the dual space (or conjugate space) of $C_o^\infty(\Omega)$, where $C_o^\infty(\Omega)$ is called the space of testing functions. For any distribution $f \varepsilon D'(\Omega)$, and any testing function $\phi \varepsilon C_o^\infty(\Omega)$, we denote by $\langle f,\phi\rangle$ the value of f on ϕ. $D'(\Omega)$ is a linear space by

$$\langle f + g,\phi\rangle = \langle f,\phi\rangle + \langle g,\phi\rangle, \text{ and } \langle \alpha f,\phi\rangle = \alpha\langle f,\phi\rangle.$$

We have two theorems concerning the criteria for a linear functional to be a distribution.

Theorem 3.41.1. A linear functional f defined on $C_o^\infty(\Omega)$ is a distribution in Ω if and only if f is bounded on every bounded set of $C_o^\infty(\Omega)$ (in the inductive limit topology of $C_o^\infty(\Omega)$).

Theorem 3.41.2. A linear functional f defined on $C_o^\infty(\Omega)$ is a distribution in Ω if and only if f satisfies the condition: to every compact subset K of Ω, there corresponds a positive constant C and a positive integer m such that for any $\phi \varepsilon C_o^\infty(\Omega)$,

$$|\langle f,\phi\rangle| \leq C \sup_{|\alpha| \leq m, x\varepsilon K} |D^\alpha \phi(x)|.$$

Definition 3.41.4. The derivative of a distribution f is defined to be the element of $D'(\Omega)$, denoted by $\frac{\partial f}{\partial x_i}$, satisfying for any $\phi \varepsilon C_o^\infty(\Omega)$

$$\langle \frac{\partial f}{\partial x_i}, \phi\rangle = - \langle f, \frac{\partial \phi}{\partial x_i} \rangle.$$

Thus, a distribution in Ω is infinitely differentiable and $D^\alpha f$ is the element in $D'(\Omega)$ defined by: for any $\phi \varepsilon C_o^\infty(\Omega)$

$$\langle D^\alpha f,\phi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \phi \rangle.$$

We have the following properties of distributions:

(i) $C_o^\infty(\Omega) \subset L^2(\Omega) \subset D'(\Omega)$; each space is dense in the following where for any $u \in C_o^\infty(\Omega)$, u is associated with the distribution $(\phi \rightarrow (u,\phi)_o)$, where $(.,.)_o$ is the L^2- inner product.

(ii) For any $\phi \in C_o^\infty(\Omega)$ and $f \in D'(\Omega)$, we define the product ϕf by: for any $u \in C_o^\infty(\Omega)$

$$< \phi f, u > = < f, \phi u >.$$

3.42. Fourier Transform Space of Tempered Distributions

In this section, we will define the Fourier Transform on the space $\mathcal{S}(R^n)$ and extend this to the space $L^2(R^n)$. Then we will define the Fourier Transform on the space of tempered distributions. We need this for the definition of the spaces $H^s(R^n)$ and $H^s(\partial\Omega)$. A more detailed discussion is found in Yosida [35] and Treves [32].

Definition 3.42.1. Let $\mathcal{S}(R^n)$ be the set of all functions $\phi \in C^\infty(R^n)$ such that for any $\alpha, \beta \in R^n$ with non-negative integer components

$$\sup_{x \in R^n} |x^\beta D^\alpha \phi(x)| < \infty$$

where $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$. The topology on $\mathcal{S}(R^n)$ is defined by the family of semi-norms $q_{\alpha\beta}(\phi) = \sup_{x \in R^n} |x^\beta D^\alpha \phi(x)|$. With this topology $\mathcal{S}(R^n)$ is a locally convex linear topological space whose elements are functions said to be rapidly decreasing at ∞.

Definition 3.42.2. The Fourier Transform of $u \in \mathcal{S}(R^n)$ is the function of $\xi \varepsilon R^n$,

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{-i(x,\xi)} u(x)dx$$

Where $(x,\xi) = x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n$. We denote it by $\hat{u}(\xi)$. The Fourier Transform, FT, is an isomorphism of $\mathcal{S}(R^n)$ onto $\mathcal{S}(R^n)$ with the given topology on $\mathcal{S}(R^n)$, and the inverse mapping is given by

$$\overline{FT}: \mathcal{S}(R^n) \ni g \to \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{i(x,\xi)} g(\xi)d\xi$$

An important property of the Fourier Transform is given by the following Plancheral-Parseval theorem, which shows that FT preserves the L^2 - inner product and norm on $\mathcal{S}(R^n)$.

Theorem 3.42.1. (Plancheral-Parseval) Let $\phi, \psi \ \varepsilon \mathcal{S}(R^n)$. Then

(i) $\int_{R^n} \phi(x) \overline{\psi(x)} \, dx = \int_{R^n} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} \, d\xi$;

(ii) $\int_{R^n} |\phi(x)|^2 \, dx = \int_{R^n} |\hat{\phi}(\xi)|^2 d\xi$.

Since $\mathcal{S}(R^n)$ is dense in $L^2(R^n)$, the Fourier Transform can be extended by continuity to an isometry, denoted by FT, from $L^2(R^n)$ onto $L^2(R^n)$. We denote by $\overline{FT}$ the inverse Fourier Transform. We can see this is an isometry by theorem 3.42.1. This gives us the following result,

Theorem 3.42.2. Let $u,v \ \varepsilon \ L^2(R^n)$. Then we have Parseval's formula, $(u,v)_o = (\hat{u},\hat{v})_o$

and Plancheral's formula, $||u||_o = ||\hat{u}||_o$.

consider the following diagram

$$\begin{array}{ccccc} C_o^\infty(R^n) & \hookrightarrow & \mathcal{S}(R^n) & \hookrightarrow & C^\infty(R^n) \\ \downarrow & & \downarrow & & \downarrow \\ D'(R^n) & \hookleftarrow & \mathcal{S}'(R^n) & \hookleftarrow & [C^\infty(R^n)]' \end{array}$$

These natural injections are all continuous and each space is dense in the following space. Hence, we can regard $\mathcal{S}'(R^n)$, the dual space of $\mathcal{S}(R^n)$, as a space of distributions. We say that $\mathcal{S}'(R^n)$ is the space of tempered distributions. A characterization of a tempered distribution is found in the following theorem.

Theorem 3.42.3. A distribution in R^n is a tempered distribution if and only if it is a finite sum of derivatives of continuous functions, growing at ∞ slower than some polynomial.

Definition 3.42.3. The Fourier Transform on $\mathcal{S}'(R^n)$, is the transpose of the continuous linear map, FT, which maps $u \varepsilon \mathcal{S}(R^n)$ into the function of $\xi \varepsilon R^n$

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{-i(x,\xi)} u(x)\ dx.$$

The transpose of FT, ${}^t FT$, is the element of $\mathcal{S}'(R^n)$ defined by the following: for any $u \varepsilon \mathcal{S}'(R^n)$ and any $\phi \varepsilon \mathcal{S}(R^n)$,

$$< {}^t FT(u), \phi > = <u, FT(\phi) >.$$

This defines the Fourier Transform on $\mathcal{S}'(R^n)$, which we will denote by FT, which extends the Fourier Transform in the space of functions $L^2(R^n)$. We have the following theorem by Treves [32].

Theorem 3.42.4. The Fourier Transform is an isomorphism from the linear topological space $\mathcal{S}'(R^n)$ onto $\mathcal{S}'(R^n)$.

3.43. The Sobolev Spaces, $H^m(\Omega)$, m integer ≥ 0

We will first consider Ω and place the following restrictions on Ω (which will hold except when otherwise stated, i.e., $\Omega = R^n$).

Ω is a bounded domain in R^n. The boundary, denoted by $\partial\Omega$, is an infinitely differentiable manifold of dimension (n-1), (3-1)
Ω being locally on one side of $\partial\Omega$, i.e., we consider $\bar{\Omega}$ a variety with boundary of class C^∞ and Ω locally on one side of $\partial\Omega$.

We denote by $d(\partial\Omega)$ the surface measure on $\partial\Omega$ induced by x. We will now give a brief discussion on what is meant by (3-1). Refer to Auslander [4] and Auslander, Mackensie [5] for a more thorough discussion of the subject.

Let Ω be an open subset of R^n. The $\partial\Omega$ is a surface which can be defined by a finite number of functions $f_i(x_1,x_2,\ldots,x_n)$ $(1 \leq i \leq s)$, where for any i, $1 \leq i \leq s$, $f_i \in C^\infty(R^n)$ and $\partial\Omega$ satisfies

$$\partial\Omega = \{ x\varepsilon\bar{\Omega} \mid f_1(x) = \ldots = f_s(x) = 0\}.$$

On $\partial\Omega$ we assign the topology induced on it by the topology of R^n. Also, on $\partial\Omega$ we assume there are no singularities, i.e., for any $x\varepsilon\partial\Omega$, there exists a neighborhood, $N(x)$, such that for every $x\varepsilon N(x)\cap\partial\Omega$

$$\det \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \cdot & \cdots & \cdot \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix} = 1.$$

We can state the following more aesthetic equivalent definition of (3-1), where for simplification we will define $C^\infty(A,B)$ as the set $\{u\varepsilon C^\infty(A) \mid D(u)\subset A, R(u)\subset B\}$:

(i) For any $x \in \partial\Omega$, there exist an open subset $W \subset R^{n-1}$, and there exists an $\eta \in C^{\infty}(W, R^{n})$, η is one-to-one, such that $x \in \eta(W) = U$, an open subset of $\partial\Omega$. Also, there exists an open subset $O \subset R^{n}$, and there exists a $\zeta \in C^{\infty}(O, R^{n-1})$, such that $U \subset O$ and $\zeta \circ \eta$ is the identity on W (see figure 3.1).

(ii) Each pair (η, W) defines a coordinate neighborhood on $\partial\Omega$, at each point of $\partial\Omega$. We have the following compatability condition, if $(\eta_{\alpha}, W_{\alpha})$ and $(\eta_{\beta}, W_{\beta})$ are any two coordinate neighborhoods on $\partial\Omega$ such that $\eta_{\alpha}(W_{\alpha}) \cap \eta_{\beta}(W_{\beta}) \neq \phi$ then $\eta_{\alpha}^{-1} \cdot \eta_{\beta} \in C^{\infty}(\eta_{\beta}^{-1}(U_1 \cap U_2), \eta_{\alpha}^{-1}(U_1 \cap U_2))$.

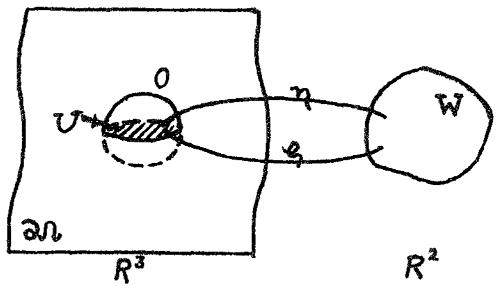

Figure 3.1

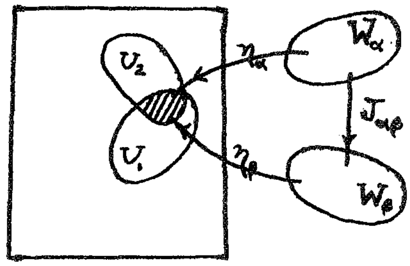

Figure 3.2

Essentially what we mean is that $\partial\Omega$ is an 'infinitely' smooth surface, and which locally looks like R^{n-1}.

Ω is locally on one side of the boundary, $\partial\Omega$, means that for any $x \in \partial\Omega$, there exists a neighborhood of $x, N(x)$, such that $N(x) \cap \bar{\Omega}$ lies entirely on one side of $\partial\Omega$, i.e., as we traverse $N(x) \cap \partial\Omega$ we see that $N(x) \cap \bar{\Omega}$ lies entirely to our left (or right). The figure below, figure 3.3, is an example of what can not occur, that is x has Ω on both sides of it.

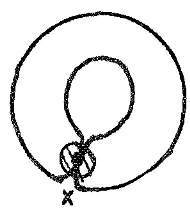

Figure 3.3.

We are now ready to define the Sobolev spaces, $H^m(\Omega)$. See Lions, Magenes [18] and Yosida [35] for more discussion on the subject. Let Ω be an open subset of the Euclidean space R^n, and m a nonnegative integer, then

$$H^m(\Omega) = \{u\epsilon D'(\Omega) | D^\alpha u\epsilon L^2(\Omega), \text{ for all } \alpha, |\alpha| \leq m\}. \tag{3-2}$$

If we define for any $u,v\epsilon H^m(\Omega)$

$$(u,v)_m = \sum_{|\alpha|\leq m} \int_\Omega (D^\alpha u)(x)\overline{(D^\alpha v)(x)}dx$$

and
$$||u||_m = (u,v)_m^{\frac{1}{2}} = (\sum_{|\alpha|\leq m} ||D^\alpha u||_o^2)^{\frac{1}{2}}$$

then $H^m(\Omega)$ becomes a Hilbert space. We note that convergence in $H^m(\Omega)$, or

$$u_n \underset{H^m(\Omega)}{\rightarrow} u \qquad \text{as } n \to \infty$$

is equivalent to the following: for any $\alpha, |\alpha| \leq m$,

$$D^\alpha u_n \underset{L^2(\Omega)}{\rightarrow} D^\alpha u \qquad \text{as } n \to \infty.$$

$H^m(\Omega)$ has the following properties, which can be found in Lions and Magenes [18].

(i) $H^o(\Omega) = L^2(\Omega)$, which motivates the denoting of the inner product in $L^2(\Omega)$ as $(.,.)_o$.

(ii) If $m_1 > m_2 \geq 0$, then $H^{m_1}(\Omega)\subset H^{m_2}(\Omega)$, and the identity injection is completely continuous.

(iii) If Ω satisfies (3-1), and $u\epsilon H^m(\Omega)$, for some integer $m\geq 0$, and if $\alpha\epsilon R^n$, such that $|\alpha| = k \leq m$ then $D^\alpha u\epsilon H^{m-k}(\Omega)$.

(iv) If Ω satisfies (3-1), then $C^\infty(\overline{\Omega})$ is dense in $H^m(\Omega)$.

It can be shown that $C_o^\infty(\Omega)$ is not dense in $H^m(\Omega)$ (except for $m = 0$), but is a proper subspace of $H^m(\Omega)$, motivating the definition of $H_o^m(\Omega)$, as the closure of $C_o^\infty(\Omega)$ in the H^m - norm.

3.44. The Boundary Spaces $H^s(\partial\Omega)$, $s \geq 0$

In this section, we will define the boundary space $H^s(\partial\Omega)$ and give some of its properties. Lions and Magenes in [18], Schechter in [28] give a more complete discussion of $H^s(\partial\Omega)$. Before defining this space, we must define the space $H^s(R^n)$. Let s be any nonnegative real number, then $H^s(R^n)$ is defined as follows:

$$H^s(R^n) = \{u\varepsilon\mathcal{S}'(R^n) \mid (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi)\varepsilon L^2(R^n)\}$$

where $\mathcal{S}'(R^n)$ is the space of tempered distributions and $\hat{u}(\xi)$ is the Fourier Transform of u. For any $u,v\varepsilon H^s(R^n)$, if we define

$$(u,v)_{H^s(R^n)} = \int_{R^n} (1 + |\xi|^2)^s \hat{u}(\xi)\overline{\hat{v}(\xi)}d$$

and $$||u||_{H^s(R^n)} = ||(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi)||_{L^2(R^n)}$$

then $H^s(R^n)$ becomes a Hilbert space. Convergence in $H^s(R^n)$ means that

$$u_n \underset{H^s(R^n)}{\rightarrow} u \qquad \text{as } n \to \infty$$

if and only if

$$(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}_n(\xi) \underset{L^2(R^n)}{\rightarrow} (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \qquad \text{as } n \to \infty.$$

It is shown in Treves [32] that $C_o^\infty(R^n)$ is dense in $H^s(R^n)$.

We are now ready to define the boundary space $H^s(\partial\Omega)$. From our assumption on Ω of (3-1) we can find a finite family of bounded open subsets of R^n, U_i($1\leq i\leq N$), covering the $\partial\Omega$ such that for every integer $i, 1\leq i\leq N$, there exists an infinitely differentiable mapping, θ_i, which maps U_i onto the sphere $\sigma = \{x'\varepsilon R^n \mid |x'|<1\}$, in such a way that

$$\theta_i(U_i\cap\Omega) = \sigma_+ \equiv \{x'\varepsilon R^n \mid x'\varepsilon\sigma, x'_n>0\}$$

$$\theta_i(U_i\cap\partial\Omega) = \partial_1\sigma_+ \equiv \{x'\varepsilon R^n \mid x'\varepsilon\sigma, x'_n = 0\}.$$

See figure 3.4.

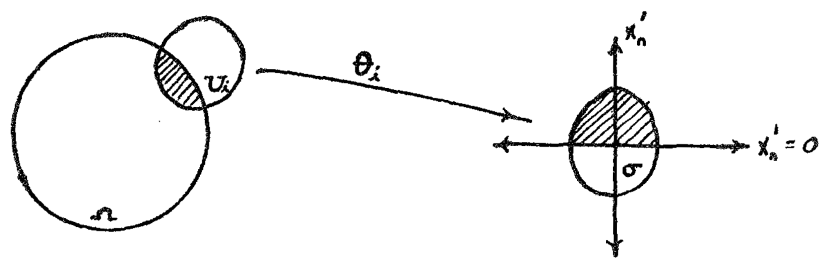

figure 3.4

We also note that θ_i is invertible, that is, if $\theta_i(x) = y$, then $\theta_i^{-1}(y) = x$, and θ_i^{-1} is also infinitely differentiable and maps σ onto U_i, and $\partial_1\sigma_+$ onto $U_i\cap\partial\Omega$. In addition, the following compatability relations are true: if $U_i\cap U_j \neq \phi$, then there exists a homeomorphism, J_{ij}, infinitely differentiable and with positive Jacobian (so we don't have any singular points, also changing from positive to negative Jacobian changes the orientation of the local coordinate system). J_{ij} satisfies the following property; J_{ij} maps $\theta_i(U_i\cap U_j)$ onto $\theta_j(U_i\cap U_j)$ in such a way that for any $x\varepsilon U_i\cap U_j$

$$\theta_j(x) = J_{ij}(\theta_i(x)).$$

See figure 3.2.

Let α_i be a partition of unity on $\partial\Omega$ with respect to the covering

$\{U_i \cap \partial\Omega\}_{i=1}^{N}$. For a complete discussion of partition of unity see Treves [32]. The family of functions, $\{\alpha_i\}$, satisfy the following properties,

(i) for any $i, 1 \leq i \leq N$, $\alpha_i \varepsilon C^{\infty}(\partial\Omega)$;

(ii) for any $i, 1 \leq i \leq N$, α_i has compact support in $U_i \cap \partial\Omega$;

(iii) for any $i, 1 \leq i \leq N$, and any $x \varepsilon \partial\Omega$, $\alpha_i(x) \geq 0$;

(iv) for any $x \varepsilon \partial\Omega$, $\sum_{i=1}^{N} \alpha_i(x) = 1$.

Note that $C^{\infty}(\partial\Omega)$ is the set of all infinitely differentiable functions defined on $\partial\Omega$.

Now if u is a real or complex-valued function defined on $\partial\Omega$ (for instance $u \varepsilon L^1(\partial\Omega)$ then we can decompose u (as a sum of L^1 functions),

$$u = \sum_{i=1}^{N} (\alpha_i u)$$

(this means that for any $x \varepsilon \partial\Omega$, $u(x) = \sum_i \alpha_i(x) u(x)$). We note here that for any $i, 1 \leq i \leq N$, $\alpha_i u$ has compact support in $U_i \cap \partial\Omega$. Now we can define a mapping $\theta_i^* (\alpha_i u)$ from $\partial_1 \sigma_+$ into the scalar field K of real or complex numbers, in the following manner: for any $y' \varepsilon \partial_1 \sigma_+$

$$\theta_i^* (\alpha_i u)(y') = \alpha_i u(\theta_i^{-1}(y')) = \alpha_i(\theta_i^{-1}(y')) u(\theta_i^{-1}(y')).$$

It can be seen that $\theta_i^* (\alpha_i u)$ has compact support in $\partial_1 \sigma_+$, and thus $\theta_i^* (\alpha_i u)$ can be considered as being defined in the Euclidean space R^{n-1}, as the extension by zero outside of $\partial_1 \sigma_+$.

<u>Definition 3.44.1.</u> Let $y \varepsilon R^{n-1}$, then

$$\theta_i^* (\alpha_i u)(y) = \begin{cases} (\alpha_i u)(\theta_i^{-1}(y)) & \text{for } y \varepsilon \partial_1 \sigma_+ \\ 0 & \text{elsewhere} \end{cases}$$

The following important theorem is due to Lions, Magenes [18].

Theorem 3.44.1. The linear mapping $u \to \phi_i^*(\alpha_i u)$ is continuous from $L^1(\partial\Omega)$ into $L^1(R^{n-1})$ and is also continuous from $C^\infty(\partial\Omega)$ into $C^\infty(R^{n-1})$, and can be extended to a continuous linear mapping from $D'(\partial\Omega)$ onto $D'(R^{n-1})$ (verified by duality).

We will now define the boundary space $H^s(\partial\Omega)$.

Definition 3.44.2. Let s be any nonnegative real number. Then

$$H^s(\partial\Omega) = \{u\varepsilon D'(\partial\Omega) \,|\, \phi_i^*(\alpha_i u) \;\varepsilon\; H^s(R^{n-1}),\; 1\leq i\leq N\} .$$

This algebraic definition is independent of local coordinates $\{(\theta_i, U_i)\}_{i=1}^N$ and partition of unity $\{\alpha_i\}_{i=1}^N$.

Let $u, v\varepsilon H^s(\partial\Omega)$. If we define

$$(u,v)_{H^s(\partial\Omega)} = \sum_{i=1}^N (\theta_i^*(\alpha_i u), \theta_i^*(\alpha_i v))_{H^s(R^{n-1})}$$

and $$||u||_{H^s(\partial\Omega)} = \langle u\rangle_{H^s(\partial\Omega)} = [\sum_{i=1}^N ||\phi_i^*(\alpha_i u)||^2_{H^s(R^{n-1})}]^{\frac{1}{2}}$$

$H^s(\partial\Omega)$ becomes a Hilbert space. We will denote the norm on $H^s(\partial\Omega)$ by $\langle\cdot\rangle_s$. Also, it can be shown that the various norms which depend on $\{\theta_i, U_i, \alpha_i\}$ are all equivalent. Convergence in $H^s(\partial\Omega)$ means that

$$u_n \underset{H^s(\partial\Omega)}{\to} u \qquad \text{as } n \to \infty$$

if and only if for any $i, 1\leq i\leq N$,

$$\theta_i^*(\alpha_i u_n) \underset{H^s(R^{n-1})}{\to} \phi_i^*(\alpha_i u) \qquad \text{as } n \to \infty.$$

$H^s(\partial\Omega)$ has the following properties, verified in Lions and Magenes [18],

(i) $C^\infty(\partial\Omega)$ is dense in $H^s(\partial\Omega)$, for any nonnegative real number s.

(ii) If Ω satisfies (3-1), and if s_1, s_2 are real numbers such that $s_1 > s_2 \geq 0$ then $H^{s_1}(\partial\Omega) \subset H^{s_2}(\partial\Omega)$ and the identity injection from $H^{s_1}(\partial\Omega)$ into $H^{s_2}(\partial\Omega)$ is completely continuous.

3.45. The Trace Theorems

In this section, we will briefly discuss the trace theorems and justify the definition of the trace of a function on the boundary and the function spaces in which they are defined. The methods we will use are more fully discussed in Peetre [26], Schechter [28] and Agmon-Douglis-Nirenberg [2]. For a different approach yielding the same results see Lions and Magenes [18]. Also, for a discussion of the trace of a function on the boundary see Volevich-Paneyakh [33].

The following two theorems which are the basic results for the trace theorems are proved in Peetre [26] and Schechter [28].

Theorem 3.45.1. Let Ω satisfy (3-1) and m be any positive integer. If $u \in C^\infty(\bar{\Omega})$ and $\gamma_o u$ denotes the restriction of u to the boundary, $\partial\Omega$, then we have the following inequality. There exists a constant, K_m, independent of u, such that,

$$\langle \gamma_o u \rangle_{m-\frac{1}{2}} \leq K_m ||u||_m.$$

Thus, since $C^\infty(\bar{\Omega})$ is dense in $H^m(\Omega)$, and $C^\infty(\partial\Omega)$ is dense in $H^{m-\frac{1}{2}}(\partial\Omega)$, the mapping

$$(u \rightarrow \gamma_o u) : C^\infty(\bar{\Omega}) \rightarrow C^\infty(\partial\Omega)$$

can be extended by continuity to a continuous linear mapping which we denote by γ_o such that,

$$(u \to \gamma_o u): H^m(\Omega) \to H^{m-\frac{1}{2}}(\partial\Omega).$$

This means that the trace of an element $u \varepsilon H^m(\Omega)$, denoted by $\gamma_o u$, can be considered as an element in $H^{m-\frac{1}{2}}(\partial\Omega)$. Conversely, for any $v \varepsilon H^{m-\frac{1}{2}}(\partial\Omega)$, there exists a $u \varepsilon H^m(\Omega)$ such that $v = \gamma_o u$, or v is the trace of u in $H^{m-\frac{1}{2}}(\partial\Omega)$.

Remark 3.45.1. Another way to look at the trace of an element is as follows: if $u \varepsilon H^m(\Omega)$, then for any sequence $u_n \varepsilon C^\infty(\overline{\Omega})$, such that

$$u_n \underset{H^m(\Omega)}{\longrightarrow} u \qquad \text{as } n \to \infty$$

and if $\gamma_o u_n$ is the restriction of u_n to the boundary, $\partial\Omega$, then

$$\gamma_o u_n \underset{H^{m-\frac{1}{2}}(\partial\Omega)}{\longrightarrow} v \qquad \text{as } n \to \infty$$

where v, denoted by $\gamma_o u$, is the trace of u in $H^{m-\frac{1}{2}}(\partial\Omega)$.

We have just defined what we mean by the trace, $\gamma_o u$, of an element $u \varepsilon H^m(\Omega)$. Now we will consider the distributional derivative, $D^\alpha u$, and define what we mean by the trace of this function on the boundary.

Theorem 3.45.2. Let Ω satisfy (3-1) and m be any positive integer. If $u \varepsilon C^\infty(\overline{\Omega})$, and $D^\alpha u$ is the distributional derivative of u of order $|\alpha| = k$ $k \leq m-1$ and $\gamma_o(D^\alpha u)$ denotes the restriction of $D^\alpha u$ to the boundary, $\partial\Omega$, then we have the following inequality. There exists a constant, K_m, independent of u, such that

$$< \gamma_o(D^\alpha u) >_{m-k-\frac{1}{2}} \leq K_m ||u||_m.$$

Thus, since $C^\infty(\overline{\Omega})$ is dense in $H^m(\Omega)$ and $C^\infty(\partial\Omega)$ is dense in $H^{m-k-\frac{1}{2}}(\partial\Omega)$, the mapping

$$(u \to \gamma_o(D^\alpha u)): \quad C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$$

can be extended by continuity to a continuous linear mapping which we denote by $\gamma_o D^\alpha$ such that

$$(u \to \gamma_o(D^\alpha u)): \quad H^m(\Omega) \to H^{m-k-\frac{1}{2}}(\partial\Omega).$$

This means that if $u \varepsilon H^m(\Omega)$, since $D^\alpha u \varepsilon H^{m-k}(\Omega)$, then the trace, $\gamma_o(D^\alpha u)$, of $D^\alpha u$ can be considered as an element in $H^{m-k-\frac{1}{2}}(\partial\Omega)$. Conversely, for any $v \varepsilon H^{m-k-\frac{1}{2}}(\partial\Omega)$, there exists a $u \varepsilon H^m(\Omega)$ such that $v = \gamma_o(D^\alpha u)$, or v is the trace of $D^\alpha u$ in $H^{m-k-\frac{1}{2}}(\partial\Omega)$.

Remark 3.45.2. As in remark 3.45.1, we can consider the trace of $D^\alpha u$ in the following way: if $u \varepsilon H^m(\Omega)$, then for any sequence $u_n \varepsilon C^\infty(\overline{\Omega})$ such that

$$u_n \underset{H^m(\Omega)}{\longrightarrow} u \qquad \text{as } n \to \infty$$

and if $\gamma_o(D^\alpha u_n)$ is the restriction of $D^\alpha u_n$ to the boundary, $\partial\Omega$, then

$$\gamma_o(D^\alpha u_n) \underset{H^{m-k-\frac{1}{2}}(\partial\Omega)}{\longrightarrow} v \qquad \text{as } n \to \infty$$

where v, denoted by $\gamma_o(D^\alpha u)$, is the trace of $D^\alpha u$ in $H^{m-k-\frac{1}{2}}(\partial\Omega)$.

In studying the Dirichlet problem we need to consider the normal derivatives to the boundary, $(\frac{\partial}{\partial n})^j$, $0 \leq j \leq m-1$, where n is the normal to the boundary, interior to the surface. The following result by Lions and Magenes [18] shows how we define $(\frac{\partial}{\partial n})^j$.

Theorem 3.45.3. Let Ω satisfy (3-1) and m be any positive integer.

Then the mapping

$$(u \to \{(\frac{\partial}{\partial n})^o u, (\frac{\partial}{\partial n})u, \ldots, (\frac{\partial}{\partial n})^{m-1}u\}): \quad C^\infty(\bar{\Omega}) \to [C^\infty(\partial\Omega)]^m$$

can be extended by continuity to a continuous linear mapping denoted by

$$(u \to \{(\frac{\partial}{\partial n})^o u, (\frac{\partial}{\partial n})u, \ldots, (\frac{\partial}{\partial n})^{m-1}u\}): \quad H^{2m}(\Omega) \to \prod_{j=0}^{m-1} H^{2m-j-\frac{1}{2}}(\partial\Omega).$$

Conversely, there exists a continuous linear mapping denoted by

$$(\{v_o, v_1, \ldots, v_{m-1}\} \to u): \prod_{j=0}^{m-1} H^{2m-j-\frac{1}{2}}(\partial\Omega) \to H^{2m}(\Omega)$$

such that for any set of functions, $\{v_j\}_{j=0}^{m-1}$, where $v_j \varepsilon H^{2m-j-\frac{1}{2}}(\partial\Omega)$, there exists a $u \varepsilon H^{2m}(\Omega)$, such that $(\frac{\partial}{\partial n})^j u = v_j$, $0 \leq j \leq m-1$.

<u>Remark 3.45.3.</u> If $u \varepsilon H^{2m}(\Omega)$, and for any sequence $u_n \varepsilon C^\infty(\bar{\Omega})$, such that

$$u_n \underset{H^{2m}(\Omega)}{\longrightarrow} u \qquad \text{as } n \to \infty$$

where $(\frac{\partial}{\partial n})^j u_n \varepsilon C^\infty(\partial\Omega)$ is the j^{th} - normal derivative at the boundary, $\partial\Omega$, then

$$(\frac{\partial}{\partial n})^j u_n \underset{H^{2m-j-\frac{1}{2}}(\partial\Omega)}{\longrightarrow} v \qquad \text{as } n \to \infty;$$

and if v is denoted by $(\frac{\partial}{\partial n})^j u$, then the j^{th} - normal derivative is considered as an element of the boundary space $H^{2m-j-\frac{1}{2}}(\partial\Omega)$.

Now let us consider, for $u \varepsilon C^\infty(\bar{\Omega})$, the linear formal partial differential boundary operator

$$B_j(x,D)u(x) = \sum_{|h| \le m_j} b_{jh}(x)\gamma_o(D^h u(x)) \qquad x\varepsilon\partial\Omega,\ 0\le j\le m-1$$

where $h\varepsilon\mathbb{R}^n$ with nonnegative integer components, $b_{jh}(x)\varepsilon C^\infty(\partial\Omega)$, m_j is the order of B_j, $m_j \le 2m-1$ and $\gamma_o(D^h u)$ is the restriction of $D^h u$ on $\partial\Omega$. Unless otherwise stated the boundary operator, $B_j(x,D)u$, will be denoted by $B_j u$. We have the following theorem due to Gerd Grebb [12], which will explain what is meant by the trace of $B_j u$ on the boundary, $\partial\Omega$.

<u>Theorem 3.45.4.</u> Let Ω satisfy (3-1), $u\varepsilon C^\infty(\overline{\Omega})$, m any positive integer and $B_j u$ defined above. Then there exists a constant, K_m, independent of u such that

$$< B_j u >_{2m-m_j-\frac{1}{2}} \le K_m ||u||_{2m}.$$

Hence, the mapping

$$(u \to B_j u):\quad C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$$

can be extended by continuity to a continuous linear mapping denoted by B_j, such that

$$(u \to B_j u):\quad H^{2m}(\Omega) \to H^{2m-m_j-\frac{1}{2}}(\partial\Omega).$$

<u>Remark 3.45.4.</u> Another equivalent way to look at the trace of $B_j u$ on $\partial\Omega$ is the following: Let $u\varepsilon H^{2m}(\Omega)$. If $u_n \varepsilon C^\infty(\overline{\Omega})$, such that

$$u_n \xrightarrow[H^{2m}(\Omega)]{} u \qquad \text{as } n \to \infty$$

and since $B_j u_n \varepsilon C^\infty(\partial\Omega)$, then

$$B_j u_n \xrightarrow[H^{2m-m_j-\frac{1}{2}}(\partial\Omega)]{} v_j \qquad \text{as } n \to \infty$$

where v_j is denoted by $B_j u$ and is in $H^{2m-m_j-\frac{1}{2}}(\partial\Omega)$. Hence, the trace, $B_j u$, of u can be considered as an element of $H^{2m-m_j-\frac{1}{2}}(\partial\Omega)$.

We have shown that for $u\varepsilon H^{2m}(\Omega)$, $B_j u (0\leq j\leq m-1)$ can be considered as an element of $H^{2m-m_j-\frac{1}{2}}(\partial\Omega)$. In this work we are interested in the boundary conditions: for $u\varepsilon H^{2m}(\Omega)$,

$$B_j u = 0 \qquad \text{on } \partial\Omega \quad (0\leq j\leq m-1) \tag{3-3}$$

The following result will explain it's meaning.

<u>Theorem 3.45.5.</u> Let Ω satisfy (3-1), $u\varepsilon C^{\infty}(\bar{\Omega})$ and B_j be the linear boundary operator defined above. For the mapping

$$B_j : C^{\infty}(\bar{\Omega}) \ni u \rightarrow 0 \varepsilon C^{\infty}(\partial\Omega)$$

the extension is a continuous linear map denoted by

$$B_j : H^{2m}(\Omega) \ni u \rightarrow 0 \varepsilon H^{2m-m_j-\frac{1}{2}}(\partial\Omega)$$

This shows that for $u\varepsilon H^{2m}(\Omega)$, the boundary condition (3-3) means that $B_j u$ is the zero element in $H^{2m-m_j-\frac{1}{2}}(\partial\Omega)$, which we will denote by

$$< B_j u >_{2m-m_j-\frac{1}{2}} = 0 \qquad (0\leq j\leq m-1) \ .$$

Note that the kernel of the map B_j is defined as

$$\text{Ker } B_j = \{u\varepsilon H^{2m}(\Omega) \mid < B_j u >_{2m-m_j-\frac{1}{2}} = 0\} \ .$$

<u>Remark 3.45.5.</u> We restrict our discussion to the Dirichlet problem, defined by: for any $u\varepsilon C^{\infty}(\bar{\Omega})$, and any $x\varepsilon\partial\Omega$

$$(\frac{\partial}{\partial n})^j u(x) = 0 \qquad (0 \leq j \leq m-1).$$

From the results above we see the generalized Dirichlet boundary conditions have the following form: for any $j\,(0 \leq j \leq m-1)$, the linear mapping

$$(\frac{\partial}{\partial n})^j: \quad C^\infty(\bar{\Omega}) \ni u \to 0 \in C^\infty(\partial\Omega)$$

has an extension which we denote by $(\frac{\partial}{\partial n})^j$, such that

$$(\frac{\partial}{\partial n})^j: \quad H^{2m}(\Omega) \ni u \to 0 \in H^{2m-j-\frac{1}{2}}(\partial\Omega).$$

Thus the generalized Dirichlet problem can be written

$$< (\frac{\partial}{\partial n})^j u >_{2m-j-\frac{1}{2}} = 0 \qquad (0 \leq j \leq m-1)$$

where $(\frac{\partial}{\partial n})^j u$ is the zero element in $H^{2m-j-\frac{1}{2}}(\partial\Omega)$. Note the Kernel of the map

$$(u \to \{(\frac{\partial}{\partial n})^o u, \ldots, (\frac{\partial}{\partial n})^{m-1} u\}): H^{2m}(\Omega) \ni u \to (0,\ldots,0) \in \prod_{j=0}^{m-1} H^{2m-j-\frac{1}{2}}(\partial\Omega)$$

is defined by

$$\text{Ker}\,\{(\frac{\partial}{\partial n})^o, \ldots, (\frac{\partial}{\partial n})^{m-1}\} = \{u \in H^{2m}(\Omega) \,|\, < (\frac{\partial}{\partial n})^j u > = 0, 0 \leq j \leq m-1\} = H^{2m}(\Omega) \cap H_o^m(\Omega).$$

This follows directly from a theorem found in Lions and Magenes [18] which characterizes $H_o^m(\Omega)$;

$$H_o^m(\Omega) = \{u \in H^m(\Omega) \,|\, < (\frac{\partial}{\partial n})^j u > =_{2m-j-\frac{1}{2}} = 0, \; 0 \leq j \leq m-1\}.$$

3.46. The Spaces $H_B^{2m}(\Omega), H_B^m(\Omega)$

In our study of the stability problem for the abstract operator equation

$$\frac{du(t)}{dt} + Au(t) = f(u)$$

we need to define explicitly the domain of the operator A, so that it satisfies the conditions of being dense in the base Hilbert space, H, and it can be utilized in solving the partial differential equation

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u).$$

In this section, we will define the spaces $H_B^{2m}(\Omega)$ and $H_B^{m}(\Omega)$ which are vital to the definition of D(A).

First, we must define the function spaces $C_B^{2m}(\Omega)$ and $C_B^{\infty}(\Omega)$, where $B = \{B_j\}_{j=0}^{m-1}$ is the system of boundary operators defined in 3.45. Let Ω satisfy (3-1).

$$C_B^{2m}(\Omega) = \{u\epsilon C^{2m}(\overline{\Omega}) \,|\, B_j(x,D)u = 0 \quad \text{on } \partial\Omega, \ (0 \leq j \leq m-1)\}$$

$$C_B^{\infty}(\Omega) = \{u\epsilon C^{\infty}(\overline{\Omega}) \,|\, B_j(x,D)u = 0 \quad \text{on } \partial\Omega, \ (0 \leq j \leq m-1)\}$$

The following definitions are equivalent:

(i) $H_B^{2m}(\Omega) = \{u\epsilon H^{2m}(\Omega) \,|\, < B_j u >_{2m-m_j-\frac{1}{2}} = 0 \quad (0 \leq j \leq m-1)\};$

(ii) $H_B^{2m}(\Omega) = \text{Ker } \{B_o, B_1, \ldots, B_{m-1}\}$

where

$$\{B_o, B_1, \ldots, B_{m-1}\}: \ H^{2m}(\Omega) \to \prod_{j=0}^{m-1} H^{2m-m_j-\frac{1}{2}}(\partial\Omega);$$

(iii) $H_B^{2m}(\Omega)$ = completion of $C_B^{\infty}(\Omega)$ in the H^{2m} - norm.

If we place restrictions on $(A(x,D),\{B_j\},\Omega)$ of $A(x,D)$ being 'properly elliptic', and $\{B_j\}$ being a 'normal system' and satisfying the 'complementary condition' found in section (5-21), then (iii) is equivalent to (iv) below,

(iv) $H_B^{2m}(\Omega)$ = completion of $C_B^{2m}(\Omega)$ in the H^{2m} - norm.

Note that Lions and Magenes [18], use definitions (i), (ii), and (iii) and Agmon [1], Schechter [28] and Friedman [11] uses definition (iv). The equivalence of (i), (ii), and (iii) follows directly from the definition of B_j, theorem 3.45.4 and remarks 3.45.4 and 3.45.5. The proof of the equivalence of (iii) and (iv) is left to the Appendix.

We will now define $H_B^m(\Omega)$, as the completion of $C_B^\infty(\Omega)$ in the H^m - norm.

Remark 3.46.1. From [18] we can characterize the spaces $H_B^m(\Omega)$ with the following equivalent definition,

$$H_B^m(\Omega) = \{u\varepsilon H^m(\Omega) | < B_j u >_{m-m_j-\frac{1}{2}} = 0, \text{ for all } B_j \text{ such that } m_j<m\}$$

where $\{B_j\}_{j=0}^{m-1}$ is the system of boundary operators defined in section 3.45.

Remark 3.46.2. If we restrict $\{B_j\}_{j=0}^{m-1}$ to be the Dirichlet boundary operators,

$$B_j = (\frac{\partial}{\partial n})^j \qquad (0\leq j\leq m-1)$$

then we have the definition for the space $H_B^{2m}(\Omega)$, which can be found in Friedman [11], and Dunford-Schwartz [9],

$$H_B^{2m}(\Omega) = \{u\varepsilon H^{2m}(\Omega) | < (\frac{\partial}{\partial n})^j >_{2m-j-\frac{1}{2}} = 0 \qquad (0\leq j\leq m-1)\}$$

$$= H^{2m}(\Omega) \cap H_o^m(\Omega)$$

and the space $H_B^m(\Omega)$, since the order of B_j is $m_j = j < m$ for every $j, 0\leq j\leq m-1$,

$$H_B^m(\Omega) = \{u\varepsilon H^m(\Omega) | < (\frac{\partial}{\partial n})^j u >_{m-j-\frac{1}{2}} = 0, \qquad (0\leq j\leq m-1)\}$$
$$= H_o^m(\Omega).$$

4.0. STABILITY THEORY FOR OPERATOR DIFFERENTIAL EQUATIONS IN A REAL HILBERT SPACE

It was shown in Chapter 2 that certain linear partial differential equations can be placed in the form of an operator differential equation, see (4-1) below, and the problem of the existence, uniqueness and stability of the solution to the partial differential equation can be solved by considering the related stability problem for the operator equation (4-1). In this chapter we will consider the stability problem for the operator differential equation, where A is, in general, an unbounded linear or nonlinear operator with domain and range both contained in the real Hilbert space, H.

4.1. Stability Theory of Linear Differential Equations in a Real Hilbert Space

In this section we will be concerned with the existence, uniqueness and stability of a solution to the operator differential equation

$$\frac{du(t)}{dt} + Au(t) = 0 \qquad (t \geq 0) \qquad (4\text{-}1)$$

where A is, in general, an unbounded linear operator with domain, D(A), and range, R(A), both contained in a real Hilbert space H, and the unknown function, u(t), is a vector valued function defined on $[0,\infty)$ to H.

We note that A can be considered as the extension of a linear partial differential operator.

We desire to study the stability of a solution of (4-1) without actually finding the solution. This can be done by considering the properties of a semi-group, because if A is the infinitesimal generator of the semi-group $\{T_t | t \geq 0\}$ of bounded linear operators on a Hilbert space H, then a solution to (4-1) with the initial value $u(t_o) = u_o \varepsilon D(A)$ is given by $u(t;u_o,t_o) = T_t u_o$ for all $t \geq t_o$, with $u(t_o;u_o,t_o) = u_o$. Hence, it suffices to impose conditions on the operator A so that it will be the infinitesimal generator of a semi-group, which will ensure the existence of a solution. Then, the stability of this solution can be established from the semi-group properties. This has been done by Pao [23] and Buis [7], where a more complete discussion is given. We will state some basic definitions and some results.

Definition 4.1.1. A solution, u(t), of the equation (4-1) with initial condition $u(o) = u_o \varepsilon D(A)$ means:

(i) u(t) is uniformly continuous in t, for all $t \geq 0$, with $u(o) = u_o$;

(ii) $u(t) \varepsilon D(A)$, for all $t \geq 0$, and Au(t) is continuous in t for all $t \geq 0$;

(iii) The derivative of u(t) exists (in the strong topology), for every $t \geq 0$, and equals $(-A)u(t)$.

Definition 4.1.2. An equilibrium solution of (4-1), denoted by $u(t) = u_e$, is a solution u(t) of (4-1) such that

$$||u(t) - u(o)||_H = 0 \qquad \text{for all } t \geq 0$$

Definition 4.1.3. An equilibrium solution u_e of (4-1) is said to be stable (with respect to initial perturbations) if given any $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$||u_o - u_e||_H < \delta \quad \text{implies } ||u(t) - u_e||_H < \varepsilon \quad \text{for all } t \geq 0.$$

u_e is said to be asymptotically stable if

(i) it is stable; and

(ii) $\lim_{t \to \infty} ||u(t) - u_e||_H = 0$,

where u(t) is any solution of (4-1) with $u(o) = u_o \varepsilon D(A)$.

In this definition, stability and asymptotic stability are taken with respect to the H - norm. It is clear from the above definition that if $0 \varepsilon D(A)$, then $u \equiv 0$, the null solution, is an equilibrium solution of (4-1). Since the domain of the operator A contains the zero vector, it follows that the study of the stability problem of an equilibrium solution to the linear equation is equivalent to the study of the stability properties of the null solution. We should note that the theory is not limited to equilibrium solutions, but is also valid by starting from any initial element, $u_o \varepsilon D(A)$, with solution $u(t;u_o,t_o)$ which is not an equilibrium solution (such as a periodic solution or any unperturbed solution).

We are also interested in the region of stability.

Definition 4.1.4. Let u(t) be a solution to (4-1) with $u(o) = u_o$. A subset D of H is said to be a stability region of the equilibrium solution u_e if for any $\varepsilon>0$, there exists a $\delta>0$ such that $u \varepsilon D$ and $||u - u_e||<\delta$ imply $||u(t) - u_e||_H<\varepsilon$, for all $t\geq 0$.

As can be seen from theorem 3.3.3, to ensure stability it is required that A be dissipative with respect to the inner product of the space H. However, Buis in [7] proved that if A is dissipative with respect to any inner product equivalent to the one defined on H, then A is the infinitesimal generator of a contraction semi-group and stability is ensured by the equivalence of the norms and the properties of semi-groups. The following theorem found in Buis [7] gives N.A.S.C. to ensure that two inner products are equivalent.

Theorem 4.1.1. Let $H_1 = (H,(.,.)_1)$ be a real Hilbert space. An inner product $(.,.)_2$ defined on the linear space H is equivalent to the inner product $(.,.)_1$ if and only if there exists a symmetric, bounded, positive definite, linear operator $S \varepsilon L(H_1,H_1)$ such that

$$(u,w)_2 = (u,Sw)_1 \qquad \text{for all } u,w\varepsilon H.$$

We will now define Lyapunov functionals which are used extensively in the study of stability theory, as we saw in Chapter 2. These definitions are found in Pao [23] and Buis [7].

Definition 4.1.5. A Lyapunov functional on a real Hilbert space H is defined through the symmetric bilinear form

$$V(u,w) = (u,Sw)_1 = (w,Su)_1 \qquad \text{for all } u,w\varepsilon H_1$$

where $S\varepsilon L(H_1,H_1)$ is a symmetric (self-adjoint), bounded, positive definite, linear operator. The Lyapunov functional is defined by

$$v(u) = V(u,u) \qquad \text{for all } u\varepsilon H_1.$$

It follows from the above definition and theorem 4.1.1 that V(u,w) defines an inner product equivalent to $(.,.)_1$, the inner product defined on H_1. We now have the following result due to Pao [23] and Buis [7] which gives sufficient conditions on A to ensure the existence and stability of the null solution of (4-1). The notation has been changed to fit into the context of this discussion.

Theorem 4.1.2. Let A be a linear operator with domain D(A) dense in H_1, range R(A) in H_1 and $R(I-(-A)) = H_1$. Then the null solution of (4-1) is asymptotically stable if there exists a Lyapunov functional, v(u), such that, for some $\beta > 0$,

$$\dot{v}(u) = 2V(u,(-A)u) \leq - \beta||u||^2_{H_1} \qquad \text{for all } u\varepsilon D(A)$$

(where β is the dissipativity constant).

Remark 4.1.1. From this theorem, we can see that in order to study the stability problem for (4-1), it is not necessary to construct the solution to the differential equation, but it suffices to construct a Lyapunov functional satisfying the conditions of theorem 4.1.2, which is the same as finding an inner product equivalent to the one defined on the Hilbert space H, with respect to which the linear operator -A is strictly dissipative, that is, since $V(u,w) = (u,w)_2$, we need only show that A satisfies, in addition to the hypothesis of theorem 4.1.2,

$$(u,(-A)u)_2 \leq - \beta||u||^2_2. \qquad \text{for all } u\varepsilon H.$$

4.2. Stability Theory of Nonlinear Differential Equations in a Real Hilbert Space

We also discuss the nonlinear differential equation

$$\frac{du(t)}{dt} + Au(t) = f(u) \tag{4-2}$$

where A is a closed, linear, unbounded operator with domain and range in a real Hilbert space H, and f is, in general, a nonlinear operator defined on all of H into H. In order to discuss this nonlinear operator equation, we need the following basic definitions of nonlinear semi-groups and infinitesimal generator of a nonlinear contraction semi-group. These concepts are discussed in Pao [23].

Definition 4.2.1. Let H be a real Hilbert space. The family $\{T_t | t \geq 0\}$ is called a continuous semi-group of nonlinear contraction operators on H or simply (nonlinear) contraction semi-group on H if the following conditions hold:

(i) for any fixed $t \geq 0$, T_t is a continuous (nonlinear) operator defined on H into H;

(ii) for any fixed $u_o \varepsilon H, T_t u_o$ is strongly continuous in t;

(iii) $T_s T_t = T_{s+t}$ for $s,t \geq 0$ and $T_o = I$ (the identity operator);

(iv) $||T_t u - T_t v|| \leq ||u - v||$ for all $u,v \varepsilon H$, and all $t \geq 0$.

If (iv) is replaced by,

(iv)' $||T_t u - T_t v|| \leq e^{-\beta t}||u - v||$ $(\beta > 0)$ for all $u,v \varepsilon H$, and all $t \geq 0$;

then $\{T_t | t \geq 0\}$ is called a (nonlinear) negative contraction semi-group on H.

Definition 4.2.2. The infinitesimal generator, A, of the nonlinear semi-group $\{T_t | t \geq 0\}$ is defined by

$$Au = w - \lim_{h \downarrow 0} \frac{T_h u - u}{h}$$

for all $u \varepsilon H$, such that the limit on the right side exists in the sense of weak convergence.

Definition 4.2.3. An operator (nonlinear) A with domain, D(A), and range, R(A), both contained in a real Hilbert space H is said to be dissipative with respect to the inner product on H, if

$$(Au - Av, u-v)_H \leq 0 \qquad \text{for all } u,v \varepsilon D(A)$$

and A is strictly dissipative with respect to the inner product on H $u,v \varepsilon D(A)$, such that

$$(Au - Av, u-v)_H \leq - \beta(u-v, u-v)_H \qquad \text{for all } u,v\varepsilon D(A).$$

Note that when A is linear these conditions coincide with the usual definitions of dissipativity (see definition 3.2.3).

Definition 4.2.4. u(t) is said to be a solution of (4-2) if it satisfies the following conditions:

(i) For each $u(o)\varepsilon D(A)$, $u(t)\varepsilon D(A)$ for all $t\geq 0$;

(ii) u(t) is uniformly Lipschitz continuous in t;

(iii) the weak derivative of u(t) exists for all $t\geq 0$ and equals $(-A)u(t) + f(u(t))$;

(iv) the strong derivative, $\frac{du(t)}{dt} = (-A)u(t) + f(u(t))$, exists and is strongly continuous except at a countable number of values t.

Definition 4.2.5. An equilibrium solution of (4-2) is an element, $u_e\varepsilon D(A)$, satisfying (4-2) (in the weak topology) such that for any solution u(t) of (4-2) with $u(o) = u_e$

$$||u(t) - u_e||_H = 0 \qquad \text{for all } t\geq 0.$$

By considering the operator $A_1 = -A + f$ we obtain the following nonlinear operator differential equation

$$\frac{du(t)}{dt} = A_1 u(t) \tag{4-3}$$

with the nonlinear operator A_1 having both domain and range in the real Hilbert space H. As in section 4.1, we need to find conditions on A_1 which will ensure that A_1 is the infinitesimal generator of a nonlinear semi-group, which in turn ensures the existence of a solution to (4-3). The stability of the solution can be established from the properties of the nonlinear semi-group.

We have the following result by Pao [23].

<u>Theorem 4.2.1.</u> Let A_1 be a nonlinear operator with domain and range both contained in a real Hilbert space $H_1 = (H,(.,.)_1)$ such that $R(I-A_1) = H$. Then A_1 is the infinitesimal generator of a nonlinear contraction (negative contraction) semi-group $\{T_t | t \geq 0\}$ if and only if any one of the following is true

(i) The Lyapunov functional $v(u) = (u,u)_2$, where $(.,.)_2$ is an inner product equivalent to $(.,.)_1$, satisfies

$$\dot{v}(u-w) = 2(A_1u-A_1w,u-w)_2 \leq 0 \;\; (\dot{v}(u-w) = 2(A_1u-A_1w,u-w)_2 \leq -\beta$$

$$||u-w||_2^2) \qquad \text{for any } u,w\varepsilon D(A_1), \text{ and } \beta > 0.$$

(ii) A is dissipative (strictly dissipative) with respect to $(.,.)_2$ which is any inner product equivalent to $(.,.)_1$.

In this work, we will consider the case where $A_1 = -A + f$ where $-A$ is the infinitesimal generator of a linear contraction semi-group and satisfies the conditions of section 4.1. Conditions must be placed on the nonlinear function $f(u)$ so theorem 4.2.1 can be applied to the operator $A + f$, and ensure the existence, uniqueness and stability, or asymptotic stability of a solution to (4-2). Pao showed in [23] that if f satisfies (4-4) below, and A satisfies the hypothesis of theorem 4.1.2 then there exists a solution to (4-2) which is stable, or asymptotically stable if $\beta > 0$, where β is the constant in theorem 4.1.2.

f maps all of $L^2(\Omega)$ into $L^2(\Omega)$, where f is continuous from the strong topology of $L^2(\Omega)$ to the weak topology of $L^2(\Omega)$, and f maps all bounded subsets of $L^2(\Omega)$ into bounded sets. Also, there exists a constant $k < \beta$, k can be negative and β is the dissipativity constant in (6-7), such that for every $u,v\varepsilon L^2(\Omega)$ (4-4)

$$(f(u)-f(v),u-v)_o \leq k||u-v||_o^2.$$

5.0. FORMAL PARTIAL DIFFERENTIAL OPERATORS

The linear abstract operator, A, discussed previously in connection with the operational differential equation

$$\frac{du(t)}{dt} + Au(t) = f(u) \qquad (t \geq 0)$$

is the extension of a certain concrete partial differential operator, A(x,D), and the functions that this operator, A, acts on are in a prescribed function space, D(A), characterized by the boundary conditions which these functions satisfy.

In this chapter, we will define more explicitly what is meant by the concrete partial differential operator and discuss the properties that this operator has. The boundary conditions which characterize the function space, D(A), can themselves be considered as linear partial differential operators. Certain restrictions will be placed on these boundary operators and the significance of these restrictions explained. The function space on which the abstract operator is defined is a key to this result and we will discuss this space and some of its properties.

5.1. Elliptic Formal Partial Differential Operators

We will define what is meant by an elliptic formal partial differential operator, and we will place certain restrictions on this operator, that of being strongly elliptic or properly elliptic. We will also discuss some properties of this operator which should give a better insight into these restrictions. Before defining an elliptic partial differential operator, let us denote the following conventional notations: $x = (x_1, x_2, \ldots, x_n)$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ are real vectors in R^n; $|\alpha| = \sum_{i=1}^{n} \alpha_j$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with the components being nonnegative integers; $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} (\frac{\partial}{\partial x_2})^{\alpha_2} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$;

if $|\alpha| = 0$ the operator D^α denotes the identity operator; $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2}, \ldots, \xi_n^{\alpha_n}$ is a real number; $a_\alpha(x)$ is the function $a_{\alpha_1 \alpha_2 \ldots \alpha_n}(x)$.

Let us first define a formal partial differential operator.

Definition 5.1.1. Let the operator

$$A(x,D) = \sum_{|\alpha| \leq \ell} a_\alpha(x) D$$

where ℓ is a positive integer and the coefficients, $a_\alpha(x)$, are infinitely differentiable functions defined on an open set $\Omega \subset R^n$. (The conditions on Ω will be made more explicit later, in (5-9). Then $A(x,D)$ is called a formal partial differential operator of order ℓ. The associated polynomial

$$A_o(x,\xi) = \sum_{|\alpha| = \ell} a_\alpha(x) \xi^\alpha$$

is called the characteristic polynomial, associated with the principal part, $A_o(x,D)$, of $A(x,D)$.

We will now define an elliptic formal partial differential operator.

Definition 5.1.2. Let

$$A(x,D) = \sum_{|\alpha|\leq\ell} a_\alpha(x)D^\alpha \tag{5-1}$$

be a formal partial differential operator of order ℓ, defined in a bounded domain $\Omega \subset R^n$. $A(x,D)$ is said to be elliptic at $x_o \varepsilon \Omega$, if for every nonzero real vector ξ in R^n

$$A_o(x_o,\xi) \neq 0. \tag{5-2}$$

The following theorem justifies the fact that the order of $A(x, D)$ is even, that is, $\ell = 2m$, for some integer m.

Theorem 5.1.1. If $n > 2$, of if $a_\alpha(x)$ are real for the case $n = 2$, then every elliptic operator is of even order.

Proof. Let ξ, $\xi' \varepsilon R^n$ be linearly independent. Consider the polynomial, of the complex variable τ, $A_o(x, \xi + \tau\xi')$. We have

$$A_o(x,\xi+\tau\xi') = \sum_{|\alpha|=\ell} a_\alpha(x)(\xi+\tau\xi')$$

$$= \tau^\ell A_o(x,\xi') + \tau^{\ell-1}A_1(x,\xi,\xi')+\ldots+A\ (x,\xi)$$

where the A_ℓ are polynomials in ξ, ξ'. The equation in τ,

$$A_o(x,\xi+\tau\xi') = 0 \tag{5-3}$$

does not admit real roots. Otherwise, if it were zero for some τ, real and nonzero, then this would imply

$$A_o(x,\eta) = 0, \quad \text{for some } \eta \text{ real and nonzero}$$

contradicting the fact that $A(x,D)$ is elliptic. Note, that if $a_\alpha(x)$

is real we now have the result, since the equation above has no real roots, there must be 2m roots which come in complex conjugate pairs. Therefore, there are m roots with positive imaginary part, and m with negative imaginary part.

Now, equation (5-3) does not admit real roots when ξ' is fixed and ξ runs through J, where

$$J = \{\xi \in R^n \mid \xi \notin \text{line which passes through } \{0\} \text{ and } \xi'\}.$$

Since $A_o(x,\xi')$ is not zero and does not depend on ξ, then the roots of (5-3) depend continuously on $\xi \in J$. Also, the number of zeros of $A_o(x, \xi + \tau\xi')$ is constant, for every $\xi \in J$. Since J is connected, the number of roots of (5-3) with positive imaginary part is constant (say m), for every $\xi \in J$. Similarly, the number of roots with negative imaginary part is constant (ℓ-m), for every $\xi \in J$. The proof of this statement is by contradiction. Assume there exists ξ_1, $\xi_2 \in J$, unequal, such that the number of roots with positive imaginary part of

$$A_o(x, \xi_1 + \tau\xi') = 0$$

is not equal to the number of roots with positive imaginary part of

$$A_o(x, \xi_2 + \tau\xi') = 0.$$

Since the number of zeros is constant, one of the roots, say $\tau_i(\xi_1)$, must change from having positive imaginary part, to one having negative imaginary part, say $\tau_i(\xi_2)$. But, since the roots depend continuously on ξ, then $\tau_i(\xi)$ must pass through a real zero, contradicting the fact there are no real roots. Hence, the assumption is false and we have proved the statement. Now, whenever $\xi \in J$, then $-\xi \in J$, and

$$A_0(x, -\xi + \tau\xi') = (-1)^{\ell} A_0(x, \xi - \tau\xi').$$

Therefore

$\ell - m$ = the number of roots of $A(x, -\xi+\tau\xi')$ with imaginary part < 0

= the number of roots of $A(x, \xi-\tau\xi')$ with imaginary part < 0

= the number of roots of $A(x, \xi+\tau\xi')$ with imaginary part > 0

= m.

The next to last step follows since

$$A_0(x, \xi-\tau\xi') = \prod_{i=1}^{m} (\tau-\tau_i^{+}(\xi)) \prod_{i=m+1}^{\ell} (\tau-\tau_i^{-}(\xi))$$

where $\tau_i^{-}(\xi)$ are the roots with negative imaginary part, and

$$A_0(x,\xi+\tau\xi') = A_0(x,\xi-(-\tau)\xi') = \prod_{i=1}^{m} (-\tau-\tau_i^{+}(\xi)) \prod_{i=m+1}^{\ell} (-\tau-\tau_i^{-}(\xi))$$

$$= \prod_{i=1}^{m} (\tau+\tau_i^{+}(\xi)) \prod_{i=m+1}^{\ell} (\tau+\tau_i^{-}(\xi))$$

where $\tau_i^{-}(\xi)$ are now the roots with positive imaginary part. Hence, we have shown $\ell = 2m$, and the proof is complete.

qed

Remark 5.1.1. This theorem fails for n = 2, as can be seen by considering the operator of Cauchy-Riemann,

$$\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}$$

which is elliptic, but of order 1.

Remark 5.1.2. It suffices to assume ellipticity of the operator when we discuss general properties of solutions to the Dirichlet problem. But for $n = 2$, when we discuss the theory for more general boundary conditions, the property in the proof of theorem 5.1.1, on the number of roots of (5-3), becomes very important.

We will now formally define this root condition.

Definition 5.1.3. The formal partial differential operator

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$$

is said to be properly elliptic (or satisfies the root condition) if

(i) $A(x,D)$ is elliptic, for all points x in Ω, and

(ii) for every $x \in \overline{\Omega}$, and for every ξ, $\xi' \in R^n$, linearly independent, the polynomials $A_o(x, \xi+\tau\xi')$ of the complex variable τ have exactly m roots with positive imaginary part.

Definition 5.1.4. The formal partial differential operator, $A(x,D)$, of order $2m$, is strongly elliptic at $x_o \in \Omega$ if for every non-zero vector $\xi \in R^n$

$$(-1)^m \mathrm{Re}\{A_o(x_o,\xi)\} = (-1)^m \mathrm{Re}\{\sum_{|\alpha|=2m} a_\alpha(x_o)\xi^\alpha\} > 0$$

Definition 5.1.5. $A(x,D)$ is said to be strongly elliptic (elliptic) in Ω, if $A(x,D)$ is strongly elliptic (elliptic) for all x in Ω.

Remark 5.1.3. If we define $\Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2$, then $-\Delta$ is a strongly elliptic operator. Also, $(-1)^k \Delta^k$ are strongly elliptic operators, for every integer $k \geq 1$.

We will now see how these properties of ellipticity, proper ellipticity and strong ellipticity are related.

Theorem 5.1.2. Let $A(x,D)$ be a formal partial differential operator.

(i) If $A(x,D)$ is strongly elliptic, then $A(x,D)$ is properly elliptic.

(ii) If $n \geq 3$, or if $n = 2$ and the coefficients, $a_\alpha(x)$, are real, then $A(x,D)$ is elliptic if and only if $A(x,D)$ is properly elliptic.

Proof. The proof of (i) follows the proof of theorem 5.1.1. (ii) is also a direct consequence of theorem 5.1.1, since if $n \geq 3$, or if $n = 2$ with real coefficients, then any elliptic operator is shown to satisfy the root condition.

qed

Remark 5.1.4. The example seen in remark 5.1.1, shows that there exists elliptic operators which are not properly elliptic. Also, Schechter in [28] showed that if we let

$$A(x,D) = \left(\frac{\partial^4}{\partial x_1^4} + \frac{\partial^4}{\partial x_2^4} - \frac{\partial^4}{\partial x_3^4}\right) + i \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \frac{\partial^2}{\partial x_3^2}$$

then, $A(x,D)$ is properly elliptic but not strongly elliptic.

The above theorem and succeeding remark show that an elliptic operator is also properly elliptic, except for the case $n = 2$ with the operator having complex coefficients. Also, all strongly elliptic operators are properly elliptic.

The following theorem gives a stronger result than theorem 5.1.2, that is, if $A(x,D)$ is strongly elliptic, not only is it properly elliptic but the polynomial $(-1)^m A_o(x, \xi+\tau\xi') + \lambda$, for $\lambda > 0$, satisfies the root condition. The importance of this theorem will be more apparent when we discuss the general boundary conditions in section 5.2.

Theorem 5.1.3. Let

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$$

be a strongly elliptic partial differential operator. Then for every $x \in \Omega$ and for every $\xi, \xi' \in R^n$, linearly independent, the polynomial in the complex variable τ,

$$(-1)^m A_o(x, \xi+\tau\xi') + \lambda \;, \quad \text{with } \lambda > 0$$

has exactly m roots with positive imaginary part.

Proof. We first must show that there are no real zeros, i.e., for any real τ

$$(-1)^m A_o(x, \xi+\tau\xi') + \lambda \neq 0.$$

The proof of this assertion is as follows. Let τ be a real number. Hence, $\xi + \tau\xi'$ is a real vector in R^n. Then

$$(-1)^m A_o(x,\xi+\tau\xi') + \lambda = (-1)^m[\text{Re}A_o(x,\xi+\tau\xi') + i\ \text{Im}A_o(x,\xi+\tau\xi')] + \lambda$$

$$= [(-1)^m \text{Re}A_o(x,\xi+\tau\xi')+\lambda] + i[(-1)^m \text{Im}A_o(x,\xi+\tau\xi')].$$

Then, since $A(x,D)$ is strongly elliptic and $\lambda > 0$

$$(-1)^m \text{Re}[A_o(x,\xi+\tau\xi')] + \lambda > \lambda > 0$$

Therefore our first assertion is proved. The proof now follows directly as the proof in theorem 5.1.1, since the roots, $\tau_i(\xi)$, depend continuously on ξ, and the number of zeros of $(-1)^m A_o(x, \xi+\tau\xi') + \lambda$ is a constant. The proof that the number of roots of $(-1)^m A_o(x, \xi+\tau\xi') + \lambda$ with positive imaginary part is a constant, m, and the number of roots with negative imaginary part is a constant, ℓ-m, utilizes the fact that there are no real zeros.

qed

<u>Remark 5.1.5</u>. In placing the appropriate restrictions on the boundary conditions, see section 5.21, it is necessary that the partial differential operator, $A(x,D)$, satisfies the condition that for every $x \in \overline{\Omega}$, and for every ξ, $\xi' \in R^n$, linearly independent, the polynomial in the complex variable τ,

$$(-1)^m A_o(x, \xi+\tau\xi') + \lambda , \quad \text{with } \lambda > 0$$

has exactly m roots with positive imaginary part. Theorem 5.1.3 shows that if $A(x,D)$ is strongly elliptic we have the desired result. By considering the following example in R^2

$$A(x,D) = \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2$$

it can be seen that $A(x,D)$ is properly elliptic, but does not satisfy the required condition on the polynomial $(-1)^m A_o(x, \xi+\tau\xi') + \lambda$. The proof follows from the definition of the root condition. Indeed, let $\xi = (\xi_1,\xi_2)$ and $\xi' = (\xi_1',\xi_2')$ be perpendicular, or equivalently,

$$\xi_1\xi_1' + \xi_2\xi_2' = 0 \quad \text{where } {\xi_1}^2 + {\xi_2}^2 \neq 0, \text{ and } {\xi_1'}^2 + {\xi_2'}^2 \neq 0.$$

Then

$$\begin{aligned} A_o(x,\xi_1+\tau\xi') &= A_o(\xi_1+\tau\xi_1', \xi_2+\tau\xi_2') \\ &= (\xi_1+\tau\xi_1')^2 + (\xi_2+\tau\xi_2')^2 \\ &= (\xi_1^2+\xi_2^2) + 2\tau(\xi_1\xi_1'+\xi_2\xi_2') + \tau^2({\xi_1'}^2+{\xi_2'}^2) \\ &= (\xi_1^2+\xi_2^2) + \tau^2({\xi_1'}^2+{\xi_2'}^2) \end{aligned}$$

Hence, we can see readily that there is one root with positive imaginary part, or $A(x,D)$ is properly elliptic. But

$$(-1)^m A_o(x, \xi+\tau\xi') + \lambda = -[(\xi_1^2+\xi_2^2) + \tau^2({\xi_1'}^2+{\xi_2'}^2)] + \lambda = 0$$

has 2 equal real roots, namely zero, if we let $\lambda = \xi_1^2 + \xi_2^2 > 0$. Therefore, $(-1)^m A_o(x, \xi+\tau\xi') + \lambda$ fails to satisfy the required root condition. This tells us that proper ellipticity is not sufficient to get the desired result, and we must assume that $A(x,D)$ is strongly elliptic.

Let us now consider the formal partial differential operator

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha.$$

Since the coefficients are in $C^\infty(\bar{\Omega})$, then, as shown in Friedman [11], one can rewrite the operator in the divergence form

$$A(x,D)(\cdot) = \sum_{|\rho|,|\sigma|\leq m} (-1)^{|\rho|} D^{\rho}(a_{\rho\sigma}(x)D^{\sigma}(\cdot)). \tag{5-4}$$

The following result is found in [11].

Theorem 5.1.4. Let A(x,D) be the formal partial differential operator defined in (5-4). Then A(x,D) is strongly elliptic at $x_o \in \overline{\Omega}$ if and only if for every $\xi \in R^n$

$$\mathrm{Re}[\sum_{|\rho|=|\sigma|=m} \xi^{\rho} a_{\rho\sigma}(x_o)\xi^{\sigma}] \geq C_o|\xi|^{2m} = C_o(|\xi|^2)^m, \text{ for some } C_o > 0. \tag{5-5}$$

Remark 5.1.6. Friedman showed in [11] that if A(x,D) is strongly elliptic at every point of Ω, since $a_{\rho\sigma}(x) \in C^{\infty}(\overline{\Omega})$, then C_o can be taken independently of $x \in \overline{\Omega}$.

Now we will give the definition of the formal adjoint of A(x,D).

Definition 5.1.6. Let

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_{\alpha}(x)D^{\alpha}$$

be a formal partial differential operator. Then, the formal adjoint of A(x,D), designated A*(x,D), is given by

$$A^*(x,D)(\cdot) = \sum_{|\alpha|\leq 2m} (-1)^{|\alpha|} D^{\alpha}(\overline{a_{\alpha}(x)}(\cdot)) \tag{5-6}$$

If A(x,D) = A*(x,D), then A(x,D) is said to be formally self-adjoint.

5.2. General Boundary Conditions

In the study of the Dirichlet problem for the elliptic partial differential equation

$$A(x,D)\ u(x) = f(u) \qquad x \in \Omega$$

with boundary conditions

$$B_j(x,D)\ u(x) = \left(\frac{\partial}{\partial n}\right)^j u(x) = 0 \qquad x \in \partial\Omega \quad (0 \leq j \leq m-1)$$

where n is a normal vector at $\partial\Omega$, oriented toward the interior, the following inequality is essential in the solving of the above problem.

<u>Garding's Inequality</u>. If $A(x,D)$ is a strongly elliptic operator of order 2m defined in the domain Ω, then for every $u \in H_o^m(\Omega)$

$$(A(x,D)u,u)_o \geq C_1||u||_m^2 - C_2||u||_o^2 \text{ , for some } C_1 > 0,\ C_2 \geq 0.$$

This inequality holds only for strongly elliptic operators and is well suited for solving the Dirichlet problem, since a function with zero Dirichlet data on $\partial\Omega$ is in $H_o^m(\Omega)$. However, when one tries to extend this method to the study of partial differential equations with more general boundary conditions (for example, the Von Neumann problem) one runs into some difficulty.

(i) First of all, Garding's inequality does not hold for classes of functions outside $H_o^m(\Omega)$, especially if such classes of functions have boundary conditions of order greater than or equal to m (Note that the Dirichlet boundary conditions only have order up to m-1.).

(ii) Also, we know that the assumption of boundary values must be obtained in some way from integration by parts. Hence, in Garding's inequality $(A(x,D)u,u)_o$ must be integrated by parts, and this will yield results only for a small class of boundary conditions.

Martin Schechter in [28], solves the general boundary value problem

$$A(x,D)\ u(x) = f(u) \qquad x \in \Omega$$

with the general boundary conditions

$$B_j(x,D)\ u(x) = 0 \qquad x \in \partial\Omega \qquad (0 \leq j \leq m-1)$$

where the operators satisfy much weaker conditions than that for solving the Dirichlet problem. It is only necessary that A(x,D) be properly elliptic, and $B_j(x,D)$ be a 'normal set' and satisfy the 'complementary condition', which are defined in section 5.21. The sufficiency of Schechter's theorem is proved by utilizing the following inequality which generalizes and corresponds to Garding's inequality.

<u>Lemma 5.21.</u> Suppose $\{B_j\}_{j=0}^{m-1}$, the set of boundary operators, satisfy the complementary condition with respect to a properly elliptic operator, A(x,D), of order 2m. Then, there exists a constant K > 0, such that for any $u \in C^\infty(\bar{\Omega})$

$$||u||_{2m}^2 \leq K(||Au||_o^2 + ||u||_o^2).$$

Also B_j being a 'normal set' is a sufficient condition for us to show that

$$C_B^\infty(\Omega) = \{u \in C^\infty(\overline{\Omega}) \mid B_j u = 0 \text{ on } \partial\Omega \ (0 \leq j \leq m-1\}$$

is dense in $H_B^{2m}(\Omega)$, which is the key function space used in defining the domain, $D(A)$, of the abstract operator extension of $A(x,D)$.

5.21. General Boundary Operators, the Case $n \geq 2$

In this work we are studying the boundary value problem for the elliptic partial differential equation

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)\, u(x,t) = f(u) \qquad x \in \Omega,\ t \geq 0$$

with the boundary conditions

$$B_j(x,D)\, u(x,t) = 0 \qquad x \in \partial\Omega,\ t \geq 0 \quad (0 \leq j \leq m-1)$$

where the B_j are linear, boundary differential operators, independent of time t, f is a nonlinear function defined on the appropriate function space, and Ω is a bounded domain in R^n, $n \geq 2$. It is well known that even in the classical cases, for example, the Laplacian

$$A(x,D) = -\Delta \text{ in } \Omega \quad R^n,$$

we can not arbitrarily define the operators B_j and be sure that the problem will be well posed. Hence, in this section we will give certain conditions of admissability on the operators B_j, with respect to the operator $A(x,D)$. These restrictions will be defined and explained.

Let $B_j(x,D)$ $(0 \leq j \leq m-1)$ be m linear boundary operators defined by

$$B_j(x,D)\phi = \sum_{|h|\leq m_j} b_{jh}(x)D^h\phi \qquad (5\text{-}7)$$

where $b_{jh} \in C^\infty(\partial\Omega)$ and m_j is the order of $B_j(x,D)$. We will designate $B_{jo}(x,D)$ as the principal part of $B_j(x,D)$. Also, it must be noted that $\{B_j\}$ are independent of time t.

Remark 5.21.1. More precisely, $B_j(x,D)$ designates the operator

$$\phi \longrightarrow \sum_{|h|\leq m_j} b_{jh}(x)\gamma_o(D^h\phi)$$

where ϕ is a function defined in $\overline{\Omega}$, so that $\gamma_o(D^h\phi)$ is the trace of $D^h\phi$ on $\partial\Omega$, which can be defined in the classical sense or in the sense of the 'trace theorem', see theorem 3.45.4.

We will now give equivalent definitions of what we mean by a normal system of boundary operators which are found in Friedman in [11], and Lions and Magenes [18].

Definition 5.21.1. Let us consider the set of boundary operators $\{B_j\}_{j=0}^{m-1}$, as defined in (5-7). $\{B_j\}_{j=0}^{m-1}$ is a normal system if the following conditions are satisfied,

(i) $m_i \neq m_j$ for $i \neq j$, where m_j is the order of B_j;

(ii) for every $x' \in \partial\Omega$, if ξ' is the normal to $\partial\Omega$ at x', and ξ is any tangential vector to $\partial\Omega$ at x', then in the polynomial in the complex variable τ

$$B_{jo}(x',\xi + \tau\xi') = C_j(x')\tau^{m_j} + \sum_{k=1}^{m_j} c_{jk}(x')\tau^{m_j-k}$$

the leading coefficient $c_j(x')$ is always not equal to zero.

<u>Definition 5.21.2.</u> The system $\{B_j\}_{j=0}^{m-1}$ defined above is a normal system if

(i) $m_i \neq m_j$, for $i \neq j$;

(ii) for every $x' \in \partial\Omega$, and for every $\xi' \neq 0$, normal to $\partial\Omega$ at x'

$$\sum_{|h|=m_j} b_{jh}(x')\xi'^h \neq 0.$$

The proof that these two definitions are equivalent is seen by rewriting $B_{jo}(x,\xi + \tau\xi')$ in its expanded form, and noting that the highest order coefficients

$$c_j(x) = \sum_{|h|=m_j} b_{jh}(x)\xi'^h.$$

Therefore, since ξ' is normal to $\partial\Omega$ at x, $c_j(x) \neq 0$ if and only if

$$\sum_{|h|=m_j} b_{jh}(x)\xi'^h \neq 0$$

which completes the proof.

Gerd Grebb formulates another equivalent definition of a normal system of boundary operators in [12] which is more aesthetic and should help clarify its meaning. $B_j(x,D)$ can be written in the following equivalent form by a transformation of the variables; see section 5.22,

$$B_j(x,D)u = b_j(x)\left(\frac{\partial}{\partial n}\right)^{m_j} u + \sum_{k=0}^{m_j-1} T_{kj}\left(\frac{\partial}{\partial n}\right)^k u \qquad (5\text{-}8)$$

where $b_j(x) \in C^{\infty}(\partial\Omega)$, and T_{kj} are 'tangential' differential operators, or derivatives in the direction tangent to $\partial\Omega$, of order $\leq m_j - k$. We will now give the equivalent definition.

Definition 5.21.3. The system of boundary operators $\{B_j\}_{j=0}^{m-1}$ as defined in (5-7), is a normal system if

(i) $m_i \neq m_j$ for $i \neq j$;

(ii) the functions $b_j(x)$ have inverses, $b_j^{-1}(x)$, such that $b_j^{-1} \in C^{\infty}(\partial\Omega)$.

where $b_j(x)$ are the functions in (5-8).

Remark 5.21.1. B_j is a normal system means that the orders are distinct, and there are no purely tangential derivatives at the boundary. An example of a normal system of boundary operators is the following, and is found in Friedman [11]. Let μ be a nontangential smoothly varying direction on $\partial\Omega$ and let

$$B_j(x,D) = \left(\frac{\partial}{\partial\mu} \right)^{s+j} + \text{lower order differential boundary operators}$$

for some s, $0 \leq s \leq m$, $(0 \leq j \leq m-1)$. Then B_j forms a normal system. If μ is a purely tangential smoothly varying direction on $\partial\Omega$, then this is an example of a set of boundary operators which is not normal.

We will now define what is meant by the system of boundary operators satisfying the strong complementary condition. First, we will define what is meant by the complementary condition as is found in Agmon [1], Friedman [11], Lions and Magenes [18] and is used by Schechter in [28] to solve the general boundary value problem.

<u>Definition 5.21.4.</u> Let $\{B_j\}_{j=0}^{m-1}$ be a system of boundary operators as defined in (5-7). $\{B_j\}_{j=0}^{m-1}$ is said to satisfy the complementary condition if for any $x \in \partial\Omega$, letting ξ' denote the outward normal to $\partial\Omega$ at x, and $\xi \neq 0$ be any real vector in the tangent hyperplane to $\partial\Omega$ at x, then the polynomials in τ

$$B_{jo}(x,\xi + \tau\xi') = \sum_{|h|=m_j} b_{jh}(x)(\xi+\tau\xi')^h \qquad (0 \leq j \leq m-1)$$

are linearly independent modulo the polynomial

$$M(\xi,\tau) = \Pi_{k=1}^{m} (\tau-\tau_k^+(\xi))$$

where $\tau_k^+(\xi)$ are the roots of $A_o(x, \xi+\tau\xi')$ with positive imaginary part (where we assume $A_o(x, \xi+\tau\xi')$ has m roots with positive imaginary part).

We will now give a stronger condition than the one above on the boundary operators, B_j, which is found in Agmon [1] and Friedman [11] and is necessary when we have to show $R(\lambda I-(-A)) = H$, where A is the abstract operator extension of A(x,D). It should be noted here that Friedman allows for a larger class of values for λ in definition 5.21.5 below, that is $\arg(\lambda) = \theta$ $(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$, but for our results it suffices to consider the definition for $\theta = 0$, or $\arg(\lambda) = 0$.

<u>Definition 5.21.5.</u> The system of boundary operators $\{B_j\}_{j=0}^{m-1}$ as defined in definition 5.21.4 is said to satisfy the strong complementary condition if we let $x \in \partial\Omega$, ξ', ξ be as in definition 5.21.4; then for any $\lambda > 0$, the polynomials in τ,

$$B_{jo}(x,\xi + \tau\xi') = \sum_{|h|=m_j} b_{jh}(x)(\xi+\tau\xi')^h \qquad (0 \leq j \leq m-1)$$

are linearly independent modulo the polynomial

$$M^*(\xi,\lambda,\tau) = \Pi_{k=1}^{m} \ (\tau-\tau_k^*(\xi,\lambda))$$

where $\tau_k^*(\xi,\lambda)$ are the m roots of $(-1)^m A_o(x,\xi + \tau\xi') + \lambda$ with positive imaginary part (where we assume $(-1)^m A_o(x,\xi + \tau\xi') + \lambda$ has m roots with positive imaginary part).

As we can see now, if we assume A(x,D) is strongly elliptic, then from theorem 5.1.3, A(x,D) satisfies the condition that $(-1)^m A_o(x,\xi + \tau\xi') + \lambda$ has m roots with positive imaginary part. To help clarify the definition of the complementary condition we will define what we mean by a finite number of polynomials being linearly independent modulo another polynomial.

<u>Definition 5.21.6.</u> Let $\{P_o(\tau), P_1(\tau), \ldots, P_{m-1}(\tau)\}$ be m polynomials in τ. We say the system $\{P_j(\tau)\}_{j=0}^{m-1}$ is linearly independent modulo the polynomial $Q(\tau)$ if the system of remainders $\{r_j(\tau)\}_{j=0}^{m-1}$ is linearly independent, where $r_j(\tau)$ is the polynomial remainder after dividing $P_j(\tau)$ by $Q(\tau)$, or

$$P_j(\tau) = S_j(\tau)Q(\tau) + r_j(\tau)$$

and the orders of $r_j(\tau)$ are all less than the order of $Q(\tau)$.

From this definition, we can see that if all the orders of $\{P_j(\tau)\}_{j=0}^{m-1}$ are less than the order of $Q(\tau)$ then $r_j(\tau) \equiv P_j(\tau)$, and this definition reduces to that of linear independence. This shows that our definition of the complementary condition generalizes linear independence of the system of polynomials $\{B_j(x,\xi + \tau\xi')\}_{j=0}^{m-1}$, in that if the orders of $B_j(x,\xi + \tau\xi')$ are less than m, which is the order of $M(\xi,\tau) = \Pi_{k=1}^{m}(\tau-\tau_k^+(\xi))$, then $\{B_j\}_{j=0}^{m-1}$ satisfies the complementary condition if and only if $\{B_j(x,\xi+\tau\xi')\}_{j=0}^{m-1}$ is linearly independent. An example of this is the Dirichlet boundary conditions, where $B_j(x,D) = (\frac{\partial}{\partial n})^j$ and the orders $m_j < m$. Therefore, as we will see in example 5.21.1, these boundary operators are linearly independent and, therefore, satisfy the complementary condition.

In addition to the above assumptions on $B_j(x,D)$, we will assume that the orders of $B_j(x,D)$, $m_j \leq 2m-1$, or equivalently, the orders of $B_j(x,D)$ < the order of $A(x,D)$.

<u>Example 5.21.1</u>. Among the system of boundary operators which satisfy the conditions of this section, see (5-9), with respect to any properly elliptic operator $A(x,D)$ is the Dirichlet system

$$B_j(x,D) = (\frac{\partial}{\partial n})^j .$$

<u>Proof</u>. $(\frac{\partial}{\partial n})^j {}_{j=0}^{m-1}$ are m in number, the coefficients are in $C^\infty(\partial\Omega)$, and $m_j = j < m \leq 2m-1$. The system is normal on $\partial\Omega$. Indeed, the orders, $m_j = j$, are distinct. Also, these boundary operators are of the form (5-8), with no tangential derivatives. In other words, the highest ordered term is purely normal, and therefore, satisfies definition 5.21.3. This proves the system is normal on $\partial\Omega$.
$\{(\frac{\partial}{\partial n})^j\}_{j=0}^{m-1}$ satisfies the strong complementary condition. Indeed, after a transformation of variables, the boundary operators are of the form (see section 5.22)

$$B_j'(x,D) = (\frac{\partial}{\partial x_n'})^j$$

where the normal, n, to the boundary of x is transformed into the

x_n'-axis. Then, it can readily be seen that

$$B_j'(x,\xi + \tau\xi') = B_j'(x,(\xi_1,\ldots,\xi_{n-1},\tau)$$
$$= \tau^j.$$

Since, the order of $M(\xi,\tau)$ is m, and the orders of $B_j'(x,\xi + \tau\xi') < m$, then the remainders $r_j(\tau) = \tau^j$. Hence, the system $\{\tau^\circ,\tau,\ldots,\tau^{m-1}\}$ is linearly independent, and satisfies the strong complementary condition.

Martin Schechter in [28] gives the following example of a normal system of boundary operators which does not satisfy the strong complementary condition with respect to a properly elliptic operator, A(x,D). Let A(x,D) be the fourth order operator corresponding to the characteristic polynomial

$$A_o(\xi_1,\xi_2) = [\xi_2^2 + 2\xi_1\xi_2 + (1+\varepsilon^2)\xi_1^2][\xi_2^2 - (1+\varepsilon\sqrt{3})\xi_1\xi_2 + (1+\varepsilon^2)\xi_1^2]$$

in two dimensions, where $0 < \varepsilon < \sqrt{3}$. Let the boundary operators be defined as follows,

$$B_o(x,D) = 1, \quad B_1(x,D) = \left(\frac{\partial}{\partial n} \right)^3$$

We will summarize below the hypothesis on the system $(A(x,D),\{B_j\},\Omega)$:

(i) Ω is a bounded domain in R^n, $n \geq 2$, with boundary, $\partial\Omega \in C^\infty$, such that Ω is locally on one side of $\partial\Omega$ (see section 3.43).

(ii) The operator $A(x,D)$ is strongly elliptic in $\overline{\Omega}$, has coefficients $a_\alpha(x) \in C^\infty(\overline{\Omega})$, and is of order $2m$. (5-9)

(iii) The operators $B_j(x,D)$ are m in number, with coefficients $b_{jh}(x) \in C^\infty(\partial\Omega)$, and are independent of time t.

(iv) $\{B_j\}_{j=0}^{m-1}$ is a normal system on $\partial\Omega$ and satisfies the strong complementary condition.

(v) The order of $B_j(x,D)$, $m_j \leq 2m-1$ $(0 \leq j \leq m-1)$.

5.22. Equivalent Definitions

In order to verify the strong complementary condition for the system $(A(x,D),\{B_j\},\Omega)$, we utilize definition 5.21.5, which is quite cumbersome and difficult to use in its present form since the vectors in the definition, $\xi = (\xi_1,\ldots,\xi_n)$ and $\xi' = (\xi_1',\ldots,\xi_n')$, are not written explicitly, which does not allow for a simple verification of linear independence. In this section, we will give equivalent definitions of the strong complementary condition, and show the more simple procedure used in verifying the condition. See Lions and Magenes in [18].

Since the boundary is of class C^∞, by 'local charts' and 'partition of unity' (see section 3.44), there exists a finite covering of $\partial\Omega$ by open sets, U_i $(1 \leq i \leq N)$, such that for every integer i, there exists a diffeomorphism, an infinitely differentiable mapping θ_i, defined as follows,

$$\theta_i \text{ maps } U_i \text{ onto the sphere } \sigma = \{x' \in R^n \mid |x'| < 1\}$$

in such a way that the image

$$\theta_i(U_i \cap \Omega) = \sigma_+ = \{x' \in R^n \mid x' \in \sigma,\ x_n' > 0\}$$

$$\theta_i(U_i \cap \partial\Omega) = \partial_1\sigma_+ = \{x' \in R^n \mid x' \in \sigma,\ x_n' = 0\}.$$

See figure 3.4.

Now, for any point $x_o \in \partial\Omega$, we know that there exists a fixed integer i, $1 \leq i \leq N$, such that $x_o \in U_i$, and we can define the diffeomorphism θ_i, which depends on x_o, such that x_o is transformed into $x_o' = (0,\ldots,0)$, and for any $x \in \partial\Omega \cap U_i$ the operator $A(x,D)$ is transformed into the operator $\mathcal{A}(x',D)$ defined by

$$\mathcal{A}(x',D) = \sum_{|\alpha|\leq 2m} a_\alpha'(x')\left(\frac{\partial}{\partial x_1'}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n'}\right)^{\alpha_n} \tag{5-10}$$

where $\alpha = (\alpha_1,\ldots,\alpha_n)$ with nonnegative integer components, $x' = (x_1',\ldots,x_n') \in R^n$, and the coefficients are infinitely differentiable in $\sigma_+ \cup \partial_1\sigma_+$. Also, the system $\{B_j\}_{j=0}^{m-1}$ is transformed into a system of operators $\{\mathcal{B}_j\}_{j=0}^{m-1}$ defined by

$$\mathcal{B}_j(x',D) = \sum_{|h|\leq m_j} b_{jh}'(x')\left(\frac{\partial}{\partial x_1'}\right)^{h_1}\cdots\left(\frac{\partial}{\partial x_n'}\right)^{h_n} \tag{5-11}$$

where $h = (h_1,\ldots,h_n) \in R^n$, with nonnegative integer components, and with infinitely differentiable coefficients in $\partial_1\sigma_+$.

Remark 5.22.1. From Lions and Magenes [18] and Peetre [26], it is shown, since θ_i is a diffeomorphism, that the property of $A(x,D)$ being properly elliptic, and $\{B_j\}_{j=0}^{m-1}$ being a normal system and satisfying the strong complementary condition are invariant under these transformations.

In considering the properties mentioned above, we need only consider the principal parts of $(A(x,D),\{B_j\})$. Hence, after the transformation θ_i, $A(x,D)$ and $B_j(x,D)$ are transformed into the operators $\mathcal{A}(x',D)$ and $\mathcal{B}_j(x',D)$ with principal parts

$$\mathcal{A}_o(x',D) = \sum_{|\alpha|=2m} a'_\alpha(x')\left(\frac{\partial}{\partial x'_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x'_n}\right)^{\alpha_n} \tag{5-12}$$

$$\mathcal{B}_{jo}(x',D) = \sum_{|h|=m_j} b'_{jh}(x')\left(\frac{\partial}{\partial x'_1}\right)^{h_1}\cdots\left(\frac{\partial}{\partial x'_n}\right)^{h_n} \tag{5-13}$$

where $\mathcal{A}_o(x',D)$ and $\mathcal{B}_{jo}(x',D)$ depend on θ_i which in turn depends on $x \in \partial\Omega$. From remark 5.22.1, we have the following equivalent definition of definition 5.21.5.

Definition 5.22.1. Let $\{B_j\}_{j=0}^{m-1}$ be a system of boundary operators as defined in (5-7), and $A(x,D)$ a strongly elliptic operator in $\bar{\Omega}$. $\{B_j\}_{j=0}^{m-1}$ satisfies the strong complementary condition if for every $x \in \partial\Omega$, and for every $\lambda > 0$, after the transformation by θ_i, which depends on x, letting $\xi' = (0,\ldots,0,1)$ denote the outward normal to the boundary $\partial\Omega$ at $x' = (0,\ldots,0)$, and $\xi = (\xi_1,\ldots,\xi_{n-1},0) \neq (0,\ldots,0)$ be a real vector in the tangent hyperplane, $x_n' = 0$, at $x' = (0,\ldots,0)$, then the polynomials in the complex variable τ

$$\mathcal{B}_{j,0}(0,\xi+\tau\xi') = \mathcal{B}_{j,0}(0,(\xi_1,\ldots,\xi_{n-1},\tau)) = \sum_{|h|=m_j} b'_{jh}(0)\xi_1^{h_1}\cdots\xi_{n-1}^{h_{n-1}}\tau^{h_n}$$

$$(0 \leq j \leq m-1)$$

are linearly independent modulo the polynomial $M'^*(\xi,\lambda) = \Pi_{k=1}^{m}(\tau-\tau_k'^*(\xi,\lambda))$ where $\tau_k'^*(\xi,\lambda)$ are the m roots with positive imaginary part of the polynomial

$$(-1)^m\mathcal{A}_o(0,\xi+\tau\xi') + \lambda = (-1)^m\mathcal{A}_o(0,(\xi_1,\ldots,\xi_{n-1},\tau)) + \lambda$$

$$= (-1)^m\sum_{|\alpha|=2m} a'_\alpha(0)\xi_1^{\alpha_1}\cdots\xi_{n-1}^{\alpha_{n-1}}\tau^n + \lambda$$

Since $x \in \partial\Omega$ is arbitrary, this equivalent definition is the usual way the strong complementary condition is verified this can be seen, since $\mathcal{B}_{j,o}$ and $\mathcal{A}_o$ are more explicitly written in this definition than in definition 5.21.5 where ξ and ξ' are not concretely defined. In order to use the equivalent definition we must perform the transformation, θ_i, for each $x \in \partial\Omega$. We will now define this transformation.

Pick $X \in \partial\Omega$, arbitrary and fix it. θ_i can be written as a product of 3 transformations. Refer to Schechter [28] for a discussion on this transformation. The first two steps are a rigid motion which takes X into the origin and the tangent hyperplane into $x_n = 0$, by first translating X so that it coincides with the origin, and then rotating the axis so the new x_n'-axis coincides with the normal to at $X' = (0,\ldots,0)$, and the tangent hyperplane at $X' = (0,\ldots,0)$ coincides with the plane, $x_n = 0$. The third step is a slight twisting of the surface so the boundary, $\partial\Omega \cap U_i$, coincides with a portion of the plane $x_n' = 0$. See figure 3.4.

Let us define

$$\theta_i = \theta_{i_3} \circ \theta_{i_2} \circ \theta_{i_1}$$

where θ_{i_1}, θ_{i_2}, θ_{i_3} are defined as follows, by letting $x = (x_1,\ldots,x_n)$

(i) translation, θ_{i_1}:

$$x_h' = x_h - X_h \qquad \text{where } 1 \leq h \leq n \qquad (5\text{-}14)$$

(ii) rotation, θ_{i_2}:

$$x_h'' = \sum_{k=1}^{n} a_{hk} x_k \qquad \text{where } 1 \leq h \leq n \qquad (5\text{-}15)$$

and the coefficients satisfy

$$\sum_{k=1}^{n} a_{hk}^2 = 1 \quad (1 \leq h \leq n), \ \sum_{k=1}^{n} a_{hk} a_{\ell k} = 0, \quad \text{where} \quad h \neq \ell$$

(iii) θ_{i_3}:

Let

$$x_n'' = \psi(x_1'',\ldots,x_{n-1}'')$$

define the surface $\partial(\Omega)'' \cap U_i''$, where $0 = \psi(0,\ldots,0)$, and ψ is of class C^∞. Then θ_{i_3} is defined as follows

$$x_h''' = x_h'' \qquad (1 \le h \le n-1)$$
$$(5\text{-}16)$$
$$x_n''' = x_n'' - \psi(x_1'',\ldots,x_{n-1}'')$$

This transformation, θ_i, is a diffeomorphism which takes $X \in \partial\Omega$ into the origin and slightly deforms $\partial\Omega \cap U_i$ so it coincides with a portion of the tangent hyperplane, $x_n' = 0$, at $X' = (0,\ldots,0)$.

After the above transformation, $\theta_i = \theta_{i_3} \circ \theta_{i_2} \circ \theta_{i_1}$, we transform $(A(x,D),\{B_j\})$ into $(\mathcal{A}(x',D),\{\mathcal{B}_j\}_{j=0}^{m-1})$ and verify definition 5.22.1.

Definition 5.22.1 is the usual procedure used to verify the strong complementary condition. But the problem occurs that the transformation, θ_i, depends on $x \in \partial\Omega$, and we must verify the strong complementary condition for every $x \in \partial\Omega$. However, there are certain conditions which will allow us to say that if we can verify the strong complementary condition for only one point $x \in \partial\Omega$, then the strong complementary condition will hold for every point on the boundary. The conditions needed are that the coefficients $a_\alpha(x)$, $b_{jh}(x)$ be constant and the system $(A(x,D),\{B_j\})$ be invariant with respect to θ_{i_2}. Hence, with these conditions on the system we will be able to give a more easily verifiable definition of the strong complementary condition. Let us now assume that the coefficients $a_\alpha(x)$ and $b_{jh}(x)$ are constants, a_α, b_{jh} and we define what is meant by $(A(x,D),\{B_j\})$ being invariant with respect to θ_{i_2}.

<u>Definition 5.22.2</u>. Let θ_{i_2} be the transformation defined by (5-15). We say the system $(A(x,D),\{B_j\})$ is invariant with respect to θ_{i_2}, if for every $x \in \partial\Omega$, after transformation by θ_{i_2}, the system of

operators

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

(5-17)

$$B_j(x,D) = \sum_{|h|\leq m_j} b_{jh} \left(\frac{\partial}{\partial x_1}\right)^{h_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{h_n} \quad (0 \leq j \leq m-1)$$

is transformed into the system $(\mathcal{A}(x',D),\{\mathcal{B}_j\})$ with principal parts

$$\mathcal{A}_o(x',D) = \sum_{|\alpha|=2m} a_\alpha \left(\frac{\partial}{\partial x_1'}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n'}\right)^{\alpha_n}$$

(5-18)

$$\mathcal{B}_{jo}(x',D) = \sum_{|h|=m_j} b_{jh} \left(\frac{\partial}{\partial x_1'}\right)^{h_1} \cdots \left(\frac{\partial}{\partial x_n'}\right)^{h_n}$$

Before giving the equivalent definition of the strong complementary condition, we will discuss more fully the transformation $\theta_i = \theta_{i_3} \cdot \theta_{i_2} \cdot \theta_{i_1}$ and show that under the conditions of definition 5.22.2 we have invariance of the system $(A(x,D),\{B_j\})$ with respect to θ_i, in the sense that after the transformation θ_i, the sytem $(A(x,D),\{B_j\})$ defined in (5-17) is transformed into the system $(\mathcal{A}(x,D),\{\mathcal{B}_j\})$ with principal parts as defined in (5-18), that is, the principal parts remain invariant after θ_i.

First, we will see from the following example that, in general, the system is not invariant with respect to the transformation θ_{i_2}.

Example 5.22.1. Let us consider the following formal partial differential operator

$$A(x,D) = a_1\left(\frac{\partial}{\partial x_1}\right)^2 + a_2\left(\frac{\partial}{\partial x_2}\right)^2, \quad a_1 \neq a_2 \neq 0. \tag{5-19}$$

Let $X \in \partial\Omega$, be arbitrary and fixed, and define the transformation θ_{i_2}.

$$x_1' = \frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2$$

$$x_2' = -\frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2$$

which clearly satisfies the conditions of orthogonality. From the chain rule for partial differentiation we obtain

$$\left(\frac{\partial}{\partial x_1}\right)^2 = \frac{1}{2}\left(\frac{\partial}{\partial x_1'}\right)^2 - \frac{\partial^2}{\partial x_1' \partial x_2'} + \frac{1}{2}\left(\frac{\partial}{\partial x_2'}\right)^2$$

$$\left(\frac{\partial}{\partial x_2}\right)^2 = \frac{1}{2}\left(\frac{\partial}{\partial x_1'}\right)^2 + \frac{\partial^2}{\partial x_1' \partial x_2'} + \frac{1}{2}\left(\frac{\partial}{\partial x_2'}\right)^2$$

and after the transformation θ_{i_2}, $A(x,D)$ becomes,

$$A(x,D) = a_1\left(\frac{\partial}{\partial x_1}\right)^2 + a_2\left(\frac{\partial}{\partial x_2}\right)^2 = \left(\frac{a_1+a_2}{2}\right)\left(\frac{\partial}{\partial x_1'}\right)^2 + (a_2-a_1)\frac{\partial^2}{\partial x_1' \partial x_2'} + \left(\frac{a_1+a_2}{2}\right)\left(\frac{\partial}{\partial x_2'}\right)^2.$$

Since $a_1 \neq a_2 \neq 0$, we readily see, in general, that a system $(A(x,D),\{B_j\})$ is not invariant with respect to θ_{i_2}.

For an example of a system which is invariant with respect to θ_{i_2}, see example 5.22.2.

Now, let us consider the system $(A(x,D),\{B_j\})$ defined in (5-17) with constant coefficients a_α, b_{jh} and the system being invariant under θ_{i_2}. We show that the system is also invariant under the transformation $\theta_i = \theta_{i_3} \circ \theta_{i_2} \circ \theta_{i_1}$.

We need the following lemma.

<u>Lemma 5.22.1</u>. Pick the point $X \in \partial\Omega$, arbitrary, and let θ_i be the transformation defined by (5-14), (5-15) and (5-16). Let us assume we have already transformed $X \in \partial\Omega$ by $\theta_{i_2} \circ \theta_{i_1}$ such that $X = (0,\ldots,0)$ and the tangent hyperplane to $\partial\Omega$ at X is $x_n = 0$. Assume that the system $(A(x,D),\{B_j\})$ has constant coefficients. Then, for every nonnegative integer $m \leq$ the order of $A(x,D)$, after the transformation θ_{i_3} defined in (5-16) we obtain

$$\frac{\partial^m}{\partial x_{j_1}\cdots\partial x_{j_m}} = \frac{\partial^m}{\partial x'_{j_1}\cdots\partial x'_{j_m}} + \sum_{\substack{i(1)=1\\ j_i\neq j_{i(1)}}}^{m} \left(-\frac{\partial\psi}{\partial x_{j_{i(1)}}}\right)\frac{\partial^m}{\partial x'_{j_1}\cdots\partial x'_{j_m}\partial x'_n}$$

$$+ \sum_{\substack{i(1)=1\\ i(1)<i(2)\\ j_i\neq j_{i(1)}\neq j_{i(2)}}}^{m-1} \left(-\frac{\partial\psi}{\partial x_{j_{i(1)}}}\right)\left(-\frac{\partial\psi}{\partial x_{j_{i(2)}}}\right)\frac{\partial^m}{\partial x'_{j_1}\cdots\partial x'_{j_m}\partial x'^2_n} + \cdots$$

$$+ \left(-\frac{\partial\psi}{\partial x_{j_1}}\right)\cdots\left(-\frac{\partial\psi}{\partial x_{j_m}}\right)\frac{\partial^m}{\partial x'^m_n} + \text{(lower order terms)}.$$

<u>Proof</u>. We use the following facts, since $X = (0,\ldots,0)$ and the tangent hyperplane is $x_n = 0$,

$$\frac{\partial\psi}{\partial x_n} = 0 \qquad \text{and} \qquad (1 \leq j_i \leq n).$$

This lemma is proved by induction on m. For the case $m = 1$, we have from the chain rule

$$\frac{\partial}{\partial x_{j_1}} = \sum_{h=1}^{n} \left(\frac{\partial x'_h}{\partial x_{j_1}} \right) \frac{\partial}{\partial x'_h} = \frac{\partial}{\partial x'_{j_1}} + \left(- \frac{\partial \psi}{\partial x_{j_1}}\right) \frac{\partial}{\partial x'_n} .$$

Assume the equation is true for the case k = m, and prove true for m + 1

$$\frac{\partial^{m+1}}{\partial x_{j_{m+1}} \partial x_{j_1} \ldots \partial x_{j_m}} = \frac{\partial}{\partial x_{j_{m+1}}} \left(\frac{\partial^m}{\partial x_{j_1} \ldots \partial x_{j_m}} \right)$$

$$= \frac{\partial}{\partial x'_1} \left(\frac{\partial^m}{\partial x_{j_1} \ldots \partial x_{j_m}} \right) \frac{\partial x'_1}{\partial x_{j_{m+1}}} + \ldots + \frac{\partial}{\partial x'_n} \left(\frac{\partial^m}{\partial x_{j_1} \ldots \partial x_{j_m}} \right) \frac{\partial x'_n}{\partial x_{j_{m+1}}}$$

$$= \frac{\partial}{\partial x'_{j_{m+1}}} \left(\frac{\partial^m}{\partial x_{j_1} \ldots \partial x_{j_m}} \right) + \left(- \frac{\partial \psi}{\partial x_{j_{m+1}}} \right) \frac{\partial}{\partial x'_n} \left(\frac{\partial^m}{\partial x_{j_1} \ldots \partial x_{j_m}} \right)$$

$$= \frac{\partial}{\partial x'_{j_{m+1}}} \Bigg[\frac{\partial^m}{\partial x'_{j_1} \ldots \partial x'_{j_m}} + \sum_{\substack{i(1)=1 \\ j_i \neq j_{i(1)}}}^{m} \left(- \frac{\partial \psi}{\partial x_{j_{i(1)}}}\right) \frac{\partial^m}{\partial x_{j_1} \ldots \partial x'_{j_m} \partial x'_n}$$

$$+ \sum_{\substack{i(1)=1 \\ i(1)<i(2) \\ j_i \neq j_{i(1)} \neq j_{i(2)}}}^{m-1} \left(- \frac{\partial \psi}{\partial x_{j_{i(1)}}}\right) \left(- \frac{\partial \psi}{\partial x_{j_{i(2)}}}\right) \frac{\partial^m}{\partial x'_{j_1} \ldots \partial x'_{j_m} \partial x'^2_n} + \ldots$$

$$+ \left(- \frac{\partial \psi}{\partial x_{j_1}}\right) \ldots \left(- \frac{\partial \psi}{\partial x_{j_m}}\right) \frac{\partial^m}{\partial x'^m_n} \Bigg]$$

$$+ \left(- \frac{\partial \psi}{\partial x_{j_{m+1}}}\right) \frac{\partial}{\partial x'_n} \Bigg[\frac{\partial^m}{\partial x'_{j_1} \ldots \partial x'_{j_m}} + \sum_{\substack{i(1)=1 \\ j_i \neq j_{i(1)}}}^{m} \left(- \frac{\partial \psi}{\partial x_{j_{i(1)}}}\right) \frac{\partial^m}{\partial x'_{j_1} \ldots \partial x'_{j_m} \partial x'_n}$$

$$+ \sum_{\substack{i(1)=1 \\ i(1)<i(2) \\ j_i \neq j_{i(1)} \neq j_{i(2)}}}^{m-1} \left(-\frac{\partial \psi}{\partial x_{j_{i(1)}}}\right)\left(-\frac{\partial \psi}{\partial x_{j_{i(2)}}}\right) \frac{\partial^m}{\partial x'_{j_1} \ldots \partial x'_{j_m} \partial x'^2_n} + \ldots$$

$$+ \left(-\frac{\partial \psi}{\partial x_{j_1}}\right) \ldots \left(-\frac{\partial \psi}{\partial x_{j_m}}\right) \frac{\partial^m}{\partial x'^m_n}] + \text{(lower crder terms)}$$

$$= \frac{\partial^{m+1}}{\partial x'_{j_1} \ldots \partial x'_{j_m} \partial x'_{j_{m+1}}} + \sum_{\substack{i(1)=1 \\ j_i \neq j_{i(1)}}}^{m} \left(-\frac{\partial \psi}{\partial x_{j_{i(1)}}}\right) \frac{\partial^{m+1}}{\partial x'_{j_1} \ldots \partial x'_{j_{m+1}} \partial x'_n}$$

$$+ \sum_{\substack{i(1)=1 \\ i(1)<i(2) \\ j_i \neq j_{i(1)} \neq j_{i(2)}}}^{m-1} \left(-\frac{\partial \psi}{\partial x_{j_{i(1)}}}\right)\left(-\frac{\partial \psi}{\partial x_{j_{i(2)}}}\right) \frac{\partial^{m+1}}{\partial x'_{j_1} \ldots \partial x'_{j_{m+1}} \partial x'^2_n} + \ldots$$

$$+ \left(-\frac{\partial \psi}{\partial x_{j_1}}\right) \ldots \left(-\frac{\partial \psi}{\partial x_{j_m}}\right) \frac{\partial^{m+1}}{\partial x'_{j_{m+1}} \partial x'^m_n} + \left(-\frac{\partial \psi}{\partial x_{j_{m+1}}}\right) \frac{\partial^{m+1}}{\partial x'_{j_1} \ldots \partial x'_{j_m} \partial x'_n}$$

$$+ \sum_{\substack{i(1)=1 \\ j_i \neq j_{i(1)}}}^{m} \left(-\frac{\partial \psi}{\partial x_{j_{i(1)}}}\right)\left(-\frac{\partial \psi}{\partial x_{j_{m+1}}}\right) \frac{\partial^{m+1}}{\partial x'_{j_1} \ldots \partial x'_{j_m} \partial x'^2_n}$$

$$+ \sum_{\substack{i(1)=1 \\ i(1)<i(2) \\ j_i \neq j_{i(1)} \neq j_{i(2)}}}^{m-1} \left(-\frac{\partial \psi}{\partial x_{j_{i(1)}}}\right)\left(-\frac{\partial \psi}{\partial x_{j_{i(2)}}}\right)\left(-\frac{\partial \psi}{\partial x_{j_{m+1}}}\right) \frac{\partial^{m+1}}{\partial x'_{j_1} \ldots \partial x'_{j_m} \partial x'^3_n}$$

$$+ \ldots + \left(-\frac{\partial \psi}{\partial x_{j_1}}\right) \ldots \left(-\frac{\partial \psi}{\partial x_{j_{m+1}}}\right) \frac{\partial^{m+1}}{\partial x'^{m+1}_n} + \text{(lower order terms)}$$

$$= \frac{\partial^{m+1}}{\partial x'_{j_1} \cdots \partial x'_{j_m} \partial x'_{j_{m+1}}} + \sum_{\substack{i(1)=1 \\ j_i \neq j_{i(1)}}}^{m+1} \left(- \frac{\partial \psi}{\partial x_{j_{i(1)}}}\right) \frac{\partial^{m+1}}{\partial x'_{j_1} \cdots \partial x'_{j_{m+1}} \partial x'_n}$$

$$+ \sum_{\substack{i(1)=1 \\ i(1)<i(2) \\ j_i \neq j_{i(1)} \neq j_{i(2)}}}^{m} \left(- \frac{\partial \psi}{\partial x_{j_{i(1)}}}\right)\left(- \frac{\partial \psi}{\partial x_{j_{i(2)}}}\right) \frac{\partial^{m+1}}{\partial x'_{j_1} \cdots \partial x'_{j_{m+1}} \partial x'^2_n} + \cdots$$

$$+ \left(- \frac{\partial \psi}{\partial x_{j_1}}\right) \cdots \left(- \frac{\partial \psi}{\partial x_{j_m}}\right) \frac{\partial^{m+1}}{\partial x'_{j_{m+1}} \partial x'^m_n} + \left(- \frac{\partial \psi}{\partial x_{j_1}}\right) \cdots$$

$$\left(- \frac{\partial \psi}{\partial x_{j_{m+1}}}\right) \frac{\partial^{m+1}}{\partial x'^{m+1}_n} + \text{(lower order terms).}$$

Hence, we have proved the result by induction.

qed

We are ready to prove the following theorem which shows us the invariance of the system $(A(x,D),\{B_j\})$ with respect to θ_i.

<u>Theorem 5.22.1</u>. Let us consider the system $(A(x,D),\{B_j\})$ such that for any $X \in \partial\Omega$,

$$A_o(X,D) = \sum_{|\alpha|=2m} a_\alpha \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n}$$

$$B_{jo}(X,D) = \sum_{|h|=m_j} b_{jh} \left(\frac{\partial}{\partial x_1} \right)^{h_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{h_n}$$

with constant coefficients a_α, b_{jh}. If θ_i is the transformation defined in (5-14), (5-15) and (5-16) such that the system is invariant with respect to θ_{i_2}, then the transformed system under θ_i is $(\mathcal{A}(x,D),$ $\{\mathcal{B}_j\})$ with principal parts

$$\mathcal{A}_o(0,D) = \sum_{|\alpha|=2m} a_\alpha \left(\frac{\partial}{\partial x_1'}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n'}\right)^{\alpha_n}$$

$$\mathcal{B}_{j,o}(0,D) = \sum_{|h|=m_j} b_{jh} \left(\frac{\partial}{\partial x_1'}\right)^{h_1} \cdots \left(\frac{\partial}{\partial x_n'}\right)^{h_n}$$

or, equivalently, the system $(A(x,D),\{B_j\})$ is invariant with respect to θ_i.

Proof. Since, after the transformation $\theta_{i_2} \cdot \theta_{i_1}$, at $X'' = (0,\ldots,0)$, $x_n'' = 0$ is the tangent hyperplane, then $\frac{\partial\psi}{\partial x_k''}(0,\ldots,0) = 0$ for $1 \leq k \leq n$. Hence, from lemma 5.22.1, we have at $X'' = (0,\ldots,0)$

$$\frac{\partial^m}{\partial x_{j_1} \cdots \partial x_{j_m}} = \frac{\partial^m}{\partial x_{j_1}' \cdots \partial x_{j_m}'} + \text{(lower order terms)}.$$

We can then see that, for any $X \in \partial\Omega$, after the transformation θ_i,

$$A_o(X,D) = \sum_{|\alpha|=2m} a_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \sum_{|\alpha|=2m} a_\alpha \left(\frac{\partial}{\partial x_1'}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n'}\right)^{\alpha_n}$$

$$+ \text{(lower order terms)}$$

$$B_{jo}(X,D) = \sum_{|h|=m_j} b_{jh} \left(\frac{\partial}{\partial x_1}\right)^{h_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{h_n} = \sum_{|h|=m_j} b_{jh} \left(\frac{\partial}{\partial x_1'}\right)^{h_1} \cdots \left(\frac{\partial}{\partial x_n'}\right)^{h_n}$$

$$+ \text{(lower order terms)}.$$

Therefore, since we can ignore the lower order terms, the transformed system $(\mathcal{A}(x,D),\{\mathcal{B}_j\})$ has the principal parts,

$$\mathcal{A}_o(0,D) = \sum_{|\alpha|=2m} a_\alpha \left(\frac{\partial}{\partial x_1'}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n'}\right)^{\alpha_n}$$

$$\mathcal{B}_{jo}(0,D) = \sum_{|h|=m_j} b_{jh} \left(\frac{\partial}{\partial x_1'}\right)^{h_1} \cdots \left(\frac{\partial}{\partial x_n'}\right)^{h_n}$$

qed

Remark 5.22.2. We can see from this theorem and definition 5.22.1, that if we have the strong complementary condition holding for only one point $X \in \partial\Omega$, and if the system satisfies the hypothesis of theorem 5.22.1, then the strong complementary condition for $\{B_j\}_{j=0}^{m-1}$ must hold for every point $x \in \partial\Omega$. This proves the following equivalent definition of the strong complementary condition.

Theorem 5.22.2. Let us consider the system $(A(x,D),\{B_j\})$ defined as in theorem 5.22.1, where $A(x,D)$ is strongly elliptic, the coefficients are constant and the system is invariant with respect to $\theta_{\frac{i}{2}}$. Since $\partial\Omega \in C^\infty$, we can consider the point $X \in \partial\Omega$ such that the tangent hyperplane at X is parallel to $x_n = 0$, and the normal vector to $\partial\Omega$ at X is parallel to the x_n-axis. $\{B_j\}_{j=0}^{m-1}$ satisfies the strong complementary condition if and only if for every $\lambda > 0$, letting $\xi' = (0,\ldots,0,1)$ be the outward normal vector to the boundary at X, and $\xi = (\xi_1,\ldots,\xi_{n-1},0)$ be in the tangent hyperplane to $\partial\Omega$ at X, the polynomials in the complex variable τ,

$$B_{jo}(X,\xi+\tau\xi') = B_{jo}(X,(\xi_1,\ldots,\xi_{n-1},\tau))$$

$$= \sum_{|h|=m_j} b_{jh} \xi_1^{h_1} \cdots \xi_{n-1}^{h_{n-1}} \tau^{h_n} \qquad (0 \le j \le m-1)$$

are linearly independent modulo the polynomial

$$M^*(\xi,\lambda) = \prod_{k=1}^{m} (\tau-\tau_k^*(\xi,\lambda))$$

where $\tau_k^*(\xi,\lambda)$ are the m roots with positive imaginary part of the polynomial

$$(-1)^m A_o(X,\xi+\tau\xi') + \lambda = (-1)^m A_o(X,(\xi_1,\ldots,\xi_{n-1},\tau)) + \lambda$$

$$= (-1)^m \sum_{|\alpha|=2m} a_\alpha \xi_1^{\alpha_1} \ldots \xi_{n-1}^{\alpha_{n-1}} \tau^{\alpha_n} + \lambda.$$

Remark 5.22.3. Now we can see that if the given system $(A(x,D),\{B_j\})$ satisfies the hypothesis of theorem 5.22.1, then it suffices to verify the strong complementary condition at one point $X \in \partial\Omega$, in order to have the boundary operators $\{B_j\}$ satisfy the strong complementary condition for every point on the boundary. If the system does not satisfy the hypothesis of the theorem, then we must go back to definition 5.22.1 and use the transformation θ_i, for every $x \in \partial\Omega$, to verify the strong complementary condition.

Remark 5.22.4. Schechter showed in [28] that, after the transformation, θ_i, any boundary operator can be placed in the following form

$$B_j(x,D) = b_j(x)\left(\frac{\partial}{\partial n}\right)^{m_j} + \sum_{k=0}^{m_j-1} T_{kj}\left(\frac{\partial}{\partial n}\right)^k$$

where T_{kj} are tangential partial differential operators of order $\leq m_j-k$ involving only the variables $x_1', x_2', \ldots, x_{n-1}'$. This leads us to the equivalent definition of a normal set given by Gerd Grebb [12] and found in definition 5.21.3.

Finally, to conclude this section, I will give an example of an operator which is invariant with respect to θ_i.

Example 5.22.1. Let us consider the operator

$$A(x,D) = (-1)^k \Delta^k$$

where Δ is the Laplacian operator

$$\Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2.$$

We assert that for every positive integer k, the operator is invariant with respect to the transformation, θ_i.

Proof. It suffices to prove that this operator verifies the hypothesis of theorem 5.22.1. First, we see the coefficients are constant. We will now show that the operator is invariant with respect to θ_{i_2}. Let us define the transformation

$$x'_h = \sum_{i=1}^{n} a_{hi}(x_i - X_i) \qquad (1 \leq h \leq n)$$

where $X = (X_1,\ldots,X_n)$ is any point on $\partial\Omega$, and the coefficients, a_{hi}, satisfy

$$\sum_{i=1}^{n} a_{hi}^2 = 1 \qquad (1 \leq h \leq n)$$

$$\sum_{i=1}^{n} a_{hi} a_{\ell i} = 0 \qquad (h \neq \ell)$$

We will prove this assertion by induction on k.

Let k = 1, then from the chain rule

$$\left(\frac{\partial}{\partial x_i} \right)^2 = \sum_{h,\ell=1}^{n} a_{hi} a_{\ell i} \frac{\partial^2}{\partial x_h' \partial x_\ell'}$$

and from the definition of $\theta_{i_2} \cdot \theta_{i_1}$,

$$\Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2 = \sum_{i=1}^{n} \sum_{h,\ell=1}^{n} a_{hi} a_{\ell i} \frac{\partial^2}{\partial x_h' \partial x_\ell'}$$

$$= \sum_{i=1}^{n} \sum_{h=\ell=1}^{n} a_{hi}^2 \left(\frac{\partial}{\partial x_h'} \right)^2 + \sum_{i=1}^{n} \sum_{h\neq\ell} a_{hi} a_{\ell i} \frac{\partial^2}{\partial x_h' \partial x_\ell'}$$

$$= \sum_{h=\ell=1}^{n} \left(\sum_{i=1}^{n} a_{hi}^2 \right) \left(\frac{\partial}{\partial x_h'} \right)^2 + \sum_{h\neq\ell} \left(\sum_{i=1}^{n} a_{hi} a_{\ell i} \right) \frac{\partial^2}{\partial x_h' \partial x_\ell'}$$

$$= \sum_{h=1}^{n} \left(\frac{\partial}{\partial x_h'} \right)^2.$$

Assume that for k = m,

$$(-1)^m \Delta^m = (-1)^m \left(\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2 \right)^m = (-1)^m \left(\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i'} \right)^2 \right)^m.$$

Let k = m+1, we have

$$(-1)^{m+1} \Delta^{m+1} = (-1)^{m+1} \left[\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2 \right]^{m+1}$$

$$= (-1)^m \left[\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2 \right]^m (-1) \left[\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2 \right]$$

$$= (-1)^m \left[\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i'} \right)^2 \right]^m (-1) \left[\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i'} \right)^2 \right]$$

$$= (-1)^{m+1}\left[\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2\right]^{m+1}$$

Hence, we have proved the assertion.

qed

5.23. General Boundary Operators, the Case n = 1

In the study of the stability theory of certain elliptic partial differential equations

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u)$$

where $u(x,t)$ acts on $\Omega x[0,\infty)$ and $\Omega \subset R^n$, we need to consider the case $n = 1$. In the previous sections we discussed only the case $\Omega \subset R^n$, $n \geq 2$. For the one dimensional case, $n = 1$, we need to discuss linear differential operators with general boundary conditions, and in this section, we give the following hypothesis on the system $(A(x,D),\{B_j\})$. This discussion is found in Agmon [1].

Let us consider the elliptic linear differential operator

$$A(x, \frac{\partial}{\partial x}) = \sum_{k=0}^{2m} a_k(x) \left(\frac{\partial}{\partial x} \right)^{2m-k} \qquad x \in [a,b] \qquad (5\text{-}20)$$

where $-\infty < a < b < \infty$, and the boundary operators

$$B_j^+(\frac{\partial}{\partial x}) = \sum_{h=0}^{m_j^+} b_{jh}^+ (\frac{\partial}{\partial x})^{m_j^+-h} \qquad (0 \leq j \leq m-1) \qquad (5\text{-}21)$$

$$B_j^-(\frac{\partial}{\partial x}) = \sum_{h=0}^{m_j^-} b_{jh}^- (\frac{\partial}{\partial x})^{m_j^--h}$$

satisfying the boundary conditions

$$B_j^+\left(\frac{\partial}{\partial x} \right)u(x) \Big|_{x=b} = B_j^-\left(\frac{\partial}{\partial x} \right)u(x) \Big|_{x=a} = 0 \quad (0 \leq j \leq m-1) \tag{5-22}$$

We give the following conditions on $(A(x, \frac{\partial}{\partial x}), \{B_j^+\}, \{B_j^-\})$

(i) $a_k(x)$ are complex-valued, measurable and bounded on $[a,b]$ $(1 \leq k \leq 2m)$, and $a_o(x)$ is continuous on $[a,b]$.

(ii) $A(x, \frac{\partial}{\partial x})$ is strongly elliptic on $[a,b]$, that is, for every $x \in [a,b]$,

$$(-1)^m \mathrm{Re}[a_o(x)] > 0. \tag{5-23}$$

(iii) $B_j^+(\frac{\partial}{\partial x})$, $B_j^-(\frac{\partial}{\partial x})$, $(0 \leq j \leq m-1)$ are linear differential operators with constant coefficients of respective orders m_j^+, $m_j^- \leq 2m-1$, and are independent of time, t.

(iv) The operators, $\{B_j^+\}_{j=0}^{m-1}$ are linearly independent. Also, $\{B_j^-\}_{j=0}^{m-1}$ are linearly independent.

Remark 5.23.1. The boundary operators

$$B_j^+u(x)\Big|_{x=b} = \left(\frac{\partial}{\partial x} \right)^j u(b,t) \qquad (0 \leq j \leq m-1)(t \geq 0)$$

$$B_j^-u(x)\Big|_{x=a} = \left(\frac{\partial}{\partial x} \right)^j u(a,t)$$

which are the Dirichlet boundary operators for the case $n = 1$, satisfy (5-23) for any strongly elliptic operator with coefficients satisfying

(5-23(i)), since $\{(\frac{\partial}{\partial x})^j u\}$ are linearly independent, with orders $m_j^+ = m_j^- = j \leq 2m-1$ $(0 \leq j \leq m-1)$.

The following proposition, found in Agmon [1], shows how these conditions for the case $n = 1$, can be related to those for $n \geq 2$.

Proposition 5.23.1. Let us define

$$\mathcal{L}(x, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}) = A(x, \frac{\partial}{\partial x}) + (-1)^m (\frac{\partial}{\partial t})^{2m}.$$

If $A(x, \frac{\partial}{\partial x})$ is strongly elliptic on $[a,b]$, we have the following results:

(i) $\mathcal{L}$ is elliptic, and is properly elliptic on

$$\Omega = \{(x,t) \mid a \leq x \leq b, -\infty < t < \infty\} ;$$

(ii) $\mathcal{L}$ and the boundary system $\{B_j^+\}_{j=0}^{m-1}$, $\{B_j^-\}_{j=0}^{m-1}$, satisfy the strong complementary condition on $x = b$, $x = a$, respectively.

6.0. STABILITY OF SOLUTIONS TO THE GENERAL BOUNDARY VALUE PROBLEM: $\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u)$

Many engineering and physical problems can be formulated as an initial-boundary value problem for a partial differential equation. In this chapter, we will be considering the following nonlinear initial-boundary value problem which has many physical applications,

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x\varepsilon\Omega, t\geq 0 \qquad (6\text{-}1)$$

with the general boundary conditions

$$B_j(x,D)u(x,t) = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \quad (0\leq j\leq m-1) \qquad (6\text{-}2)$$

and the initial condition

$$u(x,0) = u_o(x)$$

where u is a vector valued function, such that for every $t\geq 0$, $u(x,t)$ is in some Hilbert space, $A(x,D)$ is a linear partial differential operator and f is a nonlinear function defined on some prescribed function space. Sufficient conditions will be given on the system $(A(x,D),\{B_j\},\Omega)$ and on the nonlinear function, $f(u)$, to ensure the existence, uniqueness and stability of a solution to the above partial differential equation. This is done by considering the extension of $A(x,D)$ to an abstract (unbounded) linear operator A defined on some base Hilbert space, H, and considering the abstract operator evolution equation

$$\frac{du(t)}{dt} + Au(t) = f(u) \qquad (t\geq 0)$$

$$u(o) = u_o$$

and utilizing the results from Pao [23] on the stability criteria for the evolution equation to ensure the solution of the stability problem for (6-1) and (6-2). If $f(u) \equiv 0$, then the abstract operator equation becomes

$$\frac{du(t)}{dt} + Au(t) = 0 \qquad (t \geq 0) \qquad (6\text{-}3)$$
$$u(o) = u_o.$$

In this chapter, we will consider the problem (6-1) and (6-2), for the cases where $f(u) \equiv 0$, and $f(u)$ is a nonzero nonlinear function given in some function space, and $\Omega \subset R^n, n \geq 1$. We will also show that these results generalize the case for the Dirichlet problem worked out by Buis [7]. Most of this discussion is restricted to the real Hilbert space $L^2(\Omega)$. The extension to the complex space can readily be done.

6.1 Preliminaries

In sections 6.2 and 6.3 we will examine the initial-boundary value problem given in (6-1) with $f(u) \equiv 0$, or

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = 0 \qquad x \in \Omega, t \geq 0 \qquad (6\text{-}4)$$

with general boundary conditions

$$B_j(x,D)u(x,t) = 0 \qquad x \in \partial\Omega, t \geq 0 \qquad (0 \leq j \leq m-1)$$

and initial condition

$$u(x,0) = u_o(x).$$

In order to examine the stability problem for (6-4), we need to use the results of Pao in [23], in which he solves the stability problem for the abstract operator equation (6-3), where A is the abstract operator extension of A(x,D) defined in the real Hilbert space, $H \equiv L^2(\Omega)$, such that D(A) is dense in H and $R(A) \subset H$. In this section, we will give these results and others needed in order to solve the stability problem for (6-4).

First, we must define what is meant by a solution of the operator evolution equation (6-3).

Definition 6.1.1. u(t) is a solution of the equation (6-3) with initial condition $u(0) = u_o \varepsilon D(A)$ if:

(i) u(t) is uniformly continuous in t, for every $t \geq 0$, with $u(\theta) = u_o$;

(ii) $u(t) \varepsilon D(A)$, for every $t \geq 0$, and Au(t) is continuous in t; for every $t \geq 0$;

(iii) the derivative of u(t) exists (in the strong topology) for every $t \geq 0$ and equals -Au(t).

Definition 6.1.2. An equilibrium solution of (6-3) is a solution u(t) of (6-3) such that

$$||u(t)-u(0)||_H = 0 \qquad \text{for every } t \geq 0.$$

The following lemma proved in [27] is very useful in establishing the sufficient conditions for the existence and stability of the equilibrium solution to the operator evolution equation (6-3).

Lemma 6.1.1. (R.S.Phillips) Let A be a linear operator with domain D(A) and range R(A) both contained in the Hilbert space H and D(A) is dense in H. Then A generates a contraction semi-group of class (C_o) in H if and only if A is dissipative with respect to an inner product equivalent to the one defined on H, and R(I-A) = H.

The following result by Pao in [23] utilizes lemma 6.1.1, and is a basis for our examination of the stability problem of equation (6-4). Pao's notation has been changed so his results could be used in the present context.

Lemma 6.1.2. (Pao) Let A be a linear operator with domain D(A) and range, R(A) both contained in the Hilbert space H and D(A) is dense in H and R(I-(-A))= H. If A satisfies the following inequality, there exists a constant $\beta \geq 0$ such that for every $u \varepsilon D(A)$

$$(u,(-A)u)_e \leq -\beta ||u||_e^2$$

where e is the inner product equivalent to the one defined on H, then for every initial element $u_o \varepsilon D(A)$, there exists a unique solution, u(t), of (6-3) such that $u(0) = u_o$ and

(i) Any unperturbed solution is asymptotically stable if $\beta > 0$ and is stable if $\beta = 0$.

(ii) A stability region is D(A) which can be extended to the whole space H.

(iv) If $\underline{0} \varepsilon D(A)$ and $A(\underline{0}) = 0$, then $\underline{0}$ is an equilibrium solution, called the null solution.

The basic conditions needed to solve the stability problem of (6-4) is dissipativity of -A and R(I-(-A))= H. To prove the second part we need to use the next two results by Friedman [11] and Komura [16], respectively.

Lemma 6.1.3. (Friedman) If $(A(x,D),\{B_j\},\Omega)$ is the system satisfying the conditions (i)-(v) in (5-9), and A is the operator extension of A(x,D) in the base Hilbert space $H = L^2(\Omega)$, then there exists a constant $\Lambda_o > 0$, such that for every $\lambda \geq \Lambda_o$

$$R(\lambda I - (-A)) = L^2(\Omega).$$

Lemma 6.1.4. (Komura) If A is the operator defined in lemma 6.1.3 and $-A$ is dissipative, and there exists a constant $\alpha > 0$ such that $R(\alpha I - (-A)) = L^2(\Omega)$, then for every $\alpha > 0$

$$R(\alpha I - (-A)) = L^2(\Omega).$$

6.2. Stability of the Solution of a Linear Initial-Boundary Value Problem for the Case $n \geq 2$

In this section, we will give sufficient conditions to ensure the existence, uniqueness and asymptotic stability or stability of the equilibrium solution to the general boundary value problem defined below in (6-5).

Let us consider the following initial-boundary value problem,

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = 0 && x\in\Omega, t\geq 0 \\ &B_j(x,D)u(x,t) = 0 && x\in\partial\Omega, t\geq 0 \quad (0\leq j\leq m-1) \qquad (6\text{-}5) \\ &u(x,0) = u_o(x) \end{aligned}$$

where $A(x,D), B_j(x,D)$ are the partial differential operators defined by

$$\begin{aligned} A(x,D) &= \textstyle\sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha \\ B_j(x,D) &= \textstyle\sum_{|h|\leq m_j} b_{jh}(x)D^h \qquad (0\leq j\leq m-1) \qquad (6\text{-}6) \end{aligned}$$

such that the system $(A(x,D),\{B_j\},\Omega)$ satisfies the conditions (5-9) and (6-7).

There exists a constant $\beta\geq 0$, such that for every $u\varepsilon C_B^\infty(\Omega)$

$$(u,-A(x,D)u)_o \leq -\beta||u||_o^2. \qquad (6\text{-}7)$$

First, let us define $H \equiv L^2(\Omega)$, where this discussion is restricted to the real Hilbert space, $L^2(\Omega)$. Let us now define the abstract operator T_o,

$$D(T_o) = C_B^\infty(\Omega)$$

$$(T_ou)(x) = A(x,D)u(x) \qquad u\varepsilon D(T_o). \qquad (6\text{-}8)$$

Then, T_o is a linear operator such that $D(T_o)$ is dense in $L^2(\Omega)$, since $C_o^\infty(\Omega)\subset C_B^\infty(\Omega)\subset L^2(\Omega)$ and we know from Dunford-Schwartz [9], $C_o^\infty(\Omega)$ is dense in $L^2(\Omega)$.

Let us now define the abstract operator A,

$$D(A) = H_B^{2m}(\Omega) \qquad (6\text{-}9)$$

and Au is the function in $L^2(\Omega)$ defined by

$$(Au)(x) = A(x,D)u(x) \qquad u\varepsilon D(A).$$

Since A(x,D) is linear, we see that A is a linear operator. We will show that A is the smallest closed linear extension of T_o in $L^2(\Omega)$, where we have defined $H\equiv L^2(\Omega)$.

We then obtain the following evolution equation

$$\frac{du(t)}{dt} + Au(t) = 0 \qquad (6\text{-}10)$$

$$u(0) = u_o$$

where u(t) is a vector valued function defined on $[0,\infty)$ with values in $L^2(\Omega)$. Thus, for each $t\geq 0$, u can be regarded as a function $u(x,t)\varepsilon L^2(\Omega)$. A is the linear unbounded operator with domain and range both contained in $L^2(\Omega)$, defined by (6-9).

We must define what is meant by a solution to equation (6-5), which is found in Friedman in [11].

<u>Definition 6.2.1.</u> u(x,t) is a generalized solution of (6-5) if u(t) is a solution of (6-10) in the sense of definition 6.1.1.

<u>Definition 6.2.2.</u> u(x,t) satisfies the boundary condition

$$B_j(x,D)u(x,t) = 0 \qquad (0\leq j\leq m-1)$$

in a generalized sense if for each $t\geq 0$, $u(x,t)\varepsilon H_B^{2m}(\Omega)$.

We will need the following two very important inequalities in order to prove that A is a closed operator. The first inequality is found in [11], and the second shows that A is a bounded operator on its domain.

<u>Lemma 6.2.1.</u> (Friedman) If the system $(A(x,D),\{B_j\},\Omega)$ satisfies (5-9), then there exists a constant $C_o > 0$, such that for every $u\varepsilon H_B^{2m}(\Omega)$

$$||u||_{2m} \leq C_o\ (||A(x,D)u||_o + ||u||_o).$$

<u>Lemma 6.2.2.</u> If the operator A is defined as in (6-9), then there exists a constant $C_1 > 0$, such that for every $u\varepsilon D(A)$

$$||Au||_o \leq C_1||u||_{2m}.$$

<u>Proof</u>. First, I must prove that for every $u,v\varepsilon L^2(\Omega)$,

$$||u + v||_o^2 \leq 2[||u||_o^2 + ||v||_o^2].$$

This is true from fact that for $a,b>0$, $2ab \leq a^2 + b^2$, and

$$||u + v||_o^2 \leq [||u||_o + ||v||_o]^2 = ||u||_o^2 + 2||u||_o||v||_o + ||v||_o^2$$

$$\leq ||u||_o^2 + (||u||_o^2 + ||v||_o^2) + ||v||_o^2$$

$$= 2(||u||_o^2 + ||v||_o^2).$$

If we let $C_o = \max_{\substack{|\alpha| \leq 2m \\ x\varepsilon\overline{\Omega}}} |a_\alpha(x)|$, then for some integer k independent of u, we have for every $u\varepsilon D(A)$

$$||Au||_o^2 = ||A(x,D)u||_o^2 = ||\textstyle\sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha u||_o^2$$

$$\leq 2^k \textstyle\sum_{|\alpha|\leq 2m} ||a_\alpha(x)D^\alpha u||_o^2$$

$$\leq 2^k C_o^2 \textstyle\sum_{|\alpha|\leq 2m} ||D^\alpha u||_o^2$$

$$= 2^k C_o^2 ||u||_{2m}^2 .$$

Therefore,

$$||Au||_o \leq C_1||u||_{2m}, \text{ where } C_1 = 2^{\frac{k}{2}} C_o. \qquad \text{qed}$$

We will now prove that A is the smallest closed linear extension of T_o in $L^2(\Omega)$.

Lemma 6.2.3. The operator A defined in (6-9) is a closed operator in $L^2(\Omega)$.

Proof. We must show that if there exists a sequence $u_i \varepsilon D(A)$ such that

$$u_i \underset{L^2(\Omega)}{\rightarrow} u, \text{ and } Au_i \underset{L^2(\Omega)}{\rightarrow} w \qquad \text{as } i \to \infty$$

then $u \varepsilon D(A)$ and $Au = w$. First, we will show there exists $v \varepsilon H^{2m}(\Omega)$, such that

$$u_i \underset{H^{2m}(\Omega)}{\rightarrow} v \qquad \text{as } i \to \infty.$$

Indeed, let us consider $u_i - u_j \varepsilon D(A)$. We know, since u_i converges in $L^2(\Omega)$,

$$||u_i - u_j||_o \underset{R^1}{\rightarrow} o \qquad \text{as } i,j \to \infty.$$

From lemma 6.2.1, there exists a constant C_o, such that

$$||u_i - u_j||_{2m} \leq C_o[||Au_i - Au_j||_o + ||u_i - u_j||_o].$$

Since u_i converges in $L^2(\Omega)$ and Au_i converges in $L^2(\Omega)$, the right side converges to 0 as $i,j \to \infty$. Hence,

$$u_i - u_j \underset{H^{2m}(\Omega)}{\rightarrow} 0 \qquad \text{as } i,j \to \infty .$$

Since $H^{2m}(\Omega)$ is complete and we have Cauchy convergence, this implies there exists a $v \varepsilon H^{2m}(\Omega)$ such that

$$u_i \underset{H^{2m}(\Omega)}{\rightarrow} v \qquad \text{as } i \to \infty.$$

We will now show $u \varepsilon D(A) = H^{2m}_B(\Omega)$. To see this, we need to show there exists a sequence $u_n \varepsilon C^\infty_B(\Omega)$ such that

$$u_n \underset{H^{2m}(\Omega)}{\rightarrow} u \qquad \text{as } n \to \infty.$$

Indeed, we have the following inequality, since the identity injection from

$H^{2m}(\Omega)$ into $L^2(\Omega)$ is continuous, or $||\cdot||_o \leq C_2||\cdot||_{2m}$,

$$||u-v||_o \leq ||u-u_i||_o + ||u_i-v||_o \leq ||u-u_i||_o + C_2||u_i-v||_{2m}.$$

We have proved that the right-hand side converges to zero, as $i \to \infty$. Therefore, $u = v\varepsilon H^{2m}(\Omega)$, which implies

$$||u_i-u||_{2m} = ||u_i-v||_{2m} \underset{R^1}{\to} 0 \qquad \text{as } i \to \infty$$

Where $u_i \varepsilon D(A)$. Let us pick, arbitrarily, $\varepsilon > 0$. There exists a $u_i \varepsilon D(A)$, such that $||u_i-u||_{2m} < \frac{\varepsilon}{2}$. By definition of $D(A) = H_B^{2m}(\Omega)$, there exists $u_{n(i)} \varepsilon C_B^{\infty}(\Omega)$, such that $||u_{n(i)} - u_i||_{2m} < \frac{\varepsilon}{2}$.

Therefore,

$$||u_{n(i)} - u||_{2m} \leq ||u_{n(i)} - u_i||_{2m} + ||u_i - u||_{2m} < \varepsilon.$$

This shows that $u\varepsilon D(A) = H_B^{2m}(\Omega)$. Finally, we must show $Au = w\varepsilon L^2(\Omega)$. Indeed, let us consider $u_i - u\varepsilon D(A)$. From lemma 6.2.2,

$$||Au_i - Au||_o = ||A(u_i - u)||_o \leq C_1||u_i - u||_{2m}.$$

Since the right hand side converges to zero as $i \to \infty$, we see

$$Au_i \underset{L^2(\Omega)}{\to} Au \qquad \text{as } i \to \infty.$$

From the inequality

$$||Au - w||_o \leq ||Au - Au_i||_o + ||Au_i - w||_o$$

where we have seen that the right hand side converges to zero as $i \to \infty$, it follows that $Au = w\varepsilon L^2(\Omega)$. Hence, I have shown that A is a closed operator in $L^2(\Omega)$. qed

<u>Lemma 6.2.4.</u> A is the smallest closed linear extension of T_o in $L^2(\Omega)$.

<u>Proof.</u> By the definition of the extension of T_o in $L^2(\Omega)$, A is the smallest extension of T_o in $L^2(\Omega)$ if and only if the domain of A,D(A), is the set of all $u \varepsilon L^2(\Omega)$ such that there exists a sequence $u_i \varepsilon D(T_o)$, such that

$$||u_i - u||_o \underset{R^1}{\rightarrow} 0, \; ||T_o u_i - Au||_o \underset{R^1}{\rightarrow} 0 \qquad \text{as } i \rightarrow \infty.$$

Since A is a closed extension of T_o in $L^2(\Omega)$, it is obvious that A contains the smallest closed extension of T_o in $L^2(\Omega)$. Therefore, it suffices to show that for any $u \varepsilon D(A)$, there exists a sequence $u_i \varepsilon D(T_o)$, such that

$$||u_i - u||_o \underset{R^1}{\rightarrow} 0, \qquad \text{and } ||T_o u_i - Au||_o \underset{R^1}{\rightarrow} 0 \quad \text{as } i \rightarrow \infty.$$

To prove this, let $u \varepsilon D(A) = H_B^{2m}(\Omega)$. Then, by definition of $H_B^{2m}(\Omega)$, there exists a sequence $u_i \varepsilon C_B^{\infty}(\Omega) = D(T_o)$ such that $||u_i - u||_{2m} \rightarrow 0$ as $i \rightarrow \infty$. But we know, since $u_i - u \varepsilon D(A)$, from lemma 6.2.2,

$$||Au_i - Au||_o = ||A(u_i - u)||_o \leq C_1||u_i - u||_{2m}.$$

Therefore,

$$||Au_i - Au||_o \underset{R^1}{\rightarrow} 0 \qquad \text{as } i \rightarrow \infty.$$

Since the identity injection from $H^{2m}(\Omega)$ into $L^2(\Omega)$ is continuous

$$||u_i - u||_o \leq C_2||u_i - u||_{2m}$$

and we have the following inequality

$$||T_o u_i - Au||_o \leq ||T_o u_i - Au_i||_o + ||Au_i - Au||_o.$$

Since $u_i \varepsilon D(T_o) \subset D(A)$, $T_o u_i = Au_i$. Hence, we have the result

$$||u_i - u||_o \underset{R^1}{\to} 0 \quad \text{and} \quad ||T_o u_i - Au||_o \underset{R^1}{\to} 0 \qquad \text{as } i \to \infty$$

and this shows that A is the smallest closed linear extension of T_o in $L^2(\Omega)$. qed

Now we will prove the facts needed to use Lemma 6.1.2, namely the dissipativity of $-A$, and the fact that $R(I-(-A)) = H$.

<u>Lemma 6.2.5.</u> For the closure, A, of T_o in $L^2(\Omega)$, $-A$ is strictly dissipative with respect to the L^2 - norm if $\beta > 0$, and is dissipative if $\beta = 0$.

<u>Proof.</u> From the hypothesis (6-7), we see that there exists a constant $\beta \geq 0$, such that for any $u \varepsilon C_B^\infty(\Omega) = D(T_o)$

$$(u,(-T_o)u)_o \leq -\beta \; ||u||_o^2.$$

Since $D(T_o)$ is dense in $D(A)$, we know that for any $u \varepsilon H_B^{2m}(\Omega) = D(A)$, there exists a sequence $u_i \varepsilon D(T_o)$ such that

$$u_i \underset{H^{2m}(\Omega)}{\to} u \qquad \text{as } i \to \infty.$$

We must now prove that

$$(u,Au)_o = \lim_{i\to\infty} (u_i,Au_i)_o.$$

To show this, we utilize the following two inequalities,

$$||u_i - u||_o \leq C_o||u_i - u||_{2m}$$

$$||Au_i - Au||_o = ||A(u_i - u)||_o \leq C_1||u_i - u||_{2m}.$$

This shows us that

$$u_i \underset{L^2(\Omega)}{\rightarrow} u \quad \text{and } Au_i \underset{L^2(\Omega)}{\rightarrow} Au \quad \text{as } i \to \infty.$$

Due to the continuity of the inner product, we have proved that

$$(u_i, Au_i)_o \underset{R^1}{\to} (u, Au)_o \qquad \text{as } i \to \infty.$$

Since $u \varepsilon D(A)$, $u_i \varepsilon D(T_o) \subset D(A)$ and $Au_i = T_o u_i$

$$(u,(-A)u)_o = \lim_{i\to\infty} (u_i,(-A)u_i)_o = \lim_{i\to\infty} (u_i,(-T_o)u_i)_o$$

$$\leq -\beta \lim_{i\to\infty} ||u_i||_o^2$$

$$= -\beta\ ||u||_o^2.$$

From this inequality, we can see that if $\beta > 0$, $-A$ is strictly dissipative, and if $\beta = 0$, $-A$ is dissipative. qed

<u>Lemma 6.2.6.</u> A is a linear operator, such that the domain D(A) and range R(A) are both contained in the Hilbert space $L^2(\Omega)$ and D(A) is dense in $L^2(\Omega)$. Also, $R(I - (-A)) = L^2(\Omega)$.

<u>Proof.</u> It must first be noted that we have defined our base Hilbert space to be $H \equiv L^2(\Omega)$. A is a linear operator, since $A(x,D)$ is linear. Also, it can readily be seen that

$$D(A) = H_B^{2m}(\Omega) \subset L^2(\Omega), \text{ and } R(A) \subset L^2(\Omega).$$

The fact that D(A) is dense in $L^2(\Omega)$ follows since $D(A) = H_B^{2m}(\Omega)$, and $C_o^\infty(\Omega) \subset C_B^\infty(\Omega) \subset H_B^{2m}(\Omega) \subset L^2(\Omega)$, where $C_o^\infty(\Omega)$ is dense in $L^2(\Omega)$. It suffices now to show $R(I - (-A)) = L^2(\Omega)$. We know from lemma 6.1.3, that by the definition of the operator A, there exists a constant $\Lambda_o > 0$, such that for any $\lambda > \Lambda_o$

$$R(\lambda I - (-A)) = L^2(\Omega),$$

and since $-A$ is dissipative, we utilize lemma 6.1.4, to show that for any $\lambda > 0$,

$$R(\lambda I - (-A)) = L^2(\Omega).$$

Therefore, if we let $\lambda = 1$, we have the desired result. qed

We are now ready to prove the nain result of this section.

Theorem 6.2.1. Let us consider the strongly elliptic partial differential operator, A(x,D), and the boundary operators, $\{B_j\}_{j=0}^{m-1}$, defined by (6-6). Let the system $(A(x,D),\{B_j\},\Omega)$ satisfy conditions (5-9) and inequality (6-7). If we consider the initial-boundary value problem (6-5), then for any given initial value function $u_o(x) \varepsilon H_B^{2m}(\Omega)$, there exists a unique generalized solution, u(x,t), such that

(i) for every $t \geq 0$, $u(x,t) \varepsilon H_B^{2m}(\Omega)$;

(ii) u(x,t) is a generalized solution of (6-5);

(iii) $u(x,0) \equiv u_o(x)$, and satisfies the boundary conditions in a generalized sense;

(iv) the null solution is stable

if $\beta = 0$ (and is asymptotically stable if $\beta > 0$) with respect to the L^2 - norm, where β is the dissipativity constant in (6-7).

Proof. We define the evolution equation as we did before

$$\frac{du(t)}{dt} + Au(t) = 0$$
$$u(0) = u_o$$

where the abstract operator, A, is defined by

$$D(A) = H_B^{2m}(\Omega)$$

$$(Au)(x) = A(x,D)u(x) \qquad u \in D(A)$$

and is the smallest closed linear extension of the operator T_o defined in (6-8).

From lemma 6.2.6, A is a linear operator, such that the domain D(A) and range R(A) are contained in the base Hilbert space $H \equiv L^2(\Omega)$, D(A) is dense in $L^2(\Omega)$, and $R(I-(-A)) = L^2(\Omega)$. From lemma 6.2.5, -A is strictly dissipative if $\beta > 0$ and is dissipative if $\beta = 0$, where β is the dissipativity constant in (6-7). Now, we have satisfied the hypothesis in lemma 6.1.2, and by applying Pao's results, we see that for every $u_o \in D(A) = H_B^{2m}(\Omega)$, there exists a unique solution u(t) of (6-10) satisfying $u(0) = u_o$. Therefore, from definition 6.1.1, and 6.2.1, u(x,t) is a generalized solution of (6-5), and for every $t \geq 0$, $u(x,t) \in H_B^{2m}(\Omega)$. This implies, from definition 6.2.2, that u(x,t) satisfies the boundary conditions in a generalized sense. Since $u(0) = u_o$, we see $u(x,0) = u_o(x)$, the given initial value function in $H_B^{2m}(\Omega) = D(A)$. Since $u(t) \equiv 0 \in H_B^{2m}(\Omega) = D(A)$ and $A(0) = 0$, we see the null solution is stable if $\beta = 0$ (and is asymptotically stable if $\beta > 0$) with respect to the L^2 - norm. qed

6.3. Stability of the Solution of a Linear Initial-Boundary Value Problem for the Case n = 1

In this section, we will give sufficient conditions to ensure the existence, uniqueness and the stability of the null solution to the general boundary value problem defined below for the case where $\Omega = (a,b) \subset R^1$, and $-\infty < a < b < \infty$.

Let us consider the following initial-boundary value problem,

$$\frac{\partial u(x,t)}{\partial t} + A(x,\frac{\partial}{\partial x})u(x,t) = 0 \qquad x\varepsilon(a,b), t\geq 0$$

$$B_j^+(\frac{\partial}{\partial x})u(x,t)\Big|_{x=b} = B_j^-(\frac{\partial}{\partial x})u(x,t)\Big|_{x=a} = 0 \qquad (t\geq 0) \qquad (6\text{-}11)$$

$$u(x,0) = u_o(x)$$

where $A(x,\frac{\partial}{\partial x})$, $\{B_j^+\}_{j=0}^{m-1}$, and $\{B_j^-\}_{j=0}^{m-1}$ are the partial differential operators as defined in Agmon [1],

$$A(x,\frac{\partial}{\partial x}) = \sum_{k=0}^{2m} a_k(x) \ (\frac{\partial}{\partial x})^{2m-k}$$

$$B_j^+(\frac{\partial}{\partial x}) = \sum_{h=0}^{m_j^+} b_{jh}^+(\frac{\partial}{\partial x})^{m_j^+-h} \qquad (6\text{-}12)$$

$$B_j^-(\frac{\partial}{\partial x}) = \sum_{h=0}^{m_j^-} b_{jh}^-(\frac{\partial}{\partial x})^{m_j^--h}$$

such that the system $(A(x,\frac{\partial}{\partial x}), \{B_j^+,B_j^-\})$ satisfies the conditions (5-23). Let us now define the following Hilbert spaces needed to extend $A(x,\frac{\partial}{\partial x})$ to the abstract operator A,

$H^{2m}[a,b] = \{u\varepsilon C^{2m-1}[a,b] \mid (\frac{d}{dx})^{2m-1}u(x)$ is absolutely continuous in $[a,b]$, and $(\frac{d}{dx})^{2m}u(x)\varepsilon L^2[a,b]\}$

$H_B^{2m}[a,b] = \{u\varepsilon H^{2m}[a,b] \mid B_j^+(\frac{\partial}{\partial x})u(x,t)\Big|_{x=b} = B_j^-(\frac{\partial}{\partial x})u(x,t)\Big|_{x=a} = 0,$

$(0\leq j\leq m-1) \quad (t\geq 0)\}$. (6-13)

We can now define the abstract operator A

$$D(A) = H_B^{2m}[a,b] \qquad (6\text{-}14)$$

$$(Au)(x) = A(x,\frac{\partial}{\partial x})u(x) \qquad u\varepsilon D(A)$$

Remark 6.3.1. From Agmon [1], we see that A is a closed linear operator such that the spectrum is the whole plane or a discrete sequence of eigenvalues.

We then obtain the evolution equation

$$\frac{du(t)}{dt} + Au(t) = 0 \qquad (6\text{-}15)$$

$$u(0) = u_o$$

where A is defined as in (6-14), and D(A) and R(A) are both contained in the real Hilbert space $H \equiv L^2[a,b]$.

Let the operator A(x,D) satisfy the following inequality : there exists a constant $\beta \geq 0$, such that for every $u\varepsilon H_B^{2m}[a,b]$

$$(u,-A(x,\frac{\partial}{\partial x})u)_o \leq -\beta||u||_o^2 \qquad (6\text{-}16)$$

where β is called the dissipativity constant.

In order to consider the stability problem for the case n = 1, we need the following lemmas, the first being found in Agmon [1].

Lemma 6.3.1. Let A be the abstract operator extension of $A(x,\frac{\partial}{\partial x})$ defined by (6-14), where the system $(A(x,\frac{\partial}{\partial x}),\{B_j^+,B_j^-\})$ satisfies condition (5-23). Then, there exists a constant $\Lambda_o>0$, such that for any $\lambda>\Lambda_o$,

$$R(\lambda I-(-A)) = L^2[a,b].$$

In order to use Lemma 6.1.2 to solve the stability problem, we must now show the dissipativity of -A, and R(I-(-A)) = H.

Lemma 6.3.2. Let the operator A be defined as in (6-14). The operator -A is strictly dissipative with respect to the L^2 - norm if $\beta > 0$, and is dissipative if $\beta = 0$, where β is the dissipativity constant in (6-16).

Proof. This follows readily from the definition of A, and condition (6-16), since for every $u \in D(A) = H_B^{2m}[a,b]$,

$$(u,(-A)u)_o = (u,-A(x,\frac{\partial}{\partial x})u)_o$$

$$\leq -\beta||u||_o^2. \qquad \text{qed}$$

Lemma 6.3.3. The abstract operator A is a linear operator such that the domain D(A) and range R(A) are both contained in the real Hilbert space $H \equiv L^2[a,b]$, D(A) is dense in $L^2[a,b]$, and $R(I-(-A)) = L^2[a,b]$.

Proof. It can be readily seen from the definition of A(x,D), that A is linear, and the domain D(A) and range R(A) are both contained in $L^2[a,b]$

Also, $D(A) = H_B^{2m}[a,b]$ is dense in $L^2[a,b]$, since $C_o^\infty[a,b] \subset H_B^{2m}[a,b] \subset L^2[a,b]$ and $C_o^\infty[a,b]$ is dense in $L^2[a,b]$. Finally, we must show $R(I-(-A)) = L^2[a,b]$. From lemma 6.3.1, and since -A is dissipative, from Lemma 6.1.4 we have, since there exists a constant $\Lambda_o > 0$, such that for every $\lambda \geq \Lambda_o$

$$R(\lambda I-(-A)) = L^2[a,b]$$

then it must be true that for every $\lambda > 0$

$$R(\lambda I-(-A)) = L^2[a,b].$$

This leads us to our result by letting $\lambda = 1$. qed

We are now ready to prove our main stability result for the case n = 1.

Theorem 6.3.1. Let us consider the strongly elliptic linear differential operator, $A(x,\frac{\partial}{\partial x})$, and boundary operators $\{B_j^+\}_{j=0}^{m-1}$ and $\{B_j^-\}_{j=0}^{m-1}$ defined by (6-12). Let the system $(A(x,\frac{\partial}{\partial x}),\{B_j^+,B_j^-\})$ satisfy (5-23) and inequality (6-16). If we consider the initial-boundary value problem (6-11), then for any initial value function $u_o(x)\varepsilon H_B^{2m}[a,b]$, there exists a unique generalized solution, $u(x,t)$, such that

(i) for every $t\geq 0, u(x,t)\varepsilon H_B^{2m}[a,b]$;

(ii) $u(x,t)$ is a generalized solution of (6-11);

(iii) $u(x,0) \equiv u_o(x)$, and $u(x,t)$ satisfies the boundary conditions in the classical sense;

(iv) the null solution, is stable if $\beta = 0$ (and is asymptotically stable if $\beta > 0$) with respect to the L^2-norm, where β is the dissipativity constant in the equality (6-16).

Proof. We define the evolution equation as we did in (6-15) where the abstract operator, A, is defined by (6-14) and A is a closed linear operator. From lemma 6.3.3. we see that $D(A)\subset L^2[a,b], R(A)\subset L^2[a,b]$, $D(A)$ is dense in $L^2[a,b]$ and $R(I-(-A)) = L^2[a,b]$. From lemma 6.3.2, $-A$ is strictly dissipative if $\beta > 0$ and is dissipative if $\beta = 0$, where β is the dissipativity constant in (6-16). We can now apply the results of lemma 6.1.2, and see there exists a unique solution $u(t)$ of (6-15), such that $u(0) = u_o$ and from definition 6.1.1, for every $t\geq 0$, $u(t)\varepsilon D(A)$. Therefore, from definition 6.2.1, $u(x,t)$ is a generalized solution of (6-11) such that for every $t\geq 0$, $u(x,t)\varepsilon D(A) = H_B^{2m}[a,b]$ and $u(x,0) = u_o(x)$. From the definition of $H_B^{2m}[a,b]$, we see that $u(x,t)$ satisfies the boundary conditions in the classical sense. Since $u(t) \equiv 0\varepsilon H_B^{2m}[a,b] = D(A)$, and $A(0) = 0$, we see the null solution is stable if $\beta = 0$ (and is asymptotically stable for $\beta > 0$) with respect to the L^2-norm. qed

6.4. Stability of the Solution of a Nonlinear Initial-Boundary Value Problem

In this section, we will give sufficient conditions to ensure the existence, uniqueness and stability of the solution to the nonlinear initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x\varepsilon\Omega, t\geq 0$$

$$B_j(x,D)u(x,t) = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \qquad (0\leq j\leq m-1)$$

$$u(x,0) = u_o(x)$$

where $A(x,D)$ is a strongly elliptic partial differential operator, and $\{B_j\}_{j=0}^{m-1}$ satisfies general boundary conditions, and $f(u)$ is, in general, a nonlinear function satisfying certain conditions in some prescribed Hilbert space. In order to study this nonlinear problem, we need to utilize the results of Pao [23] in solving the stability problem for the nonlinear evolution equation

$$\frac{du(t)}{dt} + Au(t) = f(u)$$

$$u(0) = u_o$$

where A is the abstract operator extension of $A(x,D)$, such that the domain and range of A are contained in some Hilbert space H. In 6.41, we will consider the case $\Omega\subset R^n$, $n\geq 2$ and in 6.42 we will look at the one dimensional case $\Omega = (a,b)\subset R^1$.

6.41. The Nonlinear Stability Problem for the Case $n \geq 2$

In this section, stability criteria will be given in order to solve the nonlinear initial-boundary value problem defined on $\Omega \subset R^n$, for the case $n \geq 2$.

First, we must state some preliminary lemmas which will give sufficient conditions to ensure the existence, uniqueness and stability of the solution to the operator evolution equation

$$\frac{du(t)}{dt} + Au(t) = f(u) \qquad (6\text{-}17)$$
$$u(0) = u_o \qquad u_o \varepsilon D(A)$$

where A is the operator representation of a strongly elliptic partial differential operator, A(x,D) with general boundary conditions and f(u) is, in general, a nonlinear function. We must define what is meant by a solution to (6-17).

Definition 6.41.1. u(t) is a solution of (6-17) if

(i) for every $t \geq 0$, $u(t) \varepsilon D(A)$ and $u(0) = u_o$;

(ii) u(t) is uniformly Lipschitz continuous in t;

(iii) the weak derivative of u(t) exists for every $t \geq 0$, and equals $(-A)u(t) + f(u)$;

(iv) the strong derivative

$$\frac{du(t)}{dt} = (-A)u(t) + f(u)$$

exists and is strongly continuous except at a countable number of values of t.

The following key result and solution of the nonlinear stability problem for the evolution equation (6-17) is due to Pao [23].

Theorem 6.41.1. Let A be a linear operator with domain and range both contained in the same Hilbert space H, such that D(A) is dense in H and R(I-(-A)) = H. If A satisfies the following inequality: there exists a constant $\beta \geq 0$, such that for any $u \varepsilon D(A)$

$$(u,(-A)u)_e \leq -\beta||u||_e^2 \tag{6-18}$$

where $(.,.)_e$ is an inner product equivalent to the one defined on H; and f satisfies:

(i) f maps all of H into H, where f is continuous from H with the strong topology to the weak topology and f is bounded on every bounded subset of H.

(ii) There exists a constant $k \leq \beta$, k can be negative, such that for any $u,v \varepsilon H$

$$(f(u)-f(v),u-v)_e \leq k||u-v||_e^2 \tag{6-19}$$

where β is the dissipativity constant in (6-18), and $(.,.)_e$ is the inner product equivalent to the one on H. Then, for any $u_o \varepsilon D(A)$, there exists a unique solution u(t) of (6-17) with $u(0) = u_o$, and the null solution, if f(0)=0, is asymptotically stable if $k < \beta$, and is stable if $k = \beta$, with respect to the H - norm.

An important subclass of nonlinear functions, f, satisfying the conditions in theorem 6.41.1, is the class of Lipschitz continuous functions. The following result is a corollary in [23].

Corollary 6.41.1. Let A be the abstract operator satisfying the hypothesis of theorem 6.41.1, and f satisfies:

f is defined on all of H into H, and f is Lipschitz continuous with Lipschitz constant $k \leq \beta$, that is, there exists a constant $k \leq \beta$, such that for any $u,v \varepsilon H$

$$||f(u)-f(v)||_e \leq k||u-v||_e.$$

Then, for any initial value function $u_o \varepsilon D(A)$, there exists a unique solution, $u(t)$, of (6-17) with $u(0) = u_o$, and the null solution, if it exists, is asymptotically stable if $k < \beta$ and is stable if $k = \beta$, with respect to the H - norm.

Let us now consider the following nonlinear initial-boundary value problem

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) && x\varepsilon\Omega, t>0 \\ &B_j(x,D)u(x,t) = 0 && x\varepsilon\partial\Omega, t\geq 0, (0\leq j\leq m-1) \\ &u(x,0) = u_o(x) && \end{aligned} \tag{6-20}$$

where $A(x,D)$, $B_j(x,D)$ are the partial differential operators defined by

$$A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$$

$$B_j(x,D) = \sum_{|h|\leq m_j} b_{jh}(x)D^h \qquad (0\leq j\leq m-1) \tag{6-21}$$

such that the system $(A(x,D),\{B_j\},\Omega)$ satisfies (5-9) and inequality (6-7), and the nonlinear function, f, satisfies the condition:

f maps all of $L^2(\Omega)$ into $L^2(\Omega)$, where f is continuous from the strong topology of $L^2(\Omega)$ to the weak topology of $L^2(\Omega)$, and f maps all bounded subsets of $L^2(\Omega)$ into bounded sets. Also, there exists a constant $k \leq \beta$, k can be negative and β is the dissipativity constant in (6-7), such that for every $u,v\varepsilon L^2(\Omega)$ (6-22)

$$(f(u)-f(v),u-v)_o \leq k||u-v||_o^2.$$

Let us now define the abstract operator A, which acts on the real base Hilbert space $H \equiv L^2(\Omega)$. First, we define the operator T_o,

$$D(T_o) = C_B^\infty(\Omega) \tag{6-23}$$

$$(T_o u)(x) = A(x,D)u(x) \qquad u \varepsilon D(T_o).$$

Then, we define the abstract operator A,

$$D(A) = H_B^{2m}(\Omega) \tag{6-24}$$

$$(Au)(x) = A(x,D)u(x) \qquad u \varepsilon D(A).$$

We see from lemma 6.2.4 that A is the smallest closed linear extension of T_o in $L^2(\Omega)$. We obtain the following evolution equation

$$\frac{du(t)}{dt} + Au(t) = f(u) \tag{6-25}$$
$$u(0) = u_o \qquad u_o \varepsilon D(A)$$

where A is defined in (6-24), and f is the nonlinear function defined above in (6-22). We must define what is meant by a solution to (6-20).

Definition 6.41.2. u(x,t) is a generalized solution of (6-20) if u(t) is a solution of (6-25) in the sense of definition 6.41.1.

We are now ready to solve the nonlinear stability problem.

Theorem 6.41.2. Let us consider the strongly elliptic partial differential operator, A(x,D), and the boundary operators $\{B_j\}_{j=0}^{m-1}$ defined in (6-21), such that the system $(A(x,D),\{B_j\},\Omega)$ satisfies conditions (5-9) and inequality (6-7), and the nonlinear function, f, satisfies (6-22). If we consider the nonlinear initial-boundary value problem (6-20), then for any initial value function $u_o(x) \varepsilon H_B^{2m}(\Omega)$ there exists a unique generalized solution, u(x,t), such that

(i) for every $t \geq 0, u(x,t) \varepsilon H_B^{2m}(\Omega)$;

(ii) $u(x,t)$ is a generalized solution of (6-20);

(iii) $u(x,0) \equiv u_o(x)$, and $u(x,t)$ satisfies the boundary conditions in a generalized sense;

(iv) any equilibrium solution (if $f(0) = 0$, the null solution), is asymptotically stable if $k < \beta$, is stable if $k = \beta$, with respect to the L^2 - norm, where k is the constant in (6-22) and β is the dissipativity constant in (6-7).

Proof. We define the abstract operator equation

$$\frac{du(t)}{dt} + Au(t) = f(u)$$

$$u(0) = u_o \qquad\qquad u_o \varepsilon D(A)$$

where A is defined in (6-24) and is the smallest closed extension of the operator T_o, which is defined in (6-23). From lemma 6.2.6, we see that the domain D(A) and range R(A) are both contained in the real Hilbert space $H \equiv L^2(\Omega)$ such that D(A) is dense in $L^2(\Omega)$ and $R(I-(-A)) = L^2(\Omega)$. From lemma 6.2.5, we have the following inequality: there exists a constant $\beta \geq 0$, such that for any $u \varepsilon D(A)$

$$(u,(-A)u)_o \leq -\beta||u||_o^2.$$

Now, since f satisfies (6-22) with $H \equiv L^2(\Omega)$, we can apply theorem 6.41.1. Since $u_o \varepsilon D(A)$ is given, there exists a unique solution, u(t), of (6-25) such that $u(0) = u_o$, and the equilibrium solution (if $f(0) = 0$, the null solution) if it exists, is asymptotically stable if $k < \beta$, and is stable if $k = \beta$, with respect to the L^2 - norm. From definitions 6.41.1 and 6.41.2 we know that there exists a generalized solution, u(x,t), of (6-20) such that for every $t \geq 0$, $u(x,t) \varepsilon H_B^{2m}(\Omega)$, which implies that u(x,t) satisfies the boundary conditions in a generalized sense. Since $u(0) = u_o$, we see that $u(x,0) \equiv u_o(x) \varepsilon \dot{H}_B^{2m}(\Omega)$. Finally, the equilibrium solution (if $f(0) = 0$, the null solution), if it exists, is asymptotically stable if $k < \beta$, and is stable if $k = \beta$, with respect to the L^2 - norm.

Remark 6.41.1. From this theorem, we can see that even if $\beta = 0$, and for every $u \varepsilon D(A)$

$$(u,(-A)u)_o \leq 0,$$

we will have asymptotic stability of the solution to (6-20) if the constant in inequality (6-22), $k < 0$.

6.42. The Nonlinear Stability Problem for the Case n = 1

In this section, we consider the nonlinear stability problem where the space variable $x \varepsilon \Omega = (a,b) \subset R^1$ and $-\infty < a < b < \infty$. Consider the following initial-boundary value problem,

$$\frac{\partial u(x,t)}{\partial t} + A(x,\frac{\partial}{\partial x})u(x,t) = f(u) \qquad x \varepsilon [a,b], t \geq 0$$

$$B_j^+(\frac{\partial}{\partial x})u(x,t)|_{x=b} = B_j^-(\frac{\partial}{\partial x})u(x,t)|_{x=a} = 0 \qquad t \geq 0, 0 \leq j \leq m-1 \quad (6\text{-}26)$$

$$u(x,0) = u_o(x)$$

where $A(x,\frac{\partial}{\partial x})$, $\{B_j^+\}_{j=0}^{m-1}$ and $\{B_j^-\}_{j=0}^{m-1}$ are the linear partial differential operators defined by (6-12), such that the system $(A(x,\frac{\partial}{\partial x}), \{B_j^+, B_j^-\})$ satisfies (5-23) and inequality (6-16), and the nonlinear function, f, satisfies (6-22).

Now, we define the abstract operator A:

$$D(A) = H_B^{2m}[a,b]$$

$$(Au)(x) = A(x,\frac{\partial}{\partial x})u(x) \qquad u\varepsilon D(A)$$

where $H_B^{2m}[a,b]$ is defined in (6-13). We then obtain the evolution equation

$$\frac{du(t)}{dt} + Au(t) = f(u)$$
$$u(0) = u_o \qquad (6\text{-}27)$$

where A is defined above, and D(A) and R(A) are both contained in the real Hilbert space $H \equiv L^2[a,b]$. We are now ready for the main result in this section.

Theorem 6.42.1. Let us consider the nonlinear initial-boundary value problem (6-26), where $A(x,\frac{\partial}{\partial x})$, $\{B_j^+\}$ and $\{B_j^-\}$ are the linear partial differential operators defined in (6-12), such that the system $(A(x,\frac{\partial}{\partial x}), \{B_j^+,B_j^-\})$ satisfies (5-23) and (6-16), while the nonlinear function satisfies (6-22). Then, for any initial value function $u_o(x)\varepsilon H_B^{2m}[a,b]$, there exists a unique generalized solution, u(x,t), such that

(i) for every $t \geq 0$, $u(x,t)\varepsilon H_B^{2m}[a,b]$;

(ii) u(x,t) is a generalized solution of (6-26);

(iii) $u(x,0) \equiv u_o(x)$, and u(x,t) satisfies the boundary conditions in the classical sense;

(iv) If f(0) = 0, the null solution is asymptotically stable if $k < \beta$, is stable if $k = \beta$, with respect to the L^2-norm, where k is the constant in (6-22) and β is the dissipativity constant in (6-7).

Proof. We have defined the evolution equation in (6-27), and as we can see from lemmas 6.3.2 and 6.3.3, the abstract operator A satisfies the following conditions, $D(A) \subset L^2[a,b]$, and $R(A) \subset L^2[a,b]$, such that $D(A)$ is dense in $L^2[a,b]$ and $R(I-(-A)) = L^2[a,b]$. Furthermore, $-A$ satisfies the inequality: there exists a constant $\beta \geq 0$, such that for any $u \varepsilon D(A)$,

$$(u,(-A)u)_o \leq -\beta ||u||_o^2.$$

Since f satisfies (6-22), we can now apply theorem 6.41.1 and see that there exists a unique solution, $u(t)$, of the evolution equation (6-27) such that for every $t \geq 0$, $u(t) \varepsilon D(A) = H_B^{2m}[a,b]$, and $u(0) = u_o$. Also, the equilibrium solution (if $f(0) = 0$, the null solution), if it exists, is asymptotically stable if $k < \beta$, and is stable if $k = \beta$, with respect to the L^2 - norm. Hence, from definition 6.41.2, we see that there exists a generalized solution, $u(x,t)$, of (6-26) such that for every $t \geq 0$, $u(x,t) \varepsilon H_B^{2m}[a,b]$, and $u(x,0) \equiv u_o(x)$, and that $u(x,t)$ satisfies the boundary conditions in the classical sense. Finally, the equilibrium solution (if $f(0) = 0$, the null solution), if it exists, is asymptotically stable if $k < \beta$, and is stable if $k = \beta$, with respect to the L^2 - norm. qed

6.5. Applications to Partial Differential Equations

There is a large class of physical and engineering stability problems which fit into the theory developed in the previous sections. In this section, we consider some applications of both linear and nonlinear initial-boundary value problems which illustrates how the theory can be used to solve specific stability problems. In part 6.51 we consider the Dirichlet problem, and show that the results of Buis in [7] is just a special case of the results in 6.2 and 6.3. In the second part, 6.52, we will consider specific examples of stability problems which will show us the large class of problems that fit into the theory we developed.

6.51. The Dirichlet Problem

In this section, we will show that the results of Buis [7], in which he solved the stability problem for Dirichlet boundary conditions, is a special case of the results in sections 6.2 and 6.3. In other words, if we restrict the system of boundary operators $\{B_j\}_{j=0}^{m-1}$ to be the Dirichlet boundary conditions, our result is the same as that for Buis. We will first consider the case $n \geq 2$.

Let us consider the following initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = 0 \qquad x\varepsilon\Omega, t\geq 0$$

$$\left(\frac{\partial}{\partial n}\right)^j u(x,t) = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \, (0\leq j\leq m-1) \tag{6-28}$$

$$u(x,0) = u_o(x)$$

where $A(x,D)$ is a strongly elliptic formal partial differential operator in $\bar{\Omega}$, of order $2m$, with infinitely differentiable coefficients, and Ω is a bounded domain in R^n, $n \geq 2$, such that $\partial\Omega$ is of class C^∞ and Ω is locally on one side of $\partial\Omega$, as defined in section 3.43.

We have the following result, which will show that for the case $n \geq 2$, if the system of boundary operators are restricted to the Dirichlet boundary conditions, the results of Buis are just a special case of theorem 6.2.1.

Theorem 6.51.1. Let us consider the initial-boundary value problem (6-28) where $A(x,D)$ and Ω satisfy the conditions given in (6-28). If there exists a constant $\beta = \frac{C_1}{C_o} - C_2 > 0$, where C_1 and C_2 are the constants in Garding's inequality (see section 5.2), and C_o is the constant from the continuous injection mapping from $H^m(\Omega)$ into $L^2(\Omega)$, that is $||\cdot||_o \leq C_o||\cdot||_m$, then for any $u_o(x)\varepsilon H^{2m}(\Omega)\cap H^m_o(\Omega)$, there exists a unique generalized solution, $u(x,t)$, of (6-28) such that,

(i) for every $t \geq 0$, $u(x,t)\varepsilon H^{2m}(\Omega)\cap H^m_o(\Omega)$;

(ii) $u(x,0) \equiv u_o(x)$, and $u(x,t)$ satisfies the boundary conditions in a generalized sense;

(iii) the null solution is asymptotically stable with respect to the L^2 - norm.

Proof. From the hypothesis, we know that $A(x,D)$ is strongly elliptic in $\bar{\Omega}$ of order 2m, with infinitely differentiable coefficients, and Ω is a bounded domain in R^n, such that $\partial\Omega$ is of class C^∞ and Ω is locally on one side of $\partial\Omega$. Let us define the system of boundary operators

$$B_j(x,D) = (\frac{\partial}{\partial n})^j \qquad (0\leq j\leq m-1).$$

From example 5.21.1, we see that $\{B_j\}_{j=0}^{m-1}$ is a normal system and satisfies the strong complementary conditions with respect to $A(x,D)$, where the m boundary operators are of order $m_j = j \leq 2m-1$, and are independent of time. Also, the coefficients are constant. Therefore, the system $(A(x,D), \{(\frac{\partial}{\partial n})^j\},\Omega)$ satisfies condition (5-9). We must now show that inequality (6-7) is satisfied. Since $A(x,D)$ is strongly elliptic in $\bar{\Omega}$, it is well known that it satisfies Garding's inequality for $u\varepsilon H^m_o(\Omega)$. Therefore, for any $u\varepsilon C^\infty_B(\Omega)$

$$\begin{aligned}(u,A(x,D)u)_o &\geq C_1||u||^2_m - C_2||u||^2_o \\ &\geq \frac{C_1}{C_o}\,||u||^2_o - C_2||u||^2_o \\ &= (\frac{C_1}{C_o} - C_2)\;||u||^2_o\end{aligned}$$

$$= \beta||u||_o^2.$$

It can be seen that inequality (6-7) is satisfied, since there exists a constant $\beta > 0$, such that for any $u\epsilon C_B^\infty(\Omega)$

$$(u,-A(x,D)u)_o \leq -\beta||u||_o^2.$$

The results of theorem 6.21.1 now apply, if we note that $H_B^{2m}(\Omega) = H^{2m}(\Omega)\cap H_o^m(\Omega)$ (see remark 3.46.2). Therefore, since $u_o(x)\epsilon H^{2m}(\Omega)\cap H_o^m(\Omega)$, there exists a unique generalized solution $u(x,t)$ of (6-28), such that for any $t \geq 0, u(x,t)\epsilon H_B^{2m}(\Omega) = H^{2m}(\Omega)\cap H_o^m(\Omega)$, and $u(x,0) \equiv u_o(x)$. Also, $u(x,t)$ satisfies the boundary conditions in a generalized sense, that is, from the definition of $H_B^{2m}(\Omega)$

$$< (\frac{\partial}{\partial n})^j u >_{2m-j-\frac{1}{2}} = 0 \qquad (0\leq j\leq m-1).$$

Since $\beta > 0$, the null solution is asymptotically stable with respect to the L^2 - norm. qed

It remains to prove the theorem for the case $n = 1$. Let us consider the following initial-boundary value problem,

$$\frac{\partial u(x,t)}{\partial t} + A(x,\frac{\partial}{\partial x})u(x,t) = 0 \qquad x\epsilon[a,b],t\geq 0, -\infty < a < b < \infty$$

$$(\frac{\partial}{\partial x})^j u(b,t) = (\frac{\partial}{\partial x})^j u(a,t) = 0 \qquad \text{for each fixed } t\geq 0 \ (0\leq j\leq m-1)$$

$$u(x,0) = u_o(x) \qquad (6\text{-}29)$$

where $A(x,\frac{\partial}{\partial x})$ is a strongly elliptic linear differential operator on $[a,b]$ defined in (5-20) of order 2m, with infinitely differentiable coefficients.

We have the following result for the case $n = 1$.

<u>Theorem 6.51.2.</u> Let us consider the initial-boundary value problem (6-29). If there exists a constant $\beta = \frac{C_1}{C_o} - C_2 > 0$, where C_o, C_1, and C_2 are defined as in theorem 6.51.1, then for any

$u_o(x)\varepsilon H^{2m}[a,b]\cap H_o^m[a,b]$, there exists a unique generalized solution, $u(x,t)$ of (6-29) such that

(i) for every $t \geq 0$, $u(x,t)\varepsilon H^{2m}[a,b]\cap H_o^m[a,b]$;

(ii) $u(x,0) \equiv u_o(x)$, and $u(x,t)$ satisfies the boundary conditions in the classical sense;

(iii) the null solution is asymptotically stable with respect to the L^2 - norm.

Proof. From the hypothesis, $A(x,\frac{\partial}{\partial x})$ is strongly elliptic in $[a,b]$ with infinitely differentiable coefficients. Let us define the boundary operators, for each fixed $t \geq 0$,

$$B_j^+(\frac{\partial}{\partial x})u(x,t)\Big|_{x=b} = (\frac{\partial}{\partial x})^j u(b,t) \qquad (0\leq j\leq m-1)$$

$$B_j^-(\frac{\partial}{\partial x})u(x,t)\Big|_{x=a} = (\frac{\partial}{\partial x})^j u(a,t).$$

We can readily see that $B_j^+(\frac{\partial}{\partial x}), B_j^-(\frac{\partial}{\partial x})$ are all linear differential operators with constant coefficients of orders $m_j^+ = m_j^- = j\leq 2m-1$, and are independent of time. Also, the systems $\{B_j^+\}\{B_j^-\}$ are linearly independent. This shows that the system $(A(x,\frac{\partial}{\partial x}), \{B_j^+,B_j^-\})$ satisfies (5-23). As was proved in theorem 6.51.1, from the definition of $H_B^{2m}[a,b]$ (see (6-13)), there exists a constant $\beta > 0$, such that for any $u\varepsilon H_B^{2m}[a,b]$

$$(u,-A(x,\frac{\partial}{\partial x})u)_o \leq -\beta||u||_o^2.$$

The results of theorem 6.3.1, now apply, if we note the definition of $H_B^{2m}[a,b] = H^{2m}[a,b]\cap H_o^m[a,b]$. Hence, there exists a unique generalized solution, $u(x,t)$, of (6-29) such that for any $t \geq 0, u(x,t)\varepsilon H_B^{2m}[a,b] = H^{2m}[a,b]\cap H_o^m[a,b]$, and $u(x,0) \equiv u_o(x)$. Also, $u(x,t)$ satisfies the boundary conditions in the classical sense. Finally, since $\beta > 0$, the null solution is asymptotocally stable with respects to the L^2 - norm.

Combining theorems 6.51.1 and 6.51.2, we can see that the Dirichlet problem is a special case of the theory developed in sections 6.2 and 6.3.

6.52. Specific Examples

We will now give some specific examples which show that many physical problems can fit into the theory we developed.

Example 6.52.1. Let us consider the following diffusion equation with the boundary condition that the heat leaves normal to the surface at a rate proportional to the amount of heat at the surface, and there is no heat source.

$$\frac{\partial u(x,t)}{\partial t} - \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2 u(x,t) + bu(x,t) = 0 \qquad x\varepsilon\Omega, t\geq 0$$

$$\sigma_o(x)u(x,t) + \left(\frac{\partial}{\partial n}\right)u(x,t) = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \qquad (6\text{-}30)$$

$$u(x,0) = u_o(x)$$

where $\Omega\subset R^n$, $n\geq 2$, $b>0$, $\min_{x\varepsilon\partial\Omega} \sigma_o(x) = k_1 > 0$.

Let us define the operators $A(D)$, $B_o(x,D)$ and the spaces $C_B^\infty(\Omega)$, $H_B^2(\Omega)$:

$$A(D) = -\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2 + b$$

$$B_o(x,D) = \sigma_o(x) + \frac{\partial}{\partial n}$$

where the order of $A(D)$ is $2m=2$. Ω is a bounded domain such that $\partial\Omega$ is of class C^∞, Ω locally on one side of $\partial\Omega$.

$$C_B^\infty(\Omega) = \{u\varepsilon C^\infty(\overline{\Omega})|\sigma_o(x)u + \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$$

$$H_B^2(\Omega) = \text{completion of } C_B^\infty(\Omega) \text{ with respect to the } H^2 \text{ - norm}$$
$$= \{ u\varepsilon H^2(\Omega)|\langle B_o u\rangle_{\frac{1}{2}} = 0 \}.$$

It will be shown that problem (6-30) satisfies the conditions of theorem 6.2.1, which implies that for any $u_o(x)\varepsilon H_B^2(\Omega)$, there exists a unique generalized solution, $u(x,t)$, to (6-30), such that for every $t \geq 0$, $u(x,t)\varepsilon H_B^2(\Omega)$ with $u(x,0) \equiv u_o(x)$ and $u(x,t)$ satisfies the boundary conditions in the generalized sense. Also, the null solution is asymptotically stable with respect to the L^2 - norm.

(a) From the hypothesis, Ω is a bounded domain in R^n, $n\geq 2$, such that $\partial\Omega\varepsilon C^\infty$, Ω locally on one side of $\partial\Omega$.

(b) $A(D)$ is strongly elliptic in $\overline{\Omega}$, with infinitely differentiable coefficients. This is seen by letting $\xi = (\xi_1,\ldots,\xi_n) \neq 0\varepsilon R^n$, we have

$$(-1)^m A_o(\xi) = (-1)[-\textstyle\sum_{i=1}^n \xi_i^2] = \sum_{i=1}^n \xi_i^2 > 0 .$$

Since the coefficients are constant, $a_\alpha \varepsilon C^\infty(\overline{\Omega})$.

(c) By definition B_o is independent of time, with constant coefficients.

(d) $\{B_o\}$ is a normal system. This is obvious, since B_o is of the form (5-8) and satisfies definition 5.21.3. Also, the order of B_o is $m_o = 1 \leq 2m-1 = 1$.

(e) B_o satisfies the strong complementary condition.

Since we have shown in example 5.22.2 that the Laplacian is invariant under the transformation θ_i, we can use theorem 5.22.2 to verify the strong complementary condition, that is, it suffices to verify the condition for one point $X_o \varepsilon \partial\Omega$. Let us pick the point $X_o \varepsilon \partial\Omega$, such that the normal vector at X_o is $\xi' = (0,\ldots,0,1)$, and the tangent hyperplane at X_o is parallel to the plane $x_n = 0$, so we can let the vector in the tangent hyperplane be $\xi = (\xi_1,\ldots,\xi_{n-1},0) \neq 0$. Let $\lambda > 0$.

First, we will find the roots with positive imaginary part of the polynomial

$$(-1)^m A_o(\xi + \tau\xi') + \lambda.$$

If we define $\eta^2 = \sum_{i=1}^{n-1} \xi_i^2$, then

$$\begin{aligned}(-1)^m A_o(\xi + \tau\xi') + \lambda &= (-1)A_o(\xi_1,\ldots,\xi_{n-1},\tau) + \lambda \\ &= -\left[-\textstyle\sum_{i=1}^{n-1} \xi_i^2 - \tau^2\right] + \lambda \\ &= \tau^2 + \eta^2 + \lambda.\end{aligned}$$

The m=1 root with positive imaginary part is

$$\tau^*(\eta,\lambda) = + i\sqrt{\eta^2 + \lambda} \; .$$

Hence, we obtain the polynomial

$$M^*(\eta,\lambda) = \tau - \tau^*(\eta,\lambda) = \tau - i\sqrt{\eta^2 + \lambda} \; .$$

Next, we must show B_o is linearly independent modulo $M^*(\eta,\lambda)$. Since

$$B_o(x,D) = \left(\frac{\partial}{\partial x_n}\right) \text{ at } X_o \varepsilon \partial\Omega, \text{ we have}$$

$$B_{oo}(\xi + \tau\xi') = B_{oo}(\xi_1,\ldots,\xi_{n-1},\tau) = \tau$$

and on dividing $B_{oo}(\xi + \tau\xi')$ by M^*, we obtain the remainder

$$r_o = i\sqrt{\eta^2 + \lambda} \quad .$$

Since $\eta \neq 0 \varepsilon R^{n-1}$ and $\lambda \geq 0$, we obtain

$$\det\,(i\sqrt{\eta^2 + \lambda}) = i\sqrt{\eta^2 + \lambda} \neq 0.$$

This shows B_{oo} is linearly independent modulo M^*, which implies that B_o satisfies the strong complementary condition.

Therefore, the system $(A(D), B_o(x,D), \Omega)$ satisfies (5-9), and it remains to verify inequality (6-7). For any $u \varepsilon C_B^\infty(\Omega)$, we have the following inequality using integration by parts and the boundary conditions

$$(u,-A(D)u)_o = \int_\Omega [\sum_{i=1}^n (\frac{\partial}{\partial x_i})^2 u(x)] u(x)dx - \int_\Omega bu^2(x)dx$$

$$= -\int_\Omega \sum_{i=1}^n (\frac{\partial u(x)}{\partial x_i})^2 dx + \int_{\partial\Omega} u(x)\frac{\partial u(x)}{\partial n}\, d(\partial\Omega) - \int_\Omega bu^2(x)dx$$

$$= -\int_\Omega \sum_{i=1}^n (\frac{\partial u(x)}{\partial x_i})^2 dx - \int_{\partial\Omega} \sigma_o(x)u^2(x)d(\partial\Omega) - \int_\Omega bu^2(x)dx$$

$$\leq - \min_{x\varepsilon\Omega}\,(1,\sigma_o(x))\,\,[\int_\Omega \sum_{i=1}^n (\frac{\partial u(x)}{\partial x_i})^2 dx + \int_{\partial\Omega} u^2(x)d(\partial\Omega)]$$

from a well known inequality by Friedrich, we have for some $C_o > 0$

$$\leq -\min(1,k_1)C_o\int_\Omega u^2(x)dx$$

$$= -\beta||u||_o^2.$$

where $\beta = C_o \min\,(1,k_1) > 0$. This verifies inequality (6-7), where $\beta > 0$. Therefore, the hypothesis of theorem 6.2.1 is verified and from the results of the theorem, there exists a unique generalized solution of (6-30) such that $u(x,0) \equiv u_o(x) \epsilon H_B^2(\Omega)$, $u(x,t)$ satisfies the boundary conditions in a generalized sense and since $\beta > 0$, the null solution is asymptotically stable with respect to the L^2 - norm.

Example 6.52.2. We will consider the diffusion equation for the case $n = 1$, with the boundary condition that heat emanates from the ends of a rod at a rate proportional to the amount of heat at the rods ends,

$$\frac{\partial u(x,t)}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2 u(x,t) + bu(x,t) = 0 \qquad x\epsilon[0,1],\ t\geq 0$$

$$\frac{\partial u}{\partial x}(1,t) + \beta u(1,t) = \frac{\partial u}{\partial x}(0,t) - \alpha u(0,t) = 0 \quad \text{for each fixed } t\geq 0 \tag{6-31}$$

$$u(x,0) = u_o(x)$$

where $b > 0$, $\alpha \geq 0$, $\beta \geq 0$ and $\alpha,\beta \neq 0$ simultaneously.

Let us define the operators $A(\frac{\partial}{\partial x})$, $B_o^+(\frac{\partial}{\partial x})$, $B_o^-(\frac{\partial}{\partial x})$ and the function space $H_B^2[0,1]$,

$$A\left(\frac{\partial}{\partial x}\right) = -\left(\frac{\partial}{\partial x}\right)^2 + b$$

$$B_o^+\left(\frac{\partial}{\partial x}\right)u(1,t) = \frac{\partial u}{\partial x}(1,t) + \beta u(1,t) \qquad \text{for each fixed } t\geq 0$$

$$B_o^-\left(\frac{\partial}{\partial x}\right)u(0,t) = \frac{\partial u}{\partial x}(0,t) - \alpha u(0,t)$$

where the order of $A(\frac{\partial}{\partial x})$ is 2,

$$H_B^2[0,1] = \{u\epsilon H^2[0,1] \mid u'(1) + \beta(1) = 0, u'(0) - \alpha u(0) = 0\}.$$

We will show that problem (6-31) satisfies the hypothesis of theorem 6.3.1, which implies that for any $u_o(x)\epsilon H_B^2[0,1]$, there exists a unique generalized solution, $u(x,t)$, of (6-31), with $u(x,0) \equiv u_o(x)$, such that $u(x,t)$ satisfies the boundary conditions and the null solution is asymptotically stable with respect to the L^2 - norm.

a) As in example 6.52.1, we can see $A(\frac{\partial}{\partial x})$ is strongly elliptic in [0,1].

b) B_o^+ is linearly independent, since there is one term of order $m_o = 1 \leq 2m-1 = 1$. Similarly, B_o^- is linearly independent. This shows that the system $(A(\frac{\partial}{\partial x}), B_o^+(\frac{\partial}{\partial x}), B_o^-(\frac{\partial}{\partial x}))$ satisfies (5-23), and it remains to verify inequality (6-16). In order to do this, we need the following lemma.

<u>Lemma 6.52.1.</u> If $\alpha \geq 0, \beta \geq 0$ and $\alpha, \beta \neq 0$ simultaneously, we have the following inequality

$$\int_o^1 (\frac{du(x)}{dx})^2 dx + \beta u^2(1) + \alpha u^2(0) \geq C_o \int_o^1 u^2(x)dx, \text{ for some } C_o > 0.$$

<u>Proof.</u> Let $u \in L^2[0,1]$. We see that

$$|u(x)|^2 = |\int_o^x u'(\xi)d\xi + u(0)|^2$$

$$\leq 2|\int_o^x u'(\xi)d\xi|^2 + 2|u(0)|^2$$

$$\leq 2\int_o^x d\xi \int_o^x |u'(\xi)|^2 d\xi + 2|u(0)|^2$$

$$\leq 2\cdot 1\int_o^1 |u'(\xi)|^2 d\xi + 2|u(0)|^2.$$

For the sake of argument, we can assume $\alpha \neq 0$. Therefore,

$$\int_o^1 |u(x)|^2 dx \leq 2 \int_o^1 |u'(x)|^2 dx + 2|u(0)|^2$$

$$= 2 \int_o^1 |u'(x)|^2 dx + (\frac{2}{\alpha})\alpha |u(0)|^2$$

$$\leq C_1[\int_o^1 |u'(x)|^2 dx + \alpha u^2(0)], \text{ where } C_1 = \max (2, \frac{2}{\alpha})$$

$$\leq C_1[\int_o^1 |u'(x)|^2 dx + \beta u^2(1) + \alpha u^2(0)].$$ qed

Now we can prove inequality (6-16). For any $u\varepsilon H_B^2\,[0,1]$, from integration by parts, the boundary conditions and from lemma 6.52.1 we have

$$(u,-A(\frac{\partial}{\partial x})u)_o = \int_o^1[(\frac{d}{dx})^2u(x) - bu(x)]u(x)\,dx$$

$$= \int_o^1[(\frac{d}{dx})^2u(x)]u(x)dx - b\int_o^1 u^2(x)dx$$

$$= -\int_o^1(\frac{du(x)}{dx})^2dx + u(x)u'(x)\Big|_o^1 - b\int_o^1 u^2(x)dx$$

$$= -\int_o^1(\frac{du(x)}{dx})^2dx + u(1)u'(1) - u(0)u'(0) - b\int_o^1 u^2(x)dx$$

$$= -\int_o^1(\frac{du(x)}{dx})^2dx - \beta u^2(1) - \alpha u^2(0) - b\int_o^1 u^2(x)dx$$

$$\leq -[\int_o^1(\frac{du(x)}{dx})^2dx + \beta u^2(1) + \alpha u^2(0)]$$

$$\leq -C_o\int_o^1 u^2(x)dx$$

$$= -\beta||u||_o^2$$

where $\beta = C_o > 0$. This proves inequality (6-16). Since we have verified the hypothesis of theorem 6.3.1, we can utilize those results. Hence, there exists a unique generalized solution, $u(x,t)$, of (6-31) with $u(x,0) \equiv u_o(x)$, satisfying the boundary conditions in the classical sense. Also, since $\beta = C_o > 0$, the null solution is asymptotically stable with respect to the L^2 - norm.

Example 6.52.3. Let us consider the stability problem in example 6.52.1 with a heat source, or a nonlinear function on the right hand side. Consider the initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} - \sum_{i=1}^{n}(\frac{\partial}{\partial x_i})^2u(x,t) + bu(x,t) = f(u) \qquad x\varepsilon\Omega, t\geq 0$$

$$\sigma_o(x)u(x,t) + \frac{\partial u(x,t)}{\partial n} = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \qquad (6-32)$$

$$u(x,0) = u_o(x)$$

$$f(u) = k\frac{u^2}{\lambda^2 + u^2} \qquad \lambda^2 > 0$$

where the operators $A(D), B_o(x,D)$ are defined as in example 6.52.1, $b > 0$ and $\min_{x\varepsilon\partial\Omega} \sigma_o(x) = k_1 > 0$. I assert, if $|\frac{k}{\lambda}| \leq C_o \min(1,k_1)$, where C_o is the constant in Friedrich's inequality in example 6.52.1, then there exists a unique generalized solution of (6-32) satisfying the boundary condition and initial condition, such that if $|\frac{k}{\lambda}| < C_o \min(1,k_1)$ then the null solution is asymptotically stable, and if $|\frac{k}{\lambda}| = C_o \min(1,k_1)$ the null solution is stable with respect to the L^2 - norm. From example 6.52.1, we know that the system $(A(D), B_o(x,D), \Omega)$ satisfies (5-9) and inequality (6-7) with $\beta \equiv C_o \min(1,k_1)$, and in order to use the results of theorem 6.41.2, it must be shown that the function f satisfies (6-22). We can see readily that f maps all of $L^2(\Omega)$ into $L^2(\Omega)$ and maps bounded sets into bounded sets. It suffices to verify the following inequality

$$||f(u) - f(v)||_o \leq |\frac{k}{\lambda}| \; ||u-v||_o \qquad \text{for every } u,v\varepsilon L^2(\Omega) \qquad (6-33)$$

which shows that f is continuous on the strong topology of $L^2(\Omega)$ to the weak topology and

$$(f(u) - f(v), u-v)_o \leq ||f(u) - f(v)||_o \; ||u-v||_o \leq |\frac{k}{\lambda}| \; ||u-v||_o^2$$

which is our desired result. To see (6-33), we have

$$||f(u) - f(v)||_o^2 = \int_\Omega |f(u) - f(v)|^2 dx = \int_\Omega |\frac{ku^2}{\lambda^2 + u^2} - \frac{kv^2}{\lambda^2 + v^2}|^2 dx$$

$$= k^2\int_\Omega \left|\frac{\lambda^2u^2 + u^2v^2 - \lambda^2v^2 - u^2v^2}{(\lambda^2 + u^2)(\lambda^2 + v^2)}\right|^2 dx$$

$$= k^2|\lambda|^4\int_\Omega \left|\frac{u^2 - v^2}{(\lambda^2 + u^2)(\lambda^2 + v^2)}\right|^2 dx$$

$$= k^2|\lambda|^4\int_\Omega \left|\frac{u + v}{(\lambda^2 + u^2)(\lambda^2 + v^2)}\right|^2 |u-v|^2 dx$$

$$\leq k^2|\lambda|^4 \max_{x\varepsilon\Omega} \frac{|u(x) + v(x)|^2}{(\lambda^2 + u^2(x))^2(\lambda^2 + v^2(x))^2} \int_\Omega |u-v|^2 dx$$

$$= k^2|\lambda|^4\left(\frac{1}{|\lambda|^3}\right)^2 ||u-v||_o^2$$

$$= \frac{k^2}{|\lambda|^2} ||u-v||_o^2.$$

This follows, since it can be easily shown that for any u,v real numbers

$$\frac{|u + v|}{(\lambda^2 + u^2)(\lambda^2 + v^2)} \leq \frac{1}{|\lambda|^3} .$$

Hence, we have verified inequality (6-33) which shows that f satisfies (6-22). Since problem (6-32) satisfies the hypothesis of theorem 6.41.2, we can apply these results and see that there exists a unique generalized solution $u(x,t)$ of (6-32) with $u(x,0) \equiv u_o(x)$, such that $u(x,t)$ satisfies the boundary conditions in the generalized sense, and since $f(0) = 0$, the null solution is asymptotically stable with respect to the L^2 - norm if $|\frac{k}{\lambda}| < C_o \min(1,k_1)$, and is stable if $|\frac{k}{\lambda}| = C_o \min(1,k_1)$.

Example 6.52.4. Let us consider a more involved problem, the generalized Laplacian, found in Lions-Magenes [18]

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \Delta^2 u(x,t) = f(u) && x\varepsilon\Omega, t\geq 0 \\ &\Delta u(x,t) = 0, \frac{\partial}{\partial n}(\Delta u)(x,t) = 0 && x\varepsilon\partial\Omega, t\geq 0 \\ &u(x,0) = u_o(x) && \end{aligned} \qquad (6\text{-}34)$$

where $A(D), B_o(D), B_1(D)$, f and Ω are defined below:

$$A(D) = \Delta^2 = \sum_{i,j=1}^{n} (\frac{\partial}{\partial x_i})^2 (\frac{\partial}{\partial x_j})^2$$

where the order of A(D) is 4, and m = 2;

$$B_o(D) = \Delta, \; B_1(D) = \frac{\partial}{\partial n} (\Delta)$$

where $\Delta = \sum_{i=1}^{n} (\frac{\partial}{\partial x_i})^2$ = Laplacian;

f(u) is a nonlinear function, mapping all of $L^2(\Omega)$ into $L^2(\Omega)$, and is continuous from the strong topology on $L^2(\Omega)$ to the weak topology, mapping bounded sets into bounded sets with f(0) = 0, and there exists a constant $k \leq 0$, such that for any $u, v \varepsilon L^2(\Omega)$

$$(f(u) - f(v), u-v)_o \leq k||u-v||_o^2;$$

Ω is a bounded domain in R^n, $n \geq 2$, such that $\partial\Omega$ is of class C^∞, Ω locally on one side of $\partial\Omega$. We define the function spaces, $C_B^\infty(\Omega)$ and $H_B^4(\Omega)$,

$$C_B^\infty(\Omega) = \{u \varepsilon C^\infty(\overline{\Omega}) | \Delta u = \frac{\partial}{\partial n}(\Delta u) = 0 \text{ on } \partial\Omega\}$$

$$H_B^4(\Omega) = \text{completion of } C_B^\infty(\Omega) \text{ with respect to the } H^4 \text{ - norm}$$

$$= \{u \varepsilon H^4(\Omega) | < B_o u >_{3/2} = < B_1 u >_{1/2} = 0\}.$$

I assert that for any $u_o(x) \varepsilon H_B^4(\Omega)$, there exists a unique generalized solution, u(x,t), of (6-34) satisfying the boundary conditions and initial condition, such that if k < 0, the null solution is asymptotically stable (if k = 0, the null solution is stable) with respect to the L^2 - norm.

To prove this assertion, we will show that problem (6-34) verifies the hypothesis of theorem 6.41.2.

(a) The smoothness property on Ω is true from the above hypothesis.

(b) A(D) is strongly elliptic in $\bar{\Omega}$. Indeed, for every $\xi \neq 0 \varepsilon R^n$

$$(-1)^m A_o(\xi) = \sum_{i,j=1}^n \xi_i^2 \xi_j^2 = \xi_1^2 (\sum_{j=1}^n \xi_j^2) + \dots + \xi_n^2 (\sum_{j=1}^n \xi_j^2)$$

$$= (\sum_{j=1}^n \xi_j^2)^2 > 0 .$$

(c) $\{B_o(D), B_1(D)\}$ is a normal system. Indeed, the orders $m_o = 2, m_1 = 3$ are distinct. Now, after the transformation θ_i (see section 5.22) we obtain the transformed boundary operators

$$\mathcal{B}_o(D) = (\frac{\partial}{\partial x'_n})^2 + \sum_{i=1}^{n-1} (\frac{\partial}{\partial x'_i})^2 = (\frac{\partial}{\partial n})^2 + \sum_{i=1}^{n-1} (\frac{\partial}{\partial x'_i})^2$$

$$\mathcal{B}_1(D) = (\frac{\partial}{\partial x'_n})[(\frac{\partial}{\partial x'_n})^2 + \sum_{i=1}^{n} (\frac{\partial}{\partial x'_i})^2] = (\frac{\partial}{\partial n})^3 + [\sum_{i=1}^{n} (\frac{\partial}{\partial x'_i})^2](\frac{\partial}{\partial n}) \tag{6-35}$$

which are in the equivalent form (5-8) and satisfies definition 5.21.3.

(d) $\{B_o(D), B_1(D)\}$ satisfies the strong complementary condition. Indeed, from (6-35) and since A(D) is invariant under θ_i (see example 5.22.1), we have that the system $\{A(D), B_o(D), B_1(D)\}$ is invariant under θ_i. Therefore, we can apply theorem 5.22.2 where it suffices to verify the strong complementary condition for just one point on $\partial\Omega$. Let us consider the point $X \varepsilon \partial\Omega$ such that the tangent hyperplane to $\partial\Omega$ at X is parallel to the plane $x_n = 0$, and the normal vector to $\partial\Omega$ at X is parallel to the x_n - axis. Let $\xi' = (0,\dots,1)$ be the normal vector to $\partial\Omega$ at X and $\xi = (\xi_1,\dots,\xi_{n-1},0)$ be in the tangent hyperplane to $\partial\Omega$ at X. Let $\lambda > 0$.

First, we must find the m = 2 roots with positive imaginary part of the polynomial

$$(-1)^m A_o(\xi + \tau\xi') + \lambda.$$

Letting $\eta^2 = \sum_{i=1}^n \xi_i^2$, we have

$$\begin{aligned}(-1)^m A_o(\xi + \tau\xi') + \lambda &= A_o(\xi_1,\ldots,\xi_{n-1},\tau) + \lambda \\ &= (\xi_1^2 + \ldots + \xi_{n-1}^2 + \tau^2)^2 + \lambda \\ &= (\eta^2 + \tau^2)^2 + \lambda \\ &= \tau^4 + 2\eta^2\tau^2 + (\eta^4 + \lambda).\end{aligned}$$

It can be easily seen that

$$\tau^2 = \frac{-2\eta^2 \pm \sqrt{4\eta^4 - 4(\eta^4 + \lambda)}}{2} = -\eta^2 \pm i\sqrt{\lambda}$$

and if we let $\alpha = -\eta^2 + i\sqrt{\lambda}$, we have

$$(-1)^m A_o(\xi + \tau\xi') + \lambda = (\tau^2 - \alpha)(\tau^2 - \bar{\alpha}) = (\tau - \alpha^{1/2})(\tau + \alpha^{1/2})(\tau - \bar{\alpha}^{1/2})(\tau + \bar{\alpha}^{1/2})$$

where by a direct calculation we obtain

$$\sqrt{\alpha} = (\alpha)^{1/2} = \frac{1}{\sqrt{2}}\left[(\sqrt{\eta^4 + \lambda} - \eta^2)^{1/2} + i(\sqrt{\eta^4 + \lambda} + \eta^2)^{1/2}\right]$$

$$\sqrt{\bar{\alpha}} = (\bar{\alpha})^{1/2} = \frac{1}{\sqrt{2}}\left[-(\sqrt{\eta^4 + \lambda} - \eta^2)^{1/2} + i(\sqrt{\eta^4 + \lambda} + \eta^2)^{1/2}\right].$$

Therefore, the roots are

$$\tau_1^*(\eta,\lambda) = \sqrt{\alpha}, \quad \tau_2^*(\eta,\lambda) = \sqrt{\bar{\alpha}}$$

and we obtain the following polynomial

$$M^*(\eta,\lambda) = (\tau-\tau_1^*)(\tau-\tau_2^*) = (\tau-\sqrt{\alpha})(\tau-\sqrt{\bar\alpha}) = \tau^2 - (\sqrt{\alpha} + \sqrt{\bar\alpha})\tau+\sqrt{\alpha}\sqrt{\bar\alpha} .$$

Next, we must prove the polynomials $B_{oo}(\xi + \tau\xi')$ and $B_{1o}(\xi + \tau\xi')$ are linearly independent modulo $M^*(\eta,\lambda)$. To show this, we see that

$$B_{oo}(\xi + \tau\xi') = B_{oo}(\xi_1,\ldots,\xi_{n-1},\tau) = \tau^2 + \eta^2$$

and upon division by $M^*(\eta,\lambda)$ we get the remainder

$$r_o = (\eta^2 - \sqrt{\alpha}\sqrt{\bar\alpha}) + (\sqrt{\alpha} + \sqrt{\bar\alpha})\tau.$$

Similarly, for B_{1o} we see that

$$B_{1o}(\xi + \tau\xi') = B_{1o}(\xi_1,\ldots,\xi_{n-1},\tau) = \tau(\tau^2 + \eta^2) = \tau^3 + \eta^2\tau$$

and upon division by $M^*(\eta,\lambda)$ we get the remainder

$$r_1 = -\sqrt{\alpha}\sqrt{\bar\alpha}\,(\sqrt{\alpha} + \sqrt{\bar\alpha}) + [(\sqrt{\alpha} + \sqrt{\bar\alpha})^2 + \eta^2 - \sqrt{\alpha}\sqrt{\bar\alpha}]\tau.$$

Hence, since $\eta \neq 0$ and $\lambda > 0$

$$\det\begin{bmatrix} (\eta^2 - \sqrt{\alpha}\sqrt{\bar\alpha}) & (\sqrt{\alpha} + \sqrt{\bar\alpha}) \\ -\sqrt{\alpha}\sqrt{\bar\alpha}(\sqrt{\alpha} + \sqrt{\bar\alpha}) & (\sqrt{\alpha} + \sqrt{\bar\alpha})^2 + (\eta^2 - \sqrt{\alpha}\sqrt{\bar\alpha}) \end{bmatrix}$$

$$= (\eta^2 - \sqrt{\alpha}\sqrt{\bar\alpha})[(\sqrt{\alpha} + \sqrt{\bar\alpha})^2 + \eta^2 - \sqrt{\alpha}\sqrt{\bar\alpha}] + \sqrt{\alpha}\sqrt{\bar\alpha}(\sqrt{\alpha} + \sqrt{\bar\alpha})^2$$

$$= \eta^2(\sqrt{\alpha} + \sqrt{\bar\alpha})^2 + (\eta^2 - \sqrt{\alpha}\sqrt{\bar\alpha})^2$$

$$= \eta^2(\alpha + \bar\alpha + 2\sqrt{\alpha}\sqrt{\bar\alpha}) + \eta^4 - 2\eta^2\sqrt{\alpha}\sqrt{\bar\alpha} + \alpha\,\bar\alpha$$

$$= \eta^2(\alpha + \bar\alpha) + \eta^4 + \alpha\,\bar\alpha$$

$$= \eta^4 + \eta^2(-2\eta^2) + (\eta^4 + \lambda)$$

$$= \eta^4 - 2\eta^4 + \eta^4 + \lambda$$

$$= \lambda$$

$$\neq 0.$$

This shows that $B_{oo}(\xi + \tau\xi')$, $B_{1o}(\xi + \tau\xi')$ are linearly independent modulo $M^*(\eta,\lambda)$. Therefore, the system $\{B_o(D), B_1(D)\}$ satisfies the strong complementary condition.

We must now verify inequality (6-7). Letting $u \varepsilon C_B^\infty(\Omega)$, we have from Green's formula found in Mikhlin [21], and the boundary conditions

$$(u,-A(D)u)_o = \int_\Omega u(x)(-\Delta^2 u(x))dx$$

$$= -\int_\Omega u(x)\Delta^2 u(x)dx$$

$$= -\int_\Omega [\Delta u(x)]^2 dx - \int_{\partial\Omega} [u(x)\frac{\partial}{\partial n}(\Delta u(x)) - \Delta u(x)\frac{\partial u(x)}{\partial n}] d(\partial\Omega)$$

$$= -\int_\Omega [\Delta u(x)]^2 dx$$

$$\leq 0$$

Hence, we have verified inequality (6-7).

If $f \equiv 0$, since the system $(A(D), B_o(D), B_1(D), \Omega)$ satisfies the hypothesis of theorem 6.2.1, we have that for any initial function $u_o(x) \varepsilon H_B^4(\Omega)$, there exists a unique generalized solution, $u(x,t)$, of (6-34), such that for any $t \geq 0$, $u(x,t) \varepsilon H_B^4(\Omega)$ and $u(x,t)$ satisfies the boundary conditions in a generalized sense, and $u(x,0) \equiv u_o(x)$. Also, since $\beta = 0$, the null solution is stable with respect to the L^2 - norm.

If $f \not\equiv 0$, then f satisfies (6-22) and we have from theorem 6.41.2 the same result as above whereby, if $k < 0$, the null solution is asymptotically stable (and if $k = 0$, the null solution is stable) with respect to the L^2 - norm. An example of such a nonlinear function is the one in example 6.52.3,

$$f(u) = k \frac{u^2}{\lambda^2 + u^2}, \quad \lambda^2 > 0, k < 0.$$

These examples illustrate how the theory developed in sections 6.1 to 6.4 can be used to solve a large class of stability problems with very general boundary conditions, where these problems are of the form (6-1) and (6-2).

7.0. STABILITY OF SOLUTIONS TO THE GENERAL BOUNDARY VALUE PROBLEM:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u = f(u)$$

In the previous chapter we solved the stability problem for the initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x\varepsilon\Omega, t\geq 0$$

$$B_j(x,D)u(x,t) = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \quad (0\leq j\leq m-1)$$

$$u(x,0) = u_o(x) \qquad \frac{\partial u(x,0)}{\partial t} = v_o(x)$$

which includes as an example the heat equation, or diffusion equation. But there is a large class of physical problems which does not fit into the above theory, an example being the wave equation. In this chapter, we will consider the following elliptic partial differential equation.

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad x\varepsilon\Omega, t\geq 0 \qquad (7\text{-}1)$$

where a is a constant ≥ 0 with the general boundary conditions

$$B_j(x,D)u(x,t) = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \quad (0\leq j\leq m-1)$$

and initial condition

$$u(x,0) = u_o(x),$$

where u(x,t) is a vector valued function, such that for every $t \geq 0$,

$u(x,t)$ is in some prescribed Hilbert space, and f is a nonlinear function defined on the Hilbert space.

We solve (7-1) by reducing the problem to a system of equations, as in the linear differential equations case, of the form

$$\frac{\partial \underline{u}}{\partial t} + \underline{A}\,\underline{u} = \underline{f}(\underline{u}) \tag{7-2}$$

where $\underline{A}$ is a 2 x 2 matrix with operator elements, and $\underline{u}$ and $\underline{f}(\underline{u})$ are 2 - dimensional vectors whose elements are functions in a prescribed Hilbert space. By choosing the correct base Hilbert space we can consider the equation (7-2) as an abstract operator equation of the form

$$\frac{d\underline{u}(t)}{dt} + A\underline{u}(t) = \underline{f}(\underline{u})$$

where A is an abstract operator defined on some base Hilbert space, in this case $H \equiv H_B^m(\Omega) \times L^2(\Omega)$, $\underline{u}$ is an element of H, and $\underline{f}(\underline{u})$ is a nonlinear function defined on H into H, and again using the results of Pao [23], we obtain sufficient conditions to ensure the existence, uniqueness and stability of a solution to (7-1). We will consider the linear case, $f(u) \equiv 0$, and also the nonlinear case, for $\Omega \subset R^n$ with $n \geq 1$. The results of Pao [24] for a Dirichlet problem are shown to be a special case of the results in this chapter. Examples will be considered to see how this theory can be applied to solving specific stability problems.

7.1. Stability of the Solution of a Linear Initial-Boundary Value Problem

In this section, we will give sufficient conditions to guarantee the existence, uniqueness and stability of the null solution to the initial-boundary value problem (7-1).

Let us consider the following initial-boundary value problem,

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = 0 \qquad x\varepsilon\Omega, t\geq 0$$

$$B_j(x,D)u(x,t) = 0 \qquad x\varepsilon\partial\Omega, t\geq 0 \quad (0\leq j\leq m-1) \tag{7-3}$$

$$u(x,0) = u_o(x), \ \frac{\partial u(x,0)}{\partial t} = v_o(x)$$

where a is a constant ≥ 0,

$A(x,D)$ is a linear formal partial differential operator with infinitely differentiable coefficients in $\overline{\Omega}$, written in the divergence form

$$A(x,D)(\cdot) = \sum_{|\rho|,|\sigma|\leq m} (-1)^{|\rho|} D^\rho (a_{\rho\sigma}(x)D^\sigma(\cdot))$$

and the boundary operators, $\{B_j\}_{j=0}^{m-1}$, are written in the form

$$B_j(x,D) = \sum_{|h|\leq m_j} b_{jh}(x)D^h \qquad (0\leq j\leq m-1)$$

such that the system $(A(x,D),\{B_j\}, \Omega)$ satisfies:

(i) Ω is a bounded domain in R^n, $n \geq 1$, such that the boundary, $\partial\Omega$ is of class C^∞, Ω locally on one side of $\partial\Omega$.

(ii) $A(x,D)$ is strongly elliptic in $\overline{\Omega}$, with infinitely differentiable coefficients, $a_{\rho\sigma}$, in $\overline{\Omega}$.

(iii) $\{B_j\}_{j=0}^{m-1}$ is a normal system, satisfying the strong complementary condition, and is independent of time (if $n = 1$, $\{B_j^+\}_{j=0}^{m-1}$ is a linearly independent set, and similarly $\{B_j^-\}_{j=0}^{m-1}$ is a linearly independent set). Also, the coefficients, b_{jh}, are infinitely differ- (7-4) entiable on $\partial\Omega$.

(iv) $A(x,D)$ is formally self-adjoint, that is for any $u,v\epsilon C_B^\infty(\Omega)$

$$(u,A(x,D)v)_o = (A(x,D)u,v)_o.$$

(v) For any $u,v\epsilon C_B^\infty(\Omega)$,

$$\int_\Omega \sum_{|\rho|,|\sigma|\leq m} (-1)^{|\rho|} u(x)D^\rho(a_{\rho\sigma}(x)D^\sigma v(x))dx$$
$$= \int_\Omega \sum_{|\rho|,|\sigma|\leq m} a_{\rho\sigma}(x)D^\rho u(x)D^\sigma v(x)dx.$$

(vi) There exists a constant $k>0$, such that for any $u\epsilon C_B^\infty(\Omega)$

$$(u,A(x,D)u)_o \geq k||u||_m^2.$$

Let us define the vectors

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u \\ \frac{\partial u}{\partial t} \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where $\underline{u},\underline{v}\epsilon H \equiv H_B^m(\Omega) \times L^2(\Omega)$, where the inner product on H is given by

$$(\underline{u},\underline{v})_H = (u_1,v_1)_m + (u_2,v_2)_o$$

inducing the norm

$$||\underline{u}||_H^2 = ||u_1||_m^2 + ||u_2||_o^2$$

which makes H a real Hilbert space.

We now define the operator T_o,

$$\begin{aligned} D(T_o) &= C_B^\infty(\Omega) \\ (T_o u)(x) &= A(x,D)u(x) \qquad u\epsilon D(T_o). \end{aligned} \tag{7-5}$$

Let T denote the closure of T_o in $L^2(\Omega)$, and from lemma 6.2.4 T is defined as follows:

$$\begin{aligned} D(T) &= H_B^{2m}(\Omega) \\ (Tu)(x) &= A(x,D)u(x) \qquad u\epsilon D(T). \end{aligned} \tag{7-6}$$

We will show that the abstract operator T satisfies the conditions that

$D(T)$ and $R(T)$ are both contained in $L^2(\Omega)$, such that $D(T)$ is dense in $L^2(\Omega)$, and for any $\alpha > 0$, $R(\alpha I - (-T)) = L^2(\Omega)$. Also, $-T$ is strictly dissipative with respect to the L^2 - inner product.

<u>Lemma 7.1.1.</u> Let T_o be defined in (7-5). Then for any $u,v \varepsilon D(T_o)$

$$(v,T_o u)_o = (u,T_o v)_o.$$

Also, $-T$ is strictly dissipative with respect to the L^2 - inner product, that is, there exists a constant $k_1 > 0$, such that for any $u \varepsilon D(T)$

$$(u,(-T)u)_o \leq -k_1 ||u||_o^2.$$

<u>Proof.</u> From condition (7-4(iv)) and the definition of T_o, we have for any $u,v \varepsilon D(T_o)$

$$(v,T_o u)_o = (u,T_o v)_o.$$

Since the identity injection from $H^m(\Omega)$ into $L^2(\Omega)$ is continuous, and since (7-4(vi)) holds, we have for any $u \varepsilon D(T_o)$

$$(u,(-T_o)u)_o \leq -k||u||_m^2 \leq -kC_o||u||_o^2 = -k_1||u||_o^2.$$

Let $u \varepsilon D(T)$. Since T is the smallest closed extension of T_o in $L^2(\Omega)$, we know there exists a sequence $u_n \varepsilon D(T_o)$, such that

$$u_n \underset{L^2(\Omega)}{\rightarrow} u, \quad T_o u_n \underset{L^2(\Omega)}{\rightarrow} Tu \quad \text{as } n \rightarrow \infty.$$

Therefore, from the continuity of the L^2 - inner product

$$\begin{aligned}(u,(-T)u)_o &= \lim_{n\to\infty} (u_n,(-T_o)u_n)_o \\ &\leq -k_1 \lim_{n\to\infty} ||u_n||_o^2 \\ &= -k_1 ||u||_o^2.\end{aligned}$$

qed

<u>Lemma 7.1.2.</u> T is a linear operator such that $D(T) \subset L^2(\Omega)$, $R(T) \subset L^2(\Omega)$ and $D(T)$ is dense in $L^2(\Omega)$. Also, for any $\alpha > 0$,

$$R(\alpha I - (-T)) = L^2(\Omega).$$

<u>Proof.</u> From the definition of T and from lemma 6.2.6 we have that $D(T) \subset L^2(\Omega)$, $R(T) \subset L^2(\Omega)$ and $D(T) = H_B^{2m}(\Omega)$ is dense in $L^2(\Omega)$. Utilizing lemmas 6.1.3 and 6.1.4 and the fact that $(-T)$ is dissipative, we have for any $\alpha > 0$

$$R(\alpha I - (-T)) = L^2(\Omega).$$

(If $n = 1$, the same result holds, using lemmas 6.3.1 and 6.3.3.) qed

We now define the abstract operator A;

$$D(A) = H_B^{2m}(\Omega) \times H_B^{m}(\Omega) = D(T) \times H_B^{m}(\Omega)$$

$$A\underline{u} = \begin{bmatrix} 0 & -I \\ T & aI \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -u_2 \\ Tu_1 + au_2 \end{bmatrix} \qquad \underline{u} \in D(A) \tag{7-7}$$

where a is the constant in (7-3), and the real base Hilbert space

$$H \equiv H_B^{m}(\Omega) \times L^2(\Omega) \tag{7-8}$$

obtaining the abstract operator equation

$$\begin{aligned} \frac{d\underline{u}(t)}{dt} + A\,\underline{u}(t) &= 0 \\ \underline{u}(o) &= \underline{u}_o \qquad \underline{u}_o \in D(A). \end{aligned} \tag{7-9}$$

It must be shown that the operator, A, satisfies the conditions of lemma 6.1.2. We first show that the domain and range of A are both contained in H, such that D(A) is dense in H and $R(I - (-A)) = H$.

<u>Lemma 7.1.3.</u> Let A be the abstract operator defined in (7-7),

then $D(A) \subset H$, $R(A) \subset H$ such that $D(A)$ is dense in H and $R(I - (-A)) = H$.

Proof. $D(A) \equiv H_B^{2m}(\Omega) \times H_B^m(\Omega) \subset H_B^m(\Omega) \times L^2(\Omega) \equiv H$, since $H_B^{2m}(\Omega) \subset H_B^m(\Omega)$ and $H_B^m(\Omega) \subset L^2(\Omega)$. Also, from the definition of A, it is readily seen that, for any $\underline{u} \in D(A)$

$$A\underline{u} \in H_B^m(\Omega) \times L^2(\Omega) \equiv H.$$

It follows that $R(A) \subset H$.

$D(A)$ is dense in H. Indeed, since $C_B^\infty(\Omega) \subset H_B^{2m}(\Omega) \subset H_B^m(\Omega)$ and $C_B^\infty(\Omega)$ is dense in $H_B^m(\Omega)$, we have that $H_B^{2m}(\Omega)$ is dense in $H_B^m(\Omega)$. Similarly, since $C_o^\infty(\Omega) \subset H_B^m(\Omega) \subset L^2(\Omega)$, and $C_o^\infty(\Omega)$ is dense in $L^2(\Omega)$, we have that $H_B^m(\Omega)$ is dense in $L^2(\Omega)$. Therefore, $D(A) \equiv H_B^{2m}(\Omega) \times H_B^m(\Omega)$ is dense in $H_B^m(\Omega) \times L^2(\Omega) \equiv H$.

Finally, $R(I - (-A)) = R(I + A) = H$.

Indeed, let

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in H \equiv H_B^m(\Omega) \times L^2(\Omega).$$

We must prove that there exists a $\underline{u} \in D(A)$, such that $(I - (-A))\underline{u} = (I+A)\underline{u} = \underline{w}$, which is equivalent to showing there exists $u_1 \in D(T)$ and $u_2 \in H_B^m(\Omega)$, such that

$$(I+A)\underline{u} \equiv \begin{bmatrix} I & -I \\ T & (1+a)I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Hence, it suffices to show there exists $u_1 \in D(T)$ and $u_2 \in H_B^m(\Omega)$ such that

(i) $u_1 - u_2 = w_1$;

(ii) $Tu_1 + (1+a)u_2 = w_2$.

Substitute $u_2 = u_1 - w_1$ into (ii), which implies

$$Tu_1 + (1+a)u_1 = (1+a)w_1 + w_2$$

or, written in another form

$$[(1+a)I-(-T)]u_1 = (1+a)w_1+w_2.$$

By lemma 7.1.2, since $(1+a) > 0$, we have $R[(1+a)I-(-T)] = L^2(\Omega)$ which implies, since $(1+a)w_1+w_2 \varepsilon L^2(\Omega)$, that there exists $u_1 \varepsilon D(T) = H_B^{2m}(\Omega)$ satisfying

$$[(1+a)I-(-T)]u_1 = (1+a)w_1+w_2.$$

Let us define $u_2 \equiv u_1-w_1 \varepsilon H_B^m(\Omega)$, and we can see

$$\text{(i)}\quad u_1-u_2 = u_1 - (u_1-w_1) = w_1;$$

$$\begin{aligned}\text{(ii)}\quad Tu_1 + (1+a)u_2 &= Tu_1 + (1+a)(u_1-w_1)\\ &= [(1+a)I-(-T)]u_1 - (1+a)w_1\\ &= w_2.\end{aligned}$$

Therefore, we have shown $R(I-(-A)) = H$, and have completed the proof of this lemma.. qed

We introduce an equivalent inner product on H and show that the operator, $-A$, is dissipative (or strictly dissipative) with respect to this equivalent inner product.

Let us define the operator S:

$$D(S) = D(T_o) \times L^2(\Omega)$$

$$S(\underline{u}) = \begin{bmatrix} 2T_o + a^2 I & aI \\ aI & 2I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \underline{u} \varepsilon D(S). \qquad (7\text{-}10)$$

where we can see $D(S)$ is dense in H.

Let us define a functional $V(\underline{u},\underline{v})$ on $D(S)$ as follows: for any $\underline{u},\underline{v} \varepsilon D(S)$

$$V(\underline{u},\underline{v}) \equiv (\underline{u},S\underline{v})_o.$$

We will show that this is a continuous bilinear functional on $D(S)$, in the topology of H, which defines an equivalent inner product on D(S) and we will extend this to an equivalent inner product, $\overline{V}(u,v)$, on all of H. Then we will show, $-A$, is dissipative with respect to this equivalent inner product on H.

Lemma 7.1.4. The functional $V(\underline{u},\underline{v})$ defined above is a continuous bilinear functional on D(S) in the topology of H.

Proof. It is clear the $V(\underline{u},\underline{v})$ is bilinear. Let

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \varepsilon D(S).$$

By definition, we see that

$$V(\underline{u},\underline{v}) = \left([u_1,u_2], \begin{bmatrix} 2T_o v_1 + a^2 v_1 + a v_2 \\ a v_1 + 2 v_2 \end{bmatrix} \right)_o \tag{7-11}$$

$$= 2(u_1,T_o v_1)_o + a^2 (u_1,v_1)_o + a(u_1,v_2)_o$$

$$+ a(u_2,v_1)_o + 2(u_2,v_2)_o.$$

We must first show that there exists a constant $k \geq 0$, independent of u_1,v_1, such that

$$|(u_1,T_o v_1)_o| \leq k||u_1||_m||v_1||_m.$$

Indeed, utilizing condition (7-4(v)), the inequalities

$$\textstyle\sum_i |a_i b_i| \leq [\sum_i |a_i|^2]^{1/2} [\sum_i |b_i|^2]^{1/2}$$

$$\int_\Omega f(x)g(x)dx \leq [\int_\Omega f^2(x)dx]^{1/2} [\int_\Omega g^2(x)dx]^{1/2}$$

and letting $M = \max\limits_{\substack{x\varepsilon\bar{\Omega} \\ |\rho|,|\sigma|\leq m}} a_{\rho\sigma}(x)$, we have the following inequality

$$\begin{aligned}
|(u_1,T_o v_1)_o| &= \left|\int_\Omega \sum_{|\rho|,|\sigma|\le m} (-1)^{|\rho|} u_1(x) D^\rho (a_{\rho\sigma}(x) D^\sigma v_1(x))dx\right| \\
&= \left|\int_\Omega \sum_{|\rho|,|\sigma|\le m} a_{\rho\sigma}(x) D^\rho u_1(x) D^\sigma v_1(x)dx\right| \\
&\le M\left|\int_\Omega \sum_{|\rho|,|\sigma|\le m} D^\rho u_1(x) D^\sigma v_1(x)dx\right| \\
&\le M\int_\Omega \sum_{|\rho|,|\sigma|\le m} |D^\rho u_1(x) D^\sigma v_1(x)|dx \\
&\le M\int_\Omega [\sum_{|\rho|\le m} |D^\rho u_1(x)|^2]^{1/2} [\sum_{|\sigma|\le m} |D^\sigma v_1(x)|^2]^{1/2} dx \\
&\le M[\sum_{|\rho|\le m} \int_\Omega |D^\rho u_1(x)|^2 dx]^{1/2} [\sum_{|\sigma|\le m} \int_\Omega |D^\sigma v_1(x)|^2 dx]^{1/2} \\
&= M||u_1||_m \; ||v_1||_m .
\end{aligned}$$

Therefore, since $||\cdot||_o \le C_o||\cdot||_m$, and by letting $k_o = \max[2M+a^2C_o^2, aC_o, 2]$, we have the following inequality:

$$\begin{aligned}
|V(\underline{u},\underline{v})| &\le 2|(u_1,T_o v_1)_o| + a^2|(u_1,v_1)_o| + a|(u_1,v_2)_o| \\
&\quad + a|(u_2,v_1)_o| + 2|(u_2,v_2)_o| \\
&\le 2M||u_1||_m \; ||v_1||_m + a^2C_o^2||u_1||_m \; ||v_1||_m + aC_o||u_1||_m||v_2||_o \\
&\quad + aC_o||u_2||_o||v_1||_m + 2||u_2||_o||v_2||_o \\
&\le k_o[||u_1||_m||v_1||_m + ||u_1||_m||v_2||_o + ||u_2||_o||v_1||_m \\
&\quad + ||u_2||_o||v_2||_o \\
&= k_o(||u_1||_m + ||u_2||_o)(||v_1||_m + ||v_2||_o).
\end{aligned}$$

Squaring both sides and using the well known inequality

$$(\alpha+\beta)^2 \le 2(\alpha^2+\beta^2) \qquad \text{for } \alpha,\beta \text{ any real numbers}$$

we have

$$|V(\underline{u},\underline{v})|^2 \leq k_o^2(||u_1||_m + ||u_2||_o)^2(||v_1||_m + ||v_2||_o)^2$$

$$\leq 4k_o^2(||u_1||_m^2 + ||u_2||_o^2)(||v_1||_m^2 + ||v_2||_o^2).$$

Therefore, we have

$$|V(\underline{u},\underline{v})| \leq 2k_o(||u_1||_m^2 + ||u_2||_o^2)^{1/2}(||v_1||_m^2 + ||v_2||_o^2)^{1/2}$$

$$= 2k_o \; ||\underline{u}||_H||\underline{v}||_H \tag{7-12}$$

or $V(\underline{u},\underline{v})$ is continuous in the topology of H. qed

Lemma 7.1.5. The bilinear functional $V(\underline{u},\underline{v})$ defines an equivalent inner product on D(S) in H, and the extension, $\overline{V}(\underline{u},\underline{v})$, of $V(\underline{u},\underline{v})$ to H defines an equivalent inner product on the whole space H.

Proof. Let

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \; \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \varepsilon D(T_o) \times L^2(\Omega) = D(S)$$

and define

$$(\underline{u},\underline{v})_s \equiv V(\underline{u},\underline{v}) = (\underline{u},S\underline{v})_o.$$

It is first shown that this defines an inner product on D(S). Since V is bilinear, we have linearity in the first term of $(\underline{u},\underline{v})_s$.. From lemma 7.1.1, and (7-11) we have

$$(\underline{u},\underline{v})_s = V(\underline{u},\underline{v})$$

$$= 2\ (u_1,T_o v_1)_o + a^2(u_1,v_1)_o + a\ (u_1,v_2)_o$$

$$+ a(u_2,v_1)_o + 2(u_2,v_2)_o$$

$$= 2(v_1,T_o u_1)_o + a^2(v_1,u_1)_o + a(v_2,u_1)_o$$

$$+ a(v_1,u_2)_o + 2(v_2,u_2)_o$$

$$= V(\underline{v},\underline{u})$$
$$= (\underline{v},\underline{u})_s$$

which shows that $(\underline{u},\underline{v})_s$ is symmetric.

From condition (7-4(vi)), we have the following inequality by letting $k_2 = \min\ (2k,1)$ where k is the constant in condition (7-4(vi))

$$(\underline{u},\underline{u})_s = 2(u_1,T_o u_1)_o + a^2(u_1,u_1)_o + a(u_1,u_2)_o + a(u_2,u_1)_o + 2(u_2,u_2)_o$$

$$\geq 2k||u_1||_m^2 + \int_\Omega [a^2u_1^2(x)+2au_1(x)u_2(x)+2u_2^2(x)]dx$$

$$= 2k||u_1||_m^2 + \int_\Omega [(au_1(x)+u_2(x))^2+u_2^2(x)]dx$$

$$\geq 2k||u_1||_m^2 + \int_\Omega u_2^2(x)dx$$

$$\geq k_2[||u_1||_m^2 + ||u_2||_o^2]$$

$$= k_2||\underline{u}||_H^2.$$

Hence, we have shown that there exists a constant $k_2 > 0$, such that for any $\underline{u}\varepsilon D(S)$

$$||\underline{u}||_s^2 \geq k_2||\underline{u}||_H^2. \qquad (7\text{-}13)$$

From the inequality, we have that $(\underline{u},\underline{u})_s \geq 0$ and $(\underline{u},\underline{u})_s = 0$ if and only if $\underline{u} = 0$. Therefore, $(\underline{u},\underline{v})_s$ is an inner product on D(S). Also, from (7-12) and (7-13) we can see that the inner product $(\underline{u},\underline{v})_s$ is equivalent to the one defined on H, that is, there exists constants $k_o, k_2 > 0$, such that for any $\underline{u} \epsilon D(S)$

$$\sqrt{k_2}\ ||\underline{u}||_H \leq ||\underline{u}||_s \leq \sqrt{2k_o}\ ||\underline{u}||_H. \tag{7-14}$$

This inner product is extended to all of H by the following definition

$$(\underline{u},\underline{v})_e \equiv \overline{V}(\underline{u},\underline{v}) = \lim_{n\to\infty} V(u_n,v_n) = \lim_{n\to\infty} (\underline{u_n},\underline{v_n})_s, \underline{u},\underline{v} \epsilon H$$

where $\underline{u_n},\underline{v_n}$ are sequences in D(S) such that

$$\underline{u_n} \underset{H}{\to} \underline{u}, \qquad \underline{v_n} \underset{H}{\to} \underline{v} \qquad \text{as } n \to \infty.$$

$(\cdot,\cdot)_e$ defines an inner product on H. Indeed, from the definition of $(\underline{u},\underline{v})_e$ and the bilinearity of $(\underline{u_n},\underline{v_n})_s$, $(\underline{u},\underline{v})_e$ is linear in the first term. Also, since

$$(\underline{u},\underline{v})_e = \lim_{n\to\infty} (\underline{u_n},\underline{v_n})_s = \lim_{n\to\infty} (\underline{v_n},\underline{u_n})_s = (\underline{v},\underline{u})_e$$

then $(\underline{u},\underline{v})_e$ is symmetric. From (7-14) we have

$$(\underline{u},\underline{u})_e = \lim_{n\to\infty} (\underline{u_n},\underline{u_n})_s \geq k_2 \lim_{n\to\infty} ||\underline{u_n}||_H^2$$

$$= k_2\ ||\underline{u}||_H^2.$$

This shows us that $(\underline{u},\underline{u})_e \geq 0$, and $(\underline{u},\underline{u})_e = 0$ if and only if $\underline{u} = 0$ which proves that $(\cdot,\cdot)_e$ is an inner product on H.

Finally, we must prove that $(\cdot,\cdot)_e$ is equivalent to the inner product defined on H. This follows from the inequalities, using (7-14)

$$||\underline{u}||_e^2 = \lim_{n\to\infty} ||\underline{u}_n||_s^2 \leq 2k_o \lim_{n\to\infty} ||\underline{u}_n||_H^2 = 2k_o||\underline{u}||_H^2$$

$$||\underline{u}||_e^2 = \lim_{n\to\infty} ||\underline{u}_n||_s^2 \geq k_2 \lim_{n\to\infty} ||\underline{u}_n||_H^2 = k_2||\underline{u}||_H^2.$$

Therefore, we have defined an inner product on H, $(.,.)_e$, which is equivalent to the original inner product defined on H. qed

It remains to prove the dissipativity of the operator -A.

Lemma 7.1.6. Let A be the abstract operator defined in (7-7). Then -A is strictly dissipative with respect to the inner product $(.,.)_e$ if $a > 0$, and is dissipative with respect to $(.,.)_e$ if $a = 0$, where a is the coefficient in (7-3).

Proof. This lemma is proved in 3 steps.

(i) First, we show that -A is strictly dissipative on $D(T_o)xD(T_o)$ if $a > 0$, and is dissipative if $a = 0$. Let us pick

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \varepsilon D(T_o)xD(T_o) \subset S.$$

Since $A\underline{u}\varepsilon D(S)$ and $(.,.)_e$ coincides with $(.,.)_s$ on D(S), if we let $k_1 = \min\{k,1\}$, where k is the constant in (7-4(iv)), we have the following inequality from lemma 7.1.1, and the definitions of the operators S and A,

$$(\underline{u},-A\underline{u})_e = (\underline{u},-A\underline{u})_s = (\underline{u},S(-A)\underline{u})_o$$

$$= \left([u_1,u_2], \begin{bmatrix} 2T_o+a^2I & aI \\ aI & 2I \end{bmatrix} \begin{bmatrix} u_2 \\ -T_ou_1-au_2 \end{bmatrix}\right)_o$$

$$= \left([u_1,u_2], \begin{bmatrix} 2T_ou_2+a^2u_2-aT_ou_1-a^2u_2 \\ au_2-2T_ou_1-2au_2 \end{bmatrix}\right)_o$$

$$= \left([u_1,u_2], \begin{bmatrix} 2T_ou_2-aT_ou_1 \\ -2T_ou_1-au_2 \end{bmatrix}\right)_o$$

$$= 2(u_1,T_ou_2)_o - a(u_1,T_ou_1)_o - 2(u_2,T_ou_1)_o - a(u_2,u_2)_o$$

$$= -a(u_1,T_ou_1)_o - a(u_2,u_2)_o$$

$$\leq -ak||u_1||_m^2 - a||u_2||_o^2$$

$$\leq -ak_1[||u_1||_m^2 + ||u_2||_o^2]$$

$$= -ak_1||\underline{u}||_H^2$$

$$\leq -\frac{ak_1}{2k_o}\ ||\underline{u}||_e^2$$

$$= -\beta||\underline{u}||_e^2$$

where $\beta = \dfrac{ak_1}{2k_o} \geq 0$, and $\beta = 0$ if and only if $a = 0$. (7-15)

This proves that $-A$ is strictly dissipative on $D(T_o)xD(T_o)$ if $a > 0$ and is dissipative if $a = 0$.

(ii) Secondly, we show $-A$ is strictly dissipative on $D(T)xD(T_o)$ if $a > 0$, and is dissipative if $a = 0$. Letting

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \varepsilon D(T)xD(T_o)$$

we see that $A\underline{u}\varepsilon D(S)$ and

$$(\underline{u},(-A)\underline{u})_e = \lim_{n\to\infty} (\underline{u_n},(-A)\underline{u})_s = \lim_{n\to\infty} (\underline{u_n},S(-A)\underline{u})_o = (\underline{u},S(-A)\underline{u})_o$$

where $\underline{u_n}\ \varepsilon D(S)$ such that $\underline{u_n} \underset{H}{\to} \underline{u}$ as $n \to \infty$.

Therefore,

$$(\underline{u},(-A)\underline{u})_e = (\underline{u},S(-A)\underline{u})_o = \left([u_1,u_2], \begin{bmatrix} 2T_o+a^2I & aI \\ aI & 2I \end{bmatrix} \begin{bmatrix} u_2 \\ -Tu_1-au_2 \end{bmatrix}\right)_o$$

$$=2(u_1,T_o u_2)_o - a(u_1,Tu_1)_o - 2(u_2,Tu_1)_o - a(u_2,u_2)_o.$$

Since T is the smallest closed extension of T_o in $L^2(\Omega)$, there exists a sequence $v_n \varepsilon D(T_o)$, such that

$$v_n \underset{L^2(\Omega)}{\rightarrow} u_1, \text{ and } T_o v_n \underset{L^2(\Omega)}{\rightarrow} Tu_1 \quad \text{as } n \to \infty.$$

From lemma 7.1.1,

$$(u_2,Tu_1)_o = \lim_{n\to\infty} (u_2,T_o v_n)_o = \lim_{n\to\infty}(T_o u_2,v_n)_o = (T_o u_2,u_1)_o = (u_1,T_o u_2)_o.$$

Let us define

$$\underline{u_n} = \begin{bmatrix} v_n \\ u_2 \end{bmatrix} \varepsilon D(T_o) x D(T_o) \subset D(\mathbf{S}).$$

I assert

$$\underline{u_n} \underset{H}{\to} \underline{u} \qquad \text{as } n \to \infty.$$

Indeed, it suffices to show

$$v_n \underset{H^m(\Omega)}{\rightarrow} u_1 \qquad \text{as } n \to \infty.$$

From lemma 6.2.1, and the fact that $v_n - u_1 \varepsilon H^m_B(\Omega)$ we have the following inequality

$$||v_n - u_1||_m \leq C_o||v_n - u_1||_{2m} \leq C_o C_1[||T_o v_n - Tu_1||_o + ||v_n - u_1||_o].$$

Since the term on the right converges to 0, as $n \to \infty$

$$v_n \underset{H^m(\Omega)}{\rightarrow} u_1 \qquad \text{as } n \to \infty$$

which proves the assertion.

Since $(\cdot,\cdot)_e$ is equivalent to $(\cdot,\cdot)_H$, we have

$$||\underline{u}_n||_e \underset{R^1}{\rightarrow} ||\underline{u}||_e \text{ as } n \rightarrow \infty.$$

Therefore, from part (i)

$$\begin{aligned}
(\underline{u},(-A)\underline{u})_e &= 2(u_1,T_o u_2)_o - a(u_1,Tu_1)_o - 2(u_2,Tu_1)_o - a(u_2,u_2)_o \\
&= -a(u_1,Tu_1)_o - a(u_2,u_2)_o \\
&= \lim_{n\to\infty} [a(v_n,(-T_o)v_n)_o - a(u_2,u_2)_o] \\
&= \lim_{n\to\infty} (\underline{u}_n,(-A)\underline{u}_n)_e \\
&\leq -\beta \lim_{n\to\infty} ||\underline{u}_n||_e^2 \\
&= -\beta||\underline{u}||_e^2
\end{aligned}$$

where $\beta = \dfrac{ak_1}{2k_o}$. This shows $-A$ is strictly dissipative on $D(T)xD(T_o)$ if $a > 0$, and is dissipative if $a = 0$.

(iii) Thirdly, we show $-A$ is strictly dissipative on $D(A)$ if $a > 0$ and is dissipative if $a = 0$. Let

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \varepsilon D(A) = D(T)xH_B^m(\Omega).$$

Since $u_2 \varepsilon H_B^m(\Omega)$, there exists a sequence $w_n \varepsilon D(T_o)$, such that

$$w_n \underset{H^m(\Omega)}{\longrightarrow} u_2 \quad \text{as } n \rightarrow \infty$$

and from the continuity of the injection mapping from $H_B^m(\Omega)$ into $L^2(\Omega)$

$$w_n \underset{L^2(\Omega)}{\longrightarrow} u_2 \quad \text{as } n \rightarrow \infty.$$

If we define

$$\underline{u_n} = \begin{bmatrix} u_1 \\ w_n \end{bmatrix} \varepsilon D(T) x D(T_o)$$

we see that

$$\underline{u_n} \underset{H}{\to} \underline{u} \qquad \text{as } n \to \infty$$

and

$$A\underline{u_n} = \begin{bmatrix} -w_n \\ Tu_1 + aw_n \end{bmatrix} \underset{H}{\to} \begin{bmatrix} -u_2 \\ Tu_1 + au_2 \end{bmatrix} = A\underline{u} \quad \text{as } n \to \infty.$$

Since $(\cdot,\cdot)_e$ is equivalent to $(\cdot,\cdot)_H$, we have that

$$||\underline{u_n}||_e \underset{R^1}{\to} ||\underline{u}||_e \quad \text{and} \quad ||A\underline{u_n}||_e \underset{R^1}{\to} ||A\underline{u}||_e \text{as } n \to \infty.$$

Therefore, from part (ii)

$$(\underline{u},(-A)\underline{u})_e = \lim_{n\to\infty} (\underline{u_n},(-A)\underline{u_n})_e$$

$$\leq -\beta \lim_{n\to\infty} ||\underline{u_n}||_e$$

$$= -\beta||\underline{u}||_e, \text{ where } \beta = \frac{ak_1}{2k_o}.$$

Hence, we have proved that $-A$ is strictly dissipative with respect to $(\cdot,\cdot)_e$ if $a > 0$, and is dissipative if $a = 0$. qed

We are now ready to prove our main result, and solve the stability problem for (7-3).

<u>Theorem 7.1.1.</u> Let us consider the initial-boundary value problem (7-3) satisfying the conditions (7-4). Then, for any $u_o(x)\varepsilon D(T) = H_B^{2m}(\Omega)$ and for any $v_o(x)\varepsilon H_B^m(\Omega)$, there exists a unique generalized solution, $u(x,t)$, of (7-3) such that $u(x,t)$ satisfies the boundary conditions in a generalized

sense (if n = 1, in the classical sense), and satisfies the initial conditions

$$u(x,o) \equiv u_o(x), \quad \frac{\partial u(x,o)}{\partial t} \equiv v_o(x).$$

Furthermore, the null solution is asymptotically stable with respect to the L^2 - norm if a > 0, and is stable if a = 0.

Proof. We define the abstract operator A as in (7-7) on the real Hilbert space $H \equiv H_B^m(\Omega) x L^2(\Omega)$ obtaining the abstract operator equation in (7-9). From lemma 7.1.3 we have that A is a linear operator with domain and range both contained in H, such that D(A) is dense in H and R(I-(-A))=H. From lemma 7.1.6, -A is strictly dissipative with respect to $(.,.)_e$ if a > 0 and is dissipative if a = 0. We can now use the results of lemma 6.1.2, that is, for any

$$\underline{u}_o = \begin{bmatrix} u_o \\ v_o \end{bmatrix} \varepsilon D(A) = H_B^{2m}(\Omega) x H_B^m(\Omega)$$

there exists a unique solution $\underline{u}(t)$ of (7-9), such that for any $t \geq 0$, $\underline{u}(t) \varepsilon D(A)$, and $\underline{u}(o) = \underline{u}_o$. Also, since $\underline{0} \varepsilon D(A)$ and $A(\underline{0}) = 0$, the null solution is asymptotically stable if β > 0, and is stable if β = 0. From this result, we have for any $u_o(x) \varepsilon H_B^{2m}(\Omega)$ and $v_o(x) \varepsilon H_B^m(\Omega)$, there exists a unique generalized solution, u(x,t), of (7-3), such that for any $t \geq 0$, $u(x,t) \varepsilon H_B^{2m}(\Omega)$ and $u(x,o) = u_o(x)$ and $\frac{\partial u(x,0)}{\partial t} = v_o(x)$. Also, u(x,t) satisfies the boundary conditions in a generalized sense (if n=1, in the classical sense). Finally, the null solution is asymptotically stable if a > 0 (and is stable if a = 0) with respect to the L^2 - norm. qed

7.2. Stability of the Solution of a Nonlinear Initial-Boundary Value Problem

In the previous section, sufficient conditions were given on the system $(A(x,D),\{B_j\},\Omega)$ to guarantee the existence, uniqueness and stability of the solution to a linear initial-boundary value problem. In this section, we generalize the results of section 7.1 to the nonlinear case and sufficient conditions are given to ensure the stability of the solution of the nonlinear problem described below.

Let us consider the following initial-boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(x,u,\frac{\partial u}{\partial t}) \quad x\varepsilon\Omega,\ t\geq 0\\
&B_j(x,D)u(x,t) = 0 \qquad x\varepsilon\partial\Omega,\ t\geq 0 \quad (0\leq j\leq m-1)\\
&u(x,o) = u_o(x), \qquad \frac{\partial u(x,o)}{\partial t} = v_o(x)
\end{aligned} \tag{7-16}$$

where $A(x,D)$ and $B_j(x,D)$ are defined as follows,

$$A(x,D)(\cdot) = \sum_{|\rho|,|\sigma|\leq m} (-1)^{|\rho|} D^{\rho}(a_{\rho\sigma}(x)D^{\sigma}(\cdot))$$

$$B_j(x,D) = \sum_{|h|\leq m_j} b_{jh}(x)D^h \qquad (0\leq j\leq m-1)$$

where the system $(A(x,D),\{B_j\},\Omega)$ satisfies condition (7-4) and is defined on the real base Hilbert space $H \equiv H_B^m(\Omega)xL^2(\Omega)$ and $f(x,u,\frac{\partial u}{\partial t})$ satisfies the following condition:

f is defined on all of $H_B^m(\Omega)xL^2(\Omega)$ into $L^2(\Omega)$, and there exists a constant $k \geq 0$, such that for any $u_1,v_1\varepsilon H_B^m(\Omega)$, and for any $u_2,v_2\varepsilon L^2(\Omega)$

$$||f(x,u_1,u_2)-f(x,v_1,v_2)||_o \leq k[\sum_{|\alpha|\leq m}||D^{\alpha}u_1-D^{\alpha}v_1||_o^2+||u_2-v_2||_o^2]^{1/2}. \tag{7-17}$$

As in section 7.1, we define

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u \\ \frac{\partial u}{\partial t} \end{bmatrix} \varepsilon H \equiv H_B^m(\Omega)xL^2(\Omega)$$

$$\underline{f}(\underline{u}) = \begin{bmatrix} 0 \\ f(x,u_1,u_2) \end{bmatrix} \varepsilon H \tag{7-18}$$

and A is the abstract operator defined in (7-7) which leads to the abstract evolution equation

$$\frac{d\underline{u}(t)}{dt} + A\underline{u}(t) = \underline{f}(\underline{u})$$
$$\underline{u}(o) = \underline{u}_o. \tag{7-19}$$

Utilizing the results of section 7.1 we are ready for the main result of this section.

Theorem 7.2.1. Let us consider the initial-boundary value problem (7-16) such that the system $(A(x,D),\{B_j\},\Omega)$ satisfies (7-4) and the nonlinear function f satisfies (7-17). Then, for any $u_o(x)\varepsilon H_B^{2m}(\Omega)$ and for any $v_o(x)\varepsilon H_B^m(\Omega)$, there exists a unique generalized solution, u(x,t), of the equation (7-16), such that for any $t\geq 0, u(x,t)\varepsilon H_B^{2m}(\Omega)$ with $u(x,o) = u_o(x)$ and $\frac{\partial u(x,o)}{\partial t} = v_o(x)$, and u(x,t) satisfies the boundary conditions in a generalized sense (if n=1, in the classical sense). Furthermore, if f(o) = 0, the null solution, is asymptotically stable if $k < (\frac{k_2}{2k_o})^{1/2}\beta$, and is stable if $k = (\frac{k_2}{2k_o})^{1/2}\beta$, with respect to the L^2 - norm, where k_o, k_2 are defined in (7-14) and β is defined in (7-15).

Proof. We define the abstract operator A in (7-7) and $\underline{f}$ in (7-18) with the corresponding abstract operator equation (7-19) defined on the real Hilbert space $H \equiv H_B^m(\Omega)xL^2(\Omega)$. From Lemmas 7.1.3 and 7.1.6 we have that A is a linear operator, with domain and range contained in H such that D(A) is dense in H, and R(I-(-A)) = H. Also, -A, satisfies the inequality: there exists a constant $\beta \geq 0$, such that for any $\underline{u}\varepsilon D(A)$

$$(\underline{u},-A\underline{u})_e \leq -\beta||\underline{u}||_e^2$$

where $(.,.)_e$ is the equivalent inner product on H, defined in lemma 7.1.5, and β is the dissipativity constant defined in (7-15). Now, let us consider

$$\underline{f}(\underline{u}) = \begin{bmatrix} 0 \\ f(x,u_1,u_2) \end{bmatrix}.$$

$\underline{f}$ is defined on all of $H \equiv H_B^m(\Omega)xL^2(\Omega)$ into $H_B^m(\Omega)xL^2(\Omega)$ and for any $\underline{u},\underline{v}\varepsilon H$ such that

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \varepsilon H,$$

since

$$\underline{f}(\underline{u})-\underline{f}(\underline{v}) = \begin{bmatrix} 0 \\ f(x,u_1,u_2)-f(x,v_1,v_2) \end{bmatrix}$$

we have the following inequality, from (7-17)

$$\begin{aligned} ||\underline{f}(\underline{u})-\underline{f}(\underline{v})||_e &\leq (2k_o)^{1/2}||\underline{f}(\underline{u})-\underline{f}(\underline{v})||_H \\ &= (2k_o)^{1/2}||f(x,u_1,u_2)-f(x,v_1,v_2)||_o \\ &\leq (2k_o)^{1/2}k[\textstyle\sum_{|\alpha|\leq m}||D^\alpha u_1-D^\alpha v_1||_o^2+||u_2-v_2||_o^2]^{1/} \\ &= (2k_o)^{1/2}k[||u_1-v_1||_m^2+||u_2-v_2||_o^2]^{1/2} \\ &= (2k_o)^{1/2}k||\underline{u}-\underline{v}||_H \\ &\leq (\frac{2k_o}{k_2})^{1/2}k||\underline{u}-\underline{v}||_e. \end{aligned}$$

Hence, the hypothesis of lemma 6.4.1 is satisfied and applying the results, if we let

$$\underline{u}_o = \begin{bmatrix} u_o(x) \\ v_o(x) \end{bmatrix} \varepsilon D(A) = H_B^{2m}(\Omega)xH_B^m(\Omega)$$

then, there exists a unique solution, $\underline{u}(t)$ of (7-19) with $\underline{u}(o) = \underline{u}_o$, and if $\underline{f}(0) = 0$ the null solution is asymptotically stable if $k < (\frac{k_2}{2k_o})^{1/2}\beta$, and is stable if $k = (\frac{k_2}{2k_o})^{1/2}\beta$, with respect to the H-norm, where k_o, k_2 are defined in (7-14) and β is defined in (7-15). From this result, we have that there exists a unique generalized solution, $u(x,t)$, of (7-16), such that, for any $t \geq 0$, $u(x,t) \varepsilon H_B^{2m}(\Omega)$, $\frac{\partial u(x,t)}{\partial t} \varepsilon H_B^m(\Omega)$, with $u(x,o) = u_o(x)$ and $\frac{\partial u(x,o)}{\partial t} = v_o(x)$. Also, $u(x,t)$ satisfies the boundary conditions in a generalized sense, if $n = 1$ in the classical sense. Finally, if $f(o) = 0$, the null solution is asymptotically stable if $k < (\frac{k_2}{2k_o})^{1/2}\beta$, and is stable if $k = (\frac{k_2}{2k_o})^{1/2}\beta$, with respect to the L^2-norm. qed

7.3. Applications to Partial Differential Equations

There is a large class of physical problems which fit into the theory developed in sections 7.1 and 7.2, among these being the wave equation. In this section, we consider some applications which illustrate how the theory can be applied to solving specific problems. In section 7.31, we consider a Dirichlet problem and show that the result obtained by Pao and Vogt in [24] is just a special case of the results in 7.1 and 7.2. In section 7.32, we consider some specific examples which show how the theory can be applied to solving a large class of initial-boundary value problems.

7.31. Dirichlet Problem

In this example, we show that the Dirichlet problem worked out by Pao and Vogt in [24] is just a special case of theorems 7.1.1 and 7.2.1.

Pao considered the following initial-boundary value problem

$$\begin{aligned}
&\frac{\partial^2 u(x,t)}{\partial t^2} + a\frac{\partial u(x,t)}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x,t)}{\partial x_j}\right) - c(x)u(x,t) \\
&\qquad = f\left(x,u,\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial x_j}\right) \qquad x \in \Omega,\ t \geq 0 \\
&u(x,t) = 0 \qquad x \in \partial\Omega,\ t \geq 0 \qquad\qquad (7\text{-}20) \\
&u(x,0) = u_o(x),\ \frac{\partial u(x,0)}{\partial t} = v_o(x)
\end{aligned}$$

where we define the operators

$$A(x,D)(\cdot) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}(\cdot)) - c(x)$$

$$B_o(D) = I \qquad \text{(where I is the identity operator)}$$

and the system $(A(x,D),B_o,\Omega)$ satisfies the following conditions,

(i) $a \geq 0$, Ω is a bounded domain in R^n, $n \geq 1$, such that $\partial\Omega$ is of class C^∞, locally on one side of Ω.

(ii) $a_{ij}(x) \equiv a_{ji}(x) \in C^\infty(\overline{\Omega})$ $(1 \leq i,j \leq n)$. $c(x) \in C^\infty(\overline{\Omega})$ such that $\max\limits_{x\in\overline{\Omega}} c(x) < 0$, and we let $c_m = \min\limits_{x\in\overline{\Omega}}(-c(x))$, $c_M = \max\limits_{x\in\overline{\Omega}}(-c(x))$.

(iii) $A(x,D)$ is strongly elliptic in $\overline{\Omega}$ of order $2m = 2$, that is, there exists a constant $\alpha > 0$, such that for any $\xi = (\xi_1,\ldots,\xi_n) \in R^n$, and any $x \in \overline{\Omega}$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha\sum_{i=1}^{n} \xi_i^2.$$

(iv) f is a nonlinear function defined on all of $H_o^1(\Omega)\times L^2(\Omega)$ into $L^2(\Omega)$, and there exists a constant $k_1 \geq 0$, such that for any u_1, $v_1 \in H_o^1(\Omega)$, and any u_2, $v_2 \in L^2(\Omega)$

$$||f(x,u_1,u_2) - f(x,v_1,v_2)||_o$$
$$\leq k_1[||u_1-v_1||_o^2 + \sum_{i=1}^{n} ||\frac{\partial u_1}{\partial x_i} - \frac{\partial v_1}{\partial x_i}||_o^2 + ||u_2-v_2||_o^2]^{1/2}.$$

We show that for any $u_o(x) \in H^2(\Omega) \cap H_o^1(\Omega)$, and any $v_o(x) \in H_o^1(\Omega)$ there exists a unique generalized solution, $u(x,t)$, of (7-20) such that for every $t \geq 0$, $u(x,t) \in H^2(\Omega) \cap H_o^1(\Omega)$, satisfying the boundary conditions in the generalized sense (if $n = 1$, in the classical sense), with

$$u(x,0) = u_o(x), \qquad \frac{\partial u(x,0)}{\partial t} = v_o(x).$$

Also, if $f(0) = 0$, the null solution is asymptotically stable if $k_1 < \left(\frac{k_2}{2k_o}\right)^{1/2}\beta$ and is stable if $k_1 = \left(\frac{k_2}{2k_o}\right)^{1/2}\beta$ with respect to the L^2-norm, where k_o, k_2 are the constants in (7-14), and $\beta = \frac{a\ \min(k,1)}{2k_o}$, where $k = \min(\alpha, c_m)$.

We assert that the system $(A(x,D), B_o, \Omega)$ satisfies (7-4). It should be noted in this example that $H^1_B(\Omega) = H^1_o(\Omega)$ (see remark 3.46.1).

From the hypothesis, we have the smoothness condition on Ω and the strong ellipticity of $A(x,D)$. It was shown in example 5.21.1 that $\{B_o\}$ is a normal system and satisfies the strong complementary condition (if $n = 1$, B^+_o and B^-_o are each linearly independent).
The coefficients of $A(x,D)$, $B_o(D)$ are infinitely differentiable.

$A(x,D)$ is formally self-adjoint, in the sense that for any $u,v \in C^\infty_B(\Omega)$

$$(u, A(x,D)v)_o = (A(x,D)u, v)_o.$$

Indeed, from integration by parts and the boundary condition

$$(u,A(x,D)v)_o = \int_\Omega u(x)\Big[-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial v(x)}{\partial x_j}\Big) - c(x)v(x)\Big]dx$$

$$= \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,dx - \int_\Omega c(x)u(x)v(x)dx$$

$$= -\int_\Omega \sum_{i,j=1}^{n} v(x) \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial u(x)}{\partial x_i})dx - \int_\Omega c(x)v(x)u(x)dx$$

$$= \int_\Omega v(x)[-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} (a_{ji}(x) \frac{\partial u(x)}{\partial x_i}) - c(x)u(x)]dx$$

$$= (v,A(x,D)u)_o.$$

It is clear from the above equality, that for any $u,v \varepsilon C_B^\infty(\Omega)$

$$\int_\Omega u(x)[-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial v(x)}{\partial x_j}) - c(x)v(x)]dx$$

$$= \int_\Omega [\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} - c(x)u(x)v(x)]dx$$

It remains to show that there exists a constant $k > 0$, such that for any $u \varepsilon C_B^\infty(\Omega)$

$$(u,A(x,D)u)_o \geq k||u||_1^2,$$

where $||u||_1^2$ is the norm on $H_o^1(\Omega)$. Indeed, from integration by parts, the boundary conditions and the strong ellipticity of $A(x,D)$, we have for any $u \varepsilon C_B^\infty(\Omega)$,

$$(u,A(x,D)u)_o = -\int_\Omega \sum_{i,j=1}^{n} u(x) \frac{\partial}{\partial x_i}(a_{ij}(x) \frac{\partial u(x)}{\partial x_j})dx - \int_\Omega c(x)u^2(x)dx$$

$$= \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j}dx + \int_\Omega [-c(x)]u^2(x)dx$$

$$\geq \alpha \int_\Omega \sum_{i=1}^{n} (\frac{\partial u(x)}{\partial x_i})^2 dx + c_m \int_\Omega u^2(x)dx$$

$$\geq \min(\alpha, c_m)[\int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 dx + \int_\Omega u^2(x)dx]$$

$$= \min(\alpha, c_m)||u||_1^2.$$

Finally, we see that f satisfies (7-17). Indeed, from the hypothesis, f is defined on all of $H_o^1(\Omega)\times L^2(\Omega)$ into $L^2(\Omega)$ and there exists a constant $k_1 \geq 0$, such that for any $u_1, v_1 \in H_o^1(\Omega)$ and any $u_2, v_2 \in L^2(\Omega)$

$$||f(x,u_1,u_2) - f(x,v_1,v_2)||_o$$

$$\leq k_1[||u_1-v_1||_o^2 + \sum_{i=1}^{n} || \frac{\partial u_1}{\partial x_i} - \frac{\partial v_1}{\partial x_i} ||_o^2 + ||u_2-v_2||_o^2]^{1/2}$$

$$= k_1[\sum_{|\alpha|\leq 1} ||D^\alpha u_1 - D^\alpha v_1||_o^2 + ||u_2-v_2||_o^2]^{1/2}.$$

Therefore, we have verified the hypothesis of theorem 7.2.1 (if $f(x,u, \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial t}) \equiv 0$, we can use theorem 7.1.1) and the results apply. This gives us our desired conclusion and shows that the result in [24] is a special case of theorem 7.1.1 and 7.2.1.

7.32. Specific Examples

In this section, we give specific examples which show how certain stability problems fit into the theory developed in sections 7.1 and 7.2.

Example 7.32.1. Consider the following nonlinear initial-boundary value problem stated in Lions and Magenes [18]

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) + bu = f(u, \frac{\partial u}{\partial t}) \qquad x \varepsilon \Omega, \ t \geq 0$$

$$\frac{\partial u(x,t)}{\partial n} = 0 \qquad x \varepsilon \partial\Omega \quad t \geq 0 \tag{7-21}$$

$$u(x,0) = u_o(x), \quad \frac{\partial u(x,0)}{\partial t} = v_o(x)$$

where $A(D)$, $B_o(D)$, Ω and the function spaces $C_B^\infty(\Omega)$, $H_B^2(\Omega)$, and $H_B^1(\Omega)$ are defined as follows:

$$A(D) = -\Delta + b = -\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2 + b$$

where the order of $A(D)$ is $2m = 2$, and $b > 0$,

$$B_o(D) = \frac{\partial}{\partial n},$$

Ω is a bounded domain in R^n, $n \geq 2$, such that the boundary, $\partial\Omega$, is of class C^∞, locally on one side of Ω,

$$C_B^\infty(\Omega) = \{u \ \varepsilon \ C^\infty(\bar{\Omega}) \ | \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$$

$H_B^2(\Omega)$ = completion of $C_B^\infty(\Omega)$ with respect to the H^2-norm

$$= \{u \ \varepsilon \ H^2(\Omega) \ | \ < B_o u >_{1/2} = 0\}$$

$H_B^1(\Omega)$ = completion of $C_B^\infty(\Omega)$ with respect to the H^1-norm.

f is, in general, a nonlinear function which maps all of $H_o^1(\Omega) \times L^2(\Omega)$ into $L^2(\Omega)$, and there exists a constant $k_1 \geq 0$ such that for any $u_1, v_1 \ \varepsilon \ H_o^1(\Omega)$ and $u_2, v_2 \ \varepsilon \ L^2(\Omega)$

$$||f(u_1,u_2)-f(v_1,v_2)||_o \le k_1[||u_1-v_1||_o^2+\sum_{i=1}^{n}||\frac{\partial u_1}{\partial x_i}-\frac{\partial v_1}{\partial x_i}||_o^2+||u_2-v_2||_o^2]^{1/2}.$$

We assert that problem (7-21) verifies the hypothesis of theorem 7.2.1 from which we have the following result: for any $u_o(x) \in H_B^2(\Omega)$ and any $v_o(x) \in H_o^1(\Omega)$, there exists a unique generalized solution, $u(x,t)$, of (7-20), such that for every $t \ge 0$, $u(x,t) \in H_B^2(\Omega)$, with $u(x,0) = u_o(x)$ and $\frac{\partial u(x,0)}{\partial t} = v_o(x)$. Also $u(x,t)$ satisfies the boundary conditions in a generalized sense and if $f(0) = 0$, the null solution is asymptotically stable if $k < (\frac{k_2}{2k_o})^{1/2}\beta$, and if $k = (\frac{k_2}{2k_o})^{1/2}\beta$ is stable with respect to the L^2-norm, where k_o, k_2 are the constants in (7-14) and $\beta = \min[1,b]C_o$, where C_o is the constant in the inequality $||u||_o \le C_o||u||_1^2$.

First, we must show that the system $(A(D),B_o,\Omega)$ satisfies (7-4), where we let $H \equiv H_B^1(\Omega)\times L^2(\Omega)$.

The smoothness of $\partial\Omega$ is seen from the hypothesis. The fact that $A(D)$ is strongly elliptic in $\bar{\Omega}$ is proved in example 6.52.1. $\{B_o\}$ is a normal system since the order is $m_o = 1$, and is of the form (5-8) and satisfies the equivalent definition 5.21.3. Since $B_o = \frac{\partial}{\partial n}$, we see as in example 6.52.1, that $\{B_o\}$ satisfies the strong complementary condition. Also $\{B_o\}$ is independent of time.

We must now show that $A(x,D)$ is formally self-adjoint, that is, for any $u,v \in C_B^\infty(\Omega)$

$$(u,A(D)v)_o = (A(D)u,v)_o.$$

This follows from Green's formula, as found in Mikhlin [21], and using the boundary conditions, letting $u,v \in C_B^\infty(\Omega)$

$$
\begin{aligned}
(u,A(D)v)_o &= \int_\Omega u(x)[-\Delta v(x)+bv(x)]dx \\
&= -\int_\Omega u(x)\Delta v(x)dx + b\int_\Omega u(x)v(x)dx \\
&= \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)\left(\frac{\partial v(x)}{\partial x_i} \right)dx - \int_{\partial\Omega} u(x) \frac{\partial v(x)}{\partial n} d(\partial\Omega) \\
&\qquad + b\int_\Omega u(x)v(x)dx \\
&= -\int_\Omega v(x)\Delta u(x)dx + \int_{\partial\Omega} v(x) \frac{\partial u(x)}{\partial n} d(\partial\Omega) + b\int_\Omega u(x)v(x)dx \\
&= \int_\Omega v(x)[-\Delta u(x)+bu(x)]dx \\
&= (v,A(D)u)_o.
\end{aligned}
$$

Also, from the above equality, we have for any $u,v \in C_B^\infty(\Omega)$

$$\int_\Omega u(x)[-\Delta v(x)+bv(x)]dx = \int_\Omega \left[\sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i}\right)\left(\frac{\partial v(x)}{\partial x_i}\right) + bu(x)v(x)\right]dx .$$

Let $u \in C_B^\infty(\Omega)$, we have

$$(u,A(D)u)_o \geq k||u||_1^2$$

where $k = \min[1,b]$. Indeed, from Green's formula and the boundary conditions

$$
\begin{aligned}
(u,A(D)u)_o &= -\int_\Omega u(x)\Delta u(x)dx + b\int_\Omega u^2(x)dx \\
&= \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 dx - \int_{\partial\Omega} u(x) \frac{\partial u(x)}{\partial n} d(\partial\Omega)
\end{aligned}
$$

$$+ b\int_{\Omega} u^2 dx$$

$$= \int_{\Omega}\sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 dx + b\int_{\Omega} u^2(x)dx$$

$$\geq \min[1,b]\{\int_{\Omega}\sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 dx + \int_{\Omega} u^2(x)dx\}$$

$$= k\ ||u||_1^2, \quad k = \min[1,b] > 0.$$

The nonlinear function, f, satisfies (7-17) as was proved in section 7.31, and the hypothesis of theorem 7.11 has been verified. Now, applying those results, we obtain the desired conclusion. The same results can be verified readily for the case n = 1.

Example 7.32.2. We will now solve the problem considered by Movchan in a paper [22], where he considered the stability of elastic systems. We will rigorously show this problem fits into our theory. The system satisfies the following equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \left(\frac{\partial}{\partial x} \right)^4 u(x,t) - b\left(\frac{\partial}{\partial x} \right)^2 u(x,t) = 0 \qquad x\varepsilon[0,1], t\geq 0$$

with the boundary conditions

$$u(0,t) = \left(\frac{\partial}{\partial x} \right)^2 u(0,t) = 0,\ u(1,t) = \left(\frac{\partial}{\partial x} \right)^2 u(1,t) = 0 \quad t \geq 0 \tag{7-22}$$

where b > 0, and we define the operators $A(\frac{\partial}{\partial x})$, $B_o^+(\frac{\partial}{\partial x})$, $B_1^+(\frac{\partial}{\partial x})$, $B_o^-(\frac{\partial}{\partial x})$, $B_1^-(\frac{\partial}{\partial x})$ and the function spaces, $H_B^4[0,1]$, and $H_B^2[0,1]$.

$$A\left(\frac{\partial}{\partial x} \right) = \left(\frac{\partial}{\partial x} \right)^4 - a\left(\frac{\partial}{\partial x} \right)^2$$

where the order of $A(\frac{\partial}{\partial x})$ is 4, and $m = 2$,

$$B_o^+(\frac{\partial}{\partial x})u(1,t) = u(1,t), \qquad B_1^+(\frac{\partial}{\partial x})u(1,t) = (\frac{\partial}{\partial x})^2 u(1,t)$$

$$B_o^-(\frac{\partial}{\partial x})u(0,t) = u(0,t), \qquad B_1^+(\frac{\partial}{\partial x})u(0,t) = (\frac{\partial}{\partial x})^2 u(0,t)$$

$H_B^2[0,1]$ = completion of $C_B^\infty[0,1]$ with respect to the H^2-norm

$$H_B^4[0,1] = \{u \in H^4[0,1] \mid u(1) = (\frac{d}{dx})^2 u(1) = u(0) = (\frac{d}{dx})^2 u(0) = 0\}$$

and we let the base space be $H \equiv H_B^2[0,1] \times L^2[0,1]$.

We will show that problem (7-22) satisfies the hypothesis of theorem 7.1.1, which gives us the result that for any $u_o(x) \in H_B^4[0,1]$, and $v_o(x) \in H_B^2[0,1]$, there exists a unique generalized solution, $u(x,t)$, of (7-22), such that for every $t \geq 0$, $u(x,t) \in H_B^4[0,1]$, with $u(x,0) = u_o(x)$ and $\frac{\partial u(x,0)}{\partial t} = v_o(x)$. Also, $u(x,t)$ satisfies the boundary conditions in the classical sense and the null solution is stable with respect to the L^2-norm.

We must prove that the system $(A(\frac{\partial}{\partial x}), \{B_j^+, B_j^-\})$ satisfies (7-4). $A(D)$ is obviously strongly elliptic and the boundary operators $\{B_o^+, B_1^+\}$ are linearly independent, since the orders are distinct. The same applies to the system $\{B_o^-, B_1^-\}$.

$A(\frac{\partial}{\partial x})$ is formally self-adjoint. Indeed, letting $u,v \in H_B^4[0,1]$, we have from integration by parts and the boundary conditions

$$\begin{aligned}(u, A(\frac{\partial}{\partial x})v)_o &= \int_o^1 u(x)[(\frac{d}{dx})^4 v(x) - b(\frac{d}{dx})^2 v(x)]dx \\ &= \int_o^1 (\frac{d}{dx})^2 u(x)(\frac{d}{dx})^2 v(x)dx\end{aligned}$$

$$+ \left[u(x)\left(\frac{d}{dx}\right)^3 v(x) - \frac{du(x)}{dx}\left(\frac{d}{dx}\right) v(x)\right]\Big|_0^1$$

$$+ b\int_0^1 \left(\frac{du(x)}{dx}\right)\left(\frac{dv(x)}{dx}\right)dx - a\, u(x)\left(\frac{du(x)}{dx}\right)\Big|_0^1$$

$$= \int_0^1 \left(\frac{d}{dx}\right)^4 u(x)\cdot v(x)dx - b\int_0^1 \left(\frac{d}{dx}\right)^2 u(x)\cdot v(x)dx$$

$$= \int_0^1 \left[\left(\frac{d}{dx}\right)^4 u(x) - b\left(\frac{d}{dx}\right)^2 u(x)\right]v(x)dx$$

$$= \left(A\left(\frac{\partial}{\partial x}\right)u, v\right)_0.$$

From the above equality, we have for any $u,v \in H_B^4[0,1]$

$$\int_0^1 u(x)\left[\left(\frac{d}{dx}\right)^4 v(x) - \left(\frac{d}{dx}\right)^2 v(x)\right]dx = \int_0^1 \left[\left(\frac{d}{dx}\right)^2 u(x)\left(\frac{d}{dx}\right)^2 v(x) + b\frac{du(x)}{dx}\frac{dv(x)}{dx}\right]dx.$$

Finally, for any $u \in H_B^4[0,1]$

$$\left(u, A\left(\frac{\partial}{\partial x}\right)u\right)_0 \geq k_1 ||u||_2^2 \qquad \text{for some } k_1 > 0.$$

Indeed, from integration by parts, the boundary conditions, and the well known inequality since $u(0) = u(1) = 0$

$$\int_0^1 \left(\frac{du(x)}{dx}\right)^2 dx \geq \Pi^2 \int_0^1 u^2(x)dx$$

we have

$$(u, A(\frac{\partial}{\partial x})u)_o = \int_o^1 u(x)[(\frac{d}{dx})^4 u(x) - b(\frac{d}{dx})^2 u(x)]dx$$

$$= \int_o^1 [(\frac{d}{dx})^2 u(x)]^2 dx + b\int_o^1 (\frac{du(x)}{dx})^2 dx$$

$$\geq \int_o^1 [(\frac{d}{dx})^2 u(x)]^2 dx + \frac{b}{2}\int_o^1 (\frac{du(x)}{dx})^2 dx + \frac{\pi^2 b}{2}\int_o^1 u^2(x)dx$$

$$\geq \min[1, \frac{b}{2}, \frac{\pi^2 b}{2}]\int_o^1 \{[(\frac{d}{dx})^2 u(x)]^2 + (\frac{du(x)}{dx})^2 + u^2(x)\}dx$$

$$= k||u||_2^2.$$

Hence, we have shown that the system $(A(\frac{\partial}{\partial x}), \{B_j^+, B_j^-\})$ satisfies the hypothesis of theorem 7.1.1 and we obtain the desired result, since $a = 0$.

8.0. CONCLUSIONS

8.1. The Objective of the Research

The objective of the dissertation is to establish some criteria for the existence and uniqueness as well as the stability of the solution to linear and nonlinear partial differential equations with general boundary conditions. The following initial-boundary value problems are considered

$$\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad (8\text{-}1)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\,\frac{\partial u(x,t)}{\partial t} + A(x,D)u(x,t) = f(u) \qquad (8\text{-}2)$$

with general boundary conditions

$$B_j(x,D)u(x,t) = 0 \qquad (0 \le j \le m-1)$$

and initial condition

$$u(x,0) = u_o(x).$$

With the correct definition of the base Hilbert space, H, equations (8-1) and (8-2) with the boundary conditions and initial condition are reduced to the abstract operator equation

$$\frac{du(t)}{dt} + Au(t) = f(u) \qquad (8\text{-}3)$$

$$u(o) = u_o$$

where A is the abstract, linear, unbounded operator extension of $A(x,D)$ defined on part of the real Hilbert space H, and f is a nonlinear function on all of H into itself. Pao in [23] developed a stability theory for the abstract equation (8-3), in which sufficient conditions were given to ensure the existence, uniqueness and stability or asymptotic stability of the solution of (8-3). Stability criteria is then established for the problems

(8-1) and (8-2) from the results obtained for the abstract equation (8-3).

First, the initial-boundary value problem (8-1) is considered, for the linear case, $f(u) \equiv 0$. By defining the base Hilbert space as $H \equiv L^2(\Omega)$, and defining the appropriate abstract operator, the abstract operational equation (8-3) is formed and utilizing the results of Pao [23], a stability criteria for the system $(A(x,D),\{B_j\},\Omega)$ is established. The nonlinear case, $f(u) \not\equiv 0$, is considered, and by placing additional restrictions on the function f, criteria for the existence, uniqueness and stability of a solution are obtained. Since the boundary conditions for the cases $n \geq 2$ and $n = 1$ differ , these cases are treated separately.

Next, the partial differential equation (8-2) with general boundary conditions is considered. For the linear case, $f(u) \equiv 0$, by defining the base Hilbert space as $H \equiv H^m_B(\Omega) x L^2(\Omega)$ and the correct abstract operator, A, as a 2 x 2 matrix with operator elements, the abstract operator equation (8-3) is formed, and by defining an equivalent inner product on H, stability criteria is established for the abstract equation (8-3), and from these results, sufficient conditions are placed on the system$(A(x,D),$ $\{B_j\},\Omega)$ which guarantees the existence and stability of the solution to (8-2). Placing additional assumptions on the nonlinear function $f \not\equiv 0$, guarantees a stability criteria for the nonlinear problem (8-2).

Applications are given which show how the theory can be applied to many physical and engineering problems. In particular, it is shown that the Dirichlet problem is just a special case of the general theory, and how we have generalized the theory to include a much larger class of problems. In the following section, a brief description of the main results of this research are given.

8.2. The Main Results

The initial-boundary value problem (8-1) with general boundary conditions are investigated in Chapter 6.0. The linear case, $f(u) \equiv 0$, is considered first. By constructing the abstract operational differential equation (8-3), the operator extension, A, of A(x,D) is shown in lemmas 6.2.3 and 6.2.4 to be the smallest closed linear extension of A(x,D) defined on $C_B^\infty(\Omega)$. Sufficient conditions are given in theorem 6.2.1, to ensure the existence, uniqueness, stability or asymptotic stability of the solution to the linear problem. The one-dimensional case, $n = 1$, is considered separately and stability criteria is established in theorem 6.3.1. With additional assumptions on the nonlinear function, $f(u) \not\equiv 0$, the nonlinear stability problem is then solved for the case $n \geq 2$, in theorem 6.41.2. For the case $n = 1$, the results are found in theorem 6.42.1. In theorem 6.51.1, it is shown that the Dirichlet problem worked out by Buis in [7] is a special case of theorems 6.2.1 and 6.3.1. Specific applications are worked out in examples 6.52.1 and 6.52.2.

The initial-boundary value problem (8-2) with general boundary conditions are investigated in Chapter 7.0. The linear case is first studied. With an inner product $(.,.)_e$ on H, defined in lemma 7.1.5, which is proved in this lemma to be equivalent to the original inner product defined on H, conditions for existence, uniqueness, stability and asymptotic stability are established in theorem 7.1.1. By imposing additional restrictions on the nonlinear function, $f(u) \not\equiv 0$, the nonlinear problem is solved in theorem 7.2.1. In example 7.31.1, it is shown that a Dirichlet problem worked out by Pao and Vogt in [24] is just a special case of theorem 7.2.1. Specific Applications are worked out in examples 7.32.1 and 7.32.2.

8.3. Some Suggested Further Research

In this work, we have studied the initial-boundary value problems (8-1) and (8-2) with homogeneous boundary conditions. Lions-Magenes in [18] solved the existence and uniqueness problem for the nonhomogeneous elliptic equation

$$A(x,D)u = f(u) \qquad \text{on } \Omega$$

$$B_j(x,D)u = g_j \qquad \text{on } \partial\Omega \quad (0 \le j \le m-1).$$

The methods used here and the work done in [18] suggest an approach to establishing a stability criteria for the nonhomogeneous initial-boundary value problem, which include the homogeneous problem as a special case.

The stability problems (8-1) and (8-2) have been solved in a Hilbert space context. Pao in [23] established a stability criteria for the operational differential equation (8-3), in a Banach space using semi-scalar products, suggesting this work can be done in a Banach space setting.

Stability is a norm property, and stability criteria is established with respect to the L^2-norm. From the Sobolev Imbedding theorem which states that if Ω is smooth enough and m is a large enough integer, $H^m(\Omega) \subset C^o(\bar{\Omega})$, and if $u \in H^m(\Omega) \subset C^o(\Omega)$, then pointwise stability can be considered. This suggests that possibility of defining a different base Hilbert space, say $H^m(\Omega)$, and establishing a stability criteria with respect to the H^m-norm.

Appendix A

In this Appendix we will prove that the following two definitions are equivalent:

$$H_B^{2m}(\Omega) = \text{completion of } C_B^{\infty}(\Omega) \text{ in the } H^{2m}\text{-norm.}$$
$$H_B^{2m}(\Omega) = \text{completion of } C_B^{2m}(\Omega) \text{ in the } H^{2m}\text{-norm.}$$

The proof uses the following facts found in Schechter [44], and properties of $\{A(x,D), B_j(x,D), \Omega\}$ satisfying (5-9). Letting

$$R = \{f \epsilon L^2(\Omega) \mid \text{there exists a } u \epsilon H^{2m}(\Omega), \text{ such that } Au=f, \text{ and } B_j u = 0 \text{ on } \partial\Omega \ (0 \le j \le m-1)\}$$

$$N = \{u \epsilon H^{2m}(\Omega) \mid Au=0,\ B_j u = 0 \text{ on } \partial\Omega (0 \le j \le m-1)\}$$

From the definition of $R^{\perp} = \{g \epsilon L^2(\Omega) \mid (g,f)_o = o, \text{ for all } f \epsilon R\}$, and similarly $N^{\perp}$ we have from Schechter [24] the following facts:

(i) $N \subset C^{\infty}(\bar{\Omega})$

(ii) $R \subset C^{\infty}(\bar{\Omega})$

(iii) There exists a $k_1 > 0$, such that for every $u \epsilon N^{\perp}$,

$$||u||_{2m} \le K_1[||Au||_o + \sum_{j=0}^{m-1} \langle B_j u\rangle_{2m-m_j-\frac{1}{2}}]$$

(iv) there exists $K_2 > 0$, such that for every $u \epsilon H^{2m}(\Omega)$

$$||Au||_o \le K_2 ||u||_{2m} .$$

Let $u \epsilon C_B^{2m}(\Omega)$, and $\varepsilon > 0$ be given. We will show by construction there exists a $z \epsilon C_B^{\infty}(\Omega)$ such that $||z-u||_{2m} < \varepsilon$. From the General Projection theorem in [24], since N is a closed subspace we can let

$$u = u' + u'' \quad , \text{ where } u' \epsilon N^{\perp} \text{ and } u'' \epsilon N.$$

There exists a $v \epsilon C^{\infty}(\bar{\Omega})$, such that $||u'-v||_{2m} < \varepsilon$. Also, from the General

Projection theorem in [24], since R is a closed subspace in $L^2(\Omega)$, we have

$$Av = f + g, \qquad \text{where } f\varepsilon R \text{ and } g\varepsilon R^{\perp}.$$

We can see since $v,g\varepsilon C^{\infty}(\overline{\Omega})$, that $f\varepsilon C^{\infty}(\overline{\Omega})$. By definition of R, and from remark 2.1, theorem 2.1 in [24], we can see there exists a $w\varepsilon C^{\infty}(\overline{\Omega})\cap N$ such that

$$Aw = f, \quad B_j w = 0 \quad \text{on } \partial\Omega.$$

We can now see since $Au'' = 0$, and $u\varepsilon C_B^{2m}(\Omega)$

$$A(w-u') = Aw-A(u-u'') = f - Au$$

$$B_j(w-u') = B_j w-B_j(u-u'') = 0 \quad \text{on } \partial\Omega.$$

I now assert that

$$||w-u'||_{2m} \leq \varepsilon K_1 K_2.$$

To prove this we note that since $f\varepsilon R, Au' = Au\varepsilon R$, $f-Au'\varepsilon R$ and $g\varepsilon R^{\perp}$, from Yosida [35], $(f-Au') + g\varepsilon R + R^{\perp}$ imply

$$||f-Au'+g||_o^2 = ||f-Au'||_o^2 + ||g||_o^2$$

thus, from facts (iii) and (iv),

$$\begin{aligned} ||w-u'||_{2m} &\leq K_1||A(w-u')||_o = K_1||f-Au||_o = K_1||f-Au'||_o \\ &\leq K_1[||f-Au'||_o^2 + ||g||_o^2]^{1/2} \\ &= K_1||f + g - Au'||_o \\ &= K_1||Av-Au'||_o \\ &\leq K_1K_2||v-u'||_{2m} \\ &< \varepsilon K_1 K_2 \end{aligned}$$

Now if we define $z = w+u''$, we see that since $w\varepsilon C^{\infty}(\overline{\Omega})\cap N$ and $B_j w = 0$ on $\partial\Omega$, and $u''\varepsilon N$, that $z\varepsilon C_B^{\infty}(\overline{\Omega})$. Finally

$$||z-u||_{2m} = ||w+u''-u||_{2m} = ||w-u'||_{2m} < \varepsilon K_1 K_2. \qquad \text{qed}$$

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FINAL REPORT

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Stability of Solutions to Partial Differential Equations

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Prepared by

William G. Vogt, Principal Investigator

Professor of Electrical Engineering

January 15, 1971

The Research Results

The present status of the research is described in the technical report "Stability of the Solutions of Elliptic Partial Differential Equations with General Boundary Conditions" by Eugene Jacob Reiser, 10 copies of which are being forwarded to NASA Headquarters.

Possibilities For Future Research

Future research should concentrate on extending the class of nonlinearities which can be handled within this framework and to broaden the class of Lyapunov functionals considered. Also, specific applications should be considered, and in connection with this, certain computing techniques should be implemented.