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SIMPLIFIED SOLUTIONS FOR TWO-PERSON PERCENTILE GAMES

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Technical Report No. 94
Department of Statistics ONR Contract

DEPARTMENT OF STATISTICS
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SIMPLIFIED SOLUTIONS FOR TWO-PERSON PERCENTILE GAMES*

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ABSTRACT

Consider solution of a two-person game in which the players use percentile criteria. For player i, the stepwise procedure is to mark positions of the game outcomes (pairs of payoffs, one to each player) in his payoff matrix according to decreasing desirability level (i = 1,2). To be determined is the smallest marked set such that, for percentile $100\alpha_i$ used by player i, an outcome of this set can be assured with probability at least α_i. Also, an optimum mixed strategy is to be determined (for accomplishing this assurance). In general, the probability with which a marked set can be assured is evaluated by solution of a specialized zero-sum game with an expected-value basis. However, easily evaluated upper and lower bounds for this probability can be obtained from the matrix locations of the markings. Use of these bounds can substantially reduce the effort in the stepwise solution of a game. Moreover, equality of the bounds can occur. Then, the probability is determined without solution of a zero-sum game, and a corresponding optimum strategy is readily identified. The probability value is approximately determined when the bounds are nearly equal, and an approximately optimum strategy is easily identified. Indications are that many percentile games can be solved, exactly or approximately, by this simplified method.

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INTRODUCTION AND SOME RESULTS

Considered is the case of two players with finite numbers of strategies, where each player selects his strategy separately and independently of the strategy choice by the other player. Mixed strategies are used. That is, a player assigns probabilities to his strategies (sum to unity, with a unit probability possible) and randomly selects the strategy used according to these probabilities.

The possible game outcomes are the pairs of payoffs, one to each player, that occur for the possible combinations of strategy selection by the two players. The payoffs can be of a very general nature but are such that the outcomes can be ranked according to desirability level separately by each player. Use of a matrix form is convenient for considering the possible payoffs to a player, where rows represent his strategies and columns represent the other player's strategies.

For percentile game theory, player i specifies a probability α_i which represents the assurance with which he wants to obtain an outcome with reasonably high desirability (i = 1,2). A largest level of desirability occurs among the outcomes such that player i can assure, with probability at least α_i, that an outcome having at least this desirability level occurs. The symbol O_i designates the outcome(s) having this largest desirability level.

Given α_i, a game solution for player i consists in determining O_i and an optimum strategy for the combination α_i and O_i. This determination can be made by a stepwise procedure in which, for player i, positions of outcomes are marked in his payoff matrix according to decreasing desirability level (outcomes with the same level are marked simultaneously). O_i is determined as the outcome(s) having the smallest desirability level in the smallest set of marked outcomes that player i can assure with probability at least α_i. In general, the probability with which a stated marked set of outcomes can be assured is evaluated as the value of a zero-sum game with an expected-value basis. The payoff matrix for player i in this game has ones at the marked positions and zeroes at the unmarked positions. An optimum strategy for player i in the zero-sum game corresponding to the smallest marked set containing O_i is an optimum strategy for the combination α_i and O_i. Ref. 1 contains a detailed statement of this general method for solution of two-person percentile games.

A couple of one-sided bounds on the probability with which a marked set can be assured (one bound used for $\alpha_i \leq 1/2$, the other for $\alpha_i > 1/2$) are given in ref. 1. These bounds are helpful in reducing the effort needed for solving a game.

This paper develops a class of upper and lower bounds such that both an upper and lower bound is available for the probability with which a marked set can be assured by player i (a lower bound may have the trivial value zero, or an upper bound the trivial value unity, in some cases). Equality of the upper and lower bounds occurs in a number of cases, with the probability being directly determined without solution of a zero-sum game. At least approximate equality of upper and lower bounds occurs in many cases. Then the probability with which a marked set can be assured is at least approximately determined without solution of a zero-sum game. Moreover, when equality of bounds occurs, an optimum (mixed) strategy for accomplishing this probability is readily determined. Also, an approximately optimum (mixed) strategy is easily determined when the bounds are approximately equal. These results, which apply to any marked set in the payoff matrix for player i, can be exceedingly helpful in reducing the effort for solving a game.

Specifically, for a given marked set, suppose that at least M marks in every column are obtainable from R rows, and that at least U unmarked positions in every row are obtainable from C columns. Then, player i can assure an outcome of the marked set with probability at least M/R and at most 1 - U/C. If R rows and U columns with these properties satisfy

$$M/R = 1 - U/C,$$

then an optimum mixed strategy for player i is to choose one of these R rows with probability 1/R for each row (and probability zero for the other rows). If M/R approximately equals 1 - U/C, this mixed strategy is approximately optimum and the assurance probability with this strategy is at least M/R.

Use of these results to obtain simplified solutions for two-person percentile games is considered in the next section. An example of determination of upper and lower bounds on assurance probabilities is given in the next to last section. The final section contains the basic theorems and their verification.

SIMPLIFIED SOLUTION METHOD

The same solution method applies to each player and is stated for player i. A preferred assurance probability α_i, $0 < \alpha_i \leq 1$, is specified by player i. First, the solution method is stated for this given value of α_i. Then, advantages of making small changes in preferred values for α_i are discussed.

The method is stated in terms of a marking of outcome positions in the payoff matrix for player i. The r rows of this matrix correspond to the r strategies for player i, and the c columns are the strategies for the other player ($r, c \geq 2$).

As the initial step, mark the position(s) in the payoff matrix for player i of the outcome(s) with the highest level of desirability to player i. Determine the smallest value of $1 - U/C$ for this marking, where U and C are such that at least U unmarked positions in every row are obtainable from C columns.

Next, also mark the position(s) of the outcome(s) with the next to highest desirability level and determine the smallest value of $1 - U/C$ for the overall marking. Continue this marking, according to decreasing desirability level, until the first time that α_i is at most equal to the smallest value of $1 - U/C$ (for the overall marking). Also determine the largest value of M/R for this marking, where M and R are such that at least M marks in every column are obtainable from R rows. If

$$\text{largest } M/R \leq \alpha_i \leq \text{smallest } (1 - U/C) \tag{1}$$

and the largest M/R equals, or approximately equals, the smallest $(1 - U/C)$, a usable solution is obtained (exact or approximate). Then, O_i is determined as the outcome(s) with smallest desirability level in this marked set. An optimum (or approximately optimum) strategy consists of randomly selecting one of the R rows for which largest M/R occurs so that each row has probability 1/R of being chosen.

If (1) holds but the bounds are not approximately equal, continue the marking until (1) holds with the bounds equal or approximately equal, or until α_i is at most equal to the largest M/R. When the situation is that (1) holds with the bounds equal or nearly equal, a usable solution is obtained (as described in the preceding paragraph). However, this solution can be approximate even when the bounds are equal, since the marked set may not be the smallest set that can be assured with probability at least α_i. This possible difference in set size is usually unimportant but the method of ref. 1 could be used to determine whether a smaller set satisfies the requirements.

Finally, suppose that a marking has been reached (without first obtaining a usable solution) where α_i is at most equal to the largest M/R. Then, remove the mark(s) for the outcome(s) with lowest desirability level among the outcomes that have received marks. Then, by the following procedure, determine whether some one of the remaining marked outcomes can be assured with probability at least α_i. The procedure (used in ref. 1) is to replace every marked position in the matrix of player i by unity and all other positions by zero. The resulting matrix of ones and zeroes is considered to be the payoff matrix to player i for a zero-sum game with an expected-value basis, and is solved for the value of the game to player

i. If the game value is less than α_i, then O_i consists of the outcome(s) with mark(s) removed.

Otherwise, remove the mark(s) for the outcome(s) with least desirable level among the outcomes still having marks and, using the same procedure, determine the probability with which player i can assure a marked outcome. If this probability is less than α_i, then O_i consists of the outcome(s) with mark(s) removed last. If not, continue until the first time that some one of the remaining marked outcomes cannot be assured with probability at least α_i. Then, O_i consists of the outcome(s) with mark(s) removed last.

For the cases starting with a marking such that α_i is at most equal to the largest M/R, the same way is used to determine an optimum mixed strategy for player i. Mark the matrix positions of all outcomes whose desirability level is at least that of O_i and replace marked positions by unity and unmarked positions by zero. Treat the resulting matrix of ones and zeroes as the payoff matrix for player i in a zero-sum game with an expected-value basis. An optimum strategy for player i in this zero-sum game is α_i-optimum for him.

Now, consider some advantages of making small changes in the value preferred for α_i. Markings sometimes occur such that the smallest (1 - U/C) equals the largest M/R. If this occurs for a value near α_i, substantial solution effort can be avoided by letting α_i equal this common value for the bounds. At least approximate equality of the bounds can happen in many cases, especially when r and c are of at least moderate size. Change of the value for α_i to a nearby value which is between two approximately equal bounds also can result in substantially less solution effort (when approximate solutions are acceptable). Often, use of the arithmetic average of two approximately equal bounds provides a suitable value for α_i.

EXAMPLE OF BOUNDS DETERMINATION

To illustrate how largest M/R and smallest (1 - U/C) change as marking continues, a payoff matrix with r = 10 and c = 8 is considered. No ties in desirability level occur for this example and the numbers 1, 2,...,80 in the matrix show the locations of the most desirable outcome, the next to most desirable outcome,..., the least desirable outcome, respectively. The first mark occurs at the location of 1, the second at the location of 2, etc. Thus, a total of t marks have occurred at the time the t-th most desirable outcome is marked (t = 1,...,80). The values of largest M/R and of smallest (1 - U/C) are listed as functions of t.

The matrix for player i, with the position numbers 1,...,80 entered, is provided by Figure 1. The values of the largest M/R and of the smallest (1 - U/C) are stated in pairs for t = 1,...,80, with the largest M/R listed first:

(0,1/8), for t = 1,...,7; (0,1/7), for t = 8,9;

(1/7,1/7), for t = 10,11,12; (1/6,1/6), for t = 13,14,15;

Figure 1. Matrix for the Example

	1	2	3	4	5	6	7	8
1	63	38	15	77	35	11	51	55
2	1	75	33	43	21	36	52	67
3	57	42	2	76	28	14	70	17
4	27	31	73	48	68	6	8	44
5	69	13	20	3	37	62	30	53
6	50	78	19	29	59	66	26	7
7	23	12	61	47	71	9	49	32
8	79	41	54	18	10	34	46	80
9	45	4	39	65	24	72	22	58
10	5	64	60	25	40	74	56	16

(1/5,1/5), for t = 16,17; (1/5,1/4), for t = 18,19;

(1/4,1/4), for t = 20,21; (1/4,2/7), for t = 22;

(1/4,1/3), for t = 23; (2/7,1/3), for t = 24;

(1/3,1/3), for t = 25,26; (1/3,3/8), for t = 27;

(3/8,3/7), for t = 28,29; (2/5,3/7), for t = 30;

(2/5,1/2), for t = 31; (1/2,1/2), for t = 32,...,39;

(1/2,4/7), for t = 40,...,43; (1/2,3/5), for t = 44,45;

(5/9,5/8), for t = 46; (4/7,5/8), for t = 47;

(4/7,2/3), for t = 48; (3/5,2/3), for t = 49,50;

(5/8,2/3), for t = 51; (2/3,2/3), for t = 52;

(2/3,5/7), for t = 53,...,56; (3/4,3/4), for t = 57,...,61;

(3/4,4/5), for t = 62,63; (4/5,4/5), for t = 64,...,67;

(5/6,5/6), for t = 68; (1,1), for t = 69,...,80.

The upper and lower bounds are seen to be near each other in almost all cases and to be equal in some cases. Equality of bounds occurs for probability values 1/7, 1/6, 1/5, 1/4, 1/3, 1/2, 2/3, 3/4, 4/5, 5/6.

THEOREMS AND PROOFS

The results stated in the previous sections are based on two theorems.

THEOREM 1. For a given set of markings of outcomes in the payoff matrix for player i, at least M marks in every column are obtainable from R rows and also at least U unmarked positions in every row are obtainable from C columns. Then, player i can assure an outcome of the marked set with probability at least M/R and at most 1 - U/C.

Proof. First, it is shown that a probability of at least M/R can be assured. Let $p_1, \ldots, p_r$ and $q_1, \ldots, q_c$ be the mixed strategies used. Then, the probability of obtaining a marked outcome is

$$\sum_{i=1}^{r} p_i Q_i,$$

where Q_i is the sum of the q's for the columns that have marked positions in the i-th row. The largest value of this probability that player i can assure, by choice of $p_1, \ldots, p_r$, is

$$G = \min_{q_1, \ldots, q_c} (\max_i Q_i).$$

Let $i(1), \ldots, i(R)$ be R rows that together contain marked positions in all columns. For any minimizing choice of the values for $q_1, \ldots, q_c$, all of $Q_{i(1)}, \ldots, Q_{i(R)}$ are at most G. Hence,

$$RG \geq Q_{i(1)} + \ldots + Q_{i(R)} \geq M,$$

so that a probability of at least M/R can be assured by player i.

Similarly, considering columns and unmarked positions, the other player can assure an unmarked outcome with probability at least U/C. Thus, player i can assure a marked outcome with probability at most 1 - U/C.

Theorem 2. Under the circumstances stated in Theorem 1, use of a mixed strategy where each of the R rows is randomly selected with probability 1/R (and the other rows have zero probability) assures player i that an outcome of the marked set occurs with probability at least M/R.

Proof. Let $p_{i(1)} = \ldots = p_{i(R)} = 1/R$ while the other p's are zero. Then, for any given $q_1, \ldots, q_c$, the probability of obtaining a marked outcome is

$$(1/R) \sum_{j=1}^{R} Q_{i(j)} \geq M/R.$$

In particular, this inequality holds for any minimizing set of values for $q_1,\ldots,q_c$.

REFERENCE

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Consider solution of a two-person game in which the players use percentile criteria. For player i, the stepwise procedure is to mark positions of the game outcomes (pairs of payoffs, one to each player) in his payoff matrix according to decreasing desirability level ($i = 1,2$). To be determined is the smallest marked set such that, for percentile $100\alpha_i$ used by player i, an outcome of this set can be assured with probability at least α_i. Also, an optimum mixed strategy is to be determined (for accomplishing this assurance). In general, the probability with which a marked set can be assured is evaluated by solution of a specialized zero-sum game with an expected-value basis. However, easily evaluated upper and lower bounds for this probability can be obtained from the matrix locations of the markings. Use of these bounds can substantially reduce the effort in the stepwise solution of a game. Moreover, equality of the bounds can occur. Then, the probability is determined without solution of a zero-sum game, and a corresponding optimum strategy is readily identified. The probability value is approximately determined when the bounds are nearly equal, and an approximately optimum strategy is easily identified. Indications are that many percentile games can be solved, exactly or approximately, by this simplified method.

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