NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1513

Method of Averages Expansions for Artificial Satellite Applications

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Preface

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Abstract

Formulas for the averaged potential in artificial satellite theory are derived. The potential due to gravity harmonics is developed for the general nm harmonic. That due to a third body is developed up to the fifth degree in R/R_3, the small distance-ratio.

The expressions given differ from ones generally available in the literature in that they do not depend on expansions in either eccentricity or inclination.

In addition, there is included a discussion of the use of the method of averages for tesseral harmonics. In the case of the rapidly rotating planet Mars, the method is constrained to the evaluation of zonal harmonics. For more slowly rotating bodies such as the moon, all tesseral harmonics can be included.

Method of Averages Expansions for Artificial Satellite Applications

I. Introduction

Processing data obtained from the tracking of spacecraft orbiting Mars or the moon poses some special problems, not the least of which is the amount of data involved. However, the basic difficulty in data handling for orbiting (as opposed to cruising) spacecraft is the fact that the spacecraft ephemeris is very costly to compute. The rapidly changing velocities and accelerations require small step size in the numerical integration, resulting in long computer time and rapid accumulation of round-off error. This problem bears particularly strongly on the celestial mechanics experiment for the forthcoming *Mariner* Mars 1971 mission (Ref. 1).

Various methods of handling this problem have been used, the best known of which is the use of Fourier expansions as described by Kaula (Ref. 2.). The disturbing function is expanded in multiple series based on a spherical harmonic expansion for the coordinates with power series expansions in e and $\sin i$. Using the complete series, one obtains all orbit information. Secular and long-period trends can be identified by suppressing terms containing the mean anomaly explicitly.

An alternate method based on the method of averages (Ref. 3.) was used for the lunar orbiter long-arc analysis (Ref. 4). Since for that analysis, only the secular and long-period trends were of interest, it was felt that averaging would be more effective. Averaging provides a direct procedure for obtaining the long-term and secular effects expressed in equations using closed forms in e and $\sin i$. The results are applicable to highly eccentric and/or highly inclined orbits as well as ordinary ones. We describe the method of averages briefly in Section II, and then in Section V we examine some of the errors associated with it.

In developing the computer software for the lunar orbiter work (Ref. 4), we derived formulas for the averaged equations corresponding to various disturbing functions, including gravity-field anomalies expressed in spherical harmonics and third bodies. Since these results have been found so useful, we feel that they should be made available in the literature.

We obtain expressions for the averaged disturbing function only, not for the forces. Because of the reversibility of the order of differentiation and integration (legitimate here), it is possible then to use the averaged disturbing function in the Lagrange planetary equations to obtain the averaged motion.

Expressions for the gravity harmonics are given in Section III. A recursion formula is obtained that is applicable for arbitrary degree and order. Expressions for the third-body effects are derived in Section IV, and are carried up to fifth degree in $1/R_3$ as required by the *Lunar Orbiter* earth perturbation. For planetary orbiters, it is unlikely that this many terms would be needed.

The method of averages, as we use it, applies to equations representing small perturbations to a harmonic oscillator. We must be careful that our formulation of the satellite problem is appropriate. Specifically, the smallness of the perturbation should be evaluated in terms of the error we are prepared to tolerate. Both the rotation of the primary and the motion of the third bodies couple with the high-degree tesseral harmonics of the gravity field to introduce just such an error, which we evaluate in Section V.

Second-order effects are small, as can be seen from the analysis in Ref. 4. We have, therefore, restricted ourselves to the first-order analysis.

In the application, which we do not go into in this report, we form the differential equations of the Kepler elements and integrate numerically. The results for Lunar Orbiter (as described in Ref. 4) are very gratifying, since they exhibit the power of the method in fitting data over long arcs (several hundreds of revolutions).

II. Method of Averages

We review briefly the concepts underlying the method of averages. Consider the equations for the vector x of a perturbed harmonic oscillator written in the form (Ref. 3)

$$\frac{dx}{dt} = \epsilon X(x, t), \tag{1}$$

where ϵ is a small parameter and $X(x, t)$ is a periodic vector function of t of period τ. We look for a solution of the form

$$x = \xi + \epsilon F(\xi, t), \tag{2}$$

where ξ satisfies the equation

$$\dot{\xi} = \epsilon A(\xi). \tag{3}$$

Such a solution (derived in Section V) is given by Eq. (2) when

$$A(\xi) = \frac{1}{\tau}\int_0^\tau X(\xi, t)\, dt, \tag{4}$$

in which ξ is held fixed for the integration, and

$$F(\xi, t) = \int^t X_p(\xi, t)\, dt, \tag{5}$$

in which X_p is the periodic part of X. This solution, called the first improved approximation, satisfies the differential equation to first order in ϵ. The secular part, ξ, is called simply the first approximation.

In the artificial satellite problem, the long-arc information is contained in the averaged parameter ξ, as obtained from Eq. (3). For the Lunar Orbiter work of Ref. 4, we used the satellite equations in terms of Kepler elements, such that ξ represented the vector of averaged elements $\bar{a}$, $\bar{e}$, $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$, and $\bar{\chi}$. The bulk of the work in setting up the equations involved evaluation of the functions $A(\xi)$ with the use of Eq. (4), as will be shown in Sections III and IV.

III. Gravity Harmonics

We represent the gravity field of the moon or other planetary body by a potential, $-U$, expanded in spherical harmonics as follows:

$$U = \frac{\mu}{R} + F = \frac{\mu}{R}\left\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{1}{R^n}P_n^m(\sin\phi)\,[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]\right\}. \tag{6}$$

Then, corresponding to Eq. (1), the satellite motion is described by the Lagrange equations

$$\dot{a} = \frac{2}{Na}\frac{\partial F}{\partial \chi}, \tag{7}$$

$$\dot{e} = \frac{1}{Na^2 e}\left\{(1-e^2)\frac{\partial F}{\partial \chi} - \sqrt{1-e^2}\frac{\partial F}{\partial \omega}\right\}, \quad (8)$$

$$\dot{\chi} = -\frac{1-e^2}{Na^2 e}\frac{\partial F}{\partial e} - \frac{2}{Na}\frac{\partial F}{\partial a}, \quad (9)$$

$$\dot{\Omega} = \frac{1}{Na^2\sqrt{1-e^2}\sin i}\frac{\partial F}{\partial i}, \quad (10)$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{Na^2 e}\frac{\partial F}{\partial e} - \frac{\cot i}{Na^2\sqrt{1-e^2}}\frac{\partial F}{\partial i}, \quad (11)$$

$$\frac{di}{dt} = \frac{1}{Na^2\sqrt{1-e^2}}\left\{\cot i\frac{\partial F}{\partial \omega} - \csc i\frac{\partial F}{\partial \Omega}\right\}. \quad (12)$$

Substitution of F from Eq. (6) into Eqs. (7) through (12) leads to expressions involving the small parameters C_{nm} (the ϵ of Eq. 1), the slowly varying quantities a, e, χ, Ω, ω, and i (the vector x of Eq. 1), and the quantities ϕ and λ, which depend on the satellite position in orbit.

In the averaging integration (Eq. 4), the Kepler elements are held fixed. Thus, since in Eqs. (7) through (12) the time t occurs only in F, and since the order of the mathematical operations of integration with respect to t and differentiation with respect to the Kepler elements can be reversed, we can accomplish the averaging merely by replacing F by its average,

$$\bar{F} = \frac{1}{\tau}\int_0^\tau F\,dt. \quad (13)$$

We now proceed to evaluate the various terms in $\bar{F}$.

The first step is to express ϕ and λ in terms of the orbit parameters. In order to account for rotation of the primary, we replace λ in Eq. (6) by $\lambda - \theta$, where $\theta = \dot{\theta}t$ and $\dot{\theta}$ is the rotation rate of the primary.

Next, we replace $N\,dt$ by dM, which is acceptable to first order, and proceed as follows. We write

$$\bar{F} = \frac{1}{2\pi}\int_0^{2\pi} F\,dM = \mu \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(C_{nm}\bar{F}^C_{nm} + S_{nm}\bar{F}^S_{nm}\right), \quad (14)$$

$$\bar{F}_{nm} = \bar{F}^C_{nm} + j\bar{F}^S_{nm} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{R^{n+1}}P^m_n(\sin\phi)\,e^{jm(\lambda-\theta)}\,dM. \quad (15)$$

Since the integral is more easily expressed in true anomaly f than in mean anomaly, we change the variable of integration to f with the use of

$$\frac{dM}{df} = \left(\frac{R}{p}\right)^2 (1-e^2)^{3/2}. \quad (16)$$

Then, noting that

$$\frac{p}{R} = (1 + e\cos f), \quad (17)$$

we find

$$\bar{F}_{nm} = \frac{(1-e^2)^{3/2}}{p^{n+1}}\frac{1}{2\pi}\int_0^{2\pi}(1+e\cos f)^{n+1}\frac{P^m_n(\sin\phi)}{\cos^m\phi}\left[\cos^m\phi\, e^{jm(\lambda-\theta)}\right]df. \quad (18)$$

Expansions of the individual factors in the integrand of Eq. (18) are given in Appendixes A, B, and C.

To expand the entire integrand in Fourier series, use is made of Eqs. (B-11), (C-4), and (C-9), and Equation (18) then becomes

$$\bar{F}_{nm} = \frac{(1-e^2)^{3/2}}{p^{n+1}}\frac{1}{2\pi}\int_0^{2\pi}\sum_{s=t=l=0}^{\substack{l=m\\ t=n-m\\ s=n-1}} A^{n-1}_s B^{nm}_t C^m_l \cos sf \cos t\left(u - \frac{\pi}{2}\right)\exp\{j[(m-2l)u + m(\Omega-\theta)]\}\,df, \quad (19)$$

in which A^{n-1}_s is a function of e, given by Eqs. (C-5), (C-6), (C-7); B^{nm}_t is a function of i, given by Eqs. (C-11), (C-12), (C-13), (C-14); and C^m_l is a function of i, given by Eq. (B-13).

The integral in Eq. (19) is to be evaluated with all quantities held constant except f, which appears both explicitly and in $u = \omega + f$. Since θ is time-dependent, its variation must be accounted for also. However, in the case of a close satellite of the moon, the change in θ over one orbit is of the order of 0.01 rad, which is negligible in the first approximation. For other celestial bodies, the effect can be significant. An evaluation of the error in these calculations due to rotation of the primary is given in Section V.

Under these assumptions the only terms in Eq. (19) whose integrals do not vanish are those for which the coefficient of f vanishes. These terms are identified as having indices s, t, l, which satisfy one of the four relations

$$\left.\begin{aligned} s + t + m - 2l &= 0, \\ s - t + m - 2l &= 0, \\ -s + t + m - 2l &= 0, \\ -s - t + m - 2l &= 0. \end{aligned}\right\} \tag{20}$$

It is only for such values of s, t, l, then, that it is necessary to compute the product $A_s B_t C_l$ in order to compute $\overline{F}$.

The final form for $\overline{F}_{nm}$ is*

$$\overline{F}_{nm} = \frac{(1-e^2)^{3/2}}{4p^{n+1}} \sum_{s=t=l=0}^{\substack{l=m \\ t=n-m \\ s=n-1}} A_s^{n-1} B_t^{nm} C_l^m \exp\left\{i\left[(t+m-2l)\,\omega + m(\Omega-\theta) \pm \frac{t}{2}\pi\right]\right\}, \tag{21}$$

subject to the restriction that s, t, l satisfy at least one of the four Eqs. (20).

When the values of $\overline{F}^c_{nm}$ and $\overline{F}^s_{nm}$ as determined by Eqs. (20) are substituted in Eqs. (7) through (12), the resulting solution of these equations will describe the secular and long-period perturbations of the orbit due to the nm harmonic.

Since χ does not appear in $\overline{F}_{nm}$, it follows that

$$\dot{\bar{a}} = 0. \tag{22}$$

The rates of change of the other averaged elements can be obtained by direct substitution from Eq. (21).

IV. Third-Body Perturbations

The disturbing potential due to a third body (Fig. 1) located at point $Q\,(R_3, \phi_3, \lambda_3)$ for a satellite located at point $P\,(R, \phi, \lambda)$ can be expressed as a power series in the ratio of the radial distance R to the planet-centered distance of the third body R_3. The expansion results in the expression

$$F^3 = \frac{\mu_3}{R_3}\left[1 + \sum_{n=2}^{\infty}\left(\frac{R}{R_3}\right)^n P_n(\cos\psi)\right]. \tag{23}$$

The quantity $P_n(\cos\psi)$ is the nth order Legendre polynomial with the cosine of the angle ψ between the position vectors $\mathbf{r}$ and $\mathbf{r}_3$ as the argument. The factor μ_3 is the product of the universal gravitational constant and the mass of the third body.

In Eq. (23), the zero-order term in R does not depend on the coordinates of the satellite and can thus be neglected. The first-order term is absent. Thus, F^3 can be written in the following form as a series of spherical harmonics:

$$F^3 = \frac{\mu_3}{R_3} \sum_{n=2}^{\infty} F^3_n, \tag{24}$$

where

$$F^3_n = \left(\frac{R}{R_3}\right)^n P_n(\cos\psi). \tag{25}$$

The small parameter for the expansion in Eq. (24) is the ratio of the satellite distance R to the distance of the perturbing planet, R_3. However, R being a function of position is not appropriate as the small parameter of Eq. (1). Rather, we consider R and R_3 measured in units of the planet (or moon) radius, and then take $(1/R_3^n)$ as the small parameter in F^3_n.

The object here is to find $\overline{F}^3$ for the third body (Fig. 1) as defined by Eq. (13).

*See Appendix D for a listing of $\overline{F}_{nm}$ for specific values of n and m.

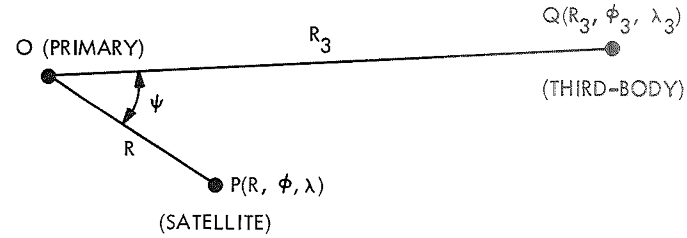

Fig. 1. Third-body geometry

In order to express the polynomials $P_n(\cos\psi)$ in terms of the spherical coordinates of the satellite (R, ϕ, λ) and of the disturbing third body (R_3, ϕ_3, λ_3), one uses the addition formula for the Legendre polynomials given in Ref. 2, as follows:

$$P_n(\cos\psi) = P_n(\sin\phi) P_n(\sin\phi_3) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\sin\phi) P_n^m(\sin\phi_3) \cos m(\lambda - \lambda_3). \tag{26}$$

We now expand the potential due to a point mass located at position R_3, ϕ_3, λ_3 as a series of spherical harmonics referenced to the origin. We arrive at an expression similar to Eq. (6) by first defining third-body coefficients C^3_{nm} and S^3_{nm} as follows:

$$\left.\begin{aligned} C^3_{n0} &= P_n(\sin\phi_3), \\ C^3_{nm} &= 2\frac{(n-m)!}{(n+m)!} P_n^m(\sin\phi_3)\cos\lambda_3, \\ S^3_{nm} &= 2\frac{(n-m)!}{(n+m)!} P_n^m(\sin\phi_3)\cos\lambda_3. \end{aligned}\right\} \tag{27}$$

The expression for F^3 now becomes

$$F^3 = \frac{\mu_3}{R_3} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{R_3}\right)^n P_n^m(\sin\phi)\{C^3_{nm}\cos m\lambda + S^3_{nm}\sin m\lambda\}. \tag{28}$$

We now derive an expression for $\bar{F}^3$, the average value of F^3 over one anomalistic period of the satellite by integrating with respect to the mean anomaly over one period. Changing the variable of integration to true anomaly f, and using complex variable representation as in Eq. (15), we have

$$\bar{F}^3_{nm} = \bar{F}^{3c}_{nm} + j\bar{F}^{3s}_{nm} = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{R}{R_3}\right)^n P_n^m(\sin\phi)\, e^{jm\lambda} \frac{dM}{df}\, df, \tag{29}$$

$$= \frac{1}{2\pi}\left(\frac{p}{R_3}\right)^n (1-e^2)^{3/2} \int_0^{2\pi} (1 + e\cos f)^{-(n+2)} \left[\frac{P_n^m}{\cos^m\phi}\right][\cos^m \phi e^{jm\lambda}]\, df. \tag{30}$$

Taking advantage of Eqs. (C-10) and (B-11), we find that Eq. (30) becomes

$$\bar{F}^3_{nm} = \frac{1}{2\pi}\left(\frac{p}{R_3}\right)^n (1-e^2)^{3/2} \int_0^{2\pi} (1+e\cos f)^{-(n+2)} \sum_{\substack{t=0\\l=0}}^{\substack{l=m\\t=n-m}} B_t^{nm} C_l^m \cos t\left(u - \frac{\pi}{2}\right) \exp j\,[(m-2l)\,u + m(\Omega)]\, df. \tag{31}$$

We next perform the integration of Eq. (31) by the method of residues, holding fixed C^3_{nm} and S^3_{nm}, as defined by Eqs. (27), during the integration.* Performing the indicated integration, we arrive at

$$\bar{F}^3_{nm} = (-1)\left(\frac{1}{2}\right)^{2n+2}\left(\frac{a}{R_3}\right)^n e^{n+1} \sum_{\substack{s=1 \\ t=l=0}}^{\substack{l=m \\ t=n-m \\ s=2}} B^{nm}_l C^m_l A^{3n}_s \alpha^{p_s}(j\eta_s), \tag{32}$$

where α is defined as

$$\alpha = -\frac{1}{e}(1-\sqrt{1-e^2}), \tag{33}$$

A^{3n}_s is defined as

$$A^{3n}_s = \sum_{q=0}^{n+1}\binom{p_s}{q}\binom{2n+2-q}{n+1-q}\left(\frac{2\sqrt{1-e^2}}{1-\sqrt{1-e^2}}\right)^q, \tag{34}$$

and p_s and η_s are defined by

$$\left.\begin{aligned} p_1 &= m-2l+t+n+1, \\ p_2 &= m-2l-t+n+1, \\ \eta_1 &= (m-2l+t)\,\omega + m(\Omega) - t\frac{\pi}{2}, \\ \eta_2 &= (m-2l-t)\,\omega + m(\Omega) + t\frac{\pi}{2}. \end{aligned}\right\} \tag{35}$$

For the majority of practical applications the expansion to $\bar{F}^3_2$ is sufficient. For lunar orbiters, however, more terms are required because of the proximity to the earth and its large mass.

To ease the burden of computation, and in view of the fact that only a few terms were required, we resorted to a direct averaging of Eq. (25), and thus avoided the use of more complicated expressions such as Eq. (32). We now find expressions for $\bar{F}^3$ up to $\bar{F}^3_4$ only.

Let **P** be a unit vector to the orbiter perifocus and **Q** be a unit vector in the orbit plane orthogonal to **P**. Let $\mathbf{U}_3$ be a unit vector pointing to the disturbing body.

*The statement in the introduction regarding the neglect of the rotation of the moon for the gravity-field analysis applies here. If the satellite is close to the moon, then during one revolution of the orbiter, the earth will have moved on the order of 0.02 rad, which is neglected for the first approximation.

We then define

$$\left.\begin{aligned} A &= \mathbf{P}\cdot\mathbf{U}_3, \\ B &= \mathbf{Q}\cdot\mathbf{U}_3, \end{aligned}\right\} \tag{36}$$

and note that A and B remain constant in the averaging process. We expand in combinations of R and $R\cos\psi$ using the equations

$$R = a(1 - e\cos E), \tag{37}$$

$$R\cos\psi = a\{A(\cos E - e) + B\sqrt{1-e^2}\sin E\} \tag{38}$$

or

$$R\cos\psi = a\{-K_1 + K_2\cos E + K_3\sin E\}, \tag{39}$$

where E is the eccentric anomaly and

$$K_1 = Ae,$$
$$K_2 = A,$$
$$K_3 = B\sqrt{1-e^2}.$$

Next, we transform the integration from mean anomaly to eccentric anomaly by using

$$\frac{dM}{dE} = \frac{R}{a} = (1 - e\cos E). \tag{40}$$

For $\bar{F}^3_2$, for example, we have

$$\bar{F}^3_2 = \left(\frac{1}{R_3}\right)^2\left\{-\frac{1}{2}\bar{R}^2 + \frac{3}{2}\overline{R^2\cos^2\psi}\right\}. \tag{41}$$

Expanding Eq. (37) and performing the averaging with respect to E by using Eq. (40), we find for $\bar{R}^2$,

$$\bar{R}^2 = a^2\left(1 + \frac{3}{2}e^2\right), \tag{42}$$

and expanding Eq. (37), we have for $\overline{R^2\cos\psi^2}$,

$$\overline{R^2\cos\psi^2} = a^2\left[A^2\left(2e^2+\frac{1}{2}\right) + \frac{B^2}{2}(1-e^2)\right]. \tag{43}$$

Combining Eqs. (42), (43), and (41), we find $\bar{F}_2^3$ to be

$$\bar{F}_2^3 = \left(\frac{1}{2}\right)\left(\frac{a}{R_3}\right)^2 \left\{-\left(1+\frac{3}{2}e^2\right) + 3A^2\left(2e^2+\frac{1}{2}\right) + \frac{3B^2}{2}(1-e^2)\right\}. \quad (44)$$

Extending the analysis to $\bar{F}_3^3$ and $\bar{F}_4^3$, we have

$$\bar{F}_3^3 = \left(\frac{5}{4}\right)\left(\frac{a}{R_3}\right)^3 Ae \left\{3\left(1+\frac{3}{4}e^2\right) - 5A^2\left(e^2+\frac{3}{4}\right) - \frac{15}{4}B^2(1-e^2)\right\}, \quad (45)$$

$$\bar{F}_4^3 = \left(\frac{3}{8}\right)\left(\frac{a}{R_3}\right)^4 \left\{\phi_1 + 35A^4\phi_2 - 5A^2\phi_3 + \frac{35}{8}B^4\phi_4 - 5B^2\phi_5 + 35A^2B^2\phi_6\right\}, \quad (46)$$

where

$$\phi_1 = \left(1 + 5e^2 + \frac{15}{8}e^4\right),$$

$$\phi_2 = \left(\frac{1}{8} + \frac{3}{2}e^2 + \frac{3}{4}e^4\right),$$

$$\phi_3 = \left(1 + \frac{41}{4}e^2 + \frac{9}{2}e^4\right),$$

$$\phi_4 = (1-e^2)^2,$$

$$\phi_5 = (1-e^2)\left(1+\frac{3}{4}e^2\right),$$

$$\phi_6 = (1-e^2)\left(\frac{1}{2}+3e^2\right).$$

Thus $\bar{F}^3$ up to $\bar{F}_4^3$ is merely

$$\bar{F}^3 = \frac{\mu_3}{R_3}(\bar{F}_2^3 + \bar{F}_3^3 + \bar{F}_4^3). \quad (47)$$

Substituting the appropriate partial derivatives of $\bar{F}^3$ from either Eq. (47) or Eqs. (32) and (24) into Eqs. (29–34) will yield the average rates of the orbiter elements due to the perturbation of a discrete third body.

V. Effect of Rotation of the Primary

In setting up our equations for the averaged variables and the evaluation of the averaged potential (see, e.g., Eq. 19), we assumed that the angular rate of the primary, $\dot{\theta}$, was so small as to be negligible over one orbit of the satellite. We now take a closer look at the implications of this assumption by examining the method of averages in some detail.

Starting with Eq. (1) and appropriate initial conditions

$$\frac{dx}{dt} = \epsilon X(x,t), \qquad x = x_0 \text{ at } t = t_0, \quad (48)$$

we look for a solution of the form of Eq. (2)

$$x = \xi + \epsilon F(\xi, t), \quad (49)$$

where

$$\frac{d\xi}{dt} = \epsilon A(\xi), \qquad \xi = \xi_0 \text{ at } t = 0. \quad (50)$$

The initial value ξ_0 and the functions $A(\xi)$ and $F(\xi, t)$ are to be determined. Substitution in Eq. (48) gives

$$\frac{d\xi}{dt} + \epsilon\left(\frac{\partial F}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial F}{\partial t}\right) = \epsilon X(\xi,t) + \frac{\partial X}{\partial \xi}(x-\xi) + \cdots, \quad (51)$$

where it has been assumed that X is expandable in a Taylor series about $x = \xi$. Using Eq. (49), and neglecting terms of order ϵ^2, we obtain from Eq. (51)

$$A(\xi) + \frac{\partial F}{\partial t} = X(\xi, t), \quad (52)$$

whence

$$A(\xi) = \text{nonperiodic part of } X = \frac{1}{\tau}\int_0^\tau X(\xi,t)\,dt, \quad (53)$$

and

$$\frac{\partial F}{\partial t} = \text{periodic part of } X$$
$$= X_p(\xi, t). \tag{54}$$

Thus

$$F(\xi, t) = \int^t X_p(\xi, s)\, ds. \tag{55}$$

The value of ξ_0 to first order is obtained from Eq. (50) as

$$\xi_0 = x_0 - \epsilon F(x_0, 0). \tag{56}$$

The solution through first-order terms, or rather the solution that satisfies the differential equation through first order, is given by Eq. (49), with the use of Eqs. (50), (53), and (55) to define ξ, A, and F.

The nonperiodic part of the solution, $x = \xi$, is referred to as the *first approximation* in the method of averages. The complete first-order solution, $x = \xi + \epsilon F$, is referred to as the *improved first approximation.* In applications of interest, the relation between these two approximations resembles that in Fig. 2. The trend in the solution is given by ξ, while the improved approximation merely supplies very small local fluctuations. For real-data application, it is possible to work with ξ directly provided that the data, and particularly the compressed data, are referenced to ξ. The information lost by rejecting x can be recouped by short-arc processing.

*Example:** Consider the differential equation system

$$\frac{dx}{dt} = \epsilon \cos^2 t, \tag{57}$$

*Suggested by H. Lass.

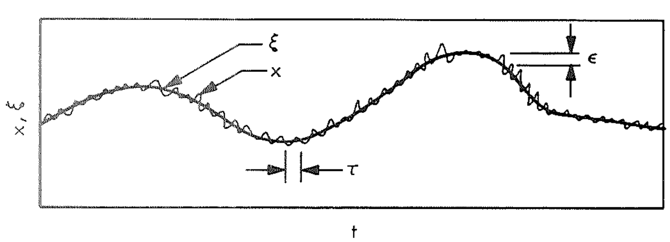

Fig. 2. First approximation ξ and improved first approximation x

$$\frac{dy}{dt} = \epsilon \sin kx \cos^2 t, \tag{58}$$

where $t = 0$, $x = x_0$, $y = y_0$.

The exact solution is

$$x - x_0 = \frac{\epsilon}{2}\left(t + \frac{1}{2}\sin 2t\right), \tag{59}$$

$$y - y_0 = -\frac{1}{k}\left[\cos k\left(\frac{\epsilon}{2}t + x_0\right) - \cos k x_0\right] + \frac{\epsilon}{4}\sin 2t \sin k\left(\frac{\epsilon}{2}t + x_0\right). \tag{60}$$

The first approximation is obtained from

$$\dot{\xi} = \frac{\epsilon}{2}, \qquad \dot{\eta} = \frac{\epsilon}{2}\sin k\xi. \tag{61}$$

The periodic parts of the forcing functions in Eqs. (57) and (58) are

$$X_p = \frac{1}{2}\cos 2t \qquad \text{and} \qquad Y_p = \frac{1}{2}\sin kx \cos 2t, \tag{62}$$

with the indefinite integrals (see Eq. 55)

$$\left.\begin{aligned} F(\xi, \eta, t) &= \frac{1}{4}\sin 2t, \\ G(\xi, \eta, t) &= \frac{1}{4}\sin k\xi \sin 2t. \end{aligned}\right\} \tag{63}$$

Then, according to Eq. (56), we get

$$\xi_0 = x_0 \qquad \text{and} \qquad \eta_0 = y_0. \tag{64}$$

Equations (61) can be integrated directly, giving the *first approximation*

$$\left.\begin{aligned} \xi &= x_0 + \frac{\epsilon}{2}t, \\ \eta &= y_0 - \frac{1}{k}\cos k\left[\left(\frac{\epsilon}{2}t + x_0\right) - \cos k x_0\right], \end{aligned}\right\} \tag{65}$$

and the *first improved approximation*

$$\left.\begin{aligned}\xi &= x_0 + \frac{\epsilon}{2}t + \frac{\epsilon}{4}\sin 2t,\\ \eta &= y_0 - \frac{1}{k}\left[\cos k\left(\frac{\epsilon}{2}t + x_0\right) - \cos k x_0\right]\\ &\quad + \frac{\epsilon}{4}\sin 2t \sin k\left(\frac{\epsilon}{2}t + x_0\right)\end{aligned}\right\} \tag{66}$$

We note that in this example the first improved approximation coincides with the exact solution (Eqs. 59 and 60).

We point out that the error in the *first approximation* has a periodic factor of period π, and that its magnitude is small, of the order of ϵ. In fact, the error is

$$\left.\begin{aligned}x - \xi &= \frac{\epsilon}{4}\sin 2t,\\ y - \eta &= \frac{\epsilon}{4}\sin 2t \sin k\left(\frac{\epsilon}{2}t + x_0\right).\end{aligned}\right\} \tag{67}$$

Thus, not only is the differential equation satisfied to first order by the first approximation, but also the solution is approximated uniformly to first order for all value of t.

We are now ready to examine the equations of celestial mechanics, as used in Section III. For the Lunar Orbiter data analysis, the method of averages was applied directly, without resorting to Fourier series expansions. However, in order better to evaluate the procedure, we now expand in series. We focus attention on one specific contribution to the forcing function, namely that due to the harmonic coefficient C_{lm}, realizing that the complete forcing function consists of a sequence of such terms. Thus, following Ref. 2, we can write the Ω equation in the form

$$\frac{d\Omega}{dM} = \frac{a_e^l}{a^l(1-e^2)^{1/2}\sin i}\sum_{p=0}^{l} F'_{lmp}(i) \sum_{q=-\infty}^{\infty} G_{lpq}(e)\, S_{lmpq}(\omega, M, \Omega, \theta), \tag{68}$$

where

$$S_{lmpq} = C_{lm}\binom{\cos}{\sin}\left[(l-2p)\,\omega + (l-2p+q)\,M + m(\Omega - \theta)\right], \tag{69}$$

with cos for $l - m$ even, sin for $l - m$ odd.

Similar expressions apply for the rates of the remaining Kepler elements, a, e, i, and ω. For a complete explanation of the notation, the reader is referred to Ref. 2. Suffice it to say here that F and G are complicated functions of i and e respectively.

To apply the method of averages, we note firstly that the equations are of the correct form corresponding to Eq. (48), where C_{lm} is the small parameter. In the case of the moon, for most values of l and m, C_{lm} is of the order of 10^{-4} or smaller. Secondly, we note that besides C_{lm}, two nondynamic parameters appear in Eq. (65), namely a_e, the moon's radius, and θ the moon's rotation angle. Since a_e is an absolute constant, it causes no problem. However, θ is of the form $\theta = \dot{\theta}t + \theta_0$, where $\dot{\theta}$ is the constant rate of rotation of the moon. Hence, the forcing function is not periodic as required by the theory.

It is a simple matter to recast the equations into proper form. Let θ be a sixth state-variable, defined by the equation

$$\begin{aligned}\frac{d\theta}{dM} &= \frac{1}{n}\dot{\theta}, \qquad \theta = \theta_0 \text{ at } M = 0,\\ &= C_{lm}\left(\frac{\dot{\theta}}{nC_{lm}}\right).\end{aligned} \tag{70}$$

The accuracy of the results will depend on the size of the quantity $\dot{\theta}/n$. For a lunar satellite of period 3 hours, the ratio $\dot{\theta}/n$ is approximately 1/220.

Let us next apply the averaging procedure to Eqs. (68) and (70). Using bars to denote the averaged variables, we have, corresponding to Eq. (50),

$$\frac{d\bar{\Omega}}{dM} = \Sigma H_p(\bar{a}, \bar{e}, \bar{i})\, C_{lm}\binom{\cos}{\sin}(l-2p)\,\bar{\omega} + m(\bar{\Omega} - \bar{\theta}), \tag{71}$$

in which H_p is obtained by setting $q = l - 2p$. Also,

$$\frac{d\bar{\theta}}{dM} = \frac{1}{n}\dot{\theta}. \tag{72}$$

For initial conditions, we have

$$\left.\begin{aligned}\bar{\Omega}_0 &= \Omega_0 - C_{lm} P(a_0, e_0, i_0, \omega_0, M_0, \Omega_0, \theta_0),\\ \bar{\theta}_0 &= \theta_0.\end{aligned}\right\} \quad (73)$$

At this point, in order to evaluate the effect of large values of $\dot{\theta}/n$, we shall consider a simpler system of equations, which however have the same general structure as Eqs. (68) and (70). We write

$$\frac{dx}{dt} = \epsilon\{1 + \cos[m(x - y) + t]\}, \quad (74)$$

$$\frac{dy}{dt} = \epsilon k, \quad (75)$$

where $t = 0$: $x = x_0$, $y = y_0$.

The exact solution is

$$y = \epsilon k t + y_0, \quad (76)$$

$$x = y_0 - \frac{(1 - m\epsilon k)\,t}{m} + \frac{1}{m}\sin^{-1}\left\{\frac{\sqrt{1 - \gamma^2}\sin\psi}{1 - \gamma\sin\psi}\right\}, \quad (77)$$

where

$$\gamma = \frac{m\epsilon}{\alpha},$$

$$\psi = \alpha t + A,$$

$$\alpha = 1 + m\epsilon(1 - k),$$

$$A = \frac{1}{\sqrt{1 - \gamma^2}}\tan^{-1}\frac{\sqrt{1 - \gamma^2}\sin\omega_0}{\gamma + \cos\omega_0}$$

$$\omega_0 = m(x_0 - y_0)$$

To first order in γ, we have

$$A = \omega_0 - \gamma\sin\omega_0 \quad (78)$$

and

$$\sin^{-1}\left(\frac{\sqrt{1 - \gamma^2}\sin\psi}{1 - \gamma\cos\psi}\right) = \psi + \gamma\sin\psi. \quad (79)$$

Thus, to first order in γ,

$$\begin{aligned} x &= y_0 - \frac{(1 - m\epsilon k)}{m}t + \frac{\alpha t + A + \gamma\sin\psi}{m}\\ &= y_0 + \epsilon t + y_0 - x_0 + \frac{\gamma(\sin\psi - \sin\omega_0)}{m}\\ &= x_0 + \epsilon t - \epsilon\sin\omega_0 + \epsilon\sin\psi \end{aligned} \quad (80)$$

The first approximation for the method of averages is obtained from

$$\frac{d\xi}{dt} = \epsilon,$$

$$\frac{d\eta}{dt} = \epsilon k,$$

$$F(\xi, \eta, t) = \sin[m(\xi - \eta) + t],$$

$$G(\xi, \eta, t) = 0,$$

$$\xi_0 = x_0 - \epsilon\sin[m(x_0 - y_0)],$$

$$\eta_0 = y_0.$$

Therefore,

$$\xi = \epsilon t + x_0 - \epsilon\sin m(x_0 - y_0), \quad (81)$$

$$\eta = \epsilon k t + y_0. \quad (82)$$

Comparing the exact solutions x with the first approximation ξ (Eqs. 80 and 81), we see that they differ only in the term $\epsilon\sin\psi$ of order ϵ and periodic of period $2\pi/\alpha$.

The only conditions under which this solution breaks down is for large values of γ (i.e., not satisfying $\gamma << 1$). Consider, then, the values of γ for a typical situation. We have (identifying k with $\dot{\theta}/n\epsilon$, as in Eq. 70),

$$\gamma = \frac{m\epsilon}{1 + m\epsilon(1 - k)} \sim \frac{m\epsilon}{1 - m\epsilon\dfrac{\dot{\theta}}{n\epsilon}} = \frac{m\epsilon}{1 - m\dfrac{\dot{\theta}}{n}},$$

since $m\epsilon$ is small.

For the Lunar Orbiter analysis, we have $\dot{\theta}/n \sim 1/220$ and, in the worst case considered, $m = 15$. Thus $1 - m\dot{\theta}/n = 0.85$ and with $\epsilon \sim 10^{-4}$ corresponding to the largest value of a normalized fifteenth order harmonic coefficient, we get

$$\gamma = 17 \times 10^{-4}$$

In this case, the solution should be valid to the order $\gamma^2 \sim 10^{-5}$. On the other hand, in the case of a 12-hour Mars orbiter, we have $\dot{\theta}/n \sim 1/2$. Then

$$1 + m\epsilon(1 - k) \cong 1 - m\left(\frac{\dot{\theta}}{n}\right)$$

$$\cong 1 - \frac{m}{2},$$

so that even for $m = 2$ the solution breaks down. Thus, the first approximation in the method of averages would not be applicable in this fashion for Mars orbiter studies.

In using this method, we work with an averaged state parameter ξ in place of the actual state parameter x. From the theory, it follows readily that the error $(x - \xi)$ satisfies the differential equation up to terms of order ϵ^2 in the perturbing parameter ϵ, except for short-period terms of order ϵ. Insofar as the differential equations of orbiter motion are concerned, the method of averages solution not only satisfies the differential equation to first order, but also agrees with the true solution to first order. This assertion is based on numerical experience and on analogy with simpler systems that can be integrated analytically. For our argument, therefore, we shall assume this to be the case.

By working with the averaged parameter ξ, we avoid the short-period error of order ϵ, and the results of the data analysis are good to order ϵ. In particular, the normal point $\bar{x}$ computed for perfect data agrees with the computed value $\bar{\xi}$, to terms of order ϵ.

In the case of a rotating primary, rate $\dot{\theta}$, and a perturbing gravity harmonic C_{lm} of order m, the error parameter γ is of the form

$$\gamma = \frac{m C_{lm}}{1 - \dfrac{m\dot{\theta}}{n}}$$

Thus, for the moon, with $m = 15$, $\dot{\theta}/n = 1/220$, and $C_{lm} = 10^{-4}$, the value of γ is 17×10^{-4}, whence the numerically induced errors in the analysis for node Ω are of the order

$$\Delta\Omega \sim (17 \times 10^{-4})^2 \text{ rad} \sim 1.7 \times 10^{-4} \text{ deg.}$$

Nomenclature

a	semimajor axis
A_s^n	defined Eqs. (C-5, 6, 7)
B_t^{nm}	defined Eqs. (C-11, 12, 13, 14)
C_l^m	defined Eq. (B-13)
C_{nm}	gravity field coefficient (Eq. 6)
e	eccentricity
f	true anomaly
$\bar{F}_{nm}$	gravity field harmonic (Eq. 15)
$\bar{F}_{nm}^c$	real part of $\bar{F}_{nm}$
$\bar{F}_{nm}^s$	imaginary part of $\bar{F}_{nm}$
i	inclination angle
M	mean anomaly
N	mean motion
p	$a(1 - e^2)$
P_n^m	Legendre polynomial (Eq. C-8)
R	radial distance to center of mass
S_{nm}	gravity field coefficient (Eq. 6)
u	$\omega + f$
U	gravity potential
θ	celestial longitude of prime meridian
λ	celestial longitude
μ	gravity constant
ϵ	small parameter
ϕ	latitude
χ	$= M - \int N dt$
ω	apse angle
Ω	node angle
ψ	angle: third body–primary–spacecraft

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Appendix A
Interchange of Summation Order

Consider the function

$$\phi(n,f) = \sum_{\kappa=0}^{n} \psi(\kappa) \sum_{l=0}^{\kappa} \binom{\kappa}{l} e^{i(\kappa-2l)f}. \tag{A-1}$$

It will be shown that

$$\phi(n,f) = \sum_{s=0}^{n} H_s \cos sf, \tag{A-2}$$

where

$$H_0 = + \sum_{k=0}^{[n/2]} \binom{2k}{k} \psi(2k), \tag{A-3}$$

$$s\text{-even:} \quad H_s = 2 \sum_{k=(s/2)}^{[n/2]} \binom{2k}{k - \frac{s}{2}} \psi(2k), \qquad s \neq 0, \tag{A-4}$$

$$s\text{-odd:} \quad H_s = 2 \sum_{k=(s-1)/2}^{[(n-1)/2]} \binom{2k+1}{k - \frac{s-1}{2}} \psi(2k+1). \tag{A-5}$$

Proof:

In the inner sum of Eq. (A-1), consider the two cases, κ even and κ odd. When κ is even, write $\kappa = 2k$, $l = r + (\kappa/2)$, and the sum becomes

$$\sum_{r=-k}^{k} \binom{2k}{r+k} e^{-2rfi}. \tag{A-6}$$

Then, we can write successively the following equivalent sums:

$$= \sum_{r=1}^{k} \binom{2k}{r+k} e^{-2rfi} + \sum_{r=1}^{-k} \binom{2k}{r+k} e^{-2rfi} + \binom{2k}{k}, \tag{A-7}$$

$$= 2 \sum_{r=1}^{k} \binom{2k}{r+k} \cos 2rf + \binom{2k}{k}. \tag{A-8}$$

When κ is odd, write $\kappa = 2k + 1$ and $l = r + (\kappa - 1)/2$. Then the inner sum in Eq. (A-1) becomes

$$\sum_{r=-k}^{k+1} \binom{2k+1}{r+k} e^{-(2r-1)fi} \tag{A-9}$$

$$= \sum_{r=1}^{k+1} \binom{2k+1}{k+1-r} e^{(2r-1)fi}$$

$$+ \sum_{r=1}^{k+1} \binom{2k+1}{k+1-r} e^{-(2r-1)fi} \tag{A-10}$$

$$= 2 \sum_{r=0}^{k} \binom{2k+1}{k-r} \cos(2r+1)f. \tag{A-11}$$

Thus, Eq. (A-1) may be rewritten as

$$\phi(n,f) = -\sum_{k=0}^{[n/2]} \binom{2k}{k} \psi(2k) + 2 \sum_{k=0}^{[n/2]} \sum_{r=0}^{k} \psi(2k) \binom{2k}{r+k}$$
$$\times \cos 2rf + 2 \sum_{k=0}^{[(n-1)/2]} \sum_{r=0}^{k} \psi(2k+1) \binom{2k+1}{k-r}$$
$$\times \cos(2k+1)f. \tag{A-12}$$

in which $[x]$ is the largest integer less than or equal to x.

To obtain Eq. (A-2) from Eq. (A-12), it is sufficient to note that the order of summation can be interchanged according to the formula

$$\sum_{k=0}^{\alpha} \sum_{r=0}^{k} F(k,r) = \sum_{r=0}^{\alpha} \sum_{k=r}^{\alpha} F(k,r). \tag{A-13}$$

Appendix B
Kepler Elements of the Orbit in Terms of ϕ and λ

In Fig. B-1, Q represents the projection of the satellite position on the unit sphere, centered at the center of mass of the central body; Q_0 is the projection of perigee position. Let ϕ, λ be the latitude and longitude of Q, referred to an inertial coordinate system x, y, z, and let the orbital parameters be

i = inclination,

ω = argument of pericenter,

Ω = longitude of the node,

f = true anomaly,

$\delta = \lambda - \Omega$,

θ = longitude of reference meridian fixed in central body,

$\Omega' = \Omega - \theta$,

$u = \omega + f$.

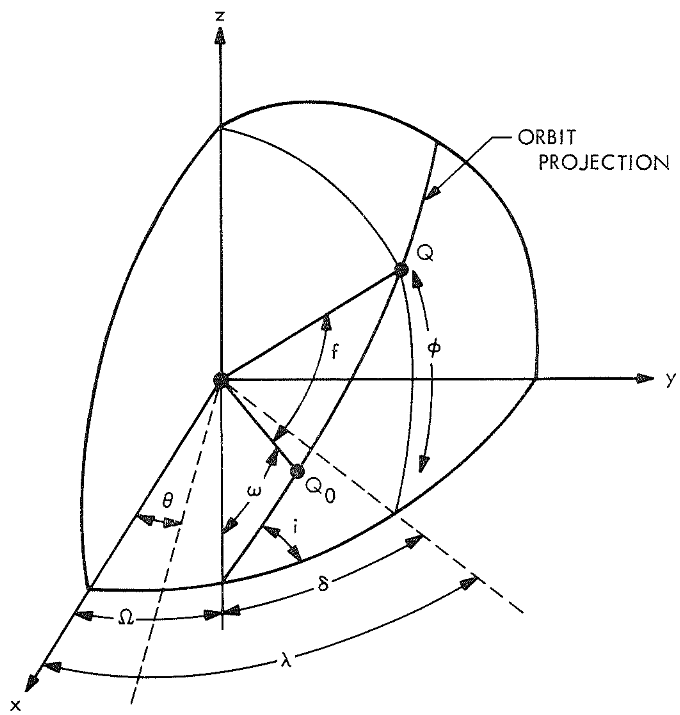

Fig. B-1. Orbit projection on unit sphere

Then, from spherical trigonometry,

$$\sin \phi = \sin u \sin i, \tag{B-1}$$

$$\cos \delta \cos \phi = \cos u, \tag{B-2}$$

$$\sin \delta \cos \phi = \sin u \cos i, \tag{B-3}$$

$$\cos (\lambda - \theta) = \cos \Omega' \cos \delta - \sin \Omega' \sin \delta, \tag{B-4}$$

$$\sin (\lambda - \theta) = \sin \Omega' \cos \delta - \cos \Omega' \sin \delta. \tag{B-5}$$

Therefore

$$\cos (\lambda - \theta) = \frac{1}{\cos \phi} [\cos \Omega' \cos u - \sin \Omega' \sin u \cos i], \tag{B-6}$$

$$\sin (\lambda - \theta) = \frac{1}{\cos \phi} [\sin \Omega' \cos u + \cos \Omega' \sin u \cos i]. \tag{B-7}$$

Next, we obtain expressions for $\cos n (\lambda - \theta), \sin n (\lambda - \theta)$ in terms of $\cos ku, \sin ku$. Write

$$e^{j(\lambda-\theta)} = \cos(\lambda - \theta) + j\sin(\lambda - \theta)$$

$$= \frac{e^{j\Omega'}}{\cos\phi}\left[e^{ju}\cos^2\left(\frac{1}{2}i\right) + e^{-ju}\sin^2\left(\frac{1}{2}i\right)\right] \tag{B-8}$$

Then

$$e^{jm(\lambda-\theta)} = \frac{1}{\cos^m\phi}\sum_{k=0}^{m}\binom{m}{k}\left(\cos\frac{1}{2}i\right)^{2(m-k)}\left(\sin\frac{1}{2}i\right)^{2k}\exp\{j[(m-2k)u + m\Omega']\}, \tag{B-9}$$

which is the required formula.

In particular, for $m = 2$,

$$e^{2j(\lambda-\theta)} = \frac{1}{\cos^2\phi}\left\{\cos^4\frac{1}{2}i\,e^{j(2u+2\Omega')} + \frac{\sin^2 i}{2}e^{j(2\Omega')} + \sin^4\frac{1}{2}i\,e^{j(2\Omega'-2u)}\right\} \tag{B-10}$$

Equation (B-9) may be written more concisely in the form

$$e^{jm(\lambda-\theta)}\cos^m\phi = \sum_{l=0}^{m} C_l^m(i)\,\exp\{j[(m-2l)u + m\Omega']\}, \tag{B-11}$$

with

$$C_l^m(i) = \binom{m}{l}\left(\cos\frac{1}{2}i\right)^{2(m-l)}\left(\sin\frac{1}{2}i\right)^{2l} = \binom{m}{l}\frac{1}{2^{2m}}\sum_{k=0}^{2m-2l}\sum_{r=0}^{2l}(-1)^{l-r}\binom{2m-2l}{k}\binom{2l}{r}\exp[j(m-k-r)(i)] \tag{B-13}$$

Appendix C
Expansions

The most direct expansion of the integral in Eq. (18) yields a series in which each trigonometric term contains m and n plus five summation indices. However, it is possible to change the order of summation so as to reduce this number by two, and still retain a comparatively simple form for the coefficients, as shown in Eq. (19).

To effect this interchange of summation order, the formulas of Appendix A are applied to each of the three factors in the integrand of Eq. (14). Then, using Eqs. (C-1) and (C-2) below, we obtain the required expansions of the integral factors as follows:

$$\cos^n f = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos(n-2k) f = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \exp\left[i(n-2k)f\right], \tag{C-1}$$

$$\sin^n f = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \exp\left[i(n-2k)\left(f - \frac{\pi}{2}\right)\right]. \tag{C-2}$$

The binomial expansion yields

$$(1 + \epsilon \cos f)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{\epsilon^k}{2^k} \sum_{l=0}^{k} \binom{k}{l} e^{i(k-2l)f}. \tag{C-3}$$

and then interchanging the summation order, as in Eqs. (A-1–A-5), we obtain

$$(1 + \epsilon \cos f)^n = \sum_{s=0}^{n} A_s^n \cos sf, \tag{C-4}$$

in which

$$A_0^n = + \sum_{k=0}^{[n/2]} \binom{2k}{k} \binom{n}{2k} \frac{\epsilon^{2k}}{2^{2k}}, \tag{C-5}$$

$$A^n_{\substack{s\,(\mathrm{even}) \\ s \neq 0}} = 2 \sum_{k=s/2}^{[n/2]} \binom{2k}{k - \frac{s}{2}} \binom{n}{2k} \frac{\epsilon^{2k}}{2^{2k}}, \tag{C-6}$$

$$A^n_{s\,(\mathrm{odd})} = 2 \sum_{k=(s-1)/2}^{[(n-1)/2]} \binom{2k+1}{k - \frac{s-1}{2}} \binom{n}{2k+1} \frac{\epsilon^{2k+1}}{2^{2k+1}}. \tag{C-7}$$

Starting with the Legendre polynomial,* we find

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n} \sum_{r=0}^{[(n-m)/2]} \frac{(-1)^r (2n-2r)!\, x^{n-m-2r}}{r!\,(n-r)!\,(n-m-2r)!} \tag{C-8}$$

The expansion of the polynomial in latitude in terms of the Kepler elements i (inclination) and $u\,(=\omega + f =$ apse angle plus true anomaly) is

$$\frac{2^n P_n^m(\sin\phi)}{\cos^m\phi} = \sum_{r=0}^{[(n-m)/2]} \frac{(-1)^r (2n-2r)!\,(\sin i)^{n-m-2r}}{r!\,(n-r)!\,(n-m-2r)!} (\sin u)^{n-m-2r} \tag{C-9}$$

Then, with the application of the summation inversion formulas of Appendix A it follows that

$$\frac{P_n^m(\sin\phi)}{\cos^m\phi} = \sum_{s=0}^{n-m} B_s^{nm} \cos s\left(u - \frac{\pi}{2}\right). \tag{C-10}$$

in which the coefficients B_s^{nm} are defined as follows:

When $(n-m)$ is even:

$$2^n B_0^{nm} = \sum_{k=0}^{[(n-m)/2]} (-1)^{\{(n-m-2k)/2\}} \binom{2k}{k} \frac{(n+m+2k)!\,(\sin i)^{2k}}{\left(\frac{n+m+2k}{2}\right)!\left(\frac{n-m-2k}{2}\right)!\,(2k)!\,2^{2k}}. \tag{C-11}$$

When $(n-m)$ is even, and s is even, but $\neq 0$:

$$2^n B_s^{nm} = 2 \sum_{k=s/2}^{[(n-m)/2]} (-1)^{\{(n-m-2k)/2\}} \binom{2k}{k-\frac{s}{2}} \frac{(n+m+2k)!\,(\sin i)^{2k}}{\left(\frac{n+m+2k}{2}\right)!\left(\frac{n-m-2k}{2}\right)!\,(2k)!\,2^{2k}}. \tag{C-12}$$

When $(n-m)$ is odd and s is odd:

$$2^n B_s^{nm} = 2 \sum_{k=(s-1)/2}^{[(n-m)/2]} (-1)^{\{(n-m-2k-1)/2\}} \binom{2k+1}{k-\frac{s-1}{2}} \frac{(n+m+2k+1)!\,(\sin i)^{2k+1}}{\left(\frac{n+m+2k+1}{2}\right)!\left(\frac{n-m-2k-1}{2}\right)!\,(2k+1)!\,2^{2k+1}}. \tag{C-13}$$

Furthermore,

$$\underset{(n-m)\,(\text{odd})}{B_0^{nm}} = \underset{(n-m)\,(\text{odd})}{B_{s\,(\text{even})}^{nm}} = \underset{(n-m)\,(\text{even})}{B_{s\,(\text{odd})}^{nm}} = 0\,. \tag{C-14}$$

*The bracket $[(n-m)/2]$ means the greatest integer less than or equal to $(n-m)/2$.

Appendix D

Gravity Harmonics for Specific Values of *n* and *m*

$$\bar{F}_{20} = \frac{(1-\epsilon^2)^{3/2}}{4p^3}(1 - 3\cos^2 i)$$

$$\bar{F}_{21} = \frac{(1-\epsilon^2)^{3/2}}{p^3}\cdot\frac{3}{4}\sin 2i \exp j\left(\Omega' + \frac{\pi}{2}\right)$$

$$\bar{F}_{22} = \frac{3}{2}\frac{(1-\epsilon^2)^{3/2}}{p^3}\sin^2 i\, e^{2j\Omega'}$$

$$\bar{F}_{30} = \frac{3\epsilon(1-\epsilon^2)^{3/2}}{2p^4}\sin\omega \sin i\left(\frac{5}{4}\sin^2 i - 1\right)$$

$$\bar{F}_{31} = \frac{-3}{64}\frac{\epsilon(1-\epsilon^2)^{3/2}}{p^4}\left[(6 + \cos i + 10\cos 2i + 15\cos 3i)\, e^{j(\omega+\Omega')} + 4(4 + \cos i - 5\cos 3i)\, e^{j(-\omega+\Omega')}\right]$$

$$\bar{F}_{32} = \frac{15\epsilon}{16}\frac{(1-\epsilon^2)^{3/2}}{p^4}\sin i\left\{(1 + 4\cos i + 3\cos 2i)\exp\left[j\left(\omega + 2\Omega' + \frac{\pi}{2}\right)\right] + (1 - 4\cos i + 3\cos 2i) \times \exp\left[j\left(-\omega + 2\Omega' - \frac{\pi}{2}\right)\right]\right\}$$

$$\bar{F}_{33} = \frac{45\epsilon}{4}\frac{(1-\epsilon^2)^{3/2}}{p^4}\sin^2 i\left\{\cos^2\frac{i}{2}\exp\left[i(\omega + 3\Omega')\right] + \sin^2\frac{i}{2}\exp\left[j(-\omega + 3\Omega')\right]\right\}$$

$$\bar{F}_{40} = \frac{3}{8}\frac{(1-\epsilon^2)^{3/2}}{p^5}\left\{\left(1 + \frac{3}{2}\epsilon^2\right)\left(1 - 5\sin^2 i + \frac{35}{8}\sin^4 i\right) + \frac{5\epsilon^2}{8}(6 - 7\sin^2 i)\sin^2 i\cos 2\omega\right\}$$

$$\bar{F}_{41} = \frac{15}{16}\frac{(1-\epsilon^2)^{3/2}}{p^5}e^{j\Omega'}\sin i\left\{(-4 + 7\sin^2 i)\cos i\left(1 + \frac{3}{2}\epsilon^2\right)e^{\pi j/2} - \epsilon^2(-3 + 7\sin^2 i)\sin 2\omega - \epsilon^2\cos i\left(-3 + \frac{7}{2}\sin^2 i\right)\cos 2\omega\, e^{\pi j/2}\right\}$$

$$\bar{F}_{42} = \frac{15(1-\epsilon^2)^{3/2}e^{2j\Omega'}}{4p^5}\left\{-\left(1 + \frac{3}{2}\epsilon^2\right)\sin^2 i\left(1 - \frac{7}{2}\sin^2 i + \frac{7}{2}\sin^2 i\cos i\right) - \frac{3}{2}\epsilon^2\cos i\left(1 - \frac{7}{2}\sin^2 i\right)e^{\pi j/2}\sin 2\omega + \frac{3\epsilon^2}{2^8} \times (30 - 151\cos 2i + 14\cos 4i + 7\cos 6i)\cos 2\omega\right\}$$

$$\bar{F}_{43} = \frac{315\epsilon^2(1-\epsilon^2)^{3/2}}{8p^5}\exp\left[j\left(3\Omega' + \frac{\pi}{2}\right)\right]\sin i\left\{\cos^6\frac{i}{2}e^{2j\omega} + 3\cos^4\frac{i}{2}\sin^6 i\left(2 + \frac{4}{3\epsilon^2} - e^{2j\omega}\right) - 3\cos^2\frac{i}{2}\sin^4\frac{i}{2} \times \left(2 + \frac{4}{3\epsilon^2} - e^{-2j\omega}\right) - \sin^6\frac{i}{2}e^{-2j\omega}\right\}$$

$$\bar{F}_{44} = \frac{315(1-\epsilon^2)^{3/2}e^{4\Omega' j}}{4}\sin^2 i\left\{\epsilon^2\cos^4\frac{i}{2}e^{2\omega j} + \epsilon^2\sin^4\frac{i}{2}e^{-2\omega j} + \frac{1}{2}\left(1 + \frac{3}{2}\epsilon^2\right)\sin^2 i\right\}$$

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