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TECHNICAL MEMORANDUM

MULTIPLE SCATTERING CALCULATIONS: GEOMETRY FOR SPHERICAL ATMOSPHERES

Bellcomm

BELLCOMM, INC.
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ABSTRACT

This memorandum describes the geometric relationships that arise in carrying out atmospheric calculations for a spherical planet. These relationships were developed for the specific case of light scattering from a planetary atmosphere or haze, as a part of Bellcomm's participation in the Mariner Mars '71 mission. However, the geometrical relationships are of general use in a wide range of problems that involve a spherical surface.

In order to obtain the apparent brightness in a scattering atmosphere, one must solve an integro-differential equation (the equation of radiative transfer). This can be accomplished by an iterative process using a digital computer. The brightness in the atmosphere is represented by its value for a set of directions at each one of an array of points in the atmosphere. Cylindrical symmetry about the sun-planet axis reduces significantly the number of points necessary to provide an adequate description of the brightness function.

A point in the atmosphere is described by two variables: its height above the surface of the (spherical) planet, and the angle between the local vertical through the point and the sun-planet axis. The direction of a line of sight from the local point is described by its polar angle to the local vertical and its azimuthal angle measured about the local vertical. Geometric relationships have been developed which give the coordinates of a new local point located a specified distance along the line of sight, as well as the directions of the line of sight at the new point.

Part of the atmosphere is shielded from the incident solar radiation by the planet itself. The equation of radiative transfer has a discontinuity if the line of sight passes into the shadow region. Geometric relationships that yield the distances, if any, at which the line of sight enters or leaves the shadow region have been developed. A computer subroutine, written in the FORTRAN V language, carries out the necessary shadow calculations.

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SUBJECT: Multiple Scattering Calculations: Geometry for Spherical Atmospheres Case 235

DATE: April 16, 1971

FROM: E. N. Shipley

TECHNICAL MEMORANDUM

1. Introduction

A computer program has recently been developed to carry out multiple scattering calculations for a spherical atmosphere illuminated by the sun.(1) The computer program and its subsequent modifications, which calculate with high accuracy the intensity of light scattered from Martian hazes and atmosphere, were developed as part of Bellcomm's participation in the Mariner Mars '71 mission.

One of the problems that arises in carrying out such calculations is the definition of a convenient coordinate system and the determination of the relationships among the variables. These topics form the subject matter for this memorandum. In addition, the calculations themselves have been described (Section 2) in order to provide a basis for understanding the relevance of the quantities that are studied.

The purpose of this memorandum is twofold. First, it is desired to document the computational procedures that have been developed. Second, the geometric definitions are sufficiently broad that the coordinate systems and the relationships among the variables will find use in other contexts. Thus it was desired to make them readily available. The calculations are somewhat tedious, but they are presented in sufficient detail to make the results accessible for further development. Significant equations have been enclosed in boxes to make them more obvious.

Section 2 of this memorandum contains a description of the multiple scattering problem and the approximation procedure that has been used to solve the problem. Section 3 describes the coordinate system and symmetries that are used, and Section 4 describes the relationships among the variables. Section 5 contains an analysis of the shadow region. A subroutine for carrying out the shadow calculations is given in Appendix B.

2. Description of the Multiple Scattering Problem

The apparent brightness due to multiple scattering in an atmosphere obeys the relation

$$-\frac{d\omega(\ell,0,0)}{d\ell} = -\sigma(\ell)\,\omega(\ell,0,0) + \beta\left[\ell,\theta_s(\ell)\right]\pi F e^{-\tau_s(\ell)} + \iint \beta(\ell,\Theta)\;\omega(\ell,\theta,\phi)\;\sin\theta d\theta d\phi \qquad (1)$$

where $\omega(\ell,\theta,\phi)$ is the luminous intensity incident on a volume element from a direction whose polar coordinates are θ,ϕ (see Figure 1). The direction $\theta=0$ corresponds to light travelling toward the observer. The quantity ℓ measures distance along the line of sight from the observer to the volume element, πF is the flux in the incident solar radiation, Θ_s is the polar angle to the direction of the incident sunlight, and τ_s is the optical thickness along the path from the volume element to the source of incident radiation. The negative sign on the left-hand side of Equation (1) arises because ℓ increases in a direction opposite to the motion of light toward the observer.

$\beta(\ell,\Theta)$ is the fraction of the incident light in a unit solid angle that is scattered into a unit solid angle centered at an angle θ to the initial direction, per unit length travelled by the light. The function β describes completely the scattering properties of the atmosphere or haze. The function $\sigma(\ell)$ represents the extinction coefficient for the atmosphere. Both β and σ are linearly proportional to the density of the atmosphere, and thus they are functions of height above the surface. Their dependence on ℓ arises from the relationship between ℓ and the height above the surface.

In Equation (1), the first term on the right-hand side accounts for the attenuation of light already travelling along the line of sight toward the observer; the attenuation arises from absorption and scattering in the atmosphere at the point ℓ. The second term, $\beta(\ell,\theta_S)\pi F e^{-\tau_S(\ell)}$, represents the contribution of the incident sunlight that is scattered toward the observer. At the point ℓ the sunlight has an intensity $\pi F e^{-\tau_S}$, its intensity having been reduced by the factor $e^{-\tau_S}$ due to absorption and scattering in the atmosphere.

The third term on the right-hand side, the integral in Equation (1), gives rise to the multiple scattering effects. At any point in the atmosphere, light that has been previously scattered and/or reflected from the surface is incident from all directions. At the point ℓ, some of this light is scattered toward the observer. This contribution is represented by the integral, which is just an integration of the product of the scattering coefficient and the incident intensity over the complete solid angle surrounding the point.

The solution to Equation (1) may be obtained as follows. Suppose we have an approximate solution to the luminous intensity, $\omega_j(\vec{r},\theta,\phi)$, where the subscript j indicates the order of the approximation. The vector $\vec{r}$ has been used to indicate that the function is known for all points in the atmosphere. This approximate solution may be used in the integral of Equation (1), giving

$$-\frac{d\omega_{j+1}}{d\ell} = -\sigma\omega_{j+1} + \beta(\ell,\theta_s)\pi F e^{-\tau_s} + \int \beta(\ell,\theta)\omega_j(\ell,\theta,\phi)\,\sin\theta d\theta d\phi \qquad (2)$$

The subscript j+1 has been used to indicate that the solution of Equation (2) is an improved approximation to the solution of the original equation.

Since ω_j is assumed to be known, the second and third terms in Equation (2) are known functions depending only on ℓ (for a given line of sight). Then Equation (2) may be written in the form

$$-\frac{d\omega_{j+1}(\ell,0,0)}{d\ell} = -\sigma(\ell)\omega_{j+1}(\ell,0,0) + G(\ell) \qquad (3)$$

where $G(\ell)$ has replaced the second and third terms of Equation (2). Now Equation (3) can be solved by numerical integration for a set of local points and lines of sight to obtain a new function, $\omega_{j+1}(\vec{r},\theta,\phi)$.

The zeroth approximation function, $\omega_0(\vec{r},\theta,\phi)$, may be taken quite simply as zero everywhere. Sufficient iterations, using Equation (2), are then carried out to insure that the function $\omega_n(\vec{r},\theta,\phi)$ is an adequate approximation according to the needs of the problem.*

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*Formal questions relative to the convergence of the iterative procedure are deferred for the present.

3. Coordinate System

In carrying out the calculations, we must represent the function $\omega(\vec{r},\theta,\phi)$ by its value at a finite set of points and directions. Such a set of values for ω will be referred to as a data base. The number of points that are selected depends on the accuracy required of the solution and on the calculation speed and storage capability of the computer that carries out the computations.

The number of points required to define the function ω also depends on the symmetry properties of the problem, since symmetry can significantly reduce the number of points necessary to achieve the desired accuracy.

In all of the calculations, the incident sunlight is represented by plane parallel rays. In addition, the following three assumptions are made about the structure of the surface and the atmosphere.

A. The surface of the planet is spherical and homogeneous.

B. The atmosphere has spherical symmetry, that is, the density of the atmosphere is a function only of height above the surface.

C. The scattering law for the surface of the planet is symmetric under reflection in the plane containing the local vertical and the sun.

It is worth noting that these assumptions are not essential to the technique. However, the coordinate systems and the geometry that are developed in this and subsequent sections depend, in some circumstances, on the symmetries that arise from these assumptions. Such occasions are pointed out in the text.

A point in space, where values of the function ω are stored for various directions, is defined to be a local point. The various directions for which the values are stored are called local directions.

A local point may be defined in terms of the variables h, height above the surface; λ, the angle between the radius vector to the local point and the sun direction; and ψ, an azimuthal angle measured about the sun-planet axis. This is shown in Figure 2. Because of Assumptions A, B and C, there is cylindrical symmetry about the sun-planet axis,* and no results from the calculation depend on the angle ψ. Thus we may think

*Cylindrical symmetry can be maintained with somewhat less restrictive assumptions.

of the local points as being confined to a plane; each point is defined by two numbers, h and λ (see Figure 3). We will denote the sun-planet axis by the term <u>symmetry axis</u>.

The local directions are defined by the angles θ and ϕ in a local coordinate system as shown in Figure 3. The value of the function $\omega(h,\lambda,\theta,\phi)$ is the intensity of the light incident on the local point h, λ <u>from</u> the direction θ, ϕ.

There is a further symmetry property. The luminous intensity is invariant to reflection in the plane in which the angle λ is measured. This follows from the cylindrical symmetry of the incident light, the atmosphere and the surface and from the fact that light scattering is invariant to such reflections.* Rayleigh scattering is invariant under reflection in any plane containing the incident ray, and Assumption C requires that scattering at the surface of the planet does not violate the symmetry. The reflection invariance is explicitly demonstrated for Equation (1) in Appendix A. In the coordinate system shown in Figure 3, reflection in the λ plane corresponds to the transformation

$$\phi \rightarrow -\phi$$

and the symmetry relationship requires

$$\omega(h,\lambda,\theta,\phi) \equiv \omega(h,\lambda,\theta,-\phi) \tag{4}$$

As a consequence, the local directions may be chosen in the hemisphere defined by

$$0 \leq \phi \leq \pi \tag{5}$$

Brightness values in the other hemisphere may be obtained by invoking Equation (4).

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*Another way of stating the reflection invariance for light is that for light travelling toward the observer, as much is scattered to the right as to the left. This invariance property holds for all electromagnetic effects, but is violated by certain processes such as beta decay. In order to observe the violation of reflection invariance, it is necessary to use polarized initial states.

In the technique that has been developed to carry out multiple scattering calculations, the local points can be chosen with great generality, so that, for example, the density of local points can be made greater than the average in regions where the value of ω is changing more rapidly. Similarly, it is possible to choose a different set of local directions for each local point, the directions being chosen to minimize the computational error. However, such stratagems have not been found necessary for achieving adequate computational accuracy, and simpler methods have been used.

The local points have been chosen to be the intersection of lines of constant height and lines of constant sun angle (λ). Further, the same set of local directions is used for each local point. The geometric relations that are developed in subsequent sections do not depend on the choice of local points or directions, but are quite general.

4. Geometric Relationships

Suppose we wish to calculate the luminous intensity incident on a local point h_1,λ_1 from the direction θ_1,ϕ_1. In order to carry out the calculation, we must solve Equation (1) along a line of sight in the direction θ_1,ϕ_1 from the local point. At various distances χ along the line of sight, Equation (1) must be evaluated, and it is necessary to know the location and orientation of the coordinate system at the new point.

The situation is depicted in Figure 4. After moving a distance χ, which may be taken positive or negative, we are at a new local point h_2,λ_2, moving in a direction θ_2,ϕ_2. The angle ψ measures the rotation about the symmetry axis, but just because of the symmetry this angle has no essential significance. What are required are the values of the quantities h_2,λ_2,θ_2 and ϕ_2 as a function of $h_1,\lambda_1,\theta_1,\phi_1$ and χ.

The distance r_1 is defined to be h_1+R, the distance from the center of the planet to the local point. A similar definition applies to r_2. The vectors $\vec{r}_1$ and $\vec{r}_2$ denote the vectors from the center of the planet to the respective local points. The vector $\vec{\chi}$ extends to the second local coordinate system from the first.

Some of the required relationships may be obtained from the law of cosines. The quantity h_2 may be obtained through the relation (compare Figure 4)

$$(h_2 + R)^2 = (h_1 + R)^2 + \chi^2 + 2\chi(h_1 + R)\cos\theta_1 \tag{6}$$

whence

$$\boxed{h_2 = \left[(h_1 + R)^2 + \chi^2 + 2\chi(h_1 + R)\cos\theta_1\right]^{1/2} - R} \tag{7}$$

Similarly, we have the relationship

$$(h_1 + R)^2 = (h_2 + R^2) + \chi^2 - 2\chi(h_2 + R)\cos\theta_2$$

and

$$\boxed{\cos\theta_2 = \left[(h_2 + R)^2 + \chi^2 - (h_1 + R)^2\right] \Big/ 2\chi(h_2 + R)} \tag{8}$$

In order to obtain the remaining quantities, it is convenient to use vector relationships. We need the components, in the ξ,η,ζ coordinate system, of the vectors $\vec{r}_1$, $\vec{r}_2$, and $\vec{\chi}$. ξ,η,ζ is the planet-centered system shown in Figure 4. We have

$$\vec{r}_1 = (h_1 + R)\ (\sin\lambda_1\vec{\xi} + \cos\lambda_1\vec{\zeta}) \tag{9}$$

and similarly

$$\vec{r}_2 = (h_2 + R)\ \left((\sin\lambda_2\cos\psi)\vec{\xi} + (\sin\lambda_2\sin\psi)\vec{\eta} + \cos\lambda_2\vec{\zeta}\right) \tag{10}$$

where $\vec{\xi},\vec{\eta}$ and $\vec{\zeta}$ are unit vectors along the respective axes.

The components of the vector $\vec{\chi}$ may be obtained by first obtaining the coordinates in a local coordinate system and then projecting these components onto the $\xi\eta\zeta$ coordinate system. Using the x_1, y_1, z_1 coordinate system as an intermediary, we find

$$\vec{X}_1 = X(\cos\theta_1 \sin\lambda_1 - \sin\theta_1 \cos\phi_1 \cos\lambda_1)\vec{\xi}$$
$$-X(\sin\theta_1 \sin\phi_1)\vec{\eta} \tag{11}$$
$$+X(\cos\theta_1 \cos\lambda_1 + \sin\theta_1 \cos\phi_1 \sin\lambda_1)\vec{\zeta}$$

where the subscript on $\vec{X}$ indicates that the reduction occurred through the first local coordinate system. Similarly

$$\vec{X}_2 = X(\sin\lambda_2 \cos\theta_2 \cos\psi - \cos\lambda_2 \sin\theta_2 \cos\phi_2 \cos\psi$$
$$+ \sin\theta_2 \sin\phi_2 \sin\psi)\vec{\xi}$$
$$+X(\sin\lambda_2 \cos\theta_2 \sin\psi - \cos\lambda_2 \sin\theta_2 \cos\phi_2 \sin\psi \tag{12}$$
$$- \sin\theta_2 \sin\phi_2 \cos\psi)\vec{\eta}$$
$$+X(\cos\lambda_2 \cos\theta_2 + \sin\lambda_2 \sin\theta_2 \cos\phi_2)\vec{\zeta}$$

Now, we have the vector relationship

$$\vec{r}_1 + \vec{X} = \vec{r}_2 \tag{13}$$

Taking the ζ component, we have

$$(h_1 + R)\cos\lambda_1 + X(\cos\theta_1 \cos\lambda_1 + \sin\theta_1 \cos\phi_1 \sin\lambda_1) = (h_2 + R)\cos\lambda_2$$

Whence we find

$$\boxed{\cos\lambda_2 = \frac{1}{(h_2+R)}\left[(h_1+R)\cos\lambda_1 + X(\cos\theta_1 \cos\lambda_1 + \sin\theta_1 \cos\phi_1 \sin\lambda_1)\right] \quad (14)}$$

Finally, we have

$$\vec{X}_1 \equiv \vec{X}_2 \tag{15}$$

and by equating the ζ components we obtain

$$\cos\theta_1\cos\lambda_1 + \sin\theta_1\cos\phi_1\sin\lambda_1 = \cos\theta_2\cos\lambda_2 + \sin\theta_2\cos\phi_2\sin\lambda_2$$

giving the result

$$\boxed{\cos\phi_2 = \frac{1}{(\sin\theta_2\sin\lambda_2)}\left[\cos\theta_1\cos\lambda_1+\sin\theta_1\cos\phi_1\sin\lambda_1-\cos\theta_2\cos\lambda_2\right] \quad (16)}$$

In Equations 8, 14 and 16, angles have been defined by a cosine. In each of the cases, the result is unambiguous, since the angle is restricted to the range between 0 and π.

In carrying out the multiple scattering calculation, it is also necessary in certain circumstances to calculate the distance from a local point h_1,λ_1, in the direction θ_1,ϕ_1, to another local point having a specified height h_2, or a specified sun angle, λ_2. The distance to a point of height h_2 can be obtained from Equation 6.

$$\chi = -(h_1+R)\cos\theta_1 \pm \left[(h_1+R)^2\cos^2\theta_1-(h_1+R)^2+(h_2+R)^2\right]^{1/2} \tag{17}$$

Care must be exercised to obtain the desired solution from the two in Equation (17).

It may be seen that in Equation (17), if $h_2>h_1$, there is always one positive and one negative solution. The positive answer for the distance is the desired one in our case. For positive χ we have

$$\chi = -(h_1 + R)\cos\theta_1 + \left[(h_2 + R)^2 - (h_1 + R)^2\sin^2\theta_1\right]^{1/2} \tag{18}$$

$$\text{for } h_2 > h_1$$

When $h_2<h_1$, there exists the possibility that an answer does not exist. This occurs when the closest approach of the line of sight to the surface is at a height greater than h_2. An examination of Equation (17) reveals that, for $h_2<h_1$, there are two positive solutions, two negative solutions, or two complex solutions. The complex solutions correspond to the case mentioned above. Our interest is restricted to the smaller positive solution. The requirement that the solutions be positive is that the term $-(R + h_1)\cos\theta_1$ be positive; this necessitates $\theta_1 > \pi/2$. The desired result is

$$\chi = -(h_1 + R)\cos\theta_1 - \left[(h_2 + R)^2 - (h_1 + R)^2\sin^2\theta_1\right]^{1/2}$$

$$\text{for } h_2 < h_1 \qquad \theta_1>\pi/2 \tag{19}$$

Now, suppose λ_2 is given in addition to the local point h_1, λ_1 and the local direction θ_1,ϕ_1. We can obtain the distance to the second local point by using Equations (7) and (14) simultaneously to eliminate h_2. We obtain a quadratic equation with solutions of the form

$$\chi = \frac{-b \pm (b^2-4ac)^{1/2}}{2a} \tag{20}$$

where

$$a = \cos^2\lambda_2 - (\cos\theta_1\cos\lambda_1 + \sin\theta_1\cos\phi_1\sin\lambda_1)^2$$

$$\frac{b}{2} = (h_1 + R)\left[\cos\theta_1(\cos^2\lambda_2 - \cos^2\lambda_1) - \sin\theta_1\cos\phi_1\sin\lambda_1\cos\lambda_1\right] \tag{21}$$

$$c = (h_1 + R)^2 (\cos^2\lambda_2 - \cos^2\lambda_1)$$

One of the solutions in Equation (20) is extraneous. The correct solution must be determined by seeing which of the distances provides the correct angle λ_2 in Equation 14.

5. Shadow Region

The planet shields a region of space from the sun's rays; this region is called the shadow region. A line of sight may begin in the shadow region, pass through the shadow region, or terminate there by intersecting the surface. For the portion of the path where the line of sight is in the shadow region, the second term on the right-hand side of Equation (1) is zero. More important, the boundary of the shadow region represents a discontinuity in the differential equation because of the change in the term representing the incident sunlight. Consequently, it is essential to the calculations that the distance to the boundaries of the shadow zone be known for a given local direction at a given local point.

The shadow zone is shown in Figure 5. Since the sun is considered to be a source of plane parallel rays, the shadow zone is contained in a cylinder of radius R, situated along the symmetry axis of the problem. The cylinder is divided by the terminator on the planet's surface; the portion of the cylinder away from the sun represents the shadow zone.

Consider a local point h,λ and a line of sight from that point in the direction θ,ϕ. We first ask at what distances, if any, from the local point does the line of sight intersect the surface of the cylinder. Subsequently, we will determine if those points are located in the shadow portion of the cylinder.

It is convenient to carry out the shadow calculations relative to the point at which the line of sight is closest to the center of the planet. The point of closest approach, which we call P, is a distance L_ζ from the local point (L_ζ may be positive or negative), and at the point of closest approach the line of sight is a distance ζ from the center of the planet. Straightforward geometric relationships give the results (see Figure 6)

$$L_\zeta = -(h + R)\cos\theta \tag{22}$$

$$\zeta = (h + R)\sin\theta \tag{23}$$

At the point P we establish a coordinate system with the z axis along the local vertical and the x axis along the line of sight (see Figure 6). In this coordinate system, the angles λ' and ϕ', which define the direction of the sun, can be obtained through Equations (14) and (16) respectively.*

The first task is to establish where the axis through the center of the planet toward the sun (the symmetry axis) penetrates the x,y plane. For $\lambda' = \pi/2$, the axis is parallel to the x,y plane; this special situation will be treated separately later in this section.

The point x_o, y_o, where the symmetry axis penetrates the x,y plane, is a distance $\zeta \tan \lambda'$ from the point P (see Figure 6). Resolving this into components, we find

$$x_o = \zeta \tan\lambda' \cos\phi'$$
$$y_o = \zeta \tan\lambda' \sin\phi' \tag{24}$$

The cylinder of radius R cuts the x-y plane in an ellipse centered on the point x_o, y_o. To find the equation of the ellipse, we carry out a series of transformations as illustrated in Figure 7. The x_1, y_1, z_1 coordinate system has its origin at x_o, y_o, the z_1 axis is parallel to the z axis (at the point P), and the x_1 axis is an extension of the line from P to x_o, y_o. The y_1 axis is chosen to complete a right-handed orthogonal coordinate system; the y_1 axis lies in the x,y plane defined by the coordinate system at the point P. The x_2, y_2, z_2 coordinate system is defined so that its origin is at x_o, y_o, the z_2 axis points parallel to the symmetry axis, and y_2 is coincident with y_1. The x_2 axis is chosen to complete a right-handed orthogonal coordinate system.

The equation for the <u>cylinder</u> about the symmetry axis is

$$x_2^2 + y_2^2 = R^2 \tag{25}$$

in the x_2, y_2, z_2 coordinate system. In x_1, y_1, z_1 coordinate system, Equation (25) is transformed to

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*There is a difference between the coordinate system in which λ' and ϕ' are defined, and the coordinate system used in Section 4. However, a close comparison will reveal that λ' is the same angle as λ_2 in Equation (14). ϕ_2 in Equation (16) is the angle, measured about the local vertical, from the sun direction to the line of sight. In this section, ϕ' is the angle, measured about the local vertical, from the line of sight to the sun direction. Hence ϕ_2 and ϕ' differ by a sign, which, however, does not affect the calculation.

$$(x_1\cos\lambda' - z_1\sin\lambda')^2 + y_1^2 = R^2 \tag{26}$$

We are interested in the ellipse generated by the intersection of the cylinder and the x,y plane, where $z_1 = 0$; the x_1 and y_1 axes lie in the x,y plane. The equation of the ellipse is

$$(x_1\cos\lambda')^2 + y_1^2 = R^2 \tag{27}$$

In order to make the x_1 axis parallel to the x axis, the x_1,y_1,z_1 coordinate system must be rotated by an angle $-\phi'$ about the z_1 axis. Remembering that the center of the ellipse is at the point x_o,y_o in the x,y,z coordinate system, we have the equation of the ellipse as

$$\left[(x-x_o)\cos\phi' + (y-y_o)\sin\phi'\right]^2\cos^2\lambda' + \left[(y-y_o)\cos\phi' - (x-x_o)\sin\phi'\right]^2 = R^2 \tag{28}$$

Our interest is in the points A and B where the ellipse intersects the line of sight. The line of sight has y=0 in the x,y plane (the geometry is shown in Figure 8 for the special case $\lambda'=\pi/2$). Therefore, the points of intersection of line of sight with the cylinder about the symmetry axis are the solutions to the equation

$$\begin{aligned}(x-x_o)^2&(\cos^2\phi'\cos^2\lambda' + \sin^2\phi')\\ &+ 2(x-x_o)y_o(\cos\phi'\sin\phi'\sin^2\lambda')\\ &+ y_o^2\ (\cos^2\phi' + \sin^2\phi'\cos^2\lambda') = R^2\end{aligned} \tag{29}$$

In Equation (29), distances are measured relative to the point P. The distance L_ζ must be added so that the distances are relative to the local point. Finally, then, we find the intersections of the line of sight with the cylinder to be

$$\begin{aligned}A &= L_\zeta + \frac{1}{a}\left[-\left(\frac{b}{2}\right) - \left(\frac{b^2}{4} - ac\right)^{1/2}\right]\\ B &= L_\zeta + \frac{1}{a}\left[-\left(\frac{b}{2}\right) + \left(\frac{b^2}{4} - ac\right)^{1/2}\right]\end{aligned} \tag{30}$$

The symbols A and B have been used to denote both the points of intersection and the distances from the local point to the intersections.

where

$$a = \cos^2\phi'\cos^2\lambda' + \sin^2\phi'$$

$$\frac{b}{2} = y_o(\cos\phi'\sin\phi'\sin^2\lambda') - x_o(\cos^2\phi'\cos^2\lambda' + \sin^2\phi') \tag{31}$$

$$c = y_o^2(\cos^2\phi' + \sin^2\phi'\cos^2\lambda') - 2x_oy_o(\cos\phi'\sin\phi'\sin^2\lambda') + x_o^2(\cos^2\phi'\cos^2\lambda' + \sin^2\phi') - R^2$$

We note that A and B must both be complex or both be real. If they are complex, it indicates that the line of sight does not intersect the cylinder. If A and B are real and equal, then the line of sight is tangent to the surface of the cylinder. If A and B are real but not equal, then because of the definition in Equation (30),

$$A < B \tag{32}$$

All the following comments refer to the case when A and B are real and distinct.

It must be remembered that A and B give the points at which the line of sight (taken to mean a line extending in both positive and negative directions along the local direction from the local point) intersects the cylinder about the symmetry axis. In order to determine if the points refer to the shadow portion of the cylinder (see Figure 5), one may calculate the sun angles λ_A and λ_B at the points A and B, respectively (using Equation 14). The points refer to the shadow zone if their sun angle, λ_A or λ_B, is greater than $\pi/2$. However, this requires a relatively long calculation, and a quicker algorithm is available in certain circumstances (see Case I below).

The values of the quantities L_ζ, ζ, A and B carry implications about the geometry of the shadow zone relative to the line of sight. This can be organized into three distinct cases.

Case I $\zeta \geq R$

The line of sight lies in a plane that is perpendicular to a radius vector at the point of closest approach. For $\lambda' = \pi/2$, a special case to be discussed later, the radius vector is at the terminator (see Figure 5) and the plane does not intersect the cylinder except if $\zeta \leq R$. If $\lambda' < \pi/2$, the plane intersects the cylinder above the terminator, and consequently any intersections between the line of sight and the cylinder occur in the sunlit region. Conversely, if $\lambda' > \pi/2$, any intersections between the line of sight and the cylinder occur in the shadow zone.

Case II $\zeta < R$, $L_\zeta > 0$

The point of closest approach lies inside the planet and hence inside the cylinder. Because of this and Equation (32), there is the relationship

$$A < L_\zeta < B \tag{33}$$

The line of sight penetrates the planet; we may confine our interest to that portion of the line of sight on the near side of the planet, that is, between the local point and the point where the line of sight enters the surface. The point B lies beyond the surface of the planet and is not of interest. The implication of $L_\zeta > 0$ is that the local direction points toward the planet's surface. If $A < 0$, then the local point is in the shadow zone, as is the entire region between the local point and the surface. If $A > 0$, then the local point is outside the shadow zone, and the line of sight enters the shadow zone at a distance A from the local point.

Case III $\zeta < R$, $L_\zeta < 0$

This case differs from Case II in that the line of sight looks away from the planet; and the point B is of concern. If $B < 0$, then the local point is outside the shadow zone and the line of sight does not enter the shadow zone for positive displacements from the local point. If $B > 0$, then the local point is in the shadow, and the line of sight emerges from the shadow zone at a distance B from the local point.

It remains to consider the situation when $\lambda' = \pi/2$. The geometry is shown in Figure 8. If $\zeta > R$, then the line of sight does not intersect the cylinder. If $\zeta = R$, the line of sight is either tangent to the cylinder at a point, or, for $\phi' = 0$ or π, imbedded in the surface of the cylinder. For both these cases it is convenient to assign the line of sight to being outside the shadow zone.*

If $\zeta < R$, then the x,y plane intersects the shadow cylinder along two lines, since $\lambda' = \pi/2$. The perpendicular distance between the lines is (see Figure 8)

$$V = 2(R^2 - \zeta^2)^{1/2} \tag{34}$$

and the line of sight intersects the lines at distances $\pm \frac{V}{2 \sin\phi'}$. Thus the points A and B have the values

$$\begin{aligned} A &= L_\zeta - (R^2 - \zeta^2)^{1/2}/\sin \phi' \\ B &= L_\zeta + (R^2 - \zeta^2)^{1/2}/\sin \phi' \end{aligned} \tag{35}$$

These equations fail at $\phi' = 0$ or $\phi' = \pi$. In that circumstance the line of sight is parallel to the symmetry axis and does not intersect the cylinder. For $\zeta < R$, the relevant portion of the line of sight is within the shadow zone if the local point has $\lambda > \pi/2$; for $\lambda < \pi/2$, the relevant portion of the line of sight is in the unshadowed portion of the cylinder. If $\lambda = \pi/2$, the entire line of sight must lie in the plane of the terminator. since $\lambda' = \pi/2$. In this case, no portion of the line of sight outside the surface of the planet can be shadowed.

Further development of the equations, including a computer program that caries out the shadow calculations, is given in Appendix B.

1011-ENS-dly

E. N. Shipley

Attachments
  Reference
  Appendices A and B
  Figures 1-8

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*Since the cylinder represents a discontinuity, a point on the surface can reasonably be chosen to have either the characteristics of the inside or the outside of the cylinder.

REFERENCE

1. E. N. Shipley, "Multiple Scattering Calculation for Planetary Atmospheres," Bellcomm TM-70-1014-2, May 15, 1970.

APPENDIX A

Reflection Symmetry

The purpose of this appendix is to give a formal derivation of the reflection symmetry that has been used in the multiple scattering calculations. While the arguments given in Section 3.0 are adequate, further insight (and greater confidence) can be obtained from a more formal derivation. The appendix is divided into two parts. Section A1.0 contains the derivation of the scattering angle, which is required for the symmetry argument. The symmetry argument itself is contained in Section A2.0.

A1.0 The Scattering Angle

The scattering angle is the angle between the incident light ray and the scattered light ray. It is shown in Figure A-1. It can be determined easily by first constructing unit vectors in the direction of the incident light ($\vec{I}$) and the scattered light ($\vec{S}$). We have

$$\begin{aligned} \vec{I} &= -\sin\theta_i\cos\phi_i\,\hat{i} - \sin\theta_i\sin\phi_i\,\hat{j} - \cos\theta_i\,\hat{k} \\ \vec{S} &= -\sin\theta_s\cos\phi_s\,\hat{i} - \sin\theta_s\sin\phi_s\,\hat{j} - \cos\theta_s\,\hat{k} \end{aligned} \qquad \text{(A1)}$$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along the x,y,z axes, respectively and θ_i, ϕ_i and θ_s, ϕ_s are the polar and azimuthal angles for the incident and scattered light rays.

Now

$$\cos\delta = \vec{I}\cdot\vec{S}$$

so that

$$\begin{aligned} \cos\delta = {} & \cos\theta_s\cos\theta_i + \\ & + (\sin\theta_s\sin\theta_i)(\cos\phi_i\cos\phi_s + \sin\phi_i\sin\phi_s) \end{aligned} \qquad \text{(A3)}$$

A2.0 Reflection Summary

It is necessary to express the integro-differential equation of transfer (Equation 1) in the coordinate system described in Section 3.

We have

$$-\frac{d\omega}{dx}(\chi,\theta,\phi) = -\sigma(\chi)\omega(\chi,\theta,\phi) + \pi F_e^{-\tau_s}\beta(\lambda,0;\ \theta,\phi)$$
$$+\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\omega(\chi,\theta',\phi')\beta(\theta',\phi';-\theta,\phi)\ \sin\theta' d\phi' d\theta' \tag{A4}$$

In this equation $\beta(\theta_i,\phi_i;\ \theta_s,\phi_s)$ is the scattering coefficient for for light incident from a direction θ_i,ϕ_i being scattered in the direction θ_s,ϕ_s (see Figure A-1). The minus sign on the left-hand side arises because distance is measured away from the observer instead of toward him as in Equation (1). The directions λ,θ,ϕ and θ',ϕ' are measured in the coordinate system at the local point located a distance χ from the observer.

As we noted in Section 3, reflection is accomplished by the transformation $\phi\rightarrow-\phi$. In carrying out this transformation on Equation (A4), care must be exercised with respect to the integral. The angles θ',ϕ' are variables of integration. Changing ϕ' to $-\phi'$ does not alter the value of the integral, but it does change the sequence in which the directions are summed. In making the transformation ϕ to $-\phi$, it is convenient also to change

$\omega(\chi,\theta',\phi')$ to $\omega(\chi,\theta',-\phi')$ and $\beta(\theta',\phi',\theta,\phi)$ to $\beta(\theta',-\phi';\ \theta,-\phi)$;

then the sequence of integration will remain the same, relative to the direction $-\phi$, as it was for the untransformed case relative to the direction $+\phi$.

The transformed equation is

$$-\frac{d\omega}{dx}(\chi,\theta,-\phi) = -\sigma(\chi)\omega(x,\theta,-\phi) + \pi F_e^{-\tau_s}\beta(\lambda,0;\ \theta,-\phi)$$
$$+\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\omega(\chi,\theta',-\phi')\beta(\theta',-\phi';\ \theta,-\phi)\ \sin\theta' d\phi' d\theta' \tag{A5}$$

For Rayleigh scattering, and most other processes, β is a function only of the angle δ, or equivalently $\cos \delta$. First, consider the second term on the right-hand side of Equation A5. It is straightforward, that

$$\beta(\lambda, 0; \theta, \phi) = \beta(\lambda, 0; \theta, -\phi)$$

since in the two cases the scattering angle is the same. This may be seen from Equation (A3), by substituting the transformed and untransformed directions.

$$\cos\lambda\cos\theta + \sin\lambda\sin\theta\cos\phi = \cos\lambda\cos\theta + \sin\lambda\sin\theta\cos(-\phi)$$

It may be noted that this equality depends on the fact that the azimuthal angle for the sun (ϕ_i in Equation A3) is zero. This in turn results since the sun lies in the reflection plane.

Second, consider the integral in Equation (A5). The scattering angles corresponding to the direction $(\theta', \phi'; \theta, \phi)$ and $(\theta', -\phi'; \theta, -\phi)$ are equal since, from Equation (A3)

$$\cos\theta'\cos\theta + \sin\theta'\sin\theta\ (\cos\phi'\cos\phi + \sin\phi\sin\phi) =$$
$$\cos\theta'\cos\theta + \sin\theta'\sin\theta\ \left\{\cos(-\phi')\cos(-\phi) + \sin(-\phi')\sin(-\phi)\right\}$$

Because of the equality of the scattering angles, we may rewrite Equations (A4) and (A5) in the forms

$$-\frac{d\omega}{dx}(\chi, \theta, \phi) = -\sigma\omega(\chi, \theta, \phi) + \pi F e^{-\tau_s} \beta(\delta_s) + \iint \omega(\chi, \theta', \phi')\beta(\delta)\sin\theta' d\phi' d\theta' \tag{A6}$$

$$-\frac{d\omega}{dx}(\chi, \theta, -\phi) = -\sigma\omega(\chi, \theta, -\phi) + \pi F e^{-\tau_s} \beta(\delta_s) + \iint \omega(\chi, \theta', -\phi)\beta(\delta)\sin\theta' d\phi' d\theta' \tag{A7}$$

Comparison of these two equations reveals that $\omega(\chi,\theta,\phi)$ satisfies the same equation as $\omega(\chi,\theta,-\phi)$. The only remaining question is that of boundary conditions on Equations (A6) and (A7). If the line of sight does not intersect the surface, the integration of the integro-differential equation may begin a very great distance from the observer, such that the density of the atmosphere is negligible, and the intensity of light travelling along the line of sight toward the observer is zero. This boundary condition is the same for Equations (A6) and (A7).

In cases where the line of sight intersects the surface, Assumption C (Section 3) guarantees that the boundary condition is the same for the transformed and untransformed equations.

Hence we may conclude that the Equation (1) is invariant under reflections in the plane containing the sun and the local vertical. That is

$$\omega(\chi,\theta,\phi) = \omega(\chi,\theta,-\phi) \tag{A8}$$

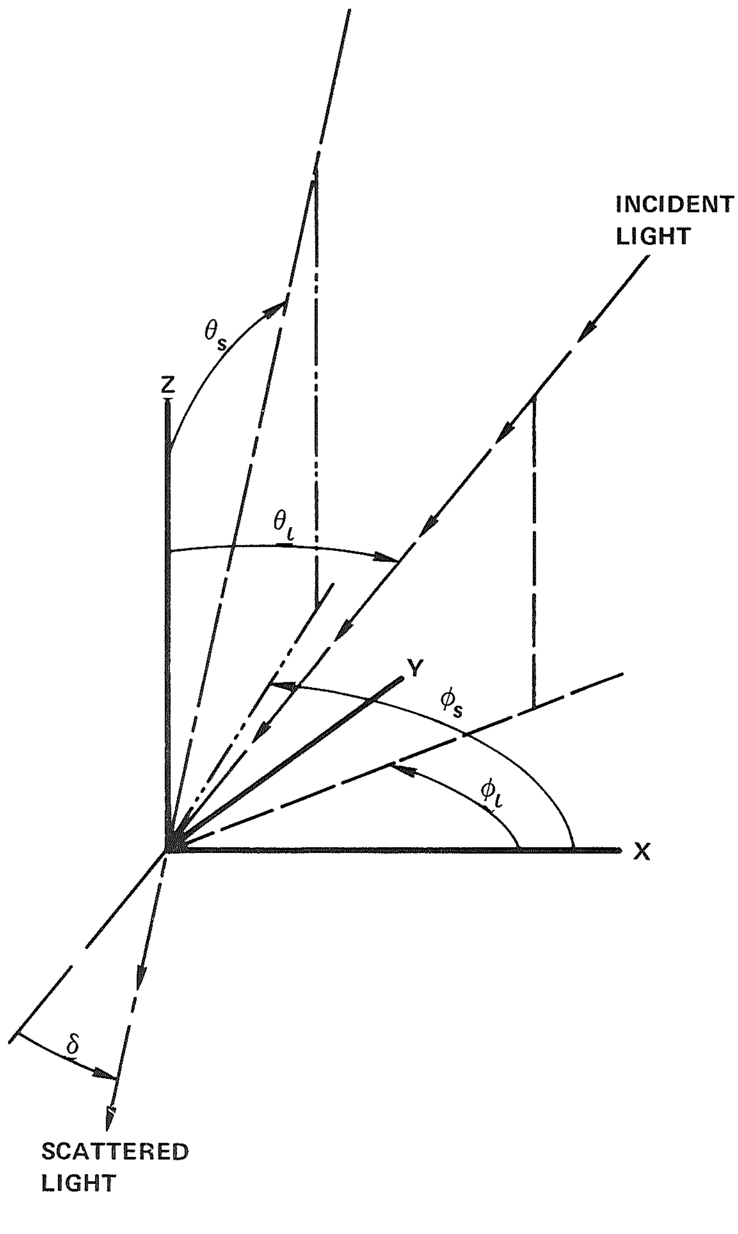

FIGURE A-1 - DEFINITION OF THE SCATTERING ANGLE δ. NOTE THAT THE POLAR AND AZIMUTHAL ANGLES FOR BOTH LIGHT RAYS ARE DEFINED FOR LIGHT MOVING TOWARD THE ORIGIN.

APPENDIX B

Shadow Calculations

A computer program subroutine was written to carry out the shadow calculations described in Section 5. In this appendix, some further development and simplification of the mathematical relations are described, and the computer program itself is presented.

B1.0 Detailed Geometry for the Shadow Calculations

B1.1 Geometry at the Point of Closest Approach

In order to carry out the shadow calculations, it is necessary to have the angles λ' and ϕ' at the point of closest approach (compare Section 5 and Figure 6). These may be obtained from Equations (14) and (16), respectively. However, the specific geometry at the point of closest approach permits some simplification of the equations. Specifically, at the point of closest approach, the line of sight is perpendicular to the local vertical; that is, $\theta' = \pi/2$.

Let ζ be the distance from the center of the planet (see Figure 6). Then, since $\sin(\pi-\theta) = \sin\theta$,

$$\zeta = (R+h)\sin\theta \tag{B1}$$

where R+h is the height of the local point, and θ is the polar angle to the line of sight, and

$$L_\zeta = -(R+h)\cos\theta \tag{B2}$$

We wish to show that

$$\cos\lambda' = \cos\lambda\sin\theta - \sin\lambda\cos\phi\cos\theta \quad \text{for } \theta' = \pi/2 \tag{B3}$$

We start with Equation (14) and recognize that

$$(R+h_2) = \zeta$$

and

$$\chi = L_\zeta$$

and we use Equation (B1) and (B2) to obtain values of ζ and L_ζ.

We find

$$\cos\lambda' = \frac{1}{(R+h)\sin\theta}\left[(R+h)\cos\lambda-(R+h)\cos\theta\ (\cos\theta\cos\lambda+\sin\theta\cos\phi\sin\lambda)\right] \quad (B4)$$

which immediately simplifies to the desired result (Equation B3).

A simplified equation for ϕ' can be obtained from Equation (16) by the straightforward substitution.

$$\sin\theta' = 1$$

giving

$$\cos\phi' = (\cos\lambda\cos\theta + \sin\lambda\cos\phi\sin\theta)/\sin\lambda' \quad (B5)$$

It may also be shown, through the straightforward use of trigonometric relations in Equation (B5), that

$$\sin\phi' = (\sin\lambda\sin\phi)/\sin\lambda' \quad (B6)$$

B1.2 Simplification of the Equations for the Ellipse

Consider Equations (30) and (31). The term $(b^2/4 - ac)$ can be simplified as follows:

Let

$$a_x = a = \cos^2\phi'\cos^2\lambda' + \sin^2\phi'$$

$$a_{xy} = \sin^2\lambda'\sin\phi'\cos\phi'$$

$$a_y = \cos^2\lambda'\sin^2\phi' + \cos^2\phi'$$

so that

$$b/2 = a_{xy}\,y_o - a_x\,x_o$$

$$c = a_y\,y_o^2 - 2\,a_{xy}\,x_o y_o + a_x\,x_o^2 - R^2$$

Let Q be the discriminant divided by four

$$Q = b^2/4 - ac$$

so that

$$Q = a_{xy}\, y_o^2 - 2a_x a_{xy} x_o y_o + a_x x_o^2 - a_x a_y\, y_o^2 + 2a_x a_{xy} x_o y_o$$

$$- a_x^{\,2}\, x_o^2 + a_x\, R^2$$

This yields

$$Q = (a_{xy} - a_x a_y)\, y_o^2 + a_x\, R^2 \qquad \text{(B7)}$$

B1.3 Shadow Free Conditions

The shadow subroutine is called many times in the course of a single multiple scattering calculation. In order to reduce the computation time as much as possible, it is desirable to recognize as soon as possible in the subroutine geometric conditions for which the line of sight cannot pass through the shadow zone. By this procedure, a great many needless calculations may be avoided. As an ancillary benefit, certain conditions that give rise to inaccurate results, because of rounding errors, are avoided.

Condition I: If the local point is inside the shadow cylinder and has $\lambda' \leq \pi/2$, the line of sight cannot pass through the shadow zone. This is illustrated in Figure B-1(a).

The shadow cylinder is a cylinder of radius R whose axis lies along the symmetry axis, which is the line from the sun through the planet center. The perpendicular distance from the local point to the sun-planet axis is

$$(R+h)\sin\lambda$$

So the condition that the local point be inside the shadow cylinder is

$$(R+h)\ \sin\lambda \leq R \tag{B8}$$

<u>Condition II</u>: If the local point is outside the shadow cylinder, and the angle γ, defined in Figure B-1(b), is less than or equal to $\pi/2$, no shadow is possible. The angle γ may be obtained through a simple rotation of the coordinate system, giving

$$\cos\gamma = \cos\theta\ \sin\lambda - \sin\theta\ \cos\phi\ \cos\lambda \tag{B9}$$

B2.0 <u>The Computer Subroutine</u>

The computer subroutine, SHASCC, is listed in Table B-1. This program has evolved over a period of time, during which the terminology has changed. Consequently, the terminology used in the computer program differs from that used in the derivations given in the text. Table B-2 gives the correspondence between the two sets of variables.

The program itself follows directly the logic described in Section 5. The following comments, which refer to the specific lines in Table B-1, are intended to clarify the procedures employed in the subroutine.

<u>Line 024</u> This includes a common statement in the subroutine. The common statement is the source of the variable HMX.

<u>Line 030 to 031</u> The functions CH(Z) and CLP(Y) are used to calculate the angle λ' at points other than the point of closest approach. These correspond to Equation (7) and (14), respectively.

<u>Line 053</u> Rounding errors can create computational difficulties when CSSG is near zero. This statement indicates that there can be no shadow if the point of closest approach is outside of the planet and has a sun angle less than or equal to $\pi/2$ (CSSG $\geq$ 0). To avoid the rounding errors for CSSG near zero, the calculations are taken as giving no shadow if

$$\text{CSSG} \geq -.003$$

This corresponds to $\lambda' \leq 90.2°$.

Line 054 If CSSG is near +1 or -1, set CSSG = ± 1 ($\lambda' = 0$ or π) and $\phi' = 0$. See Lines 119 ff.

Line 084-085 The maximum height of the atmosphere is R + HMX. For a line of sight for which the point of closest approach is a distance ζ from the center of the planet, one can go a distance XMX, given by

$$XMX = \left[(R+HMX)^2 - \zeta^2\right]^{1/2}$$

in either direction from the point of closest approach and still remain in the atmosphere. Use is made of the relationship AA < BB (Equation 32). If BB < -XMX, both AA and BB are less than -XMX, and the line of sight within the atmosphere is free of shadow. The same conclusion results if AA > +XMX. If AA < -XMX and BB > +XMX, the entire line of sight is shadowed.

Line 087 to 090 The distances AA and BB, heretofore calculated with respect to the point of closest approach, are converted so as to be given relative to the local point.

Line 093 to 094 Statement number 560 is used if there is no shadow. Statement number 550, which sets TL to .TRUE., is used if some part of the shadowed region of the line of sight exists within the atmosphere.

Line 109 to 111 For $\zeta < R$, $\lambda' = \pi/2$ and $\phi' = 0$ or π, the line of sight is parallel to the symmetry axis, and the line of sight does not penetrate the shadow cylinder. However, the entire line of sight is within the shadow cylinder. Two artificial shadow points, chosen to be well outside the atmosphere, are used to indicate to the calling program that the line of sight is shadowed.

TABLE B-1

Listing of the Subroutine SHASCC

```
      SUBROUTINE SHASCC(H,LBM,THET,PHT,AA,BB,TL)
C
C     THIS SUBROUTINE CALCULATES THE POINTS, IF ANY, AT WHICH THE
C     LINE OF SIGHT INTERSECTS THE REGION SHADOWED BY A SPHERICAL
C     PLANET ILLUMINATED BY PLANE PARALLEL LIGHT. THE LINE OF SIGHT
C     IS SPECIFIED BY THE FOLLOWING VARIABLES
C             H       HEIGHT ABOVE THE SURFACE
C             LBM     ANGLE BETWEEN SUN AND REFERENCE POINT, MEASURED
C                     AT THE CENTER OF THE PLANET
C             THET    POLAR ANGLE FROM THE REFERENCE POINT TO THE LINE
C                     OF SIGHT
C             PHT     AZIMUTHAL ANGLE MEASURED TO THE PLANE CONTAINING
C                     THE SUN
C
C     THE QUANTITIES AA AND BB ARE THE POINTS WHERE THE LINE OF SIGHT
C     ENTER AND LEAVE THE SHADOW REGION, MEASURED WITH RESPECT TO THE
C     REFERENCE POINT.
C
C     IF THERE IS NO INTERSECTION BETWEEN THE LINE OF SIGHT AND THE
C     SHADOW REGION, THE LOGICAL VARIABLE TL IS SET TO .FALSE.
C     TL IS SET .TRUE. ONLY IF SOME PART OF THE SHADOWED LINE OF SIGHT
C     IS WITHIN THE ATMOSPHERE, THAT IS, AT A HEIGHT LESS THAN HMX.
C
      INCLUDE PRFLST,LIST
      REAL LBM
      LOGICAL TL
      COMMON/GEO/R,ERR,ZETA,PSI,PI
C
C     DEFINE FUNCTIONS FOR HEIGHT (CH) AND COS LAMBDA' (CLP)
      CH(Z)=SQRT((R+H)**2+Z**2+2.*(R+H)*Z*CT)
      CLP(Y)=((R+H)*CL+Y*(CL*CT+SL*ST*CP))/CH(Y)
C
C     INITIALIZE LOGICAL VARIABLE
              TL=.FALSE.
C     CALCULATE TRIGONOMETRIC QUANTITIES
              CL=COS(LBM)
              SL=SIN(LBM)
C       RETURN IF CONDITION I IS SATISFIED
              IF((R+H)*SL.LE.R.AND.CL.GE.0.) GO TO 560
              CP=COS(PHT)
              SP=SIN(PHT)
              CT=COS(THET)
              ST=SIN(THET)
C       RETURN IF CONDITION II IS SATISFIED
              CG=CT*SL-ST*CP*CL
              IF((R+H)*SL.GE.R.AND.CG.GE.0.) GO TO 560
C
C     COMPUTE SUN ANGLE AND HEIGHT AT CLOSEST APPROACH POINT
              CSSG=CL*ST-SL*CP*CT
              ZTA=(R+H)*ST
              IF(ZTA-R) 200,,
C     NO SHADOW FOR ZTA.GE.R AND CSSG GREATER THAN (ABOUT) ZERO
              IF(CSSG+.003) ,560,560
              IF(ABS(CSSG)-(.99999)) ,401,401  @ CHECK VALID COSINE
```

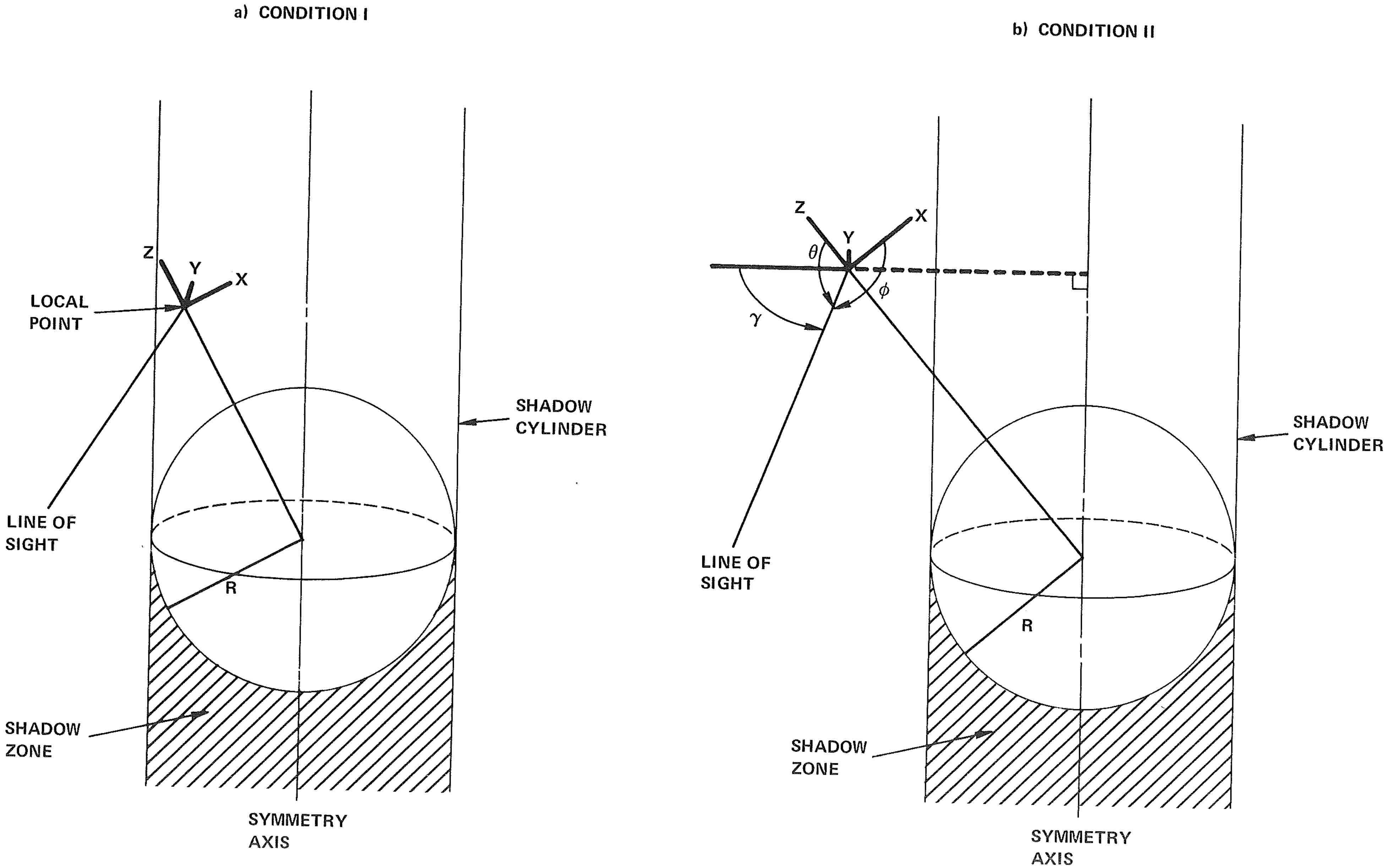

FIGURE B-1 - ILLUSTRATION OF CONDITIONS UNDER WHICH NO SHADOWING OF THE LINE OF SIGHT IS POSSIBLE

```
C       CALCULATE TRIGONOMETRIC FUNCTIONS FOR CLOSEST POINT               SHASC055
              CSG2=CSSG**2                                                SHASC056
              SNG2=1.-CSG2                                                SHASC057
              SNSG=SQRT(SNG2)                                             SHASC058
              TNSG=SNSG/CSSG                                              SHASC059
              SNPH=SL*SP/SNSG                                             SHASC060
              SNP2=SNPH**2                                                SHASC061
              CSPH=(CL*CT+SL*CP*ST)/SNSG                                  SHASC062
              CSP2=CSPH**2                                                SHASC063
C       CALULATE THE POINT X0,Y0  (XPT AND YPT)                           SHASC064
 20           XPT=ZTA*TNSG*CSPH                                           SHASC065
              YPT=ZTA*TNSG*SNPH                                           SHASC066
C       CALCULATE ELLIPSE COEFFICIENTS                                    SHASC067
              CSG2=CSSG**2                                                SHASC068
              AX=CSG2*CSP2+SNP2                                           SHASC069
              AXY=SNG2*SNPH*CSPH                                          SHASC070
              AY=CSG2*SNP2+CSP2                                           SHASC071
C                                                                         SHASC072
              ALPHA=AX                                                    SHASC073
              BETA=(AXY*YPT-AX*XPT)                                       SHASC074
C                                                                         SHASC075
              Q=(AXY**2-AX*AY)*YPT**2+AX*R**2                             SHASC076
              IF(Q) 560,560,  @ RETURN IF Q IS NEGATIVE OR ZERO           SHASC077
C                                                                         SHASC078
C       CALCULATE INTERSECTION POINTS                                     SHASC079
              Q=SQRT(Q)                                                   SHASC080
              AA=(-BETA-Q)/(ALPHA)                                        SHASC081
              BB=(-BETA+Q)/(ALPHA)                                        SHASC082
C       CHECK THAT SHADOW POINTS ARE WITHIN THE ATMOSPHERE                SHASC083
              XMX=SQRT((R+HMX)**2-ZTA**2)                                 SHASC084
              IF(BB.LT.(-XMX).OR.AA.GT.XMX) GO TO 560                     SHASC085
C       SHIFT TO BOXER COORDINATES                                        SHASC086
 50           XSHFT=(R+H)*CT                                              SHASC087
              AA=AA-XSHFT                                                 SHASC088
              BB=BB-XSHFT                                                 SHASC089
              IF(ZTA-R) 300,,                                             SHASC090
C                                                                         SHASC091
 550          TL=.TRUE.                                                   SHASC092
 560          RETURN                                                      SHASC093
C                                                                         SHASC094
```

```
C       CALCULATIONS FOR ZTA .LT. R
 200            IF(ABS(CSSG)-.00001) ,,10
C       CONTINUE FOR LAMBDA' EQUAL PI/2
                CSPH=ABS(CL*CT+SL*ST*CP)
                IF(ABS(CSPH).GE.1.00001) WRITE(6,707) CSPH
 707    FORMAT(' 707  INACCURATE COSINE(PHI). CSPH='F10.7)
                IF(ABS(CSPH).GE.(.99999)) GO TO 260
C       USE CSPH FOR SINE(PHI)
                CSPH=SQRT(1.-CSPH**2)
                V=SQRT(R**2-ZTA**2)
                AA=-V/CSPH
                BB=-AA
                GO TO 50
C       LAMBDA' EQUAL PI/2 AND PHI' EQUAL 0 OR PI
 260            AA=-2.*(R+HMX)
                BB=-AA
                GO TO 50
C       LOGIC FOR ZTA .LT. R
 300            IF(XSHFT) ,,320
                CLPA=CLP(AA)
                IF(CLPA) 550,560,560
 320            CLPB=CLP(BB)
                IF(CLPB) ,560,560
                IF(BB) 560,560,550
C       ADJUST TRIGONOMETRIC QUANTITIES IF SIN(LAMBDA')=0.
 401            SNPH=0.
                SNP2=0.
                CSPH=1.
                CSP2=1.
                SNSG=0.
                SNG2=0.
                TNSG=0.
                CSSG=SIGN(1.,CSSG)
                CSG2=1.
                GO TO 20
        END
```

TABLE B-2

Correspondence between variables used in the derivations in the text and thosed used in the computer program SHASCC.

SHASCC	Text	
LMB	λ	
H	h	
THET	θ	
PHT	ϕ	
AA	A	
BB	B	
R	R	
PI	π	
CL	$\cos\lambda$	
SL	$\sin\lambda$	
CP	$\cos\phi$	
SP	$\sin\phi$	
CT	$\cos\theta$	
ST	$\sin\theta$	
CG	$\cos\gamma$	
CSSG	$\cos\lambda'$	These variables refer to the point of closest approach.
ZTA	ζ	
CSG2	$\cos^2\lambda'$	
SNSG	$\sin\lambda'$	
SNG2	$\sin^2\lambda'$	
TNSG	$\tan\lambda'$	
SNPH	$\sin\phi'$	
SNP2	$\sin^2\phi'$	
CSPH	$\cos\phi'$	
CSP2	$\cos^2\phi'$	
XPT	x_o	
YPT	y_o	
AX	a_x	
AXY	a_{xy}	
AY	a_y	
Q	Q	
XSHFT	$-L_\zeta$	
V	V	

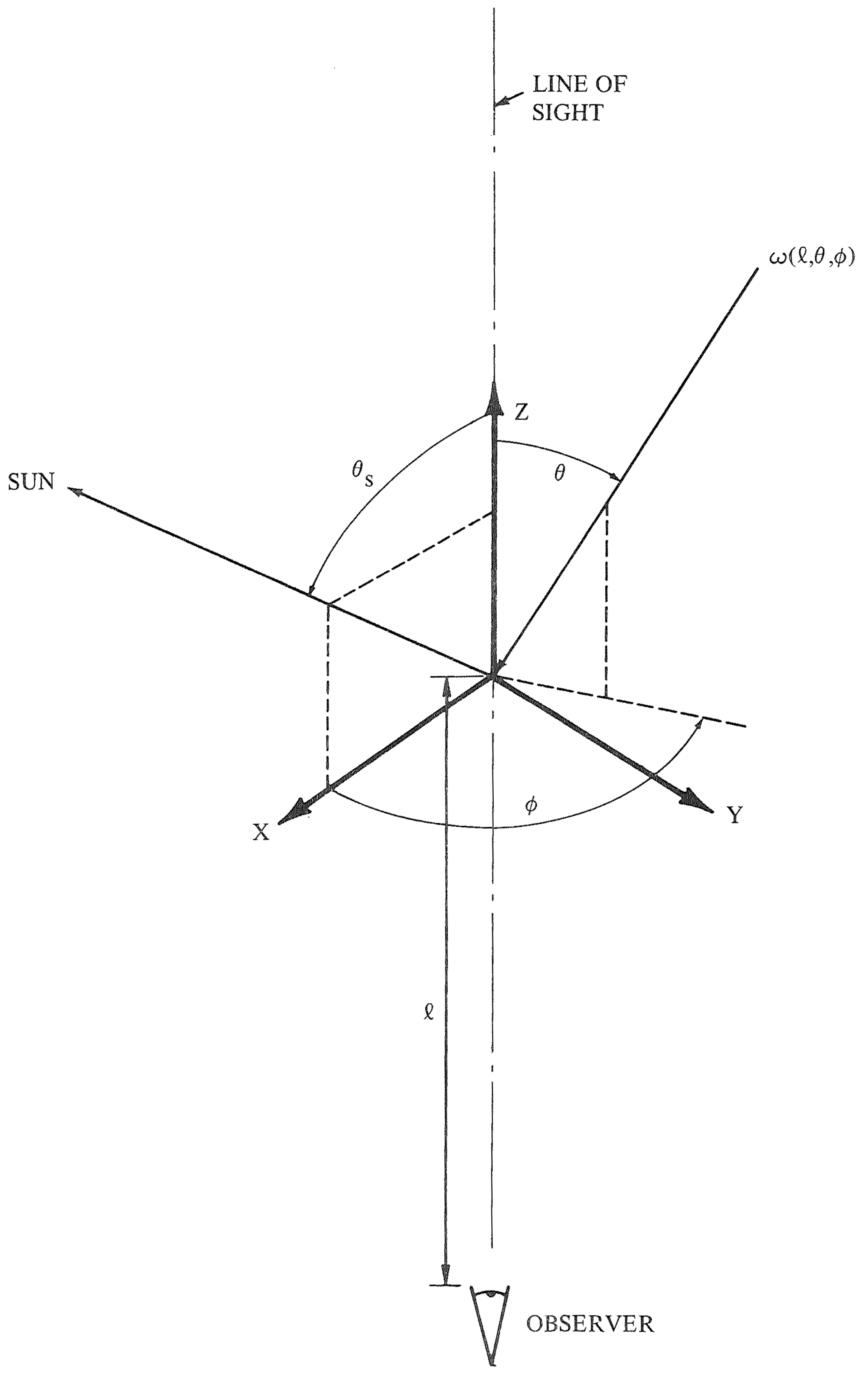

FIGURE 1 - COORDINATE SYSTEM ALONG THE LINE OF SIGHT. THE X AXIS LIES IN THE PLANE DEFINED BY THE SUN AND THE LINE OF SIGHT

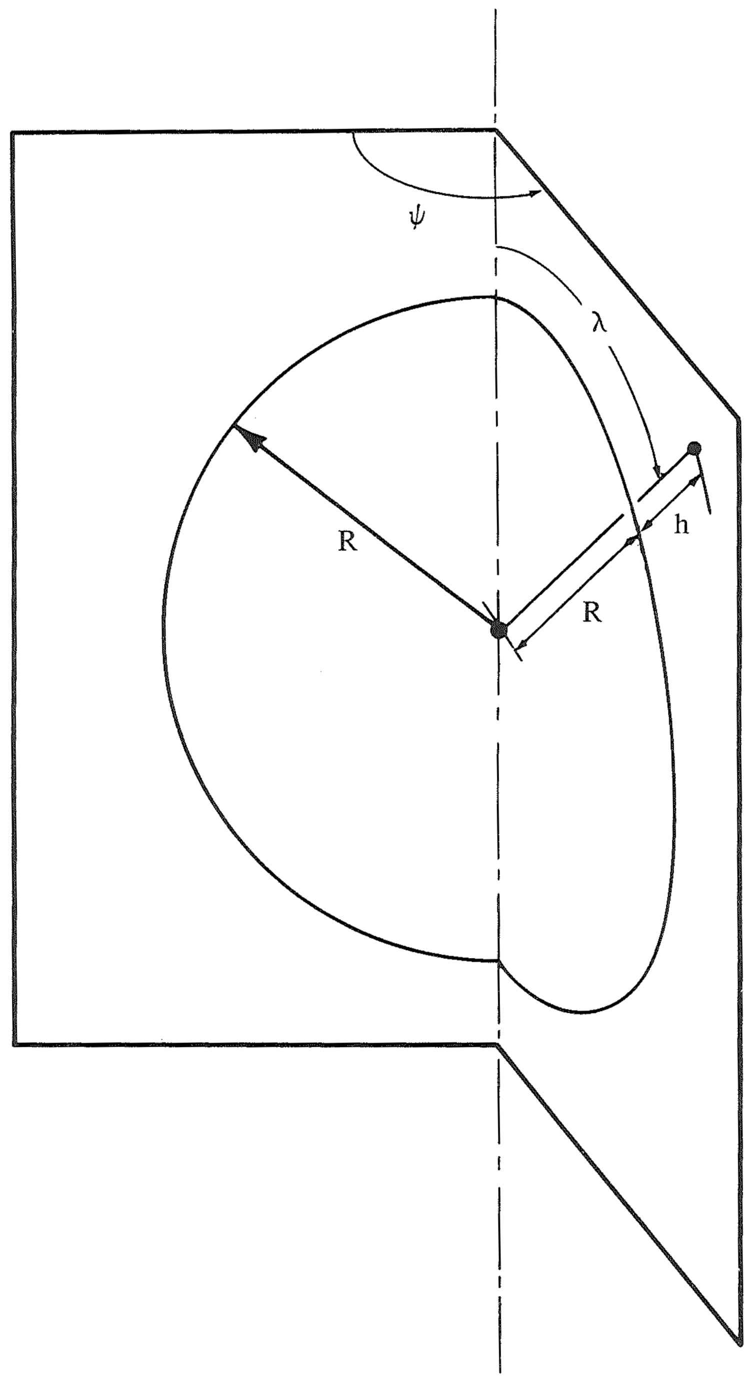

FIGURE 2 - COORDINATE SYSTEM FOR DEFINING A LOCAL POINT. R IS THE RADIUS OF THE PLANET

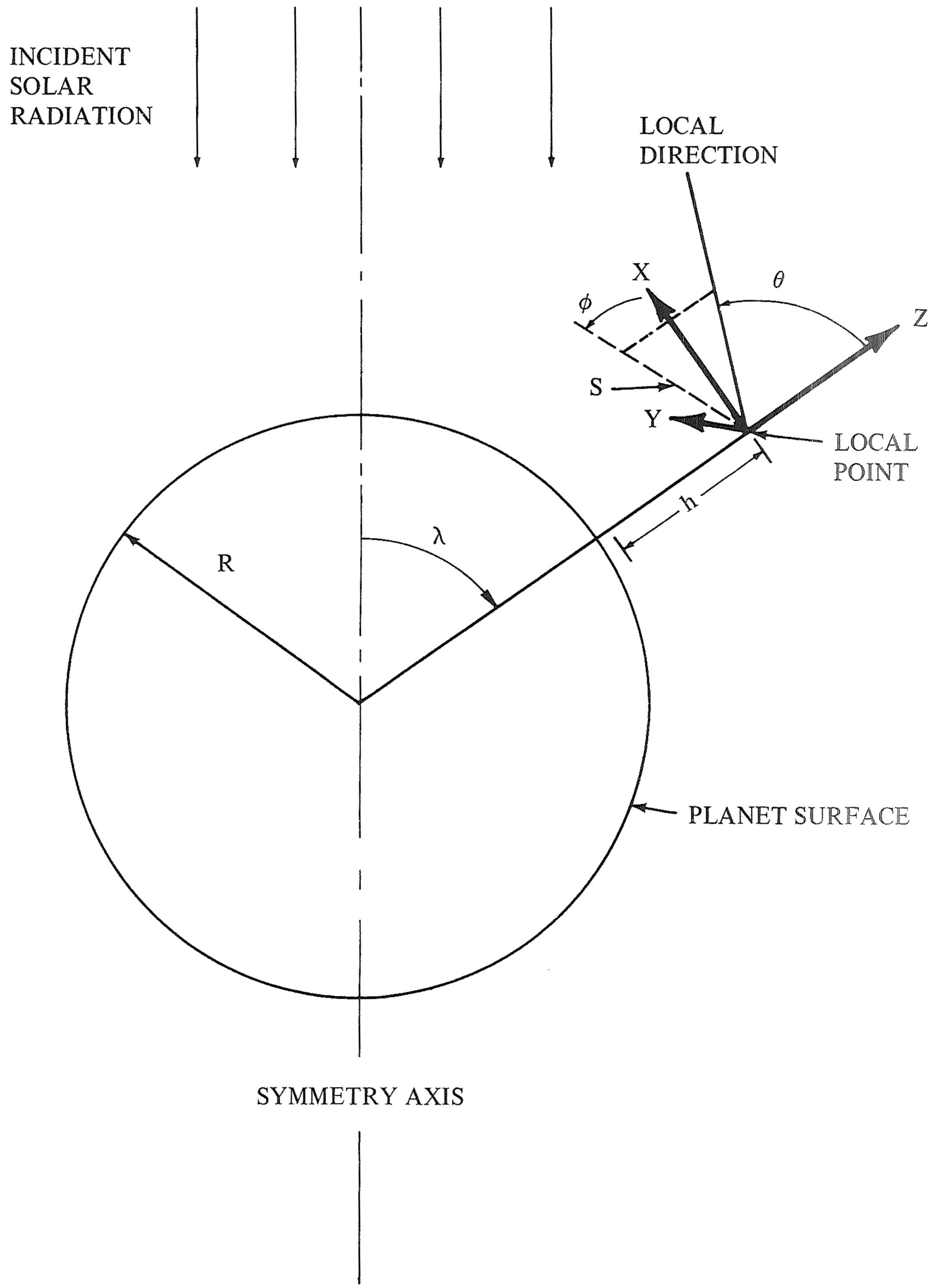

FIGURE 3 - GEOMETRY OF A LOCAL POINT AND A LOCAL COORDINATE SYSTEM. THE Z AXIS IS IN THE DIRECTION OF THE LOCAL VERTICAL, AND THE X AXIS LIES IN THE PLANE DEFINED BY Z AND THE SYMMETRY AXIS. THE Y AXIS IS CHOSEN TO COMPLETE A RIGHT-HANDED ORTHOGONAL COORDINATE SYSTEM. THE LINE MARKED S LIES IN THE X-Y PLANE

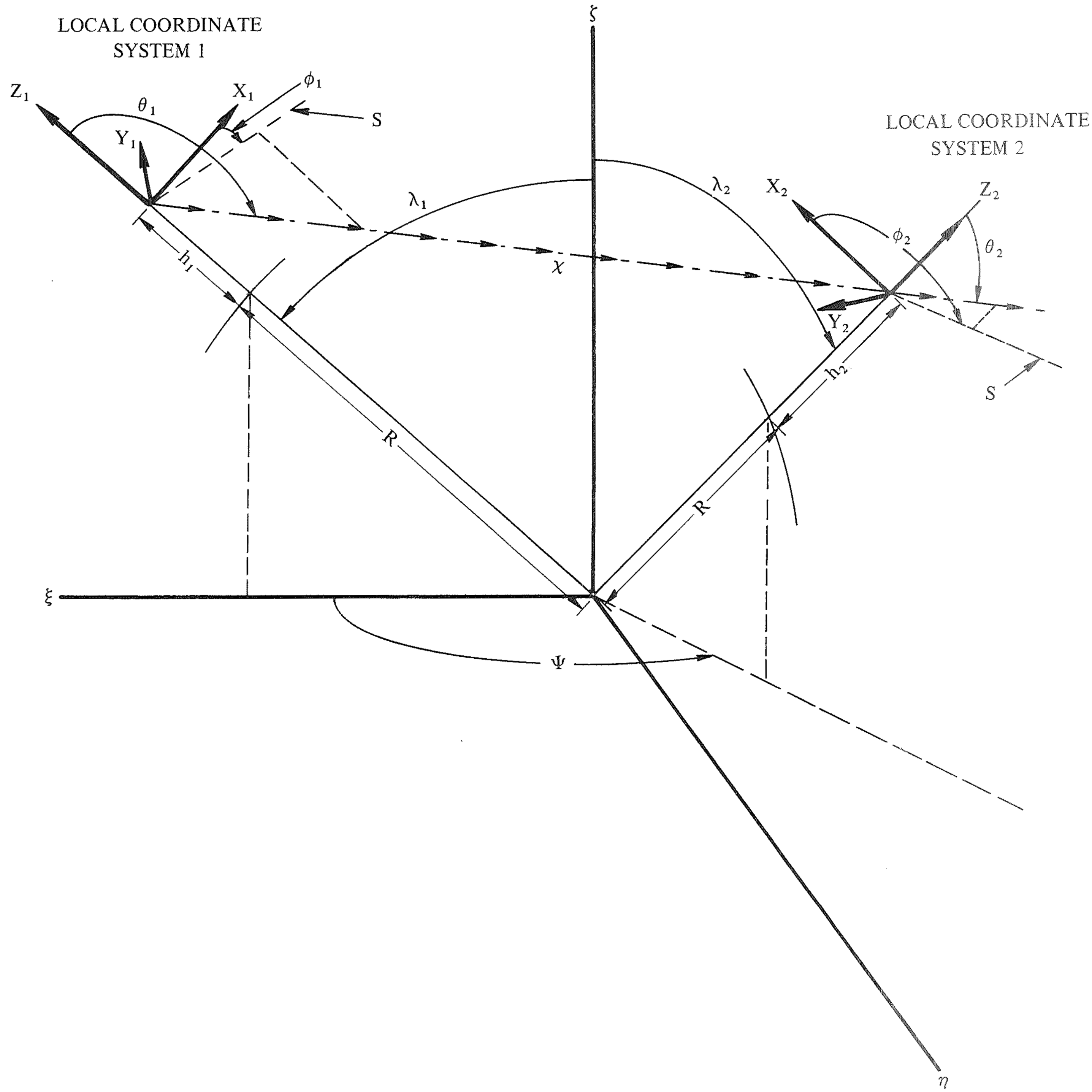

FIGURE 4 - LOCATION OF ONE LOCAL COORDINATE SYSTEM WITH RESPECT TO ANOTHER. THE ORIGIN OF THE $\xi\eta\zeta$ COORDINATE SYSTEM IS AT THE CENTER OF THE PLANET; THE ζ AXIS POINTS TOWARD THE SUN. BOTH ϕ_1 AND ϕ_2 ARE SHOWN AS NEGATIVE ANGLES. THE LINES MARKED S ARE IN THE RESPECTIVE X-Y PLANES.

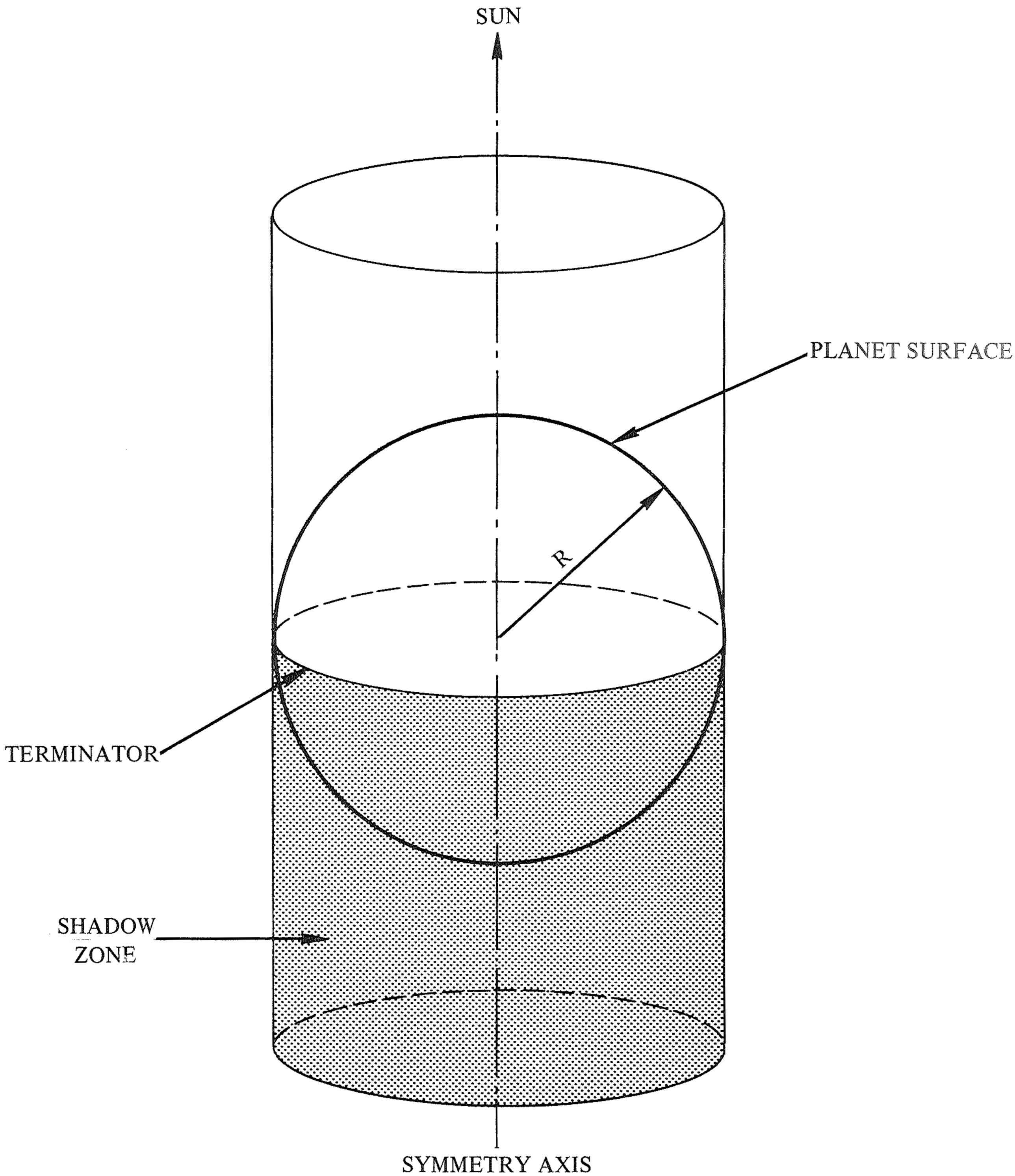

FIGURE 5 - THE SHADOW ZONE

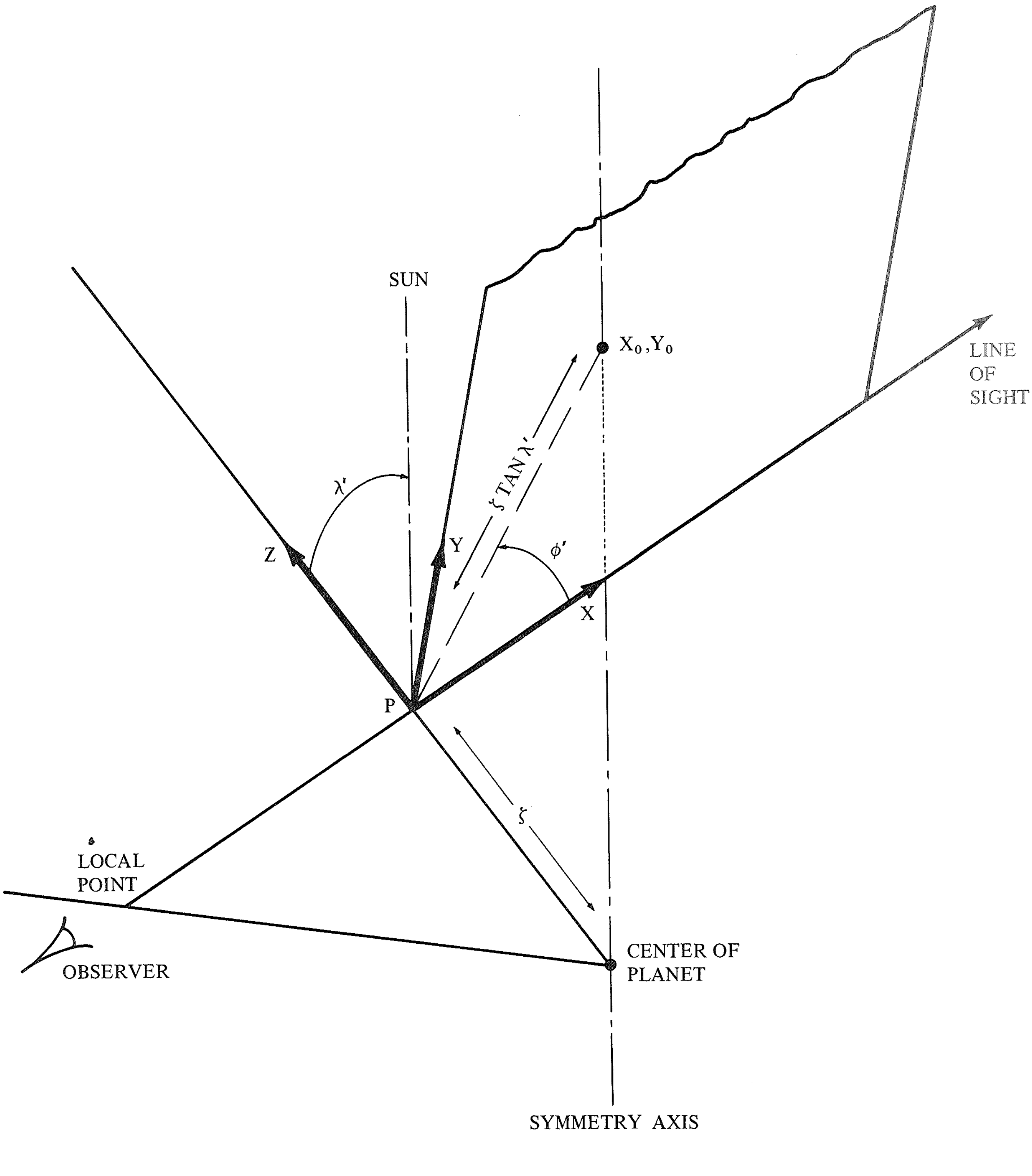

FIGURE 6 - THE POINT OF CLOSEST APPROACH, P, AND ASSOCIATED GEOMETRY

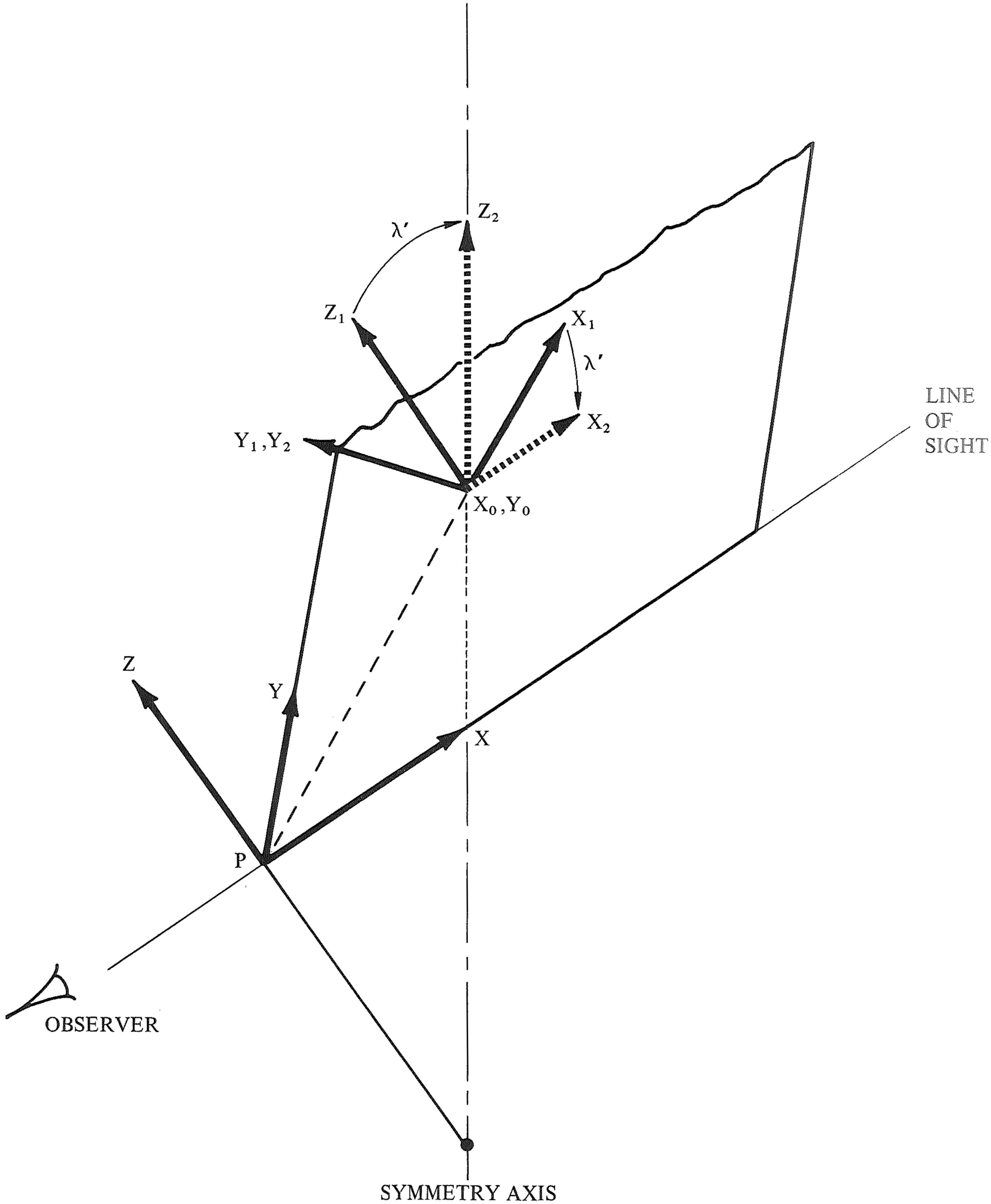

FIGURE 7 - COORDINATE TRANSFORMATIONS FOR CALCULATION OF THE EQUATION OF THE ELLIPSE

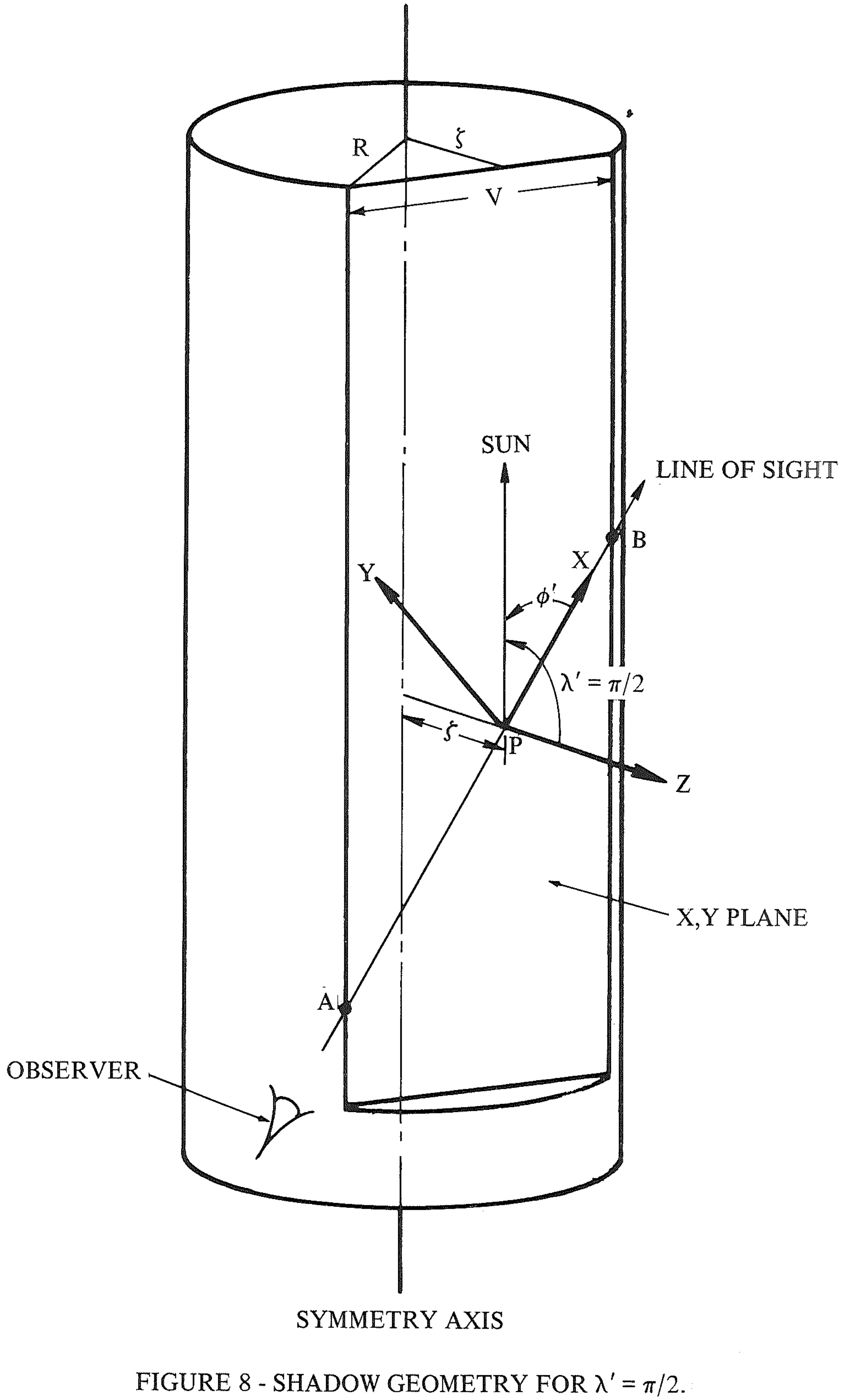

FIGURE 8 - SHADOW GEOMETRY FOR $\lambda' = \pi/2$.