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MEAN SQUARE EXCEEDANCE CHARACTERISTICS OF A SINGLE TUNED SYSTEM TO AMPLITUDE MODULATED RANDOM NOISE

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16. ABSTRACT

This report concerns exceedance characteristics in the mean square response of a single-tuned system to amplitude modulated noise of limited time duration. The input model is considered representative of gusts, earthquakes, initial ignition of boosters and automotive vehicle impacts; the results thus bear directly on the design/analysis/testing of structural systems subjected to such nonstationary random environments.

A general formulation is reviewed for an envelope modulation of arbitrary form and noise of arbitrary correlation. Three modulation functions and noise with a damped cosine correlation are treated. For correlated noise, the time-varying mean square response may exceed the stationary value by a factor in excess of two; such exceedances do not occur for white noise. This behavior is summarized by parametric plots and explained in terms of a time-varying shaping filter.

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NOMENCLATURE

Since the symbols are defined as an integral part of the text, only a partial listing is provided here. Those included are considered to be of greatest assistance to the reader in interpreting the content of this report.

a	=	damped natural frequency of the system
b	=	exponential decay coefficient in the system response to a unit impulse
c_o	=	stationary response coefficient (see Eqn. 4.23)
$e(t)$	=	envelope modulation function
$f(t)$	=	input force function
f_n	=	system natural frequency in hertz
$h(t)$	=	system response to a unit impulse (see Eqn. 4.5)
$H(\omega)$	=	frequency response function of the system
$H_o(\omega)$	=	$m\omega_n^2 H(\omega)$
$K(t, \omega)$	=	shaping filter for the unit step modulation (see Eqn. 4.14)
$K_s(t, \omega)$	=	residual shaping filter for the rectangular step modulation (see Eqn. 4.28)
$K_\gamma(t, \omega)$	=	shaping filter for the damped exponential modulation (see Eqn. 4.36)
m	=	mass of the system
$n(t)$	=	input random noise
$Q = 1/2\zeta$	=	measure of system damping
$R_n(\tau)$	=	autocorrelation function of the input noise (see Eqn. 3.4)
R_o	=	constant in the autocorrelation function for correlated noise

$S_n(\omega)$ = two-sided power spectral density function of the input noise (see Eqn. 4.11)

$S_o(\omega)$ = $\left(\frac{a}{R_o}\right) S_n(\omega)$

t_o = time duration of input pulse

$u(t)$ = unit step function

α = exponential decay coefficient of the autocorrelation function for the correlated noise input (see Eqn. 3.4)

β_o = threshold ratio for probabilistic spectra

γ = decay coefficient for the exponential decay modulation (see Eqn. 3.3)

ζ = damping factor of the system

ρ = frequency of the autocorrelation function for the correlated noise input (see Eqn. 3.4)

σ_y = stationary root-mean-square response

$\sigma_y(t)$ = nonstationary root-mean-square response

$\sigma_{\rho k}$ = root-mean-square response maximum

ω_n = $2\pi f_n$ = system natural frequency in radians per second

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1.0 INTRODUCTION

An ability to assess the response characteristics of physical systems in random force fields is fundamental to proper design of structural systems in random environments. Such an assessment is mandatory for problems of structural fatigue and/or catastrophic failure, and frequently is founded upon the results of analyses which deal with much simplified structural models. We consider here an analytical study, based largely upon previous works of the authors, which bears directly upon catastrophic failure predictions in random environments considered representative of earthquakes, gusts and pyrotechnic shocks. The study finds application to problems of structural and/or component testing with shaped random excitation of limited time duration; it relates peripherally to problems of structural fatigue.

2.0 BACKGROUND INFORMATION

Previous studies [1] have produced results identified as probabilistic shock spectra. Such results, obtained initially by analog simulation studies [2] and later verified by digital simulation, enable one to predict (as a function of probability) the maximum response of a single-tuned mechanical system to amplitude modulated Gaussian white noise of zero mean. The modulation functions used are well defined envelope expressions and of adjustable time duration so that both stationary and nonstationary properties can be examined. In the literature, studies which relate to this class of problems frequently are categorized under subject headings associated with the single highest peak (SHP) problem, extremal statistics, and the first passage problem [5].

As depicted by Figures 2.1 through 2.4, probabilistic spectra are families of curves in $\hat{P}_M(|\beta| \leq \beta_o; \Delta t)$ of the normalized response $|\beta|$ versus the dimensionless parameter $f_n t_o/Q$. The quantity $\hat{P}_M(|\beta| \leq \beta_o; \Delta t)$ should be interpreted as an estimate of the probability that the maximum value of $|\beta|$, for the input time interval Δt, is less than or equal to the threshold level β_o. The dimensionless quantity β is a ratio of the SHP response of the system to the stationary rms response for an input of white noise. The term f_n is the system natural frequency, the quality factor $Q = 1/2\zeta$ and provides a measure of the system damping, and t_o is a measure of the time duration of the input noise. With Δt as the actual time duration of the modulation function, $t_o = \Delta t$ for the rectangular step, $t_o = 2\Delta t$ for the half-sine, and $t_o = 3\Delta t$ for the ramp functions.

The modulation functions used refer to an input excitation of the form

$$f(t) = e(t)\, n(t) \qquad (2.1)$$

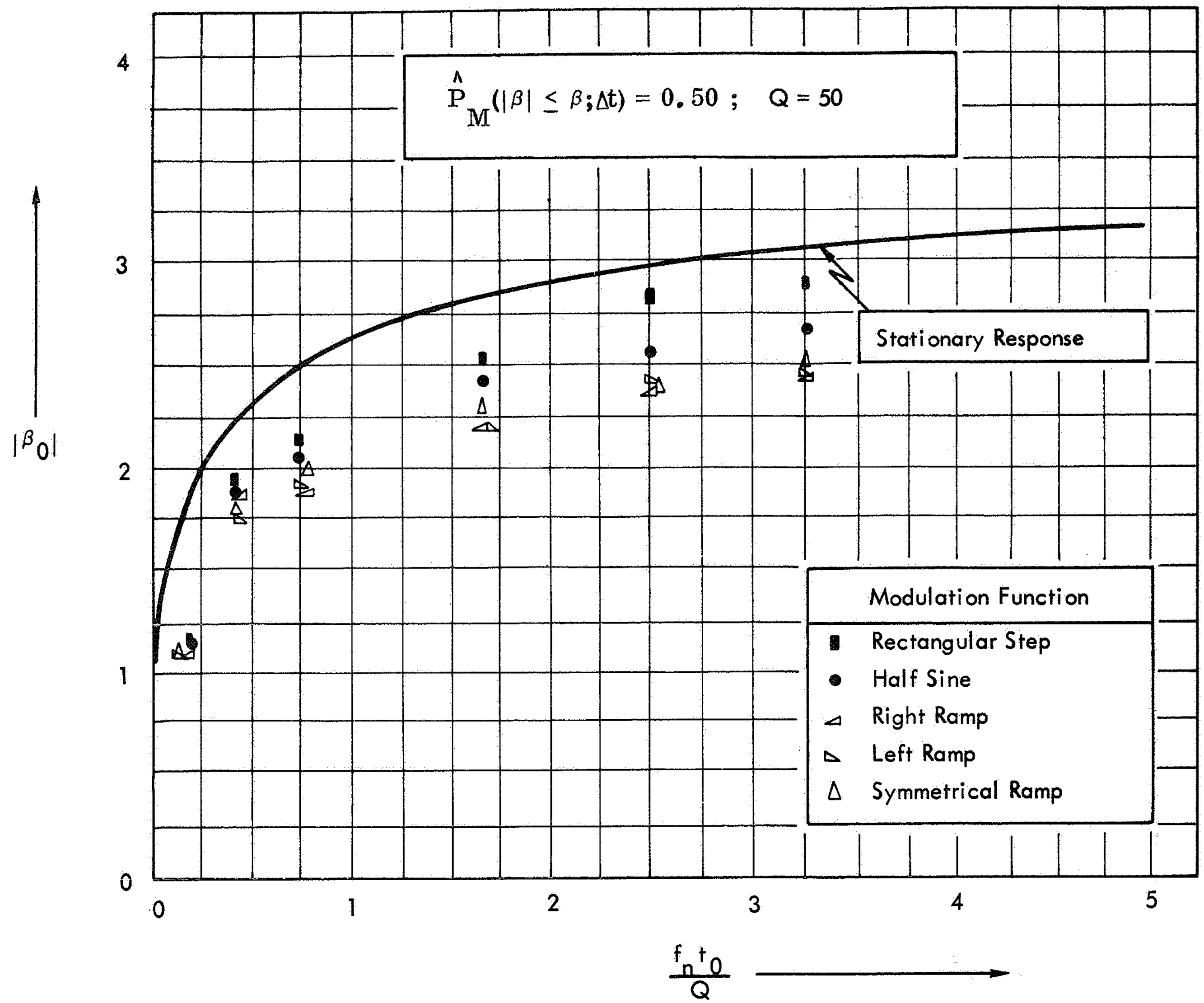

Figure 2. 1. Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise, $\hat{P}_M(|\beta| \leq \beta_o; \Delta t) = 0.50$, $Q = 50$

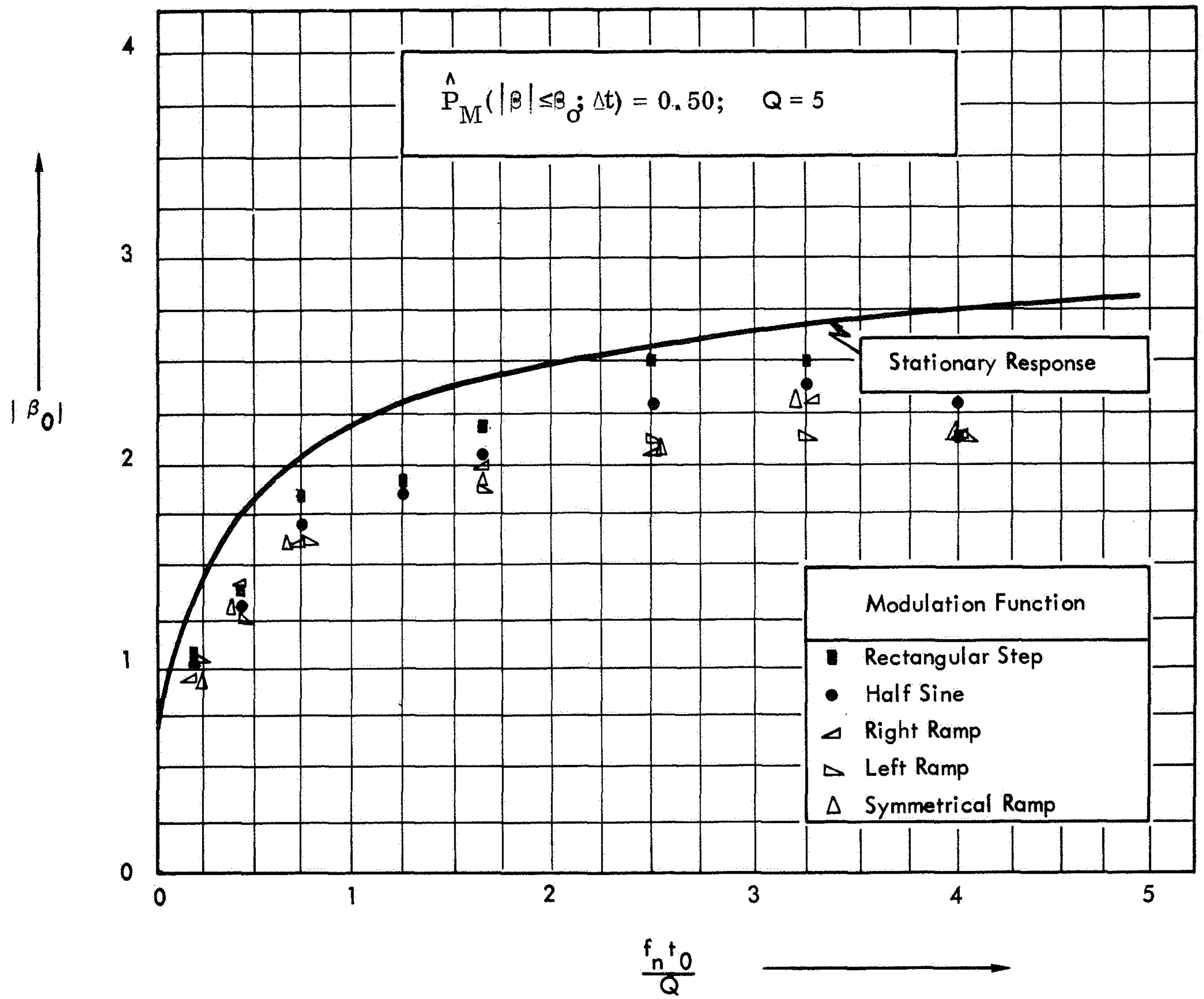

Figure 2.2. Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise, $\hat{P}_M(|\beta| \leq \beta_o; \Delta t) = 0.50$, $Q = 5$

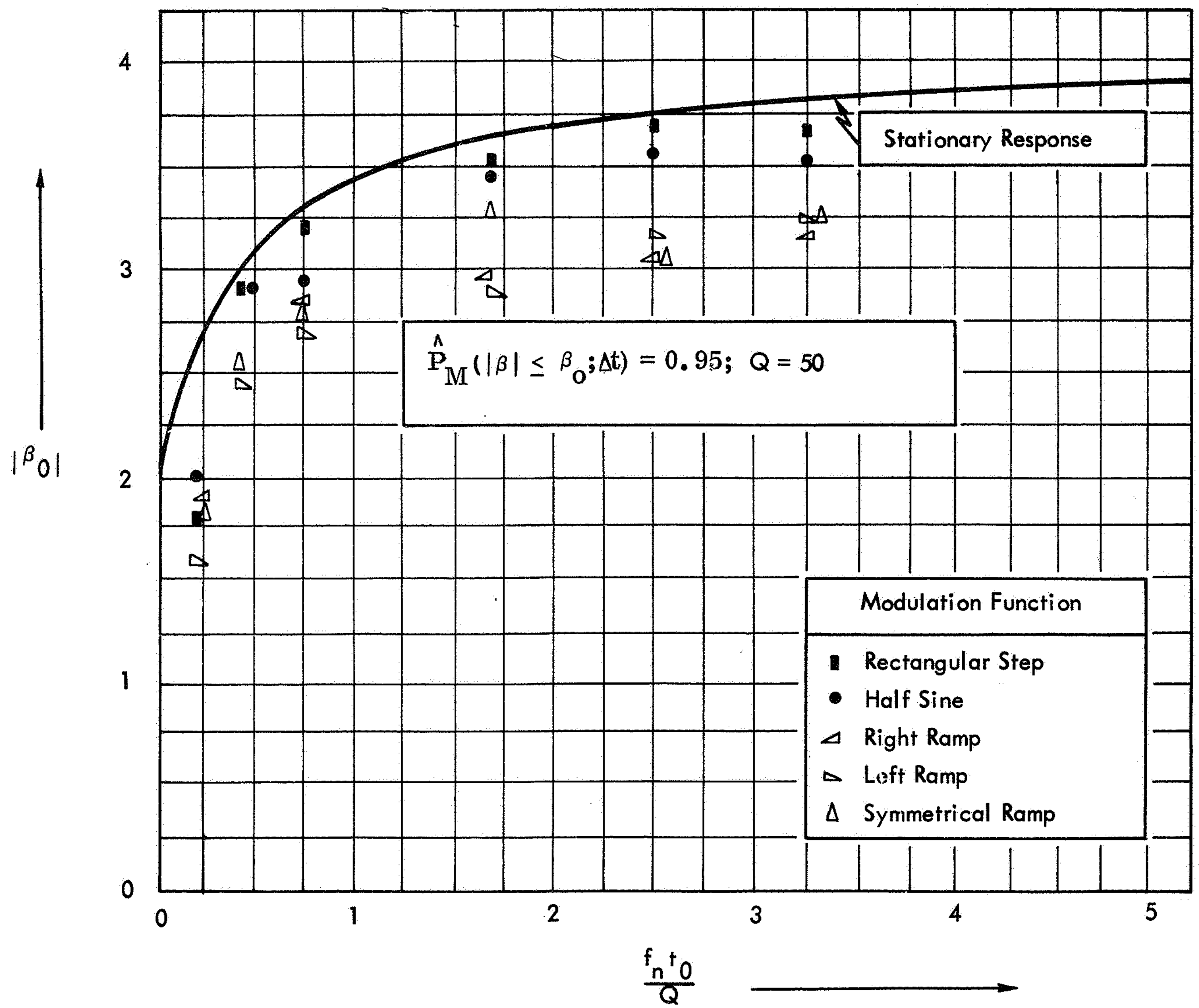

Figure 2.3. Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise, $\hat{P}_M(|\beta| \leq \beta_o;\ \Delta t) = 0.95,\ Q = 50$

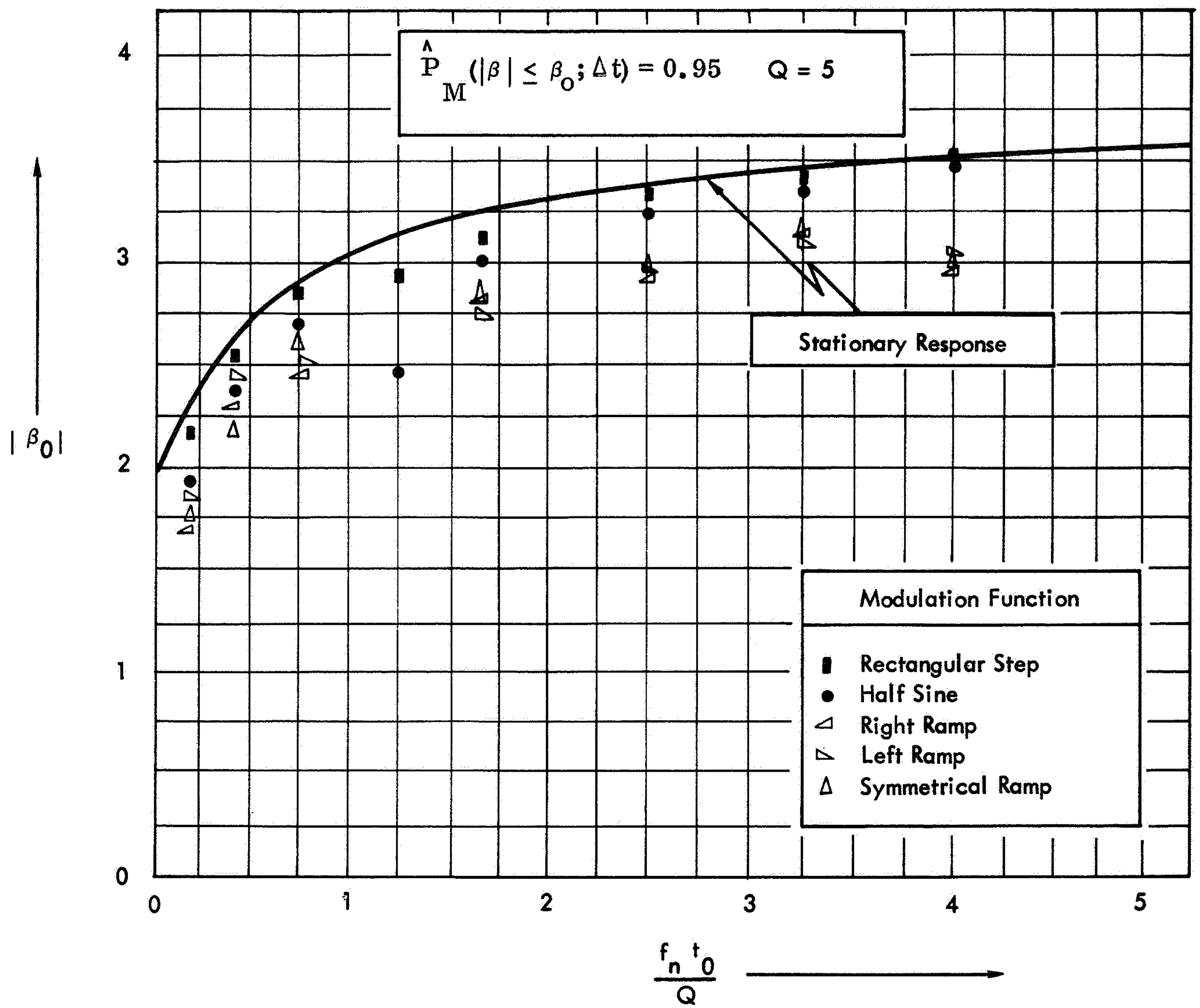

Figure 2.4. Response Maxima of a Single Degree-of-Freedom System to a Burst of Amplitude Modulated White Noise, $\hat{P}_M(|\beta| \leq \beta_o; \Delta t) = 0.95$, Q = 5

where e(t) is the shape of the modulation function and n(t) is random noise; for the shown results, n(t) is assumed broadband white noise. The stationary response curves are probabilistic spectra for a single tuned system which has achieved stationarity in its response (with e(t) assumed unity). Under such conditions, the time parameter t_o should be interpreted as the time interval T over which the system response is sampled. The system mean square response then is given by the familiar expression

$$\sigma_o^2 = \frac{\pi Q S_o}{m^2 \omega_n^3} \qquad (2.2)$$

where m is the mass of the system, ω_n is the system undamped natural frequency and S_o is the spectral magnitude of the input noise. Further, for larger numbers of response cycles (say $f_n T/Q > 1000$), the noise burst response values tend to the stationary results shown as Figures 2.5 and 2.6.

The preceding data suggest the stationary results provide somewhat conservative estimates of response maxima for shaped noise bursts of limited time duration. Fundamental to such results, however, is the normalization (both in β and in β_o) by the stationary rms response of the system. Indeed, the shown response maxima are conservative provided the time-varying rms response to a noise burst input does not exceed its corresponding stationary value. Since the rms response is influenced collectively by properties of the physical system, the modulation function and the input noise, it is not an easy matter to make categorical statements concerning such exceedance* characteristics except for limited conditions [4]. Mean square exceedance characteristics (and related topics) thus have been the subject of several recent investigations [3, 6, 7] and, in fact, are a major concern of this study. For completeness therefore,

*the word exceedance and overshoot are used interchangeably.

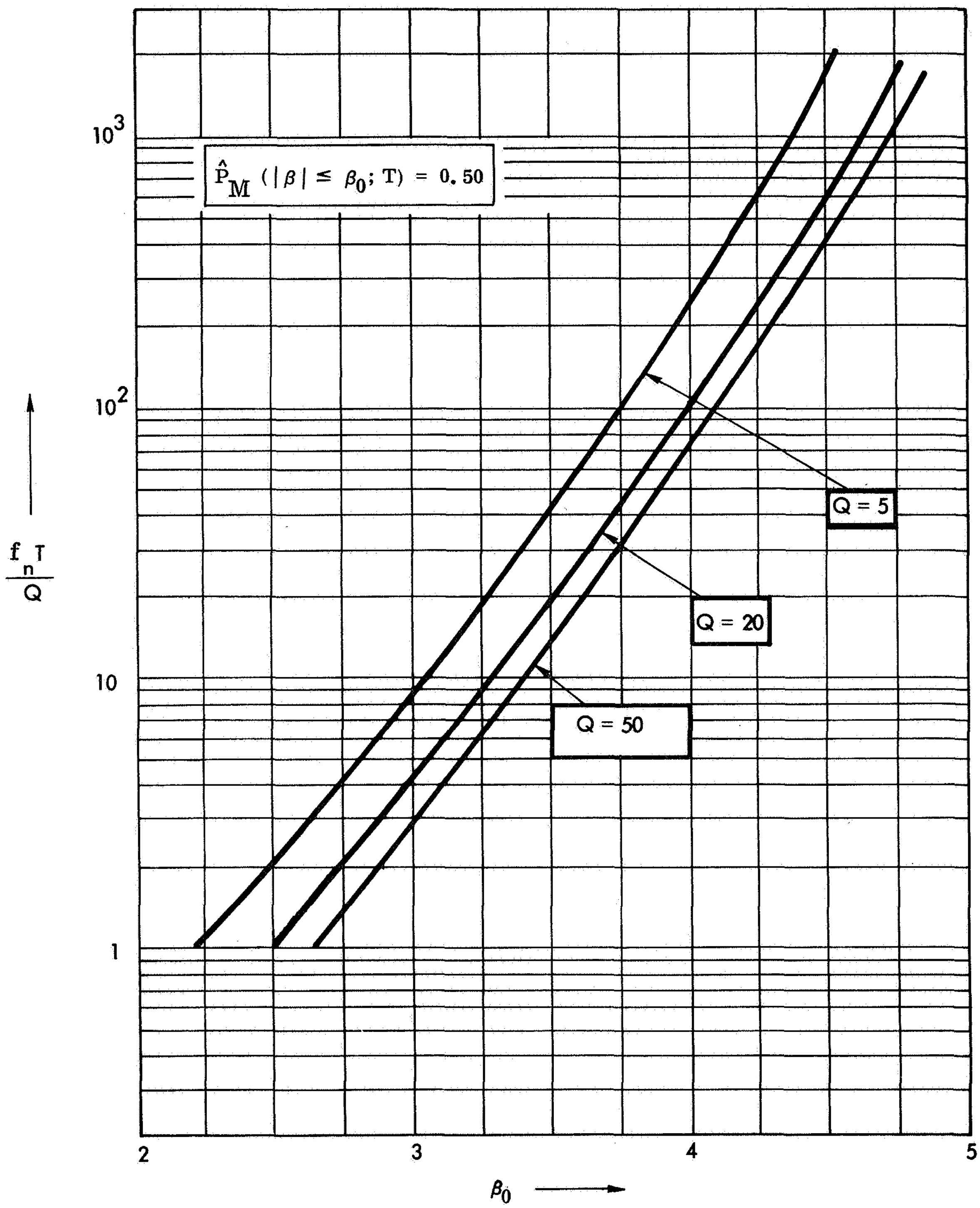

Figure 2.5. Response Maxima of a Single Degree-of-Freedom System to Stationary White Noise, $\hat{P}_M$ ($|\beta| \leq \beta_0$; T) = 0.50

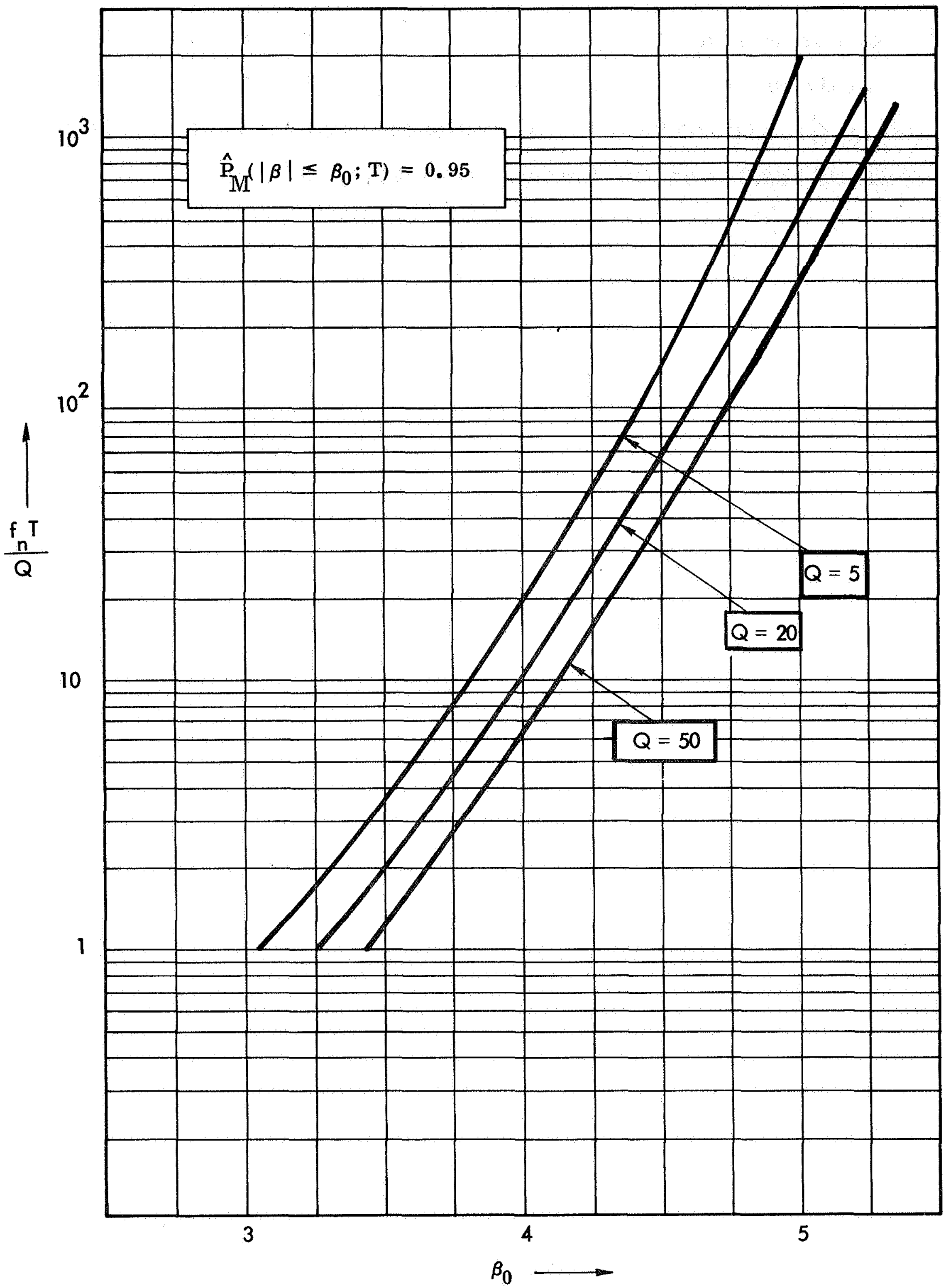

Figure 2.6. Response Maxima of a Single Degree-of-Freedom System to Stationary White Noise, $\hat{P}_M(|\beta| \leq \beta_0; T) = 0.95$

let us examine the time-varying mean square exceedance problem in abbreviated detail even though we may repeat material documented elsewhere. In what follows, we examine formulations basic to our solution and quote the results. The attendant mathematical detail is omitted for conciseness in presentation.

3.0 PROBLEM DEFINITION

Simply, the problem is to determine the variance of the variable y(t) given the equation of motion

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \frac{1}{m} f(t) \tag{3.1}$$

where the input excitation is the previously defined modulated noise expression

$$f(t) = e(t)\, n(t) \tag{3.2}$$

with n(t) assumed to have a mean value of zero. We consider the three modulation functions

$$\begin{aligned} &\bullet \quad e(t) = u(t) \\ &\bullet \quad e(t) = u(t) - u(t - t_o) \\ &\bullet \quad e(t) = u(t)\, e^{-\gamma t} \end{aligned} \tag{3.3}$$

and the two noise correlation functions

$$\begin{aligned} &\bullet \quad R_n(\tau) = 2\pi R_o \delta(\tau) \\ &\bullet \quad R_n(\tau) = R_o e^{-\alpha|\tau|} \cos \rho\tau \end{aligned} \tag{3.4}$$

where the notation u(t) represents the unit step function and τ defines the time difference $t_2 - t_1$. The delta function identifies white noise and the exponentially decaying harmonic expression defines the correlated noise.

4.0 SOLUTION FORMULATIONS

The basic procedures commonly used to determine the expectation $E[y^2(t)]$ for a modulated noise input are founded upon either unit impulse formulations and/or a spectral formulations; we concentrate here upon the latter. Since the detailed mathematics are outlined elsewhere [3], we quote only the expressions essential to our solution.

The desired expectation may be written as

$$\sigma_y^2(t) \equiv E[y^2(t)] = \int_{-\infty}^{\infty} S_n(\omega) |I(t,\omega)|^2 d\omega \tag{4.1}$$

where

$$I(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\bar{\omega}) F_e(\bar{\omega} - \omega) e^{i\bar{\omega}t} d\bar{\omega} \tag{4.2}$$

with

$$H(\omega) = \frac{1}{m\omega_n^2} \cdot \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i2\zeta\frac{\omega}{\omega_n}}$$

(4.3)

$$F_e(\bar{\omega} - \omega) = \int_{-\infty}^{\infty} e(t) e^{-i(\bar{\omega} - \omega)t} dt$$

It is worthwhile to recall that

$$h(t) \longleftrightarrow H(\omega) \qquad (4.4)$$

where the unit impulse response of the system is

$$h(t) = \frac{1}{am} e^{-bt} \sin at \qquad (4.5)$$

and

$$a = \omega_n [1 - \zeta^2]^{1/2}$$

$$b = \zeta \omega_n \qquad (4.6)$$

The quantity $H(\omega)$ is a system frequency response function, $F_e(\bar{\omega} - \omega)$ defines a transformation associated with the modulation function and $S_n(\omega)$ is the (two-sided) ordinary spectral density function* of the input noise. It should be remembered that central to the shown formulations are the assumptions of system linearity as well as separability (in product form) of the input excitation.

For the modulation functions quoted earlier, Eqn (4.1) is expressable in the form

$$\sigma_y^2(t) = \int_{-\infty}^{\infty} |H(\omega)|^2 S_n(\omega) K(t, \omega) d\omega \qquad (4.7)$$

---

*or, simply, power spectral density

where

$$|I(t,\omega)|^2 = |H(\omega)|^2 K(t,\omega) \tag{4.8}$$

The quantity $K(t, \omega)$ acts as a "shaping" filter on the stationary formulations; specifically, it provides a time-varying spectral description of the interaction between the envelope of the input excitation and the structural system. For assumptions of stationarity in the response, $K(t, \omega)$ reduces to a constant (unity) so that $\sigma_y^2(t) \longrightarrow \sigma_y^2$ and we have the expression

$$\sigma_y^2 = \int_{-\infty}^{\infty} |H(\omega)|^2 S_n(\omega)\, d\omega \tag{4.9}$$

The spectral density of the correlated noise is obtained by the Fourier transformation

$$R_n(\tau) \longleftrightarrow S_n(\omega) \tag{4.10}$$

so that

$$S_n(\omega) = \frac{R_o}{\pi} \left(\frac{\alpha(\alpha^2 + \rho^2 + \omega^2)}{(\omega^2 - s_3^2)(\omega^2 - s_4^2)} \right) \tag{4.11}$$

where

$$s_3 = \rho + i\alpha = - s_4^* \tag{4.12}$$

For white noise of spectral magnitude S_o,

$$S_o = \lim_{\alpha \to \infty} \alpha S_n(\omega) = \frac{R_o}{\pi} \qquad (4.13)$$

Normalized plots* of the system frequency response function and ordinary spectra of the correlated noise are shown as Figures 4,1 and 4.2.

4.1 UNIT STEP MODULATION

For the unit step modulation function,

$$K(t,\omega) = 1 + A(t) + B(t)\left(\frac{b^2 - a^2 + \omega^2}{a^2}\right) - 2C(t)\cos\omega t - \frac{2\omega}{a}\left(D(t)\sin\omega t\right) \qquad (4.14)$$

with the time-varying coefficients

$$\begin{aligned} A(t) &= e^{-2bt}\left(1 + \frac{b}{a}\sin 2at\right) \\ B(t) &= e^{-2bt}\left(\sin^2 at\right) \\ C(t) &= e^{-bt}\left(\cos at + \frac{b}{a}\sin at\right) \\ D(t) &= e^{-bt}\left(\sin at\right) \end{aligned} \qquad (4.15)$$

where the quantities a and b are those defined earlier.

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*Note; $H_o(\omega) = m\omega_n^2 H(\omega)$ and $S_o(\omega) = \left(\frac{a}{R_o}\right) S_n(\omega)$

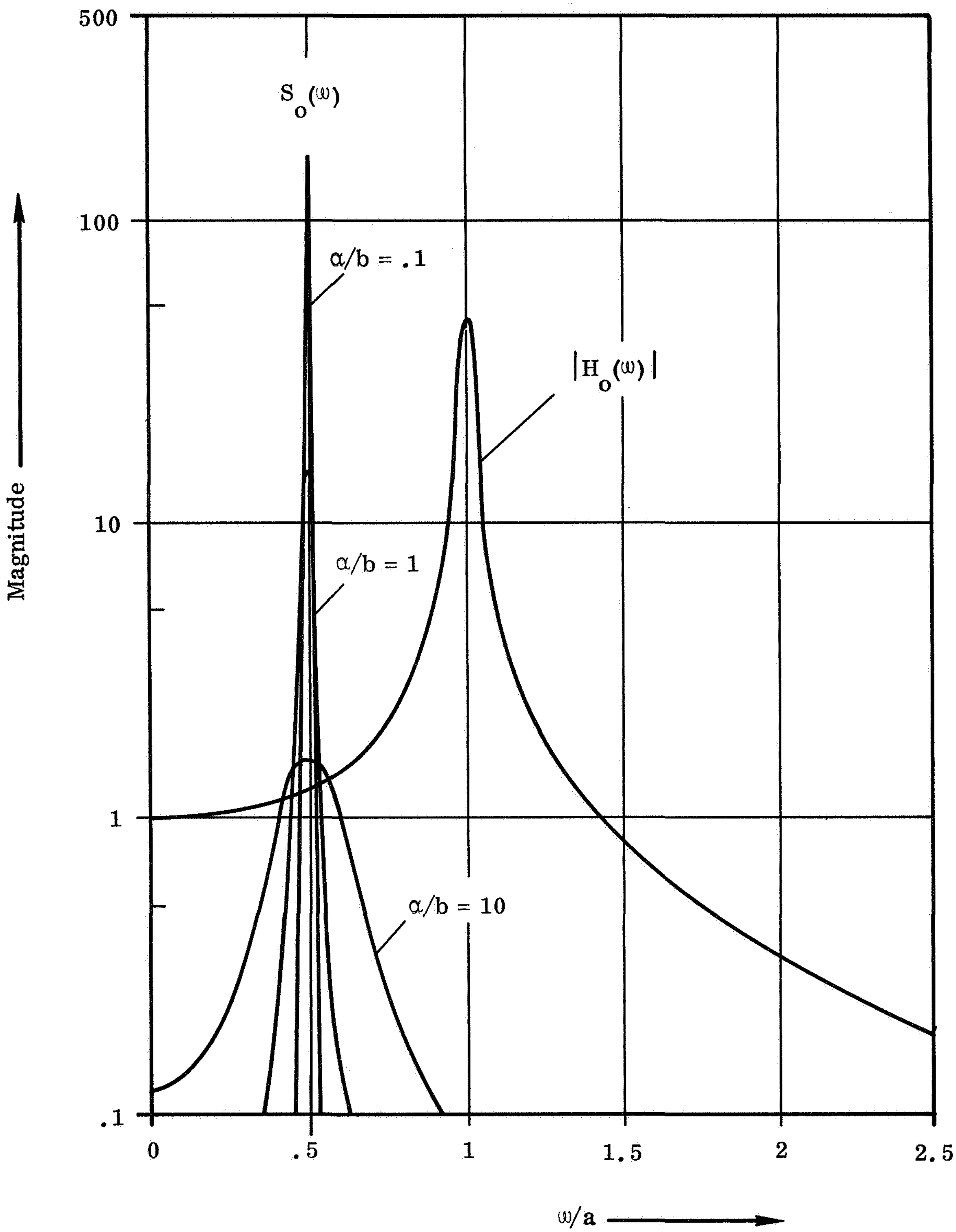

Figure 4.1. Normalized Spectral Plots of $|H_o(\omega)|$ for Q = 50 and of $S_o(\omega)$ for $\rho/a = .5$

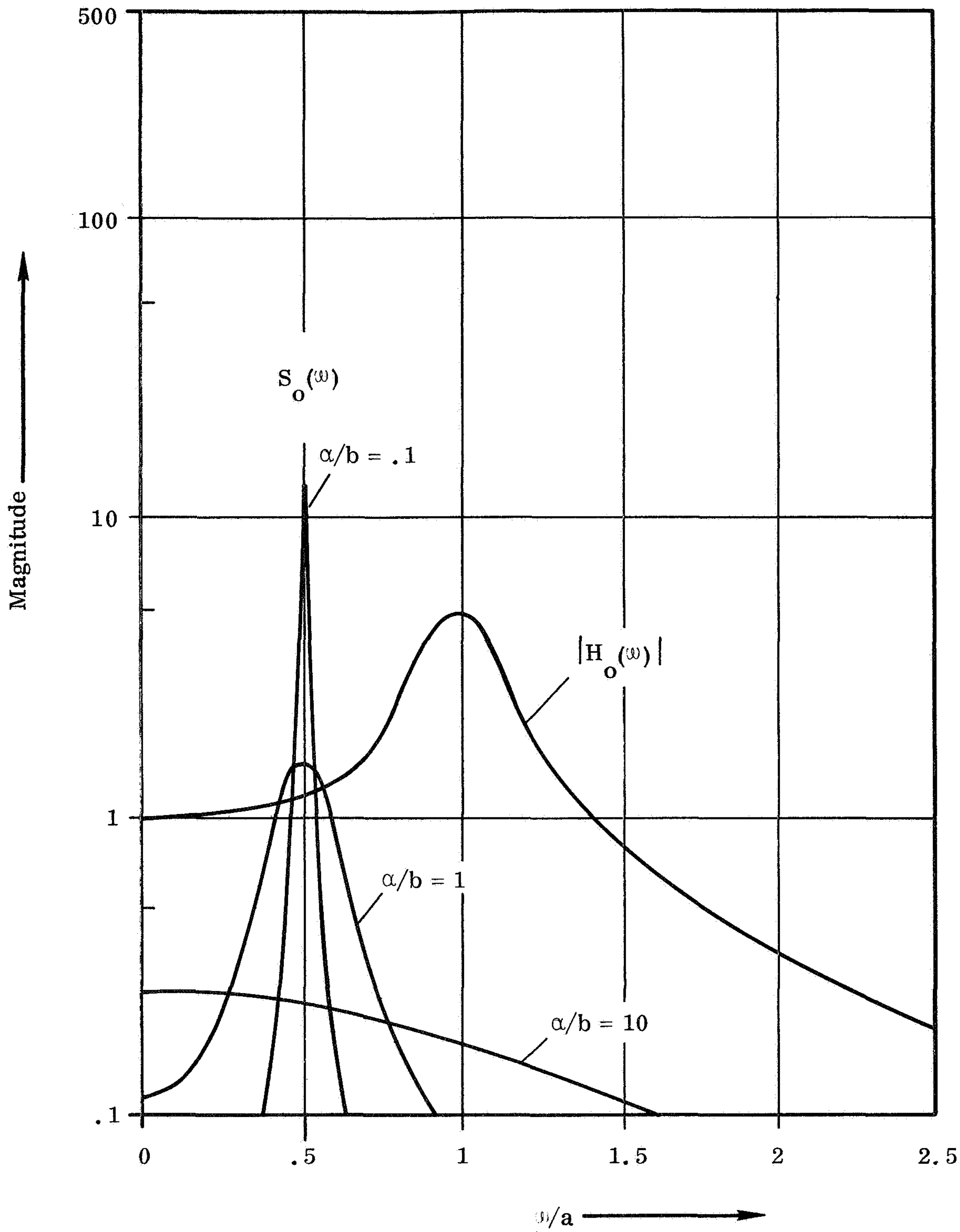

Figure 4.2. Normalized Spectral Plots of $|H_o(\omega)|$ for Q = 5 and of $S_o(\omega)$ for $\beta/a = .5$

When $S_n(\omega) \longrightarrow S_o$,

$$\sigma_y^2(t) = \frac{\pi S_o}{2\zeta m^2 \omega_n^3} \left\{ 1 - e^{-2bt} \left(1 + \frac{b}{a} \sin 2at + 2 \left(\frac{b}{a}\right)^2 \sin^2 at \right) \right\} \qquad (4.16)$$

while for the correlated noise,

$$\sigma_y^2(t) = \frac{R_o}{m^2} \left\{ R_1 T_1 - X_1 T_2 + R_3 T_3 - X_3 T_4 \right\} \qquad (4.17)$$

where we define

$$\begin{aligned} R_1 &= \mathrm{Re}\,(Z_1) \\ X_1 &= \mathrm{Imag}\,(Z_1) \\ R_3 &= \mathrm{Re}\,(Z_3) \\ X_3 &= \mathrm{Imag}\,(Z_3) \end{aligned} \qquad (4.18)$$

with

$$\begin{aligned} Z_1 &= \frac{\alpha}{a^2} \left(\frac{\rho^2 + \alpha^2 + s_1^2}{s_1 \left(s_1^2 - s_3^2\right)\left(s_1^2 - s_4^2\right)} \right) \\ Z_3 &= \frac{1}{\left(s_3^2 - s_1^2\right)\left(s_3^2 - s_2^2\right)} \end{aligned} \qquad (4.19)$$

and

$$s_1 = a + ib = -s_2^* \tag{4.20}$$

while s_3 and s_4 are defined by Eqn (4.12). The remaining terms, all time-varying coefficients, are given by

$$T_1 = \frac{a}{2b}\left[1 - A(t)\right]$$

$$T_2 = - B(t)$$

$$T_3 = \left[1 + A(t) + \left(\frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2}\right) B(t) - 2\left(C(t) + \frac{\alpha}{a} D(t)\right) e^{-\alpha t} \cos \rho t - 2\frac{\rho}{a} D(t)\, e^{-\alpha t} \sin \rho t\right] \tag{4.21}$$

$$T_4 = \left[2\left(\frac{\rho\alpha}{a^2}\right) B(t) - 2\left(C(t) + \frac{\alpha}{a} D(t)\right) e^{-\alpha t} \sin \rho t + 2\frac{\rho}{a} D(t)\, e^{-\alpha t} \cos \rho t\right]$$

where A(t), B(t), C(t) and D(t) are defined previously.

The stationary mean square response may be obtained either from an evaluation of $\sigma_y^2(t)$ as $t \rightarrow \infty$ or by the direct integration of Eqn (4.9). For white noise, we have the result quoted as Eqn (2.2). For correlated noise, the stationary response may be written as

$$\sigma_y^2 = \frac{R_o}{m^2 a^4} c_o^2 \tag{4.22}$$

where

$$c_o = \left[\left(\frac{a}{2b}\right) \overline{R}_1 + \overline{R}_3\right]^{1/2} \tag{4.23}$$

with the normalized values

$$\begin{aligned} \overline{R}_1 &= a^4 R_1 \\ \overline{R}_3 &= a^4 R_3 \end{aligned} \tag{4.24}$$

Note when $s_1 = s_3$, the solution offered by Eqn (4.17) becomes unbounded due to the second-order poles which occur in contrast to the four first-order poles when $a \neq \rho$ and $b \neq \alpha$. Under such circumstances, the coefficients of the system response to a unit impulse force are identical with those of the auto-correlation function of the correlated noise. Subsequently, the time varying mean square response becomes

$$\sigma_y^2(t) = \frac{R_o}{8m^2\omega_n^4}\left\{\left(\frac{a^2}{b^2} + 3\right)\right.$$

$$-e^{-2bt}\left(\left(\frac{a^2 + 3b^2 + 2\omega_n^2 bt}{b^2}\right) + \frac{2}{a}\left(b + \omega_n^2 t\right)\sin 2at\right. \qquad (4.25)$$

$$\left.\left. - \frac{2}{a^2}\left(a^2 - b^2 - 2bt\,\omega_n^2\right)\sin^2 at\right)\right\}$$

and has the stationary value

$$\sigma_y^2 = \frac{R_o}{8m^2\omega_n^4}\left(3 + \frac{a^2}{b^2}\right) \qquad (4.26)$$

4.2 RECTANGULAR STEP MODULATION

For the rectangular step modulation,

$$\sigma_y^2(t) = \int_{-\infty}^{\infty} |H(\omega)|^2 S_n(\omega)\, K(t,\omega)\, d\omega, \qquad \text{for } 0 \le t \le t_o$$

(4.27)

$$\sigma_y^2(t) = \int_{-\infty}^{\infty} |H(\omega)|^2 S_n(\omega)\, K_s(t,\omega)\, d\omega, \qquad \text{for } t \ge t_o$$

where $K(t, \omega)$ is that defined as Eqn (4.14) and

$$K_s(t, \omega) = E + \left(\frac{b^2 - a^2 + \omega^2}{a^2}\right) F - 2 \left(G + \left(\frac{\omega}{a}\right)^2 J\right) \cos \omega t_o$$
$$- 2 \left(\frac{\omega}{a}\right) K \sin \omega t_o \tag{4.28}$$

with

$$E = A(t) + A(t - t_o)$$

$$F = B(t) + B(t - t_o)$$

$$G = C(t)\ C(t - t_o) \tag{4.29}$$

$$J = D(t)\ D(t - t_o)$$

$$K = C(t - t_o)\ D(t) - C(t)\ D(t - t_o)$$

Since the integral given by Eqn (4.27) for $0 \le t \le t_o$ is precisely that of the unit step modulation, we use directly the work in the previous section.

For white noise over the range $0 \le t \le t_o$, $\sigma_y^2(t)$ is given by Eqn (4.16) while for $t \ge t_o$,

$$\sigma_y^2(t) = \frac{\pi S_o}{2\zeta m^2 \omega_n^3} \left\{ E + 2\left(\frac{b}{a}\right)^2 F \right.$$

$$\left. -2C(t_o)\left(G + \left(\frac{\omega_n}{a}\right)^2 J\right) + 2\left(\frac{\omega_n}{a}\right)^2 D(t_o)\left(2J - K\right) \right\} \tag{4.30}$$

which reduces to

$$\sigma_y^2(t) = \frac{\pi S_o}{2\zeta m^2 \omega_n^3} \left\{ \left(A(t - t_o) - A(t)\right) + \frac{2b^2}{a^2}\left(B(t - t_o) - B(t)\right) \right\} \tag{4.31}$$

Now for correlated noise over the range $0 \le t \le t_o$, $\sigma_y^2(t)$ is given by Eqn (4.17) while for $t \ge t_o$, we find after some mathematics that

$$\sigma_y^2(t) = \frac{R_o}{m^2} \left\{ R_1 T_a - X_1 T_b + R_3 T_c - X_3 T_d \right\} \tag{4.32}$$

where

$$T_a = \frac{a}{2b}\left[E - 2\left(G + \frac{b}{a} K + \frac{a^2 - b^2}{a^2} J\right) e^{-bt_o} \cos at_o \right.$$

$$\left. - 2\left(K - \frac{2b}{a} J\right) e^{-bt_o} \sin at_o \right] \tag{4.33}$$

$$T_b = \frac{a}{b}\left[\frac{b}{a} F - \left(G + \frac{b}{a} K + \frac{a^2 - b^2}{a^2} J\right) e^{-bt_o} \sin at_o \right.$$

$$\left. + \left(K - \frac{2b}{a} J\right) e^{-bt_o} \cos at_o \right]$$

$$T_c = \left[E + \left(\frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2}\right) F - 2 \left(G + \frac{\alpha}{a} K + \frac{\rho^2 - \alpha^2}{a^2} J\right) e^{-\alpha t_o} \cos \rho t_o \right.$$

$$\left. - \frac{2\rho}{a}\left(K - \frac{2\alpha}{a} J\right) e^{-\alpha t_o} \sin \rho t_o \right] \qquad (4.33) \text{ cont.}$$

$$T_d = 2\left[\frac{\rho\alpha}{a^2} F - \left(G + \frac{\alpha}{a} K + \frac{\rho^2 - \alpha^2}{a^2} J\right) e^{-\alpha t_o} \sin \rho t_o \right.$$

$$\left. + \frac{\rho}{a}\left(K - \frac{2\alpha}{a} J\right) e^{-\alpha t_o} \cos \rho t_o \right]$$

When $s_1 = s_3$, $\sigma_y^2(t)$ over $0 \le t \le t_o$ is given by Eqn (4.25) while for $t \ge t_o$, $\sigma_y^2(t)$ is shown by the somewhat tedious expression which appears on the following page

$$\sigma_y^2(t) = \frac{R_o}{8m^2\omega_n^4}\left\{\left(\frac{a^2}{b^2}+3\right)E + 4\left(\frac{b}{a}\right)^2 F\right.$$

$$- \left[\left(2\left(\frac{a^2}{b^2}+3\right)G + \left(\frac{\omega_n}{a}\right)^4\left(2\frac{a^2}{b^2}+1\right)J\right)\cos at_o\right.$$

$$\left. + \left(\frac{4b}{a}G + 2\frac{\omega_n^4}{a^2b^2}K - \left(\frac{\omega_n}{a}\right)^4\frac{a}{b}J\right)\sin at_o\right]e^{-bt_o} \qquad (4.34)$$

$$- 2\frac{\omega_n^2}{b}t_o\left[\left(G + \left(\frac{\omega_n}{a}\right)^2 J\right)\cos at_o\right.$$

$$\left.\left. + \left(\left(\frac{\omega_n}{a}\right)^2 K - \frac{b}{a}G - \frac{b\omega_n^2}{a^3}J\right)\sin at_o\right]e^{-bt_o}\right\}$$

4.3 EXPONENTIAL DECAY MODULATION

In this case,

$$\sigma_y^2(t) = \int_{-\infty}^{\infty} |H_\gamma(\omega)|^2 S_n(\omega)\, K_\gamma(t,\omega)\, d\omega \qquad (4.35)$$

where $H_\gamma(\omega)$ and $K_\gamma(t,\omega)$ are obtained from $H(\omega)$ and $K(t,\omega)$ simply by replacing 'b' with '$(b - \gamma)$', and multiplying the resulting $K(t,\omega)$ by $e^{-2\gamma t}$.

Thus,

$$H_\gamma(\omega) = \frac{1}{m} \cdot \frac{1}{a^2 + (b - \gamma)^2 - \omega^2 + i2(b - \gamma)\omega}$$

$$K_\gamma(t, \omega) = e^{-2\gamma t}\left[1 + A_\gamma(t) + B_\gamma(t)\left(\frac{(b - \gamma)^2 - a^2 + \omega^2}{a^2}\right)\right. \qquad (4.36)$$

$$\left. - 2C_\gamma(t) \cos \omega t - \frac{2\omega}{a}\left(D_\gamma(t) \sin \omega t\right)\right]$$

where $A_\gamma(t)$, $B_\gamma(t)$, $C_\gamma(t)$ and $D_\gamma(t)$ are obtained from A(t), B(t), C(t) and D(t) by replacing, as before, b with $(b - \gamma)$.

Note the form of Eqn (4.35) is identical to the mean square formulation for the unit step modulation except for the exponential $e^{-2\gamma t}$; again, we make direct use of the work in Section 4.1. For white noise, where $b \neq \gamma$,

$$\sigma_y^2(t) = \frac{\pi S_o}{2m^2(b - \gamma)\left(a^2 + (b - \gamma)^2\right)} e^{-2\gamma t} \cdot$$

$$\left\{1 - e^{-2(b - \gamma)t}\left(1 + \left(\frac{b - \gamma}{a}\right) \sin 2at + \frac{2(b - \gamma)^2}{a^2} \sin^2 at\right)\right\} \qquad (4.37)$$

while for $b = \gamma$,

$$\sigma_y^2(t) = \frac{\pi S_o}{m^2 a^2} e^{-2bt} \left\{ t - \frac{1}{2a} \sin 2at \right\} \tag{4.38}$$

For correlated noise with either $\gamma < b$ or $\gamma > b$, the mean square response is given by

$$\sigma_y^2(t) = \frac{R_o}{m^2} e^{-2\gamma t} \left\{ R_{1\gamma} T_{1\gamma} - X_{1\gamma} T_{2\gamma} + R_{3\gamma} T_{3\gamma} - X_{3\gamma} T_{4\gamma} \right\} \tag{4.39}$$

where

$$\begin{aligned} R_{1\gamma} &= \text{Re}\,(Z_{1\gamma}) \\ X_{1\gamma} &= \text{Imag}\,(Z_{1\gamma}) \\ R_{3\gamma} &= \text{Re}\,(Z_{3\gamma}) \\ X_{3\gamma} &= \text{Imag}\,(Z_{3\gamma}) \end{aligned} \tag{4.40}$$

with

$$\begin{aligned} Z_{1\gamma} &= \frac{\alpha}{a^2} \left(\frac{\rho^2 + \alpha^2 + s_{1\gamma}^2}{s_{1\gamma} (s_{1\gamma}^2 - s_3^2)(s_{1\gamma}^2 - s_4^2)} \right) \\ Z_{3\gamma} &= \left(\frac{1}{(s_3^2 - s_{1\gamma}^2)(s_3^2 - s_{2\gamma}^2)} \right) \end{aligned} \tag{4.41}$$

and

$$s_{1\gamma} = a + i(b - \gamma)$$
$$s_{2\gamma} = -a + i(b - \gamma) \tag{4.42}$$

Accordingly,

$$T_{1\gamma} = \frac{a}{2(b-\gamma)}\left[1 - A_\gamma(t)\right]$$

$$T_{2\gamma} = -B_\gamma(t)$$

$$\begin{aligned} T_{3\gamma} = \Bigg[& 1 + A_\gamma(t) + \left(\frac{(b-\gamma)^2 - a^2 + \rho^2 - \alpha^2}{a^2}\right) B_\gamma(t) \\ & - 2\left(C_\gamma(t) + \frac{\alpha}{a} D_\gamma(t)\right) e^{-\alpha t} \cos \rho t \\ & - \frac{2\rho}{a} D_\gamma(t)\, e^{-\alpha t} \sin \rho t \Bigg] \end{aligned} \tag{4.43}$$

$$\begin{aligned} T_{4\gamma} = 2\Bigg[& \frac{\rho\alpha}{a^2} B_\gamma(t) - \left(C_\gamma(t) + \frac{\alpha}{a} D_\gamma(t)\right) e^{-\alpha t} \sin \rho t \\ & + \frac{\rho}{a} D_\gamma(t)\, e^{-\alpha t} \cos \rho t \Bigg] \end{aligned}$$

with

$$A_{\gamma}(t) = e^{-2(b-\gamma)t}\left(1 + \frac{b-\gamma}{a}\sin 2at\right)$$

$$B_{\gamma}(t) = e^{-2(b-\gamma)t}\sin^2 at$$

$$C_{\gamma}(t) = e^{-(b-\gamma)t}\left(\cos at + \frac{b-\gamma}{a}\sin at\right) \qquad (4.44)$$

$$D_{\gamma}(t) = e^{-(b-\gamma)t}\sin at$$

For the special case when $b = \gamma$ and $s_{1\gamma} \neq s_3$,

$$\sigma_y^2(t) = \frac{R_o}{m^2}e^{-2bt}\left\{R'_{1\gamma}T'_{1\gamma} - X'_{1\gamma}T'_{2\gamma} + R'_{3\gamma}T'_{3\gamma} + X'_{3\gamma}T'_{4\gamma}\right\} \qquad (4.45)$$

where

$$R'_{1\gamma} = \mathrm{Re}\,(Z'_{1\gamma})$$

$$X'_{1\gamma} = \mathrm{Imag}\,(Z'_{1\gamma})$$

$$R'_{3\gamma} = \mathrm{Re}\,(Z'_{3\gamma}) \qquad (4.46)$$

$$X'_{3\gamma} = \mathrm{Imag}\,(Z'_{3\gamma})$$

with

$$Z'_{1\gamma} = \frac{\alpha}{a^3}\left(\frac{\rho^2 + \alpha^2 + a^2}{(a^2 - s_3^2)(a^2 - s_4^2)}\right)$$

$$Z'_{3\gamma} = \left(\frac{1}{(s_3^2 - a^2)^2}\right) \tag{4.47}$$

and

$$T'_{1\gamma} = at - \frac{1}{2}\sin 2\,at$$

$$T'_{2\gamma} = -\sin^2 at$$

$$T'_{3\gamma} = 2\left[1 + \left(\frac{\rho^2 - a^2 - \alpha^2}{a^2}\right)\sin^2 at - (\cos at + \frac{\alpha}{a}\sin at)\, e^{-\alpha t}\cos \rho t - \frac{\rho}{a} e^{-\alpha t}\sin at \sin \rho t\right] \tag{4.48}$$

$$T'_{4\gamma} = 2\left[\frac{\rho\alpha}{a^2}\sin^2 at - \left(\cos at + \frac{\alpha}{a}\sin at\right) e^{-\alpha t}\sin \rho t + (\frac{\rho}{a}\sin at)\, e^{-\alpha t}\cos \rho t\right]$$

For the case when $s_{1\gamma} = s_3$ (with $b \neq \gamma$) so that $\alpha = b - \gamma$ and $\rho = a$,

$$\sigma_y^2(t) = \frac{R_o}{8m^2\omega_n^4} e^{-2\gamma t}\left\{\left(\left(\frac{a}{b-\gamma}\right)^2 + 3\right)\right.$$

$$- e^{-2(b-\gamma)t}\left(\left(\frac{a}{b-\gamma}\right)^2 + 3 + \frac{2\omega_n^2 t}{b-\gamma} + 2\left(\frac{b-\gamma+\omega_n^2 t}{a}\right)\sin 2at\right.$$

$$\left.\left. - 2\left(1 - \left(\frac{b-\gamma}{a}\right)^2 - \frac{2(b-\gamma)\omega_n^2 t}{a^2}\right)\sin^2 at\right)\right\} \tag{4.49}$$

which is similar to Eqn (4.25), as might be expected.

5.0 RESULTS

Due to the importance of the mean-square exceedance problem to predictions of response maxima, this study has two main objectives:

(1) to predict the onset of overshoot* over a practical range of both system and noise parameters

(2) to develop an explanation of why overshoot occurs.

Upon review, the significance of the analytical results is not always evident by a cursory inspection of the shown mathematical expressions. Such is particularly true for the correlated noise in spite of somewhat more than a moderate attempt at simplification. These expressions, however, are central to an explanation of the overshoot problem. To understand their importance, we decide upon select variations of system parameters, noise parameters and modulation envelopes, then construct what are considered meaningful parametric plots.

To partially satisfy the objectives, consider first the form of the expressions which governs the nonstationary mean square response of the system. From Eqn (4.7), we may write

$$\sigma_y^2(t) = \int_{-\infty}^{\infty} S_y(t, \omega)\, d\omega \qquad (5.1)$$

*We remember that overshoot and exceedance are used synonomously, and refer to the exceedance of $\sigma_y(t)$ over the stationary value σ_y. Thus, $\sigma_{\rho k}/\sigma_y > 1$ implies an overshoot (exceedance).

where the time-varying response spectral density $S_y(t, \omega)$ is

$$S_y(t, \omega) = |H(\omega)|^2 S_n(\omega) K(t, \omega) \tag{5.2}$$

with $K(t, \omega)$ termed a "shaping" filter for the unit step modulation. The latter is so-named for it acts to alter (in time) the spectral content of what otherwise is an integrand for stationary response. This function ultimately governs the time variation of $\sigma_y(t)$ and, subsequently, response overshoot. Due to the form associated with $\sigma_y^2(t)$ in Eqns (4.27) and (4.35), such remarks are equally pertinent to the other two modulation functions.

In this work, we restrict attention to a single-tuned system and to three modulation envelopes with emphasis upon the unit step and damped exponential modulations. To assess system behavior at both high and low values of damping, Q = 50 and Q = 5 are selected arbitrarily. From Eqn (4.17), the stationary response for correlated noise is governed by the relative values of the system constants 'a' and 'b', and the noise constants α and ρ. The parameter variations used range over the values $0.1 \le \rho/a \le 10$ and $0.1 \le \alpha/b \le 10$; such reflect a broad range of practical interest as $S_o(\omega)$ extends well to either side of the system center frequency. Figures 5.1 and 4.2 show clearly this behavior and moreover, allow a complete interpretation for stationary response.

For the unit step modulation, Figures 5.1 and 5.2 provide a measure of the magnitude of the system stationary response for $0.1 \le \rho/a \le 10$ and $0.1 \le \alpha/b \le 10$. The term c_o is defined by Eqns (4.23) and (4.24), and represents a normalized value of the stationary integral*. The largest values of c_o occur when $S_o(\omega)$ and $H_o(\omega)$ are tuned to resonance; that is $\omega = \rho = a$. Since the smaller values of α/b

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* $\int_{-\infty}^{\infty} |H_o(\omega)|^2 S_o(\omega)\, d\omega$

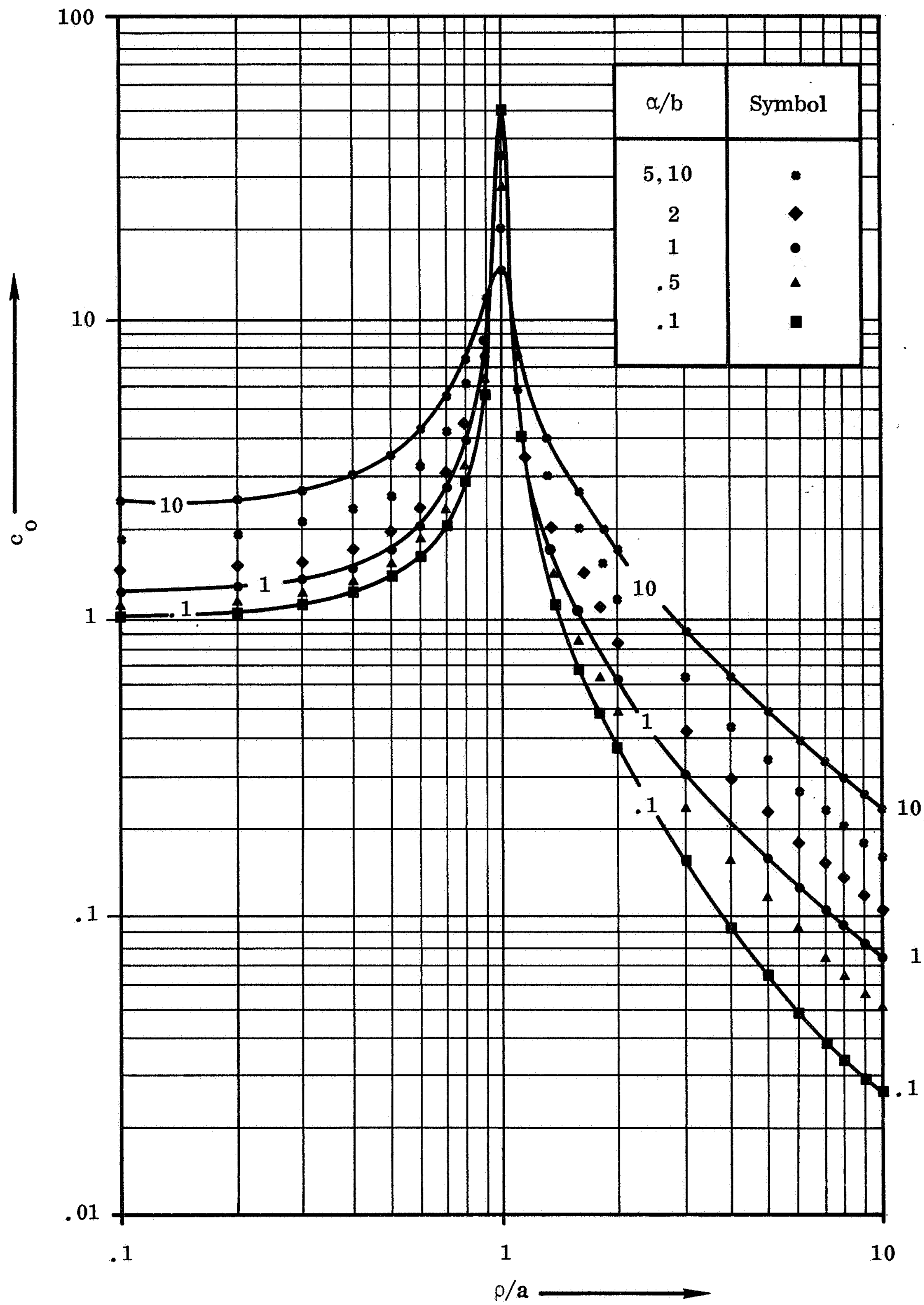

Figure 5.1. Normalized Stationary Values for Correlated Noise Inputs, Q = 50

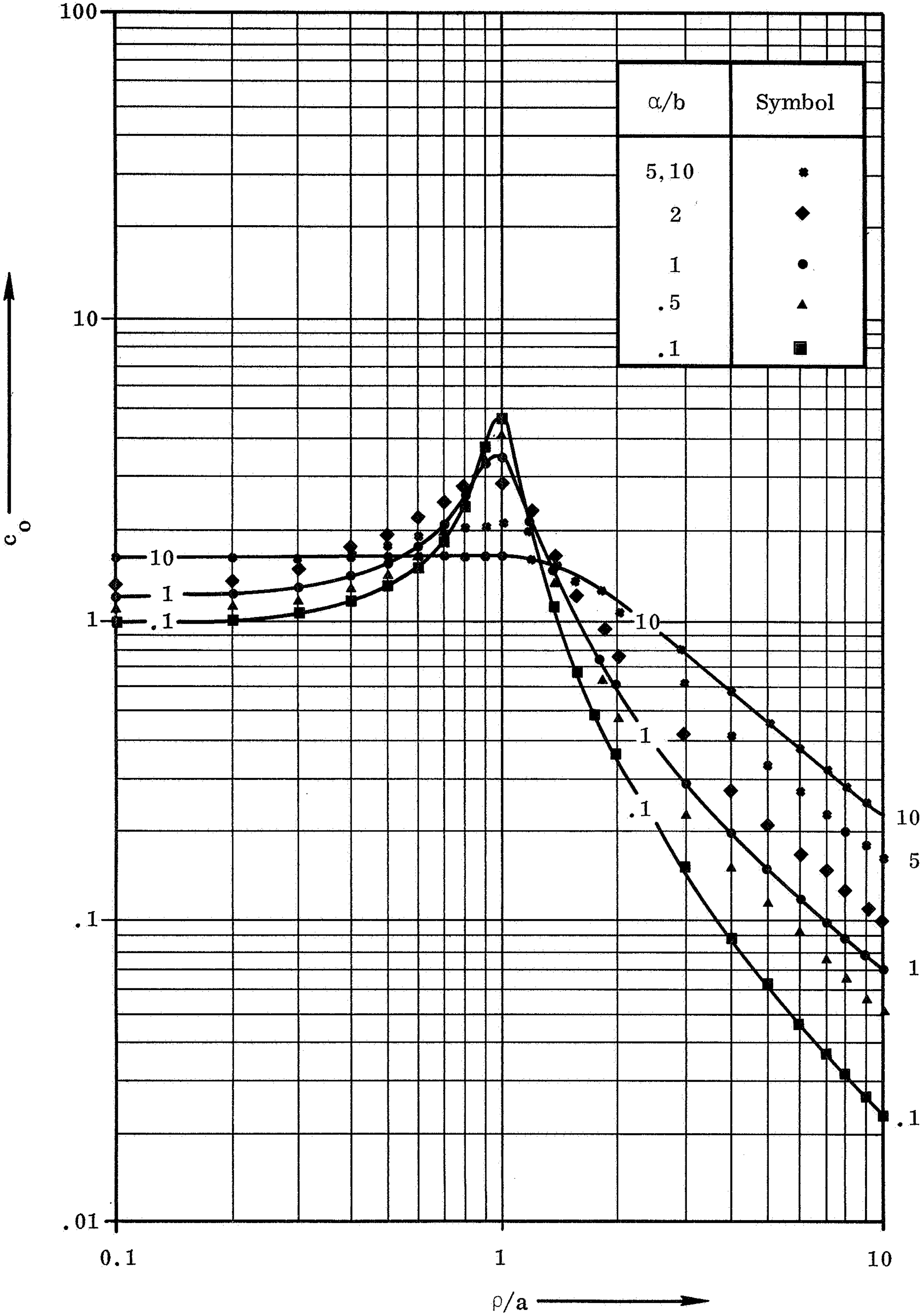

Figure 5.2. Normalized Stationary Values for Correlated Noise Inputs. Q = 5

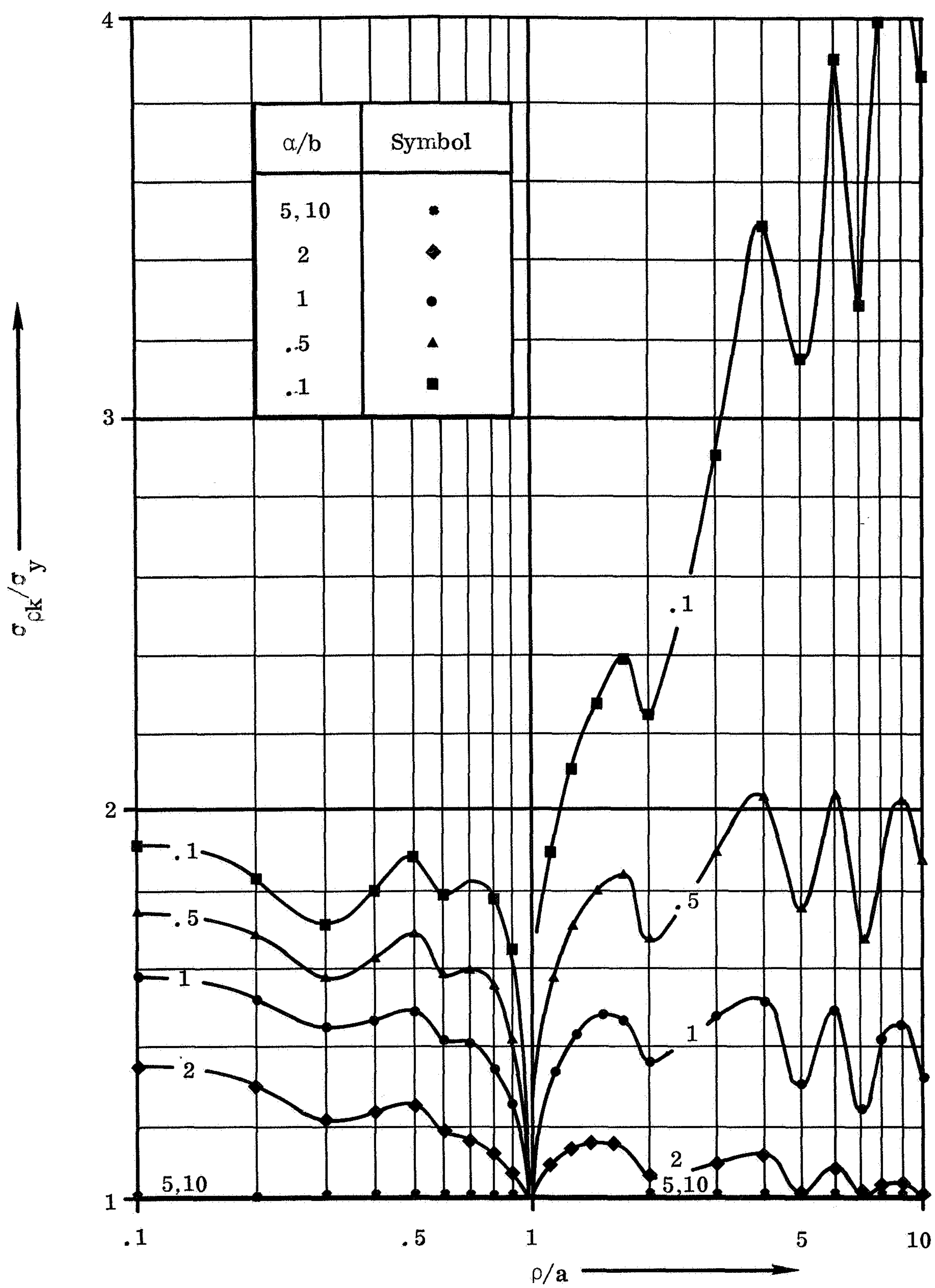

Figure 5.3. Response Overshoot for Correlated Noise Inputs Modulated by e(t) = u(t). Q = 50

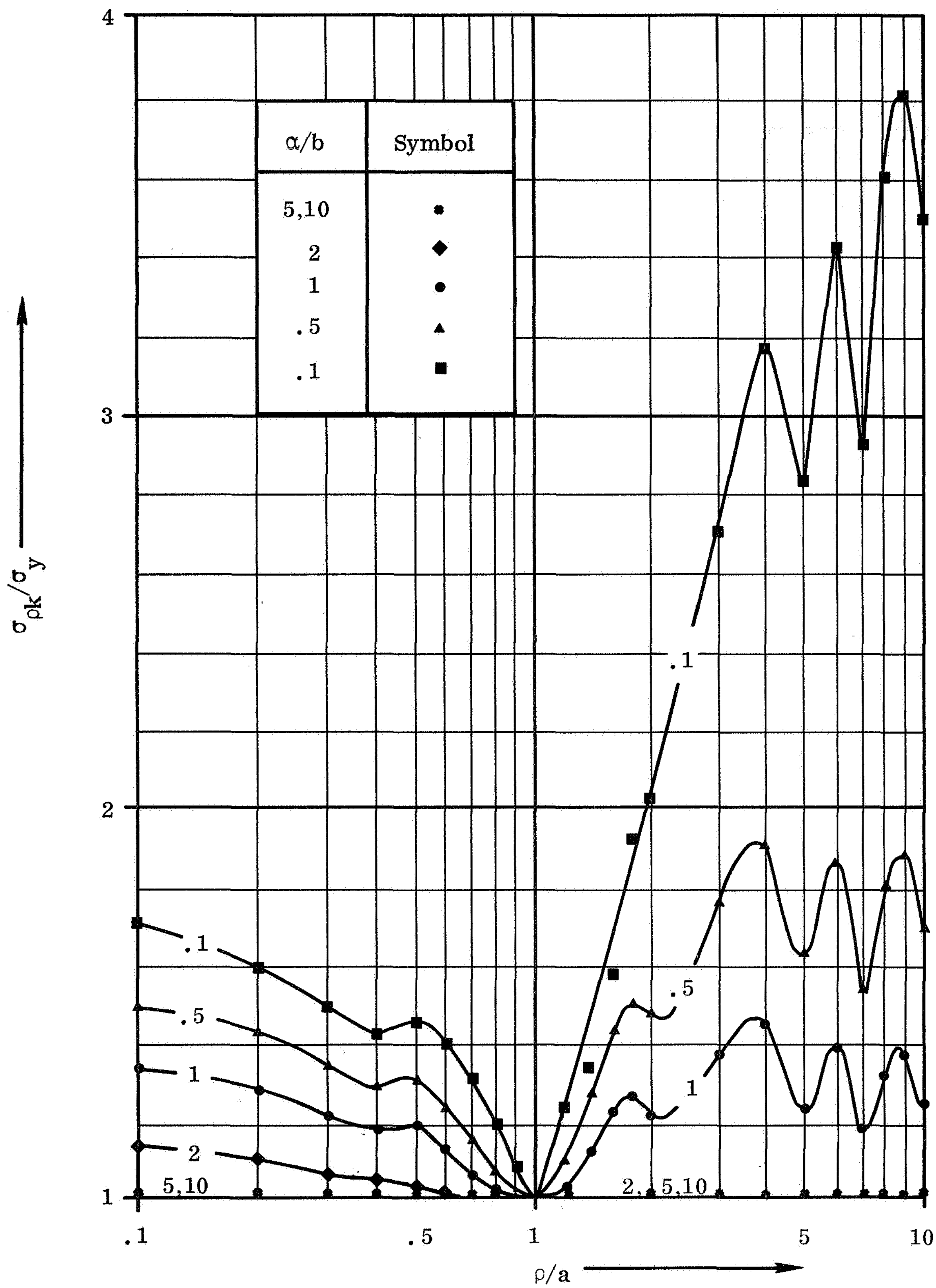

Figure 5.4. Response Overshoot for Correlated Noise Inputs Modulated by e(t) = u(t), Q = 5

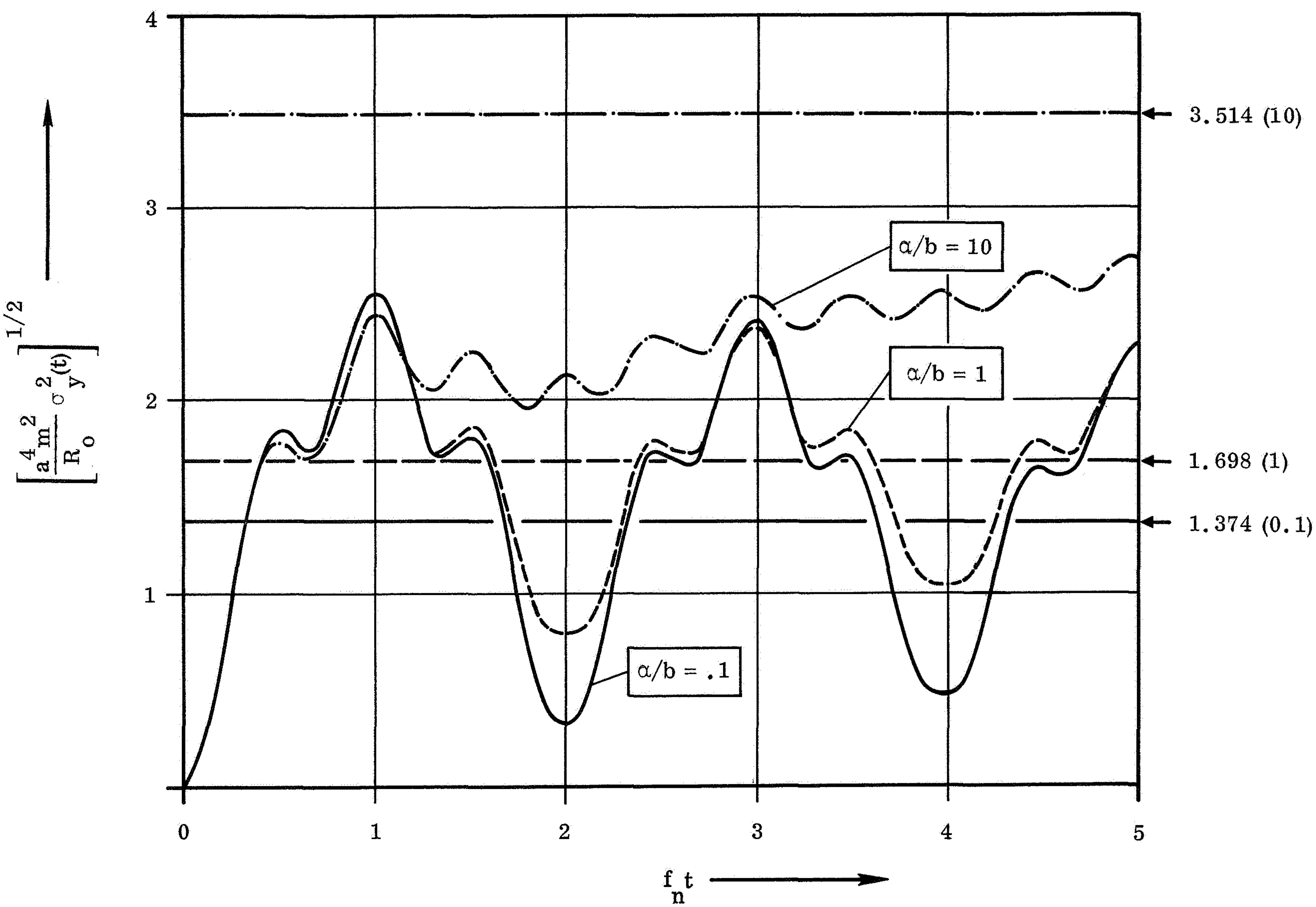

Figure 5.5. Normalized System Response to Correlated Noise Modulated by the Unit Step Function, Q = 50, ρ/a = .5

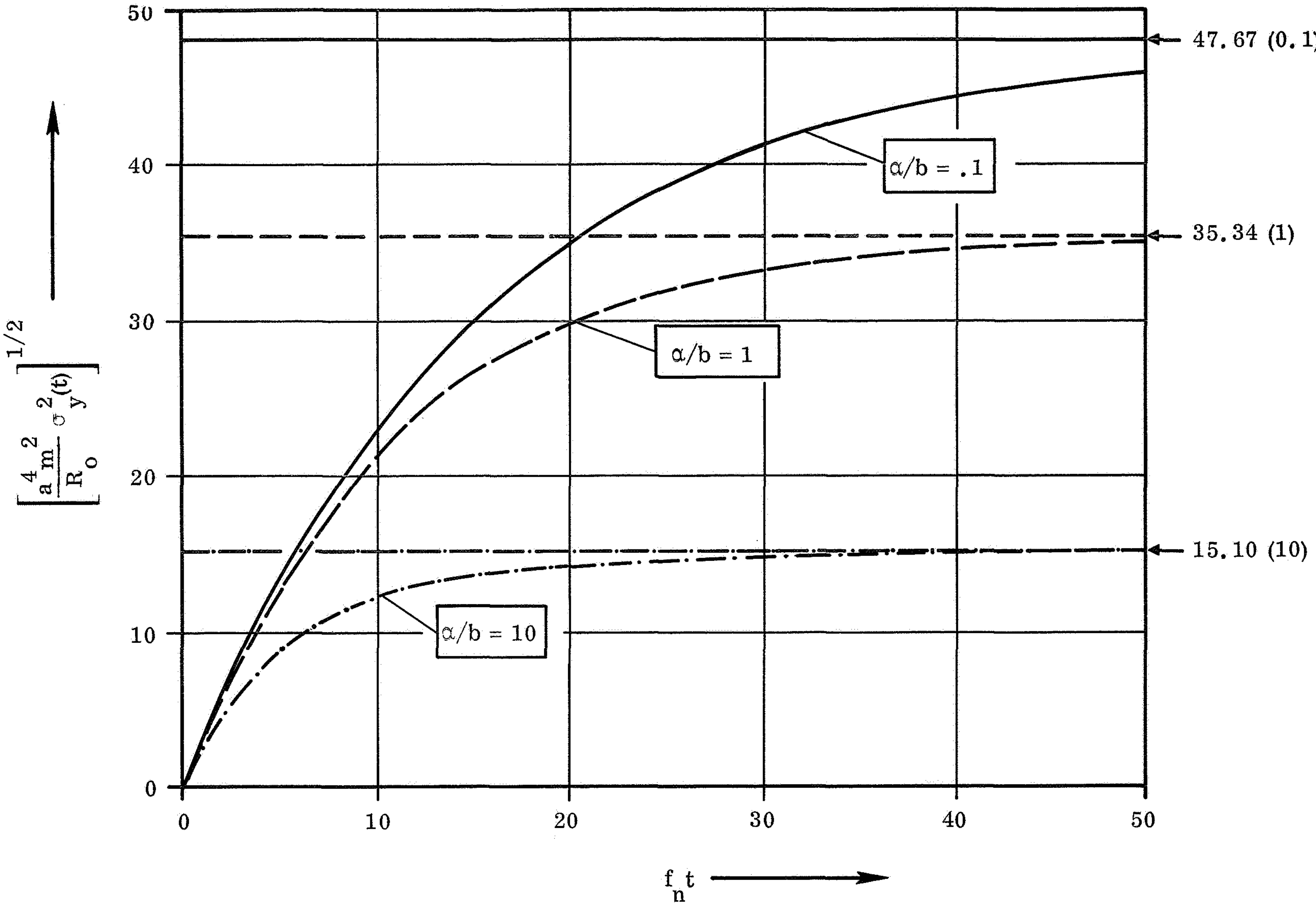

Figure 5.6. Normalized System Response to Correlated Noise Modulated by the Unit Step Function, $Q = 50$, $\rho/a = 1$

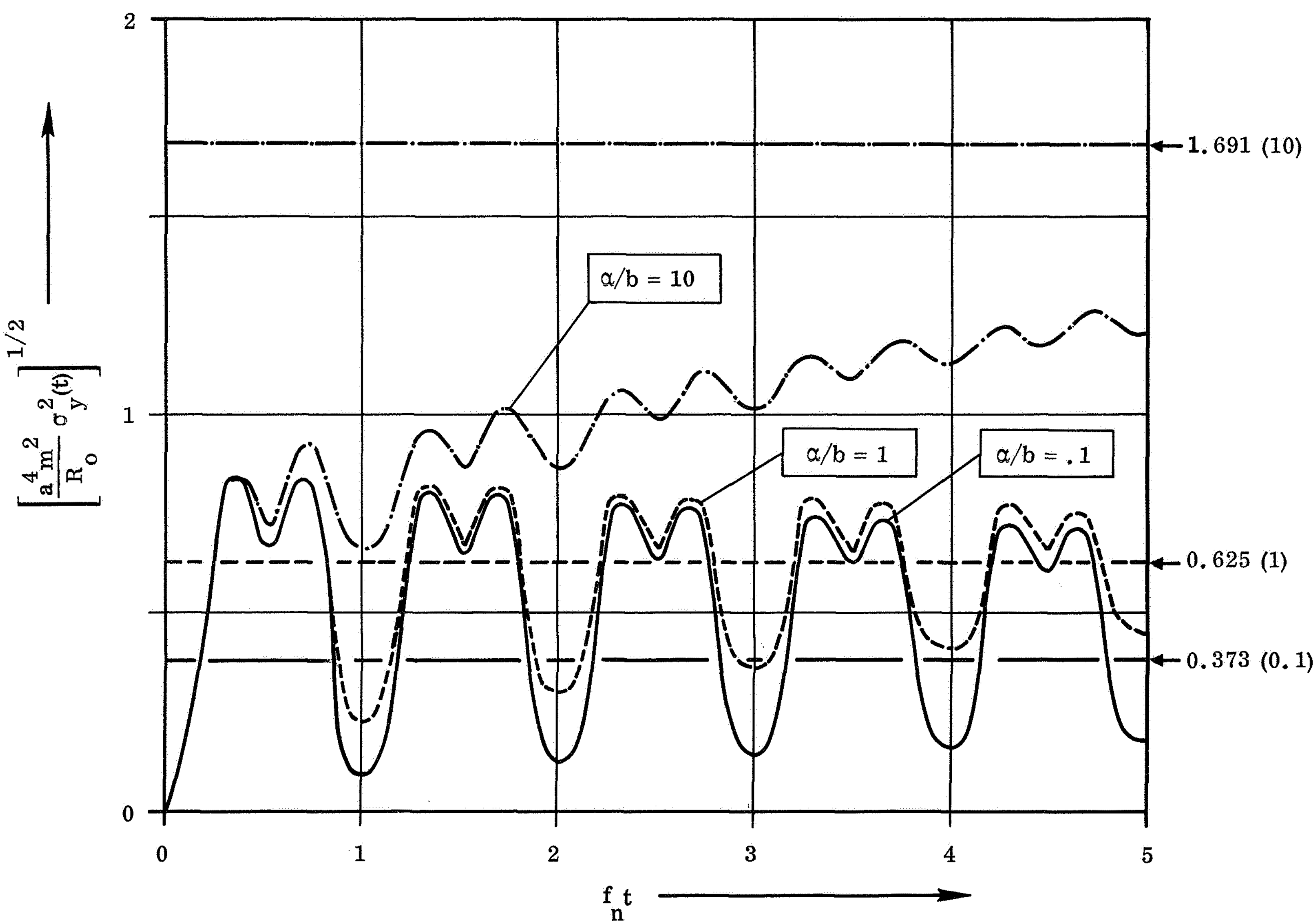

Figure 5.7. Normalized System Response to Correlated Noise Modulated by the Unit Step Function, $Q = 50$, $\rho/a = 2$

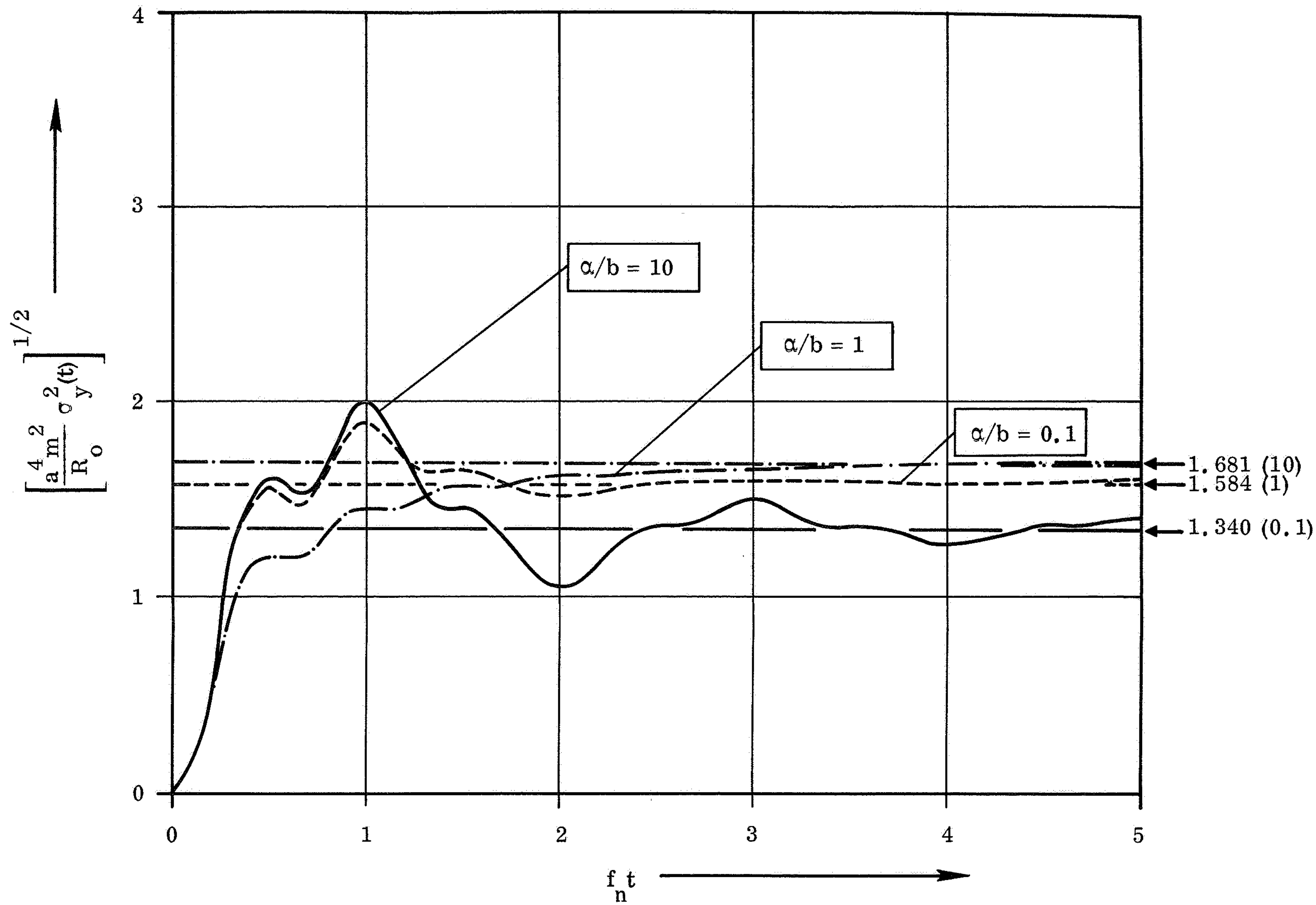

Figure 5.8. Normalized System Response to Correlated Noise Modulated by the Unit Step Function. Q = 5. $\beta/a = 0.5$

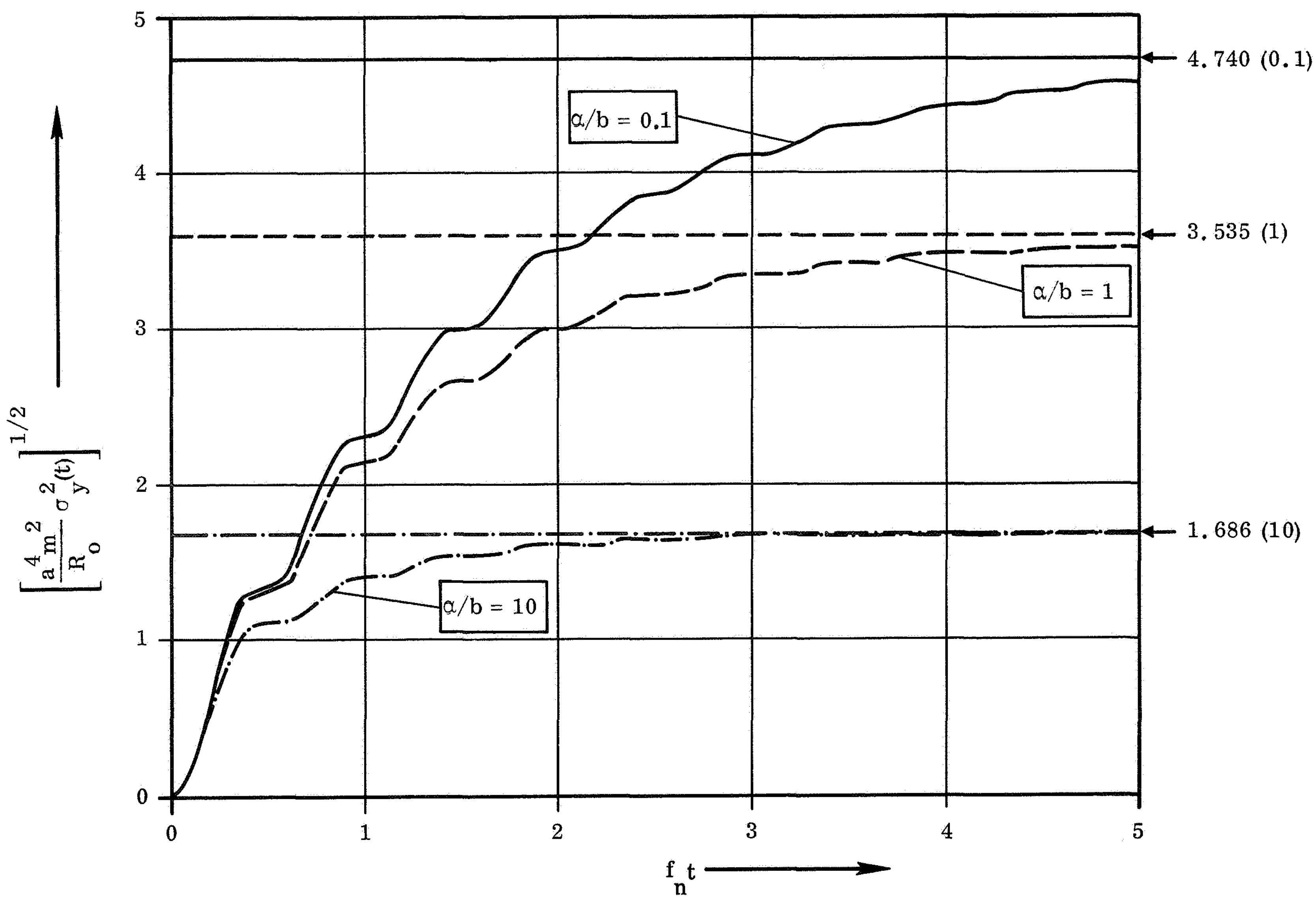

Figure 5.9. Normalized System Response to Correlated Noise Modulated by the Unit Step Function, Q = 5, ρ/a = 1

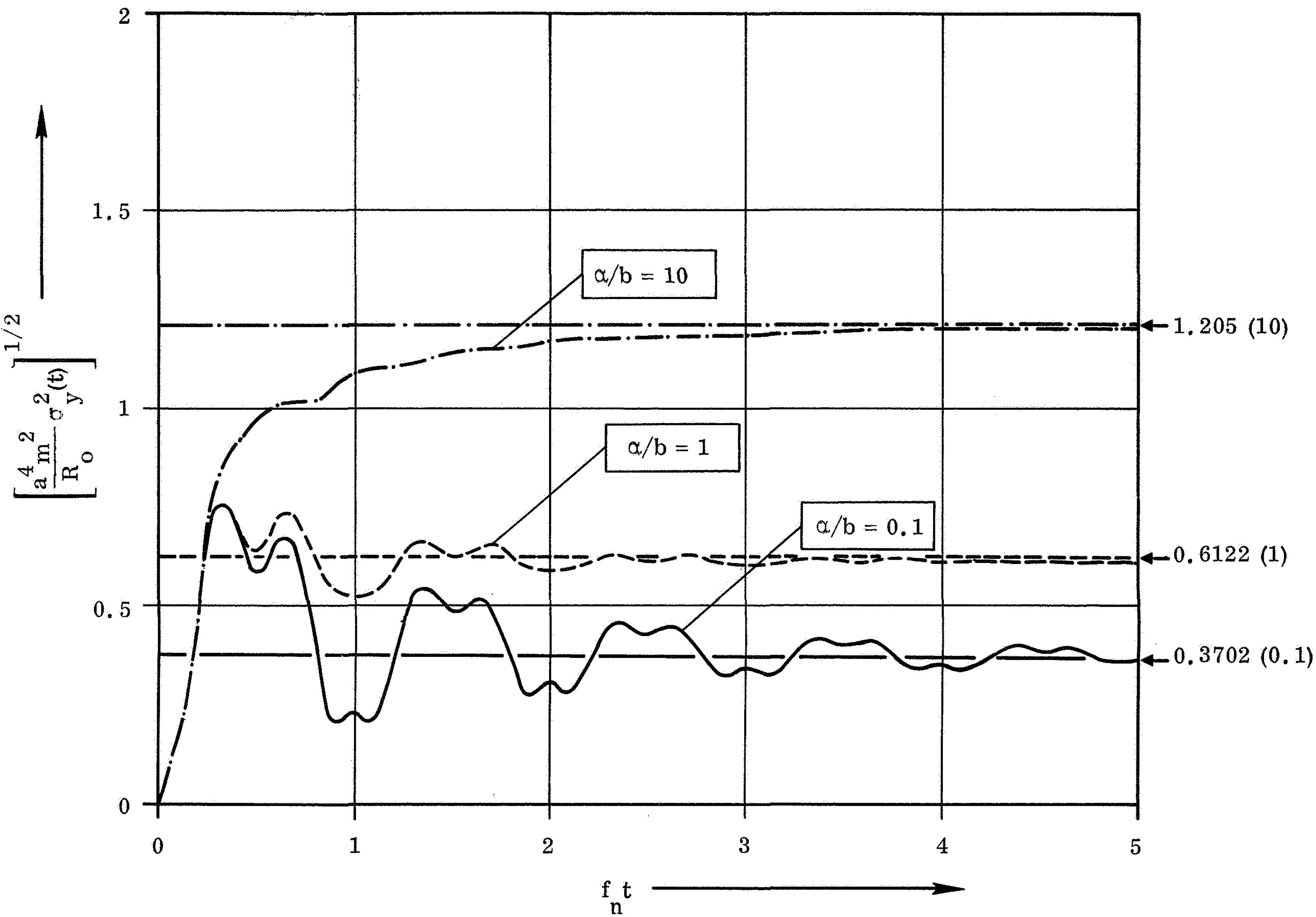

Figure 5.10. Normalized System Response to Correlated Noise Modulated by the Unit Step Function Q = 5, ρ/a = 2

imply an input with highly selective filter characteristics (for example, note the very narrow band and highly peaked characteristics of $S_o(\omega)$ for $\alpha/b = .1$), c_o is greatest at resonance for the smallest α/b. Due to the selective nature of $S_o(\omega)$ for $\alpha/b \leq 1$, c_o varies predominantly as the system function $H_o(\omega)$. For less selective characteristics of $S_o(\omega)$, the c_o values (even though the plots may be similar to those for the smaller values of α/b) are larger away from resonance and reduced in a very narrow band centered at resonance.

Both the onset and magnitude of overshoot are summarized by Figures 5.3 and 5.4. The plots are families of curves in α/b for $\sigma_{\rho k}/\sigma_y$ versus ρ/a with $0.1 \leq \rho/a \leq 10$; only the values $.1 \leq \alpha/b \leq 10$ are treated. For $\alpha/b \geq 5$, no overshoot is noted over the complete range of ρ/a for either $Q = 50$ or $Q = 5$. Equivalent mangitudes for $Q = 50$ generally are larger, as expected. In all cases, the larger overshoot values are associated with the smaller values of α/b with ρ/a centered away from resonance. Over the range $0.1 \leq \rho/a \leq 1$, no exceedance greater than two is noted while over $1 \leq \rho/a \leq 10$, exceedance values above two for $\alpha/b \leq .5$ are not uncommon. At resonance, no exceedance of the stationary value σ_y is experienced by either system for any α/b over the entire range $0.1 \leq \alpha/b \leq 10$.

Figures 5.5 through 5.10 are families of curves in α/b for a normalized form of the system response $\sigma_y(t)$. Each figure reflects three time histories ($\alpha/b = 0.1$, 1, 10) for either $Q = 50$ or $Q = 5$ and one of the ρ/a values of 0.5, 1 or 2. The horizontal lines (and/or 'labeled' arrows) correspond to normalized stationary response values for correlated noise. Such figures provide a portrait of the system time history where both overshoot and convergence to stationarity are depicted rather clearly. For $\alpha/b = 10$, no overshoot is experienced (as might be anticipated from an examination of Figures 5.3 and 5.4) and convergence is asymptotic to the stationary value. The predominant oscillation is twice the damped natural frequency of the system, a characteristic typical of $\sigma_y(t)$ when $n(t)$ is white noise (see Eqns. 4.16 and 4.37). At resonance, the asymptotic convergence to stationarity is noted for all α/b.

From Figures 5.5, 5.7, 5.8 and 5.10, with $\alpha/b = 0.1$ and $\alpha/b = 1$, we note the exceedance occurs very quickly in the time history; the response oscillates about and eventually converges to the stationary value. The convergence to stationarity is achieved more rapidly with the higher system damping ($Q = 5$), an expected behavior. As noted by Eqn (4.21), the response contains arguments of 2a, $(a + \rho)$, and $(a - \rho)$, hence the oscillatory characteristics of $\sigma_y(t)$. The particular argument(s) emphasized depend upon the relative values of a, b, α and ρ; such is apparent by comparing the time histories for $\rho/a = 0.5$ and $\rho/a = 2$.

An explanation for overshoot is contained mainly in Figures 5.11 through 5.19, with Figures 4.1 and 4.2 needed for clarity. The $K(t,\omega)$ plots are the filter shapes imposed on the system stationary response at distinct intervals of time; in particular, note the sharp selectivity for the smaller values of $f_n t$. At any time in the response therefore, the value of $\sigma_y(t)$ is obtained by the integration over ω of an integrand formed by the triple product $|H(\omega)|^2 S_n(\omega) K(t, \omega)$. The magnitude of $K(t, \omega)$ thus "shapes" the integrand and controls the $\sigma_y(t)$ value at any specific time. Over all $f_n t$ shown, $K(t, \omega)$ has a minimum ("notch") at system resonance; this quantity eventually resolves to a constant when the system response achieves stationarity. Since this notch effect (minimum) becomes gradually less predominant with time, the asymptotic convergence to the stationary value at resonance for all α/b is clear. In such circumstances, the response is governed almost entirely by the product $|H(\omega)|^2 S_n(\omega)$ near ω_n and as the value of $K(t, \omega)$ near ω_n gradually reduces to a constant, so too must $\sigma_y(t)$.

Further, consider the time history $\sigma_y(t)$ for $\alpha/b = 0.1$, $\rho/a = 2$, and $Q = 50$ which is shown in Figure 5.7. Let us now remark on the maxima and minima of the response keeping in mind Figure 4.1 with $S_o(\omega)$ centered at $\omega/a = 2$. At $f_n t = 0.3$, 0.7 and 0.8, $\sigma_y(t)$ is relatively large due mainly to contributions by $S_n(\omega)$ as $|H(\omega)|^2$ is reduced due to the notch at system resonance. At $f_n t = 1$, $\sigma_y(t)$ is near minimal due to the dual notching effect of $K(t, \omega)$ at $\omega = \omega_n$ and $\omega = 2\omega_n$, the latter corresponding to nearly the maximum of $S_n(\omega)$. For interest, included as Figures 5.20 and 5.21 are normalized plots of $|I(t, \omega)|$ for $f_n t = 1$, 3

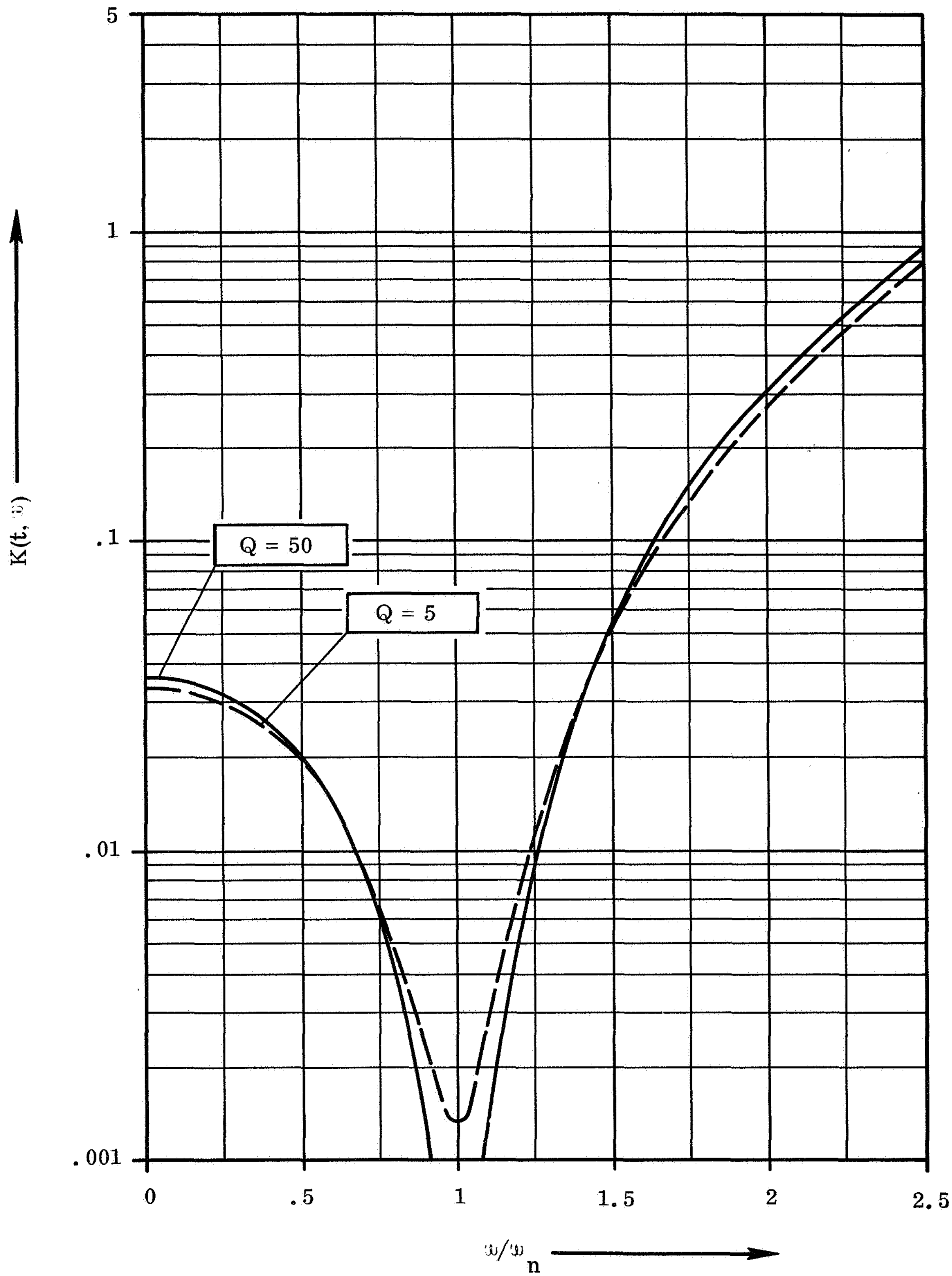

Figure 5.11. Shaping Filter K(t, ω) with $f_n t = 0.1$

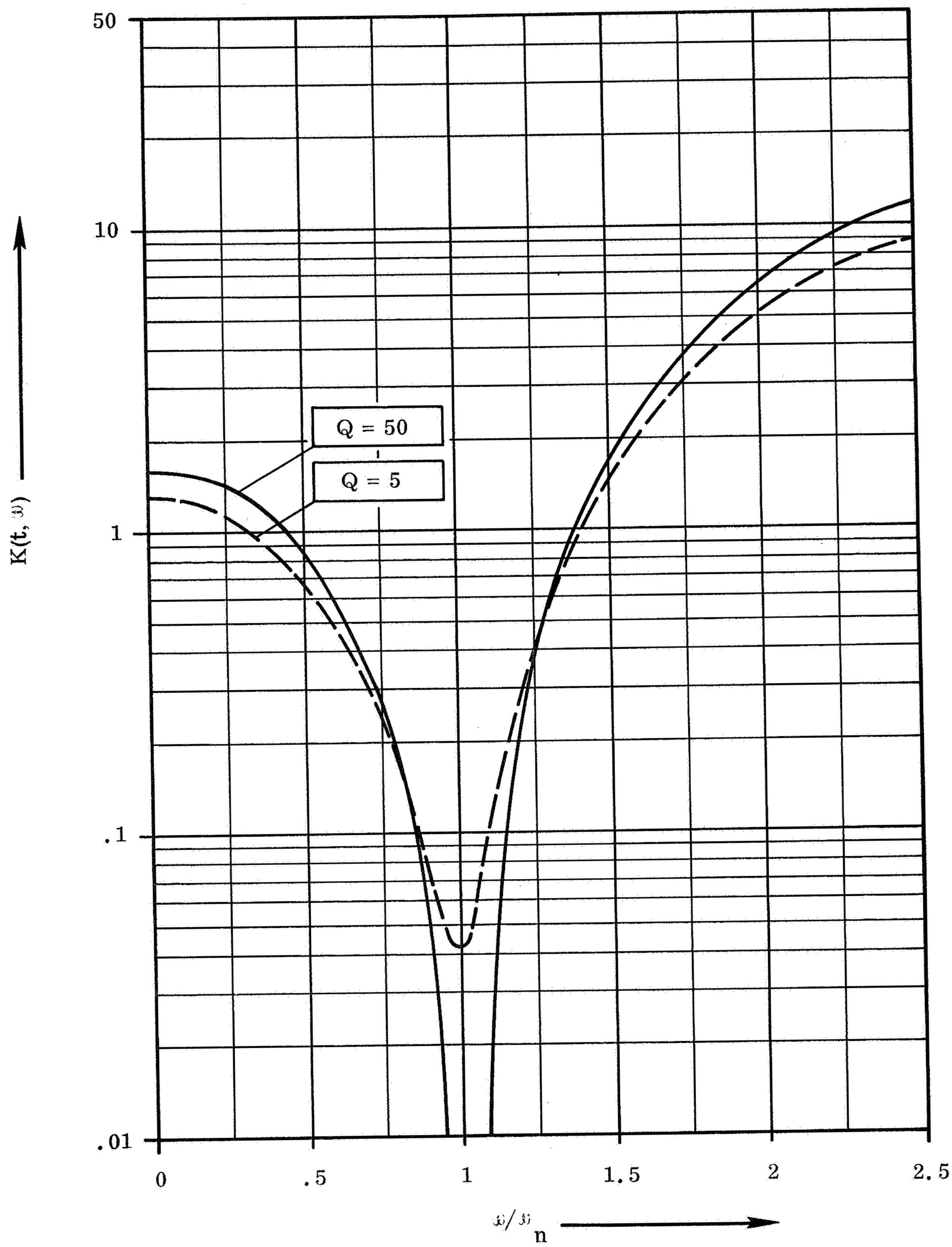

Figure 5.12. Shaping Filter $K(t, \omega)$ with $f_n t = 0.3$

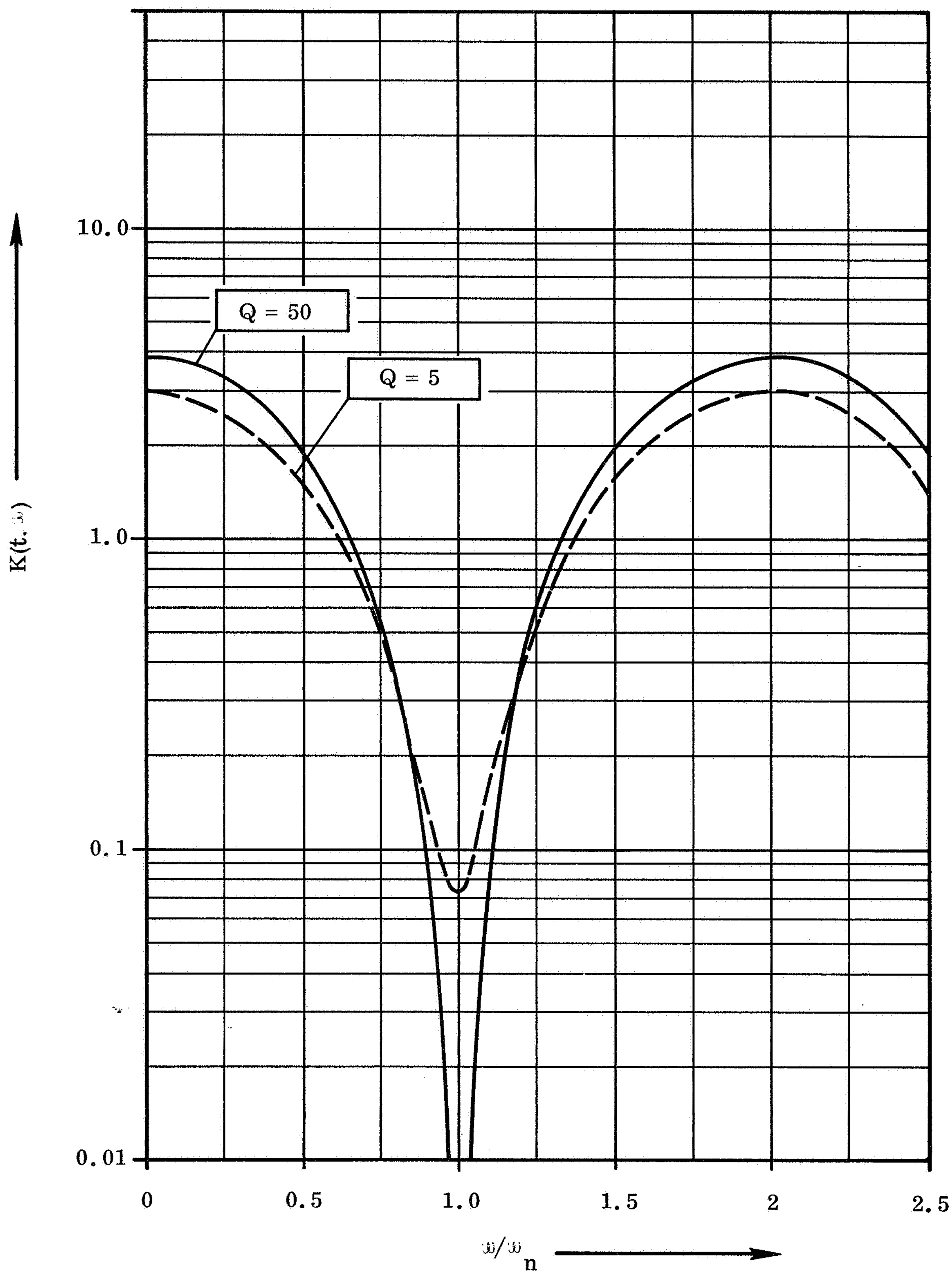

Figure 5.13. Shaping Filter K(t, ω) with $f_n t = .5$

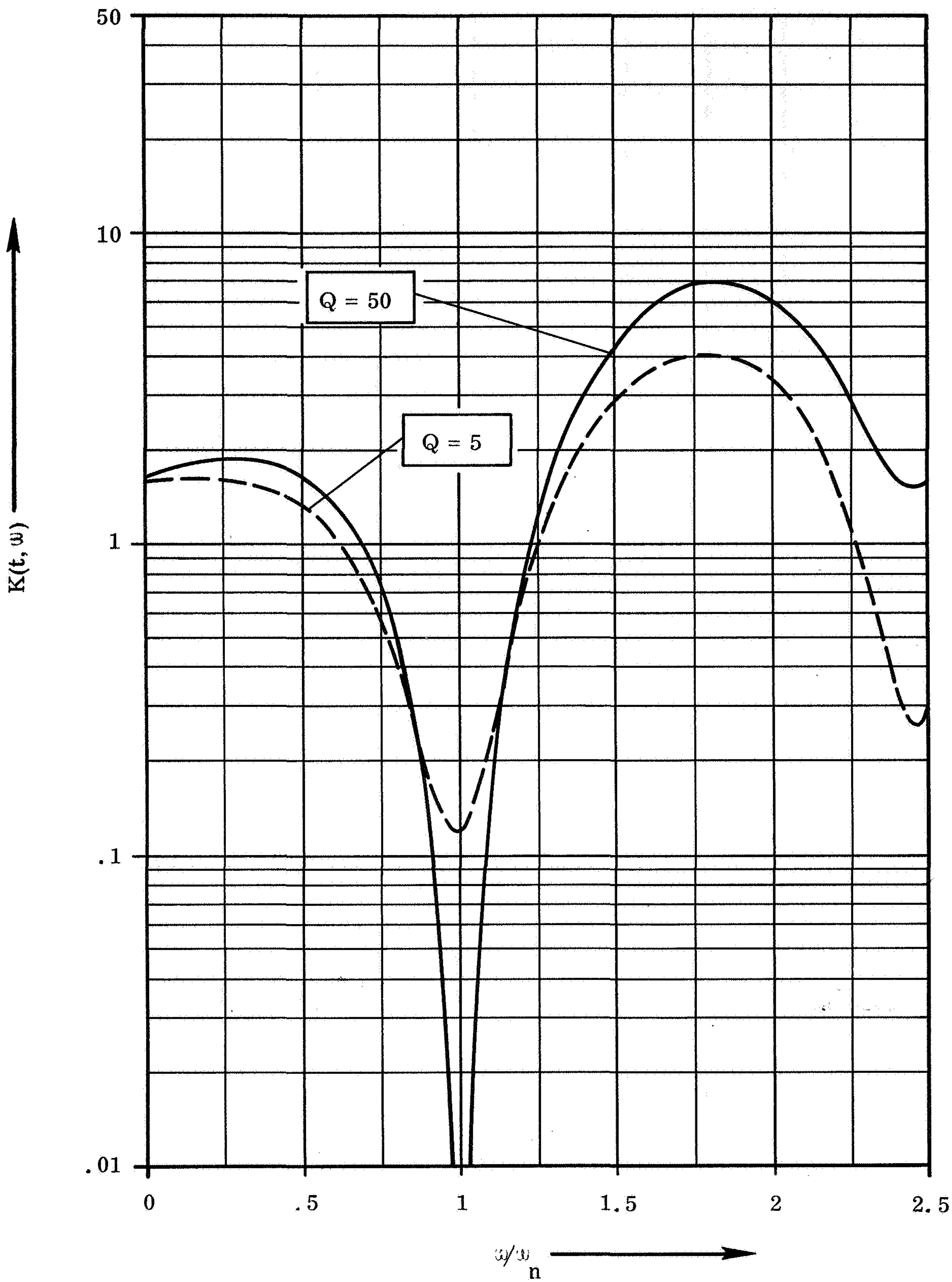

Figure 5.14. Shaping Filter K(t, ω) with $f_n t = 0.7$

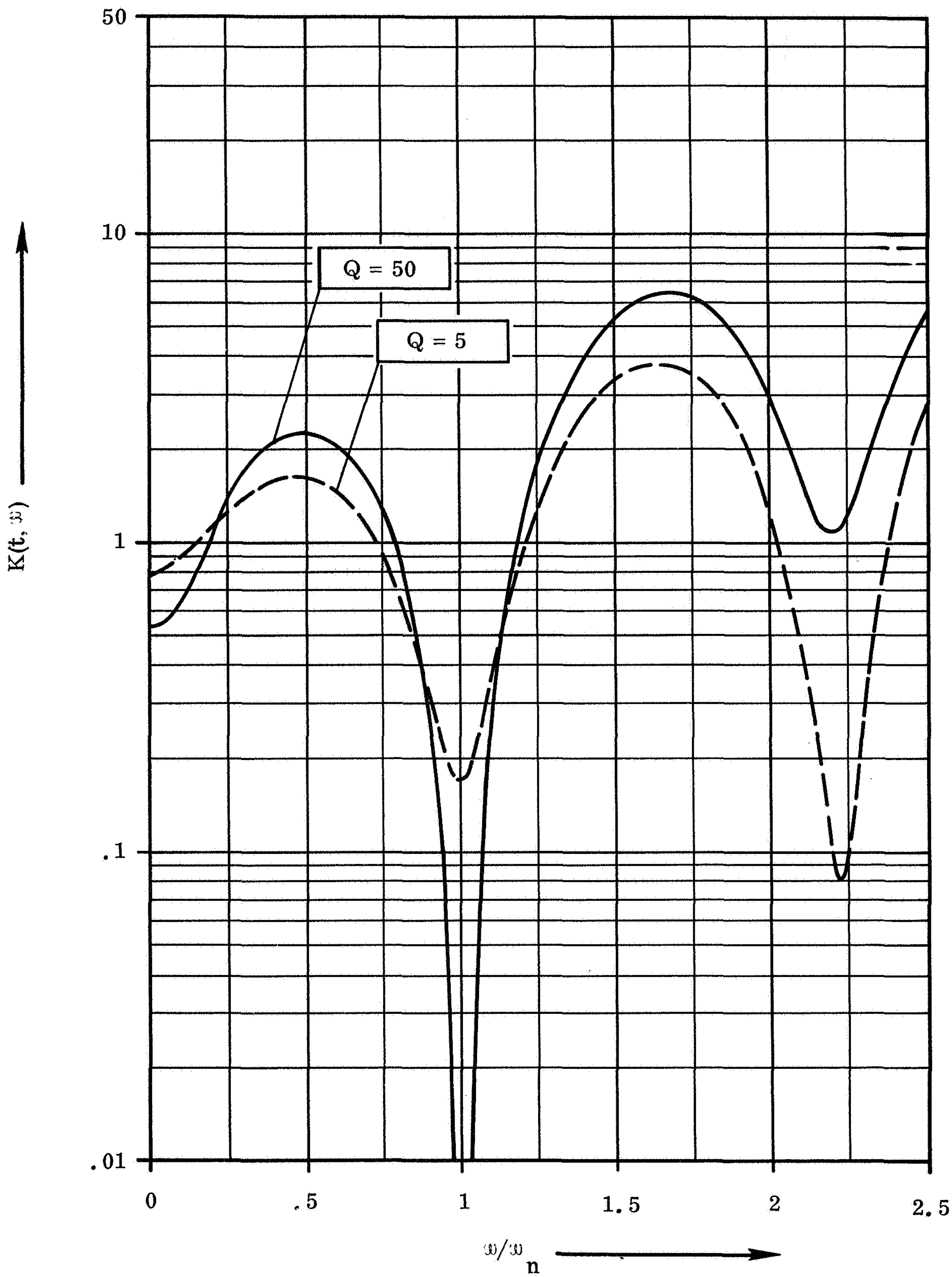

Figure 5.15. Shaping Filter K(t, ω) with $f_n t = 0.8$

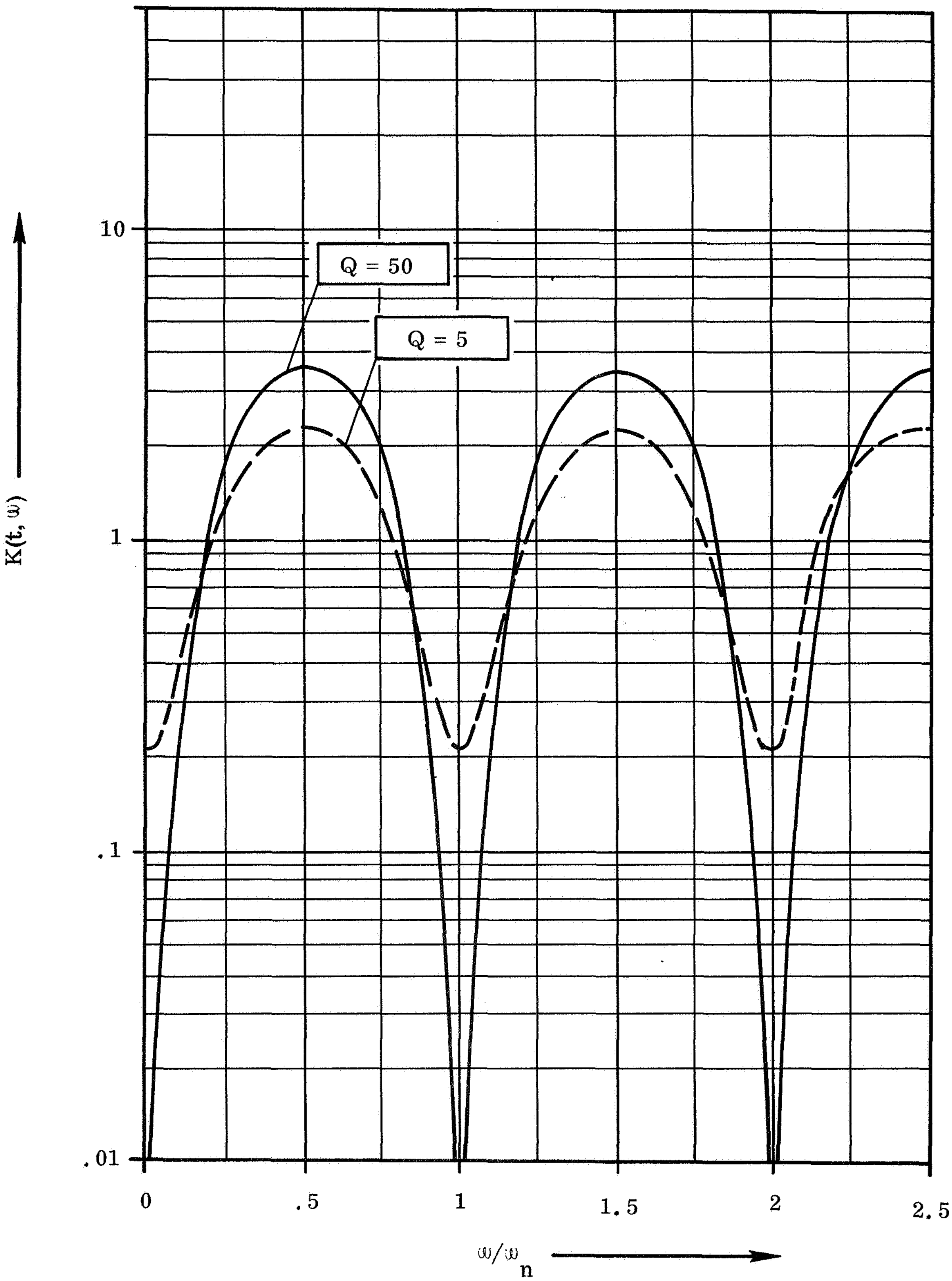

Figure 5.16. Shaping Filter K(t, ω) with $f_n t = 1$

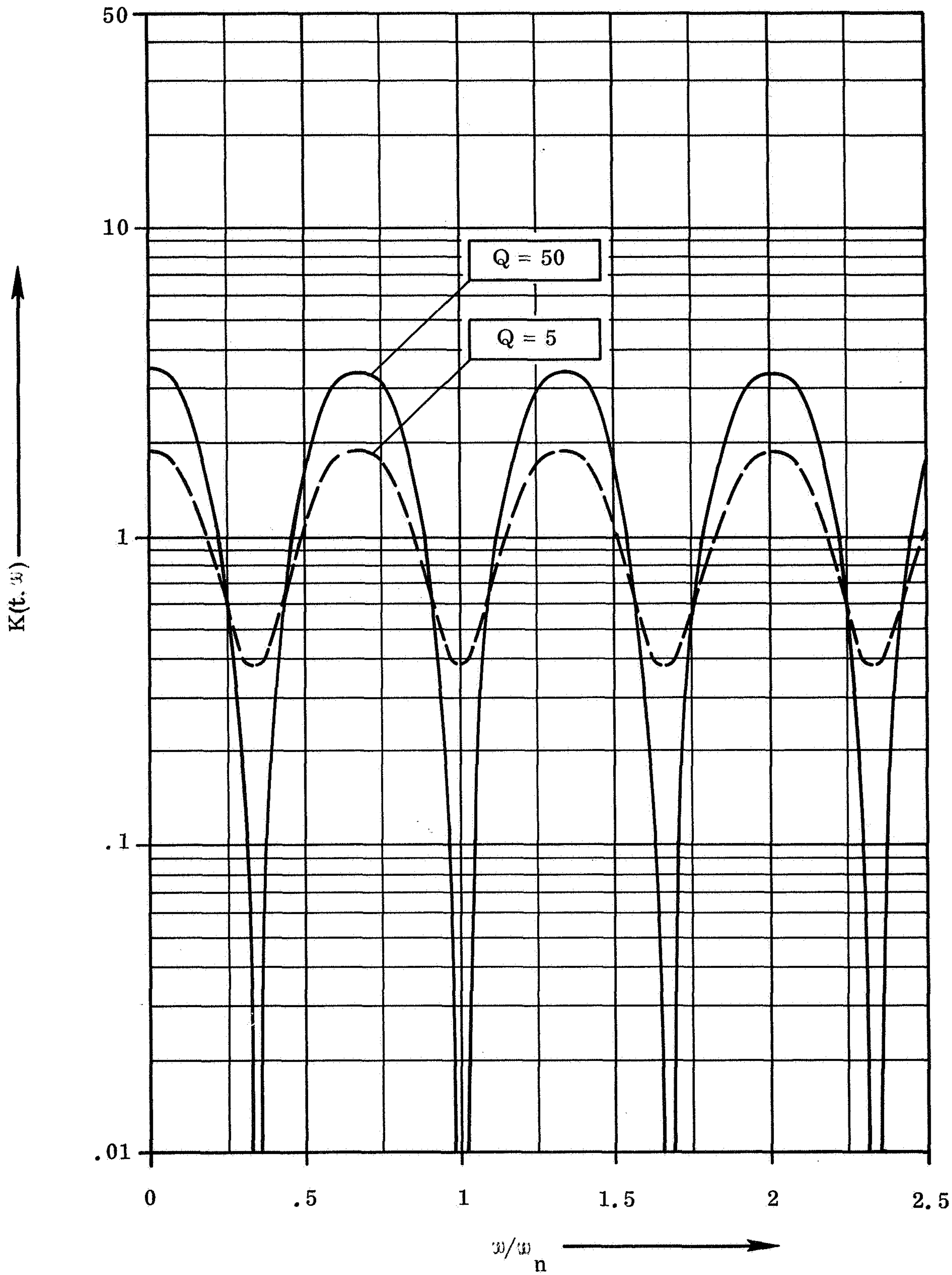

Figure 5.17. Shaping Filter K(t, ω) with $f_n t = 1.5$

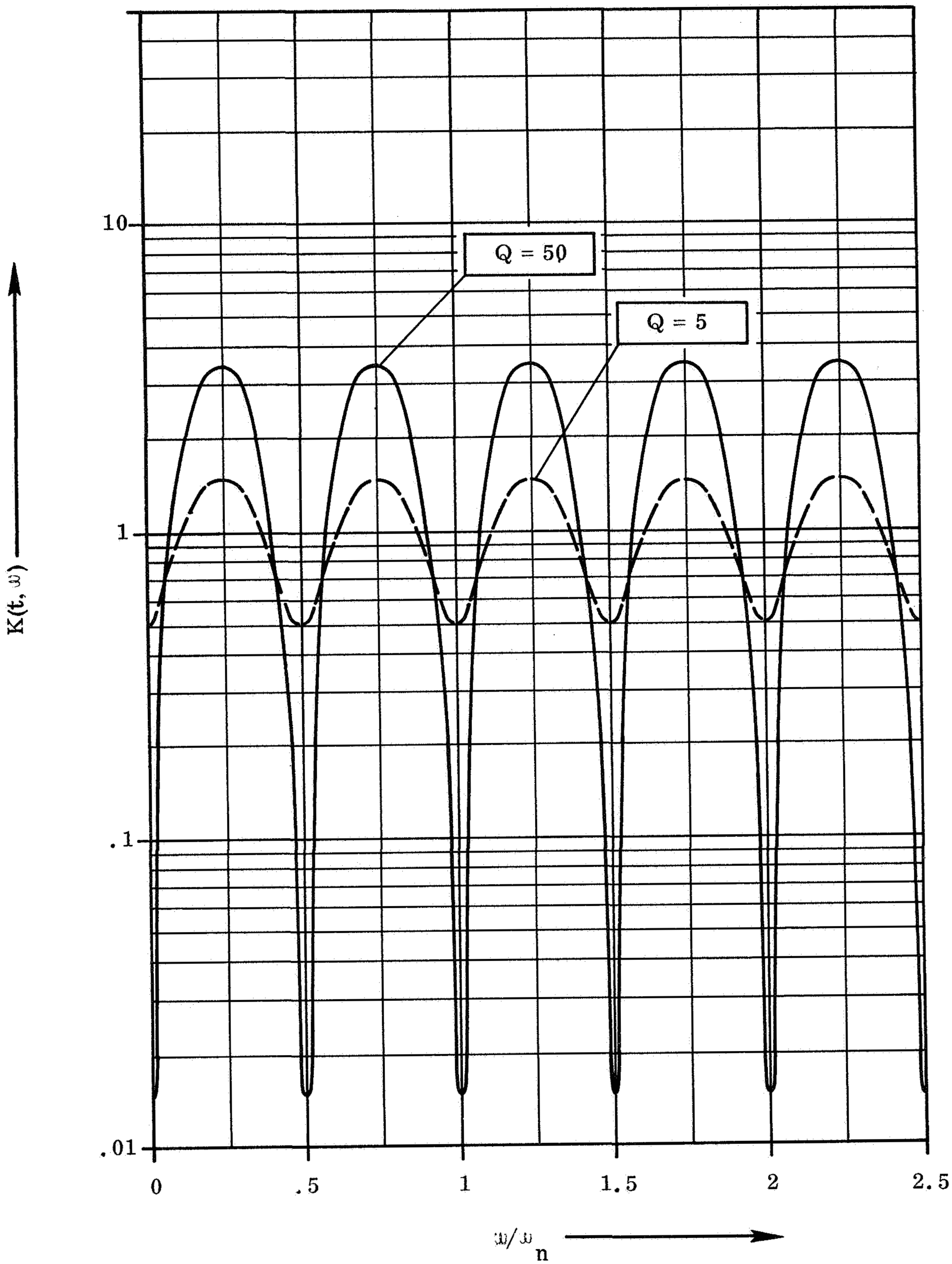

Figure 5.18. **Shaping Filter K(t, ω) with $f_n t = 2$**

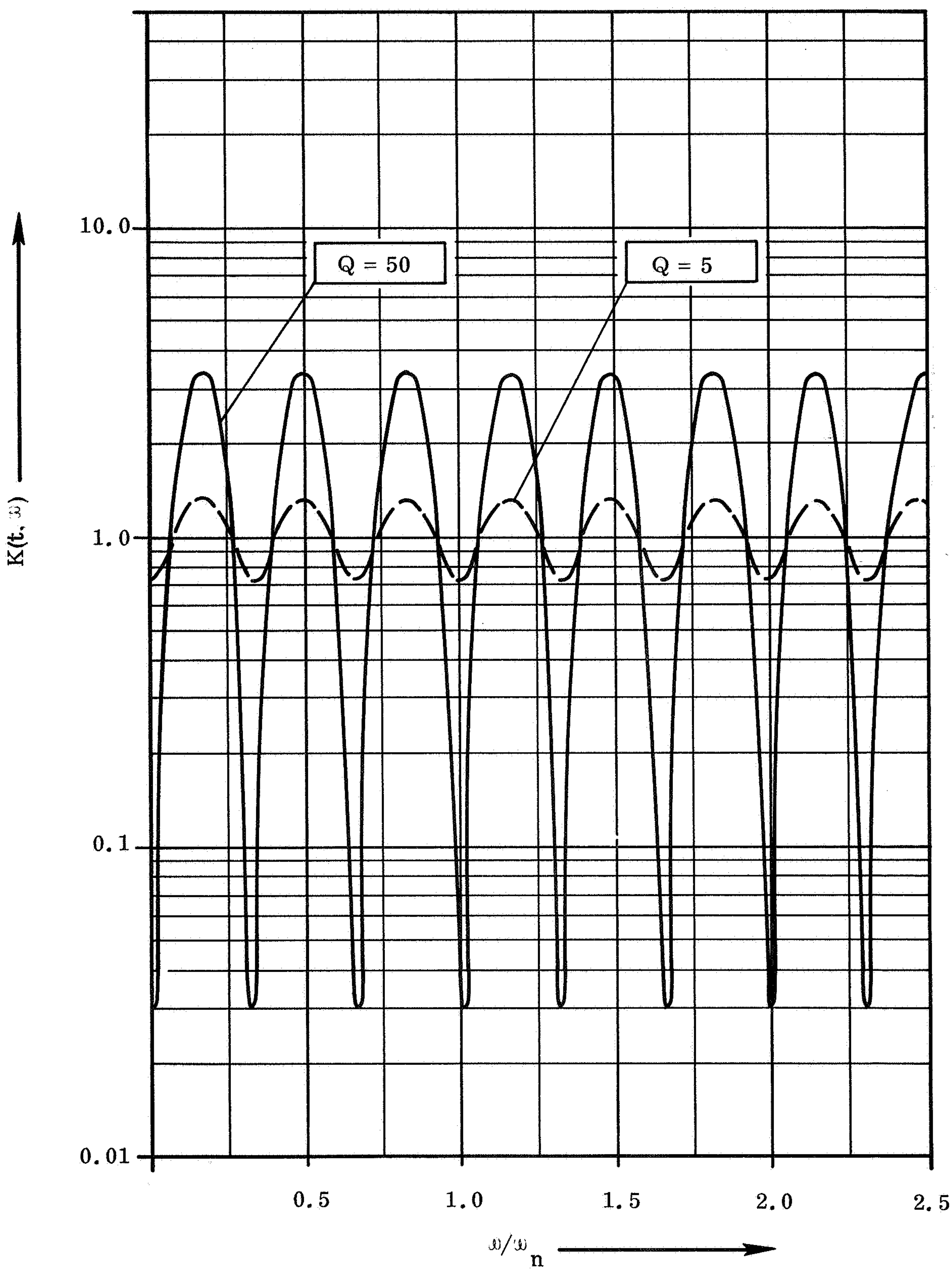

Figure 5.19. Shaping Filter K(t, ω) with $f_n t = 3.0$

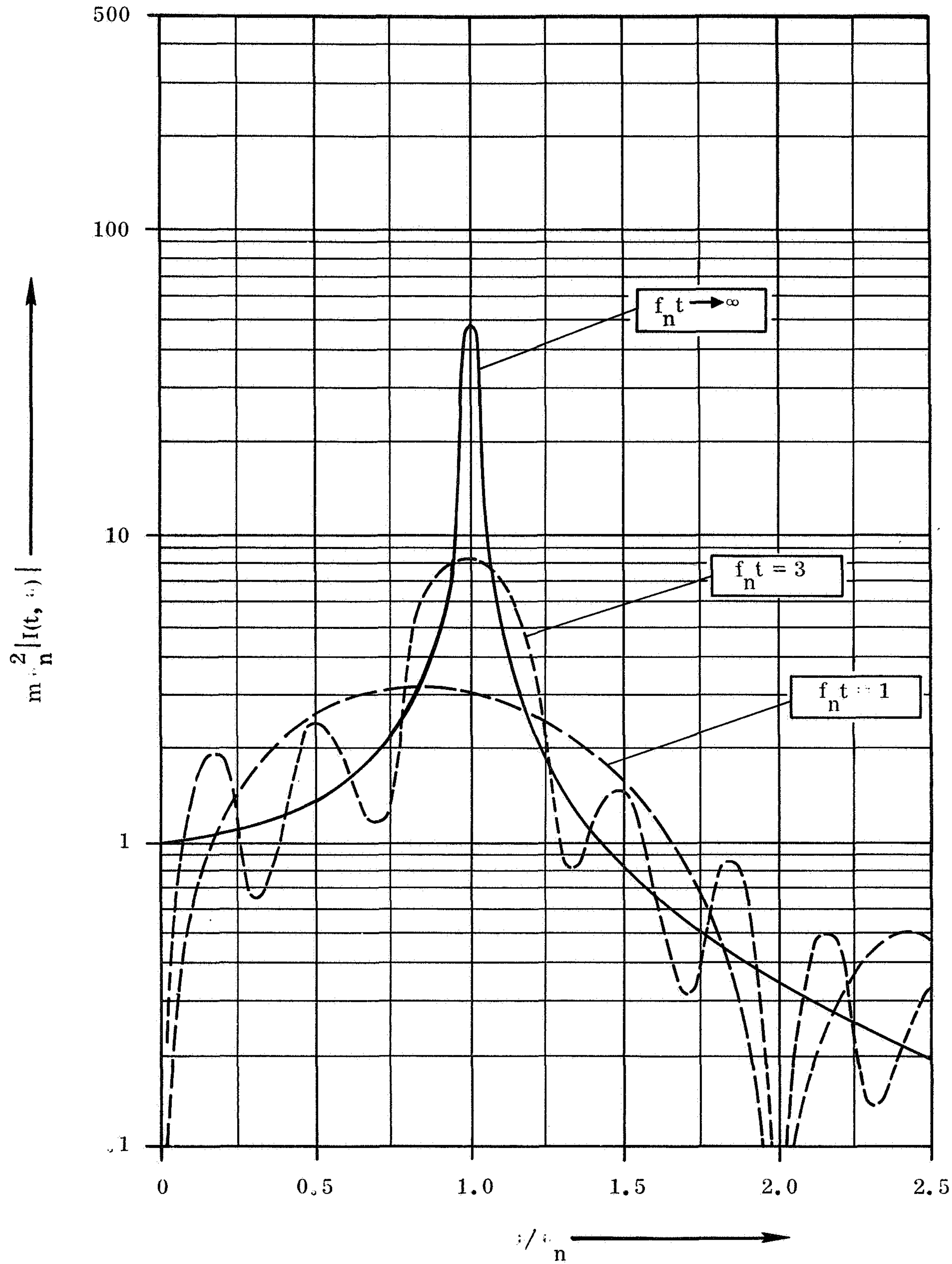

Figure 5.20. $m\omega_n^2 |I(t, \omega)| = |H_o(\omega)| K(t, \omega)$ with Q = 50

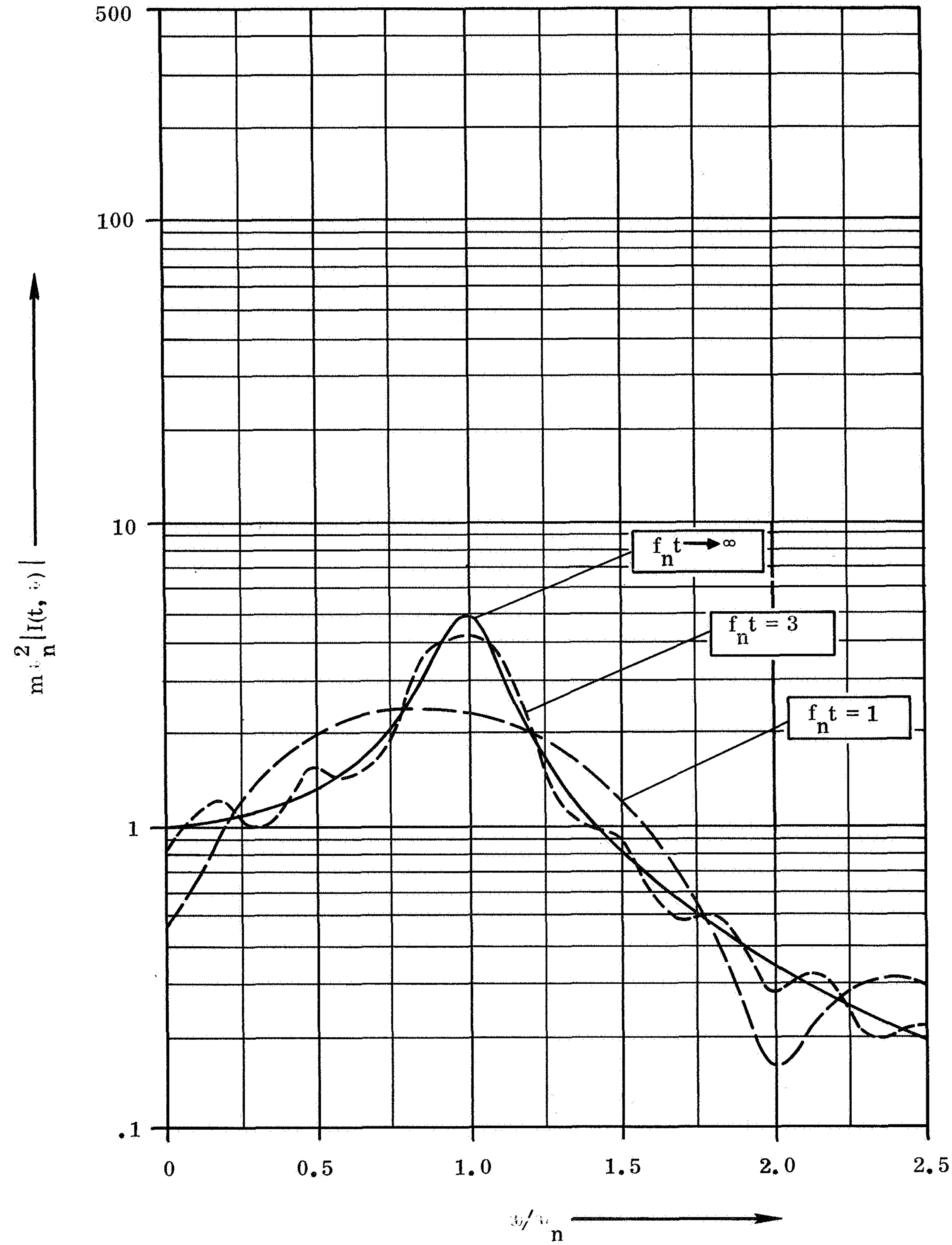

Figure 5.21. $m\omega_n^2 |I(t, \omega)| = |H_o(\omega)| K(t, \omega)$ with Q = 5

and infinity. These plots show the build-up of the mean square integrand at selected values of time where, at infinity, the integrand is that for the stationary response.

The effect of rectangular step modulation is shown by the time histories of Figures 5.22 through 5.28. Such represent the output response of a system with $Q = 50$, $\alpha/b = 1$, $\rho/a = 2$ where the time duration of the rectangular envelope is varied as $f_n t_o = 0.22$, 0.35, 0.5, 0.70, 0.80, 1.0 and 1.4. Of note is that the residual response may exceed not only the stationary value, but the peak value of $\sigma_y(t)$ as well. Given such behavior, its practical use, for example, as a modulation envelope in environmental testing, is severely limited.

The system response modulated by the exponential function is shown by Figure 5.29 through 5.34. The general effect of an increase in the envelope damping coefficient γ is to decrease $\sigma_y(t)$ in time as well as to alter the oscillatory behavior from that experienced with the unit step modulation. Although overshoot may occur, it can be eliminated beyond $f_n t = 2$ by either an increase in γ (see Figures 5.29, 5.30 and 5.31) or an increase in system damping, or both. For $\gamma/b = 10$, $\sigma_y(t)$ is nearly the same for both $\alpha/b = 0.1$ and $\alpha/b = 10$. The response for $\alpha/b = 10$ drops below the $\sigma_y(t)$ curves for the other two values of α/b; note the reverse is true for $\gamma/b = 0.1$ and $\gamma/b = 1$. A more complete set of response time histories for this modulation function is shown in Appendix C as Figures C.1 through C.18.

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*It should be noted that the response decay exponential $e^{-2\gamma t}$ (see Eqn 4.39) does not bound the system response $\sigma_y(t)$.

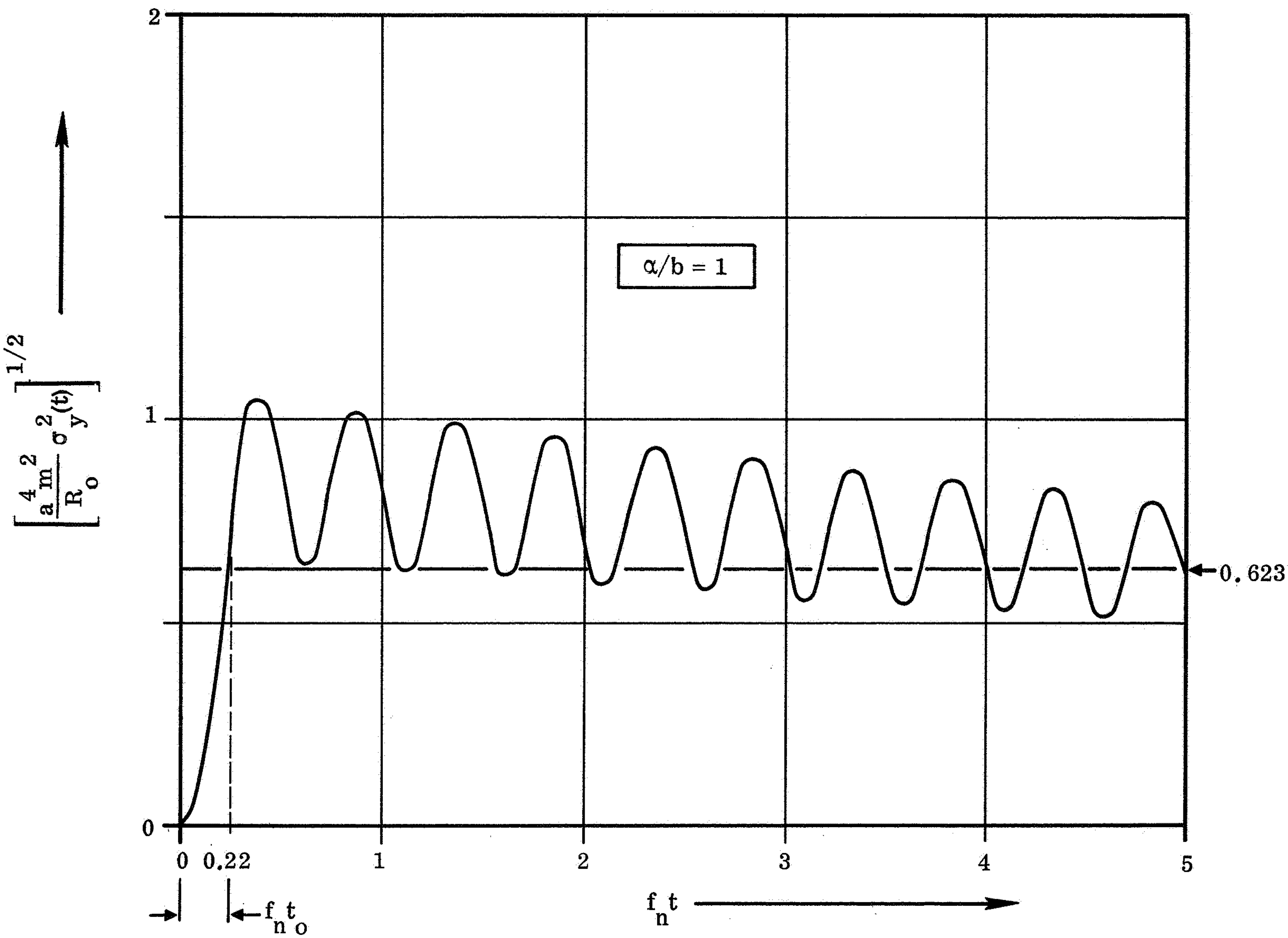

Figure 5.22. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function, $Q = 50$, $\rho/a = 2.0$

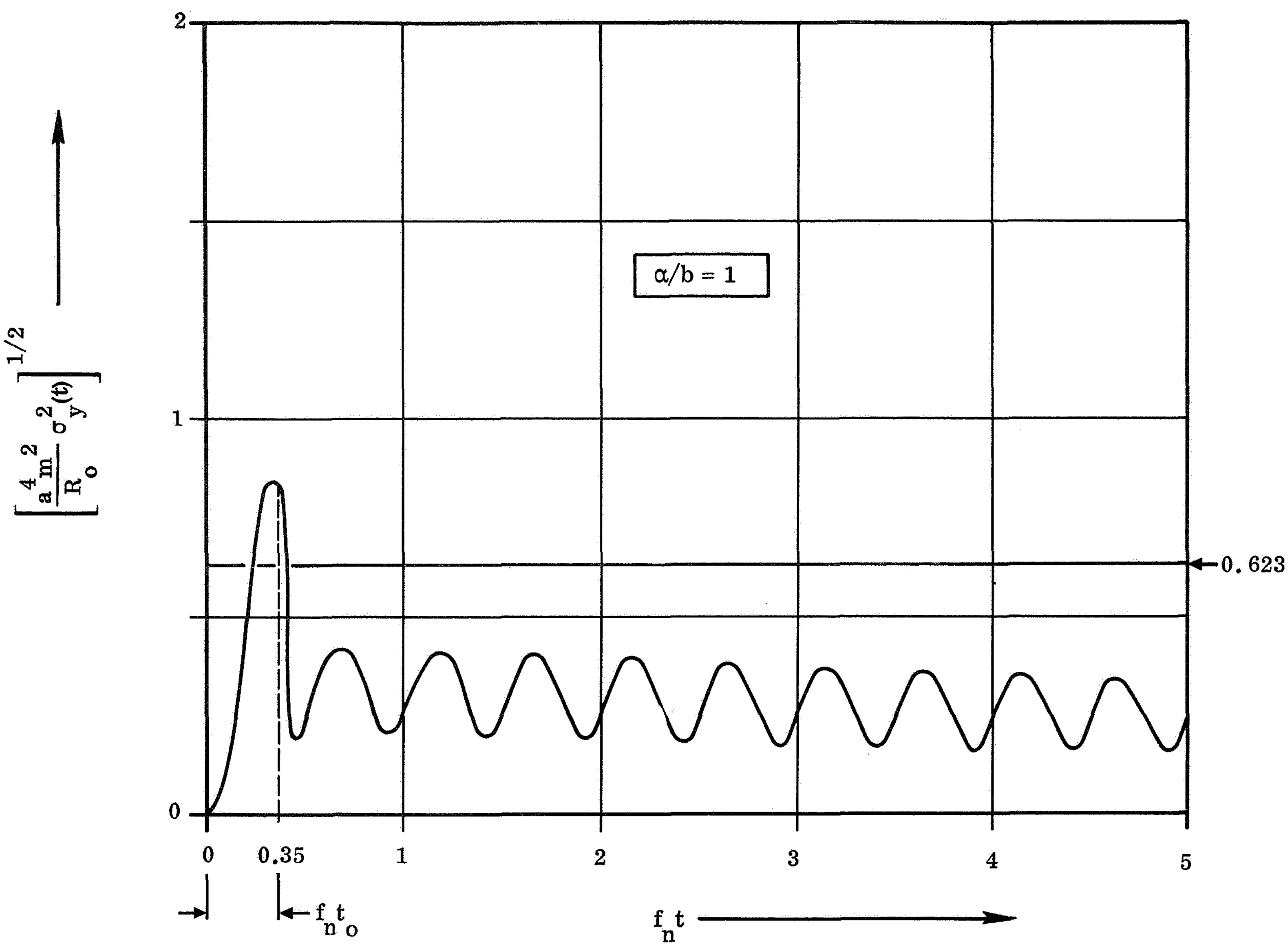

Figure 5.23. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 50, ρ/a = 2.0

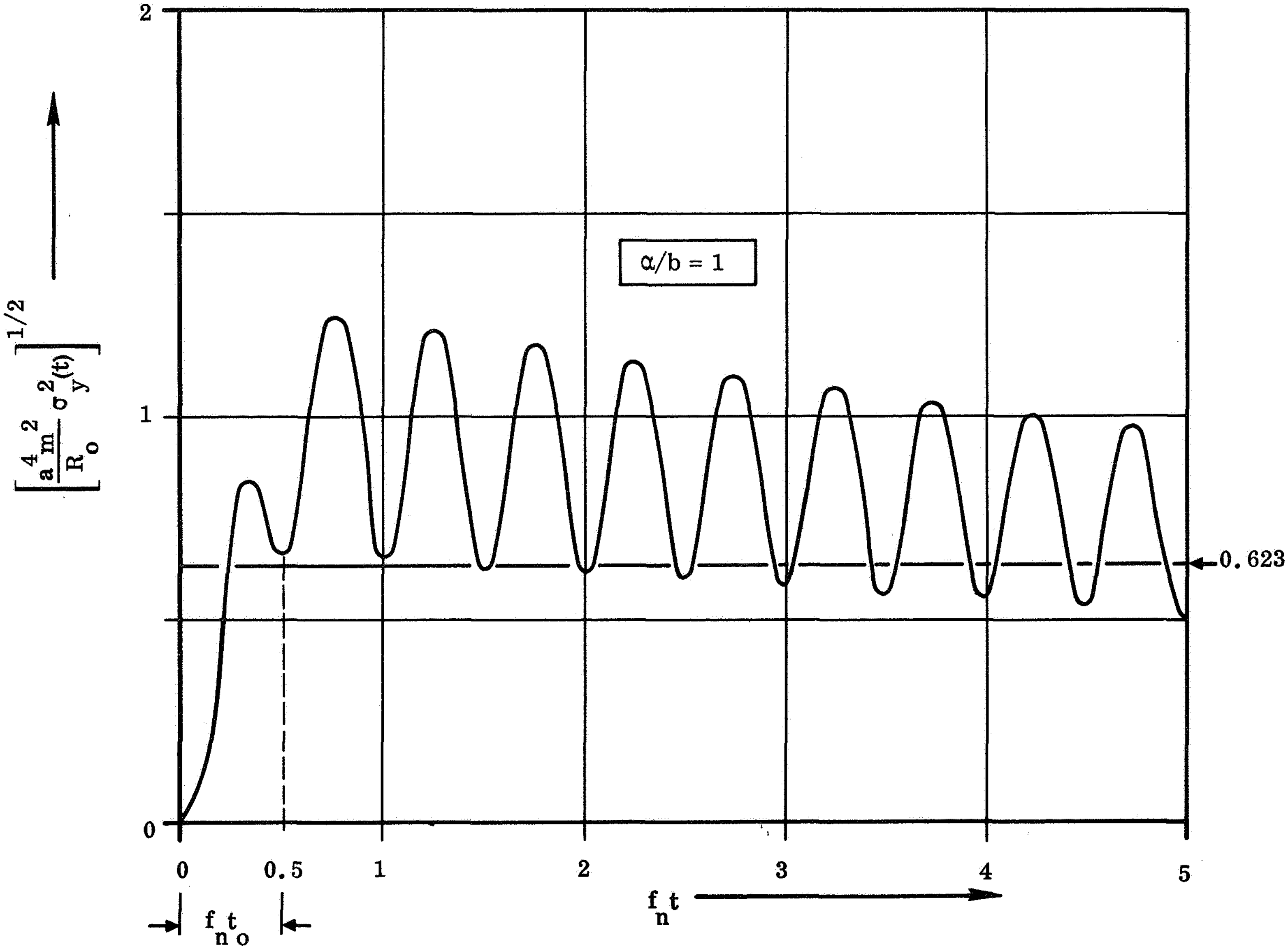

Figure 5.24. Normalized System Response to Correlated Noise Modulated by the Rectangular Step, Q = 50, ρ/a = 2

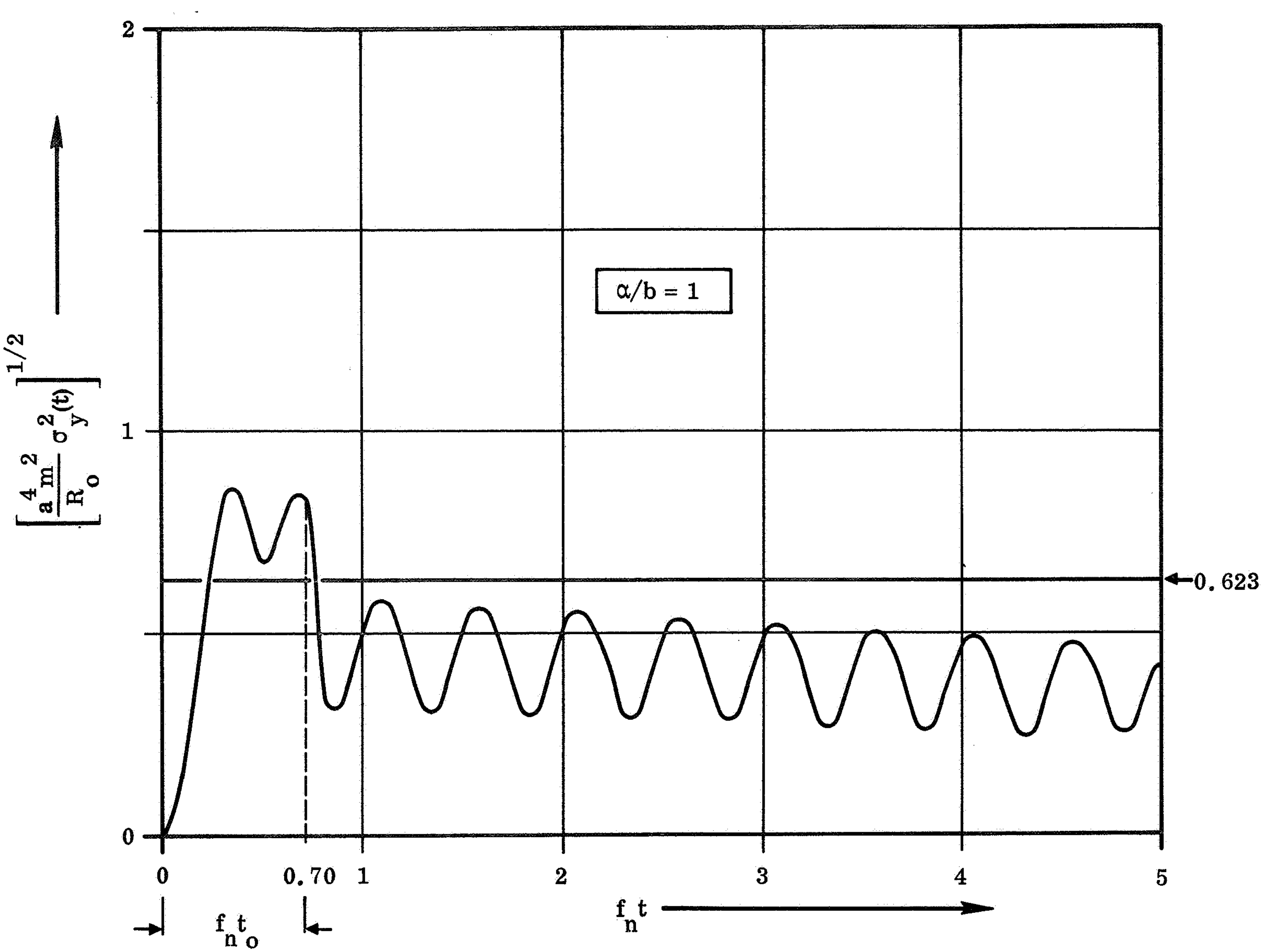

Figure 5.25. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 50, ρ/a = 2

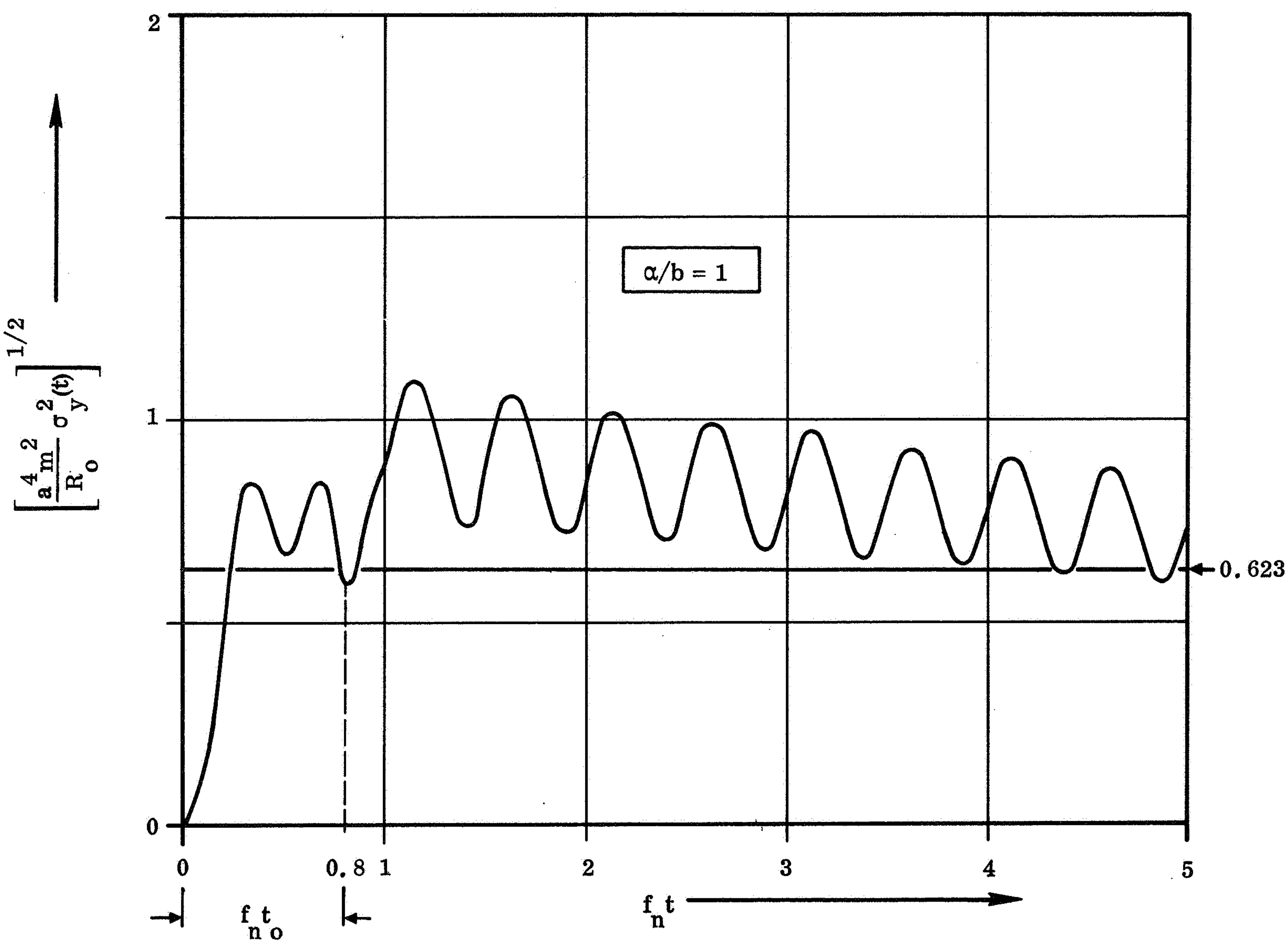

Figure 5.26. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function, Q = 50, ρ/a = 2

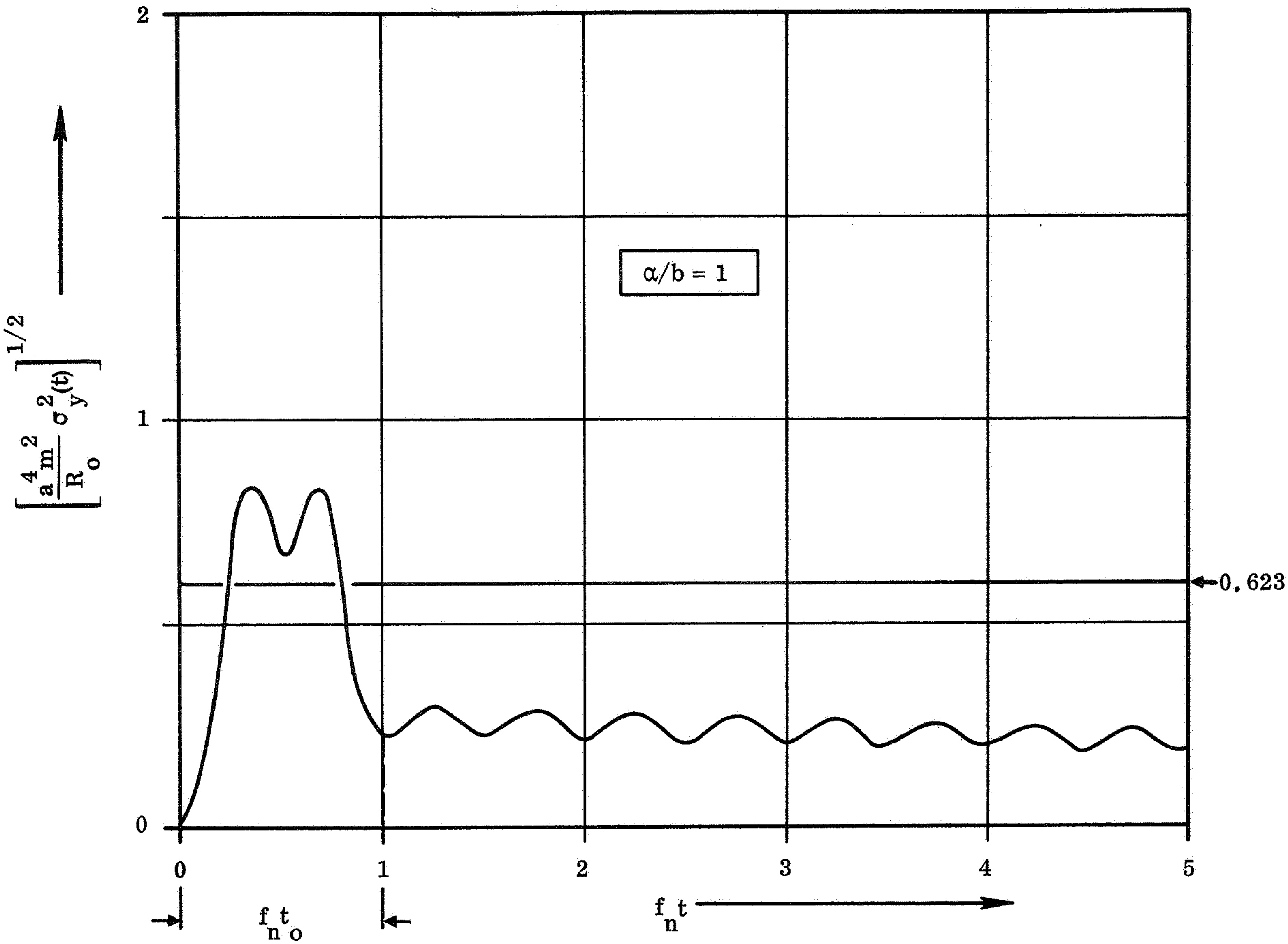

Figure 5.27. Normalized System Response to Correlated Noise Modulated by the Rectangular Step, Q = 50, ρ/a = 2

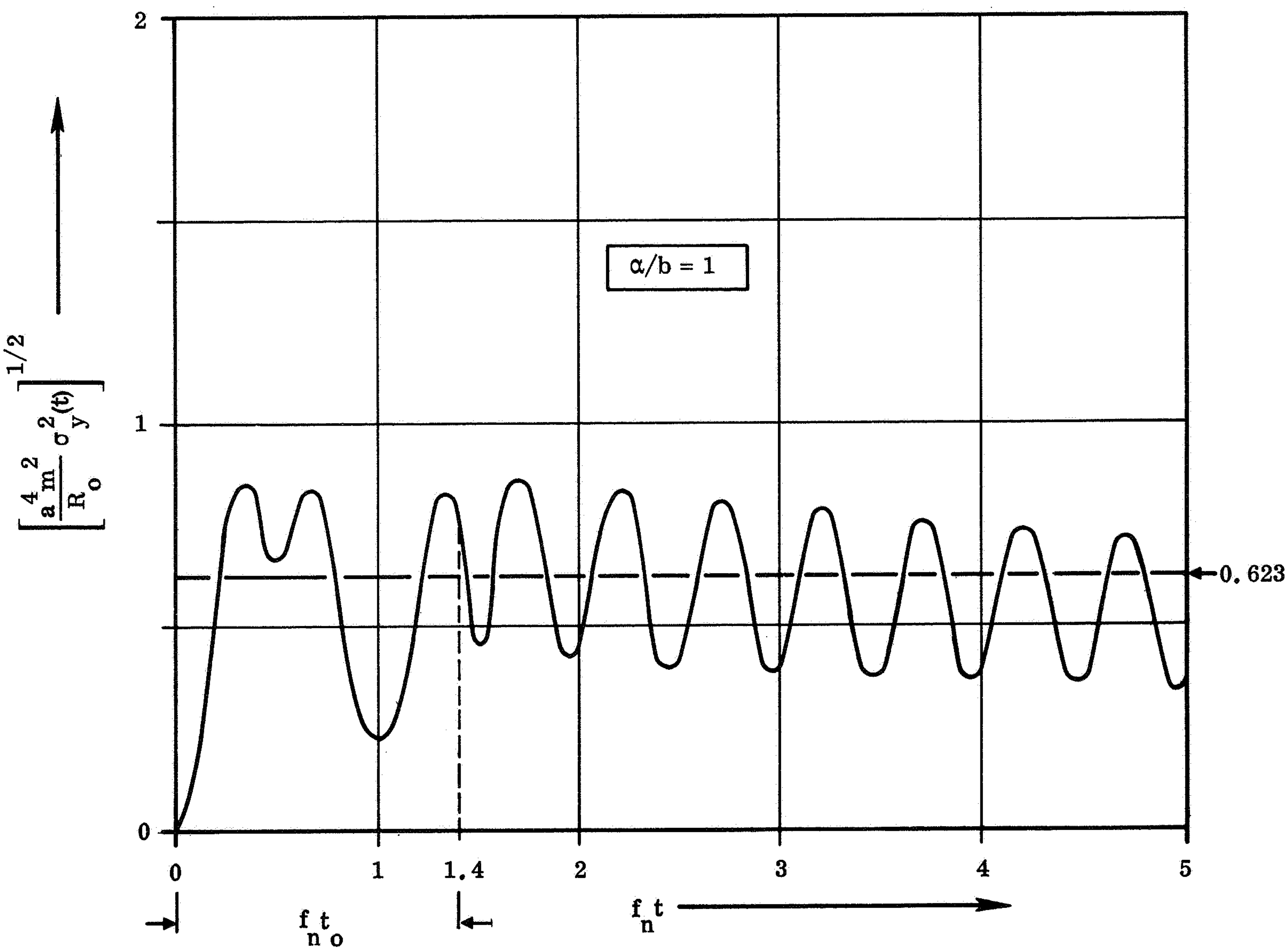

Figure 5.28. Normalized System Response to Correlated Noise Modulated by the Rectangular Step, Q = 50, ρ/a = 2

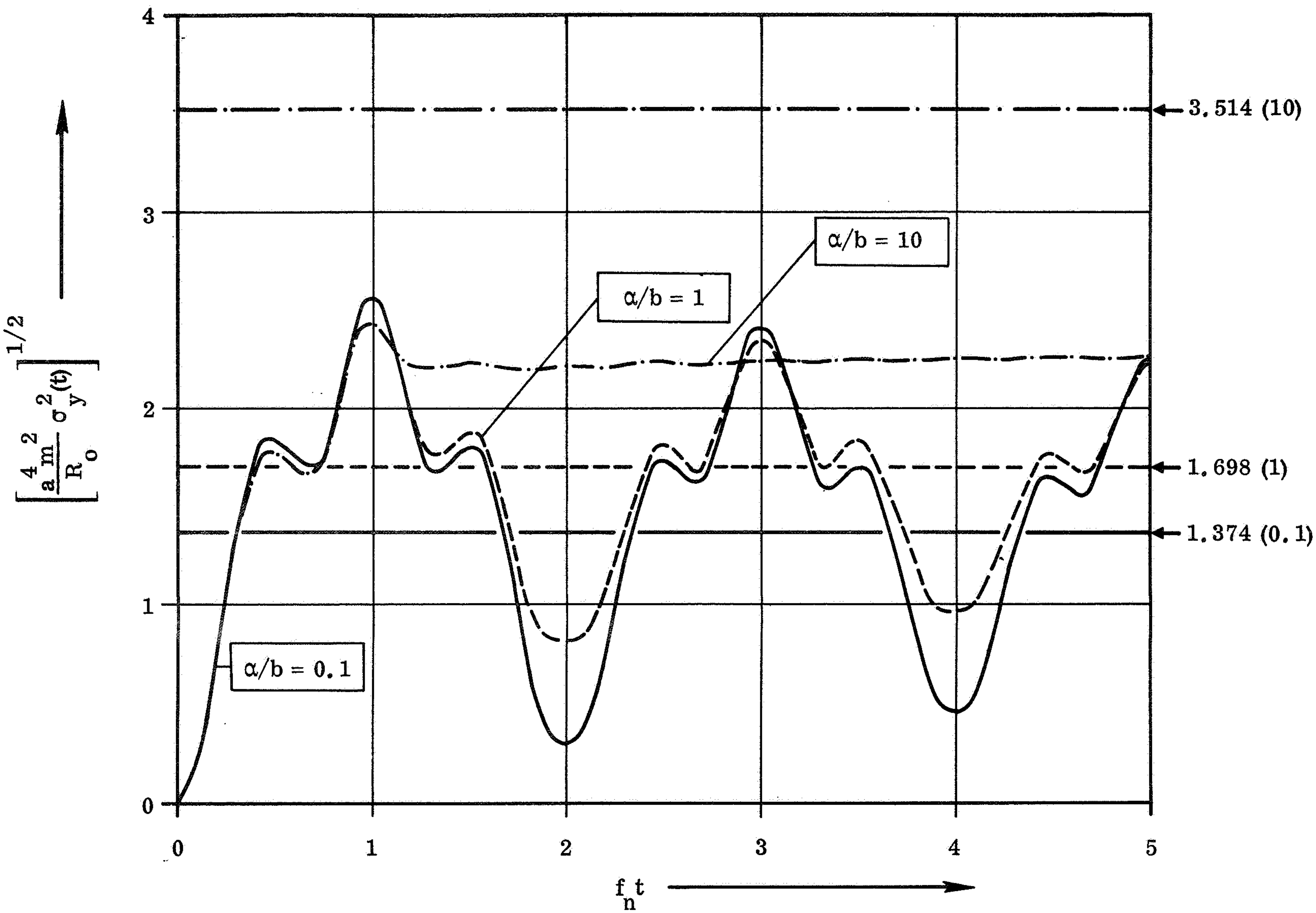

Figure 5. 29. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 0.1$, $Q = 50$, $\rho/a = 0.5$

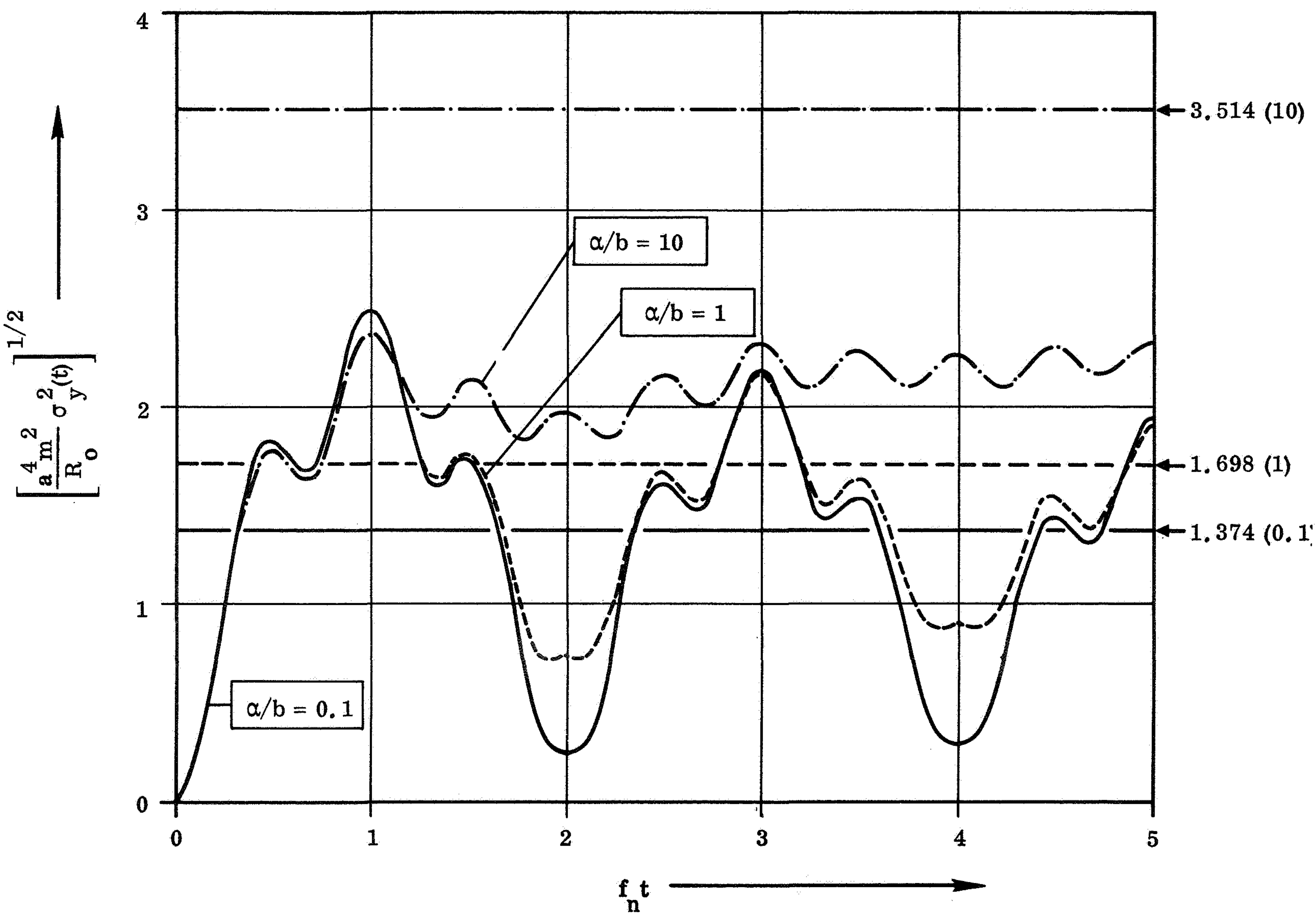

Figure 5.30. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 1$, $Q = 50$, $\rho/a = 0.5$

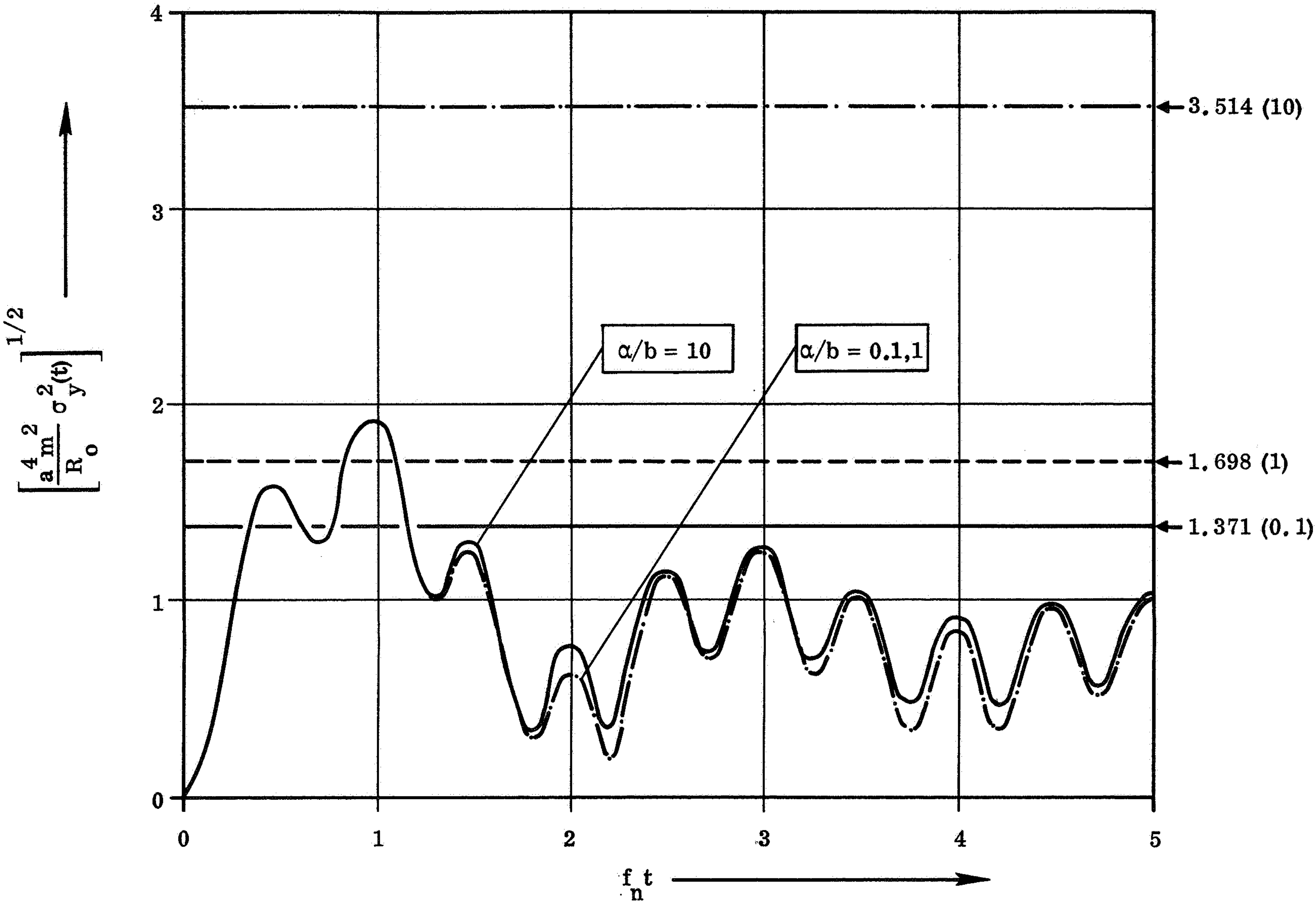

Figure 5.31. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 10$, $Q = 50$, $\rho/a = 0.5$

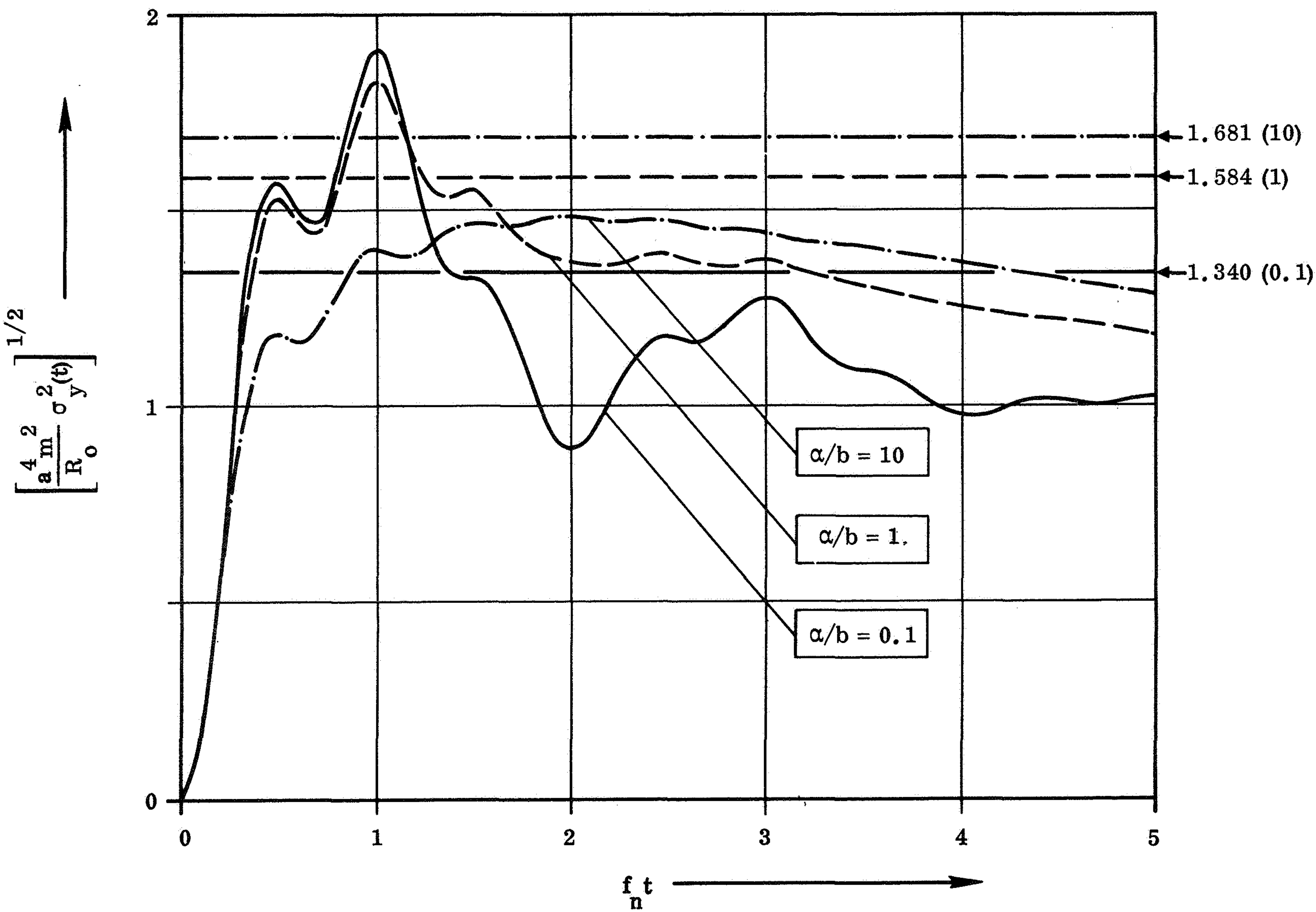

Figure 5.32. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 0.1$, $Q = 5$, $\rho/a = 0.5$

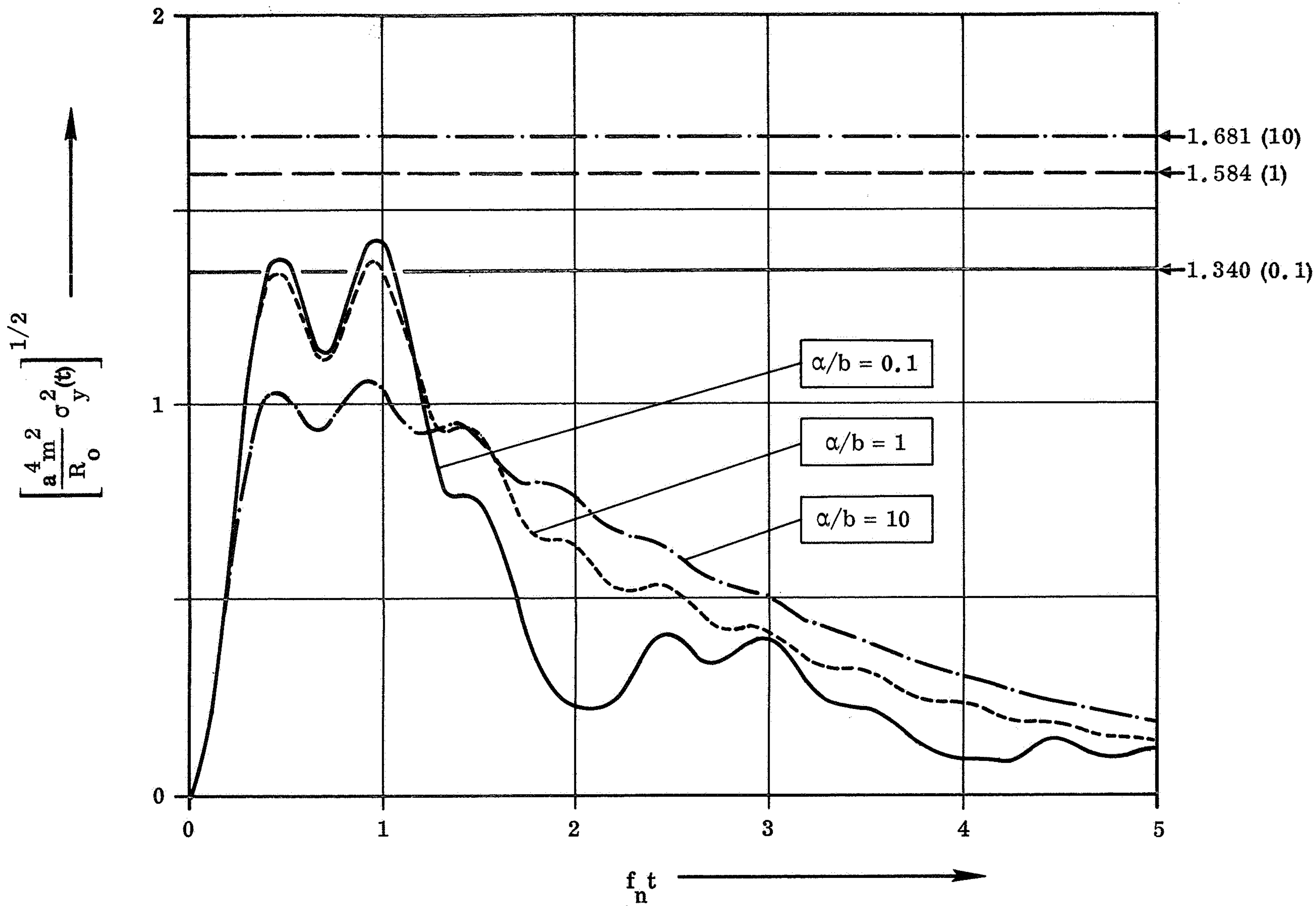

Figure 5.33. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 1$, $Q = 5$, $\rho/a = 0.5$

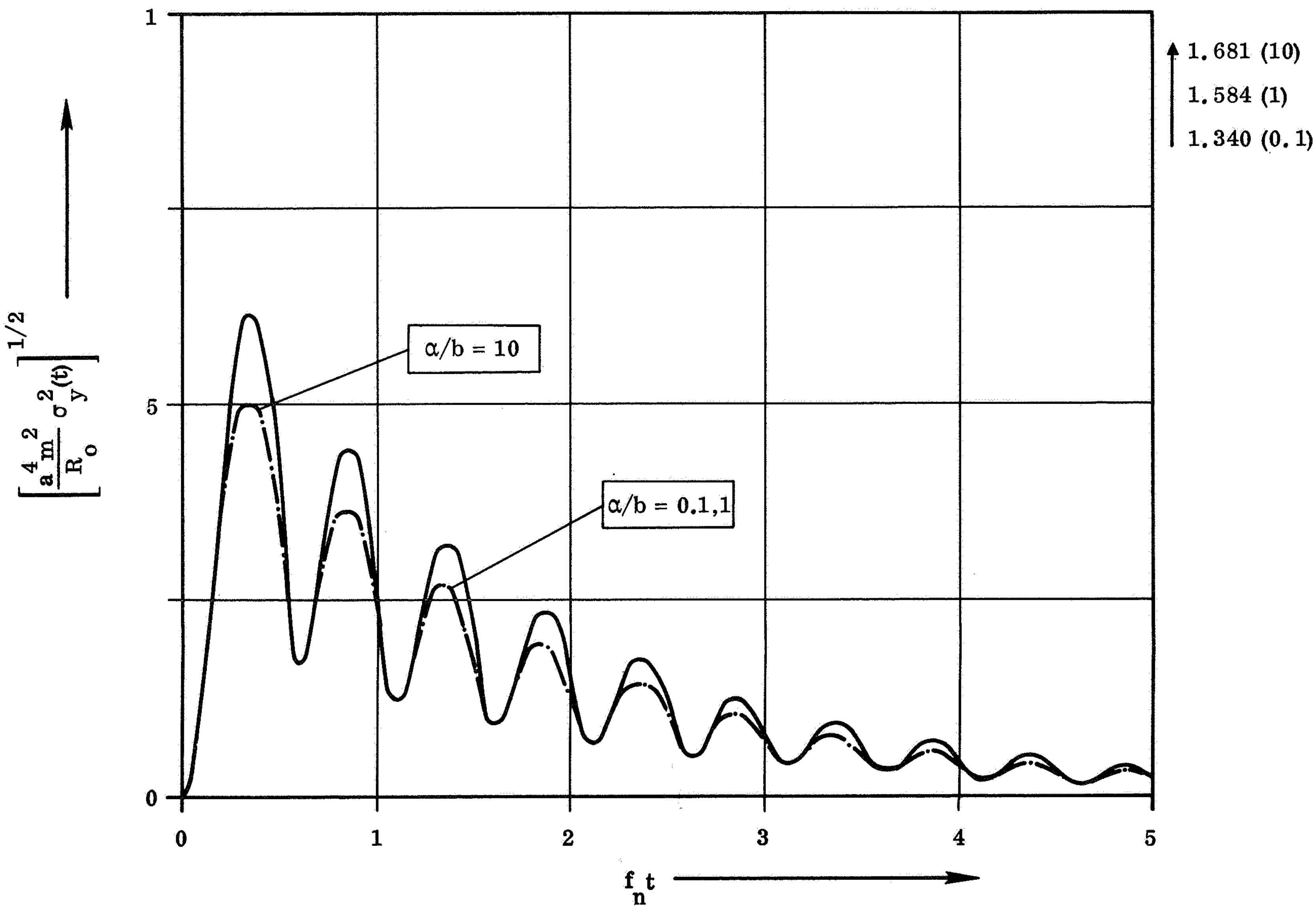

Figure 5.34. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 10$, $Q = 5$, $\rho/a = 0.5$

The companion shaping filter $K_\gamma(t, \omega)$ at selected values of both time and α/b is shown as Figures 5.35 through 5.43. With increased γ, $K_\gamma(t, \omega)$ becomes less oscillatory and tends to a constant; such is illustrated rather strikingly by a comparison of Figures 5.35, 5.36, 5.37 and Figures 5.41, 5.42, 5.43 for $Q = 5$. Note that $K_\gamma(t, \omega)$ is more selective for $\gamma/b = 1$ than for $\gamma/b = 0.1$, an apparent anomaly. This can be explained upon inspection of Eqn (4.36) where we note the terms $b - \gamma$ throughout the expression. For $\gamma/b = 1$, these terms reduce to zero and serve to lessen the effect of system damping on $\sigma_y(t)$. A more complete set of plots for $K_\gamma(t, \omega)$ is shown in Appendix C as Figures C.19 through C.45.

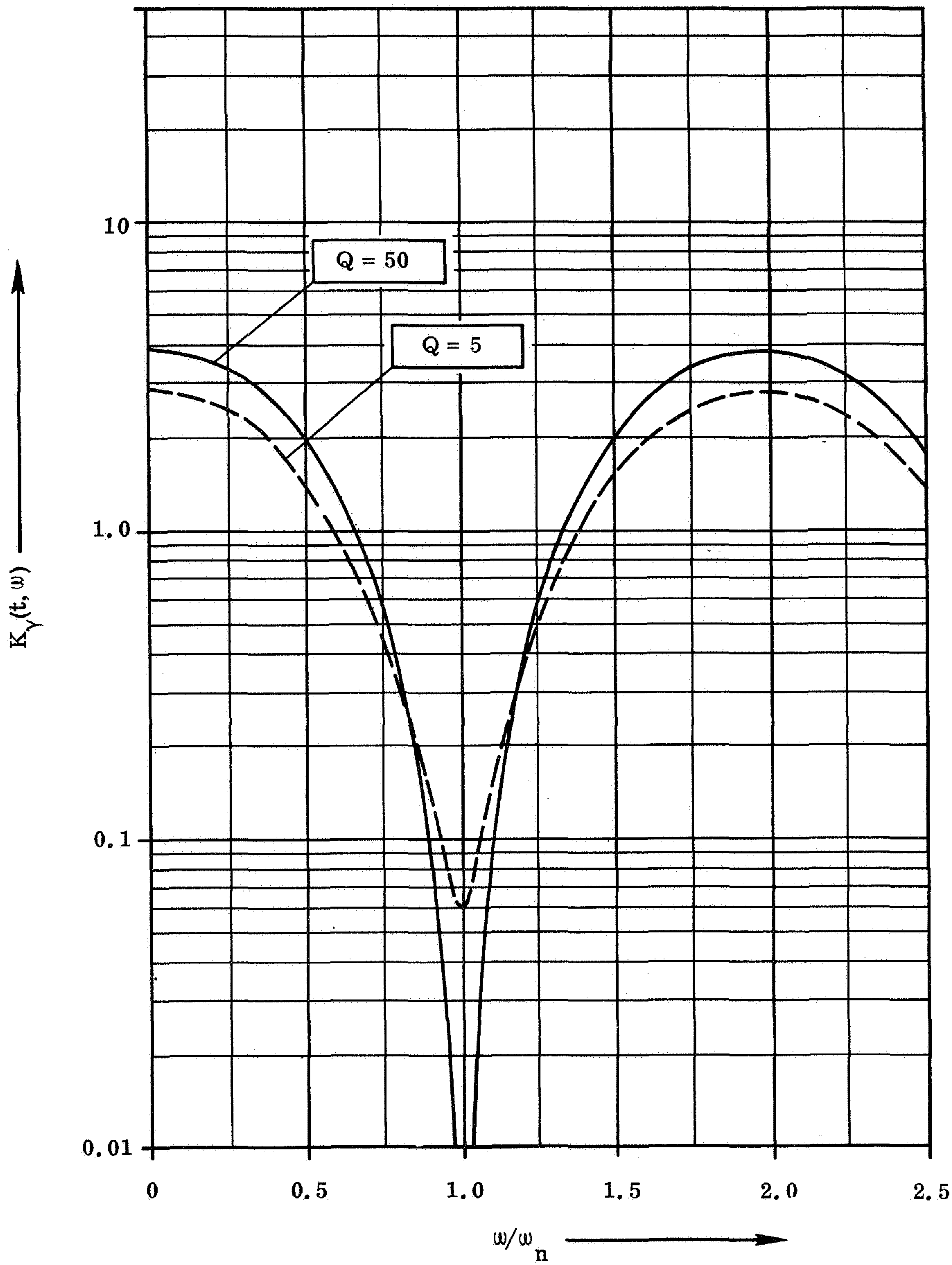

Figure 5.35. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.5$, $\gamma/b = 0.1$

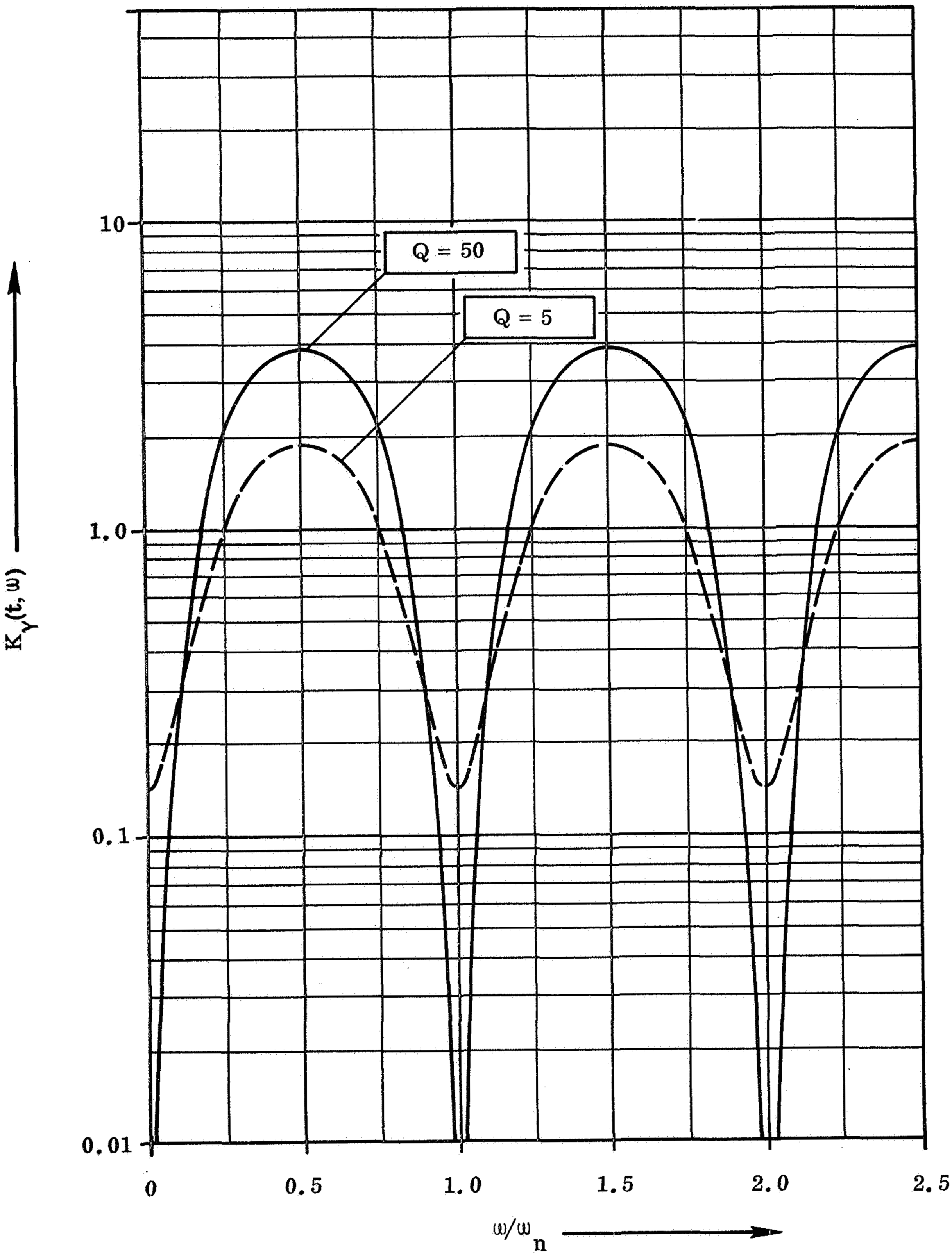

Figure 5.36. Shaping Filter $K_{\gamma}(t, \omega)$ with $f_n t = 1.0$, $\gamma/b = 0.1$

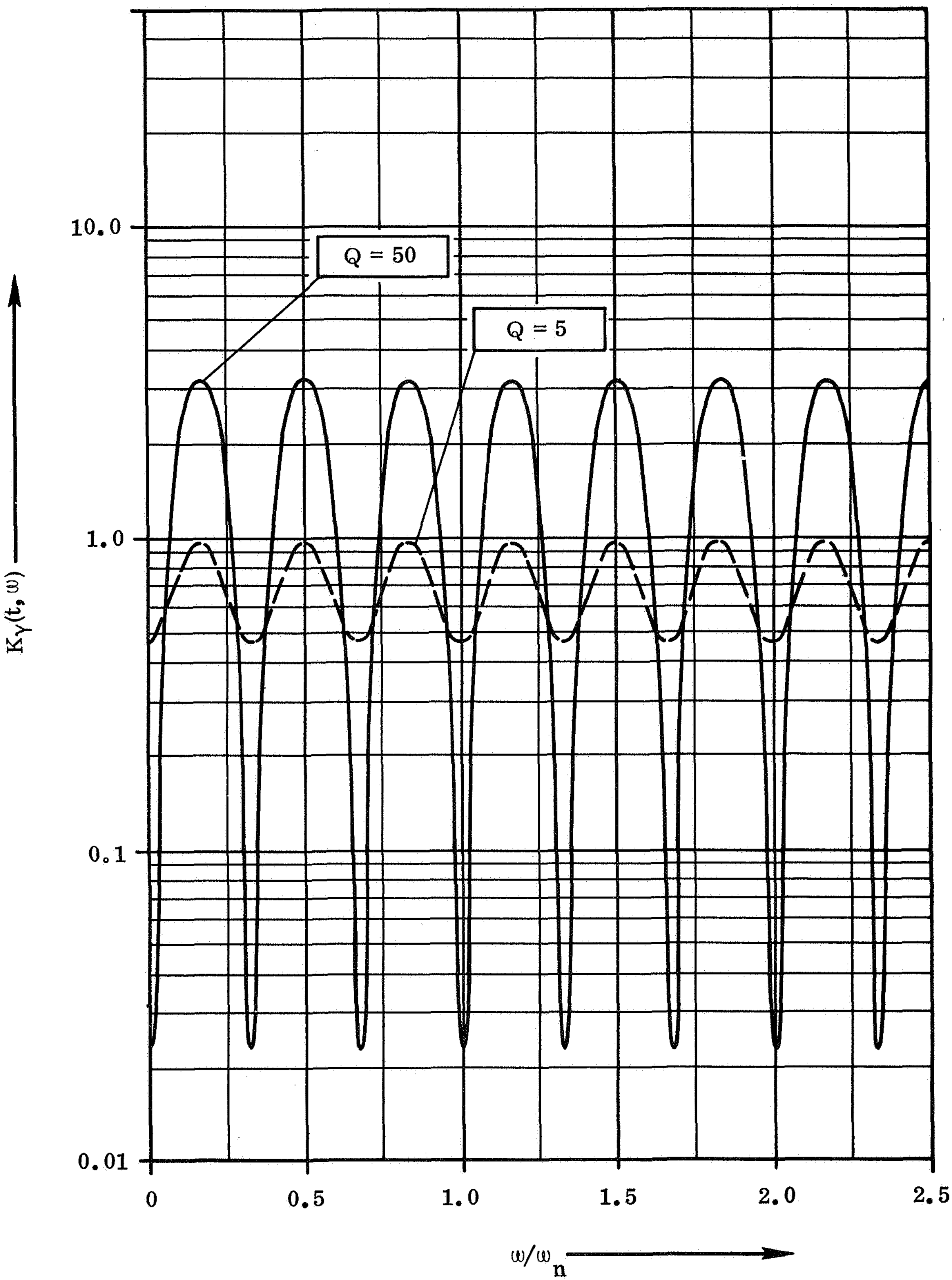

Figure 5.37. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 3.0$, $\gamma/b = 0.1$

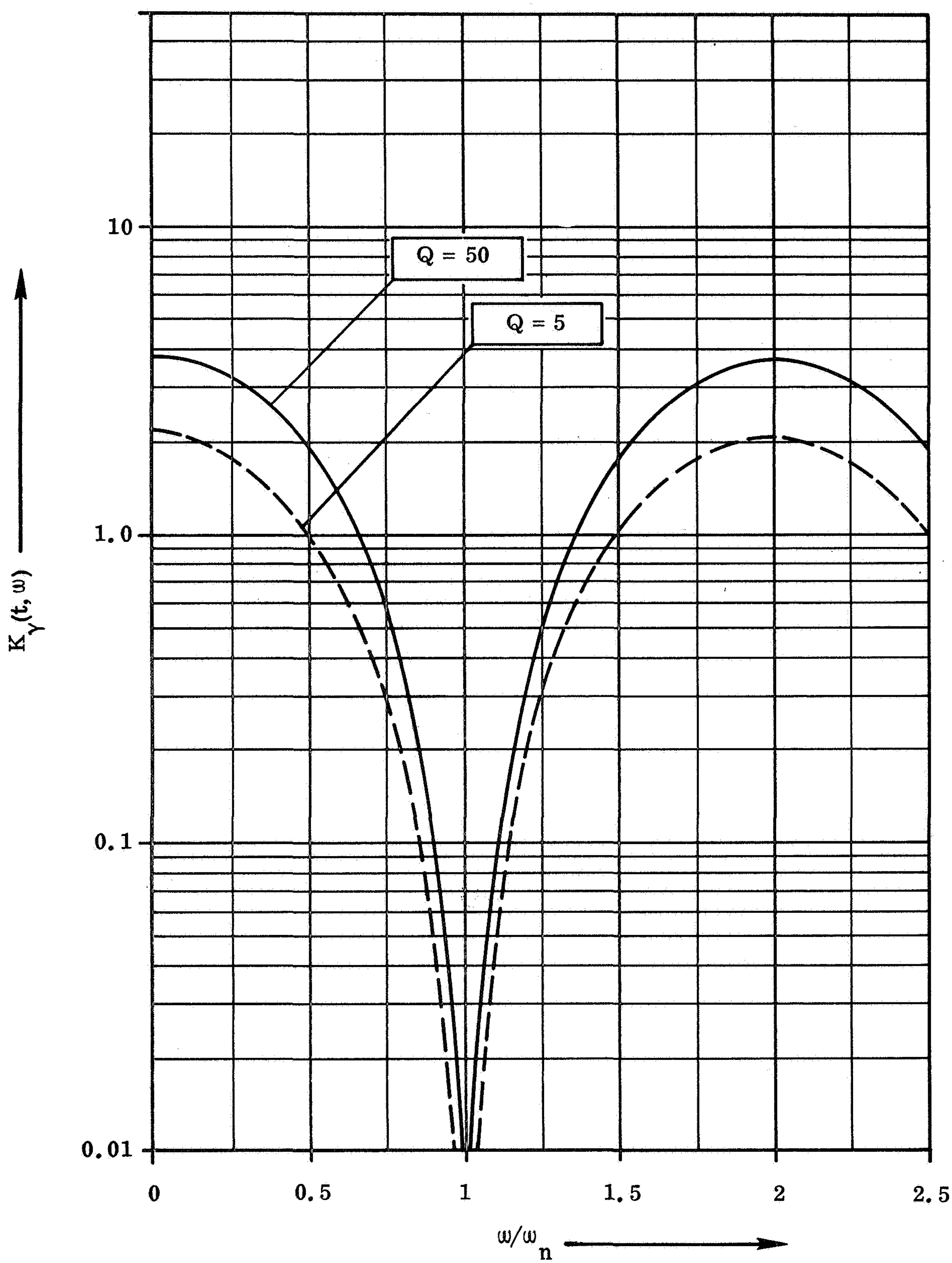

Figure 5.38. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.5$, $\gamma/b = 1.0$

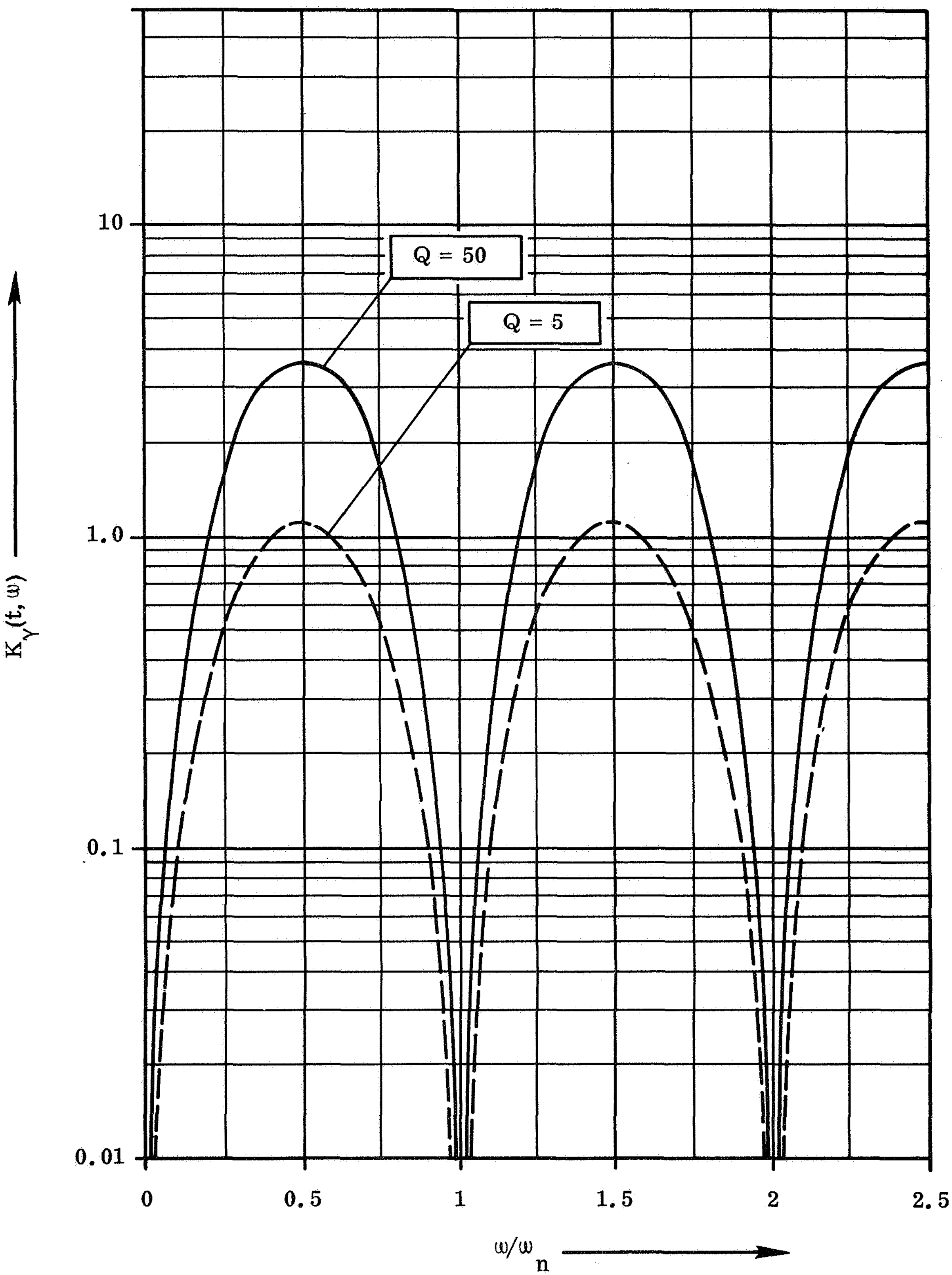

Figure 5.39. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 1.0$, $\gamma/b = 1:0$

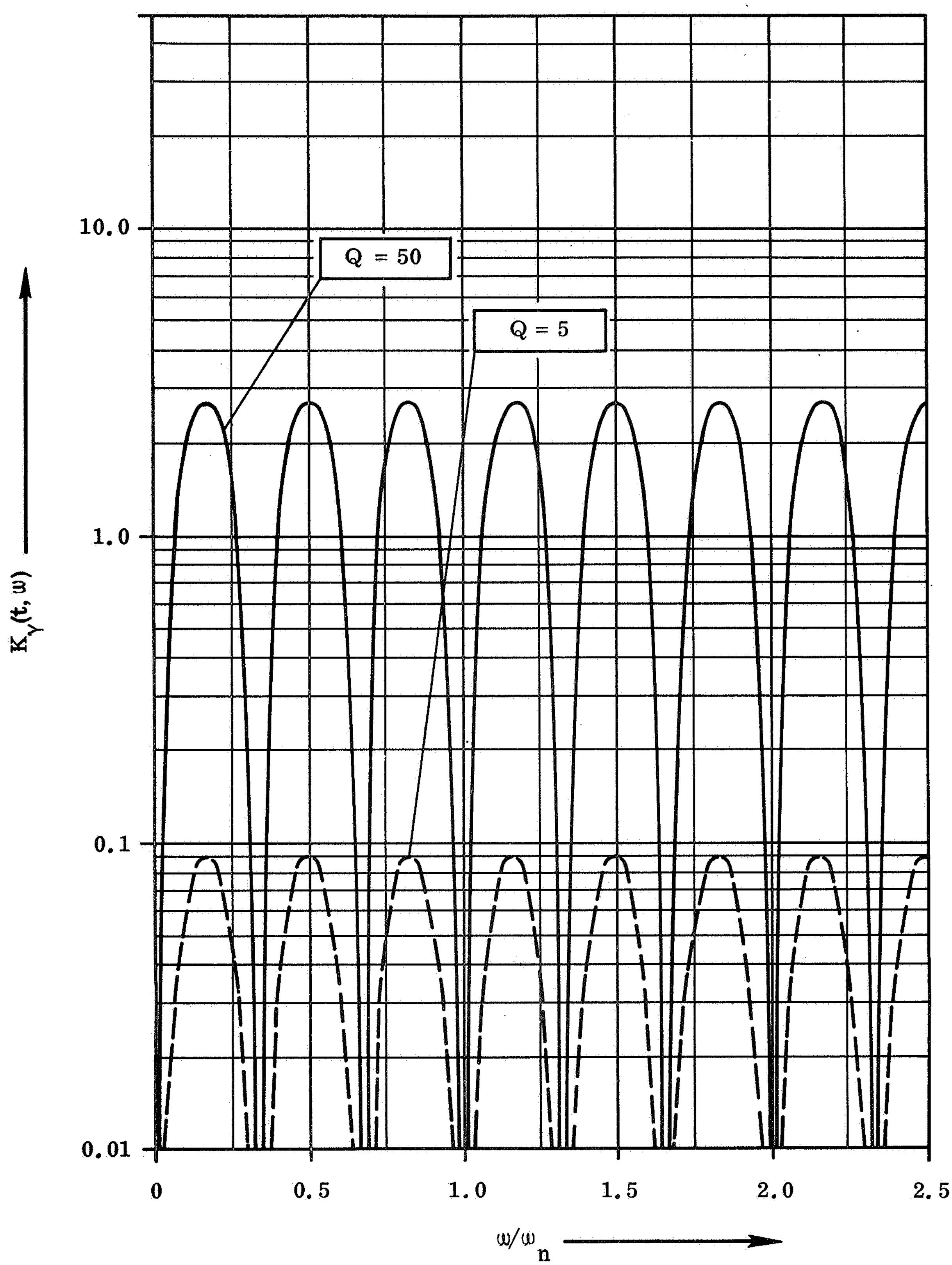

Figure 5.40. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 3.0$, $\gamma/b = 1.0$

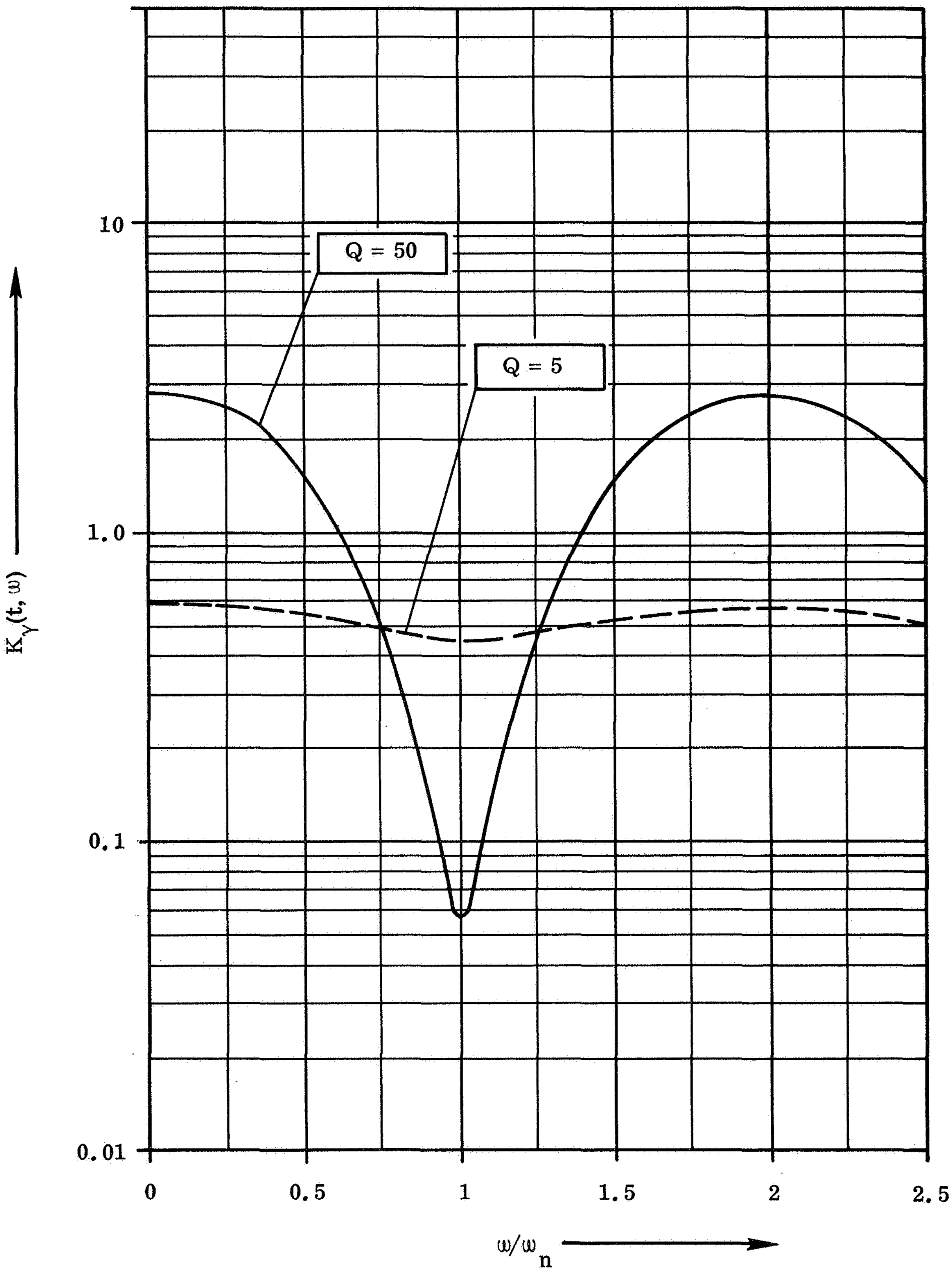

Figure 5.41. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.5$, $\gamma/b = 10$

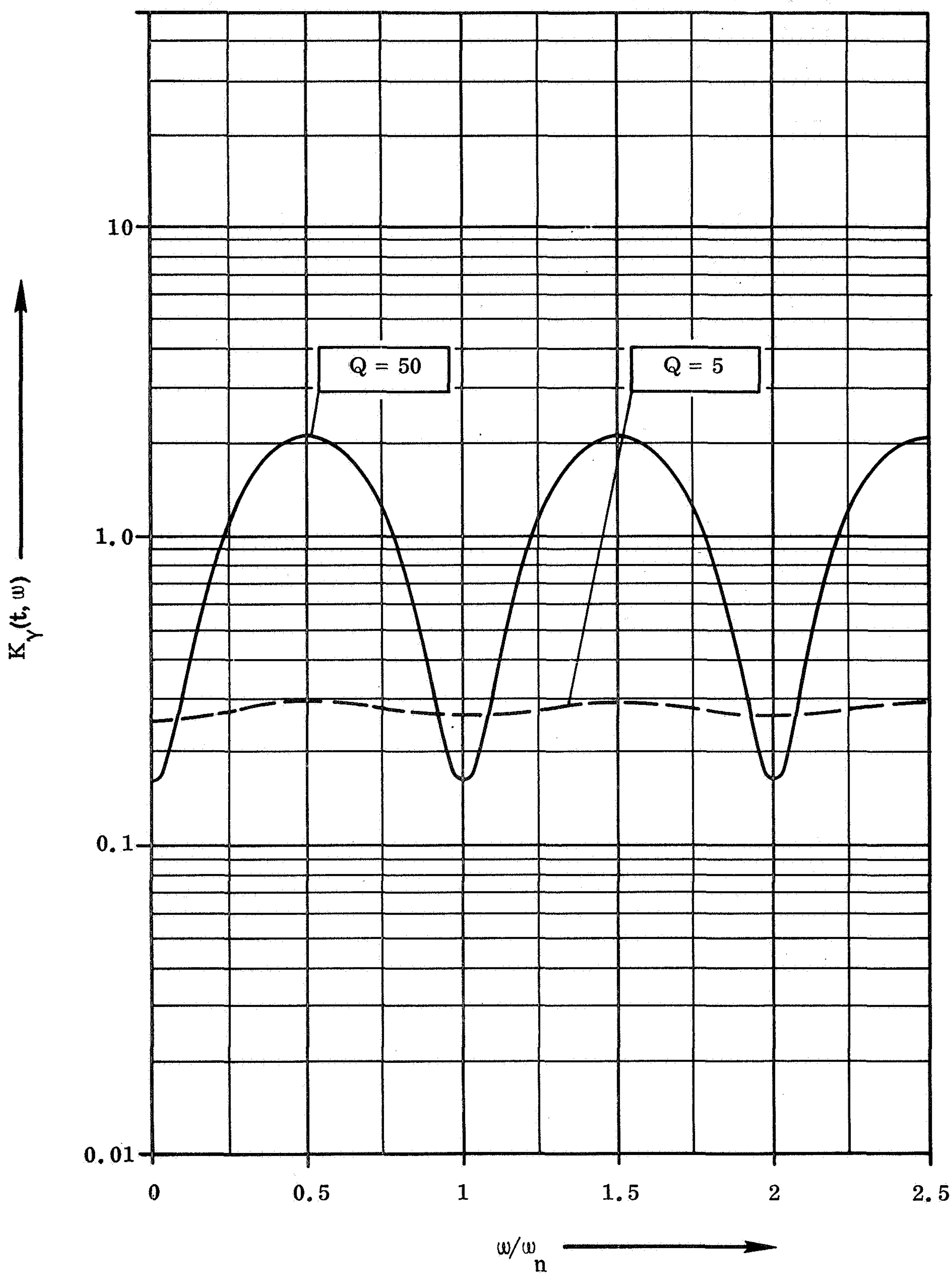

Figure 5.42. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 1.0$, $\gamma/b = 10$

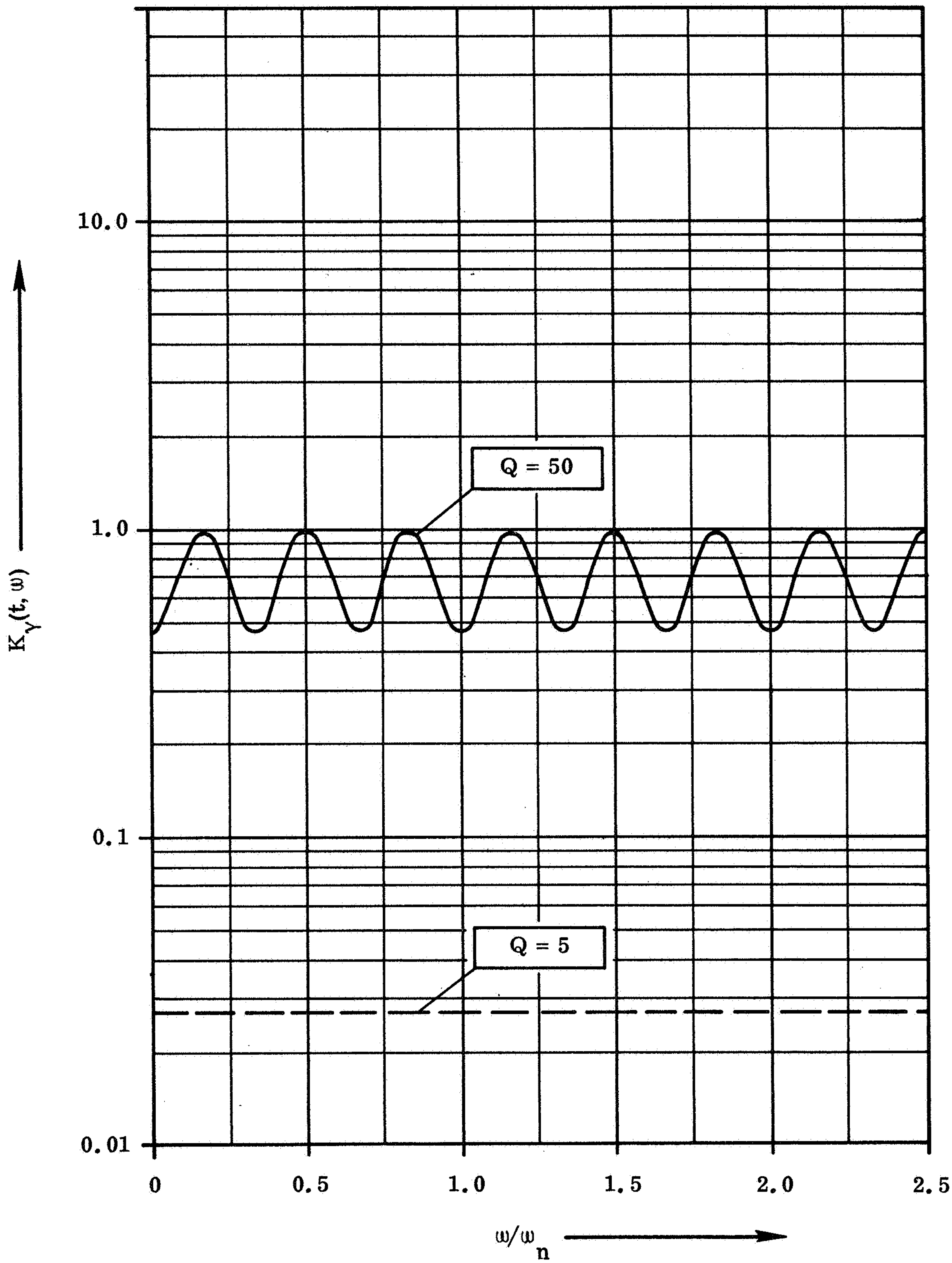

Figure 5.43. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 3.0$, $\gamma/\bar{b} = 10$

6.0 CONCLUDING REMARKS

The detailed mathematics which lead to the solutions presented here are not included as an integral part of this report as they admittedly are cumbersome and lengthy, and do not necessarily contribute to a direct understanding of the problem. One is reminded, however, that attention to and persistence with mathematical detail are necessary attributes in the evaluation of the integrals defined in the text. For the more than casually interested reader, a listing of the major integrals encountered and their solutions are included as Appendix A.

The noise burst model $f(t) = e(t)\, n(t)$ is considered representative of gusts, earthquake and pyro-shock environments. As such, its subsequent use in analytical studies has practical significance both in design and in response prediction. Since this form of input is amenable to implementation in the laboratory, its use as an "equivalent" shock test for the above environments has appeal. As this study provides much of the theoretical foundation in support of such ideas, let us make note of the more leading conclusions.

Granted the assumptions of system linearity and the product form of the input, and given the parametric variations $.1 \le \rho/a \le 10$ and $.1 \le \alpha/b \le 10$ with $5 \le Q \le 50$:

- stationary response values for correlated noise may be determined (see Eqn 4.1 along with Figures 5.1 and 5.2)
- response overshoot is not expected for a noise input which is nearly white
- for a correlated noise input, response overshoot is not expected for $\alpha/b > 5$ over the range $0.1 \le \rho/a \le 10$ (see Figures 5.3 and 5.4)

- for correlated noise where $0.1 \leq \alpha/b \leq 5$, response overshoot estimates may be made (see Figures 5.3 and 5.4)

- response maxima may be predicted for the probabilities $\hat{P}_M(|\beta| \leq \beta_o; t_o) = 0.50$ and $\hat{P}_M(|\beta| \leq \beta_o; t_o) = 0.95$; intermediate probability values may be estimated (see Figures 2.1 through 2.6)

- since the residual response of rectangular modulation may exceed the peak value of $\sigma_y(t)$, caution should be exercised in its application

- response overshoot may be controlled for exponentially damped modulation by varying either the decay coefficient γ or the system damping Q, or both

An explanation of (mean square) response overshoot due to amplitude modulated noise bursts is a nontrivial task, particularly for correlated noise. Overshoot is dependent upon the relative interaction of the system parameters, the noise parameters, and the envelope modulation function; it is governed mainly by its corresponding "shaping" filter which involves properties of both the system and the modulation envelope. For certain conditions, by shaping (in time) the manner in which correlated noise is applied, the system response $\sigma_y(t)$ can be made to exceed its stationary value by well over a factor of two.

Response overshoot may be avoided for the three modulation functions provided the input noise appears to the system as nearly white. For the parametric variation $0.1 \leq \rho/a \leq 10$, $\alpha/b \leq 5$ satisfies this requirement for $5 \leq Q \leq 50$. Overshoot also may be eliminated by "tuning" to resonance the noise correlation function and the system frequency response function; this condition produces a maximum value in the stationary response of the system to correlated noise.

Before the concepts here can be amalgamated into an effective engineering tool for system design in random shock environments, the following need be resolved:

- feasibility of using the input model $f(t) = e(t)n(t)$ with standard laboratory test equipment must be established
- ranges of practical values for the α and ρ parameters need be determined from measured data
- the shape(s) for a practical modulation envelope must be verified and, if necessary, additional analyses carried out

Recalling modulation shapes commonly used for deterministic shock inputs, it appears natural to include (1) the half-sine and (2) some form of ramp function as modulation functions in future analyses.

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3. Barnoski, R.L., and Maurer, J.R., "Mean Square Response of Simple Mechanical Systems to Nonstationary Random Excitation," Transactions of the ASME, Journal of Applied Mechanics, Vol. 36, Series E, No. 2, June 1969.

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APPENDIX A

TABLE OF SELECTED INTEGRALS

This table of integrals is a partial list of integrals encountered in using a frequency domain approach to determine the mean square response of single degree of freedom systems to random excitation. In most instances, the integrand is written in terms of basic system frequency response function $H(\omega)$, and a spectral density function $S(\omega)$, both defined below.

Basic system properties;

$$H(\omega) = \frac{1}{(\omega_o^2 - \omega^2) + i2\zeta\omega_o\omega} = -\frac{1}{(\omega - s_1)(\omega - s_2)}$$

$$\omega_o^2 = a^2 + b^2 = \frac{k}{m}$$

$$s_1 = a + ib = -s_2^*$$

$$a = \omega_o \left[1 - \zeta^2\right]^{1/2}$$

$$b = \zeta\omega_o$$

$$\zeta = \text{system damping factor}$$

$$H^*(\omega) = \text{conjugate of } H(\omega)$$

$$|H(\omega)|^2 = H(\omega)\,H^*(\omega) = \frac{1}{(\omega^2 - s_1^2)(\omega^2 - s_2^2)}$$

Spectral density function for the input noise (see note below);

$$S(\omega) = \frac{\alpha(\rho^2 + \alpha^2 + \omega^2)}{\pi(\omega^2 - s_3^2)(\omega^2 - s_4^3)}$$

$$s_3 = \rho + i\alpha = -s_4^*$$

U(t) is the unit step function

$$U(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

$\delta(t)$ is the unit impulse function defined by

$$\int_{-\infty}^{\infty} \delta(t - t_o)\, f(t)dt = f(t_o)$$

Note: $S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)\, e^{i\omega\tau}\, d\tau$ where $R(\tau) = e^{-\alpha|\tau|} \cos \rho\tau$

Notation pertinent to the Table of Integrals which begin on the following page

$$A(t) = e^{-2bt} (1 + \frac{b}{a} \sin 2at)$$

$$B(t) = e^{-2bt} \sin^2 at$$

$$C(t) = e^{-bt} (\cos at + \frac{b}{a} \sin at)$$

$$D(t) = e^{-bt} \sin at$$

$$E(t) = A(t) + A(t - t_o)$$

$$F(t) = B(t) + B(t - t_o)$$

$$G(t) = C(t)\, C(t - t_o)$$

$$J(t) = D(t)\, D(t - t_o)$$

$$K(t) = C(t - t_o)\, D(t) - C(t)\, D(t - t_o)$$

$$A_\gamma(t) = e^{-2(b - \gamma)t} (1 + \frac{b - \gamma}{a} \sin 2at)$$

$$B_\gamma(t) = e^{-2(b - \gamma)t} \sin^2 at$$

$$C_\gamma(t) = e^{-(b - \gamma)t} (\cos at + \frac{b - \gamma}{a} \sin at)$$

$$D_\gamma(t) = e^{-(b - \gamma)t} \sin at$$

TABLE OF INTEGRALS

(1) $$\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega - s_1} d\omega = 2\pi i \; e^{is_1 t} U(t)$$

$$= 2\pi i \; e^{-bt} (\cos at + i \sin at) \; U(t)$$

(2) $$\int_{-\infty}^{\infty} \left[\frac{1}{(\omega - s_1)(\omega' - s_2)} - \frac{1}{(\omega - s_2)(\omega' - s_1)} \right] e^{i\omega' t} d\omega' = -4\pi i a H(\omega) \; e^{-bt} \left(\cos at + \frac{b + i\omega}{a} \sin at\right) U(t)$$

(3) $$\int_{-\infty}^{\infty} \left[\pi\delta(\omega' - \omega) + \frac{1}{i(\omega' - \omega)} \right] e^{i\omega' t} d\omega' = 2\pi e^{i\omega t} U(t)$$

(4) $$\int_{-\infty}^{\infty} H(\omega) \; e^{i\omega t} d\omega = \frac{2\pi}{a} e^{-bt} \sin at \; U(t)$$

(5) $$\int_{-\infty}^{\infty} H(\omega') \left[\pi\delta(\omega' - \omega) + \frac{1}{i(\omega' - \omega)} \right] e^{i\omega' t} d\omega' = 2\pi H(\omega) \left[e^{i\omega t} - e^{-bt} \left(\cos at + \frac{b + i\omega}{a} \sin at\right) \right] U(t)$$

$$(6)\quad \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega = \frac{\pi}{2b\omega_o^2}$$

$$(7)\quad \int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 \, d\omega = \frac{\pi}{2b}$$

$$(8)\quad \int_{-\infty}^{\infty} (b^2 - a^2 + \omega^2) \, |H(\omega)|^2 \, d\omega = \frac{\pi b}{\omega_o^2}$$

$$(9)\quad \int_{-\infty}^{\infty} \omega^4 |H(\omega)|^2 \, d\omega = \frac{\pi}{2b} (a^2 - 3b^2)$$

$$(10)\quad \int_{-\infty}^{\infty} [C(t) \cos \omega t + \frac{\omega}{a} D(t) \sin \omega t] \, |H(\omega)|^2 \, d\omega = \frac{\pi}{2b\omega_o^2} e^{-bt} \left[C(t) \left(\cos at + \frac{b}{a} \sin at\right) + \left(\frac{\omega_o}{a}\right)^2 D(t) \sin at \right] U(t)$$

$$(11)\quad \int_{-\infty}^{\infty} (\omega^2 \cos \omega t)\ |H(\omega)|^2\, d\omega = \frac{\pi}{2b} e^{-bt} [\cos at - \frac{b}{a} \sin at]$$

$$(12)\quad \int_{-\infty}^{\infty} \left\{ 1 + A(t) + \left(\frac{b^2 - a^2 + \omega^2}{a^2} \right) B(t) - 2\,(C(t) \cos \omega t + \frac{\omega}{a} D(t) \sin \omega t) \right\} |H(\omega)|^2$$

$$= \frac{\pi}{2b\omega_o^2} \left[1 - e^{-2bt} \left(1 + \frac{b}{a} \sin 2at + \frac{2b^2}{a^2} \sin^2 at \right) \right] = \frac{\pi}{2b\omega_o^2} \left[1 - A(t) - \frac{2b^2}{a^2} B(t) \right]$$

$$(13)\quad \int_{-\infty}^{\infty} \left\{ E(t) + \frac{b^2 - a^2 + \omega^2}{a^2} F(t) - 2 \left[G(t) + \frac{\omega^2}{a^2} J(t) \right] \cos \omega t_o - \frac{2\omega}{a} K(t) \sin \omega t_o \right\} |H(\omega)|^2\, d\omega$$

$$= \frac{\pi}{2b\omega_o^2} \left[A(t - t_o) - A(t) + \frac{2b^2}{a^2} \left(B(t - t_o) - B(t) \right) \right]$$

$$(14)\quad \int_{-\infty}^{\infty} |H(\omega + i\gamma)|^2 \, d\omega = \frac{\pi}{2|b-\gamma| \left[a^2 + (b-\gamma)^2\right]}, \qquad \text{for} \quad \gamma \neq b$$

$$(15)\quad \int_{-\infty}^{\infty} \frac{(b-\gamma)^2 - a^2 + \omega^2}{a^2} |H(\omega + i\gamma)|^2 \, d\omega = + \frac{\pi |b-\gamma|}{a^2\left[a^2 + (b-\gamma)^2\right]}, \quad \text{for} \quad \gamma \neq b$$

$$(16)\quad \int_{-\infty}^{\infty} \left[C \cos \omega t + \frac{\omega}{a} D \sin \omega t\right] |H(\omega + i\gamma)|^2 \, d\omega$$

$$= \frac{\pi e^{-|b-\gamma|t}}{2|b-\gamma| \left[a^2 + (b-\gamma)^2\right]} \left[C\left(\cos at + \frac{|b-\gamma|}{a} \sin at\right) + \frac{a^2 + (b-\gamma)^2}{a^2} D \sin at\right], \quad \gamma \neq b$$

$$(17)\quad \int_{-\infty}^{\infty} \left[1 + A_\gamma(t) + \frac{(b-\gamma)^2 - a^2 + \omega^2}{a^2} B_\gamma(t) - 2C_\gamma(t) \cos \omega t - 2\frac{\omega}{a} D_\gamma(t) \sin \omega t\right] |H(\omega + i\gamma)|^2 \, d\omega$$

$$= \begin{cases} \dfrac{\pi}{2(b-\gamma)\left(a^2 + (b-\gamma)^2\right)} \left[1 - A_\gamma(t) - \dfrac{2(b-\gamma)^2}{a^2} B_\gamma(t)\right], & \text{for} \quad b \neq \gamma \\[2ex] \dfrac{\pi}{a^2} \left[t - \dfrac{1}{2a} \sin 2at\right], & \text{for} \quad b = \gamma \end{cases}$$

$$(18) \quad \int_{-\infty}^{\infty} S(\omega) \; |H(\omega)|^2 \, d\omega = \frac{a}{2b} R_1 + R_3 , \qquad \text{for } s_1 \neq s_3$$

$$\text{where:} \quad R_1 = \text{Re} \left\{ \frac{\alpha(s_1^2 + \alpha^2 + \rho^2)}{a^2 s_1 \, (s_1^2 - s_3^2) \, (s_1^2 - s_4^2)} \right\}$$

$$R_3 = \text{Re} \left\{ \frac{1}{(s_3^2 - s_1^2) \, (s_3^2 - s_2^2)} \right\}$$

$$(19) \quad \int_{\infty}^{\infty} \frac{b^2 - a^2 + \omega^2}{a^2} S(\omega) \; |H(\omega)|^2 \, d\omega = -X_1 + \frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2} R_3 - \frac{2\rho\alpha}{a^2} X_3 , \qquad \text{for } s_1 \neq s_3$$

$$\text{where:} \quad X_1 = \text{Im} \left\{ \frac{\alpha(s_1^2 + \alpha^2 + \rho^2)}{a^2 s_1 \, (s_1^2 - s_3^2) \, (s_1^2 - s_4^2)} \right\}$$

$$X_3 = \text{Im} \left\{ \frac{1}{(s_3^2 - s_1^2) \, (s_3^2 - s_2^2)} \right\}$$

[See (18) for R_3]

$$(20) \quad \int_{\infty}^{\infty} (C \cos \omega t + \frac{\omega}{a} D \sin \omega t)\, S(\omega)\, |H(\omega)|^2\, d\omega = \frac{a}{2b} e^{-bt} \Big\{ R_1 \Big[(C + \frac{b}{a} D) \cos at + D \sin at \Big]$$

$$+ X_1 \Big[D \cos at - (C + \frac{b}{a} D) \sin at \Big] \Big\} + e^{-\alpha t} \Big\{ R_3 \Big[(C + \frac{\alpha}{a} D) \cos \rho t + \frac{\rho}{a} D \sin \rho t \Big]$$

$$+ X_3 \Big[\frac{\rho}{a} D \cos \rho t - (C + \frac{\alpha}{a} D) \sin \rho t \Big] \Big\}, \qquad \text{for } s_1 \neq s_3$$

[See (18), (19) for R_1, R_3, X_1, X_3]

For $s_1 = s_3$, ($a = \rho$ and $\alpha = b$) the integrals (18), (19) and (20) reduce to equations (21), (22) and (23) respectively.

$$(21) \quad \int_{-\infty}^{\infty} \frac{b(\omega^2 + \omega_o^2)}{\pi} |H(\omega)|^4\, d\omega = \frac{a^2 + 3b^2}{8\, b^2 \omega_o^4}$$

$$(22) \quad \int_{-\infty}^{\infty} \frac{b(b^2 - a^2 + \omega^2)}{\pi a^2} (\omega^2 + \omega_o^2)\, |H(\omega)|^4\, d\omega = \frac{b^2}{2a^2 \omega_o^4}$$

$$(23) \quad \int_{-\infty}^{\infty} \left(C \cos \omega t + \frac{\omega}{a} D \sin \omega t\right) (\omega^2 + \omega_o^2) \, |H(\omega)|^4 \, d\omega = \frac{\pi}{8b\omega_o^2} e^{-bt} \left[\frac{C}{b^2} \left(\frac{a^2 + 3b^2}{\omega_o^2} + bt \right) \cos at \right.$$

$$\left. + \frac{C}{ab} \left(\frac{2b^2}{\omega_o^2} + bt \right) \sin at + \frac{\omega_o^2}{a^2 b^2} D (1 + bt) \sin at \right]$$

$$(24) \quad \int_{-\infty}^{\infty} (\omega^2 \cos \omega t) \, S_n(\omega) \, |H(\omega)|^2 \, d\omega = \frac{a}{2b} e^{-bt} \left[(a^2 - b^2) \cos at - 2ab \sin at \right] R_1$$

$$- \frac{a}{2b} e^{-bt} \left[2ab \cos at + (a^2 - b^2) \sin at \right] X_1$$

$$+ e^{-\alpha t} \left[(\rho^2 - \alpha^2) \cos \rho t - 2\rho\alpha \sin \rho t \right] R_3$$

$$- e^{-\alpha t} \left[2\rho\alpha \cos \rho t + (\rho^2 - \alpha^2) \sin \rho t \right] X_3, \quad \text{for } s_1 \neq s_2$$

[See (18), (19) for R_1, R_3, X_1, X_3]

$$(25)\quad \int_{-\infty}^{\infty} \omega^2 (\omega^2 + \omega_o^2) \cos \omega t \, |H(\omega)|^4 \, d\omega = \frac{\pi}{16b^3} e^{-bt} \left[\left(2 + \frac{b^2}{a^2} + 2bt \right) \cos at - \frac{b}{a} (1 + 2bt) \sin at \right]$$

$$(26)\quad \int_{-\infty}^{\infty} \left[1 + A(t) + \frac{b^2 - a^2 + \omega^2}{a^2} B(t) - 2\, C(t) \cos \omega t - 2 \frac{\omega}{a} D(t) \sin \omega t \right] S(\omega) \, |H(\omega)|^2 \, d\omega$$

$$= \begin{cases} \left[R_1 T_1 - X_1 T_2 + R_3 T_3 - X_3 T_4 \right], & \text{for } s_1 \neq s_3 \\ \dfrac{1}{8\omega_o^4} \left[\left(\dfrac{a^2}{b^2} + 3 \right) - \left(\dfrac{a^2}{b^2} + 3 + \dfrac{2\omega_o^2 t}{b} \right) e^{-2bt} - \dfrac{2}{a} \left(b + \omega_o^2 t \right) e^{-2bt} \sin 2at \right. & \\ \qquad \left. + 2 \left(1 - \dfrac{b^2}{a^2} - \dfrac{2b\omega_o^2 t}{a^2} \right) e^{-2bt} \sin^2 at \right], & \text{for } s_1 = s_3 \end{cases}$$

where:

$$T_1 = \frac{a}{2b} \left[1 - A(t) \right]$$

$$T_2 = -B(t)$$

$$T_3 = \left[1 + A(t) + \frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2} B(t) - 2\left(C(t) + \frac{\alpha}{a} D(t)\right) e^{-\alpha t} \cos \rho t - \frac{2\rho}{a} D(t)\, e^{-\alpha t} \sin \rho t\right]$$

$$T_4 = 2\left[\frac{\rho\alpha}{a^2} B(t) - \left(C(t) + \frac{\alpha}{a} D(t)\right) e^{-\alpha t} \sin \rho t + \frac{\rho}{a} D(t)\, e^{-\alpha t} \cos \rho t\right]$$

[See (18), (19) for R_1, R_3, X_1, X_3]

(27) $$\int_{-\infty}^{\infty} \left[E(t) + \frac{b^2 - a^2 + \omega^2}{a^2} F(t) - 2\left(G(t) + \frac{\omega^2}{a^2} J(t)\right) \cos \omega t_o - 2\frac{\omega}{a} K(t) \sin \omega t_o\right] S(\omega)\, |H(\omega)|^2\, d\omega$$

$$= \begin{cases} R_1 T_a - X_1 T_b + R_3 T_c - X_3 T_d, & \text{for } s_1 \neq s_3 \\[2ex] \dfrac{1}{8b^2\omega_o^4}\left\{\left(a^2 + 3b^2\right) E(t) + \dfrac{4b^4}{a^2} F(t) - \left[2\left(a^2 + 3b^2\right) G(t) + \dfrac{\omega_o^4}{a^4}\left(2a^2 + b^2\right) J(t)\right] e^{-bt_o} \cos at_o \right. \\ \qquad - \left[\dfrac{4b^3}{a} G(t) + \dfrac{2\omega_o^4}{a^2} K(t) - \dfrac{b\omega_o^4}{a^3} J(t)\right] e^{-bt_o} \sin at_o \\ \qquad \left. - 2b\omega_o^2 t_o e^{-bt_o}\left[\left(G(t) + \dfrac{\omega_o^2}{a^2} J(t)\right) \cos at + \left(\dfrac{\omega_o^2}{a^2} K(t) - \dfrac{b}{a} G(t) - \dfrac{\omega_o^2 b}{a^3}\right) \sin at_o\right]\right\}, & \text{for } s_1 = s_3 \end{cases}$$

where:

$$T_a = \frac{a}{2b}\left[E(t) - 2\left(G(t) + \frac{b}{a}K(t) + \frac{a^2 - b^2}{a^2}J(t)\right)e^{-bt_o}\cos at_o - 2\left(K(t) - \frac{2b}{a}J(t)\right)e^{-bt_o}\sin at_o\right]$$

$$T_b = \frac{a}{b}\left[\frac{b}{a}F(t) - \left(G(t) + \frac{b}{a}K(t) + \frac{a^2 - b^2}{a^2}J(t)\right)e^{-bt_o}\sin at_o + \left(K(t) - \frac{2b}{a}J(t)\right)e^{-bt_o}\cos at_o\right]$$

$$T_c = \left[E(t) + \frac{b^2 - a^2 + \rho^2 - \alpha^2}{a^2}F(t) - 2\left(G(t) + \frac{\alpha}{a}K(t) + \frac{\rho^2 - \alpha^2}{a^2}J(t)\right)e^{-\alpha t_o}\cos \rho t_o - \frac{2\rho}{a}\left(K(t) - \frac{2\alpha}{a}J(t)\right)e^{-\alpha t_o}\sin \rho t_o\right]$$

$$T_d = 2\left[\frac{\rho\alpha}{a^2}F(t) - \left(G(t) + \frac{\alpha}{a}K(t) + \frac{\rho^2 - \alpha^2}{a^2}J(t)\right)e^{-\alpha t_o}\sin \rho t_o + \frac{\rho}{a}\left(K(t) - \frac{2\alpha}{a}J(t)\right)e^{-\alpha t_o}\cos \rho t_o\right]$$

[See (18), (19) for R_1, R_3, X_1, X_3]

$$(28)\quad \int_{-\infty}^{\infty} \left[1 + A_\gamma(t) + \frac{(b-\gamma)^2 - a^2 + \omega^2}{a^2} B_\gamma(t) - 2\, C_\gamma(t) \cos \omega t - \frac{2\omega}{a} D_\gamma(t) \sin \omega t\right] S(\omega)\; |H(\omega + i\gamma)|^2 \, d\omega$$

(a) $s_{1\gamma} \neq s_3, \;\; \gamma \neq b$

$$\int_{-\infty}^{\infty} (\cdot)\, d\omega = \left[R_{1\gamma} T_{1\gamma} - X_{1\gamma} T_{2\gamma} + R_{3\gamma} T_{3\gamma} - X_{3\gamma} T_{4\gamma}\right]$$

where:

$$T_{1\gamma} = \frac{a}{2(b-\gamma)} \left[1 - A_\gamma(t)\right]$$

$$T_{2\gamma} = -\, B_\gamma(t)$$

$$T_{3\gamma} = \left[1 + A_\gamma(t) + \frac{(b-\gamma)^2 - a^2 + \rho^2 - \alpha^2}{a^2} B_\gamma(t) - 2\left(C_\gamma(t) + \frac{\alpha}{a} D_\gamma(t)\right) e^{-\alpha t} \cos \rho t - \frac{2\rho}{a} D_\gamma(t)\, e^{-\alpha t} \sin \rho t\right]$$

$$T_{4\gamma} = 2\left[\frac{\rho\alpha}{a^2} B_\gamma(t) - \left(C_\gamma(t) + \frac{\alpha}{a} D_\gamma(t)\right) e^{-\alpha t} \sin \rho t + \frac{\rho}{\alpha} D_\gamma(t)\, e^{-\alpha t} \cos \rho t\right]$$

$$R_{1\gamma} = \mathrm{Re}\left\{Z_{1\gamma}\right\} \qquad R_{3\gamma} = \mathrm{Re}\left\{Z_{3\gamma}\right\}$$

$$X_{1\gamma} = \mathrm{Im}\left\{Z_{1\gamma}\right\} \qquad X_{3\gamma} = \mathrm{Im}\left\{Z_{3\gamma}\right\}$$

$$Z_{1\gamma} = \frac{\alpha\left(\rho^2 + \alpha^2 + s_{1\gamma}^2\right)}{a^2 s_{1\gamma}\left(s_{1\gamma}^2 - s_3^2\right)\left(s_{1\gamma}^2 - s_4^2\right)} \qquad Z_{3\gamma} = \frac{1}{\left(s_3^2 - s_{1\gamma}^2\right)\left(s_3^2 - s_{2\gamma}^2\right)}$$

$$s_{1\gamma} = a + i(b - \gamma) = - s_{2\gamma}^*$$

(b) $s_{1\gamma} \neq s_3, \quad \gamma = b$

$$\int_{-\infty}^{\infty} (\cdot)\, d\omega = \Bigg\{ R'_{1\gamma}\left(at - \frac{1}{2}\sin 2at\right) + X'_{1\gamma}\sin^2 at$$

$$- 2R'_{3\gamma}\left[1 + \frac{\rho^2 - a^2 - \alpha^2}{a^2}\sin^2 at - \left(\cos at + \frac{\alpha}{a}\sin at\right)e^{-\alpha t}\cos \rho t \right.$$

$$\left. - \frac{\rho}{a}e^{-\alpha t}\sin at \sin \rho t\right]$$

$$+ 2X_{3\gamma}\left[\frac{\rho\alpha}{a^2}\sin^2 at - \left(\cos at + \frac{\alpha}{a}\sin at\right)e^{-\alpha t}\sin \rho t \right.$$

$$\left. + \frac{\rho}{a}e^{-\alpha t}\sin at \cos \rho t\right]\Bigg\}$$

where:

$$R'_{1\gamma} = \mathrm{Re}\left\{Z'_{1\gamma}\right\} \qquad R'_{3\gamma} = \mathrm{Re}\left\{Z'_{3\gamma}\right\}$$

$$X'_{1\gamma} = \mathrm{Im}\left\{Z'_{1\gamma}\right\} \qquad X'_{3\gamma} = \mathrm{Im}\left\{Z'_{3\gamma}\right\}$$

$$Z'_{1\gamma} = \frac{\alpha\left(\rho^2 + \alpha^2 + a^2\right)}{a^3\left(a^2 - s_3^2\right)\left(a^2 - s_4^2\right)} \qquad Z'_{3\gamma} = \frac{1}{\left(s_3^2 - a^2\right)^2}$$

(c) $s_{1\gamma} = s_3, \quad \gamma \neq b$

$$\int_{-\infty}^{\infty} (\cdot)\, d\omega = \frac{1}{8\omega_o^4}\left\{\left(\frac{a}{b-\gamma}\right)^2 + 3 - \left[\left(\frac{a}{b-\gamma}\right)^2 + 3 + \frac{2\omega_o^2 t}{b-\gamma}\right]e^{-2(b-\gamma)t} - 2\left(\frac{b-\gamma}{a} + \frac{\omega_o^2 t}{a}\right)e^{-2(b-\gamma)t}\sin 2at + 2\left[1 - \left(\frac{b-\gamma}{a}\right)^2 - \frac{2(b-\gamma)\omega_o^2 t}{a^2}\right]e^{-2(b-\gamma)t}\sin^2 at\right\}$$

APPENDIX B

ADDITIONAL RESULTS FOR THE RECTANGULAR STEP MODULATION

This appendix contains response time histories $\sigma_y(t)$ for the rectangular step modulation. The intent is to examine the effect of various pulse durations on the system response for Q = 5, although some curves are shown for Q = 50. Specifically included are:

Figure B.1:	$\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$,	$\alpha/b = 1$,	$\rho/a = 2$,	$Q = 5$,	$f_n t_o$	$= 0.22$
B.2:	$\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$,	↓	↓	↓		$= 0.35$
B.3:	$\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$,	↓	↓	↓		$= 0.50$
B.4:	$\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$,	↓	↓	↓		$= 0.7$
B.5:	$\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$,	↓	↓	↓		$= 0.8$
B.6:	$\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$,	↓	↓	↓		$= 1.0$
B.7:	$\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$,	↓	↓	↓		$= 1.4$

Figure B.8: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $\alpha/b = 1$, $\rho/a = 0.5$, $Q = 5$, $f_n t_o = 0.5$

B.9: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, ↓ ↓ ↓ $= 1.0$

Figure B.10: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $\alpha/b = 1$, $\rho/a = 0.5$, $Q = 50$, $f_n t_o = 0.5$

B.11: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, ↓ ↓ ↓ $= 1.0$

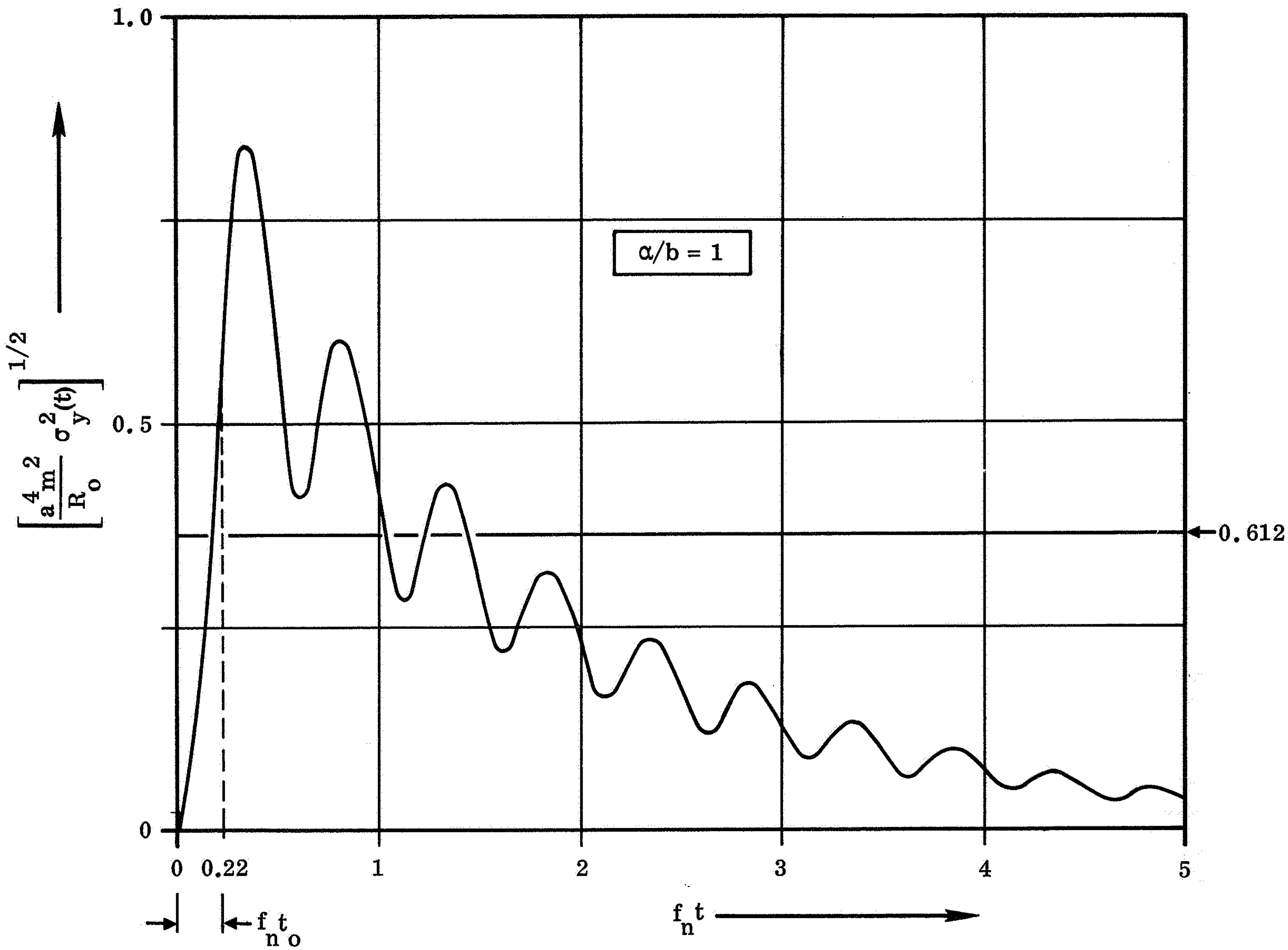

Figure B.1. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 5, ρ/a = 2

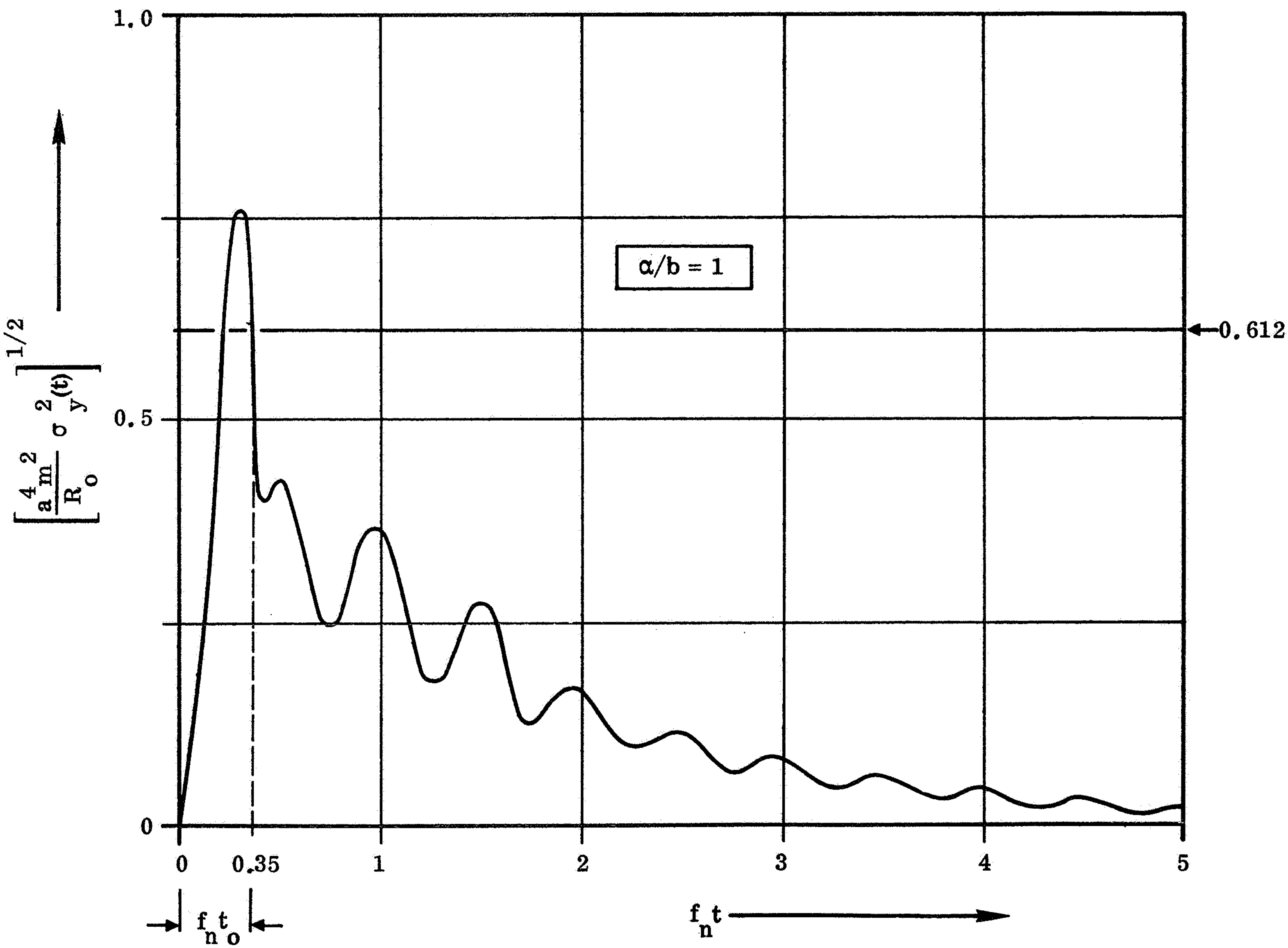

Figure B.2. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; $Q = 5$, $\rho/a = 2$

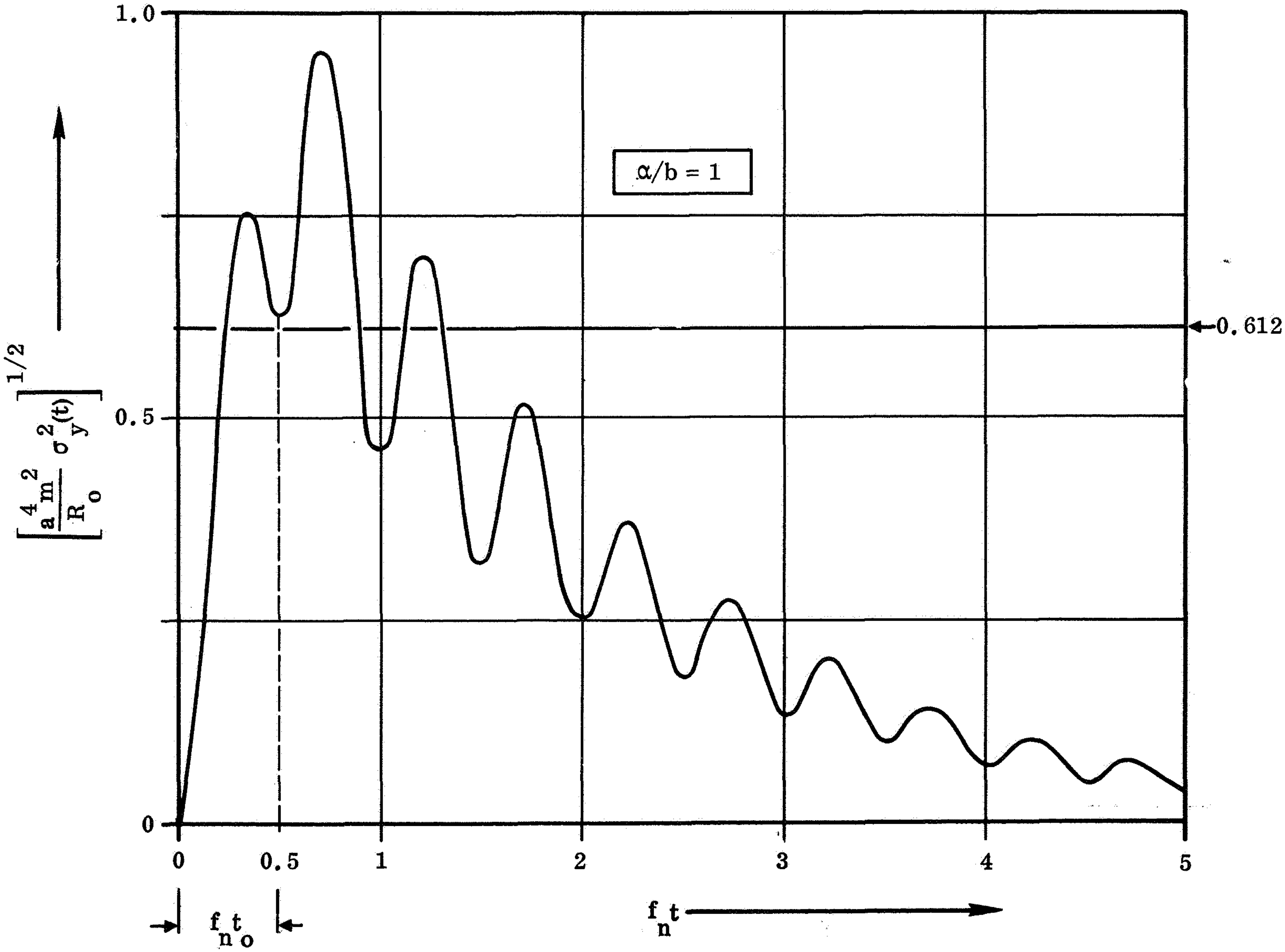

Figure B.3. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 5, ρ/a = 2

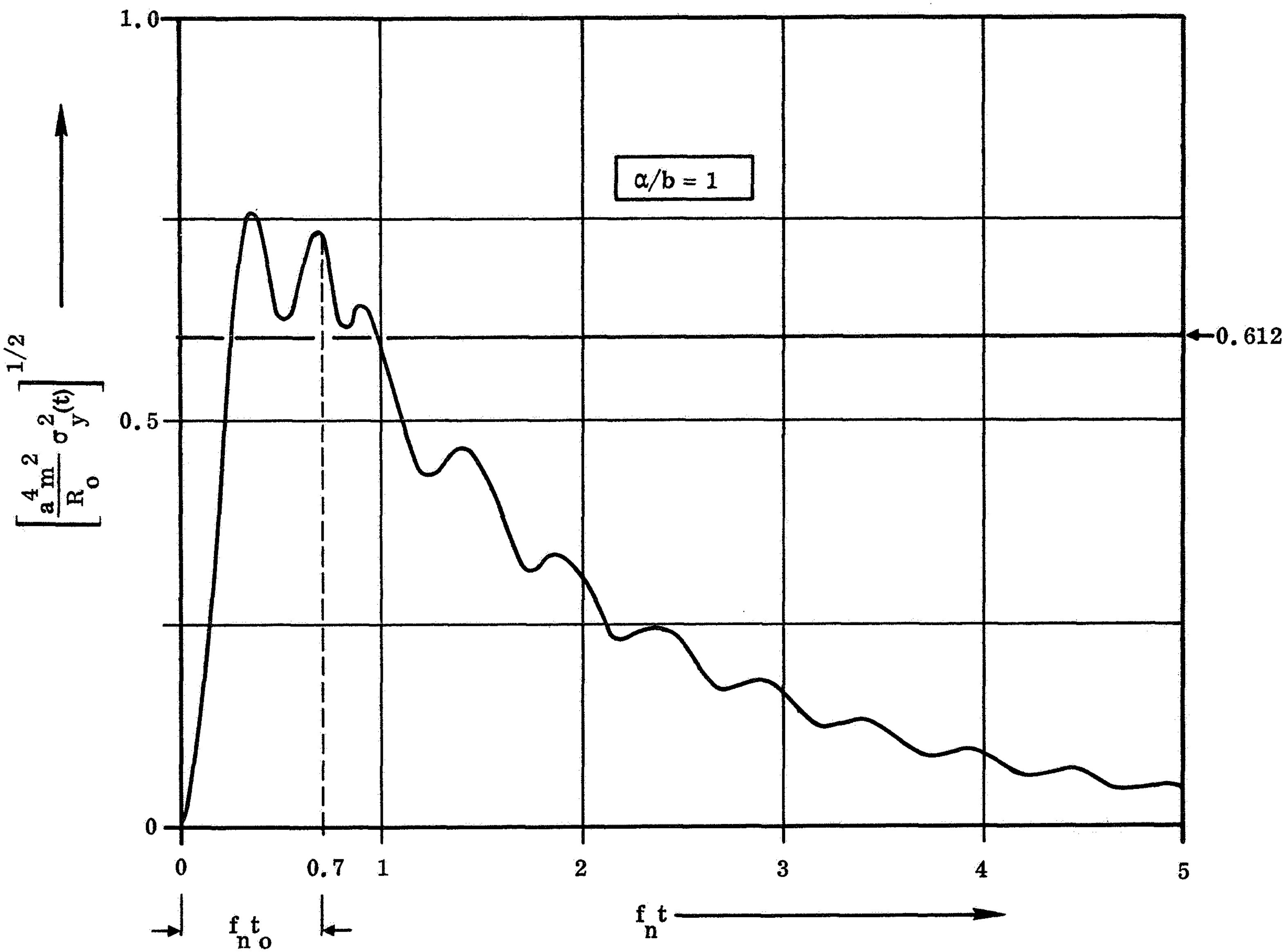

Figure B.4. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 5, ρ/a = 2

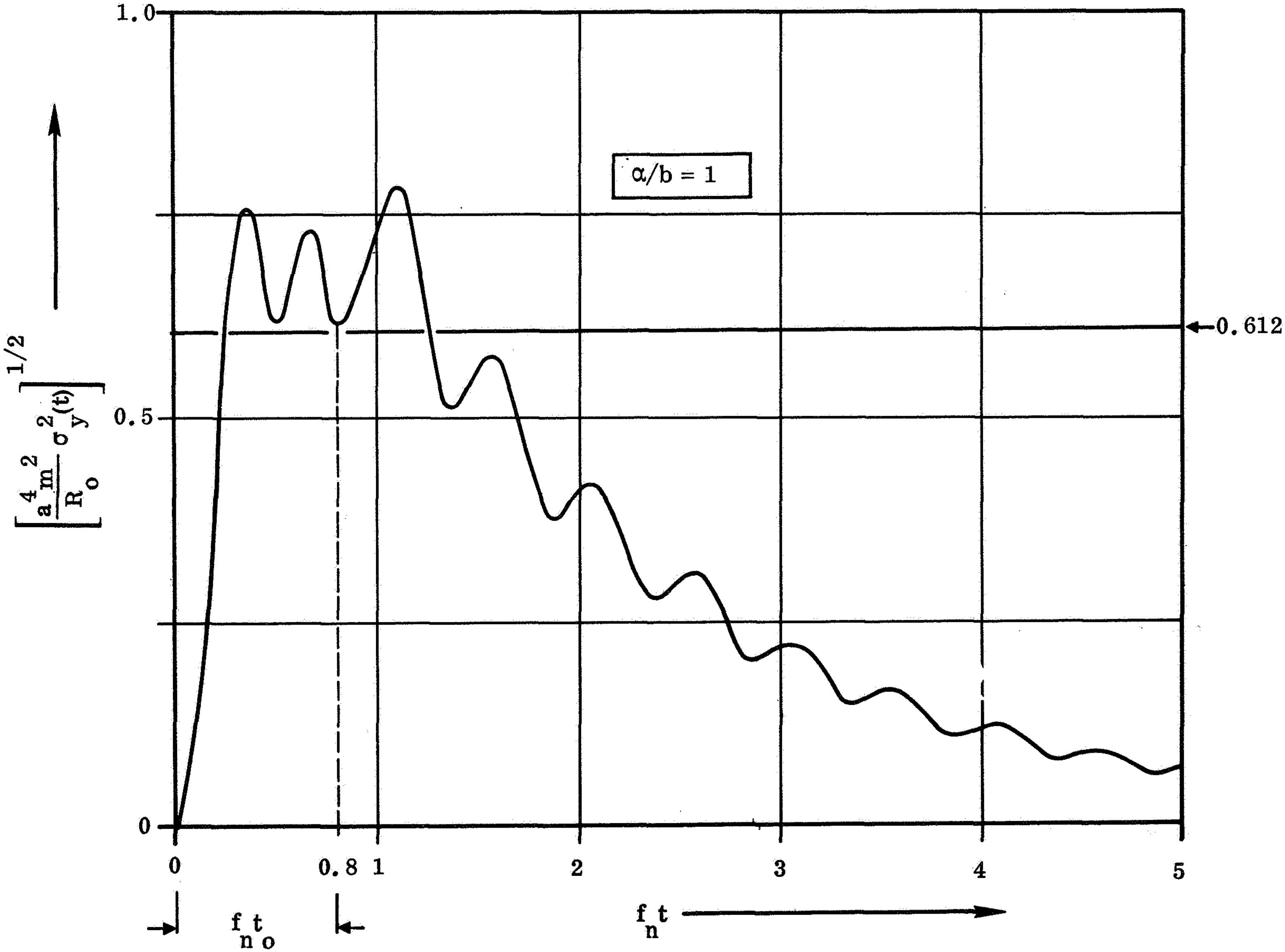

Figure B.5. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 5, ρ/a = 2

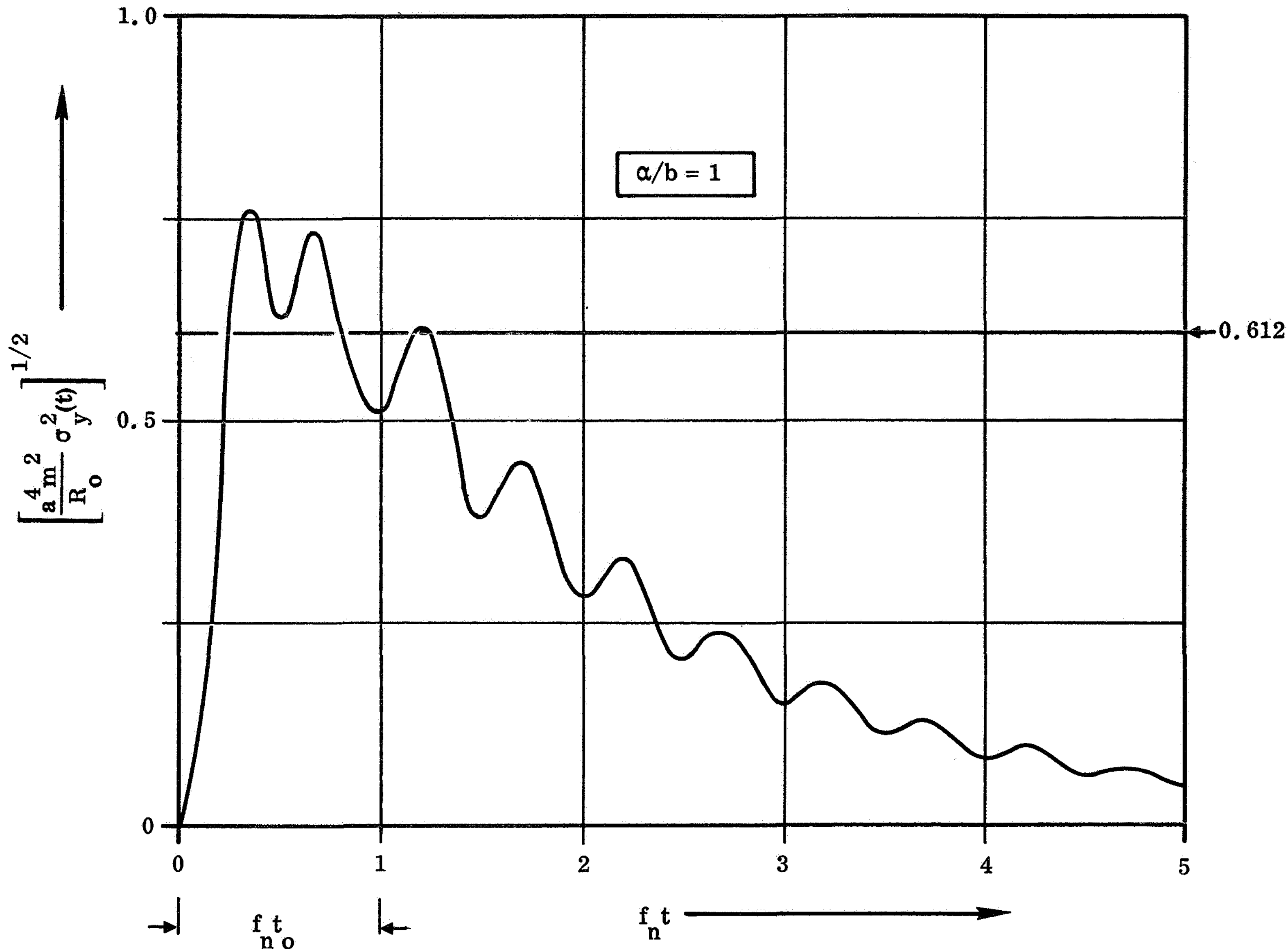

Figure B. 6. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 5, ρ/a = 2

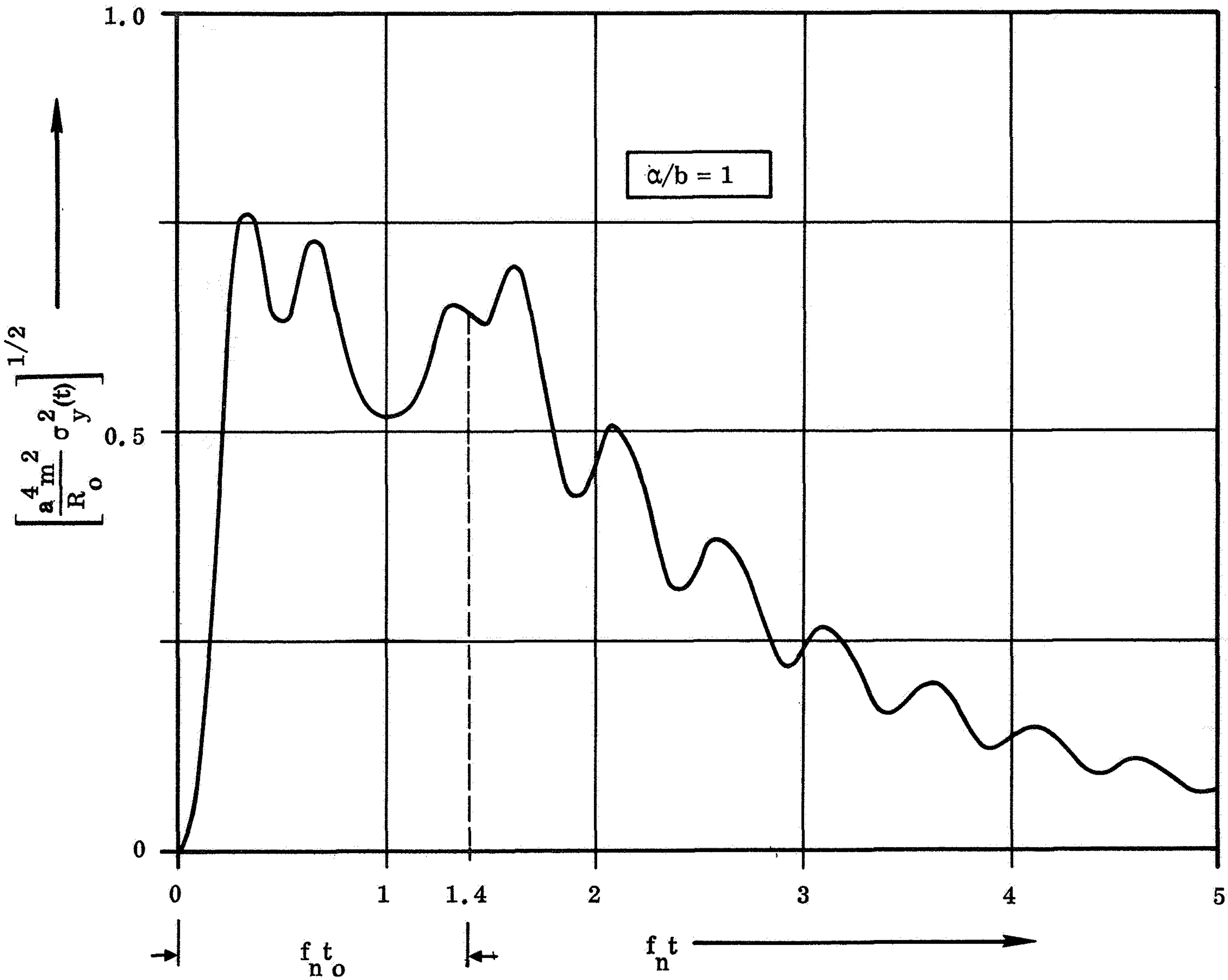

Figure B. 7. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 5, ρ/a = 2

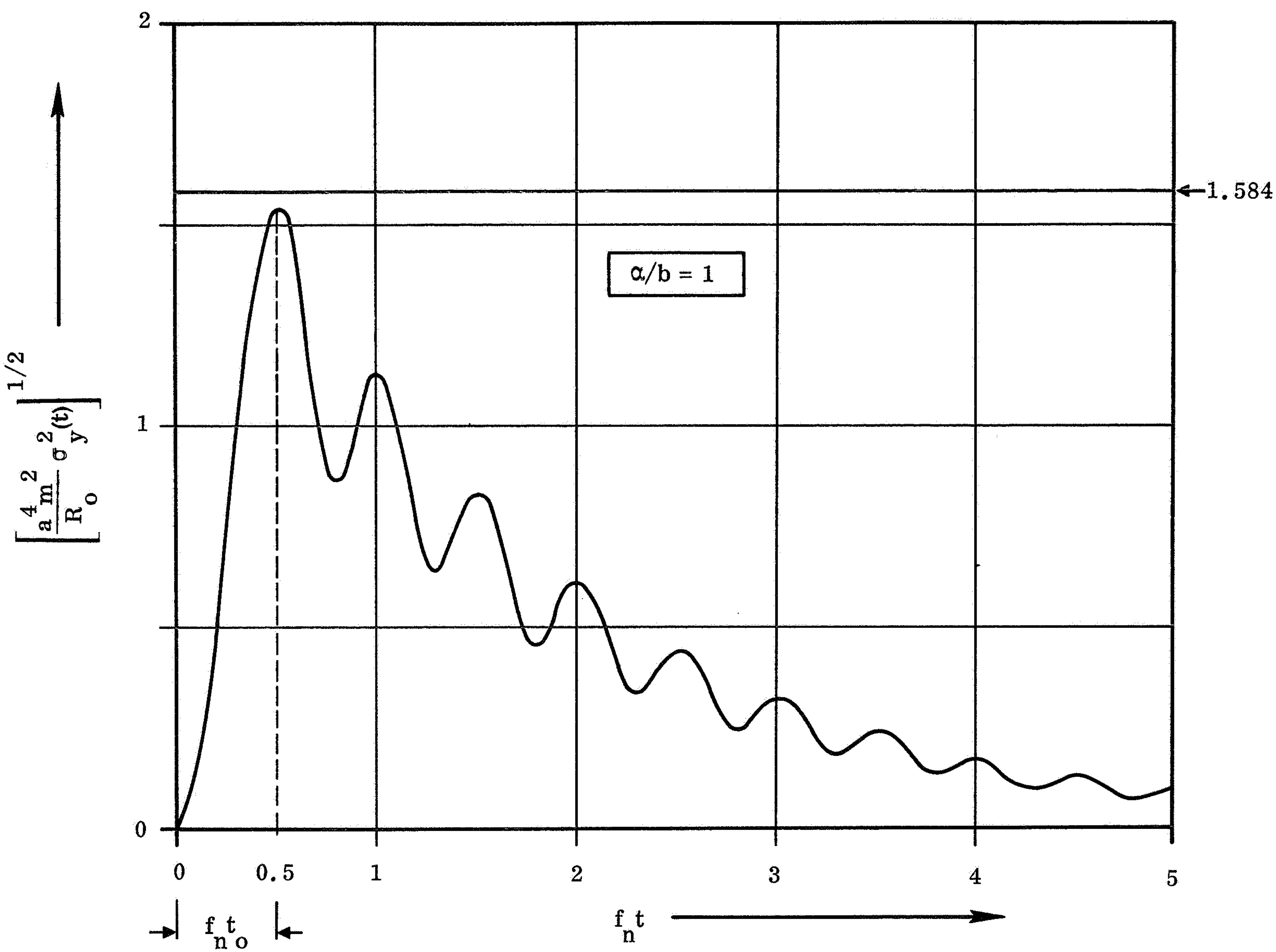

Figure B. 8. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 5, ρ/a = 0.5

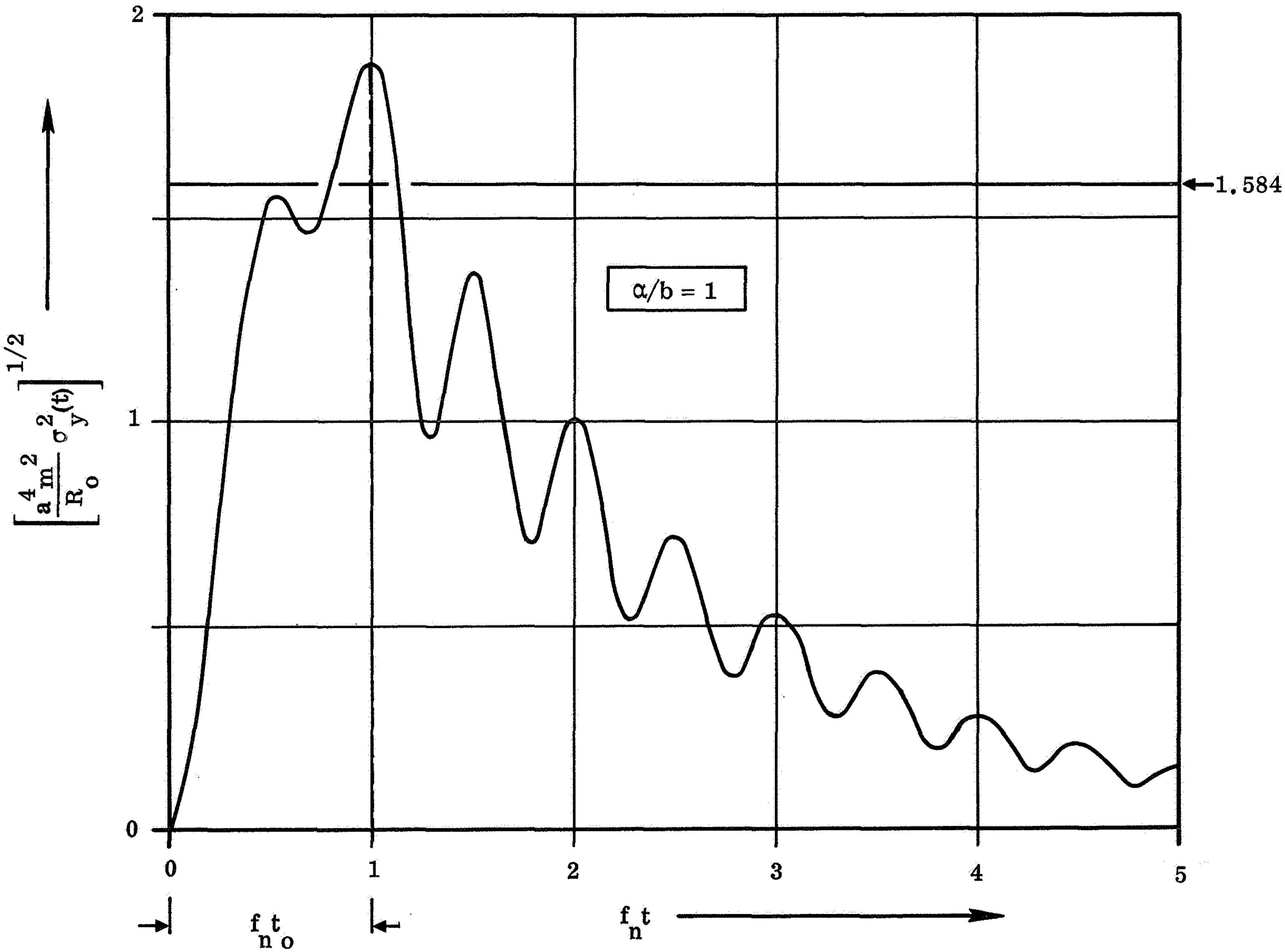

Figure B.9. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 5, ρ/a = 0.5

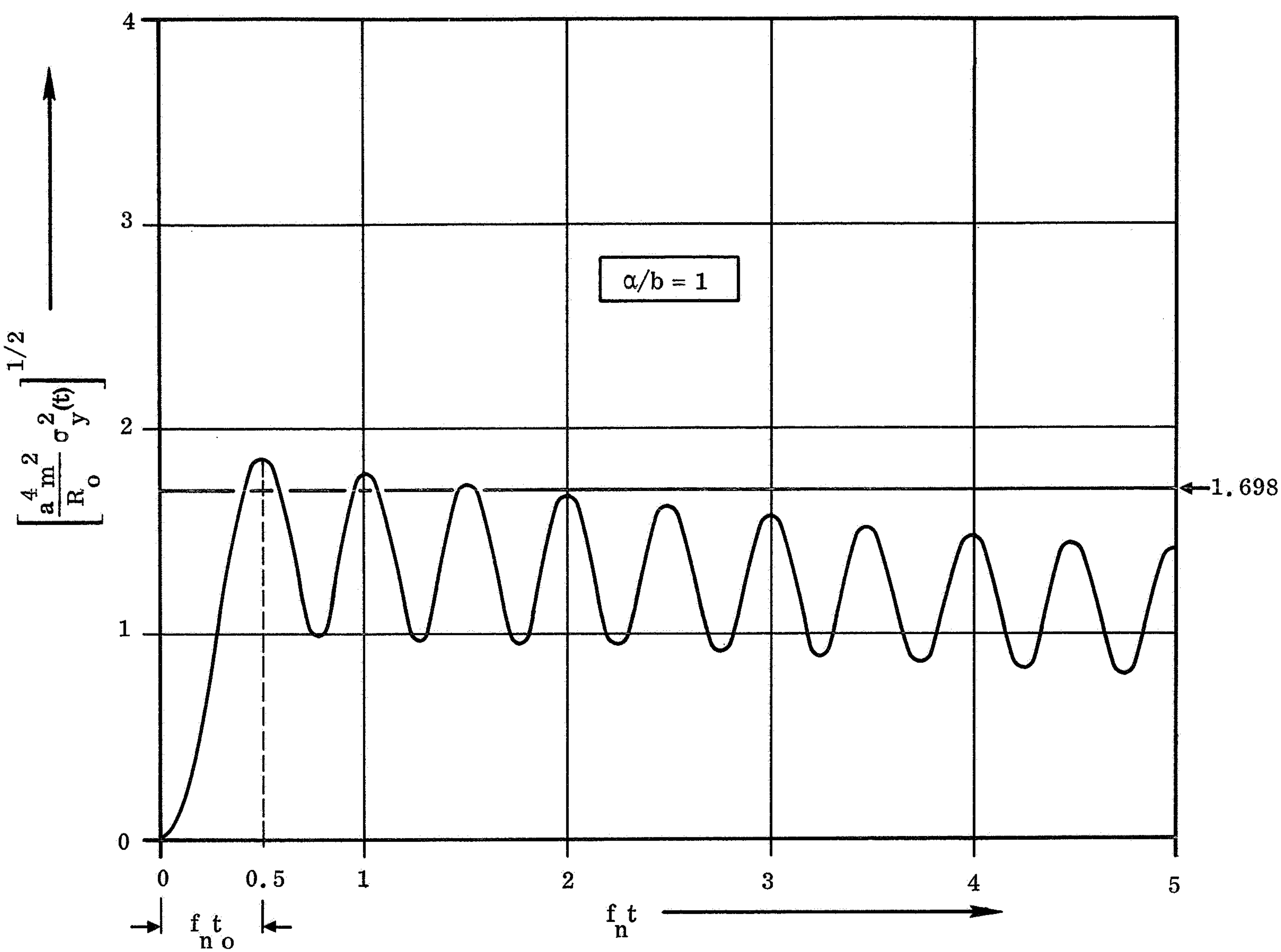

Figure B.10. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function, Q = 50, ρ/a = 0.5

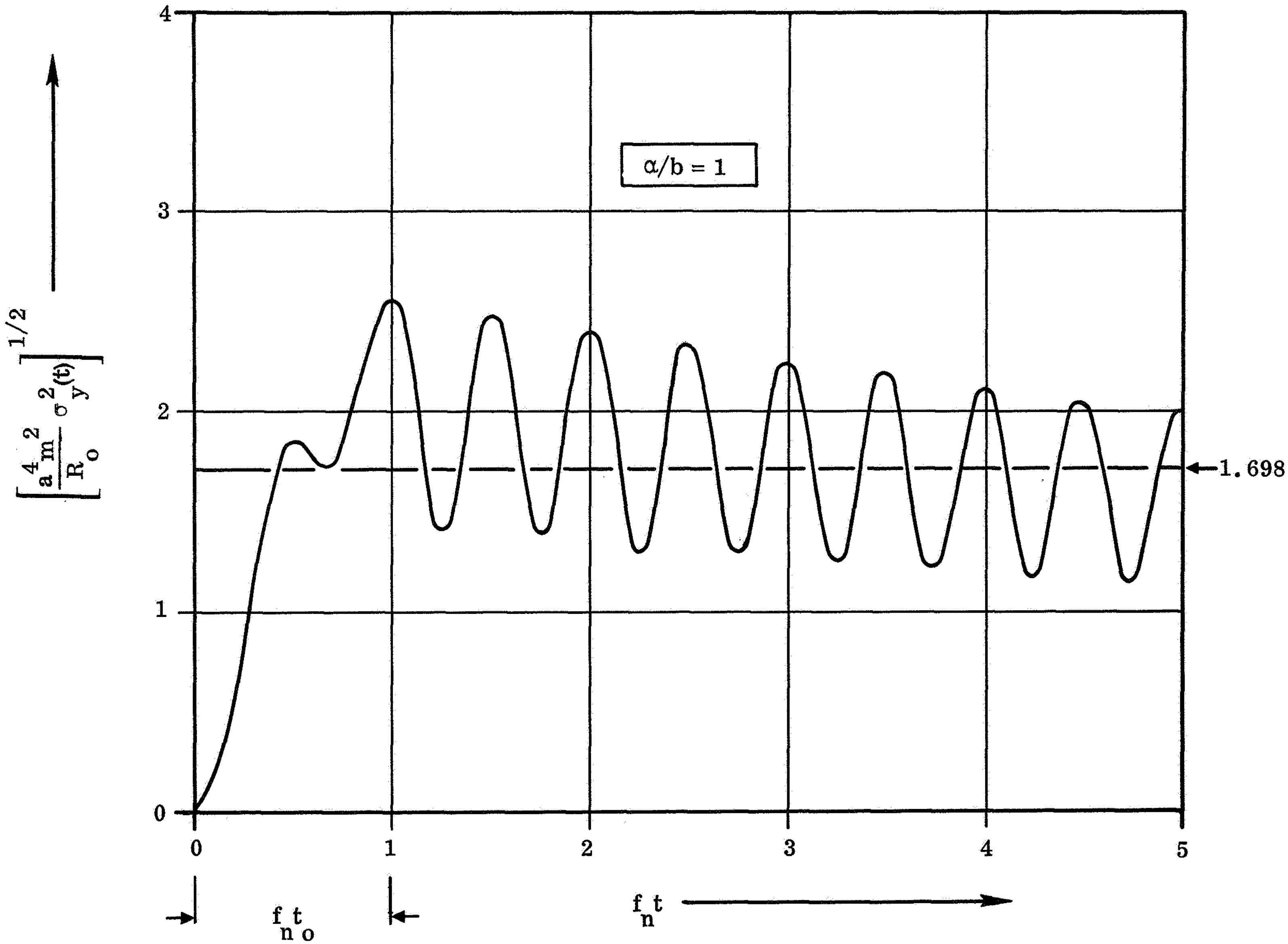

Figure B.11. Normalized System Response to Correlated Noise Modulated by the Rectangular Step Function; Q = 50, ρ/a = 0.5

APPENDIX C

ADDITIONAL RESULTS FOR THE EXPONENTIAL DECAY MODULATION

This appendix includes response time histories $\sigma_y(t)$ for the damped exponential modulation and spectral content of the shaping filter $K_\gamma(t, \omega)$. Such should be considered expository in nature and serve to reinforce an understanding of system behavior (including overshoot) for the exponential envelope function. Specifically included are:

<u>Response time histories:</u>

Figure C.1: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $\alpha/b = 0.1$, $\rho/a = 0.5$, $Q = 50$

C.2: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 1.0$

C.3: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 2.0$

C.4: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $\alpha/b = 1$, $\rho/a = 0.5$

C.5: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 1.0$

C.6: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 2.0$

Figure C.7: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $\alpha/b = 10$, $\rho/a = 0.5$, $Q = 50$

C.8: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 1.0$

C.9: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 2.0$

Figure C.10: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $\alpha/b = 0.1$, $\rho/a = 0.5$, $Q = 5$

C.11: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 1.0$

C.12: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 2.0$

C.13: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $\alpha/b = 1$, $\rho/a = 0.5$

C.14: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 1.0$

C.15: $\left[\frac{a^4 m^2}{R_o}\sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $= 2.0$

Figure C.16: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, $\alpha/b = 10$, $\rho/a = 0.5$, $Q = 5$

C.17: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, ↓ $= 1.0$ ↓

C.18: $\left[\frac{a^4 m^2}{R_o} \sigma_y^2(t)\right]^{1/2}$ vs. $f_n t$, ↓ $= 2.0$ ↓

<u>Shaping filters:</u>

Figure C.19: $K_\gamma(t, \omega)$ vs. ω/ω_n, $\gamma/b = 0.1$, $f_n t = 0.1$

C.20: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.3$

C.21: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.5$

C.22: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.7$

C.23: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.8$

C.24: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 1.0$

C.25: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 1.5$

C.26: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 2.0$

C.27: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 3.0$

Figure C.28: $K_\gamma(t, \omega)$ vs. ω/ω_n, $\gamma/b = 1$, $f_n t = 0.1$

C.29: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.3$

C.30: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.5$

C.31: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.7$

C.32: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.8$

C.33: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 1.0$

C.34: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 1.5$

C.35: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 2.0$

C.36: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 3.0$

Figure C.37: $K_\gamma(t, \omega)$ vs. ω/ω_n, $\gamma/b = 10$, $f_n t = 0.1$

C.38: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.3$

C.39: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.5$

C.40: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.7$

C.41: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 0.8$

C.42: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 1.0$

C.43: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 1.5$

C.44: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 2.0$

C.45: $K_\gamma(t, \omega)$ vs. ω/ω_n, ↓ $= 3.0$

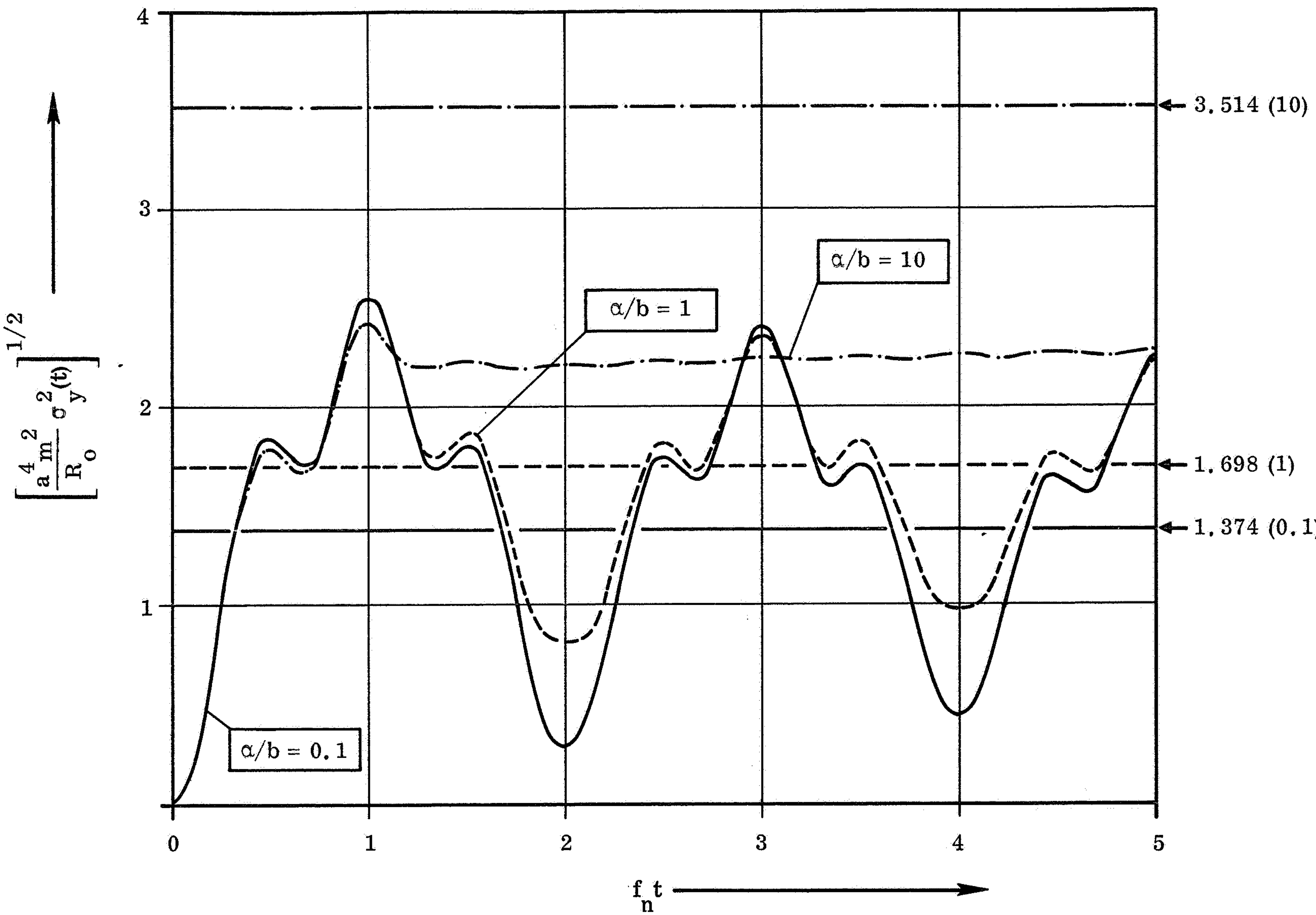

Figure C.1. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 0.1$, $Q = 50$, $\rho/a = 0.5$

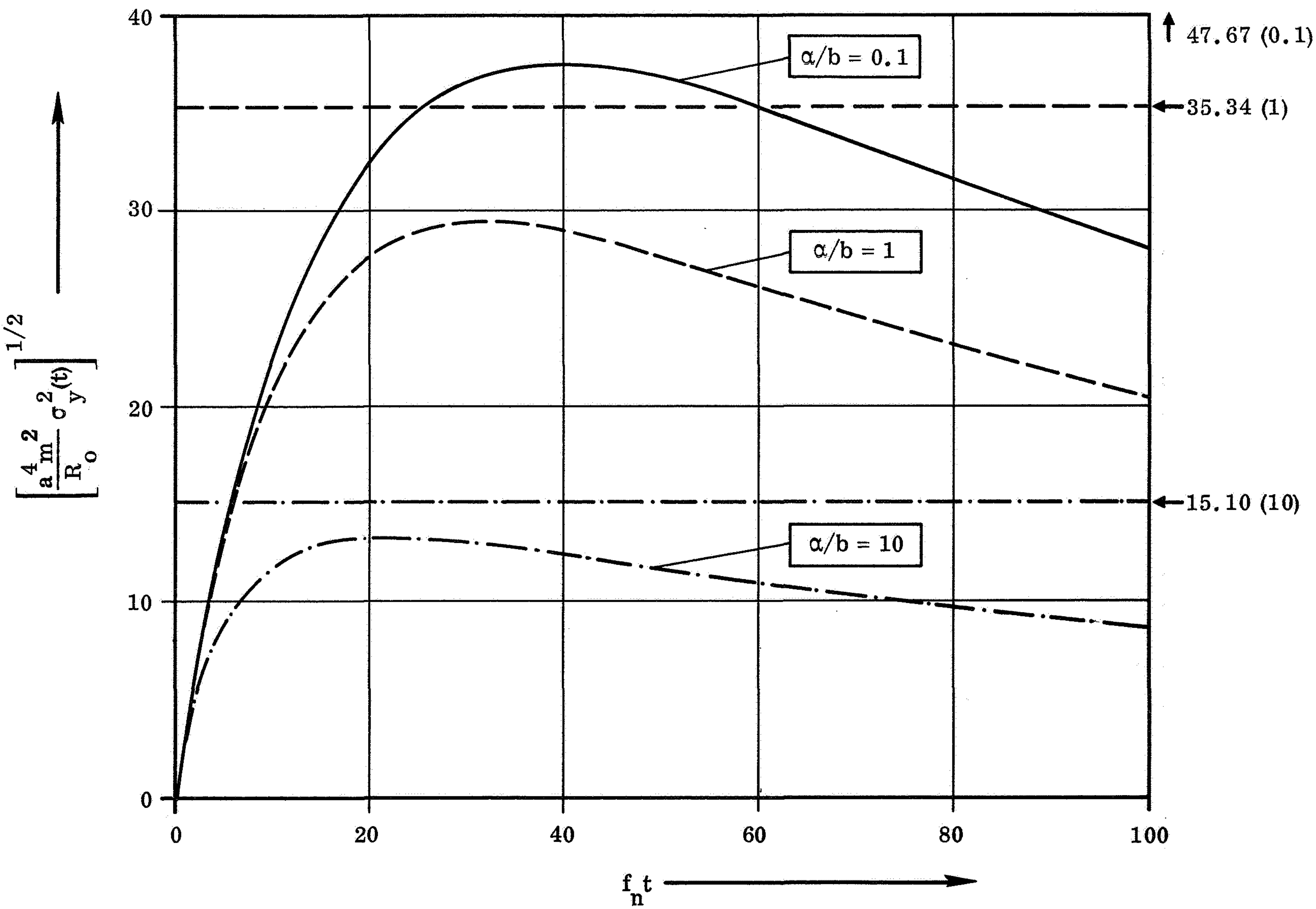

Figure C.2. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 0.1$, $Q = 50$, $\rho/a = 1$

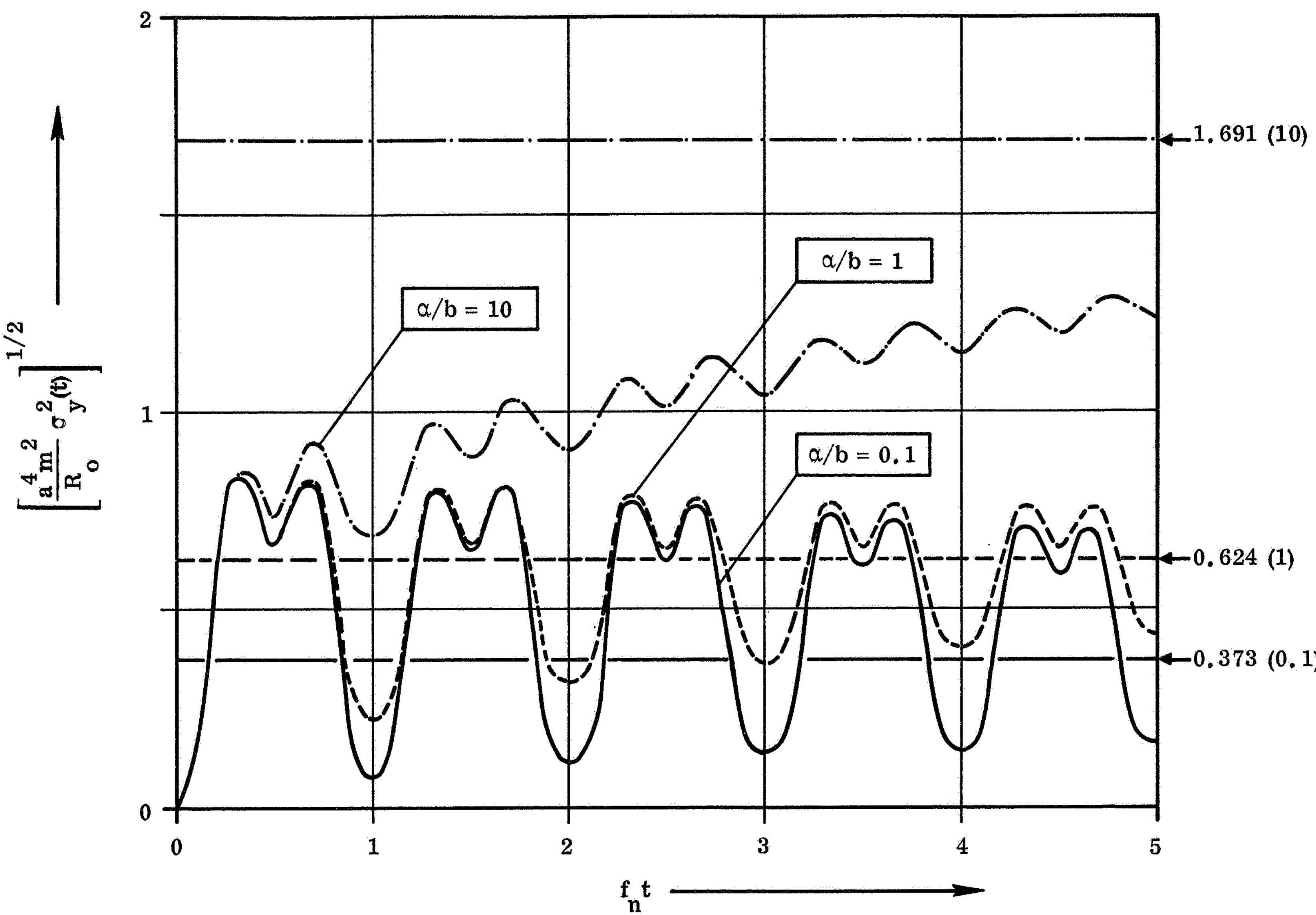

Figure C.3. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 0.1$; $Q = 50$, $\rho/a = 2.0$

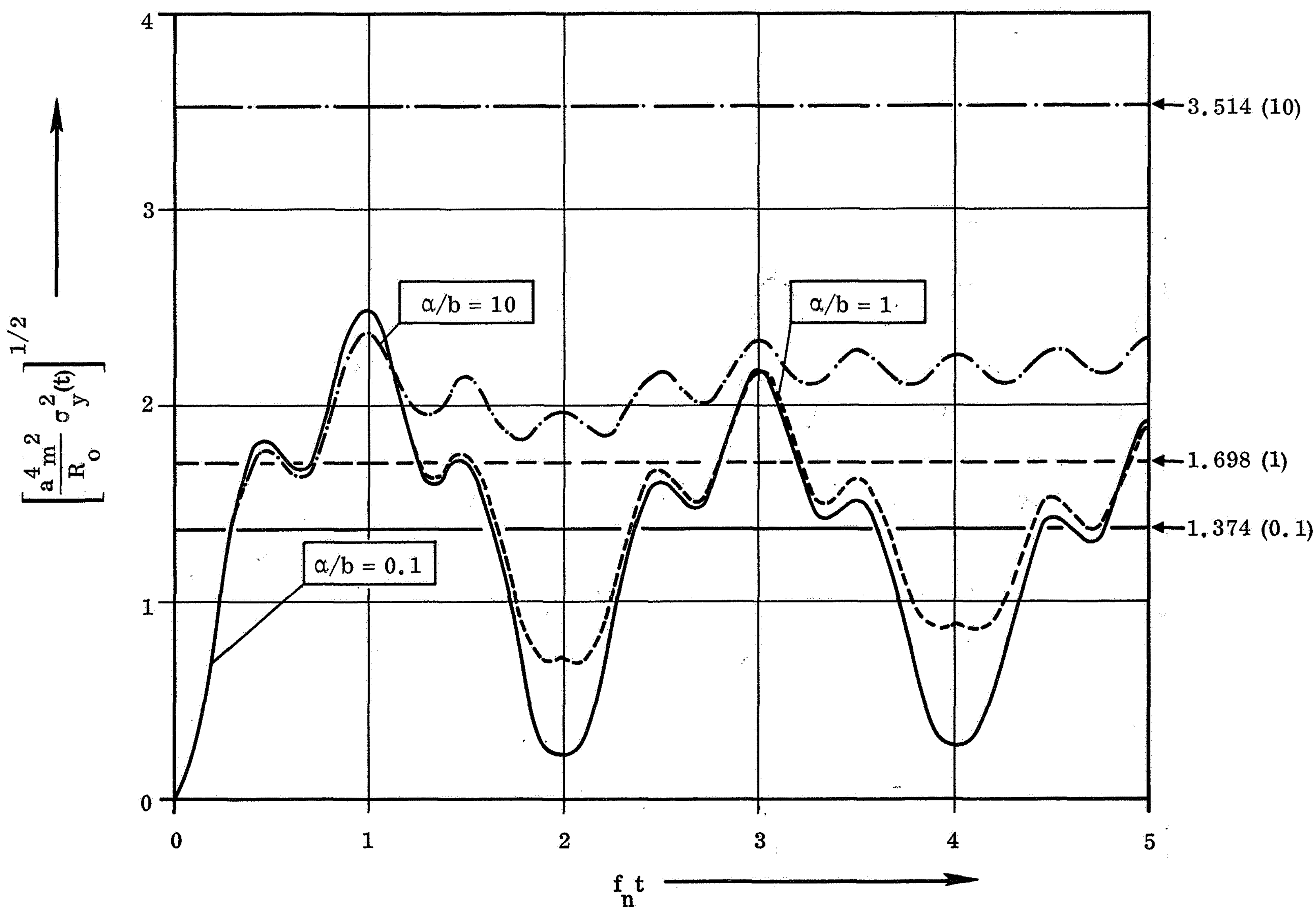

Figure C.4. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 1$; $Q = 50$, $\rho/a = 0.5$

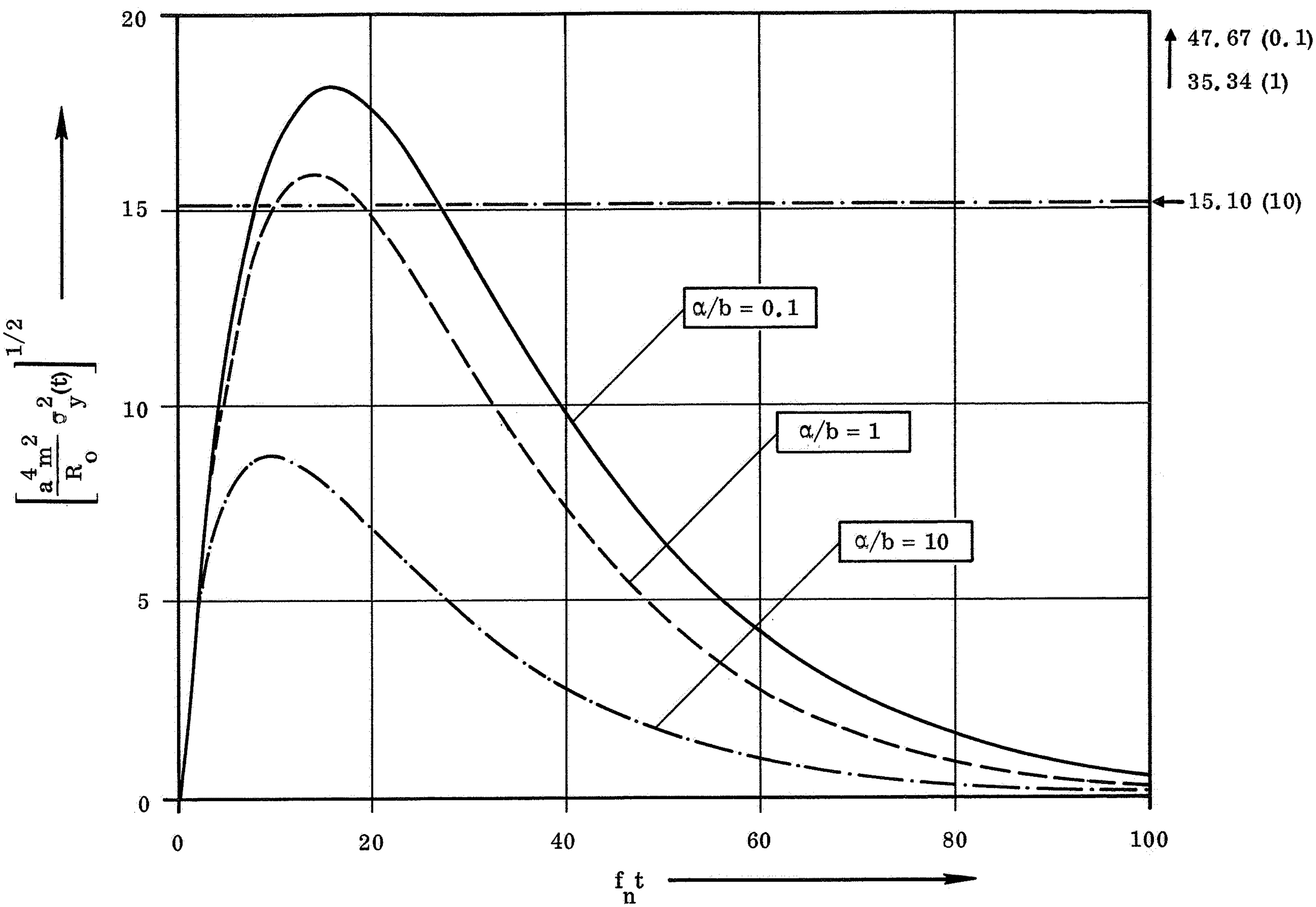

Figure C.5. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 1$, $Q = 50$, $\rho/a = 1$

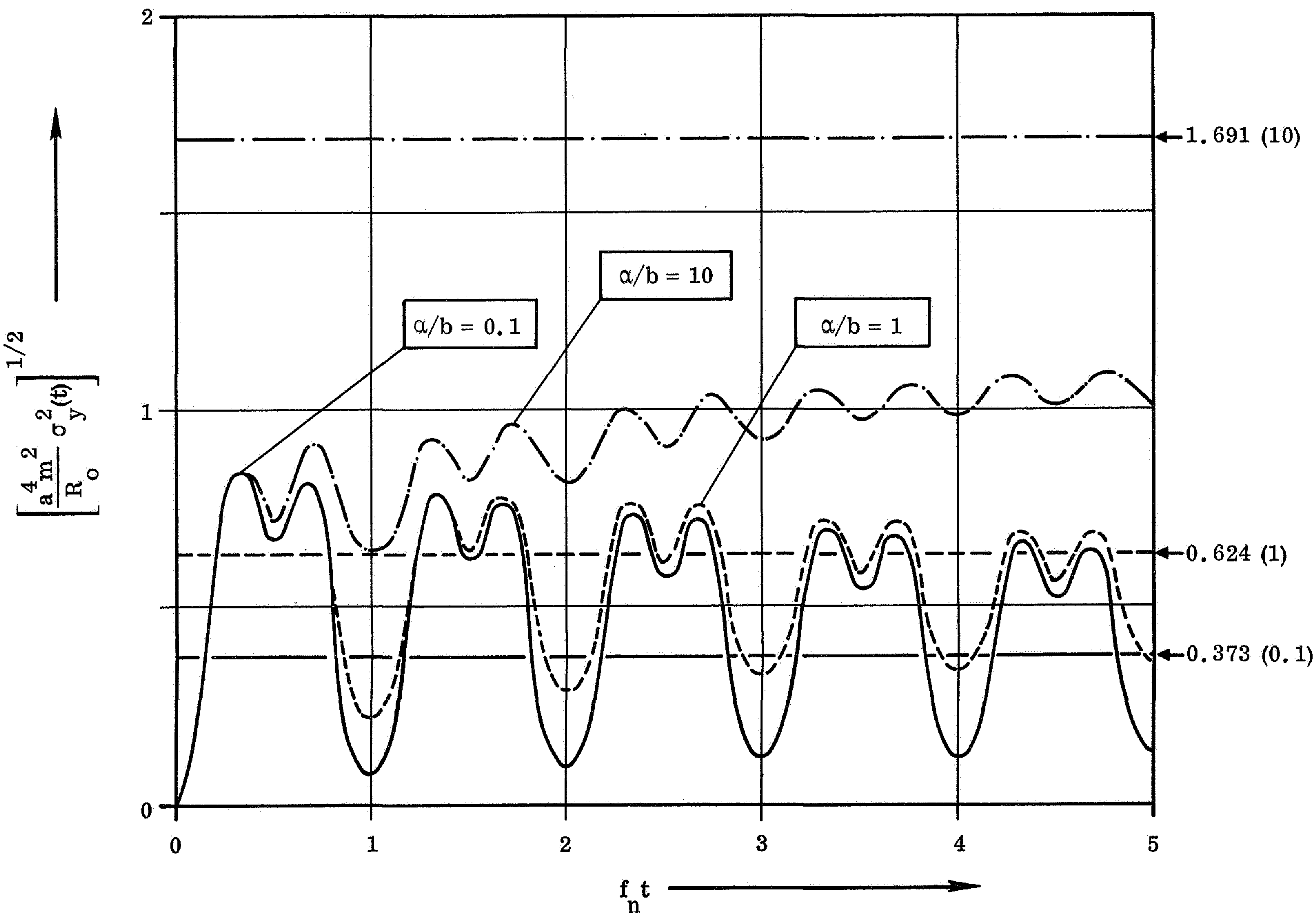

Figure C. 6. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 1.0$; $Q = 50$, $\rho/a = 2.0$

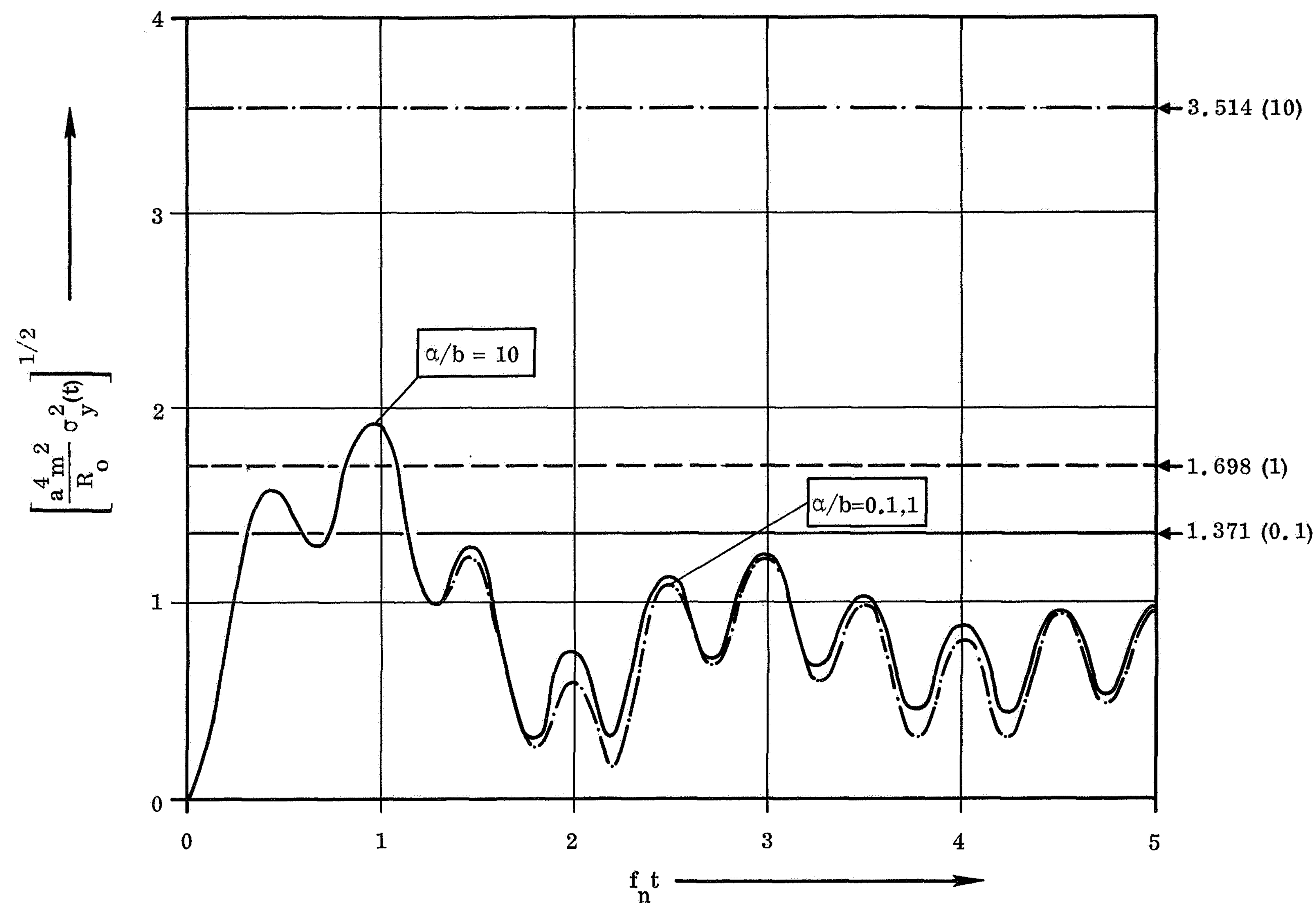

Figure C. 7. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 10.0$; $Q = 50$, $\rho/a = 0.5$

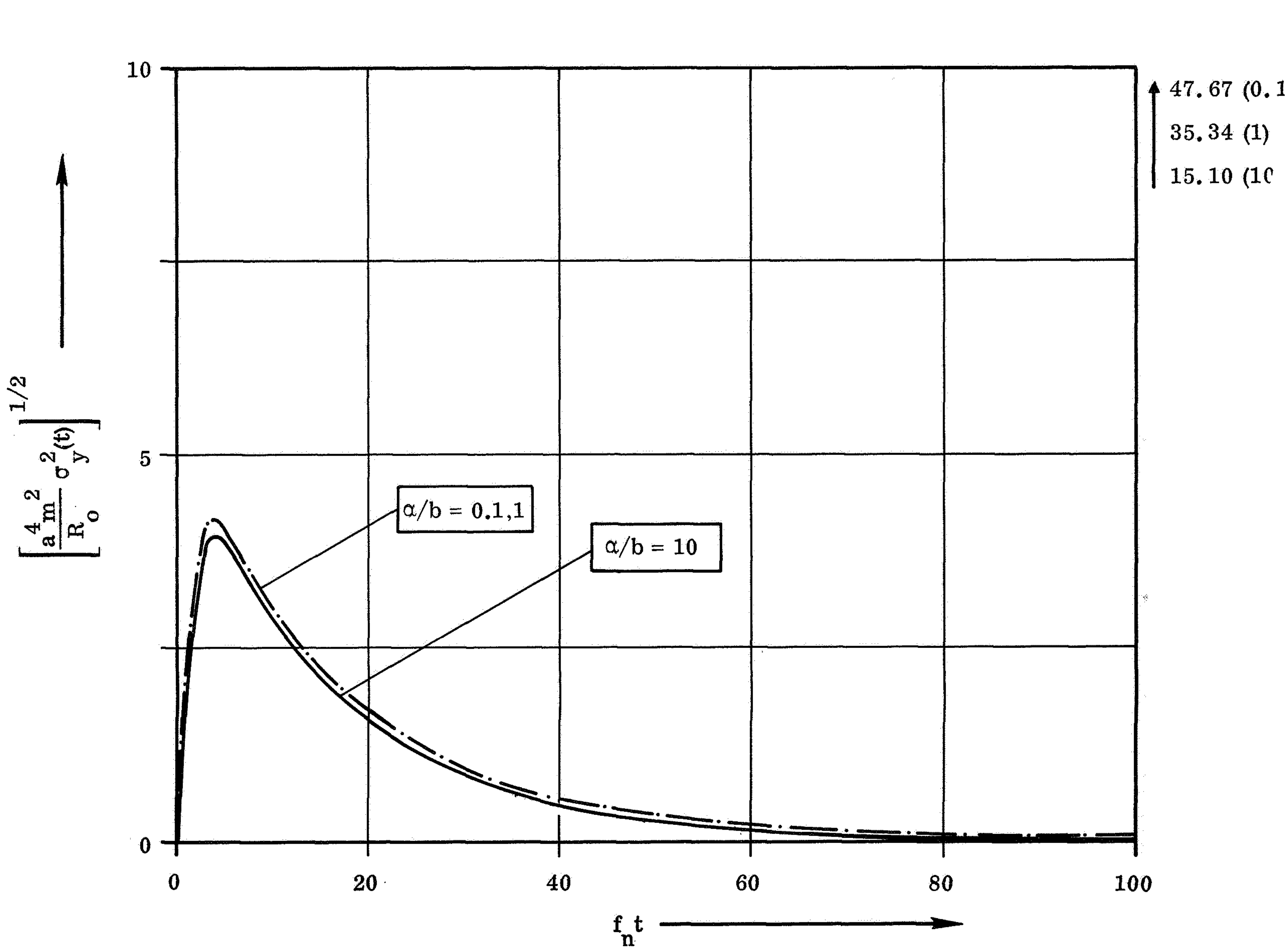

Figure C.8. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 10$, $Q = 50$, $\rho/a = 1$

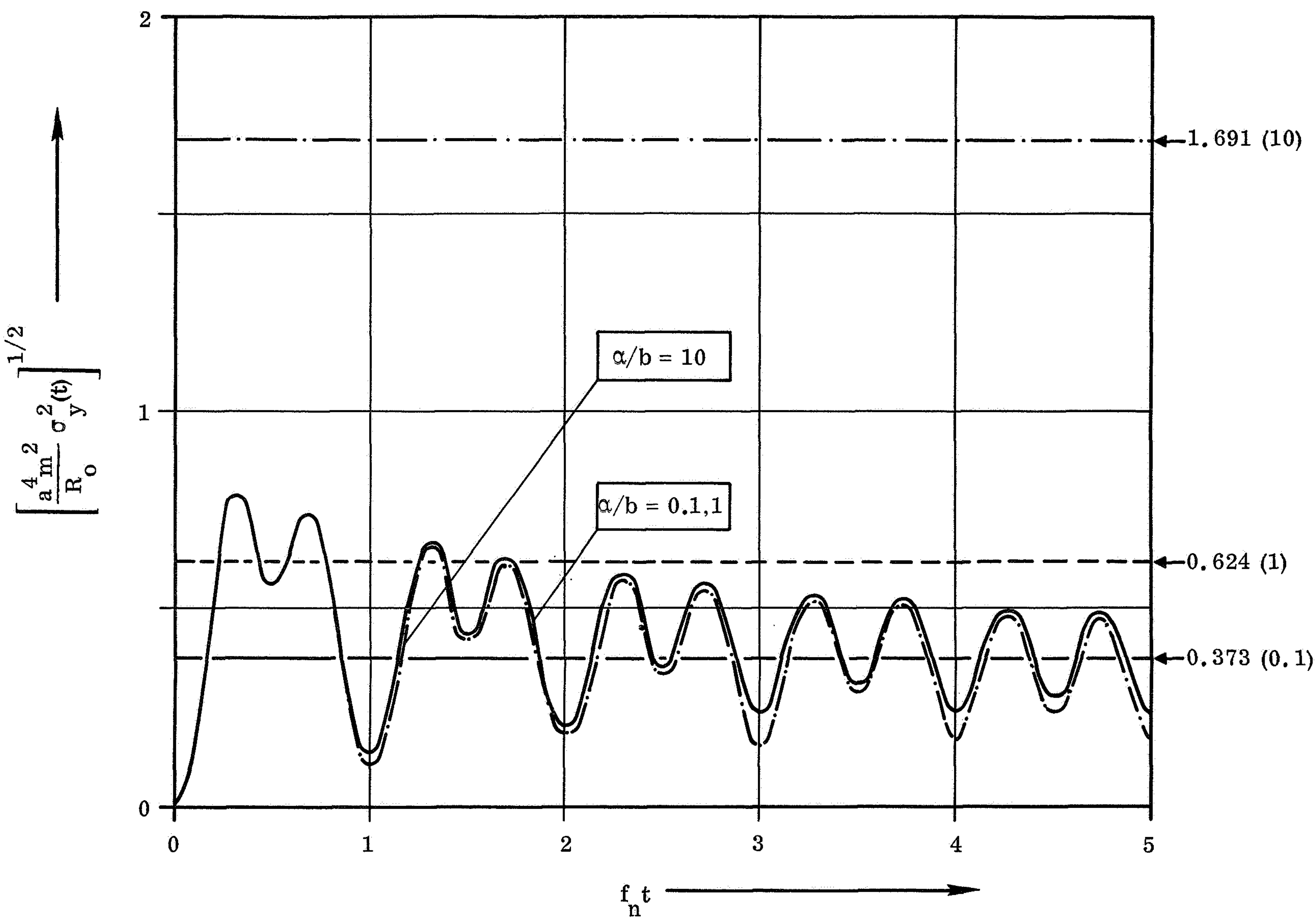

Figure C.9. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 10$; $Q = 50$, $\rho/a = 2.0$

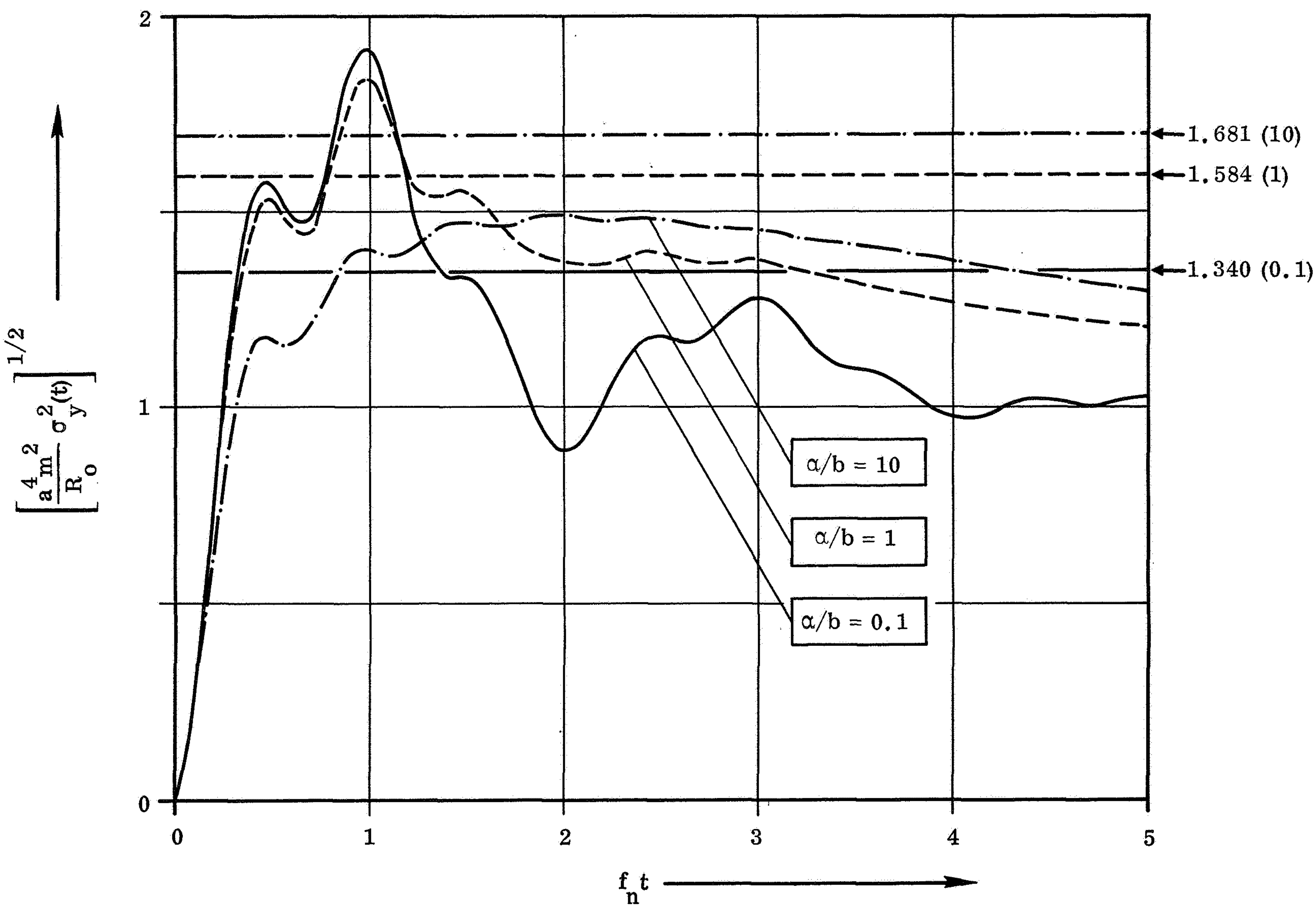

Figure C.10. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 0.1$, $Q = 5$, $\rho/a = 0.5$

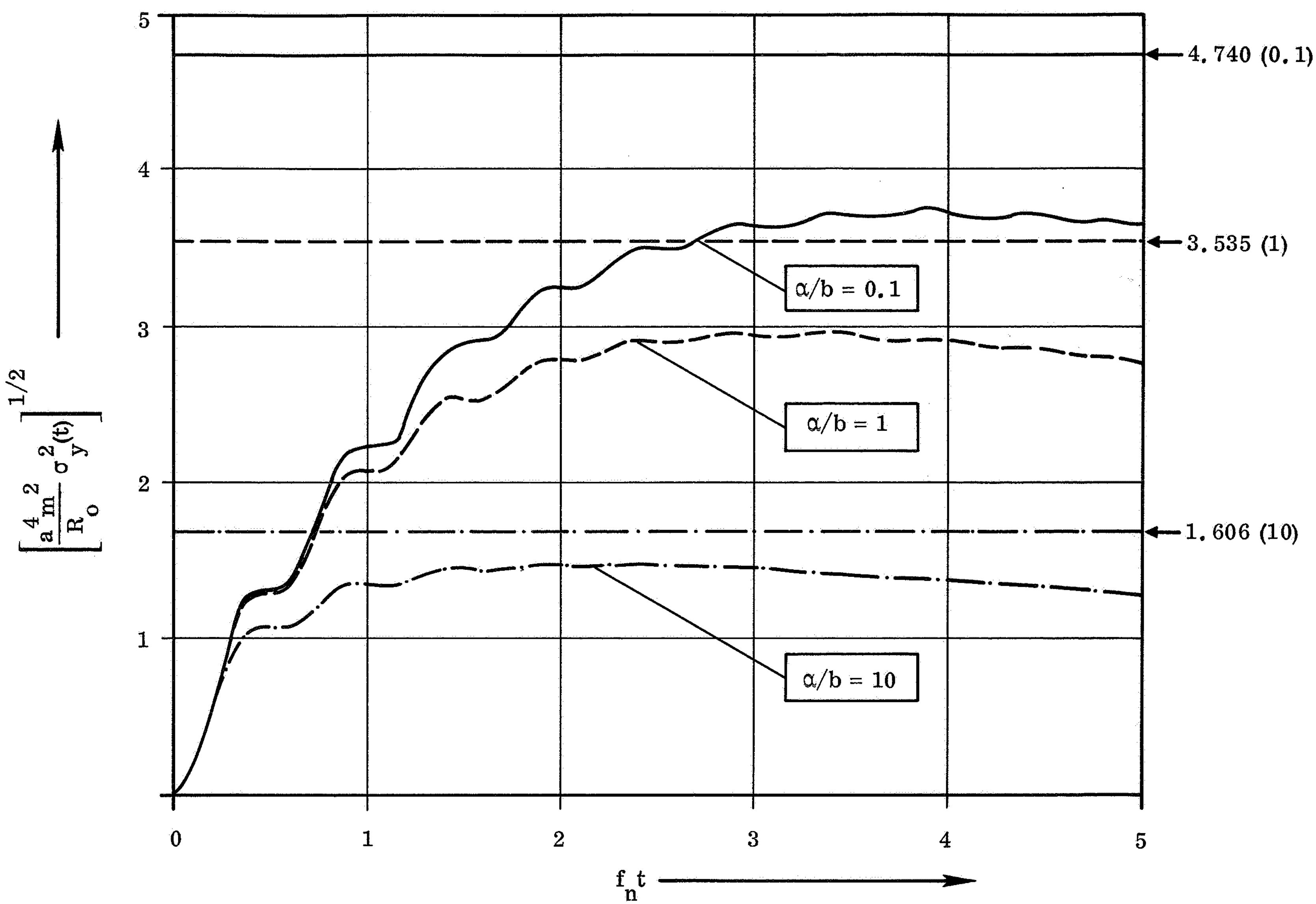

Figure C.11. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 0.1$, $Q = 5$, $\rho/a = 1$

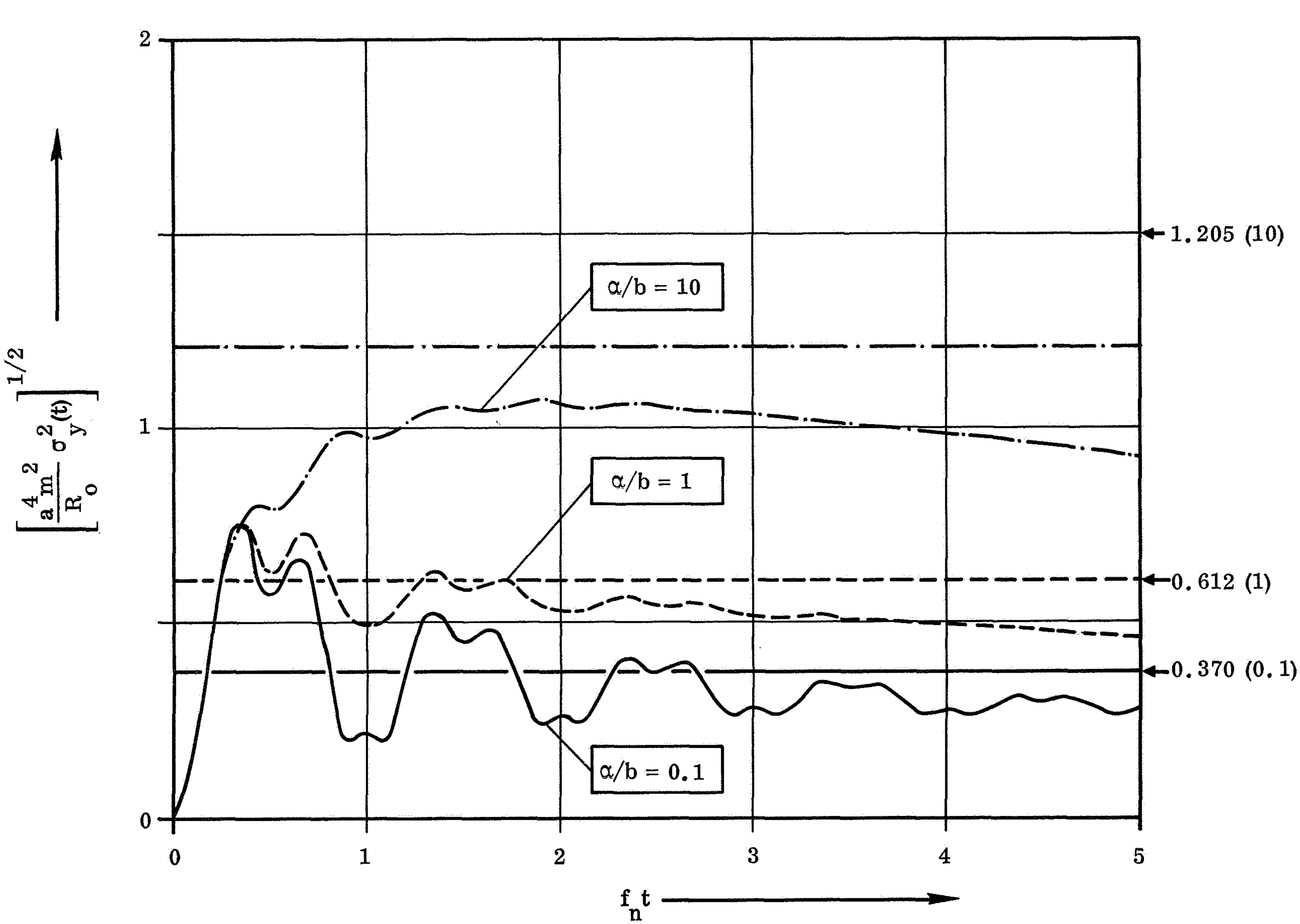

Figure C.12. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 0.1$, $Q = 5$, $\rho/a = 2$

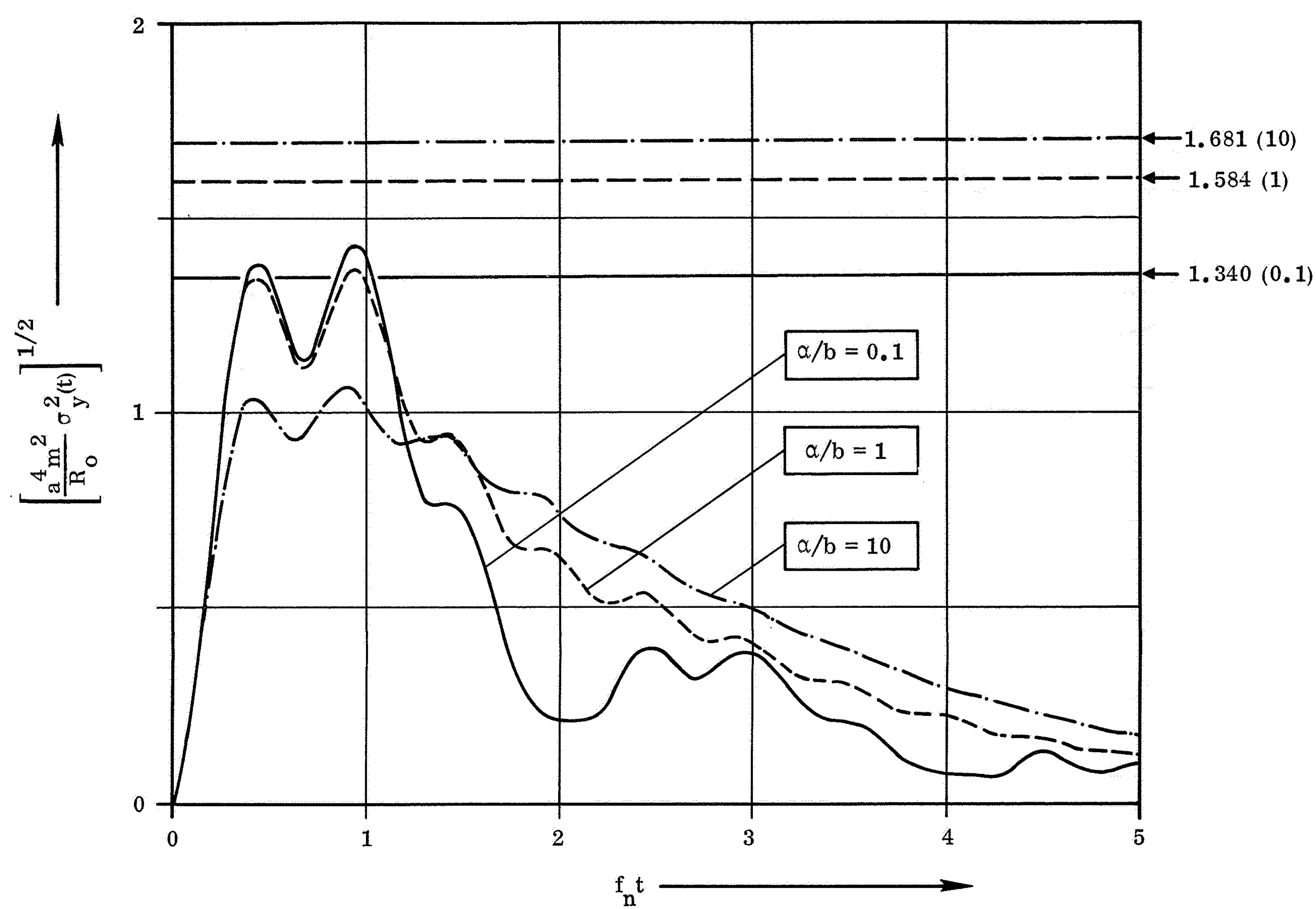

Figure C.13. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 1$, $Q = 5$, $\rho/a = 0.5$

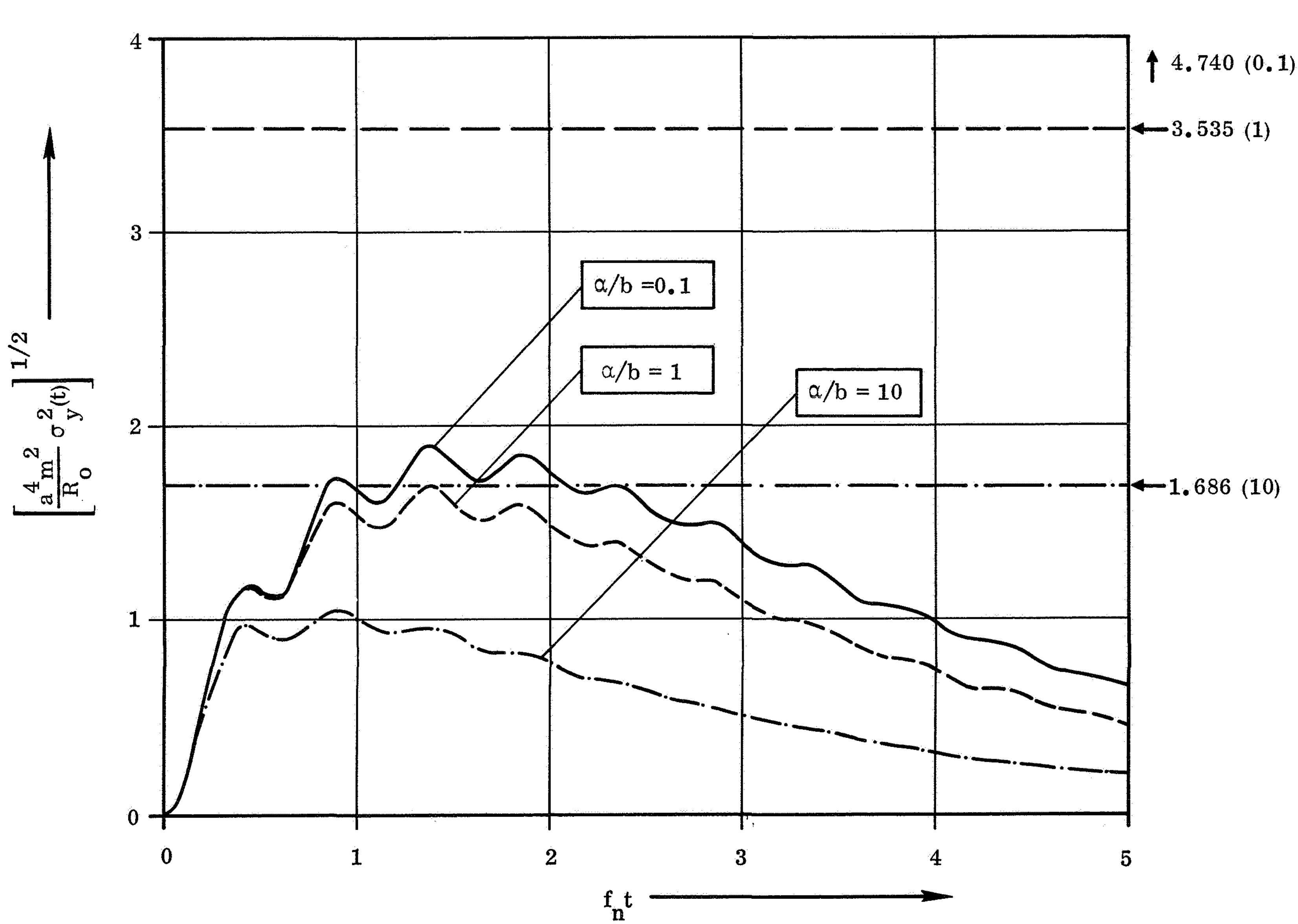

Figure C.14. Normalized System Response to Correlated Noise Modulated by the Exponential Function, $\gamma/b = 1$, $Q = 5$, $\rho/a = 1$

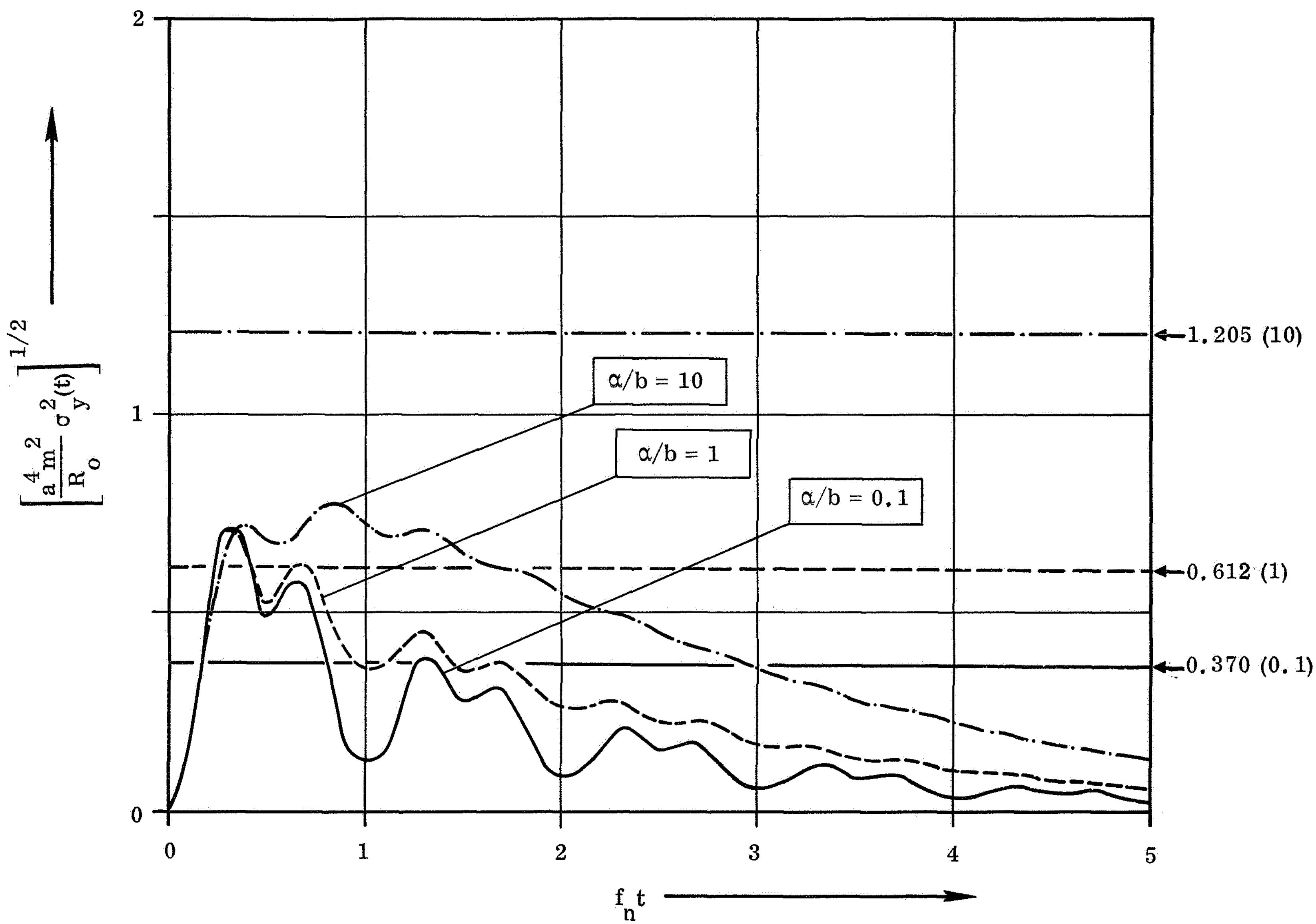

Figure C.15. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 1$, $Q = 5$, $\rho/a = 2$

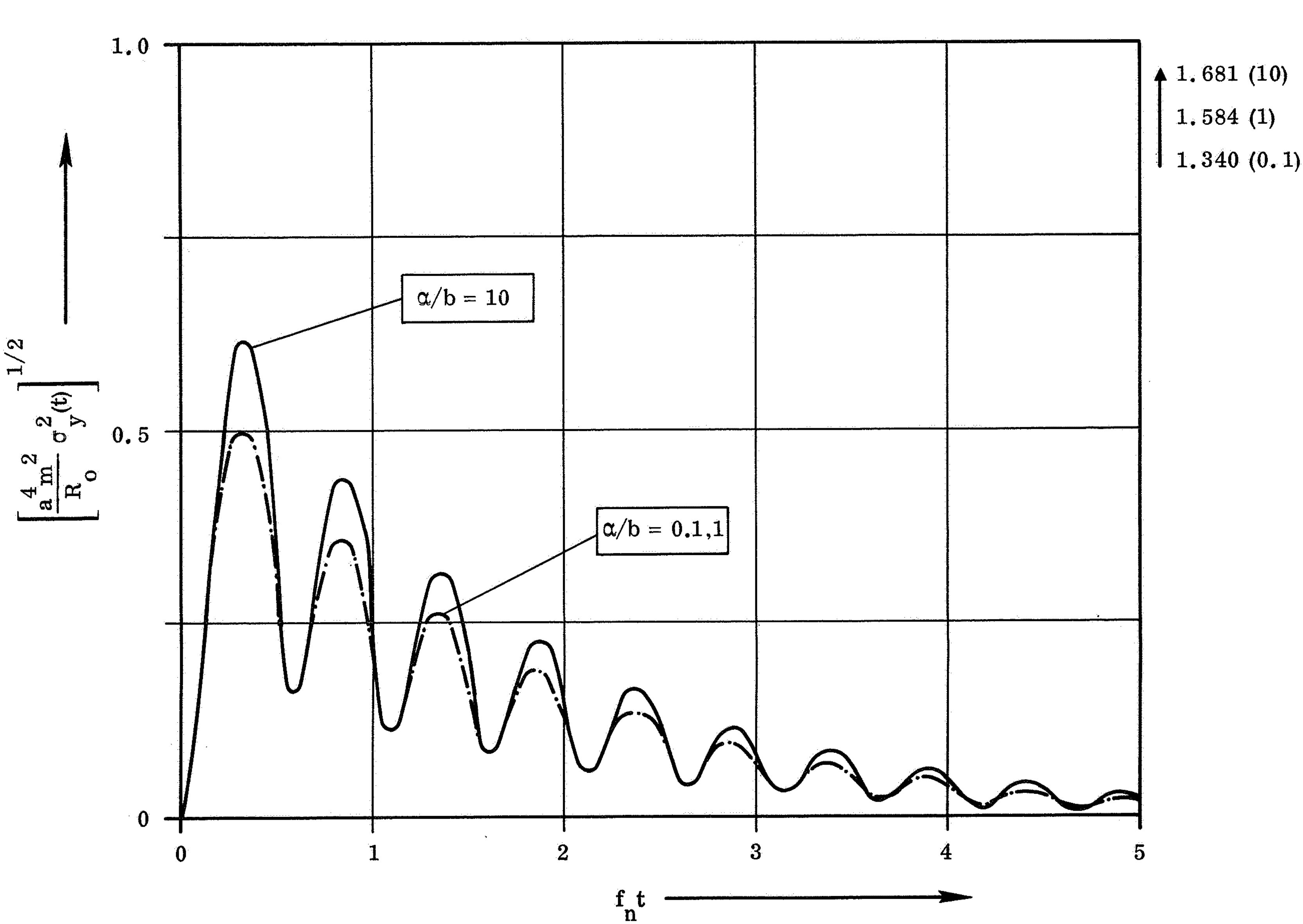

Figure C.16. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 10$, $Q = 5$, $\rho/a = 0.5$

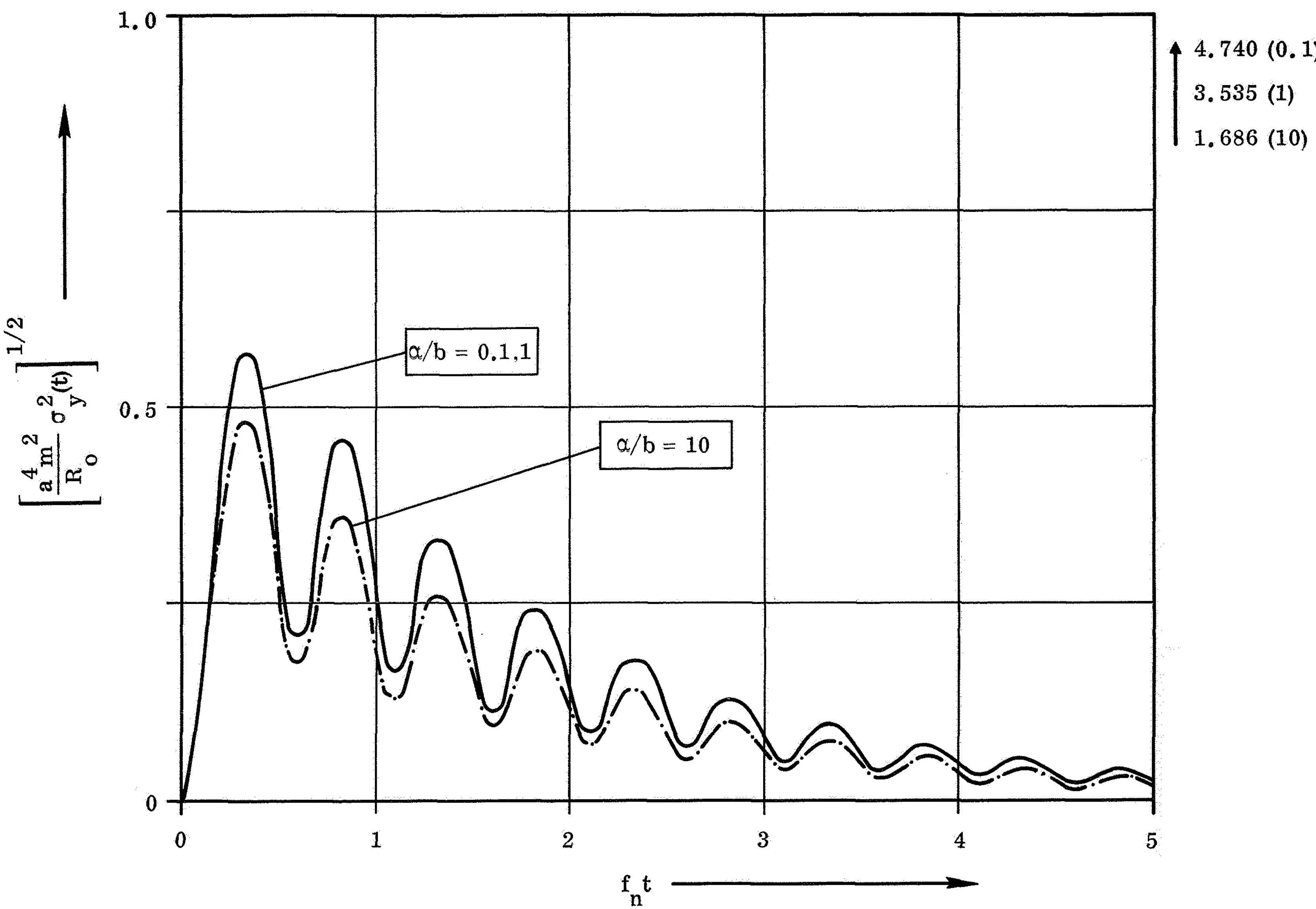

Figure C.17. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 10$, $Q = 5$, $\rho/a = 1$

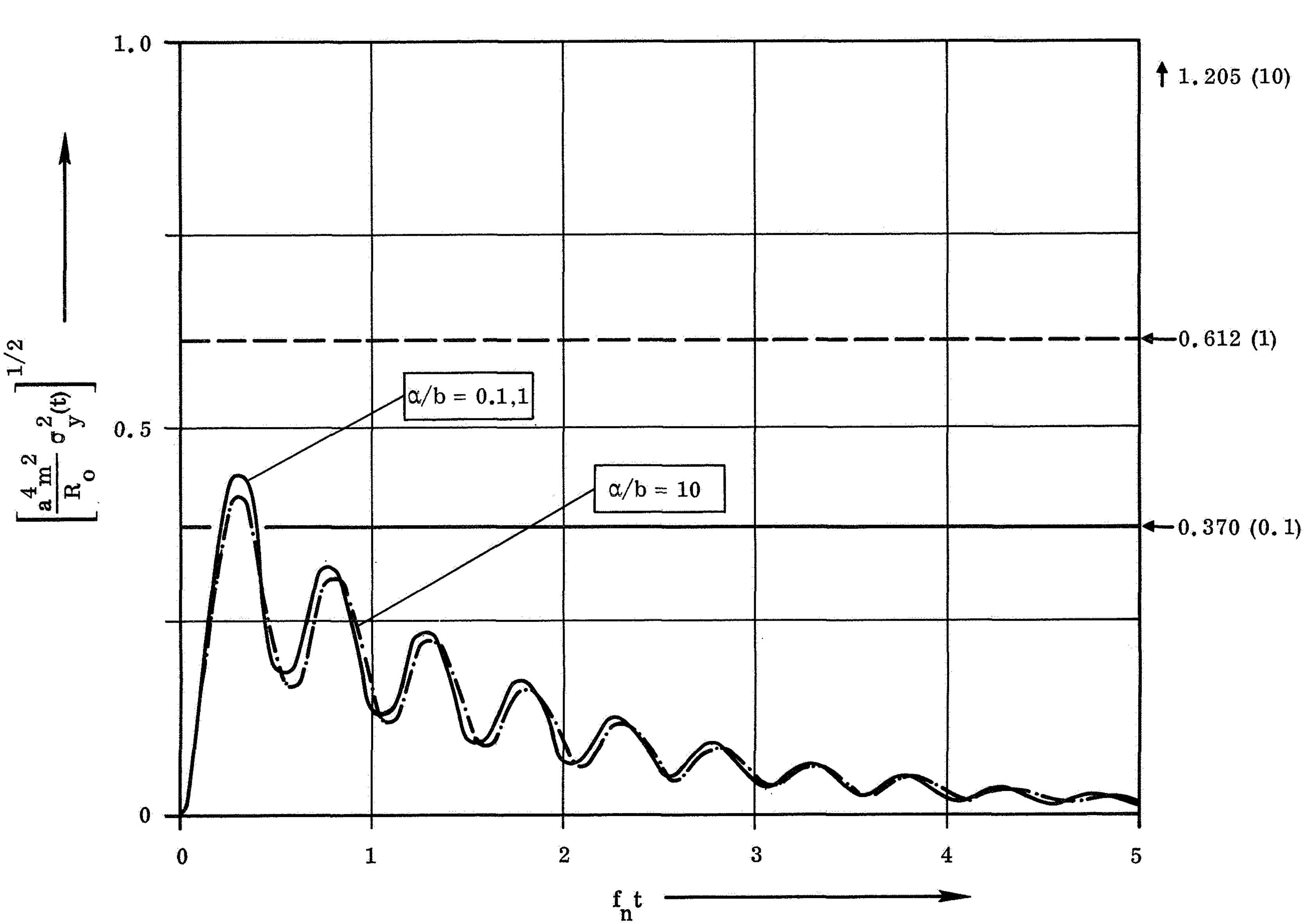

Figure C.18. Normalized System Response to Correlated Noise Modulated by the Exponential Function; $\gamma/b = 10$, $Q = 5$, $\rho/a = 2$

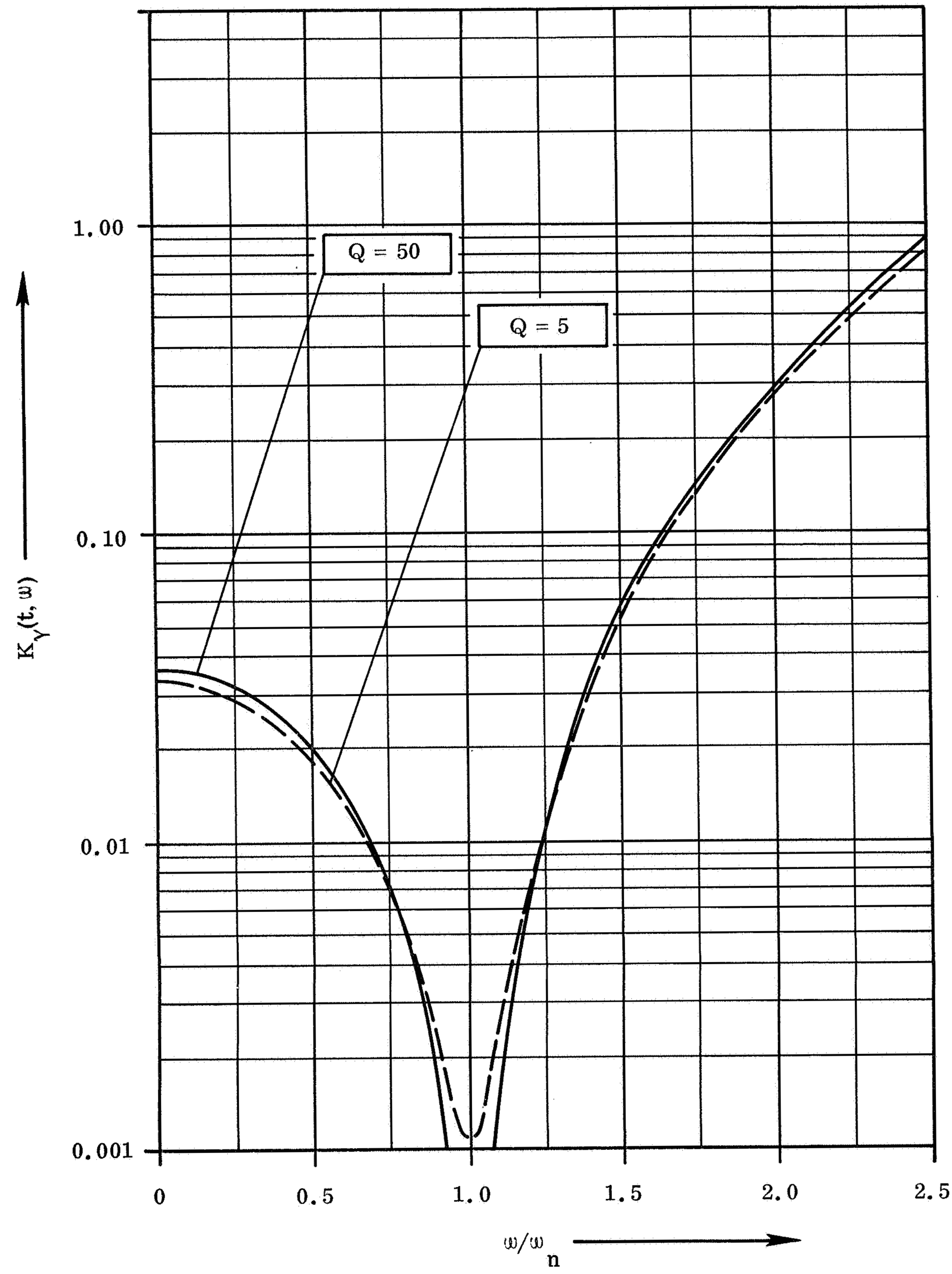

Figure C.19. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.1$, $\gamma/b = 0.1$

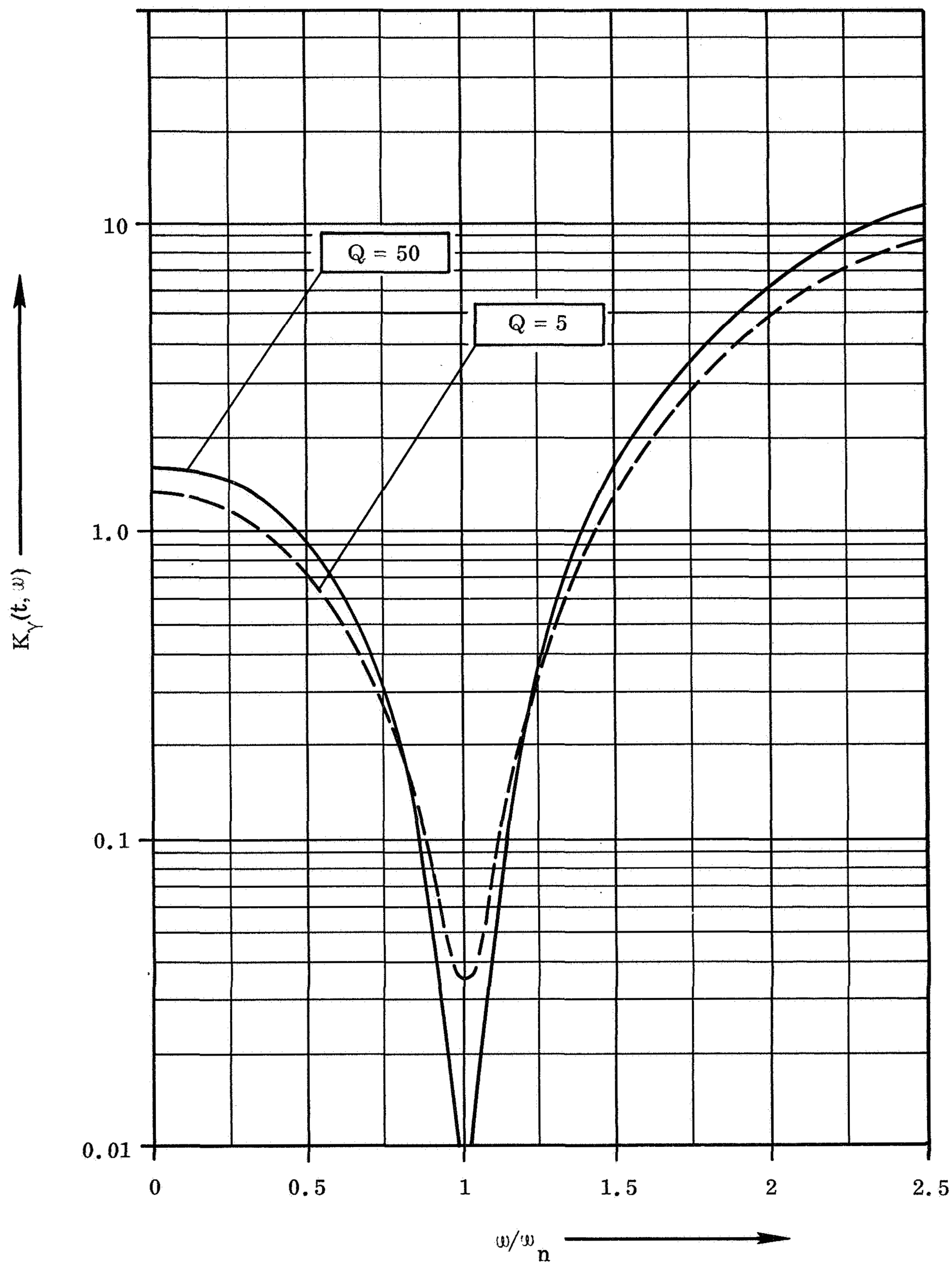

Figure C.20. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.3$, $\gamma/b = 0.1$

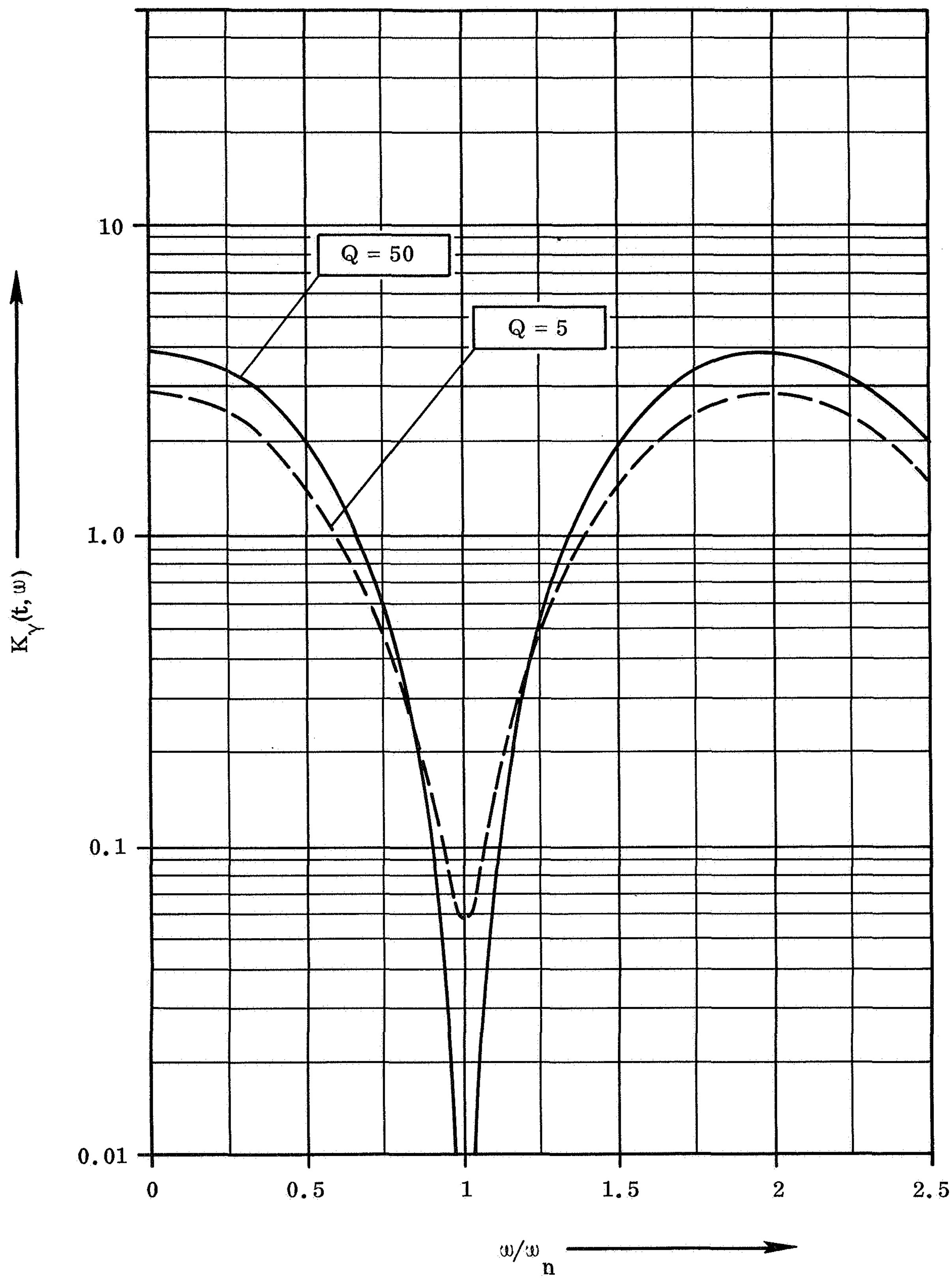

Figure C.21. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.5$, $\gamma/b = 0.1$

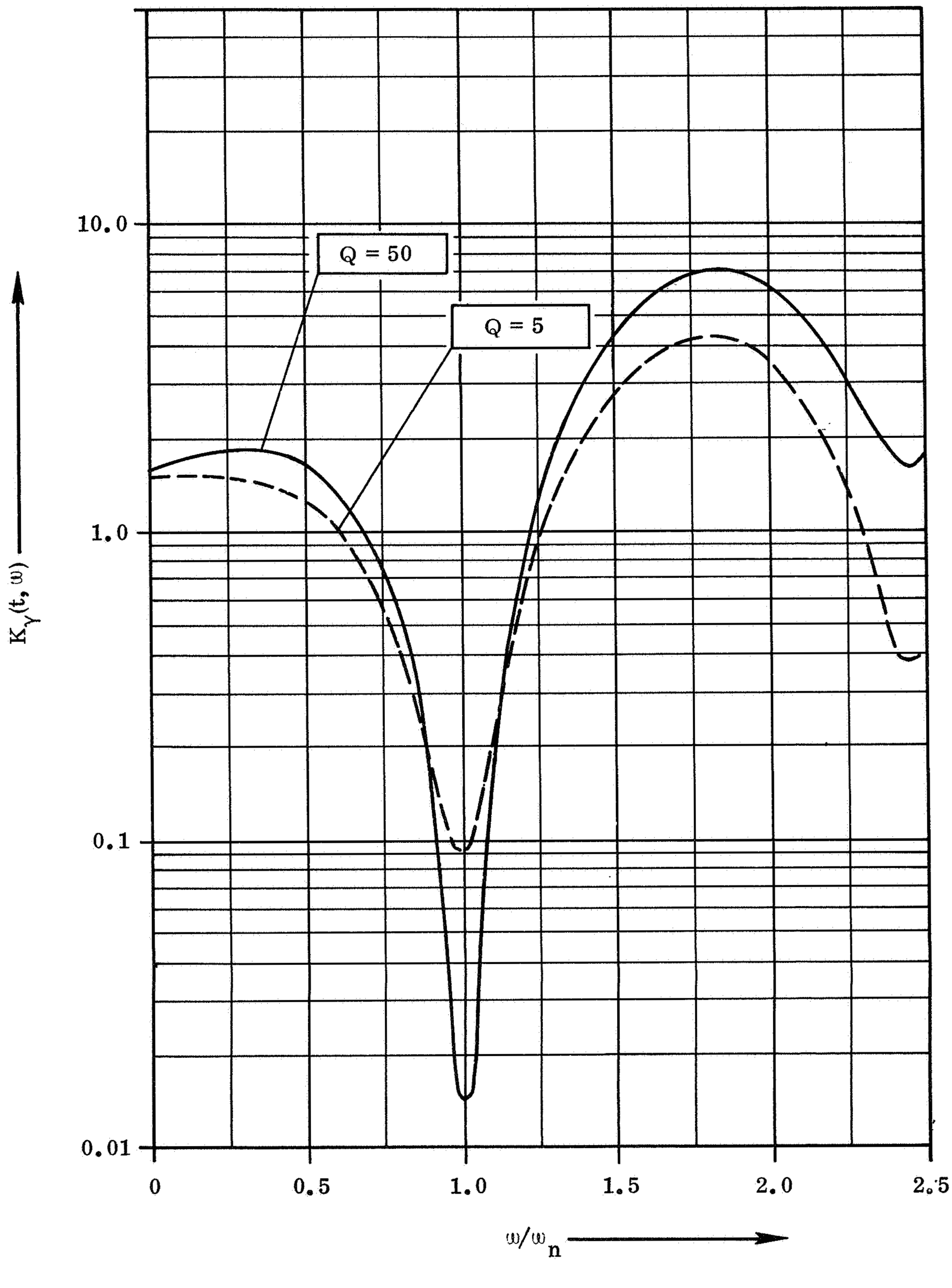

Figure C.22. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.7$, $\gamma/b = 0.1$

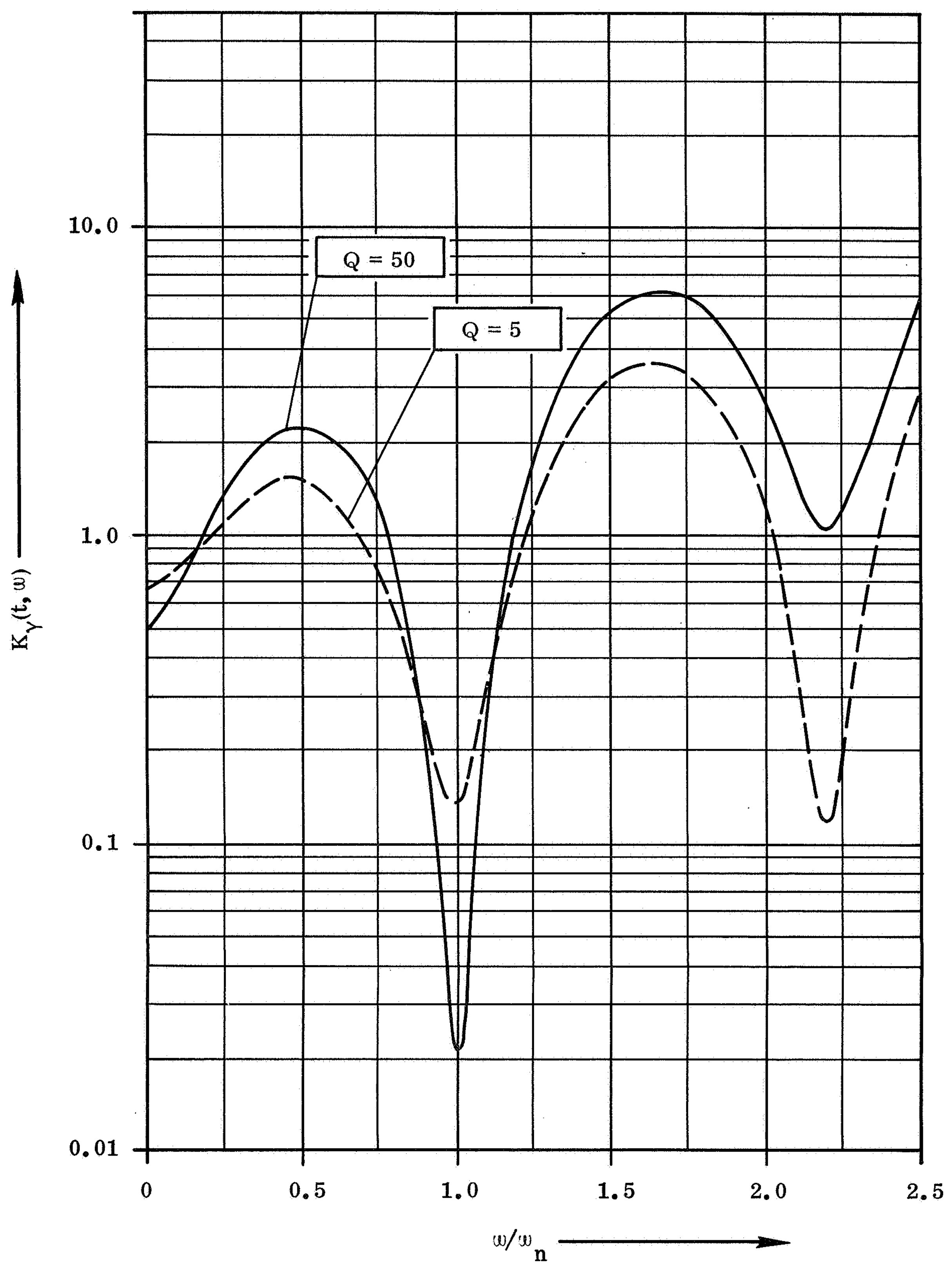

Figure C.23. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.8$, $\gamma/b = 0.1$

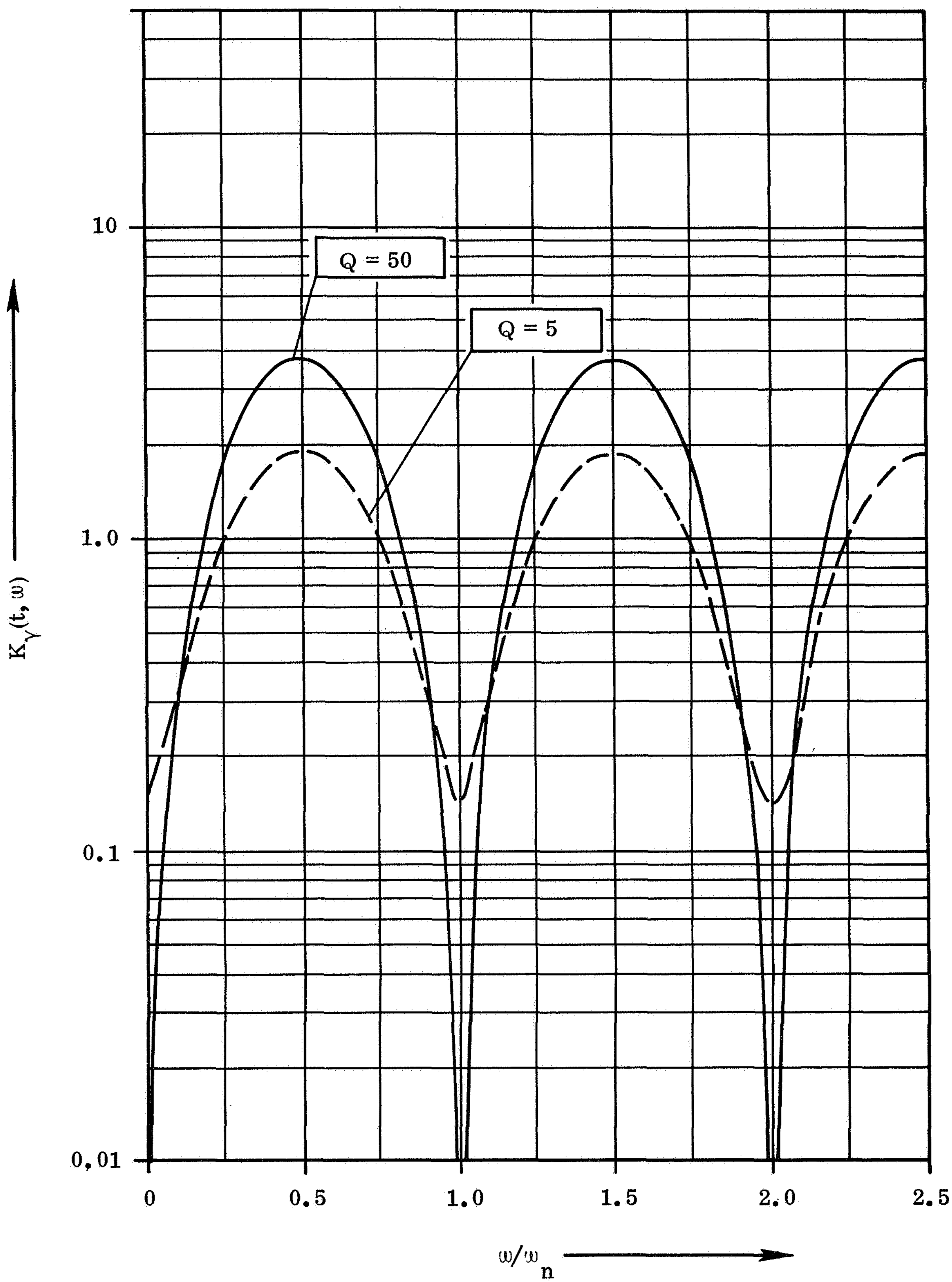

Figure C.24. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 1.0$, $\gamma/b = 0.1$

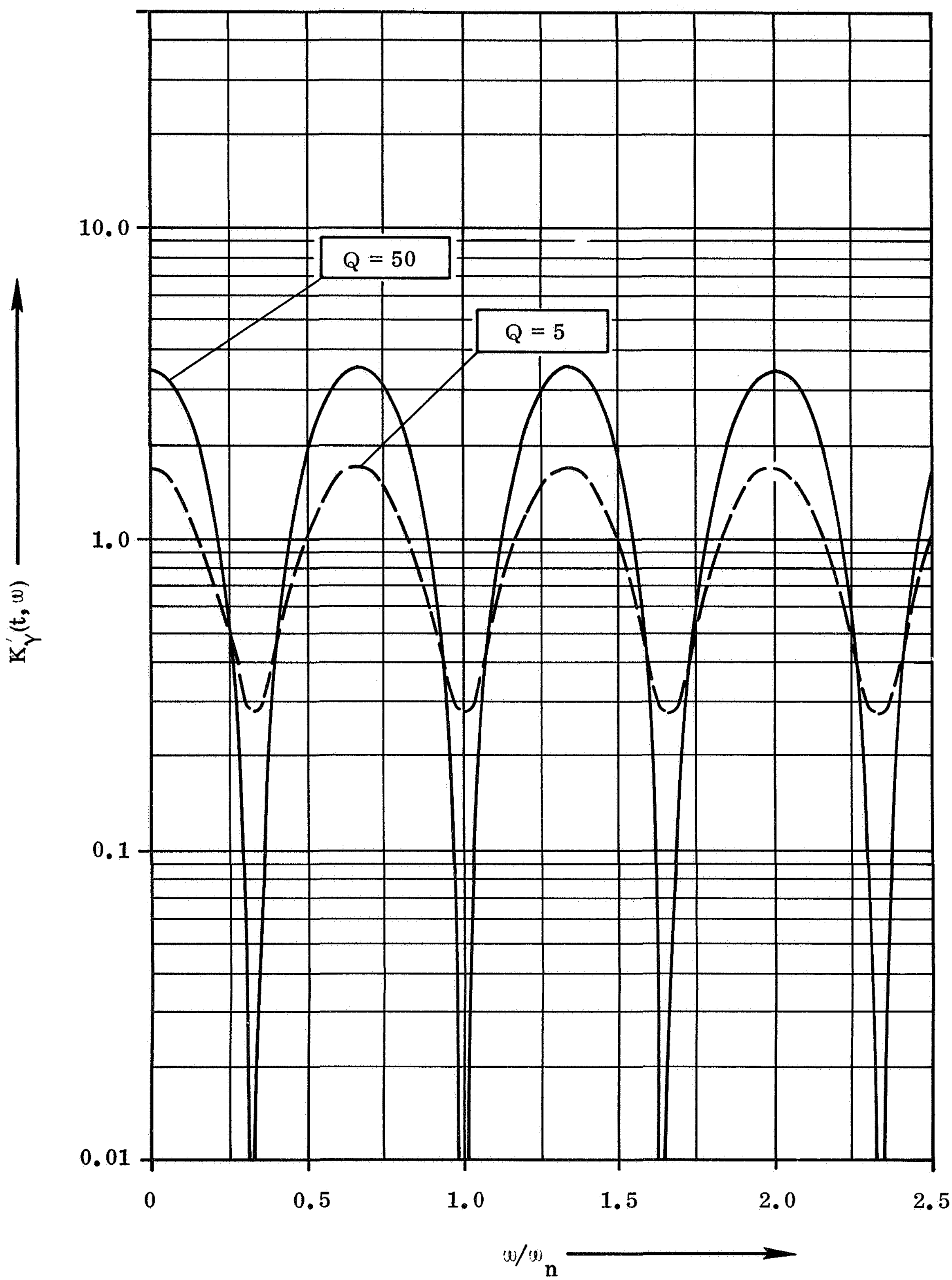

Figure C.25. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 1.5$, $\gamma/b = 0.1$

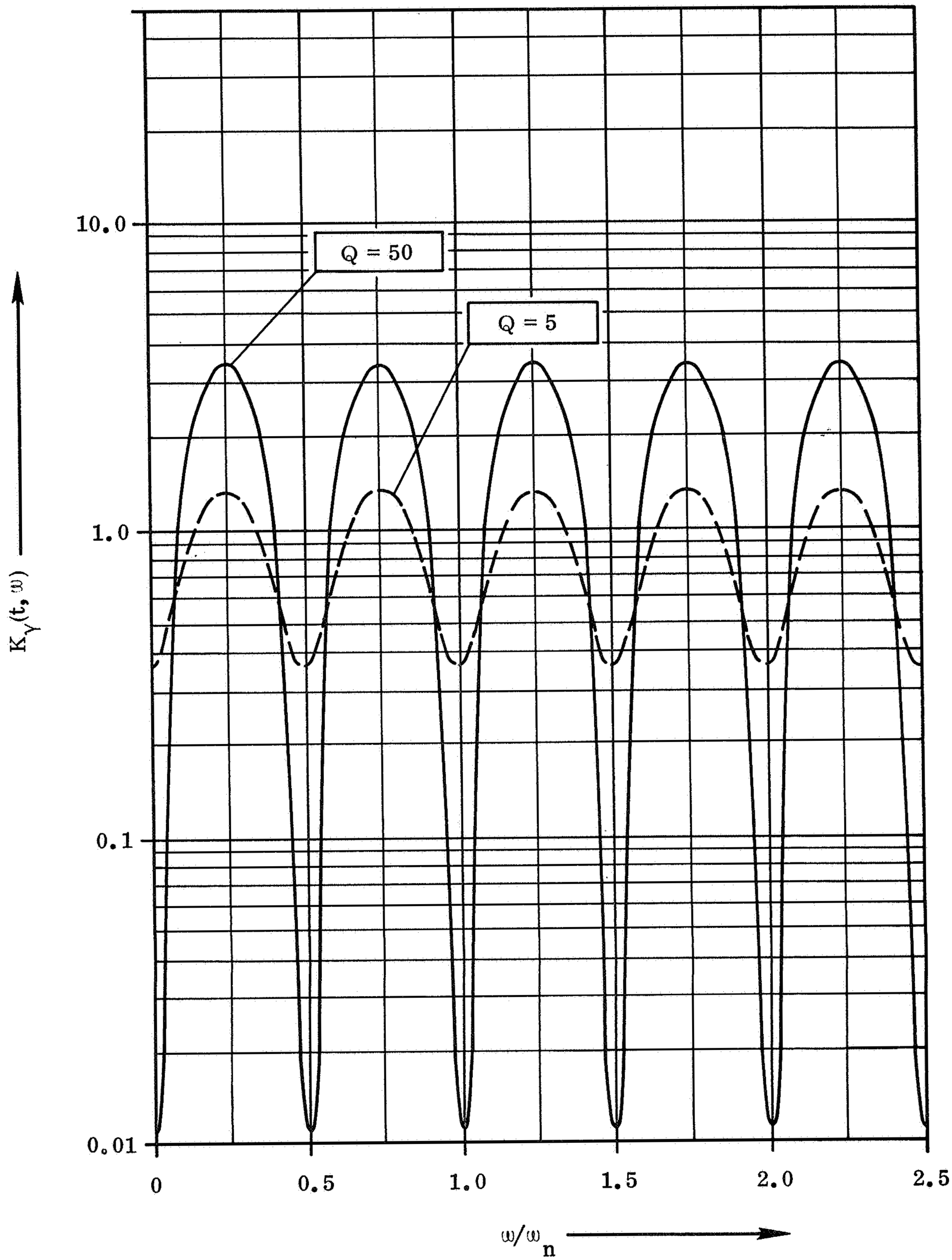

Figure C.26. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 2.0$, $\gamma/b = 0.1$

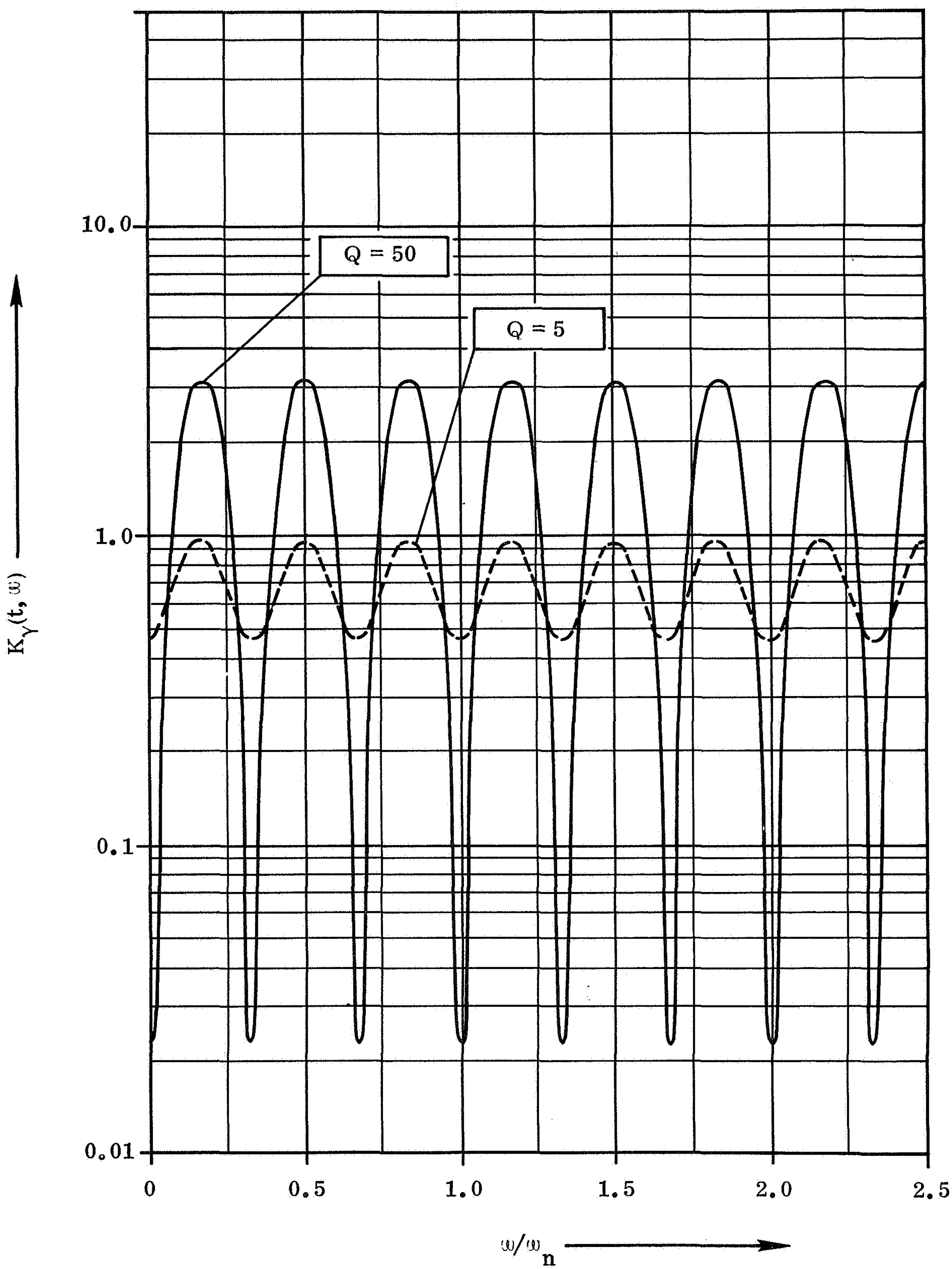

Figure C.27. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 3.0$, $\gamma/b = 0.1$

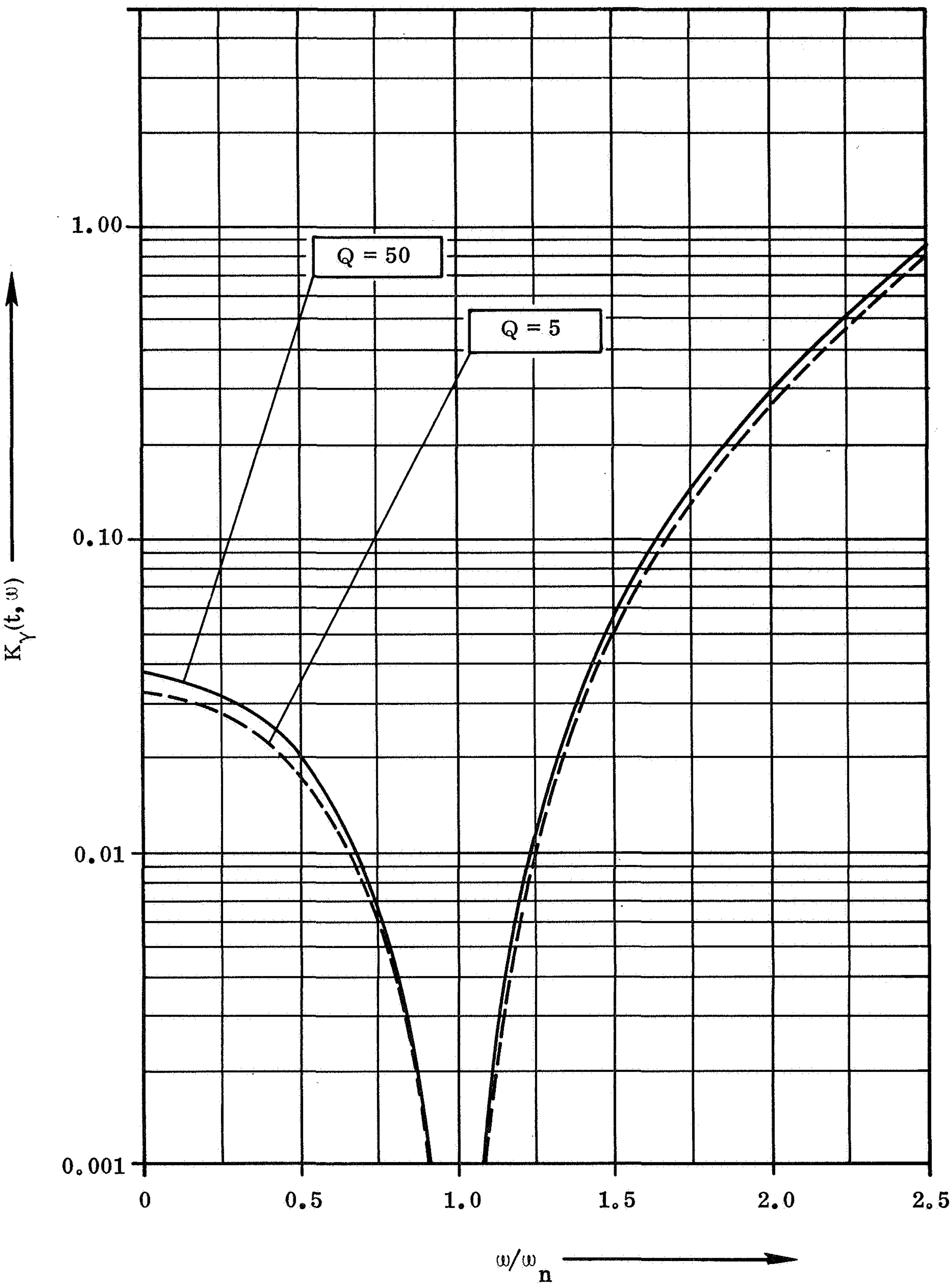

Figure C.28. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.1$, $\gamma/b = 1.0$

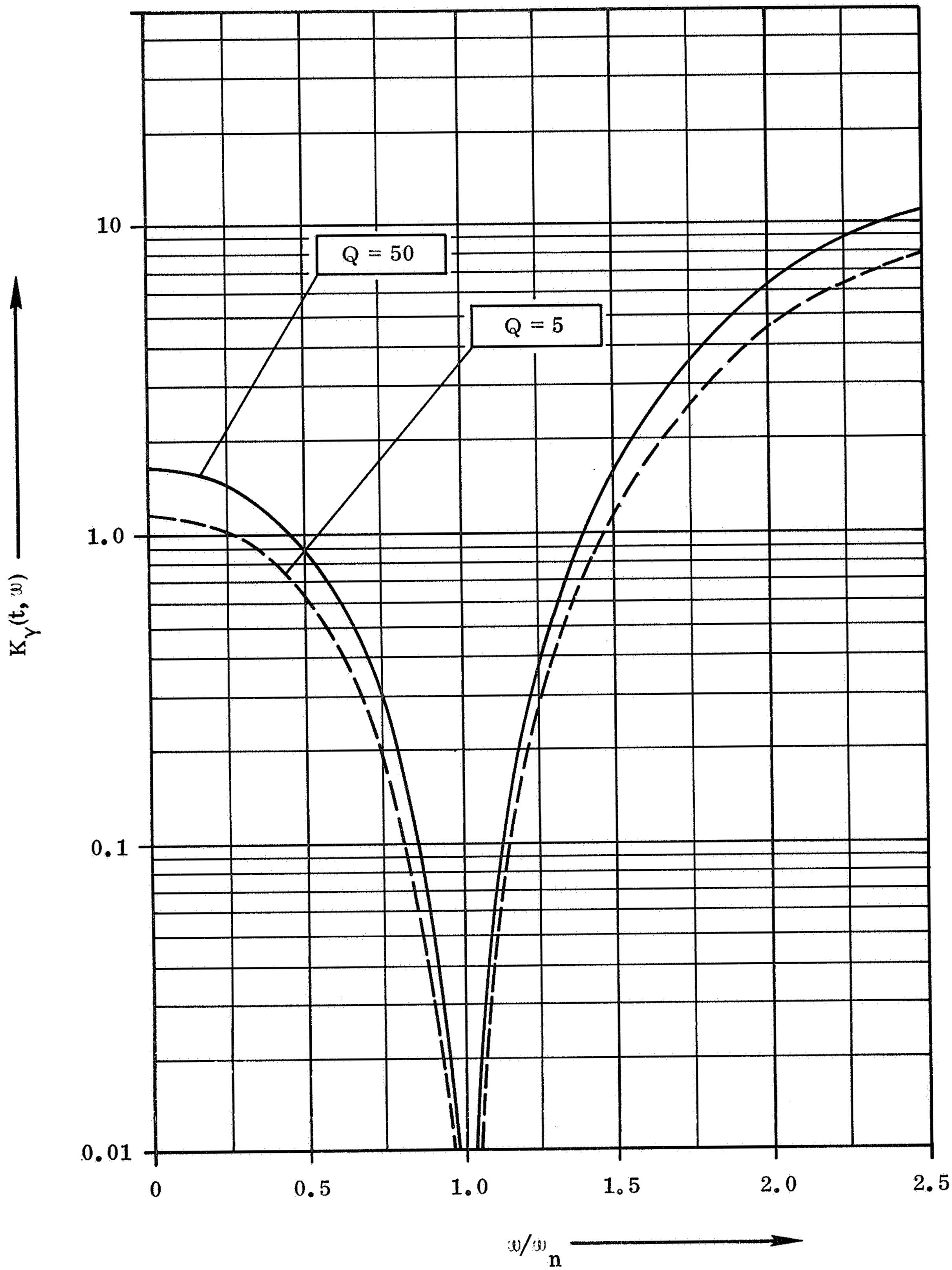

Figure C.29. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.3$, $\gamma/b = 1.0$

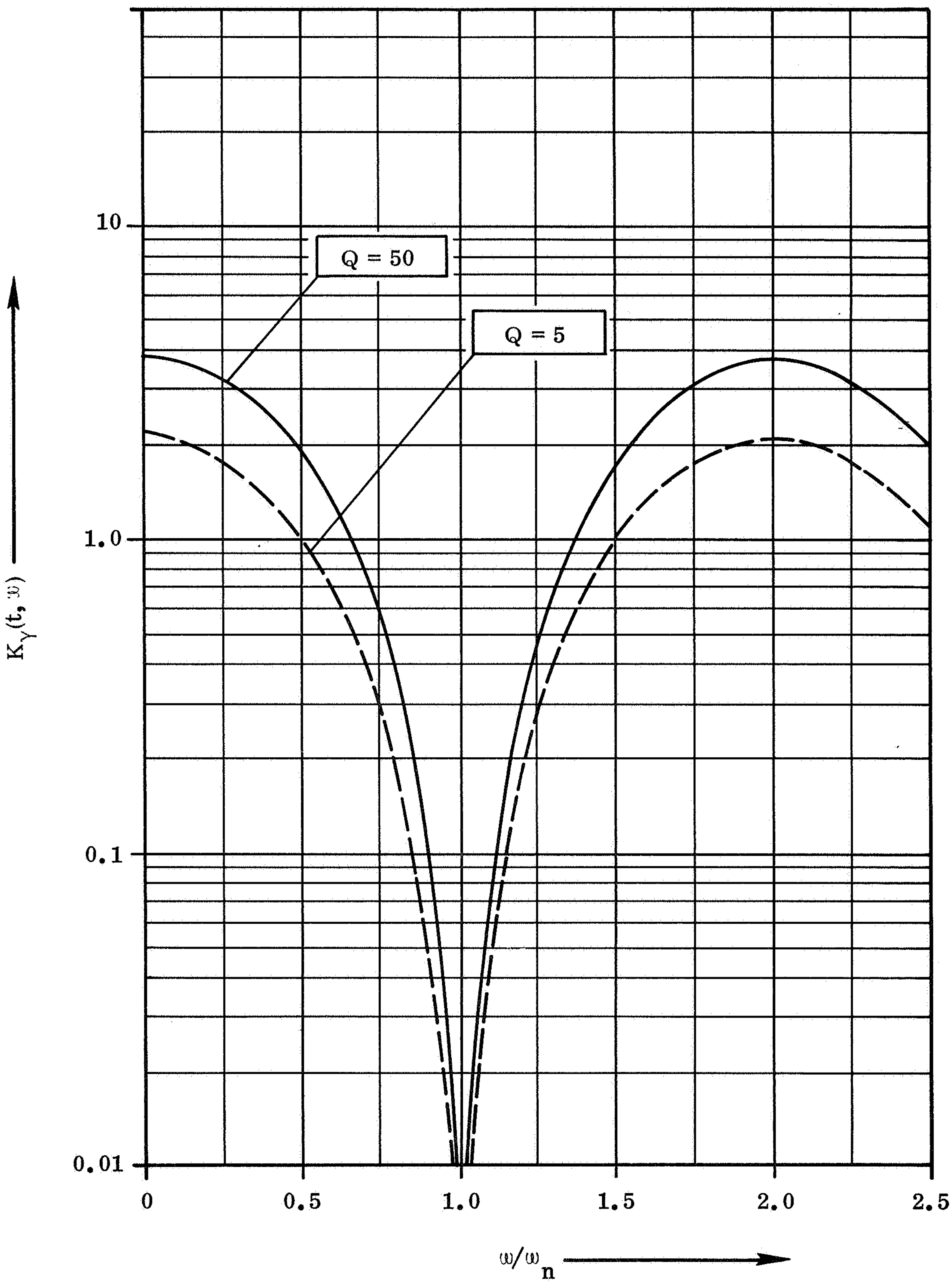

Figure C.30. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.5$, $\gamma/b = 1.0$

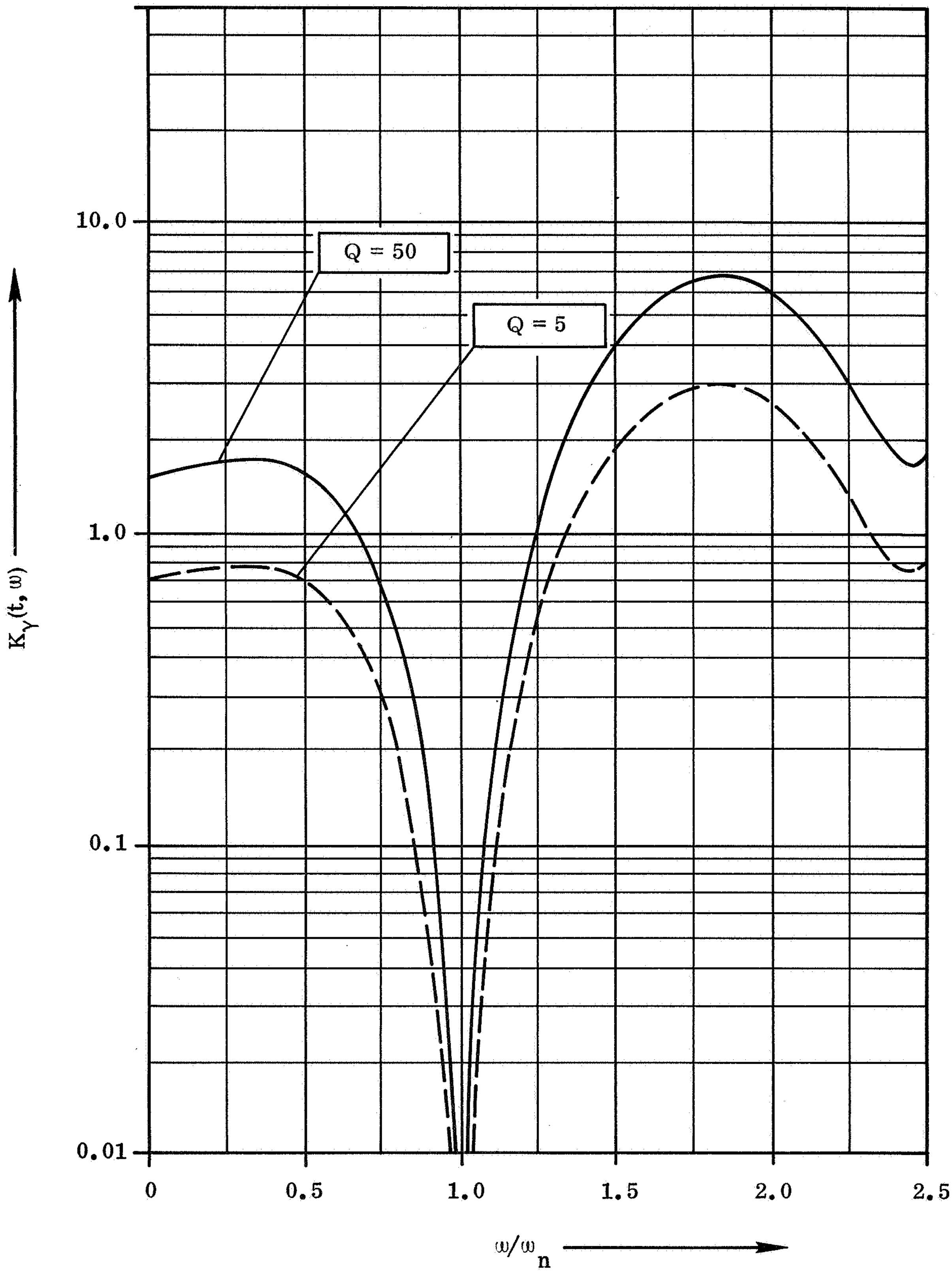

Figure C.31. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.7$, $\gamma/b = 1.0$

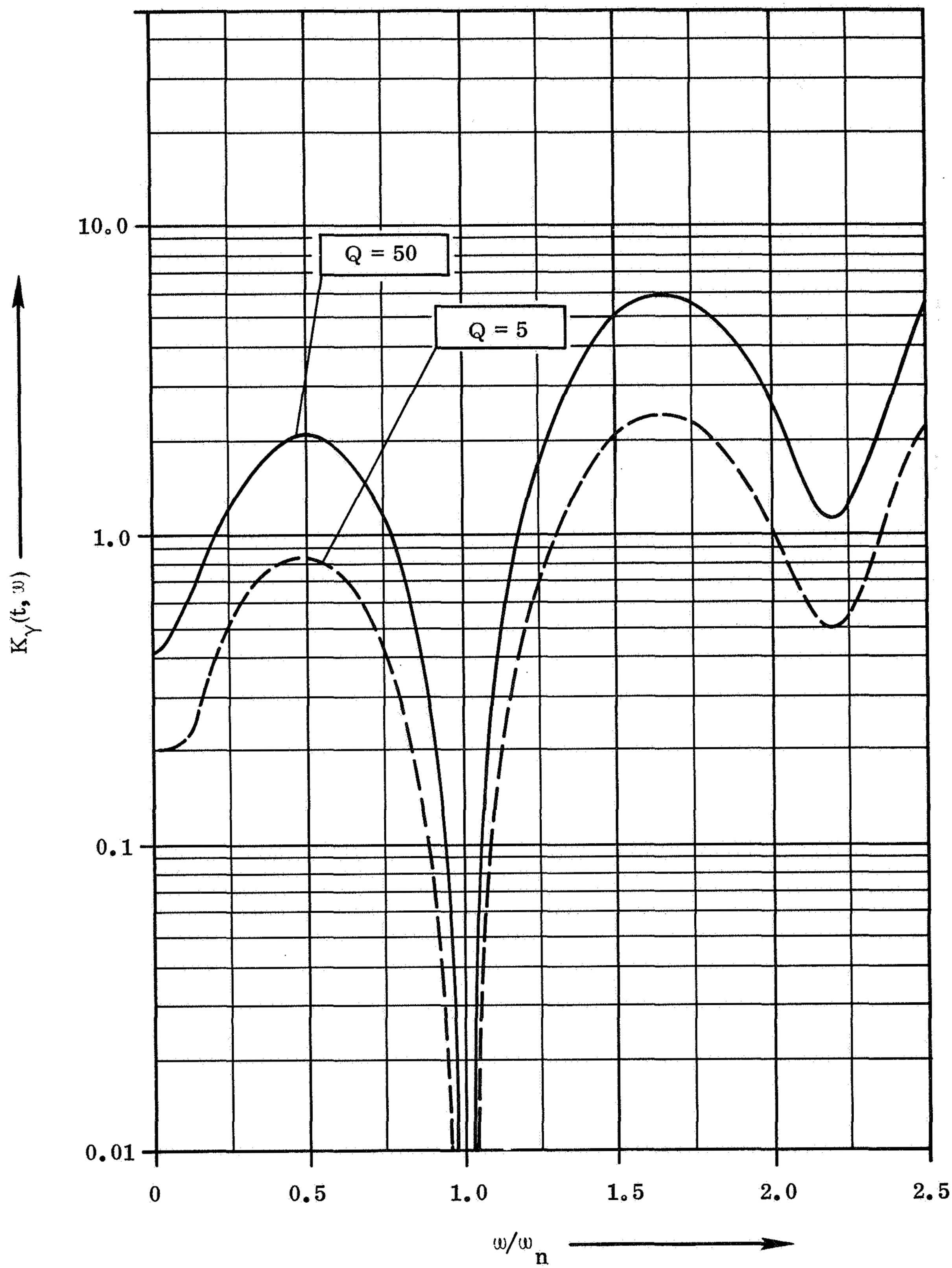

Figure C.32. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.8$, $\gamma/b = 1.0$

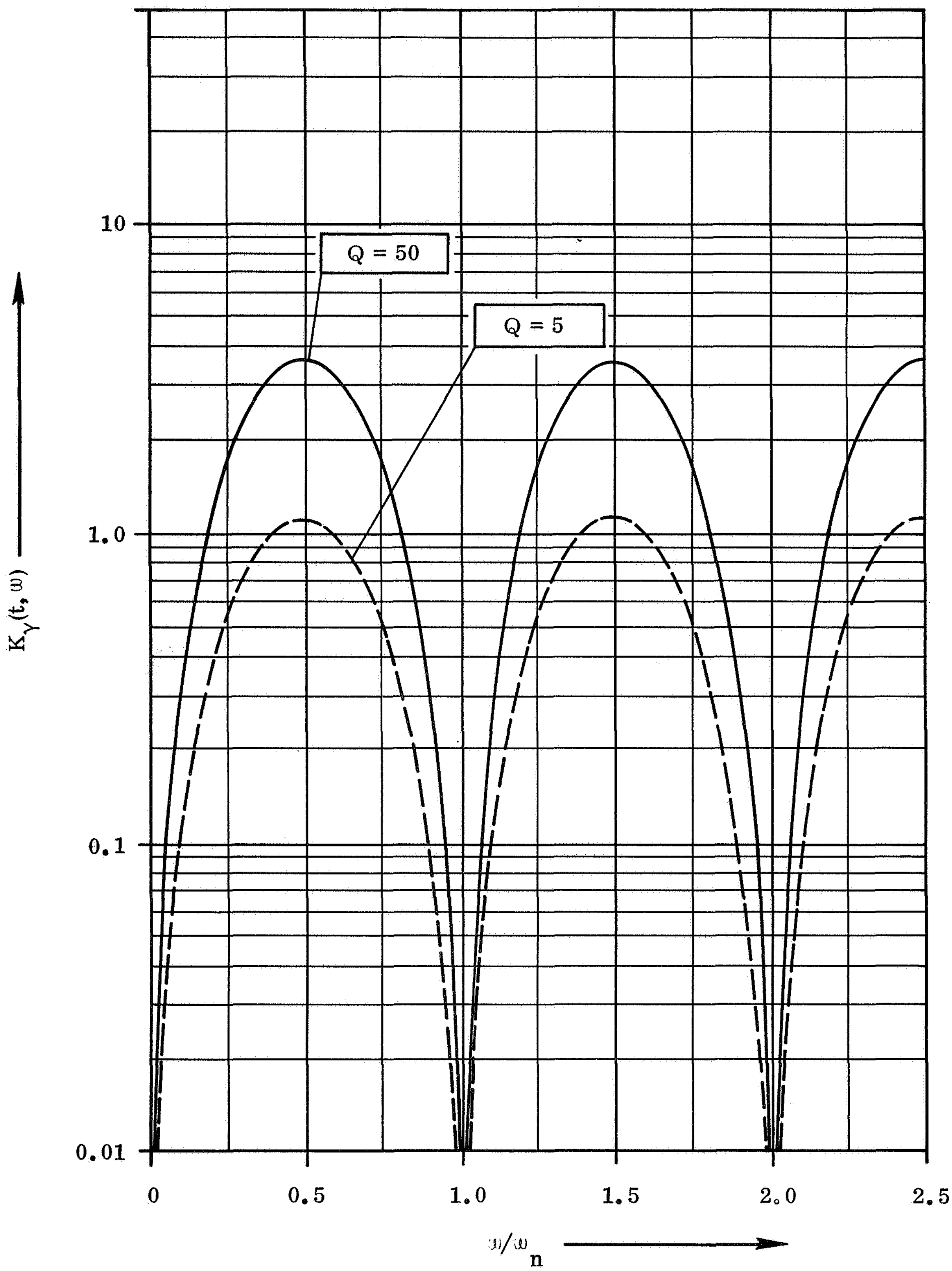

Figure C.33. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 1.0$, $\gamma/b = 1.0$

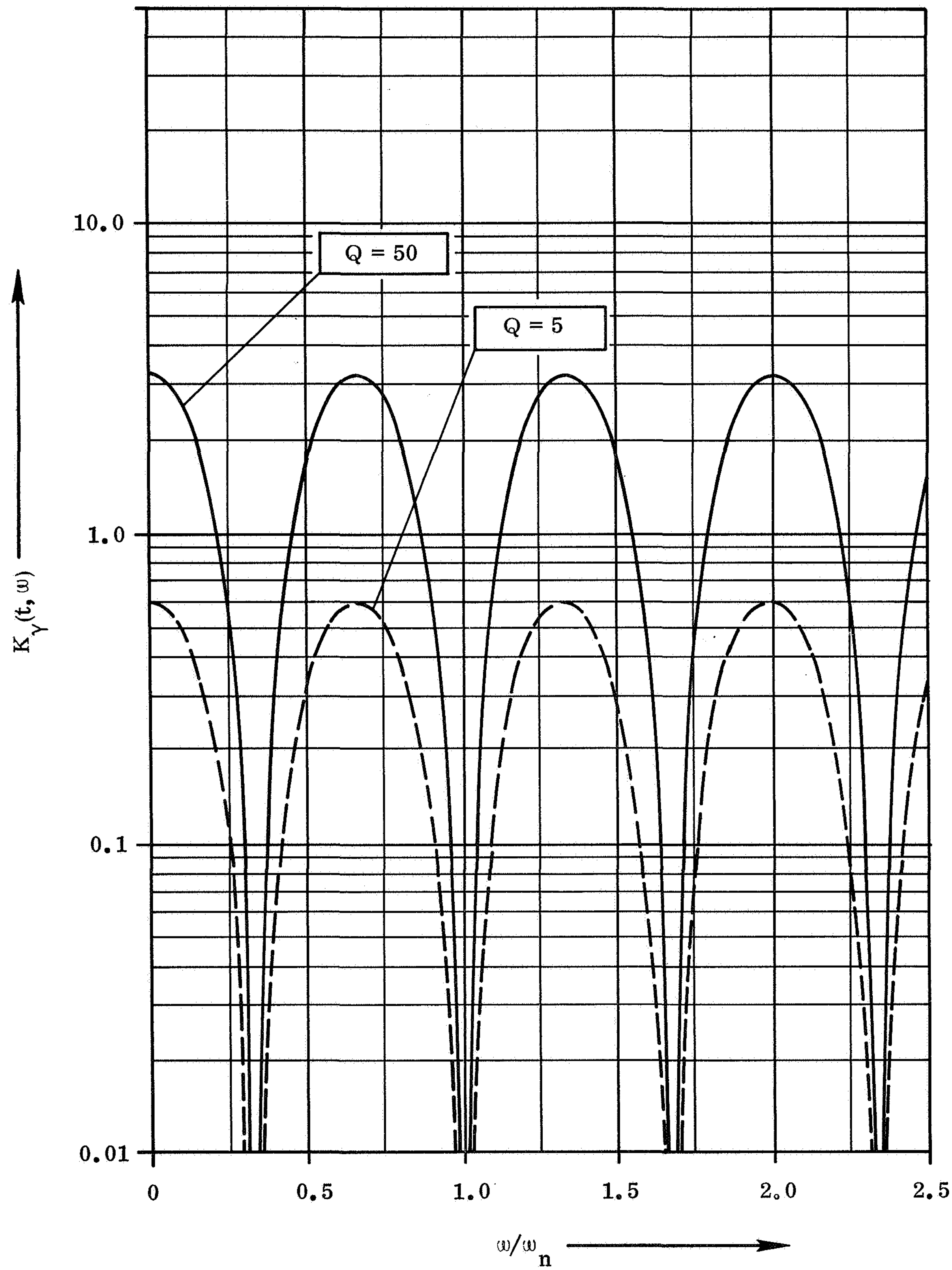

Figure C.34. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 1.5$, $\gamma/b = 1.0$

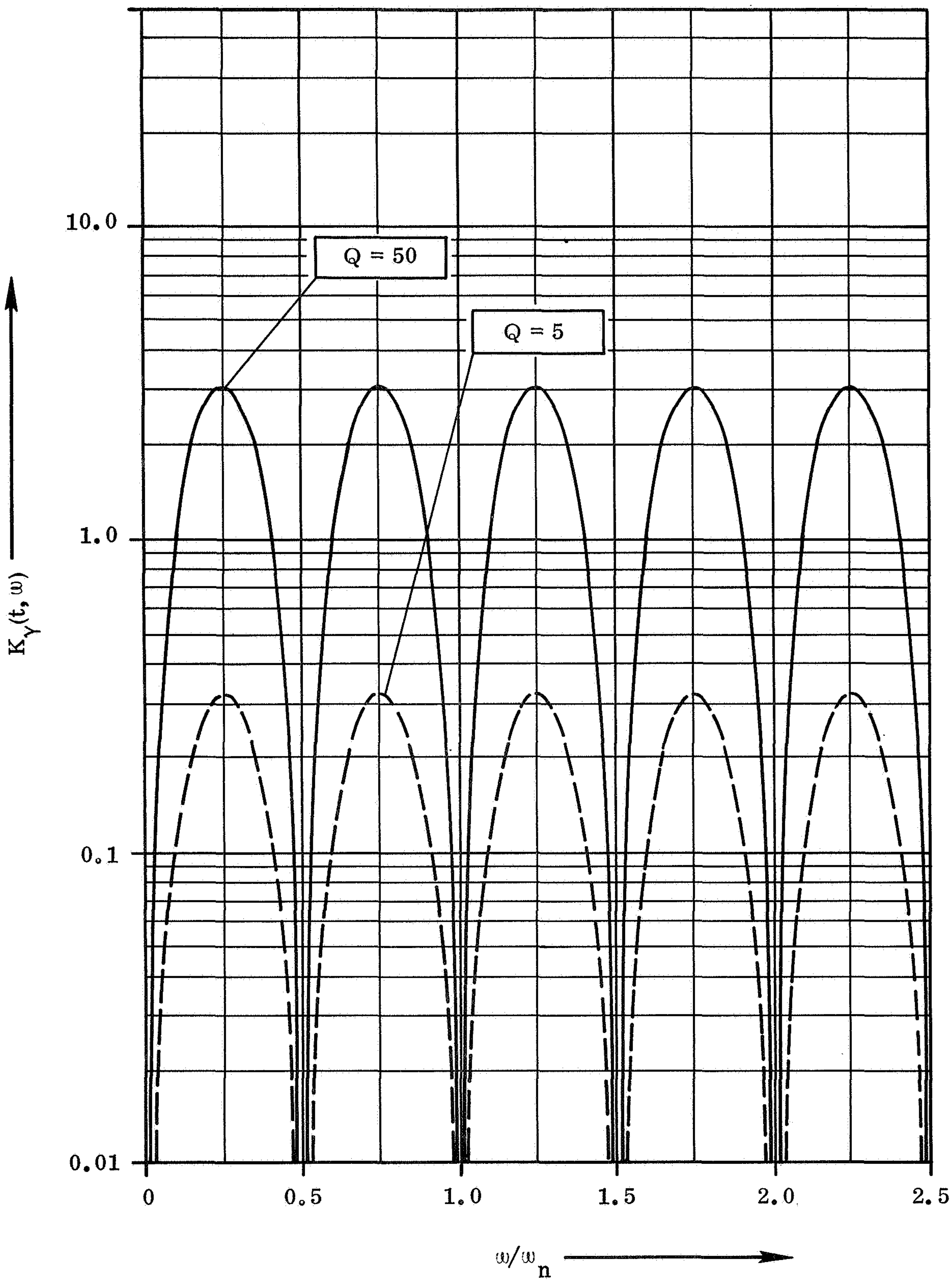

Figure C.35. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 2.0$, $\gamma/b = 1.0$

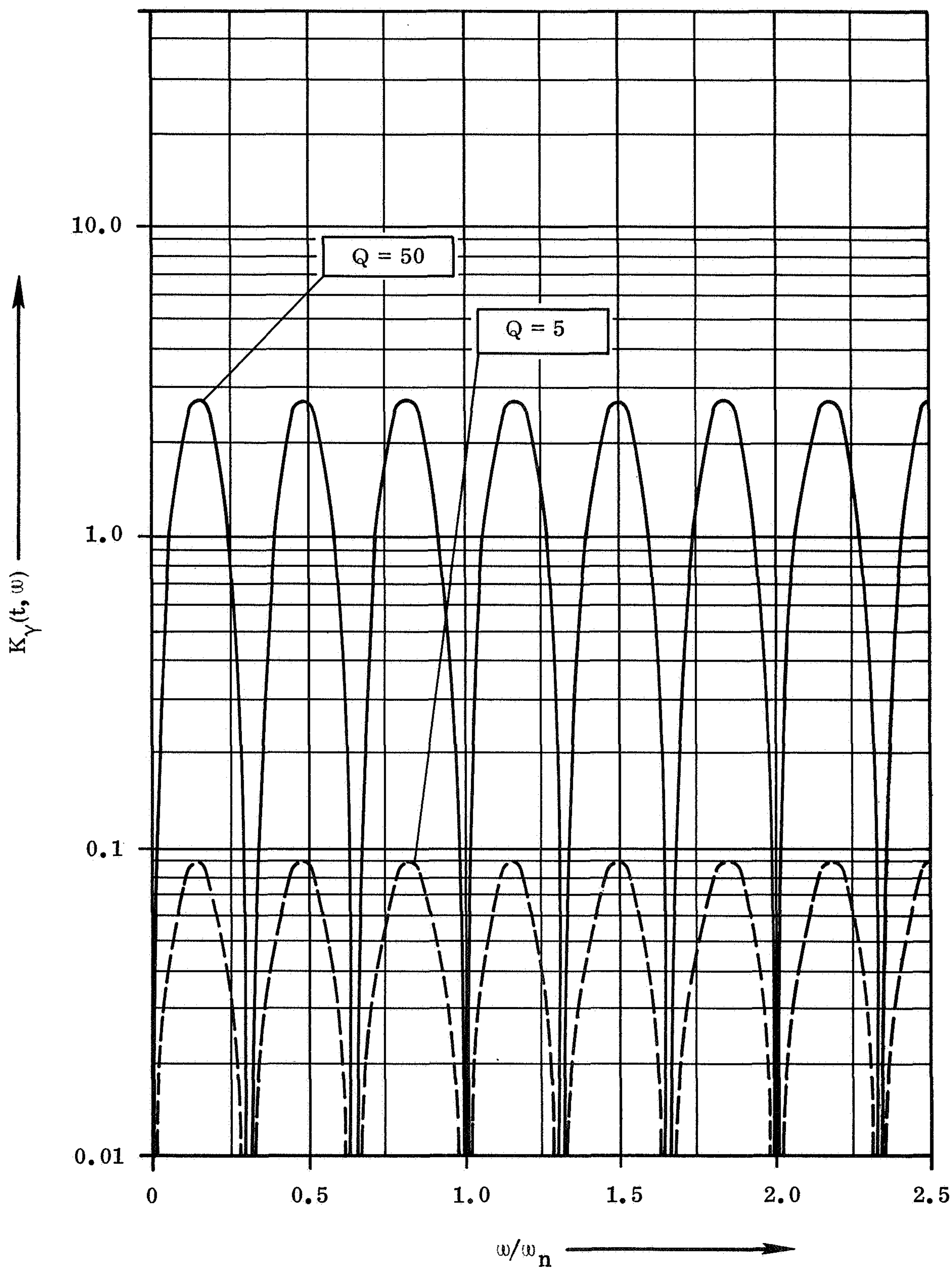

Figure C.36. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 3.0$, $\gamma/b = 1.0$

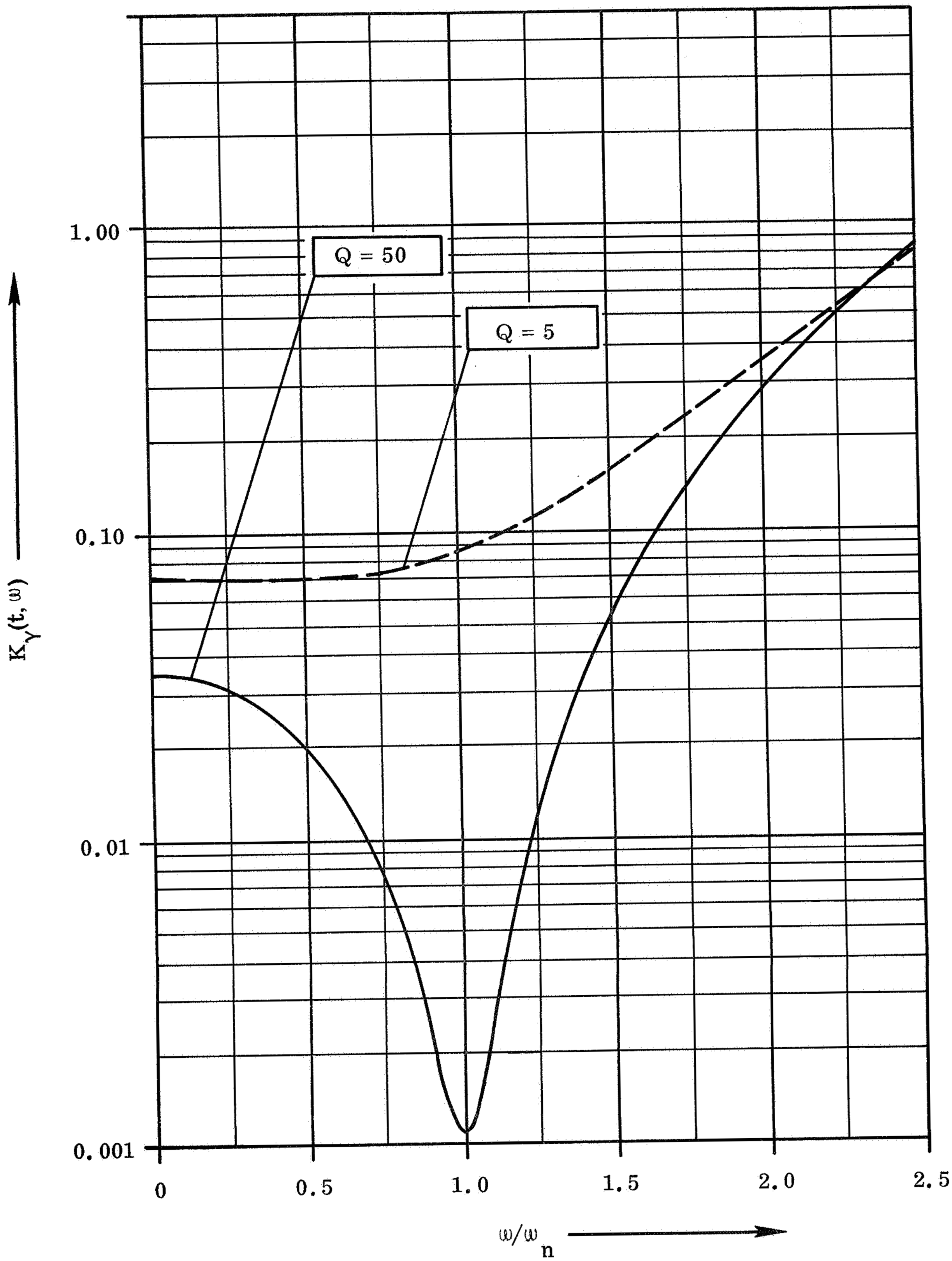

Figure C.37. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.1$, $\gamma/b = 10$

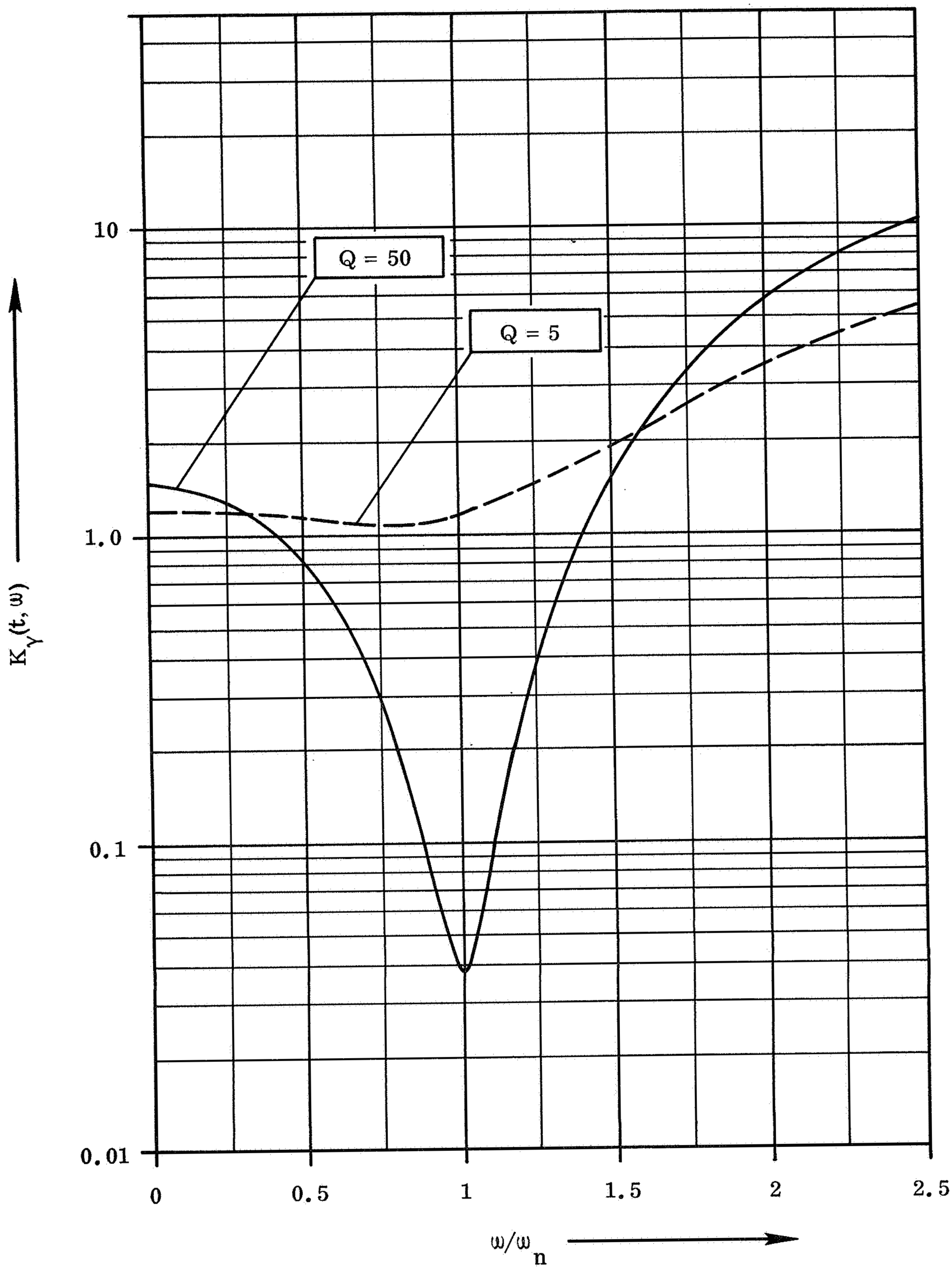

Figure C.38. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.3$, $\gamma/b = 10$

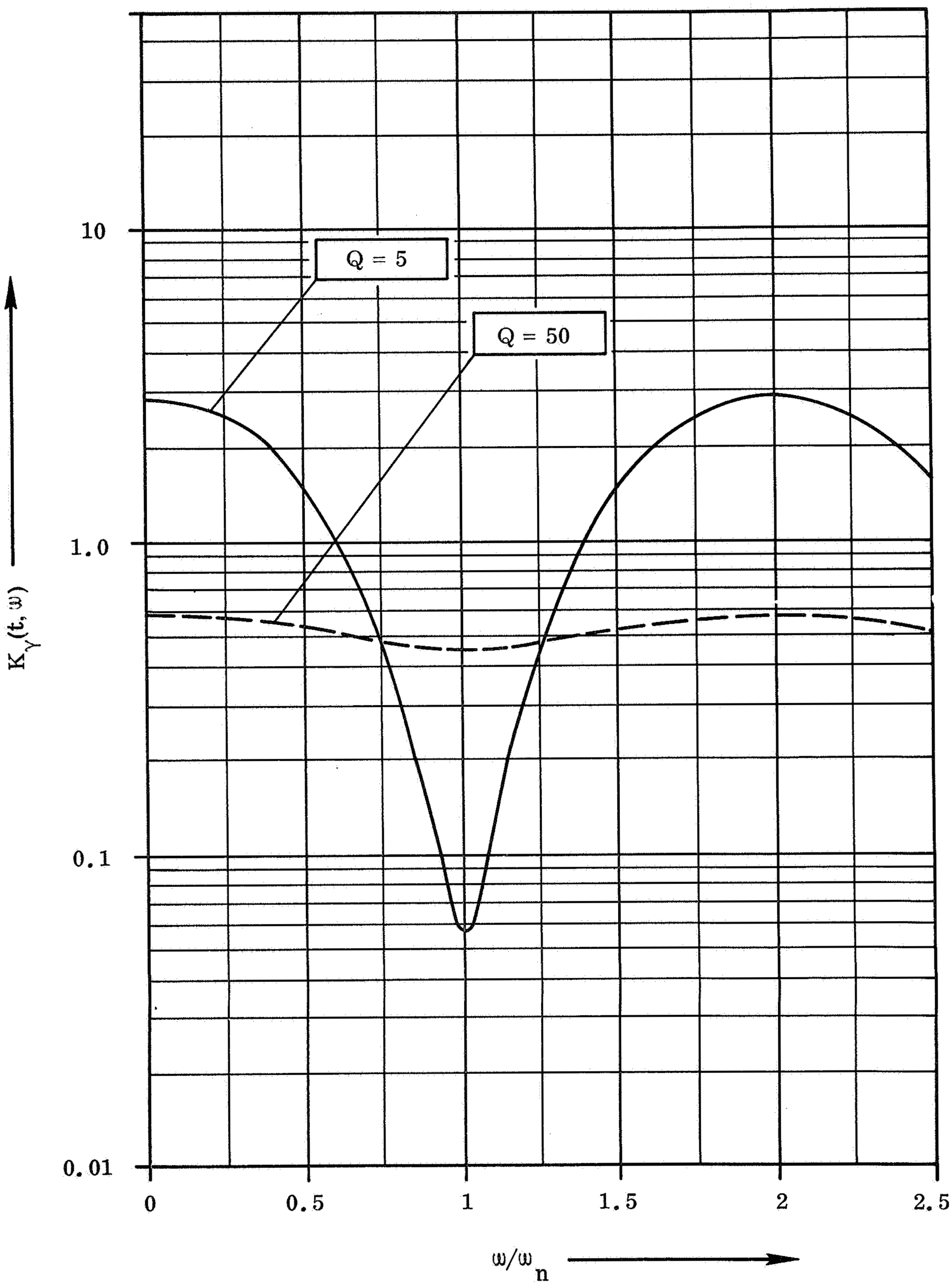

Figure C.39. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.5$, $\gamma/b = 10$

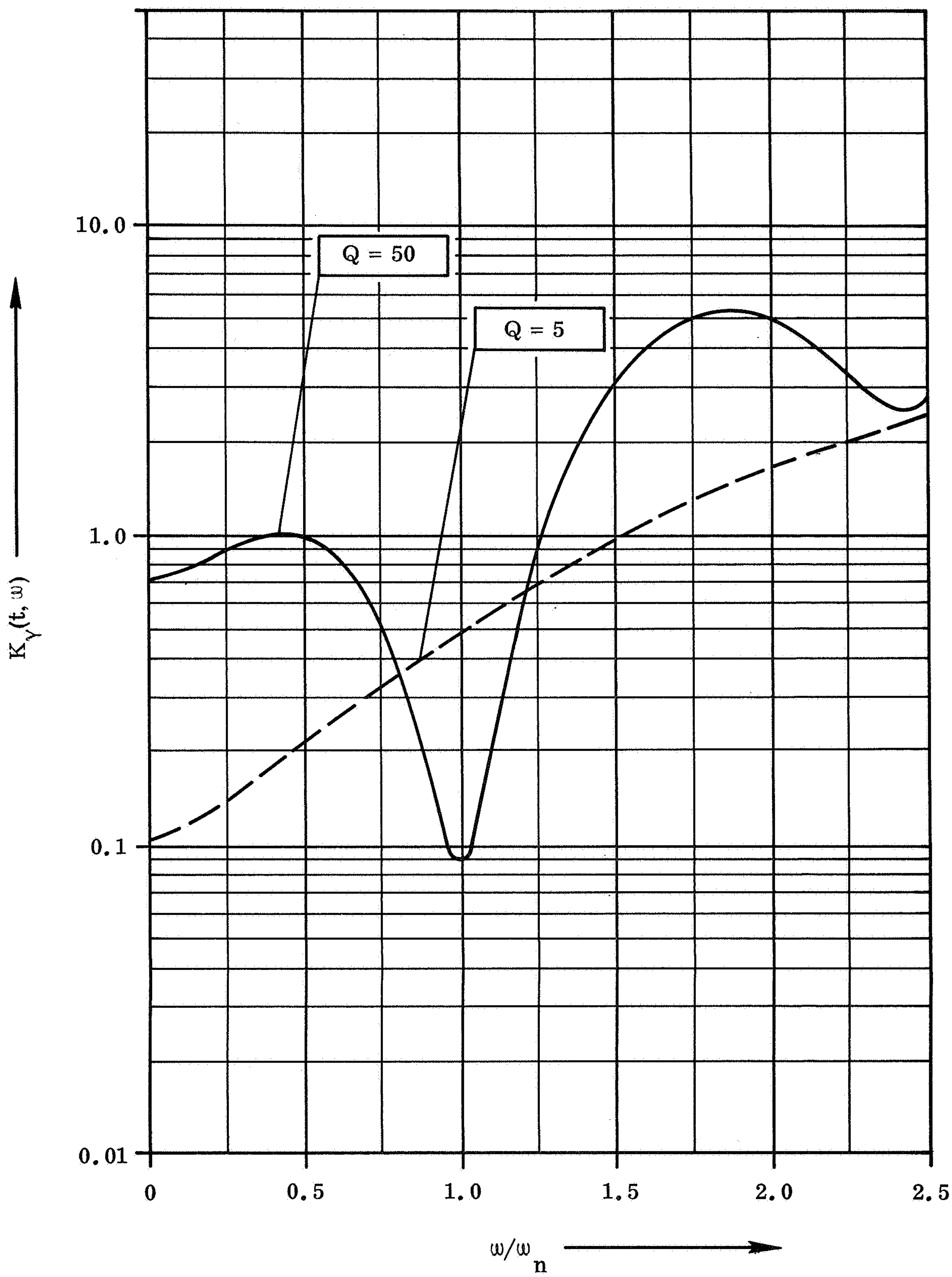

Figure C. 40. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 0.7$, $\gamma/b = 10.0$

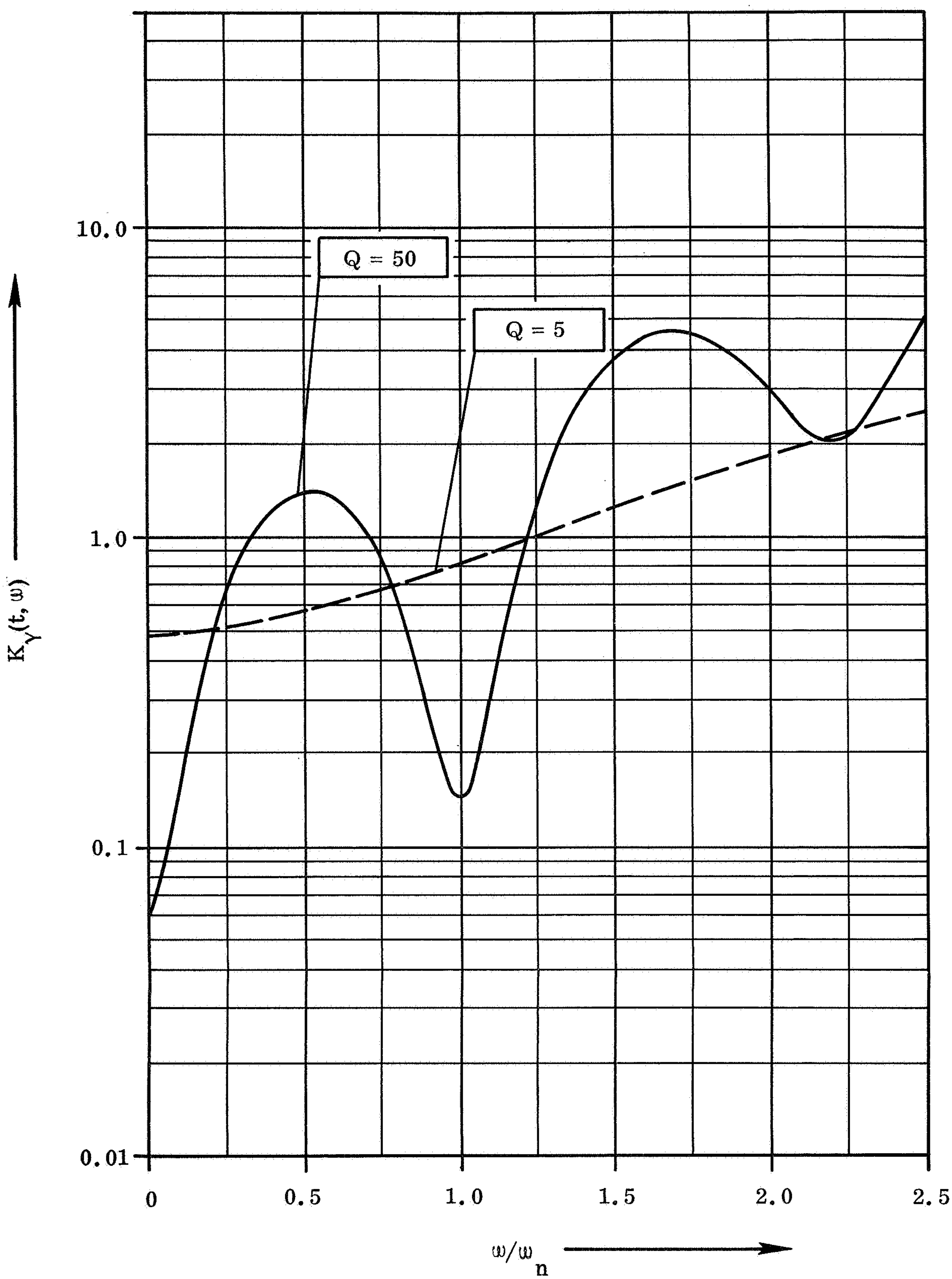

Figure C.41. Shaping Filter $K_\gamma(t,\omega)$ with $f_n t = 0.8$, $\gamma/b = 10.0$

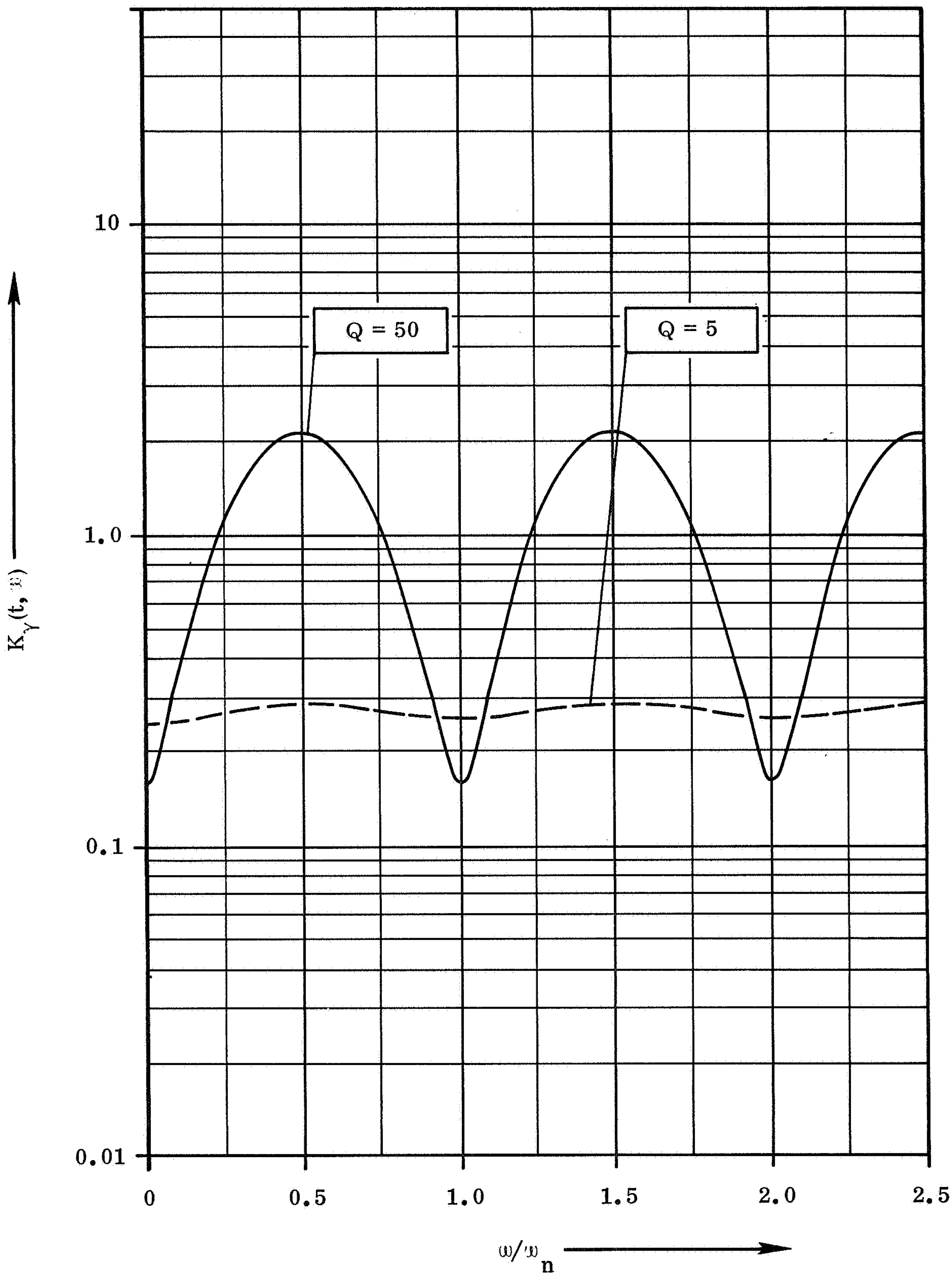

Figure C.42. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 1.0$, $\gamma/b = 10$

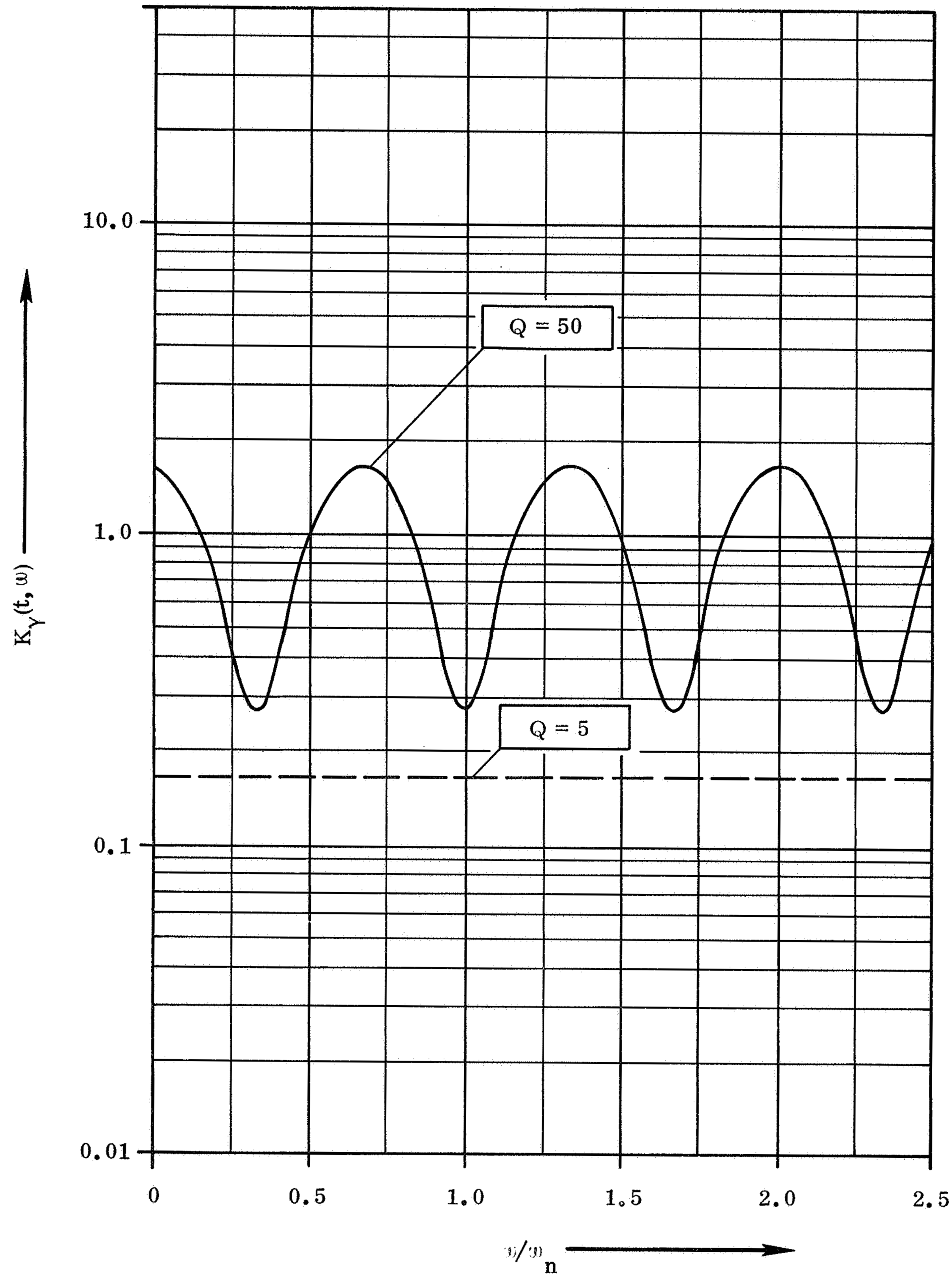

Figure C.43. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 1.5$, $\gamma/b = 10$

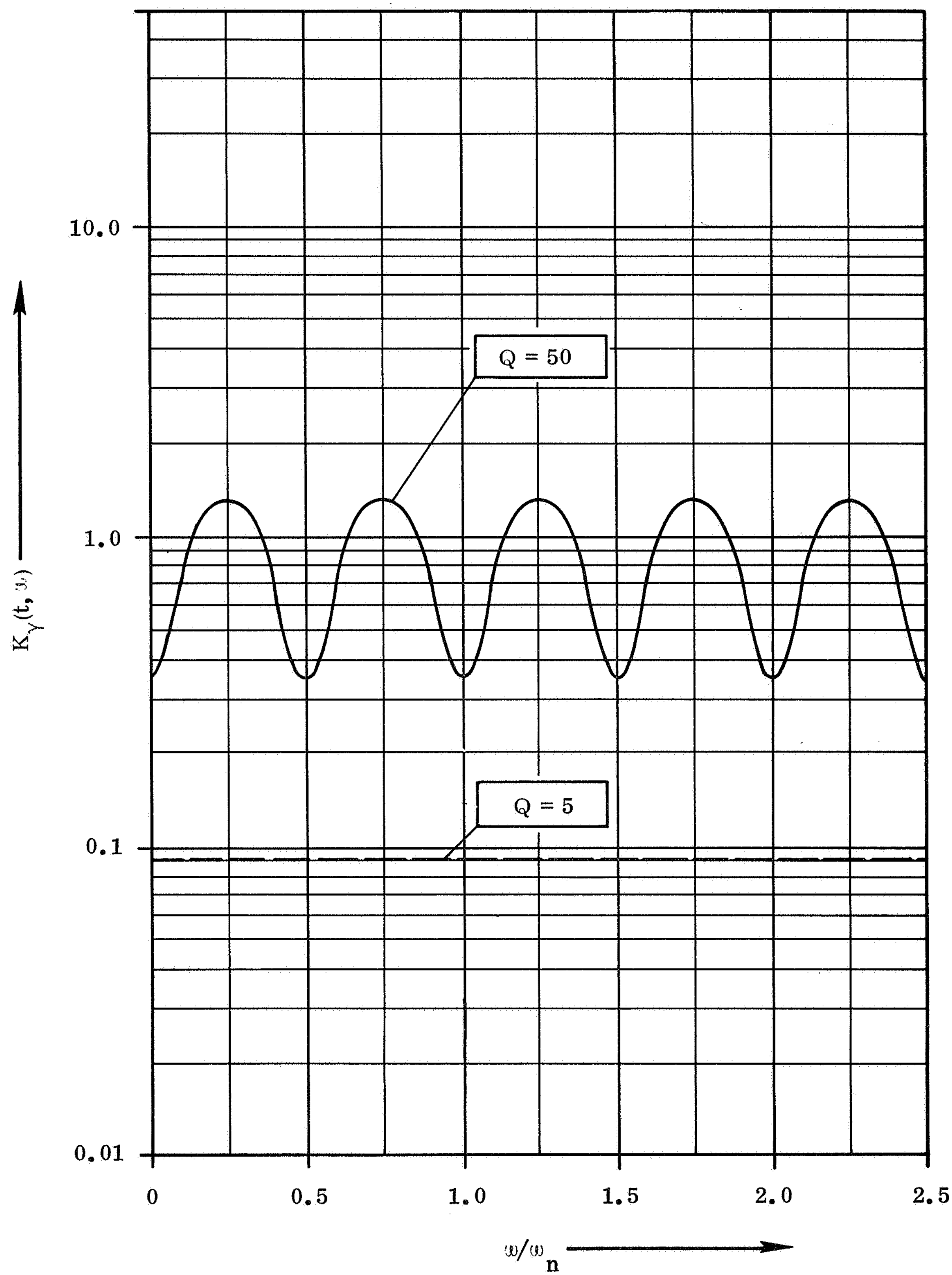

Figure C.44. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 2.0$, $\gamma/b = 10$

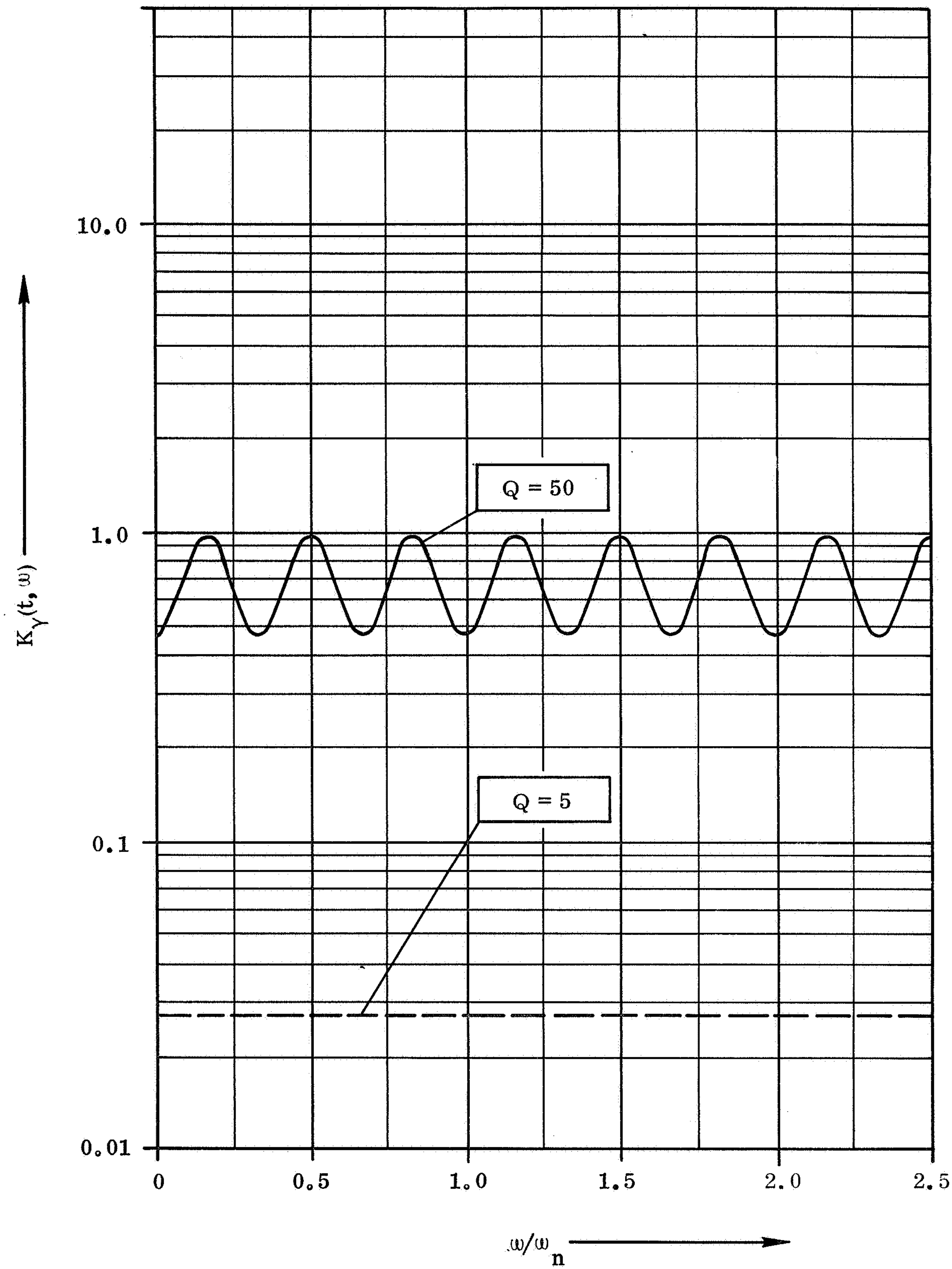

Figure C.45. Shaping Filter $K_\gamma(t, \omega)$ with $f_n t = 3.0$, $\gamma/b = 10$