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THEORY OF HIGHER ORDER OPTIMUM IMPULSIVE SOLUTIONS

by J. F. Andrus

Prepared by
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Dr. J. F. Andrus

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16. ABSTRACT

This report presents a theoretical discussion of impulsive approximations to optimum solutions to space flight problems with bang-bang thrust magnitude control. With a few exceptions (such as constant thrust intercept problems), the theory is limited to exo-atmospheric, multi-burn problems in which the thrust magnitude is a constant on each thrust arc and in which the coast arcs are of relatively long duration.

Moreover, the theory of impulsive solutions is extended to include linear and higher order corrections to impulsive solutions. Analytical methods for optimization and guidance are the end result. The detailed development stops after the second-order corrections. The higher order corrections, although probably too complicated for practical use, can be derived in an analogous manner.

The theory is applied to rendezvous and intercept problems. Concise expressions for first order corrections to the impulsive solutions to the latter problems are derived, and two numerical applications are made in which the errors in the impulsive approximations are reduced by an order of magnitude.

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FOREWORD

This report presents results of work performed by Northrop-Huntsville while under contract to the Aero-Astrodynamics Laboratory of the Marshall Space Flight Center (Contract NAS8-20082). The work was performed for the Astrodynamics and Guidance Theory Division in partial response to the requirements of Appendix E-1, Schedule Order No. 67.

The material presented in this report is primarily an elaboration of references 4 and 5 by the author. However, the solution for the adjoint variables and transition matrices contained in Appendix A and the associated computer program are the original contributions of I. F. Burns of Northrop.

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LIST OF SYMBOLS

Symbol	Definition
β	fuel burning rate magnitude
c	exhaust velocity
F	thrust magnitude ($F = c\beta$)
ε	$1/\beta$, an independent variable ($\varepsilon = 0$ corresponds to impulsive case)
t	independent variable time
t_I	initial time
$t_k(\varepsilon)$	($k = 1, 2, \ldots, N$) initial time on k^{th} burn arc
$\bar{t}_k(\varepsilon)$	($k = 1, 2, \ldots, N$) final time on k^{th} burn arc
$t_F(\varepsilon)$	final time
$y(t, \varepsilon)$	position of space vehicle, a two or three component column vector
$m(t, \varepsilon)$	mass of space vehicle
$\lambda(t, \varepsilon)$	Vector of Lagrange multipliers corresponding to $\dot{y}$
$\lambda^T(t, \varepsilon)$	transpose of column vector
$\dot{\lambda}(t, \varepsilon)$	time rate of change λ_t of λ and the negative of the Lagrange multiplier vector corresponding to y (In general the dot represents partial derivative $\partial/\partial t$ with respect to time t)
$\|\lambda\|$	magnitude of λ
$K(t, \varepsilon)$	switching function $K = (c/m)\|\lambda\| - \lambda_m$, where λ_m is the multiplier corresponding to m
$G(t, y)$	acceleration due to gravity, usually assumed in this report to be equal to $-\mu y/\|y\|^3$
$H(t,y,\dot{y},\lambda,\dot{\lambda},K,\beta)$	variational Hamiltonian which is $\lambda^T G - \dot{\lambda}^T \dot{y} + \beta K$ on a thrust arc and $\lambda^T G - \dot{\lambda}^T \dot{y}$ on a coast arc. It is ordinarily a constant of integration in a uniform gravitational field. When t_F is free, it is zero.

LIST OF SYMBOLS (Continued)

Symbol	Definition
$U(\lambda, \dot{\lambda})$	function $\lambda^T\dot{\lambda}/\lvert\lambda\rvert$ (Note: $\dot{K} = cU/m$)
$L(\lambda)$	optimal steering function $\lambda/\lvert\lambda\rvert$
$G_y(t, y)$	square matrix whose i^{th} row consists of the derivatives of the i^{th} component of G with respect to the components of y
$Q(t, y, \lambda)$	vector $G_y\lambda$ (Note: $\ddot{\lambda} = Q$ is an Euler-Lagrange equation)
$U^*(t, \varepsilon)$	$U[\lambda(t, \varepsilon), \dot{\lambda}(t, \varepsilon)]$
$G^*(t, \varepsilon)$	$G[t, y(t, \varepsilon)]$
$G_y^*(t, \varepsilon)$	$G_y[t, y(t, \varepsilon)]$
$G_t^*(t, \varepsilon)$	$G_t[t, y(t, \varepsilon)] + G_y[t, y(t, \varepsilon)]\,\dot{y}(t, \varepsilon)$ i.e., the "total" time derivative of G
$(G_t)^*$	$G_t[t, y(t, \varepsilon)]$ (i.e., the partial time derivative)
$(\dot{\lambda}L^*_{\lambda\lambda})$	$\sum_{p=1}^{3} \dot{\lambda}^{(p)} L^*_{\lambda\lambda}(p)$ where the superscript (p) indicates p^{th} component
$y_k(\varepsilon)$	$y[t_k(\varepsilon), \varepsilon]$
$\bar{y}_k(\varepsilon)$	$y[\bar{t}_k(\varepsilon), \varepsilon]$
$G_k^*(\varepsilon)$	$G^*[t_k(\varepsilon), \varepsilon]$
$y_{\varepsilon k}(\varepsilon)$	$y_\varepsilon[t_k(\varepsilon), \varepsilon]$
$y_{k\varepsilon}(\varepsilon)$	$y_\varepsilon[t_k(\varepsilon), \varepsilon] + t_{k\varepsilon}(\varepsilon)\,\dot{y}[t_k(\varepsilon), \varepsilon]$
$\ddot{y}_k^+(\varepsilon)$	$\lim_{\substack{t \to t_k(\varepsilon) \\ (t > t_k)}} \ddot{y}(t, \varepsilon)$ (righthand limit of $\ddot{y}$ at t_k)
$\ddot{y}_k^-(\varepsilon)$	$\lim_{\substack{t \to t_k(\varepsilon) \\ (t < t_k)}} \ddot{y}(t, \varepsilon)$ (lefthand limit of $\ddot{y}$ at t_k)
$\Delta\dot{y}_k(\varepsilon)$	$\dot{\bar{y}}_k(\varepsilon) - \dot{y}_k(\varepsilon)$
$\Delta V_k(\varepsilon)$	$\lvert\dot{\bar{y}}_k(\varepsilon) - \dot{y}_k(\varepsilon)\rvert$

LIST OF SYMBOLS (Concluded)

Symbol	Definition
$\Delta t_k(\varepsilon)$	$\bar{t}_k(\varepsilon) - t_k(\varepsilon)$
$\Delta m_k(\varepsilon)$	$\bar{m}_k(\varepsilon) - m_k(\varepsilon)$
$\Delta t_{k\varepsilon}(\varepsilon)$	$\bar{t}_{k\varepsilon}(\varepsilon) - t_{k\varepsilon}(\varepsilon)$
$\left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right]$	considering y_{k+1} to be a function of $\bar{y}_k$ (thereby deviating from the usual functional notation), this symbol represents a matrix whose (i,j) element is the partial derivative of the i^{th} component of y_{k+1} with respect to j^{th} component of $\bar{y}_k$ (these partial derivatives depend upon ε)
$[f(t)]_{t_k}^{\bar{t}_k}$	$f(\bar{t}_k) - f(t_k)$
$(\tilde{y}_{\varepsilon\varepsilon k})$	represents a matrix (or tensor) summation defined in the body of the report
$\sigma(\varepsilon^n)$, $\tilde{\sigma}(\varepsilon^n)$, $\theta(\varepsilon^n)$	these symbols denote terms of "order ε". Precise definitions are given in the body of the text.

Section I
INTRODUCTION

This report is concerned with impulsive approximations to exo-atmospheric, space flight optimization problems in which the powered portions of the flight are of relatively short duration. For example, the problem may be that of determining ignition and burnout times and the time-histories of the thrust control angles which provide an orbital transfer maneuver, from a given state to a given final orbit, with minimum fuel expenditure. A typical program might consist of a coast phase, followed by a powered phase to take the vehicle out of its initial orbit, followed by a long coast period to carry the vehicle to the vicinity of the final orbit, followed by another powered portion which serves to inject the vehicle into the specified orbit (Figure 1-1). Only space vehicles having constant thrust magnitude F and fuel burning rate magnitude β, on each thrust arc, are considered in this report.

In many such cases it is possible to obtain multi-impulse solutions which closely approximate the true solution, the impulsive solution being defined (ref. 1) as the limit of the optimum finite thrust solutions as β approaches infinity, where $F = c\beta$, and c is the constant exhaust velocity. The impulsive approximations to many finite thrust problems may be obtained quickly and easily (refs. 2 and 3).

It is understood throughout this report that the optimal solution for any finite thrust problem is theoretically obtained by solving the boundary condition problem arising from the necessary conditions for optimality of the calculus of variations. Such a solution requires the determination of the time histories of Lagrange multiplier variables. Once the initial values K_I, λ_I, $\dot{\lambda}_I$ of the Lagrange multipliers (or equivalent variables), the ignition times $t_1, t_2, \ldots, t_N$, the engine cutoff times $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_N$, and the final time t_F are known, then the entire optimal flight program can be completely determined.

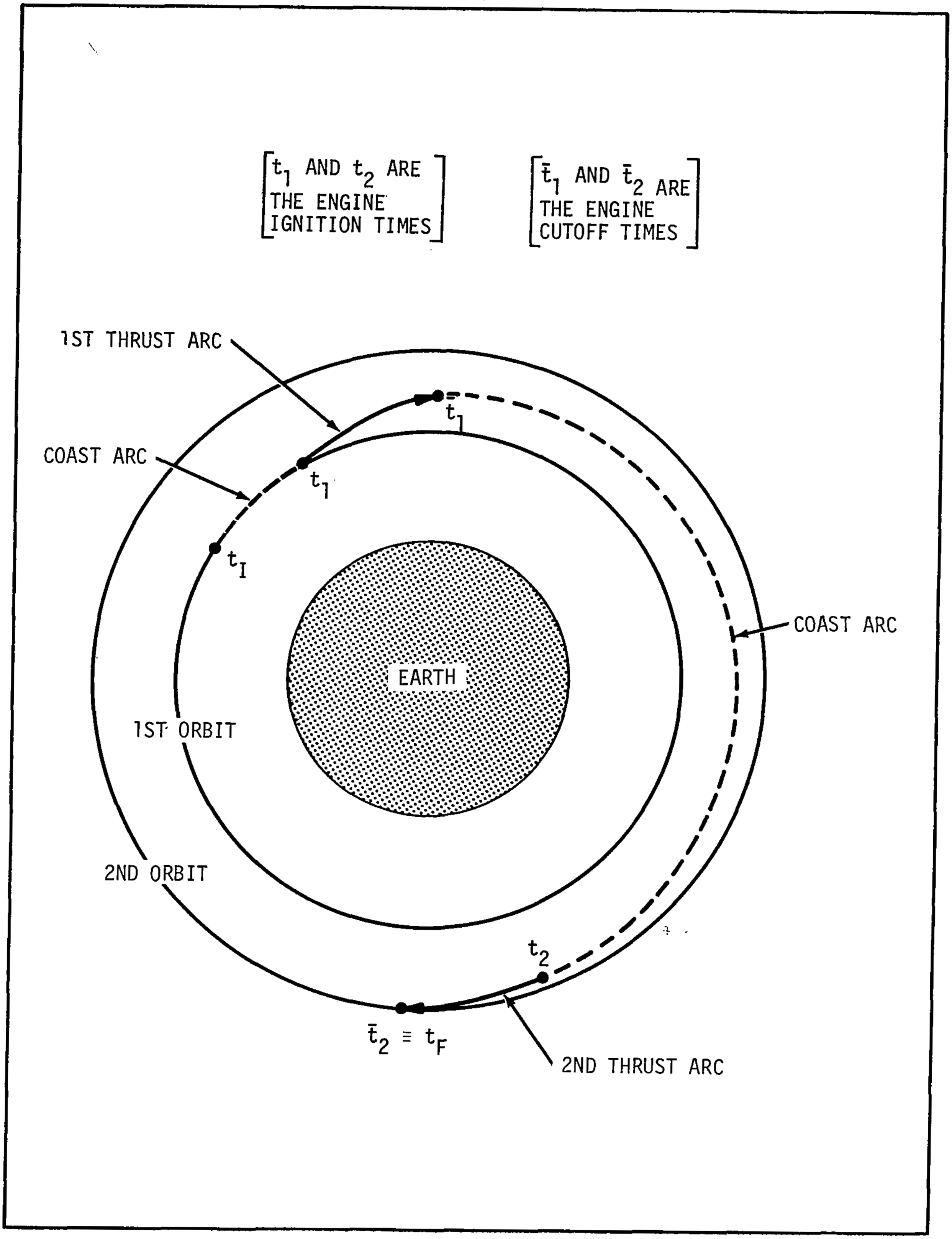

Figure 1-1. EXAMPLE ORBITAL TRANSFER PROBLEM

Roughly speaking, the method for obtaining the corrections to the impulsive solutions is to let $\varepsilon = 1/\beta$, to consider K_I, λ_I, $\dot{\lambda}_I$, $\bar{t}_k$, t_k, and t_F as functions of ε, to assume that the derivatives of these variables with respect to ε exist for $\varepsilon \geq 0$ in some neighborhood of $\varepsilon = 0$, and to develop the systems of linear equations which the derivatives, corresponding to $\varepsilon = 0$, must satisfy. If the linear equations are independent, they may be solved for the desired derivatives. Once the derivatives have been computed, Taylor series expansions, in powers of ε, of $K_I(\varepsilon)$, $\lambda_I(\varepsilon)$, etc. about $\varepsilon = 0$ (i.e., about the impulsive solution) may be determined. For example,

$$\lambda_I(\varepsilon) = \lambda_I(0) + \frac{\varepsilon}{1!}\frac{d\lambda_I}{d\varepsilon}\Bigg|_{\varepsilon=0} + \frac{\varepsilon^2}{2!}\frac{d^2\lambda_I}{d\varepsilon^2}\Bigg|_{\varepsilon=0} + \ldots$$

The linear equations determining derivatives with respect to ε are obtained by considering the boundary conditions to be identities in ε and by differentiating both members of each boundary equation with respect to ε. However, all of the derivatives with respect to ε (for example $dy_F/d\varepsilon$ where $y_F(\varepsilon)$ is the final value of y corresponding to ε) appearing in the resulting equations must be expressed in terms of derivatives of K_I, λ_I, $\dot{\lambda}_I$, t_k, $\bar{t}_k$, and t_F. Therefore, it is necessary to obtain expressions, applying at $\varepsilon = 0$, for the changes in the derivatives, with respect to ε, over coast arcs and over thrust arcs.

Section II of this report describes the general physical problem and the necessary conditions of the calculus of variations which determine optimal solutions to the problem. Section III defines the impulsive solutions to the problems discussed in Section II. Section IV gives the mathematical assumptions of differentiability to be employed as well as a few basic definitions.

The introductory material of Sections I through IV allows for the detailed description given in Section V of the overall procedure to be followed in obtaining the derivatives with respect to ε. Then there remain the problems of determining changes in the derivatives (at $\varepsilon = 0$) over coast arcs (Section VI and Appendix A) and over thrust arcs (Sections VII through IX and Appendix B).

Sections X, XI, and XII provide specific problems to which the theory is applied. Appendix C gives listings of computer programs employed in the numerical studies given in the latter sections.

This report places work presented in references 4 and 5 in one detailed and comprehensive document. It also includes a broadened discussion of the work on intercept problems reported in reference 6. Emphasis is placed on building a firm analytical basis for the generalized impulsive theory.

The problem of obtaining higher order corrections to impulsive solutions has been studied independently by the author of this report and several men associated with Princeton University. The latter work is contained in references 7 through 11. References 7 and 9 are concerned with the problem in which the thrust-acceleration F/m is a constant on each thrust arc. References 8 and 10 consider the same problem and extend the work to the problem considered in this report. However, these two papers make use of expansions in terms of two parameters rather than just the parameter ε. The parameters are initial thrust-acceleration and the rocket jet exhaust velocity. Reference 11 considers numerical applications of the other papers to the early phase of low thrust mission analysis.

Section II

PHYSICAL PROBLEM AND GOVERNING EQUATIONS

The problem under consideration is the determination of the optimal vacuum flight of a space vehicle (assumed to be a point mass) from a given point in state space (i.e., given time, mass, and position and velocity components) to a specified orbit, position, etc. The flight has at least one coasting phase. When thrusting, the vehicle has a constant thrust magnitude F and a burning rate magnitude β. The flight program is to be optimized with respect to final payload. The choice of this payoff function is somewhat arbitrary; the theory can be easily modified to cover other payoff functions.

The equations of motion are

$$\ddot{y} = \frac{F}{m} L(\lambda) + G(t, y)$$

$$\dot{m} = -\beta$$

$$(F = \beta = 0 \text{ on coast arc})$$

where y is the position vector, G is acceleration due to gravity, m is the total mass, and $L(\lambda)$ is the optimal steering function $\lambda/|\lambda|$ which may be determined from the calculus of variations (refs. 1 and 12). The vector λ is the solution of the Euler-Lagrange equations (of the calculus of variations), having the form

$$\ddot{\lambda} = Q(t, \lambda, y)$$

Let t_k and $\bar{t}_k$ be the initial and final times on the k^{th} thrust arc (k = 1, 2, ..., N). If the initial subarc is a thrust arc, then t_1 is equivalent to the known initial time t_I. If the N^{th} thrust arc is the last subarc of the trajectory, then $\bar{t}_N$ coincides with t_F. Again, the calculus of variations (refs. 1 and 12) may be used to show that the optimum t_k and $\bar{t}_k$ for each thrust arc are determined by the switching function K satisfying a differential equation*

$$\dot{K} = \frac{c}{m} U(\lambda, \dot{\lambda})$$

where

$$U = \frac{\lambda^T \dot{\lambda}}{|\lambda|}$$

λ being considered as a column vector and λ^T a row vector. It is necessary that K = 0 when $t = t_k$ (unless $t_k \equiv t_I$) and when $t = \bar{t}_k$ (unless $\bar{t}_k \equiv t_F$). Letting $\varepsilon = 1/\beta$, the total system of differential equations is

$$\ddot{y} = \frac{c}{\varepsilon m} L(\lambda) + G(t, y) \qquad \frac{c}{\varepsilon m} L \text{ term omitted on coast arcs}$$

$$\ddot{\lambda} = Q(t, \lambda, y)$$

$$\dot{K} = \frac{c}{m} U(\lambda, \dot{\lambda}) \tag{2-1}$$

$$\dot{m} = -\frac{1}{\varepsilon} \qquad \text{(on thrust arcs)}$$

$$\dot{m} = 0 \qquad \text{(on coast arcs)}$$

The problem is ultimately a multi-point boundary condition problem in which the initial values K_I, λ_I, and $\dot{\lambda}_I$, the switching times, and the final time t_F are to be determined such that K = 0 at the switch times and such that the terminal end conditions (including a scaling condition on the Lagrange multipliers and including transversality conditions from the calculus of variations) are satisfied. A somewhat arbitrary scaling condition on the Lagrange multipliers is imposed because the differential equations (2-1) are known to be homogeneous in the variables λ, $\dot{\lambda}$, and K.

The multi-point boundary condition problem is a problem fundamentally of solving a system of non-linear equations in several unknowns.

**In reference 12, it is shown that $K = \frac{c}{m}|\lambda| - \lambda_m$, where λ_m is a Lagrange multiplier satisfying the equation $\dot{\lambda}_m = \frac{F}{m^2}|\lambda|$. Therefore, $\dot{K} = \frac{c}{m}\frac{\lambda^T\dot{\lambda}}{|\lambda|}$.*

Section III

IMPULSIVE SOLUTIONS

For each $\varepsilon > 0$ in a neighborhood of $\varepsilon = 0$, assume that there is a solution

$$y(t, \varepsilon), \dot{y}(t, \varepsilon), m(t, \varepsilon), \lambda(t, \varepsilon), \dot{\lambda}(t, \varepsilon), K(t, \varepsilon)$$

$$t_k(\varepsilon), \bar{t}_k(\varepsilon), t_F(\varepsilon)$$

to the multi-point boundary condition problem discussed in Section II. We are now thinking of y, $\dot{y}$, etc. as functions of two arguments, t and ε. The limiting solution as ε approaches zero, if it exists, is called the impulsive solution. (Recall that $\beta = 1/\varepsilon$ and $F = c/\varepsilon$.)

In reference 1 and later in this report it is shown that, as $\varepsilon \to 0$,

$$\bar{t}_k(\varepsilon) \to t_k(\varepsilon)$$

$$y(\bar{t}_k, \varepsilon) \to y(t_k, \varepsilon)$$

$$\dot{y}(\bar{t}_k, \varepsilon) \to \dot{y}(t_k, \varepsilon) + c \left[\log \frac{m(t_k, \varepsilon)}{m(\bar{t}_k, \varepsilon)}\right] L[\lambda(t_k, \varepsilon)]$$

$$\Delta V_k(\varepsilon) = \left|\dot{y}(\bar{t}_k, \varepsilon) - \dot{y}(t_k, \varepsilon)\right| \to c \log \frac{m(t_k, \varepsilon)}{m(\bar{t}_k, \varepsilon)}$$

$$\lambda(\bar{t}_k, \varepsilon) \to \lambda(t_k, \varepsilon)$$

$$\dot{\lambda}(\bar{t}_k, \varepsilon) \to \dot{\lambda}(t_k, \varepsilon)$$

$$U[\lambda(t_k, \varepsilon), \dot{\lambda}(t_k, \varepsilon)] \to 0 \qquad \text{(i.e., } \dot{K} = 0\text{) at an interior impulse*}$$

The multi-point boundary condition problem for the impulsive case is a modification of that of the non-impulsive problem. In the impulsive case, one must choose K_I, λ_I, $\dot{\lambda}_I$, t_k, ΔV_k, and t_F such that

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**If the impulse is at t_I or t_F, it is not necessarily true that $\dot{K}(t_k, \varepsilon) \to 0$ as $\varepsilon \to 0$. In orbital transfer problems $K_F(\varepsilon) \to 0$, but this is not necessarily true in other problems.*

$$K(t_k, 0) = 0$$

$$\dot{K}(t_k, 0) = 0 \qquad \text{(for interior impulses)}$$

and such that the terminal end conditions are satisfied.

Section IV

ASSUMPTIONS AND DEFINITIONS

Assume that for any $\varepsilon > 0$ within some neighborhood of $\varepsilon = 0$, there are functions $y(t, \varepsilon)$, $\dot{y}(t, \varepsilon)$, $\lambda(t, \varepsilon)$, $\dot{\lambda}(t, \varepsilon)$, $K(t, \varepsilon)$, $m(t, \varepsilon)$, $t_k(\varepsilon)$, $\bar{t}_k(\varepsilon)$, and $t_F(\varepsilon)$ which satisfy the differential equations (2-1) and the boundary conditions. (It is understood that ε now appears in all of the arguments since it is to be treated as a variable.) Furthermore, all of these functions are assumed to be continuous in ε and, for $t_I \leq t \leq t_F$, to approach finite limits as ε approaches zero. For any t in the interval and for $\varepsilon \geq 0$, the latter assumptions imply that within some neighborhood of $\varepsilon = 0$, the functions are bounded.

Assume that for $\varepsilon > 0$, the derivatives (with respect to ε) of all orders of $\lambda_I(\varepsilon)$, $\dot{\lambda}_I(\varepsilon)$, $t_k(\varepsilon)$, $\bar{t}_k(\varepsilon)$, and $t_F(\varepsilon)$ exist in some neighborhood of $\varepsilon = 0$ and approach finite limits as $\varepsilon \to 0$. This implies that the derivatives of any order of these quantities are bounded for $\varepsilon \geq 0$ in some neighborhood of $\varepsilon = 0$.

The following notation is introduced:

$$y_k(\varepsilon) = y[t_k(\varepsilon), \varepsilon]$$

$$\bar{y}_k(\varepsilon) = y[\bar{t}_k(\varepsilon), \varepsilon]$$

$$y_\varepsilon = \frac{\partial y}{\partial \varepsilon}$$

$$\dot{y} = \frac{\partial y}{\partial t} \quad \text{or} \quad y_t = \frac{\partial y}{\partial t}$$

$$y_{k\varepsilon} = \frac{dy_k}{d\varepsilon} = \left.\frac{\partial y}{\partial \varepsilon}\right|_{t=t_k} + \frac{dt_k}{d\varepsilon} \left.\frac{\partial y}{\partial t}\right|_{t=t_k} = y_{\varepsilon k} + t_{k\varepsilon}\,\dot{y}_k$$

and so on for the other variables. (As a notational simplification an ε subscript is sometimes used to signify an ordinary derivative rather than a partial derivative.) When required or convenient, + and − superscripts will be used to denote right and lefthand limits with respect to time at t_k or $\bar{t}_k$; for example,

$$\ddot{\bar{y}}_k^{+}(\varepsilon) = \lim_{\substack{t \to \bar{t}_k \\ (t > \bar{t}_k)}} \ddot{y}(t, \varepsilon) = G(\bar{t}_k, \bar{y}_k)$$

We define new functions as follows:

$$L^*(t, \varepsilon) = L[\lambda(t, \varepsilon)]$$
$$G^*(t, \varepsilon) = G[t, y(t, \varepsilon)]$$
$$Q^*(t, \varepsilon) = Q[t, \lambda(t, \varepsilon), y(t, \varepsilon)]$$
$$U^*(t, \varepsilon) = U[\lambda(t, \varepsilon), \dot{\lambda}(t, \varepsilon)]$$

and similarly for other functions of t, y, and λ to be introduced later. For example,

$$G^*_y(t, \varepsilon) = G_y[t, y(t, \varepsilon)]$$

$$L^*_t(t, \varepsilon) = L_\lambda[\lambda(t, \varepsilon)]\, \dot{\lambda}(t, \varepsilon)$$

where L_λ is a matrix and $\dot{\lambda}$ is a column vector. In order that there will be no ambiguity in the symbol G^*_t, we let

$$(G_t)^* = G_t[t, y(t, \varepsilon)]$$

and

$$G^*_t = (G_t)^* + G^*_y\, \dot{y} .$$

In a uniform gravitational field, $(G_t)^* = 0$.

The symbol $\sigma(\varepsilon^n)$ will denote a finite summation of functions of the form

$$a(\varepsilon)\varepsilon^{n_1}[\Delta t_k(\varepsilon)]^{n_2}$$

where n_1, n_2, and n are non-negative integers such that $n_1 + n_2 \geq n$, and $a(\varepsilon)$ is bounded for $\varepsilon \geq 0$ within some neighborhood of $\varepsilon = 0$. Therefore, as $\varepsilon \to 0$, $\sigma(\varepsilon^n) \to 0$ if $n \geq 1$. The symbol $\tilde{\sigma}(\varepsilon^n)$ is defined in the same manner as $\sigma(\varepsilon^n)$ with each function $a(\varepsilon) \equiv$ constant.

Section V

GENERAL PROCEDURE FOR DETERMINING DERIVATIVES WITH RESPECT TO $\varepsilon = 1/\beta$

Consider a terminal end constraint

$$\phi(y_F, \dot{y}_F, m_F, \lambda_F, \dot{\lambda}_F, K_F, t_F) = 0$$

Let

$$\phi^*(\varepsilon) = \phi[y_F(\varepsilon), \dot{y}_F(\varepsilon), m_F(\varepsilon), \lambda_F(\varepsilon), \dot{\lambda}_F(\varepsilon), K_F(\varepsilon), t_F(\varepsilon)]$$

Then

$$\phi^*(\varepsilon) \equiv 0,$$

and

$$\frac{d}{d\varepsilon} \phi^*(\varepsilon) \equiv 0, \frac{d^2}{d\varepsilon^2} \phi^*(\varepsilon) \equiv 0, \ldots .$$

Applying the chain rule of differentiation to the identity corresponding to the first derivative, we obtain

$$\phi_{y_F} y_{F\varepsilon} + \phi_{\dot{y}_F} \dot{y}_{F\varepsilon} + \phi_{m_F} m_{F\varepsilon} + \phi_{\lambda_F} \lambda_{F\varepsilon} + \phi_{\dot{\lambda}_F} \dot{\lambda}_{F\varepsilon} + \phi_{K_F} K_{F\varepsilon} + \phi_{t_F} t_{F\varepsilon} \equiv 0 \quad (5\text{-}1)$$

If, for example, the terminal subarc is a coast arc, then the quantities $y_{F\varepsilon}$, $\dot{y}_{F\varepsilon}$, $m_{F\varepsilon}$, $\lambda_{F\varepsilon}$, $\dot{\lambda}_{F\varepsilon}$, and $K_{F\varepsilon}$ in equation (5-1) can be expressed in terms of $\bar{y}_{N\varepsilon}$, $\bar{\dot{y}}_{N\varepsilon}$, $\bar{m}_{N\varepsilon}$, $\bar{\lambda}_{N\varepsilon}$, $\bar{\dot{\lambda}}_{N\varepsilon}$, and $\bar{K}_{N\varepsilon}$ as shown in Section VI.

Furthermore, $\bar{y}_{N\varepsilon}(0)$, $\bar{\dot{y}}_{N\varepsilon}(0)$, $\bar{m}_{N\varepsilon}(0)$, $\bar{\lambda}_{N\varepsilon}(0)$, $\bar{\dot{\lambda}}_{N\varepsilon}(0)$, and $\bar{K}_{N\varepsilon}(0)$ can be expressed in terms of $y_{N\varepsilon}(0)$, $\dot{y}_{N\varepsilon}(0)$, $m_{N\varepsilon}(0)$, $\lambda_{N\varepsilon}(0)$, $\dot{\lambda}_{N\varepsilon}(0)$, and $K_{N\varepsilon}(0)$; i.e., in terms of the ε derivatives at the initial point of the N^{th} thrust arc, where the derivatives are evaluated at $\varepsilon = 0$. For example,

$$\bar{\dot{y}}_{N\varepsilon}(0) = \dot{y}_{N\varepsilon}(0) + \Delta\dot{y}_{N\varepsilon}(0)$$

where $\Delta\dot{y}_{N\varepsilon}$ is an expression to be derived later in this report. The function $\Delta\dot{y}_{N\varepsilon}$ depends upon $\Delta m_{N\varepsilon}(0)$, $m_{N\varepsilon}(0)$, and $\lambda_{N\varepsilon}(0)$ as well as upon the known quantities $\dot{y}_N(0)$, $\lambda_N(0)$, etc. The major problem resolved in this report is the determination of expressions for $\Delta y_{k\varepsilon}(0)$, $\Delta \dot{y}_{k\varepsilon}(0)$, $\Delta \dot{m}_{k\varepsilon}(0)$, $\Delta\lambda_{k\varepsilon}(0)$, $\Delta\dot{\lambda}_{k\varepsilon}(0)$, $\Delta K_{k\varepsilon}(0)$, and the changes in the second derivatives as well.

Proceeding in the manner described above, it is possible to work back to the initial point of the flight path and to express the lefthand member of equation (5-1) in terms of known quantities (obtained from a precomputed impulsive solution) as well as the unknowns $\lambda_{I\varepsilon}(0)$, $\dot{\lambda}_{I\varepsilon}(0)$, $K_{I\varepsilon}(0)$, $t_{k\varepsilon}(0)$ ($k = 1, 2, \ldots, N$), $\Delta m_{k\varepsilon}(0)$ ($k = 1, 2, \ldots, N$), and $t_{F\varepsilon}(0)$. A similar procedure can be followed for every boundary constraint on the flight path. The resulting system of linear equations can be solved for the unknowns $\lambda_{I\varepsilon}(0)$, $\dot{\lambda}_{I\varepsilon}(0)$, etc. In the cases of the example problems to be discussed later in this report, it is shown that it is rather easy to invert the matrix A of coefficients of the unknowns and thereby to obtain simple expressions for $\lambda_{I\varepsilon}(0)$, $\dot{\lambda}_{I\varepsilon}(0)$, etc., in terms of the impulsive solutions. Furthermore, the coefficient matrices for the unknown second derivatives $\lambda_{I\varepsilon\varepsilon}(0)$, $\dot{\lambda}_{I\varepsilon\varepsilon}(0)$, ... and the higher order derivatives are identical to A, although the column vectors of constants become increasingly complicated as the order of the derivatives increases. It is understood that the derivatives of lower order must be computed before the derivatives of the N^{th} order can be solved for.

At this point it is helpful to examine the relationship between Δt_k and Δm_k. When $\Delta m_k = \bar{m}_k - m_k$ is computed for the impulsive case, then $\Delta t_{k\varepsilon}(0) = \bar{t}_{k\varepsilon}(0) - t_{k\varepsilon}(0)$ is known, because

$$\bar{m}_k(\varepsilon) = m_k(\varepsilon) - \beta[\bar{t}_k(\varepsilon) - t_k(\varepsilon)]$$

$$\frac{\Delta t_k(\varepsilon)}{\varepsilon} = -\Delta m_k(\varepsilon)$$

$$\Delta t_{k\varepsilon}(0) = -\Delta m_k(0)$$

using L'Hospital's Rule. Therefore, $\bar{t}_{k\varepsilon}(0)$ can be determined from $t_{k\varepsilon}(0)$ and $\Delta m_k(0)$. Furthermore,

$$\bar{m}_k(\varepsilon) = m_k(\varepsilon) - \frac{\Delta t_k(\varepsilon)}{\varepsilon}$$

$$\bar{m}_{k\varepsilon}(\varepsilon) = m_{k\varepsilon}(\varepsilon) - \frac{\Delta t_{k\varepsilon} - \frac{\Delta t_k}{\varepsilon}}{\varepsilon} = m_{k\varepsilon}(\varepsilon) - \frac{\varepsilon \Delta t_{k\varepsilon} - \Delta t_k}{\varepsilon^2}$$

(last term indeterminant at $\varepsilon = 0$)

$$\bar{m}_{k\varepsilon}(0) = m_{k\varepsilon}(0) - \lim_{\varepsilon \to 0} \frac{\varepsilon \Delta t_{k\varepsilon\varepsilon} + \Delta t_{k\varepsilon} - \Delta t_{k\varepsilon}}{2\varepsilon} = m_{k\varepsilon}(0) - \frac{1}{2} \Delta t_{k\varepsilon\varepsilon}(0)$$

In general, for n = 0, 1, 2, ... and $\varepsilon = 0$, it may be shown that

$$\frac{d^n \bar{m}_k}{d\varepsilon^n} = \frac{d^n m_k}{d\varepsilon^n} - \frac{1}{n+1} \frac{d^{n+1} \Delta t_k}{d\varepsilon^{n+1}}$$

In obtaining the first derivatives with respect to ε, the unknowns - corresponding to the k^{th} thrust arc - will be $\Delta m_{k\varepsilon}$ and $t_{k\varepsilon}$ (or $\bar{t}_{k\varepsilon}$) rather than $t_{k\varepsilon}$ and $\bar{t}_{k\varepsilon}$. In general, in the determination of the n^{th} derivatives, $\frac{d^n \Delta m_k}{d\varepsilon^n}$ at $\varepsilon = 0$ will be an unknown.

Another complication can arise in the "ideal" approach which has been outlined. If $K(t_k, \varepsilon) = 0$ and $K(\bar{t}_k, \varepsilon) = 0$ (as in the cases of interior thrust arcs or the final thrust arc of an orbital injection problem), then $\dot{K}(t_k, 0) = 0$ (see Section VIII) and $U_k^*(0) = 0$. It will be evident that in these cases the conditions $K_{k\varepsilon}(0) = 0$ and $\bar{K}_{k\varepsilon}(0) = 0$ will lead to only one independent condition on the first derivatives. Similarly, the conditions $K_{k\varepsilon\varepsilon}(0) = 0$ and $\bar{K}_{k\varepsilon\varepsilon}(0) = 0$ will lead to only one independent condition on the second derivatives, and so on. In the latter cases, the condition $\bar{K}_{k\varepsilon}(0) = 0$ will be supplanted by $\bar{K}_{k\varepsilon\varepsilon}(0) = 0$ which will contain no unknown second derivatives. In general, in obtaining the derivatives in the aforementioned cases, the condition

$$\left. \frac{d^n \bar{K}_k}{d\varepsilon^n} \right|_{\varepsilon=0} = 0$$

will be replaced by

$$\left. \frac{d^{n+1}\bar{K}_k}{d\varepsilon^{n+1}} \right|_{\varepsilon=0} = 0$$

In summary, in calculating n^{th} derivatives with respect to ε at $\varepsilon = 0$, the unknowns are

$$\frac{d^n\lambda_I}{d\varepsilon^n}, \quad \frac{d^n\dot{\lambda}_I}{d\varepsilon^n}, \quad \frac{d^nK_I}{d\varepsilon^n}$$

$\dfrac{d^nt_k}{d\varepsilon^n}$ (where $k = 1, 2, \ldots, N$ unless $t_1 \equiv t_I$, in which case $k = 2, 3, \ldots, N$)

$\dfrac{d^n\Delta m_k}{d\varepsilon^n}$ $(k = 1, 2, \ldots, N)$

$\dfrac{d^nt_F}{d\varepsilon^n}$ (unless $\bar{t}_N \equiv t_F$, in which case this derivative can be determined, if so chosen, from $\dfrac{d^nt_N}{d\varepsilon^n}$ and $\dfrac{d^{n-1}\Delta m_k}{d\varepsilon^{n-1}}$)

The determining equations include the n^{th} derivatives of eight terminal conditions (in the case of three spacial dimensions); namely, a scaling condition on the Lagrange multipliers, the given physical constraints, and the transversality conditions. In addition, they include the conditions

$\dfrac{d^nK_k}{d\varepsilon^n} = 0$ (unless $k = 1$ and $t_I \equiv t_1$, in which case we take $\dfrac{d^n\bar{K}_1}{d\varepsilon^n} = 0$)

$\dfrac{d^{n+1}\bar{K}_k}{d\varepsilon^{n+1}} = 0$ (for interior thrust arcs and the final thrust arc of orbital injection problems)

It is not necessary that one choose exactly the same unknowns and conditions as have been indicated here.

Section VI

THE EQUATIONS OF VARIATION

This section is primarily concerned with the determination of changes in the derivatives, with respect to ε, over coast arcs. However, the section will first give a brief introductory discussion of the equations of variation which may be used to determine the derivatives over any arc for any gravitational field.

The derivatives y_ε, $\dot{y}_\varepsilon$, λ_ε, $\dot{\lambda}_\varepsilon$, K_ε, m_ε on a time interval $[t_k, \bar{t}_k]$ are the solution to the equations of variation:

$$\ddot{y}_\varepsilon = - c \frac{m + \varepsilon m_\varepsilon}{(\varepsilon m)^2} L + \frac{c}{\varepsilon m} L_\lambda \lambda_\varepsilon + G_y y_\varepsilon$$

$$\ddot{\lambda}_\varepsilon = Q_\lambda \lambda_\varepsilon + Q_y y_\varepsilon$$

$$\dot{K}_\varepsilon = - \frac{c m_\varepsilon}{m^2} U + \frac{c}{m} (U_\lambda \lambda_\varepsilon + U_{\dot{\lambda}} \dot{\lambda}_\varepsilon)$$

$$\dot{m}_\varepsilon = \frac{1}{\varepsilon^2}$$

with initial values $y^+_{\varepsilon k}$, $\dot{y}^+_{\varepsilon k}$, $\lambda^+_{\varepsilon k}$, $\dot{\lambda}^+_{\varepsilon k}$, $K^+_{\varepsilon k}$, $m^+_{\varepsilon k}$. The equations of variation may be obtained by differentiating equations (2-1) with respect to ε (ref. 13). The equations of variation for the second derivatives may be obtained by differentiating equations (2-1) twice (ref. 14) and so on for the higher derivatives.

On an interval $[\bar{t}_k, t_{k+1}]$, corresponding to a coast arc, the equations of variation simplify to

$$\ddot{y}_\varepsilon = G_y y_\varepsilon$$

$$\ddot{\lambda}_\varepsilon = Q_\lambda \lambda_\varepsilon + Q_y y_\varepsilon$$

$$\dot{K}_\varepsilon = - \frac{c m_\varepsilon}{m^2} U + \frac{c}{m} (U_\lambda \lambda_\varepsilon + U_{\dot{\lambda}} \dot{\lambda}_\varepsilon)$$

$$\dot{m}_\varepsilon = 0 \quad (\text{so, for } t_{k+1} \geq t \geq \bar{t}_k,\ m_\varepsilon(t, \varepsilon) = \bar{m}^+_{\varepsilon k}(\varepsilon) = m^+_{\varepsilon k}(\bar{t}_k, \varepsilon))$$

with initial values $\bar{y}^+_{\varepsilon k}$, $\bar{\dot{y}}^+_{\varepsilon k}$, $\bar{\lambda}^+_{\varepsilon k}$, $\bar{\dot{\lambda}}^+_{\varepsilon k}$, $\bar{K}^+_{\varepsilon k}$, $\bar{m}^+_{\varepsilon k}$ and similarly for the equations of variation for the second derivatives.

Of course the equations of motion, the Euler-Lagrange equations and the equations of variation may be integrated numerically over each coast arc of the impulsive solution in order to obtain the derivatives which correspond to the final point of the trajectory. However, in cases of an inverse square or constant gravitational field, closed form solutions to all of the latter differential equations may be found. References 15 through 18 and Appendix A of this report contain basic discussions of the closed form solutions. References 15 and 17 and Appendix A describe computer programs for obtaining first derivatives of y, $\dot{y}$, λ, $\dot{\lambda}$ at time t_{k+1} with respect to y, $\dot{y}$, λ, $\dot{\lambda}$ at time $\bar{t}_k$. Closed expressions for the second and higher order derivatives of these quantities can be found by differentiating the closed form expressions for the first derivatives.

Let the matrix of partial derivatives of y at time t_{k+1} with respect to y at time $\bar{t}_k$ be denoted by

$$\left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right]$$

and similarly for the other partial derivative matrices. As in reference 17, the chain rule of differentiation yields

$$\begin{aligned}
\bar{y}_{\varepsilon,k+1} &= \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^{+}_{\varepsilon k} + \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{y}^{+}_{\varepsilon k} \\
\dot{\bar{y}}_{\varepsilon,k+1} &= \left[\frac{\partial \dot{y}_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^{+}_{\varepsilon k} + \left[\frac{\partial \dot{y}_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}^{+}_{\varepsilon k} \\
\bar{\lambda}_{\varepsilon,k+1} &= \left[\frac{\partial \lambda_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^{+}_{\varepsilon k} + \left[\frac{\partial \lambda_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}^{+}_{\varepsilon k} + \left[\frac{\partial \lambda_{k+1}}{\partial \bar{\lambda}_k}\right] \bar{\lambda}^{+}_{\varepsilon k} + \left[\frac{\partial \lambda_{k+1}}{\partial \dot{\bar{\lambda}}_k}\right] \dot{\bar{\lambda}}^{+}_{\varepsilon k} \\
\dot{\bar{\lambda}}_{\varepsilon,k+1} &= \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^{+}_{\varepsilon k} + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}^{+}_{\varepsilon k} + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \bar{\lambda}_k}\right] \bar{\lambda}^{+}_{\varepsilon k} + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \dot{\bar{\lambda}}_k}\right] \dot{\bar{\lambda}}^{+}_{\varepsilon k}
\end{aligned} \tag{6-1}$$

The rather subtle reason for the appearance of the partial derivatives $\bar{y}^{+}_{\varepsilon k}$, $\dot{\bar{y}}^{+}_{\varepsilon k}$, etc. rather than the total derivatives $\bar{y}_{k\varepsilon}$, $\dot{\bar{y}}^{+}_{k\varepsilon}$, etc. in the righthand members of equations (6-1) is illustrated as follows:

$$
\begin{aligned}
y^-_{\varepsilon,k+1} &= \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] \bar{y}_{k\varepsilon} + \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}_{k\varepsilon} + \bar{t}_{k\varepsilon} \left[\frac{\partial y_{k+1}}{\partial \bar{t}_k}\right] \\
&= \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] \bar{y}_{k\varepsilon} + \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}_{k\varepsilon} - \bar{t}_{k\varepsilon}\left(\left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] \dot{\bar{y}}_k + \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] \ddot{\bar{y}}^+_k \right) \quad \text{(ref. 13)} \\
&= \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] \left(\bar{y}_{k\varepsilon} - \bar{t}_{k\varepsilon}\, \dot{\bar{y}}_k \right) + \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] \left(\dot{\bar{y}}_{k\varepsilon} - \bar{t}_{k\varepsilon}\, \ddot{\bar{y}}^+_k \right) \\
&= \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^+_{\varepsilon k} + \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}^+_{\varepsilon k}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
y^-_{\varepsilon\varepsilon,k+1} &= \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^+_{\varepsilon\varepsilon k} + \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}^+_{\varepsilon\varepsilon k} + (\tilde{y}_{\varepsilon\varepsilon k}) \\
\dot{y}^-_{\varepsilon\varepsilon,k+1} &= \left[\frac{\partial \dot{y}_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^+_{\varepsilon\varepsilon k} + \left[\frac{\partial \dot{y}_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}^+_{\varepsilon\varepsilon k} + (\dot{\tilde{y}}_{\varepsilon\varepsilon k}) \\
\lambda^-_{\varepsilon\varepsilon,k+1} &= \left[\frac{\partial \lambda_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^+_{\varepsilon\varepsilon k} + \left[\frac{\partial \lambda_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}^+_{\varepsilon\varepsilon k} + \left[\frac{\partial \lambda_{k+1}}{\partial \bar{\lambda}_k}\right] \bar{\lambda}^+_{\varepsilon\varepsilon k} + \left[\frac{\partial \lambda_{k+1}}{\partial \dot{\bar{\lambda}}_k}\right] \dot{\bar{\lambda}}^+_{\varepsilon\varepsilon k} + (\tilde{\lambda}_{\varepsilon\varepsilon k}) \\
\dot{\lambda}^-_{\varepsilon\varepsilon,k+1} &= \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \bar{y}_k}\right] \bar{y}^+_{\varepsilon\varepsilon k} + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \dot{\bar{y}}_k}\right] \dot{\bar{y}}^+_{\varepsilon\varepsilon k} + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \bar{\lambda}_k}\right] \bar{\lambda}^+_{\varepsilon\varepsilon k} + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \dot{\bar{\lambda}}_k}\right] \dot{\bar{\lambda}}^+_{\varepsilon\varepsilon k} + (\dot{\tilde{\lambda}}_{\varepsilon\varepsilon k})
\end{aligned}
$$

where the symbol $(\tilde{y}_{\varepsilon\varepsilon k})$ stands for the matrix summation

$$
\begin{aligned}
(\tilde{y}_{\varepsilon\varepsilon k}) = &\left\{ \sum_i \left(\bar{y}^{(i)+}_{\varepsilon k} \frac{\partial}{\partial \bar{y}^{(i)}_k} \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] + \dot{\bar{y}}^{(i)+}_{\varepsilon k} \frac{\partial}{\partial \dot{\bar{y}}^{(i)}_k} \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right] \right) \right\} \bar{y}^+_{\varepsilon k} \\
&+ \left\{ \sum_i \left(\bar{y}^{(i)+}_{\varepsilon k} \frac{\partial}{\partial \bar{y}^{(i)}_k} \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] + \dot{\bar{y}}^{(i)+}_{\varepsilon k} \frac{\partial}{\partial \dot{\bar{y}}^{(i)}} \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right] \right) \right\} \dot{\bar{y}}^+_{\varepsilon k}
\end{aligned}
$$

where the i superscript signifies the i^{th} component of the vector. Similar meanings hold for $(\dot{\tilde{y}}_{\varepsilon\varepsilon k})$, $(\tilde{\lambda}_{\varepsilon\varepsilon k})$, and $(\dot{\tilde{\lambda}}_{\varepsilon\varepsilon k})$. The latter quantities contain no second derivatives with respect to ε.

The derivatives of K can be derived from the equation

$$K_{k+1} = \bar{K}_k + \frac{c}{\bar{m}_k} (|\lambda_{k+1}| - |\bar{\lambda}_k|)$$

whose righthand member is the solution to

$$\dot{K} = \frac{c}{m} \frac{\lambda^T \dot{\lambda}}{|\lambda|}$$

evaluated at $t = t_{k+1}$. Then the first derivative is

$$K^-_{\varepsilon,k+1} = \bar{K}^+_{\varepsilon k} + \frac{c}{\bar{m}_k |\bar{\lambda}_k|} (\lambda^T_{k+1} \lambda^-_{\varepsilon,k+1} - \bar{\lambda}^T_k \bar{\lambda}^+_{\varepsilon k}) \tag{6-2}$$

since $K_{k+1} = \bar{K}_k = 0$ and, consequently, $|\lambda_{k+1}| = |\bar{\lambda}_k|$. Similarly,

$$\begin{aligned} K^-_{\varepsilon\varepsilon,k+1} = \bar{K}^+_{\varepsilon\varepsilon k} &+ \frac{c}{\bar{m}_k |\bar{\lambda}_k|} (\lambda^T_{k+1} \bar{\lambda}_{\varepsilon\varepsilon,k+1} - \bar{\lambda}^T_k \bar{\lambda}^+_{\varepsilon\varepsilon k}) \\ &+ \frac{c}{\bar{m}_k |\bar{\lambda}_k|} (\lambda^{-T}_{\varepsilon,k+1} \lambda^-_{\varepsilon,k+1} - \bar{\lambda}^{+T}_{\varepsilon k} \bar{\lambda}^+_{\varepsilon k}) \\ &- \frac{c}{\bar{m}_k |\bar{\lambda}_k|^3} (\lambda^T_{k+1} \lambda^-_{\varepsilon,k+1} - \bar{\lambda}^T_k \bar{\lambda}^+_{\varepsilon k}) (\bar{\lambda}^T_k \bar{\lambda}^+_{\varepsilon k}) \\ &- \frac{2c\bar{m}_{k\varepsilon}}{\bar{m}_k^2 |\bar{\lambda}_k|} (\lambda^T_{k+1} \lambda^-_{\varepsilon,k+1} - \bar{\lambda}^T_k \bar{\lambda}^+_{\varepsilon k}) \end{aligned}$$

Appendix A contains a further discussion of the partial derivative matrices, presents a new method for determining these matrices in an inverse square gravitational field, and provides a listing of a computer program for obtaining the matrices of first order derivatives.

In the expressions so far derived in this section, computation is in terms of partial derivatives $y^-_{\varepsilon k}$, $\bar{y}^+_{\varepsilon k}$, $\lambda^-_{\varepsilon\varepsilon k}$, etc. However, in the determination of changes in the ε derivatives over thrust arcs, we will employ the total derivatives $y_{k\varepsilon}$, $\bar{y}_{k\varepsilon}$, $\lambda_{k\varepsilon\varepsilon}$, etc. In order to obtain the relationships between the partial and the total derivatives, consider a composite function

$$B_k(\varepsilon) = B[t_k(\varepsilon), \varepsilon]$$

where B(t, ε) is a function of t and ε. Then, for example, we have

$$B_{k\varepsilon} \triangleq \frac{dB_k}{d\varepsilon} = \frac{\partial B}{\partial \varepsilon}\bigg|_{t_k^-} + \frac{dt_k}{d\varepsilon}\,\frac{\partial B}{\partial t}\bigg|_{t_k^-}$$

$$B_{k\varepsilon} = B^-_{\varepsilon k} + t_{k\varepsilon} B^-_{tk} \tag{6-3}$$

Therefore

$$B^-_{\varepsilon k} = B_{k\varepsilon} - t_{k\varepsilon} B^-_{tk} \tag{6-4}$$

and, similarly,

$$\bar{B}^+_{\varepsilon k} = \bar{B}_{k\varepsilon} - \bar{t}_{k\varepsilon} \bar{B}^+_{tk} \tag{6-5}$$

It follows from equation (6-3) that

$$\begin{aligned} B_{k\varepsilon\varepsilon} &= B^-_{\varepsilon k\varepsilon} + t_{k\varepsilon} B^-_{tk\varepsilon} + t_{k\varepsilon\varepsilon} B^-_{tk} \\ &= B^-_{\varepsilon\varepsilon k} + t_{k\varepsilon} B^-_{t\varepsilon k} + t_{k\varepsilon} B^-_{tk\varepsilon} + t_{k\varepsilon\varepsilon} B^-_{tk} \\ &= B^-_{\varepsilon\varepsilon k} + t_{k\varepsilon}(B^-_{tk\varepsilon} - t_{k\varepsilon} B^-_{ttk}) + t_{k\varepsilon} B^-_{tk\varepsilon} + t_{k\varepsilon\varepsilon} B^-_{tk} \end{aligned}$$

Therefore,

$$B^-_{\varepsilon\varepsilon k} = B_{k\varepsilon\varepsilon} - 2t_{k\varepsilon} B^-_{tk\varepsilon} + t^2_{k\varepsilon} B^-_{ttk} - t_{k\varepsilon\varepsilon} B^-_{tk} \tag{6-6}$$

and, similarly,

$$\bar{B}^+_{\varepsilon\varepsilon k} = \bar{B}_{k\varepsilon\varepsilon} - 2\bar{t}_{k\varepsilon} \bar{B}^+_{tk\varepsilon} + \bar{t}^2_{k\varepsilon} \bar{B}^+_{ttk} - \bar{t}_{k\varepsilon\varepsilon} \bar{B}^+_{tk} \tag{6-7}$$

Employing equations (6-4) and (6-5), it is easy to express equations (6-1) and (6-2) in terms of total derivatives. Thus,

$$y_{k+1,\varepsilon} = t_{k+1,\varepsilon}\dot{y}_{k+1} + \left[\frac{\partial y_{k+1}}{\partial \bar{y}_k}\right](\bar{y}_{k\varepsilon} - \bar{t}_{k\varepsilon}\dot{\bar{y}}_k) + \left[\frac{\partial y_{k+1}}{\partial \dot{\bar{y}}_k}\right](\dot{\bar{y}}_{k\varepsilon} - \bar{t}_{k\varepsilon}\bar{G}_k^*)$$

$$\dot{\bar{y}}_{k+1,\varepsilon} = t_{k+1,\varepsilon}G_{k+1}^* + \left[\frac{\partial \dot{y}_{k+1}}{\partial \bar{y}_k}\right](\bar{y}_{k\varepsilon} - \bar{t}_{k\varepsilon}\dot{\bar{y}}_k) + \left[\frac{\partial \dot{y}_{k+1}}{\partial \dot{\bar{y}}_k}\right](\dot{\bar{y}}_{k\varepsilon} - \bar{t}_{k\varepsilon}\bar{G}_k^*)$$

$$\lambda_{k+1,\varepsilon} = t_{k+1,\varepsilon}\dot{\lambda}_{k+1} + \left[\frac{\partial \lambda_{k+1}}{\partial \bar{y}_k}\right](\bar{y}_{k\varepsilon} - \bar{t}_{k\varepsilon}\dot{\bar{y}}_k) + \left[\frac{\partial \lambda_{k+1}}{\partial \dot{\bar{y}}_k}\right](\dot{\bar{y}}_{k\varepsilon} - \bar{t}_{k\varepsilon}\bar{G}_k^*)$$
$$+ \left[\frac{\partial \lambda_{k+1}}{\partial \bar{\lambda}_k}\right](\bar{\lambda}_{k\varepsilon} - \bar{t}_{k\varepsilon}\dot{\bar{\lambda}}_k) + \left[\frac{\partial \lambda_{k+1}}{\partial \dot{\bar{\lambda}}_k}\right](\dot{\bar{\lambda}}_{k\varepsilon} - \bar{t}_{k\varepsilon}\bar{Q}_k^*) \tag{6-8}$$

$$\dot{\lambda}_{k+1,\varepsilon} = t_{k+1,\varepsilon}Q_{k+1}^* + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \bar{y}_k}\right](\bar{y}_{k\varepsilon} - \bar{t}_{k\varepsilon}\dot{\bar{y}}_k) + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \dot{\bar{y}}_k}\right](\dot{\bar{y}}_k - \bar{t}_{k\varepsilon}\bar{G}_k^*)$$
$$+ \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \bar{\lambda}_k}\right](\bar{\lambda}_{k\varepsilon} - \bar{t}_{k\varepsilon}\dot{\bar{\lambda}}_k) + \left[\frac{\partial \dot{\lambda}_{k+1}}{\partial \dot{\bar{\lambda}}_k}\right](\dot{\bar{\lambda}}_{k\varepsilon} - \bar{t}_{k\varepsilon}\bar{Q}_k^*)$$

$$K_{k+1,\varepsilon} = \bar{K}_{k\varepsilon} + \frac{c}{\bar{m}_k}(t_{k+1,\varepsilon}U_{k+1}^* - \bar{t}_{k\varepsilon}\bar{U}_k^*)$$
$$+ \frac{c}{\bar{m}_k|\bar{\lambda}_k|}[\lambda_{k+1}^T(\lambda_{k+1,\varepsilon} - t_{k+1,\varepsilon}\dot{\lambda}_{k+1}) - \bar{\lambda}_k^T(\bar{\lambda}_{k\varepsilon} - \bar{t}_{k\varepsilon}\dot{\bar{\lambda}}_k)]$$

Likewise, equations (6-6) and (6-7) could be used to express all changes in the second derivatives (over coast arcs) in terms of total derivatives.

Section VII

INTEGRALS WHICH APPROACH ZERO

In Sections VIII and IX we will determine changes over thrust arcs in the derivatives of y, $\dot{y}$, λ, $\dot{\lambda}$, etc. with respect to ε. For example, we will write

$$\dot{\bar{\lambda}}_k = \dot{\lambda}_k + \int_{t_k}^{\bar{t}_k} Q^* dt.$$

Integration by parts will give

$$\dot{\bar{\lambda}}_k = \dot{\lambda}_k - \varepsilon [mQ^*]_{t_k}^{\bar{t}_k} + \varepsilon \int_{t_k}^{\bar{t}_k} mQ_t^* dt$$

Then differentiation yields

$$\dot{\bar{\lambda}}_{k\varepsilon} = \dot{\lambda}_{k\varepsilon} - [mQ^*]_{t_k}^{\bar{t}_k} - \varepsilon \frac{d}{d\varepsilon} [mQ^*]_{t_k}^{\bar{t}_k} + \int_{t_k}^{\bar{t}_k} mQ_t^* dt + \varepsilon \frac{d}{d\varepsilon} \int_{t_k}^{\bar{t}_k} mQ_t^* dt.$$

In preparing for Section VIII, this section will show that the latter two integral terms and similar expressions approach zero as ε approaches zero.

Consider a function $f(\lambda, \dot{\lambda}, y, \dot{y}, m)$ which has continuous partial derivatives with respect to its arguments and such that, for $\varepsilon \geq 0$, the function

$$f^*(t, \varepsilon) = f[\lambda(t, \varepsilon), \dot{\lambda}(t, \varepsilon), y(t, \varepsilon), \dot{y}(t, \varepsilon), m(t, \varepsilon)]$$

is continuous and bounded in the interval $[t_k(\varepsilon), \bar{t}_k(\varepsilon)]$ within some neighborhood of $\varepsilon = 0$. Consider the integral

$$I(\varepsilon) = \int_{t_k(\varepsilon)}^{\bar{t}_k(\varepsilon)} f^*(t, \varepsilon) dt$$

By a mean value theorem for integrals,

$$I(\varepsilon) = f^*[\tilde{t}(\varepsilon), \varepsilon] \int_{t_k(\varepsilon)}^{\bar{t}_k(\varepsilon)} dt = \Delta t_k(\varepsilon)\; f^*[\tilde{t}(\varepsilon), \varepsilon] \qquad (t_k \leq \tilde{t} \leq \bar{t}_k)$$

Since f^* is bounded, we write $I(\varepsilon) = \sigma(\varepsilon)$. (See Section IV.)

Let

$$I(\varepsilon) = \int_{t_k(\varepsilon)}^{\bar{t}_k(\varepsilon)} \int_{t_k(\varepsilon)}^{\tau} f^*(t, \varepsilon)dt\, d\tau$$

The mean value theorem for double integrals implies that

$$I(\varepsilon) = f^*(\tilde{t}, \varepsilon) \int_{t_k}^{\bar{t}_k} \int_{t_k}^{\tau} dt\, d\tau \qquad (t_k \leq \tilde{t} \leq \bar{t}_k)$$

$$= \frac{1}{2} \Delta t_k^{\ 2} f^*(\tilde{t}, \varepsilon)$$

$$= \sigma(\varepsilon^2)$$

Next the derivatives of $I(\varepsilon)$ will be examined. From the rule for differentiation under an integral sign, it follows that

$$\frac{d}{d\varepsilon} I(\varepsilon) = \bar{t}_{k\varepsilon}\bar{f}^*_k - t_{k\varepsilon}f^*_k + \int_{t_k}^{\bar{t}_k} f^*_\varepsilon\, dt$$

$$= \bar{t}_{k\varepsilon}\bar{f}^*_k - t_{k\varepsilon}f^*_k + \int_{t_k}^{\bar{t}_k} (f^*_\lambda{}^T\lambda_\varepsilon + f^*_{\dot{\lambda}}{}^T\dot{\lambda}_\varepsilon + f^*_y{}^T y_\varepsilon$$

$$+ f^*_{\dot{y}}{}^T\dot{y}_\varepsilon + f^*_m m_\varepsilon)\, dt$$

where f^*_λ for example is a vector whose components are assumed to be bounded in some neighborhood of $\varepsilon = 0$. The mean value theorem for integrals yields

$$\frac{d}{d\varepsilon} I(\varepsilon) = \bar{t}_{k\varepsilon}\bar{f}^*_k - t_{k\varepsilon}f^*_k + (f^*_\lambda{}^T\lambda_\varepsilon + f^*_{\dot{\lambda}}{}^T\dot{\lambda}_\varepsilon + f^*_y{}^T y_\varepsilon$$

$$+ f^*_{\dot{y}}{}^T\dot{y}_\varepsilon + f^*_m m_\varepsilon)\Big|_{t=\tilde{t}(\varepsilon)} \int_{t_k}^{\bar{t}_k} dt \qquad (t_k \leq \tilde{t} \leq \bar{t}_k)$$

$$= \bar{t}_{k\varepsilon}f^*_k - t_{k\varepsilon}f^*_k + [f^*_\lambda{}^T(\varepsilon\lambda_\varepsilon) + f^*_{\dot{\lambda}}{}^T(\varepsilon\dot{\lambda}_\varepsilon) + f^*_y{}^T(\varepsilon y_\varepsilon)$$

$$+ f^*_{\dot{y}}{}^T(\varepsilon\dot{y}_\varepsilon) + f^*_m(\varepsilon m_\varepsilon)]_{t=\tilde{t}} \frac{\Delta t_k}{\varepsilon}$$

At this point it must be argued that the functions $\varepsilon\lambda_\varepsilon[\tilde{t}(\varepsilon), \varepsilon]$, $\varepsilon\dot{\lambda}_\varepsilon[t(\varepsilon), \varepsilon]$, $\varepsilon y_\varepsilon[\tilde{t}(\varepsilon), \varepsilon]$, $\varepsilon\dot{y}_\varepsilon[\tilde{t}(\varepsilon), \varepsilon]$, $\varepsilon m_\varepsilon[t(\varepsilon), \varepsilon]$ are bounded for $\varepsilon > 0$ within some neighborhood of $\varepsilon = 0$. Let us examine $\varepsilon m_\varepsilon$ for example. The equation of variation for m_ε on a thrust arc is

$$\dot{m}_\varepsilon = \frac{1}{\varepsilon^2}$$

Therefore

$$m_\varepsilon[\tilde{t}(\varepsilon), \varepsilon] = m_\varepsilon[t_k(\varepsilon), \varepsilon] + \int_{t_k(\varepsilon)}^{\tilde{t}(\varepsilon)} \frac{1}{\varepsilon^2}\, dt$$

$$\varepsilon m_\varepsilon[\tilde{t}(\varepsilon), \varepsilon] = \varepsilon m_\varepsilon[t_k(\varepsilon), \varepsilon] + \frac{\tilde{t}(\varepsilon) - t_k(\varepsilon)}{\varepsilon}$$

But

$$\frac{\tilde{t}(\varepsilon) - t_k(\varepsilon)}{\varepsilon} \leq \frac{\bar{t}_k(\varepsilon) - t_k(\varepsilon)}{\varepsilon} = \frac{\Delta t_k(\varepsilon)}{\varepsilon}$$

where

$$\lim_{\varepsilon \to 0} \frac{\Delta t_k(\varepsilon)}{\varepsilon} = \Delta t_{k\varepsilon}(0) \ .$$

Therefore, $(\tilde{t} - t_k)/\varepsilon$ is bounded within some neighborhood of $\varepsilon = 0$. Also $\varepsilon m_\varepsilon(t_k, \varepsilon)$ is bounded provided $\varepsilon m_\varepsilon(\bar{t}_{k-1}, \varepsilon)$ is bounded since $\varepsilon m_\varepsilon$ is constant over a coast arc. Eventually the question of boundedness depends on whether or not $\varepsilon m_\varepsilon(t_I, \varepsilon)$ is bounded; but $\varepsilon m_\varepsilon(t_I, \varepsilon)$ is zero.

As another example consider $\dot{y}_\varepsilon$ on a thrust arc. The equation of variation (Section VI) is

$$\ddot{y}_\varepsilon = - c\, \frac{m + \varepsilon m_\varepsilon}{(\varepsilon m)^2} L + \frac{c}{\varepsilon m} L_\lambda\, \lambda_\varepsilon + G_y y_\varepsilon$$

Therefore,

$$\varepsilon\dot{y}_\varepsilon[\tilde{t}(\varepsilon), \varepsilon] = \varepsilon\dot{y}_\varepsilon[t_k(\varepsilon), \varepsilon] - \frac{c}{\varepsilon} \int_{t_k(\varepsilon)}^{\tilde{t}(\varepsilon)} \frac{1}{m} L\, dt + \text{(higher order terms)}$$

The integral

$$J = \frac{1}{\varepsilon} \int_{t_k(\varepsilon)}^{\tilde{t}(\varepsilon)} \frac{1}{m} L\, dt$$

requires the most critical study in regard to boundedness. By a mean value theorem for integrals, there is a t' in the interval $[t_k, \tilde{t}]$ such that

$$J = \frac{1}{\varepsilon} L^*(t', \varepsilon) \int_{t_k(\varepsilon)}^{\tilde{t}(\varepsilon)} \frac{1}{m}\, dt = L^*(t', \varepsilon) \log \frac{m(t_k, \varepsilon)}{m(\tilde{t}, \varepsilon)}$$

Hence

$$|J| \leq |L^*(t', \varepsilon)| \log \frac{m(t_k, \varepsilon)}{m(\bar{t}_k, \varepsilon)}$$

since $m(\bar{t}_k, \varepsilon) \leq m(\tilde{t}, \varepsilon)$. Therefore, assuming $m(t, \varepsilon)$ has a lower bound which is greater than zero, one may conclude that J is bounded for all $\varepsilon \geq 0$ within some neighborhood of $\varepsilon = 0$.

Similarly it can be argued that

$$\varepsilon^n \frac{\partial^n m}{\partial \varepsilon^n}, \quad \varepsilon^n \frac{\partial^n y}{\partial \varepsilon^n}, \quad \text{etc.} \qquad (n = 1, 2, \ldots)$$

are bounded within some neighborhood of $\varepsilon = 0$. It would be impractical to provide a full proof of the proposition in this report. The general idea, however, is simple: ε factors in the denominators of the terms in the equations of variations are removed by multiplying both members of each of the equations by ε factors, thereby allowing the solutions to be finite.

Having taken care of the problem of boundedness, one can finally conclude that

$$\varepsilon \frac{d}{d\varepsilon} I(\varepsilon) = \sigma(\varepsilon)$$

Using similar arguments, one can show for $n = 0, 1, 2, \ldots,$ that

$$\varepsilon^n \frac{d^n}{d\varepsilon^n} I(\varepsilon) = \sigma(\varepsilon)$$

$$\varepsilon^{n-1} \frac{d^n}{d\varepsilon^n} I(\varepsilon) = \sigma(\varepsilon) \qquad (7\text{-}1)$$

Now it is possible to define a symbol $\Theta(\varepsilon^m)$, $m \geq 1$, denoting a function $\tilde{\sigma}(\varepsilon^m)$ as defined in Section IV, or

$$\tilde{\sigma}(\varepsilon^{m-1})\ \varepsilon^n \frac{d^n}{d\varepsilon^n} I(\varepsilon) \qquad (n = 0, 1, 2, \ldots)$$

or

$$\tilde{\sigma}(\varepsilon^{m-1})\ \varepsilon^{n-1} \frac{d^n}{d\varepsilon^n} I(\varepsilon) \qquad (n = 0, 1, 2, \ldots)$$

or a finite sum of such functions. Employing equations (7-1), it can be shown that $\Theta(\varepsilon^m)$ has the properties, for $m > p > 0$, that

$$\frac{d^p}{d\varepsilon^p} \Theta(\varepsilon^m) = \Theta(\varepsilon^{m-p})$$

and

$$\Theta(\varepsilon) = \sigma(\varepsilon) \quad .$$

Suppose for example that $\Theta(\varepsilon^3) = \tilde{\sigma}(\varepsilon^2)\varepsilon \frac{d}{d\varepsilon} I(\varepsilon)$. Since $\frac{d}{d\varepsilon} \tilde{\sigma}(\varepsilon^n) = \tilde{\sigma}(\varepsilon^{n-1})$, we have

$$\frac{d}{d\varepsilon} \Theta(\varepsilon^3) = \tilde{\sigma}(\varepsilon)\varepsilon \frac{d}{d\varepsilon} I(\varepsilon) + \tilde{\sigma}(\varepsilon^2) \frac{d}{d\varepsilon} I(\varepsilon) + \tilde{\sigma}(\varepsilon^2)\varepsilon \frac{d^2}{d\varepsilon^2} I(\varepsilon)$$

$$= \tilde{\sigma}(\varepsilon)\varepsilon \frac{d}{d\varepsilon} I(\varepsilon) + \tilde{\sigma}(\varepsilon)\varepsilon \frac{d}{d\varepsilon} I(\varepsilon) + \tilde{\sigma}(\varepsilon)\varepsilon^2 \frac{d^2}{d\varepsilon^2} I(\varepsilon)$$

$$= \Theta(\varepsilon^2)$$

If $\Theta(\varepsilon^2) = \tilde{\sigma}(\varepsilon)\varepsilon \frac{d}{d\varepsilon} I(\varepsilon)$, then

$$\frac{d}{d\varepsilon} \Theta(\varepsilon^2) = \tilde{\sigma}(\varepsilon^0)\varepsilon \frac{d}{d\varepsilon} I(\varepsilon) + \tilde{\sigma}(\varepsilon) \frac{d}{d\varepsilon} I(\varepsilon) + \tilde{\sigma}(\varepsilon)\varepsilon \frac{d^2}{d\varepsilon^2} I(\varepsilon)$$

$$= \tilde{\sigma}(\varepsilon^0)\varepsilon \frac{d}{d\varepsilon} I(\varepsilon) + \tilde{\sigma}(\varepsilon^0)\varepsilon \frac{d}{d\varepsilon} I(\varepsilon) + \tilde{\sigma}(\varepsilon^0)\varepsilon^2 \frac{d^2}{d\varepsilon^2} I(\varepsilon)$$

$$= \sigma(\varepsilon) \quad \text{(by equations (7-1))}$$

The reader might wonder why the symbol $\sigma(\varepsilon^m)$ would not suffice without the need for $\Theta(\varepsilon^m)$. The explanation is that one can not state that

$$\frac{d}{d\varepsilon}\,\sigma(\varepsilon^m) = \sigma(\varepsilon^{m-1})$$

with assurance because $\sigma(\varepsilon)$ has a factor $a(\varepsilon)$ which is bounded within some neighborhood of $\varepsilon = 0$, but whose derivatives may not even exist. Sometimes little is known about $a(\varepsilon)$ because it may correspond to a function evaluated at some indeterminate time $t(\varepsilon)$ within the interval $[t_k(\varepsilon), \bar{t}_k(\varepsilon)]$.

Section VIII

CHANGES OVER A THRUST ARC – NON-IMPULSIVE CASE

In this section we will derive expressions of the form

$$\bar{y}_k(\varepsilon) = y_k(\varepsilon) + a_k(\varepsilon) + \theta(\varepsilon^3)$$

$$\dot{\bar{y}}_k(\varepsilon) = \dot{y}_k(\varepsilon) + b_k(\varepsilon) + \theta(\varepsilon^3)$$

etc. For example, the change in $\dot{y}$ over a thrust arc will be expressed as a derived function $b_k(\varepsilon)$ plus a third order term $\theta(\varepsilon^3)$. Later, the first and second derivatives with respect to ε will be taken in order that the jumps in these derivatives over a thrust arc may be determined; for example,

$$\dot{\bar{y}}_{k\varepsilon\varepsilon}(\varepsilon) = \dot{y}_{k\varepsilon\varepsilon}(\varepsilon) + b_{k\varepsilon\varepsilon}(\varepsilon) + \theta(\varepsilon)$$

where

$$\lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$$

The expressions for the higher derivatives, though complicated, may be obtained by similar procedures.

The first step is to find expressions for the integrals

$$\int_{t_k}^{\bar{t}_k} G^* \, dt, \quad \int_{t_k}^{\bar{t}_k}\int_{t_k}^{\tau} G^* \, dt \, d\tau$$

Since $\dot{m} = -\frac{1}{\varepsilon}$, repeated integration by parts yields:

$$\int_{t_k}^{\bar{t}_k} G^* \, dt = -\varepsilon[mG^*]_{t_k}^{\bar{t}_k} + \varepsilon \int_{t_k}^{\bar{t}_k} mG^*_t \, dt \qquad (\textstyle\int u dv = uv - \int v du, \text{ where } u = G^* \text{ and } v = -\varepsilon m)$$

$$= -\varepsilon[mG^*]_{t_k}^{\bar{t}_k} - \frac{1}{2}\varepsilon^2[m^2G^*_t]_{t_k}^{\bar{t}_k} + \frac{1}{2}\varepsilon^2 \int_{t_k}^{\bar{t}_k} m^2G^*_{tt} \, dt$$

$$(u = G^*_t, \; v = -\frac{1}{2}\varepsilon m^2)$$

But

$$G^*_t = (G_t)^* + G^*_y \dot{y}$$

$$G^*_{tt} = G^*_y \ddot{y} + f^*(t, \varepsilon)$$

$$= \frac{c}{\varepsilon m} G^*_y L^* + f^{*\prime}$$

where f^* and $f^{*\prime}$ are continuous and bounded in the interval $[t_k, \bar{t}_k]$ within some neighborhood of $\varepsilon = 0$. Therefore,

$$\frac{1}{2} \varepsilon^2 \int_{t_k}^{\bar{t}_k} m^2 G^*_{tt}\, dt = \frac{1}{2} c\varepsilon \int_{t_k}^{\bar{t}_k} mG^*_y L^*\, dt + \Theta(\varepsilon^3) \qquad \left(\begin{matrix}\Theta \text{ is defined in} \\ \text{Section VII}\end{matrix}\right)$$

Integration by parts of the integral on the right yields the desired expression

$$\int_{t_k}^{\bar{t}_k} G^*\, dt = -\varepsilon[mG^*]_{t_k}^{\bar{t}_k} - \frac{1}{2} \varepsilon^2 [m^2 G^*_t]_{t_k}^{\bar{t}_k} - \frac{1}{4} c\varepsilon^2 [m^2 G^*_y L^*]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^3)$$

The expansion of the double integral is

$$\int_{t_k}^{\bar{t}_k} \int_{t_k}^{\tau} G^*\, dt\, d\tau = \int_{t_k}^{\bar{t}_k} \left(-\varepsilon[mG^*]_{t_k}^{\tau} + \varepsilon \int_{t_k}^{\tau} mG^*_t\, dt \right) d\tau$$

$$= -\varepsilon \int_{t_k}^{\bar{t}_k} mG^*\, dt + \varepsilon \int_{t_k}^{\bar{t}_k} m_k\, G^*_k\, dt + \Theta(\varepsilon^3)$$

$$= \frac{1}{2} \varepsilon^2 [m^2 G^*]_{t_k}^{\bar{t}_k} + \varepsilon \Delta t_k m_k G^*_k + \Theta(\varepsilon^3)$$

Now we may examine the changes in $\dot{y}$ and y over a thrust arc:

$$\ddot{y} = \frac{c}{\varepsilon m} L + G$$

$$\dot{\bar{y}}_k = \dot{y}_k + \frac{c}{\varepsilon} \int_{t_k}^{\bar{t}_k} \frac{1}{m} L^*\, dt + \int_{t_k}^{\bar{t}_k} G^*\, dt \qquad \left(\text{Let } u = L^*,\ v = -\, c \log \frac{m}{\bar{m}_k}\right)$$

$$\dot{\bar{y}}_k = \dot{y}_k - c \left[\left(\log \frac{m}{\bar{m}_k} \right) L^* \right]_{t_k}^{\bar{t}_k} + c \int_{t_k}^{\bar{t}_k} \left(\log \frac{m}{\bar{m}_k} \right) L^*_t\, dt + \int_{t_k}^{\bar{t}_k} G^*\, dt$$

$$\dot{\bar{y}}_k = \dot{y}_k + c\left(\log\frac{m_k}{\bar{m}_k}\right)L^*_k - c\varepsilon\left[m\left(\log\frac{m}{\bar{m}_k} - 1\right)L^*_t\right]_{t_k}^{\bar{t}_k}$$

$$+ c\varepsilon\int_{t_k}^{\bar{t}_k} m\left(\log\frac{m}{\bar{m}_k} - 1\right)L^*_{tt}\,dt + \int_{t_k}^{\bar{t}_k} G^*\,dt$$

$$\dot{\bar{y}}_k = \dot{y}_k + c\left(\log\frac{m_k}{\bar{m}_k}\right)L^*_k + c\varepsilon\, m_k\left(\log\frac{m_k}{\bar{m}_k}\right)L^*_{tk} + c\varepsilon[mL^*_t]_{t_k}^{\bar{t}_k}$$

$$- c\varepsilon^2\left[\frac{m^2}{2}\left(\log\frac{m}{\bar{m}_k} - \frac{3}{2}\right)L^*_{tt}\right]_{t_k}^{\bar{t}_k} + \int_{t_k}^{\bar{t}_k} G^*\,dt + \Theta(\varepsilon^3)$$

$$\dot{\bar{y}}_k = \dot{y}_k + c\left(\log\frac{m_k}{\bar{m}_k}\right)L^*_k + c\varepsilon\, m_k\left(\log\frac{m_k}{\bar{m}_k}\right)L^*_{tk} + c\varepsilon[mL^*_t]_{t_k}^{\bar{t}_k}$$

$$+ \frac{1}{2}c\varepsilon^2 m_k^2\left(\log\frac{m_k}{\bar{m}_k}\right)L^*_{ttk} + \frac{3}{4}c\varepsilon^2[m^2L^*_{tt}]_{t_k}^{\bar{t}_k}$$

$$+ \int_{t_k}^{\bar{t}_k} G^*\,dt + \Theta(\varepsilon^3)$$

$$\bar{y}_k = y_k + \Delta t_k\dot{y}_k + \frac{c}{\varepsilon}\int_{t_k}^{\bar{t}_k}\int_{t_k}^{\tau}\frac{L^*}{m}\,dt\,d\tau + \int_{t_k}^{\bar{t}_k}\int_{t_k}^{\tau} G^*\,dt\,d\tau$$

$$\bar{y}_k = y_k + \Delta t_k\dot{y}_k + c\int_{t_k}^{\bar{t}_k}\left\{-\left[\left(\log\frac{m}{\bar{m}_k}\right)L^*\right]_{t_k}^{\tau} + \int_{t_k}^{\tau}\left(\log\frac{m}{\bar{m}_k}\right)L^*_t\,dt\right\}d\tau$$

$$+ \iint G^*$$

$$\bar{y}_k = y_k + \Delta t_k \dot{y}_k - c \int_{t_k}^{\bar{t}_k} \left(\log \frac{m}{\bar{m}_k}\right) L^* \, dt + c \, \Delta t_k \left(\log \frac{m_k}{\bar{m}_k}\right) L^*_k$$

$$+ c \int_{t_k}^{\bar{t}_k} \int_{t_k}^{\tau} \left(\log \frac{m}{\bar{m}_k}\right) L^*_t \, dt \, d\tau + \iint G^*$$

$$\bar{y}_k = y_k + \Delta t_k \dot{y}_k + c\varepsilon \left[m \left(\log \frac{m}{\bar{m}_k} - 1\right) L^* \right]_{t_k}^{\bar{t}_k} - c\varepsilon \int_{t_k}^{\bar{t}_k} m \left(\log \frac{m}{\bar{m}_k} - 1\right) L^*_t \, dt$$

$$+ c \, \Delta t_k \left(\log \frac{m_k}{\bar{m}_k}\right) L^*_k - c\varepsilon \int_{t_k}^{\bar{t}_k} \left[m \left(\log \frac{m}{\bar{m}_k} - 1\right) L^*_t \right]_{t_k}^{t} dt + \iint G^* + \Theta(\varepsilon^3)$$

$$\bar{y}_k = y_k + \Delta t_k \dot{y}_k - c\varepsilon \, m_k \left(\log \frac{m_k}{\bar{m}_k}\right) L^*_k - c\varepsilon [mL^*]_{t_k}^{\bar{t}_k}$$

$$- 2c\varepsilon \int_{t_k}^{\bar{t}_k} m \left(\log \frac{m}{\bar{m}_k} - 1\right) L^*_t \, dt + c \, \Delta t_k \left(\log \frac{m_k}{\bar{m}_k}\right) L^*_k$$

$$+ c\varepsilon \Delta t_k \, m_k \left(\log \frac{m_k}{\bar{m}_k} - 1\right) L^*_{tk} + \iint G^* + \Theta(\varepsilon^3)$$

$$\bar{y}_k = y_k + \Delta t_k \dot{y}_k - c\varepsilon \, m_k \left(\log \frac{m_k}{\bar{m}_k}\right) L^*_k - c\varepsilon [mL^*]_{t_k}^{\bar{t}_k}$$

$$+ 2c\varepsilon^2 \left[\frac{m^2}{2} \left(\log \frac{m}{\bar{m}_k} - \frac{3}{2}\right) L^*_t \right]_{t_k}^{\bar{t}_k} + c \, \Delta t_k \left(\log \frac{m_k}{\bar{m}_k}\right) L_k^*$$

$$+ c\varepsilon \, \Delta t_k \, m_k \left(\log \frac{m_k}{\bar{m}_k} - 1\right) L^*_{tk} + \iint G^* + \Theta(\varepsilon^3)$$

$$\bar{y}_k = y_k + \Delta t_k \mathring{y}_k + c(\Delta t_k - \varepsilon m_k)\left(\log \frac{m_k}{\bar{m}_k}\right) L^*_k - c\varepsilon [mL^*]_{t_k}^{\bar{t}_k}$$

$$+ c\varepsilon\, m_k(\Delta t_k - \varepsilon m_k)\left(\log \frac{m_k}{\bar{m}_k}\right) L^*_{tk} - \frac{3}{2} c\varepsilon^2 [m^2 L^*_t]_{t_k}^{\bar{t}_k}$$

$$- c\varepsilon\, \Delta t_k\, m_k\, L^*_{tk} + \iint G^* + \Theta(\varepsilon^3)$$

$$\bar{y}_k = y_k + \Delta t_k \mathring{y}_k - c\varepsilon\, \bar{m}_k \left(\log \frac{m_k}{\bar{m}_k}\right) L^*_k - c\varepsilon [mL^*]_{t_k}^{\bar{t}_k}$$

$$- c\varepsilon^2\, m_k\, \bar{m}_k \left(\log \frac{m_k}{\bar{m}_k}\right) L^*_{tk} - \frac{3}{2} c\varepsilon^2 [m^2 L^*_t]_{t_k}^{\bar{t}_k} - c\varepsilon\, \Delta t_k\, m_k\, L^*_{tk}$$

$$+ \int_{t_k}^{\bar{t}_k} \int_{t_k}^{\tau} G^*\, dt\, d\tau + \Theta(\varepsilon^3)$$

Next $\mathring{\lambda}$ and λ will be examined:

$$\ddot{\lambda} = Q(t, y, \lambda)$$

$$\dot{\bar{\lambda}}_k = \dot{\lambda}_k + \int_{t_k}^{\bar{t}_k} Q^*\, dt$$

$$\dot{\bar{\lambda}}_k = \dot{\lambda}_k - \varepsilon [mQ^*]_{t_k}^{\bar{t}_k} + \varepsilon \int_{t_k}^{\bar{t}_k} mQ^*_t\, dt$$

$$\dot{\bar{\lambda}}_k = \dot{\lambda}_k - \varepsilon [mQ^*]_{t_k}^{\bar{t}_k} - \frac{1}{2} \varepsilon^2 [m^2 Q^*_t]_{t_k}^{\bar{t}_k} + \frac{1}{2} \varepsilon^2 \int_{t_k}^{\bar{t}_k} m^2\, Q^*_{tt}\, dt$$

$$\dot{\bar{\lambda}}_k = \dot{\lambda}_k - \varepsilon [mQ^*]_{t_k}^{\bar{t}_k} - \frac{1}{2} \varepsilon^2 [m^2 Q^*_t]_{t_k}^{\bar{t}_k} + \frac{1}{2} \varepsilon^2 \int_{t_k}^{\bar{t}_k} m^2\, Q^*_y \ddot{y}\, dt + \Theta(\varepsilon^3)$$

$$\dot{\bar{\lambda}}_k = \dot{\lambda}_k - \varepsilon[mQ^*]_{t_k}^{\bar{t}_k} - \frac{1}{2}\,\varepsilon^2[m^2Q^*_t]_{t_k}^{\bar{t}_k} + \frac{1}{2}\,c\varepsilon \int_{t_k}^{\bar{t}_1} m\, Q^*_y\, L^*\, dt + \Theta(\varepsilon^3)$$

$$\dot{\bar{\lambda}}_k = \dot{\lambda}_k - \varepsilon[mQ^*]_{t_k}^{\bar{t}_k} - \frac{1}{2}\,\varepsilon^2[m^2Q^*_t]_{t_k}^{\bar{t}_k} - \frac{1}{4}\,c\varepsilon^2[m^2Q^*_y L^*]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^3)$$

$$\bar{\lambda}_k = \lambda_k + \Delta t_k\, \dot{\lambda}_k + \int_{t_k}^{\bar{t}_k}\int_{t_k}^{\tau} Q^*\, dt\, d\tau$$

$$\bar{\lambda}_k = \lambda_k + \Delta t_k\, \dot{\lambda}_k + \frac{1}{2}\,\varepsilon^2[m^2Q^*]_{t_k}^{\bar{t}_k} + \varepsilon\, \Delta t_k\, m_k\, Q^*_k + \Theta(\varepsilon^3)$$

following the same procedure used earlier in the expansion of $\iint G^*$.

For reasons mentioned in Section V and to be fully explained later, it will be necessary to determine derivatives of $\bar{K}_k(\varepsilon)$ up to the third order. Therefore, the next task is to derive an expression, for the change in K over a thrust arc, of the form

$$\bar{K}_k(\varepsilon) = K_k(\varepsilon) + \mathcal{K}_k(\varepsilon) + \Theta(\varepsilon^4)$$

Since

$$\dot{K} = \frac{c}{m}\, U(\lambda,\, \dot{\lambda})$$

we have

$$\bar{K}_k = K_k + c\int_{t_k}^{\bar{t}_k} \frac{U^*}{m}\, dt$$

$$\bar{K}_k = K_k - c\varepsilon\left[U^* \log \frac{m}{\bar{m}_k}\right]_{t_k}^{\bar{t}_k} + c\varepsilon \int_{t_k}^{\bar{t}_k} U^*_t \log \frac{m}{\bar{m}_k}\, dt$$

$$\bar{K}_k = K_k + c\varepsilon\, U^*_k \log \frac{m_k}{\bar{m}_k} - c\varepsilon^2\left[U^*_t\, m\left(\log \frac{m}{\bar{m}_k} - 1\right)\right]_{t_k}^{\bar{t}_k}$$

$$+\, c\varepsilon^2 \int_{t_k}^{\bar{t}_k} U^*_{tt}\, m\left(\log \frac{m}{\bar{m}_k} - 1\right) dt$$

$$\bar{K}_k = K_k + c\varepsilon\, U^*_k \log \frac{m_k}{\bar{m}_k} + c\varepsilon^2\, U^*_{tk}\, m_k \log \frac{m_k}{\bar{m}_k} + c\varepsilon^2 [U^*_t\, m]_{t_k}^{\bar{t}_k}$$

$$- c\varepsilon^3 \left[U^*_{tt} \frac{m^2}{2} \left(\log \frac{m}{\bar{m}_k} - \frac{3}{2} \right) \right]_{t_k}^{\bar{t}_k} + c\varepsilon^3 \int_{t_k}^{\bar{t}_k} U^*_{ttt} \frac{m^2}{2} \left(\log \frac{m}{\bar{m}_k} - \frac{3}{2} \right) dt$$

$$\bar{K}_k = K_k + c\varepsilon\, U^*_k \log \frac{m_k}{\bar{m}_k} + c\varepsilon^2\, U^*_{tk}\, m_k \log \frac{m_k}{\bar{m}_k} + c\varepsilon^2 [U^*_t\, m]_{t_k}^{\bar{t}_k}$$

$$+ \frac{1}{2} c\varepsilon^3\, U^*_{ttk}\, m_k^{\,2} \log \frac{m_k}{\bar{m}_k} + \frac{3}{4} c\varepsilon^3 [U^*_{tt}\, m^2]_{t_k}^{\bar{t}_k} + \tilde{I}$$

where

$$\tilde{I} = \frac{1}{2} c\varepsilon^3 \int_{t_k}^{\bar{t}_k} (U^*_{\dot{\lambda}}\, Q^*_y\, \ddot{y})\, m^2 \left(\log \frac{m}{\bar{m}_k} - \frac{3}{2} \right) dt + \Theta(\varepsilon^4)$$

Here $U^*_{\dot{\lambda}} = L^{*T}$, Q^*_y is a square matrix, and $\ddot{y}$ is a column vector. Then

$$\tilde{I} = \frac{1}{2} c^2 \varepsilon^2 \int_{t_k}^{\bar{t}_k} (L^{*T}\, Q^*_y\, L^*)\, m \left(\log \frac{m}{\bar{m}_k} - \frac{3}{2} \right) dt + \Theta(\varepsilon^4)$$

$$= - \frac{1}{2} c^2 \varepsilon^3 \left[(L^{*T}\, Q^*_y\, L^*) \frac{m^2}{2} \left(\log \frac{m}{\bar{m}_k} - 2 \right) \right]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^4)$$

$$= \frac{1}{4} c^2 \varepsilon^3\, (L_k^{*T}\, Q^*_{yk}\, L^*_k)\, m_k^{\,2} \log \frac{m_k}{\bar{m}_k} + \frac{1}{2} c^2 \varepsilon^3 [(L_k^{*T}\, Q^*_y\, L^*) m^2]_{t_k}^{\bar{t}_k}$$

$$+ \Theta(\varepsilon^4)$$

Summarizing, we have:

$$
\left.
\begin{aligned}
\dot{\bar{y}}_k &= \dot{y}_k + c\left(\log \frac{m_k}{\bar{m}_k}\right)(L^*_k + \varepsilon m_k\, L^*_{tk} + \frac{1}{2}\varepsilon^2 m_k^{\,2}\, L^*_{ttk}) \\
&\quad + c\varepsilon[mL^*_t]_{t_k}^{\bar{t}_k} + \frac{3}{4} c\varepsilon^2[m^2 L^*_{tt}]_{t_k}^{\bar{t}_k} - \varepsilon[mG^*]_{t_k}^{\bar{t}_k} \\
&\quad - \frac{1}{2}\varepsilon^2[m^2(G^*_t + \frac{1}{2} c\, G^*_y\, L^*)]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^3) \\
\bar{y}_k &= y_k + \Delta t_k \dot{y}_k - c\varepsilon\, \bar{m}_k \left(\log \frac{m_k}{\bar{m}_k}\right)(L^*_k + \varepsilon m_k\, L^*_{tk}) \\
&\quad - c\varepsilon[mL^*]_{t_k}^{\bar{t}_k} - \frac{3}{2} c\varepsilon^2[m^2 L^*_t]_{t_k}^{\bar{t}_k} - c\varepsilon\, \Delta t_k\, m_k\, L^*_{tk} \\
&\quad + \frac{1}{2}\varepsilon^2[m^2\, G^*]_{t_k}^{\bar{t}_k} + \varepsilon\, \Delta t_k\, m_k\, G^*_k + \Theta(\varepsilon^3) \\
\dot{\bar{\lambda}}_k &= \dot{\lambda}_k - \varepsilon[mQ^*]_{t_k}^{\bar{t}_k} - \frac{1}{2}\varepsilon^2[m^2(Q^*_t + \frac{1}{2} c\, Q^*_y\, L^*)]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^3) \\
\bar{\lambda}_k &= \lambda_k + \Delta t_k\, \dot{\lambda}_k + \frac{1}{2}\varepsilon^2[m^2\, Q^*]_{t_k}^{\bar{t}_k} + \varepsilon\, \Delta t_k\, m_k\, Q^*_k + \Theta(\varepsilon^3) \\
\bar{K}_k &= K_k + c\varepsilon[U^*_k + \varepsilon U^*_{tk}\, m_k + \frac{1}{2}\varepsilon^2\, U^*_{ttk}\, m_k^{\,2} \\
&\quad + \frac{1}{4} c\varepsilon^2 (L_k^{*T}\, Q^*_{yk}\, L^*_k)\, m_k^{\,2}] \log \frac{m_k}{\bar{m}_k} + c\varepsilon^2[U^*_t\, m]_{t_k}^{\bar{t}_k} \\
&\quad + \frac{1}{4} c\varepsilon^3[(3U^*_{tt} + 2cL_k^{*T}\, Q^*_y\, L^*)m^2]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^4)
\end{aligned}
\right\} \quad (8\text{-}1)
$$

Taking first derivatives of both members of equations (8-1), we have:

$$\dot{\bar{y}}_{k\varepsilon} = \dot{y}_{k\varepsilon} + c\left(\frac{d}{d\varepsilon}\log\frac{m_k}{\bar{m}_k}\right)(L^*_k + \varepsilon m_k L^*_{tk})$$

$$+ c\left(\log\frac{m_k}{\bar{m}_k}\right)(L^*_{k\varepsilon} + m_k L^*_{tk} + \varepsilon m_{k\varepsilon} L^*_{tk} + \varepsilon m_k L^*_{tk\varepsilon} + \varepsilon m_k^2 L^*_{ttk})$$

$$+ c[mL^*_t]_{t_k}^{\bar{t}_k} + c\varepsilon(\bar{m}_{k\varepsilon}\bar{L}^*_{tk} - m_{k\varepsilon} L^*_{tk})$$

$$+ c\varepsilon(\bar{m}_k \bar{L}^*_{tk\varepsilon} - m_k L^*_{tk\varepsilon}) + \frac{3}{2} c\varepsilon[m^2 L^*_{tt}]_{t_k}^{\bar{t}_k}$$

$$- [mG^*]_{t_k}^{\bar{t}_k} - \varepsilon(\bar{m}_{k\varepsilon}\bar{G}^*_k - m_{k\varepsilon} G^*_k) - \varepsilon(\bar{m}_k \bar{G}^*_{k\varepsilon} - m_k G^*_{k\varepsilon})$$

$$- \varepsilon[m^2(G^*_t + \frac{1}{2} c\, G^*_y L^*)]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^2)$$

$$\bar{y}_{k\varepsilon} = y_{k\varepsilon} + \Delta t_{k\varepsilon}\dot{y}_k + \Delta t_k \dot{y}_{k\varepsilon} - c\left(\log\frac{m_k}{\bar{m}_k}\right)(\varepsilon\bar{m}_{k\varepsilon} L^*_k + \varepsilon\bar{m}_k L^*_{k\varepsilon}$$

$$+ 2\varepsilon m_k \bar{m}_k L^*_{tk} + \bar{m}_k L^*_k) - c\varepsilon\,\bar{m}_k\left(\frac{d}{d\varepsilon}\log\frac{m_k}{\bar{m}_k}\right)L^*_k - c[mL^*]_{t_k}^{\bar{t}_k}$$

$$- c\varepsilon(\bar{m}_{k\varepsilon}\bar{L}^*_k - m_{k\varepsilon} L^*_k) - c\varepsilon(\bar{m}_k \bar{L}^*_{k\varepsilon} - m_k L^*_{k\varepsilon}) - 3c\varepsilon[m^2 L^*_t]_{t_k}^{\bar{t}_k}$$

$$- c\,\Delta t_k\, m_k L^*_{tk} - c\varepsilon\,\Delta t_{k\varepsilon}\, m_k L^*_{tk} + \varepsilon[m^2 G^*]_{t_k}^{\bar{t}_k} + \Delta t_k\, m_k G^*_k$$

$$+ \varepsilon\,\Delta t_{k\varepsilon}\, m_k G^*_k + \Theta(\varepsilon^2)$$

$$\dot{\bar{\lambda}}_{k\varepsilon} = \dot{\lambda}_{k\varepsilon} - [mQ^*]_{t_k}^{\bar{t}_k} - \varepsilon(\bar{m}_{k\varepsilon}\, \bar{Q}^*_k - m_{k\varepsilon}\, Q^*_k) - \varepsilon(\bar{m}_k\, \bar{Q}^*_{k\varepsilon} - m_k\, Q^*_{k\varepsilon})$$

$$- \varepsilon[m^2(Q^*_t + \frac{1}{2}\, c\, Q^*_y\, L^*)]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^2)$$

$$\bar{\lambda}_{k\varepsilon} = \lambda_{k\varepsilon} + \Delta t_{k\varepsilon}\, \dot{\lambda}_k + \Delta t_k\, \dot{\lambda}_{k\varepsilon} + \varepsilon[m^2 Q^*]_{t_k}^{\bar{t}_k} + \Delta t_k\, m_k\, Q^*_k$$

$$+ \varepsilon\, \Delta t_{k\varepsilon}\, m_k\, Q^*_k + \Theta(\varepsilon^2)$$

$$\bar{K}_{k\varepsilon} = K_{k\varepsilon} + c\varepsilon(U^*_k + \varepsilon U^*_{tk}\, m_k)\ \frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k}$$

$$+ c[U^*_k + \varepsilon U^*_{k\varepsilon} + 2\varepsilon U^*_{tk}\, m_k + \varepsilon^2\, U^*_{tk\varepsilon}\, m_k + \varepsilon^2\, U^*_{tk}\, m_{k\varepsilon}$$

$$+ \frac{3}{2}\, \varepsilon^2\, U^*_{ttk}\, m_k^{\ 2} + \frac{3}{4}\, c\varepsilon^2 (L_k^{*T}\, Q^*_{yk}\, L^*_k) m_k^{\ 2}]\ \log \frac{m_k}{\bar{m}_k}$$

$$+ 2c\varepsilon[U^*_t\ m]_{t_k}^{\bar{t}_k} + c\varepsilon^2(\bar{U}^*_{tk\varepsilon}\, \bar{m}_k - U^*_{tk\varepsilon}\, m_k)$$

$$+ c\varepsilon^2(\bar{U}^*_{tk}\, \bar{m}_{k\varepsilon} - U^*_{tk}\, m_{k\varepsilon})$$

$$+ \frac{3}{4}\, c\varepsilon^2[(3U^*_{tt} + 2c\, L_k^{*T}\, Q^*_y\, L^*)m^2]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon^3)$$

The second derivatives are:

$$\dot{\bar{y}}_{k\varepsilon\varepsilon} = \dot{y}_{k\varepsilon\varepsilon} + c\left(\frac{d^2}{d\varepsilon^2}\log\frac{m_k}{\bar{m}_k}\right)L^*_k + 2c\left(\frac{d}{d\varepsilon}\log\frac{m_k}{\bar{m}_k}\right)(L^*_{k\varepsilon} + m_k\, L^*_{tk})$$

$$+ c\left(\log\frac{m_k}{\bar{m}_k}\right)(L^*_{k\varepsilon\varepsilon} + 2m_{k\varepsilon}\, L^*_{tk} + 2m_k\, L^*_{tk\varepsilon} + m_k^{\,2}\, L^*_{ttk})$$

$$+ 2c(\bar{m}_{k\varepsilon}\, \bar{L}^*_{tk} - m_{k\varepsilon}\, L^*_{tk}) + 2c(\bar{m}_k\, \bar{L}^*_{tk\varepsilon} - m_k\, L^*_{tk\varepsilon})$$

$$+ \frac{3}{2}\, c[m^2\, L^*_{tt}]_{t_k}^{\bar{t}_k} - 2(\bar{m}_{k\varepsilon}\, \bar{G}^*_k - m_{k\varepsilon}\, G^*_k)$$

$$- 2(\bar{m}_k\, \bar{G}^*_{k\varepsilon} - m_k\, G^*_{k\varepsilon}) - [m^2(G^*_t + \frac{1}{2}\, c\, G^*_y\, L^*)]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon)$$

$$\bar{y}_{k\varepsilon\varepsilon} = y_{k\varepsilon\varepsilon} + \Delta t_{k\varepsilon\varepsilon}\dot{y}_k + 2\Delta t_{k\varepsilon}\dot{y}_{k\varepsilon} - 2c\left(\log\frac{m_k}{\bar{m}_k}\right)(\bar{m}_{k\varepsilon}\, L^*_k + \bar{m}_k\, L^*_{k\varepsilon}$$

$$+ m_k\, \bar{m}_k\, L^*_{tk}) - c\bar{m}_k\left(\frac{d}{d\varepsilon}\log\frac{m_k}{\bar{m}_k}\right)L^*_k - 2c(\bar{m}_{k\varepsilon}\, \bar{L}^*_k - m_{k\varepsilon}\, L^*_k)$$

$$- 2c(\bar{m}_k\, \bar{L}_{k\varepsilon} - m_k\, L^*_{k\varepsilon}) - 3c[m^2\, L^*_t]_{t_k}^{\bar{t}_k} - 2c\, \Delta t_{k\varepsilon}\, m_k\, L^*_{tk}$$

$$+ [m^2 G^*]_{t_k}^{\bar{t}_k} + 2\, \Delta t_{k\varepsilon}\, m_k\, G^*_k + \Theta(\varepsilon)$$

$$\dot{\bar{\lambda}}_{k\varepsilon\varepsilon} = \dot{\lambda}_{k\varepsilon\varepsilon} - 2(\bar{m}_{k\varepsilon}\, Q^*_k - m_{k\varepsilon}\, Q^*_k) - 2(\bar{m}_k\, \bar{Q}^*_{k\varepsilon} - m_k\, Q^*_{k\varepsilon})$$

$$- [m^2(Q^*_t + \frac{1}{2}\, cQ^*_y\, L^*)]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon)$$

$$\bar{\lambda}_{k\varepsilon\varepsilon} = \lambda_{k\varepsilon\varepsilon} + \Delta t_{k\varepsilon\varepsilon}\dot{\lambda}_k + 2\, \Delta t_{k\varepsilon}\dot{\lambda}_{k\varepsilon} + [m^2 Q^*]_{t_k}^{\bar{t}_k} + 2\, \Delta t_{k\varepsilon}\, m_k\, Q^*_k + \Theta(\varepsilon)$$

$$\bar{K}_{k\varepsilon\varepsilon} = K_{k\varepsilon\varepsilon} + 2c(U^*_k + \varepsilon U^*_{k\varepsilon} + 2\varepsilon U^*_{tk}\ m_k)\ \frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k}$$

$$+ c[2U^*_{k\varepsilon} + \varepsilon U^*_{k\varepsilon\varepsilon} + 2U^*_{tk}\ m_k + 4\varepsilon U^*_{tk\varepsilon}\ m_k + 4\varepsilon U^*_{tk}\ m_{k\varepsilon}$$

$$+ 3\varepsilon U^*_{ttk}\ m_k^{\ 2} + \frac{3}{2}\ c\varepsilon (L_k^{*T}\ Q^*_{yk}\ L^*_k)m_k^{\ 2}]\ \log \frac{m_k}{\bar{m}_k}$$

$$+ 2c[U^*_t m]_{t_k}^{\bar{t}_k} + 4\ c\varepsilon(\bar{U}^*_{tk\varepsilon}\ \bar{m}_k - U^*_{tk\varepsilon}\ m_k)$$

$$+ 4\ c\varepsilon(\bar{U}^*_{tk}\ \bar{m}_{k\varepsilon} - U^*_{tk}\ m_{k\varepsilon}) + \frac{3}{2}\ c\varepsilon[(3U^*_{tt} + 2cL_k^{*T}\ Q^*_y\ L^*)m^2]_{t_k}^{\bar{t}_k}$$

$$+ \Theta(\varepsilon^2)$$

The third derivative of $\bar{K}_k$ will be required. It is

$$\bar{K}_{k\varepsilon\varepsilon\varepsilon} = K_{k\varepsilon\varepsilon\varepsilon} + 6c(U^*_{k\varepsilon} + U^*_{tk}\ m_k)\ \frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k} + 2cU_k^*\ \frac{d^2}{d\varepsilon^2} \log \frac{m_k}{\bar{m}_k}$$

$$+ 3c[U^*_{k\varepsilon\varepsilon} + 2U^*_{tk\varepsilon}\ m_k + 2U^*_{tk}\ m_{k\varepsilon} + U^*_{ttk}\ m_k^{\ 2}$$

$$+ \frac{c}{2}\ (L_k^{*T}\ \ Q^*_{yk}\ L^*_k)\ m_k^{\ 2}]\ \log \frac{m_k}{\bar{m}_k}$$

$$+ 6c(\bar{U}^*_{tk\varepsilon}\ \bar{m}_k - U^*_{tk\varepsilon}\ m_k) + 6c(\bar{U}^*_{tk}\ \bar{m}_{k\varepsilon} - U^*_{tk}\ m_{k\varepsilon})$$

$$+ \frac{3}{2}\ c[(3U^*_{tt} + 2cL_k^{*T}\ Q^*_y\ L^*)m^2]_{t_k}^{\bar{t}_k} + \Theta(\varepsilon)$$

Section IX

CHANGES OVER A THRUST ARC AS ε APPROACHES ZERO

In this section we will be concerned with the limits of $\bar{m}_k$, $\dot{\bar{y}}_k$, $\bar{y}_k$, $\dot{\bar{\lambda}}_k$, $\bar{\lambda}_k$, $\bar{K}_k$, and their derivatives as ε approaches zero. The expressions for the derivatives for $\varepsilon > 0$ were derived in the preceding section.

From equations (8-1) it follows that

$$\dot{\bar{y}}_k \to \dot{y}_k + c\left(\log \frac{m_k}{\bar{m}_k}\right)L^*_k$$

$$\bar{y}_k \to y_k,\ \dot{\bar{\lambda}}_k \to \dot{\lambda}_k,\ \bar{\lambda}_k \to \lambda_k$$

$$\bar{K}_k \to K_k$$

as $\varepsilon \to 0$. Therefore, y, λ, $\dot{\lambda}$, and K are continuous over an impulse (as indicated in ref. 1), but

$$\Delta\dot{y}_k \to c\left(\log \frac{m_k}{\bar{m}_k}\right)L^*_k$$

Letting $\Delta V_k = |\Delta\dot{y}_k|$ and observing that $|L^*_k| = 1$, it follows that, for $\varepsilon = 0$,

$$\Delta V_k = c \log \frac{m_k}{\bar{m}_k}$$

$$\bar{m}_k = m_k\, e^{-\Delta V_k/c}$$

$$\Delta\dot{y}_k = \Delta V_k\, L^*_k$$

as indicated in reference 1.

It follows from the equations for the first derivatives that

$$\bar{K}_{k\varepsilon} = K_{k\varepsilon} + c\, U^*_k \log \frac{m_k}{\bar{m}_k} + \Theta(\varepsilon)$$

Therefore, if $K_k(\varepsilon) \equiv 0$ and $\bar{K}_k(\varepsilon) \equiv 0$, then $U^*_k(0) = 0$; i.e., $\dot{K}_k(0) = 0$. This is a more general proof that $\dot{K}_k(0) = 0$ than that provided in reference 1. It is also evident that, when $\dot{K}_k(0) = 0$, the condition $\bar{K}_{k\varepsilon} = 0$, for example, will not provide for a condition on the first derivatives (with respect to ε) independent of the condition $K_{k\varepsilon} = 0$. In this case we will resort to the condition $\bar{K}_{k\varepsilon\varepsilon} = 0$ in order to obtain an independent condition on the first derivatives, and similarly for the higher order derivatives.

Furthermore, as $\varepsilon \to 0$,

$$\dot{\bar{y}}_{k\varepsilon} \to \dot{y}_{k\varepsilon} + c\left(\frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k}\right) L^*_k + c\left(\log \frac{m_k}{\bar{m}_k}\right)(L^*_{k\varepsilon} + m_k L^*_{tk})$$

$$+ c(\bar{m}_k\, \bar{L}^*_{tk} - m_k\, L^*_{tk}) - (\bar{m}_k\, \bar{G}^*_k - m_k\, G^*_k)$$

$$\to y_{k\varepsilon} + c\left(\frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k}\right) L^*_k + \Delta V_k L^*_{k\varepsilon} + (m_k \Delta V_k + c\Delta m_k) L^*_{tk} - \Delta m_k G^*_k$$

where

$$\frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k} = \frac{\bar{m}_k\, m_{k\varepsilon} - m_k\, \bar{m}_{k\varepsilon}}{m_k\, \bar{m}_k} = \frac{\Delta m_k\, m_{k\varepsilon} - m_k\, \Delta m_{k\varepsilon}}{m_k\, \bar{m}_k}$$

and, as shown in Appendix B,

$$L^*_{k\varepsilon} = \frac{1}{|\lambda_k|} (I - L^*_k L^{*T}_k) \lambda_{k\varepsilon}$$

Moreover,

$$\bar{y}_{k\varepsilon} \to y_{k\varepsilon} + \Delta t_{k\varepsilon} \dot{y}_k - c\left(\log \frac{m_k}{\bar{m}_k}\right) \bar{m}_k\, L^*_k - c(\bar{m}_k\, \bar{L}^*_k - m_k\, L^*_k)$$

$$\bar{y}_{k\varepsilon} \rightarrow y_{k\varepsilon} - (\bar{m}_k \Delta V_k + c\Delta m_k)L^*_k - \Delta m_k \dot{y}_k$$

$$\dot{\bar{\lambda}}_{k\varepsilon} \rightarrow \dot{\lambda}_{k\varepsilon} - (\bar{m}_k \bar{Q}^*_k - m_k Q^*_k)$$

$$\rightarrow \dot{\lambda}_{k\varepsilon} - \Delta m_k Q^*_k$$

$$\bar{\lambda}_{k\varepsilon} \rightarrow \lambda_{k\varepsilon} + \Delta t_{k\varepsilon} \dot{\lambda}_k$$

$$\rightarrow \lambda_{k\varepsilon} - \Delta m_k \dot{\lambda}_k$$

$$\bar{K}_{k\varepsilon} \rightarrow K_{k\varepsilon} + U^*_k \Delta V_k$$

From the equations for the second derivatives with respect to ε it follows that as $\varepsilon \rightarrow 0$,

$$\bar{K}_{k\varepsilon\varepsilon} \rightarrow K_{k\varepsilon\varepsilon} + 2cU^*_k \frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k} + 2c(U^*_{k\varepsilon} + U^*_{tk}\, m_k) \log \frac{m_k}{\bar{m}_k}$$

$$+ 2c(\bar{U}^*_{tk}\, \bar{m}_k - U^*_{tk}\, m_k)$$

$$\bar{K}_{k\varepsilon\varepsilon} \rightarrow K_{k\varepsilon\varepsilon} + 2cU^*_k \frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k} + 2\, \Delta V_k U^*_{k\varepsilon} + 2(m_k \Delta V_k + c\Delta m_k)U^*_{tk}$$

where $U^*_{k\varepsilon}$ and U^*_{tk} are developed in Appendix B. If $\bar{K}_k(\varepsilon) \equiv 0$ and $K_k(\varepsilon) \equiv 0$, then $U^*_k(0) = 0$, $\bar{K}_{k\varepsilon\varepsilon}(\varepsilon) = K_{k\varepsilon\varepsilon}(\varepsilon) = 0$, and it is necessary at $\varepsilon = 0$ that

$$\Delta V_k (U^*_{k\varepsilon} + U^*_{tk}\, m_k) + c\Delta m_k\, U^*_{tk} = 0$$

No second derivatives are involved in the latter equation.

Summarizing, at $\varepsilon = 0$ the equations of discontinuity for the first derivatives on the k^{th} thrust arc are:

$$\left.\begin{aligned}
\Delta\dot{y}_{k\varepsilon}(0) &= \frac{c}{m_k\,\bar{m}_k}(\Delta m_k m_{k\varepsilon} - m_k \Delta m_{k\varepsilon})L^*_k + \frac{\Delta V_k}{|\lambda_k|}(I - L^*_k\, L^{*T}_k)\lambda_{k\varepsilon} \\
&\quad + a_{1k} L^*_{tk} - \Delta m_k G^*_k \\
\Delta y_{k\varepsilon}(0) &= -\,a_{2k} L^*_k - \Delta m_k\, \dot{y}_k \\
\Delta\dot{\lambda}_{k\varepsilon}(0) &= -\,\Delta m_k Q^*_k \\
\Delta\lambda_{k\varepsilon}(0) &= -\,\Delta m_k \dot{\lambda}_k \\
\Delta K_{k\varepsilon}(0) &= \Delta V_k U^*_k
\end{aligned}\right\} \quad (9\text{-}1)$$

where

$$a_{1k} \overset{\Delta}{=} m_k\, \Delta V_k + c\Delta m_k$$

$$a_{2k} \overset{\Delta}{=} \bar{m}_k\, \Delta V_k + c\Delta m_k$$

(See Appendix B for the series development of a_{1k} and a_{2k}.)

$$m_{k\varepsilon} = \bar{m}_{k-1,\,\varepsilon} = m_{k-1,\,\varepsilon} + \Delta m_{k\varepsilon} \quad \text{if } k > 1$$

$$= 0 \text{ if } k = 1.$$

The necessary conditions for the first derivatives at $\varepsilon = 0$, applying at the k^{th} thrust, are

$$K_{k\varepsilon} = 0 \quad (\text{or perhaps } \bar{K}_{k\varepsilon} = 0)$$

$$\frac{\Delta V_k}{|\lambda_k|}(\dot{\lambda}_k^T \lambda_{k\varepsilon} + \lambda_k^T \dot{\lambda}_{k\varepsilon}) = -\,a_{1k} U^*_{tk} \begin{pmatrix}\text{does not apply if} \\ U^*_k(0) \neq 0\end{pmatrix} \quad (9\text{-}2)$$

In examining the limit as $\varepsilon \to 0$ of the second derivatives, it will be necessary to take a closer look at the relationships between $\bar{G}^*_{tk}(0)$ and $G^*_{tk}(0)$, between $\bar{L}^*_{tk\varepsilon}$ and $L^*_{tk\varepsilon}$, etc. The latter functions are developed in Appendix B, where it is shown that as $\varepsilon \to 0$,

$$\bar{G}^*_k \to G^*_k \qquad \bar{L}^*_{tk\varepsilon} \to L^*_{tk\varepsilon} + \Delta L^*_{tk\varepsilon}$$

$$\bar{G}^*_{tk} \to G^*_{tk} + G^*_{yk}\Delta\dot{y}_k \qquad \bar{Q}^*_k \to Q^*_k$$

$\bar{G}^*_{yk} \to G^*_{yk}$ $\qquad$ $\bar{Q}^*_{tk} \to Q^*_{tk} + Q^*_{yk}\Delta\dot{y}_k$

$\bar{G}^*_{k\varepsilon} \to G^*_{k\varepsilon} + \Delta G^*_{k\varepsilon}$ $\qquad$ $\bar{Q}^*_{yk} \to Q^*_{yk}$

$\bar{L}^*_{k} \to L^*_{k}$ $\qquad$ $\bar{Q}^*_{k\varepsilon} \to Q^*_{k\varepsilon} + \Delta Q^*_{k\varepsilon}$

$\bar{L}^*_{tk} \to L^*_{tk}$ $\qquad$ $\bar{U}^*_{k} \to U^*_{k}$

$\bar{L}^*_{ttk} \to L^*_{ttk}$ $\qquad$ $\bar{U}^*_{tk} \to U^*_{tk}$

$\bar{L}^*_{\lambda k} \to L^*_{\lambda k}$ $\qquad$ $\bar{U}^*_{ttk} \to U^*_{ttk} + L^*_k{}^T Q^*_{yk}\Delta\dot{y}_k$

$\bar{L}^*_{k\varepsilon} \to L^*_{k\varepsilon} - \Delta m_k L^*_{tk}$ $\qquad$ $\bar{U}^*_{tk\varepsilon} \to U^*_{tk\varepsilon} + \Delta U^*_{tk\varepsilon}$

where

$$\Delta G^*_{k\varepsilon} = -\, a_{2k} G^*_{yk} L^*_k - \Delta m_k G^*_{tk}$$

$$\Delta L^*_{tk\varepsilon} = -\, \Delta m_k [(\dot{\lambda}_k L^*_{\lambda\lambda k})\dot{\lambda}_k + L^*_{\lambda k} Q^*_k]$$

$$\Delta Q^*_{k\varepsilon} = -\, \Delta m_k Q^*_{tk} - a_{2k} Q^*_{yk} L^*_k$$

$$\Delta U^*_{tk\varepsilon} = -\, a_{2k} L^*_k{}^T Q^*_{yk} L^*_k - \Delta m_k L^*_k{}^T Q^*_{tk} + \frac{3\Delta m_k}{|\lambda_k|} (-\dot{\lambda}_k{}^T Q^*_k + U^*_k U^*_{tk})$$

The symbol $(\dot{\lambda}_k L^*_{\lambda\lambda k})$ signifies

$$\sum_p \dot{\lambda}_k^{(p)} L^*_{\lambda\lambda^{(p)}k}$$

where the p superscript indicates p^{th} component.

Observe that $\Delta G^*_{k\varepsilon}$, $\Delta L^*_{tk\varepsilon}$, etc., do not depend on any of the unknown ε derivatives. Therefore, these functions can be calculated as soon as the impulsive solution has been computed.

Now we are prepared to examine the second derivatives with respect to ε as $\varepsilon \to 0$. Referring to the second derivatives developed in Section VIII we have:

$$\dot{\bar{y}}_{k\varepsilon\varepsilon} \to \dot{y}_{k\varepsilon\varepsilon} + c\left(\frac{d^2}{d\varepsilon^2}\log\frac{m_k}{\bar{m}_k}\right)L^*_k + 2c\left(\frac{d}{d\varepsilon}\log\frac{m_k}{\bar{m}_k}\right)(L^*_{k\varepsilon} + m_k\, L^*_{tk})$$

$$+ \Delta V_k(L^*_{k\varepsilon\varepsilon} + 2m_{k\varepsilon}\, L^*_{tk} + 2m_k\, L^*_{tk\varepsilon} + m_k^{\;2}\, L^*_{ttk})$$

$$+ 2\,c\,\Delta m_{k\varepsilon}\, L^*_{tk} + 2\,c\,\Delta m_k\, L^*_{tk\varepsilon} + 2\,c\,\bar{m}_k\,\Delta L^*_{tk\varepsilon}$$

$$+ \frac{3}{2}\,c\,(\bar{m}_k^{\;2} - m_k^{\;2})L^*_{ttk} - 2\,\Delta m_{k\varepsilon}\, G^*_k - 2\,\Delta m_k\, G^*_{k\varepsilon} - 2\,\bar{m}_k\,\Delta G^*_{k\varepsilon}$$

$$- (\bar{m}_k^{\;2} - m_k^{\;2})\,(G^*_{tk} + \frac{1}{2}\,c\,G^*_{yk}\, L^*_k) - \bar{m}_k^{\;2}\,\Delta G^*_{tk}$$

It is easily shown that

$$\frac{d^2}{d\varepsilon^2}\log\frac{m_k}{\bar{m}_k} = \frac{\Delta m_k\, m_{k\varepsilon\varepsilon} - m_k\Delta m_{k\varepsilon\varepsilon}}{m_k\,\bar{m}_k} + \left(\frac{\bar{m}_{k\varepsilon}}{\bar{m}_k}\right)^2 - \left(\frac{m_{k\varepsilon}}{m_k}\right)^2$$

and in Appendix B it is proven that

$$L^*_{k\varepsilon\varepsilon} = \frac{1}{|\lambda_k|}\,(I - L^*_k L^{*\,T}_k)\lambda_{k\varepsilon\varepsilon} + \frac{1}{|\lambda_k|^2}\Big[- 2(L^{*\,T}_k\lambda_{k\varepsilon})\lambda_{k\varepsilon} - (\lambda^T_{k\varepsilon}\lambda_{k\varepsilon})L^*_k$$

$$+ 3(L^{*\,T}_k\lambda_{k\varepsilon})^2 L^*_k\Big]$$

Therefore,

$$\dot{\bar{y}}_{k\varepsilon\varepsilon} \to y_{k\varepsilon\varepsilon} + c\left(\frac{\Delta m_k m_{k\varepsilon\varepsilon} - m_k\Delta m_{k\varepsilon\varepsilon}}{m_k\,\bar{m}_k}\right)L^*_k + \frac{\Delta V_k}{|\lambda_k|}\,(I - L^*_k L^{*\,T}_k)\lambda_{k\varepsilon\varepsilon} + \eta_k$$

where

$$\eta_k = c\left[\left(\frac{\bar{m}_{k\varepsilon}}{\bar{m}_k}\right)^2 - \left(\frac{m_{k\varepsilon}}{m_k}\right)^2\right]L^*_k + 2c\left(\frac{\bar{m}_k m_{k\varepsilon} - m_k\bar{m}_{k\varepsilon}}{m_k\,\bar{m}_k}\right)(L^*_{k\varepsilon} + m_k L^*_{tk})$$

$$+ \frac{\Delta V_k}{|\lambda_k|^2}\,[- 2(L^{*\,T}_k\lambda_{k\varepsilon})\lambda_{k\varepsilon} - (\lambda^T_{k\varepsilon}\lambda_{k\varepsilon})L^*_k + 3(L^{*\,T}_k\lambda_{k\varepsilon})^2 L^*_k]$$

$$+ \Delta V_k(2m_{k\varepsilon}L^*_{tk} + 2m_k L^*_{tk\varepsilon} + m_k^2 L^*_{ttk}) + 2c\Delta m_{k\varepsilon}L^*_{tk} + 2c\Delta m_k L^*_{tk\varepsilon}$$

$$- 2c\,\bar{m}_k\Delta m_k[(\dot{\lambda}_k L^*_{\lambda\lambda k})\dot{\lambda}_k + L^*_{\lambda k}Q^*_k] + \frac{3}{2}c\,(\bar{m}_k^{\,2} - m_k^{\,2})L^*_{ttk} - 2\Delta m_{k\varepsilon}G^*_k$$

$$+ 2\,\bar{m}_k[\Delta m_k(G^*_{tk} + c\,G^*_{yk}L^*_k) + \bar{m}_k\,\Delta V_k\,G^*_{yk}\,L^*_k] - 2\Delta m_k G^*_{k\varepsilon}$$

$$- (\bar{m}_k^{\,2} - m_k^{\,2})(G^*_{tk} + \frac{1}{2}c\,G^*_{yk}\,L^*_k) - \bar{m}_k^{\,2}\,\Delta V_k\,G^*_{yk}\,L^*_k$$

$$\eta_k = \left\{c\left[\left(\frac{\bar{m}_{k\varepsilon}}{\bar{m}_k}\right)^2 - \left(\frac{m_{k\varepsilon}}{m_k}\right)^2\right] + \frac{\Delta V_k}{|\lambda_k|^2}\left[-\lambda^T_{k\varepsilon}\lambda_{k\varepsilon} + 3(L^{*T}_k\lambda_{k\varepsilon})^2\right]\right\} L^*_k$$

$$- \frac{2\Delta V_k}{|\lambda_k|^2}(L^{*T}_k\lambda_{k\varepsilon})\lambda_{k\varepsilon} + 2c\left(\frac{\bar{m}_k m_{k\varepsilon} - m_k\bar{m}_{k\varepsilon}}{m_k\,\bar{m}_k}\right)L^*_{k\varepsilon}$$

$$+ \left[2c\left(\frac{\bar{m}_k m_{k\varepsilon} - m_k\bar{m}_{k\varepsilon}}{\bar{m}_k}\right) + 2\Delta V_k m_{k\varepsilon} + 2c\Delta m_{k\varepsilon}\right]L^*_{tk}$$

$$+ 2\,(\Delta V_k m_k + c\Delta m_k)L^*_{tk\varepsilon} + [\Delta V_k m_k^{\,2} + \frac{3}{2}c\,(\bar{m}_k^{\,2} - m_k^{\,2})]L^*_{ttk}$$

$$- 2c\,\bar{m}_k\Delta m_k\,[(\dot{\lambda}_k\,L^*_{\lambda\lambda k})\dot{\lambda}_k + L^*_{\lambda k}Q^*_k] - 2\Delta m_{k\varepsilon}\,G^*_k$$

$$+ [2\bar{m}_k\Delta m_k - (\bar{m}_k^{\,2} - m_k^{\,2})]G^*_{tk} - 2\Delta m_k G^*_{k\varepsilon}$$

$$+ [2(\bar{m}_k\Delta m_k c + \bar{m}_k^{\,2}\Delta V_k) - \frac{1}{2}(\bar{m}_k^{\,2} - m_k^{\,2})c - \bar{m}_k^{\,2}\,\Delta V_k]G^*_{yk}\,L^*_k$$

$$= \left\{c\left[\left(\frac{m_{k\varepsilon}}{\bar{m}_k}\right)^2 - \left(\frac{m_{k\varepsilon}}{m_k}\right)^2\right] + \frac{\Delta V_k}{|\lambda_k|^2}\left[-\lambda^T_{k\varepsilon}\lambda_{k\varepsilon} + 3(L^{*T}_k\lambda_{k\varepsilon})^2\right]\right\} L^*_k$$

$$+ \left[-\frac{2\Delta V_k}{|\lambda_k|^2}(L^{*T}_k\lambda_{k\varepsilon}) + \frac{2c}{|\lambda_k|}\left(\frac{\bar{m}_k m_{k\varepsilon} - m_k\bar{m}_{k\varepsilon}}{m_k\,\bar{m}_k}\right)(I - L^*_k L^{*T}_k)\,\lambda_{k\varepsilon}\right]$$

$$+ \frac{2}{\bar{m}_k} (a_{2k} m_{k\varepsilon} + c\Delta m_k \Delta m_{k\varepsilon}) L^*_{tk} + 2a_{1k} L^*_{tk\varepsilon}$$

$$+ [\Delta V_k m_k^2 + 3 \frac{c}{2} (\bar{m}_k^2 - m_k^2)] L^*_{ttk}$$

$$- 2c\, \bar{m}_k \Delta m_k [(\dot{\lambda}_k L^*_{\lambda\lambda k}) \dot{\lambda}_k + L^*_{\lambda k} Q^*] - 2\Delta m_{k\varepsilon} G^*_k + \Delta m_k^2 G^*_{tk} - 2\Delta m_k G^*_{k\varepsilon}$$

$$+ (a_{2k} \bar{m}_k + \frac{1}{2} c\Delta m_k^2) G^*_{yk} L^*_k$$

Observe that η_k can be computed as soon as the first derivatives have been computed. It is also noteworthy that $G^*_{yk} \lambda_k = Q^*_k$.

Likewise

$$\bar{y}_{k\varepsilon\varepsilon} \to y_{k\varepsilon\varepsilon} + \xi_k$$

where

$$\xi_k = \Delta t_{k\varepsilon\varepsilon} \dot{y}_k + 2\, \Delta t_{k\varepsilon} \dot{y}_{k\varepsilon} - 2\, \Delta V_k (\bar{m}_{k\varepsilon} L^*_k + \bar{m}_k L^*_{k\varepsilon} + m_k \bar{m}_k L^*_{tk})$$

$$- c\bar{m}_k \left(\frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k} \right) L^*_k - 2c\Delta m_{k\varepsilon} L^*_k - 2c\Delta m_k L^*_{k\varepsilon}$$

$$- 2c\bar{m}_k \Delta L^*_{k\varepsilon} - 3c(\bar{m}_k^2 - m_k^2) L^*_{tk}$$

$$+ 2c\Delta m_k m_k L^*_{tk} + \Delta m_k^2 G^*_k$$

$$= - 2\Delta m_{k\varepsilon} \dot{y}_k - 2\Delta m_k \dot{y}_{k\varepsilon} + \Delta m_k^2 G^*_k$$

$$- \left[2\Delta V_k \bar{m}_{k\varepsilon} + c \left(\frac{\bar{m}_k m_{k\varepsilon} - m_k \bar{m}_{k\varepsilon}}{m_k} \right) + 2c\Delta m_{k\varepsilon} \right] L^*_k - 2\, a_{2k} L^*_{k\varepsilon}$$

$$- (2a_{1k} \bar{m}_k + c\Delta m_k^2 - 4c\Delta m_k m_k)$$

Similarly,

$$\dot{\bar{\lambda}}_{k\varepsilon\varepsilon} \to \dot{\lambda}_{k\varepsilon\varepsilon} + \zeta_k$$

where

$$\zeta_k = -2\Delta m_{k\varepsilon} Q^*_k - 2\Delta m_k Q^*_{k\varepsilon} - 2\bar{m}_k \Delta Q^*_{k\varepsilon}$$

$$- (\bar{m}_k^{\ 2} - m_k^{\ 2}) Q^*_{tk} - \bar{m}_k^{\ 2} \Delta Q^*_{tk} - \frac{c}{2} (\bar{m}_k^{\ 2} - m_k^{\ 2}) Q^*_{yk} L^*_k$$

$$= -2\Delta m_{k\varepsilon} Q^*_k - 2\Delta m_k Q^*_{k\varepsilon} + 2\bar{m}_k (\Delta m_k Q^*_{tk} + a_{2k} Q^*_{yk} L^*_k)$$

$$- (\bar{m}_k^{\ 2} - m_k^{\ 2}) Q^*_{tk} - \bar{m}_k^{\ 2} \Delta V_k Q^*_{yk} L^*_k - \frac{c}{2} (\bar{m}_k^{\ 2} - m_k^{\ 2}) Q^*_{yk} L^*_k$$

$$= -2\Delta m_{k\varepsilon} Q^*_k - 2\Delta m_k Q^*_{k\varepsilon} + \Delta m_k^{\ 2} Q^*_{tk}$$

$$+ [2\bar{m}_k a_{2k} - \bar{m}_k^{\ 2} \Delta V_k - \frac{c}{2} (\bar{m}_k^{\ 2} - m_k^{\ 2})] Q^*_{yk} L^*_k$$

Also

$$\bar{\lambda}_{k\varepsilon\varepsilon} \rightarrow \lambda_{k\varepsilon\varepsilon} + \rho_k$$

where

$$\rho_k = -2\Delta m_{k\varepsilon} \dot{\lambda}_k - 2\Delta m_k \dot{\lambda}_{k\varepsilon} + \Delta m_k^{\ 2} Q^*_k$$

Furthermore,

$$\bar{K}_{k\varepsilon\varepsilon} \rightarrow K_{k\varepsilon\varepsilon} + \nu_k$$

where

$$\nu_k = 2cU^*_k \frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k} + c(2U^*_{k\varepsilon} + 2U^*_{tk} m_k) \log \frac{m_k}{\bar{m}_k} + 2c\Delta m_k U^*_{tk}$$

$$= 2c \left(\frac{\bar{m}_k m_{k\varepsilon} - m_k \bar{m}_{k\varepsilon}}{m_k \bar{m}_k} \right) U^*_k + 2\Delta V_k U^*_{k\varepsilon} + 2a_{1k} U^*_{tk}$$

In the cases in which $U^*_k = 0$, we will have $\Delta K_{\varepsilon\varepsilon\varepsilon}(0) = 0$. We can set the mathematical expression for $\Delta K_{\varepsilon\varepsilon\varepsilon}(0)$ equal to zero and employ it as one of the conditions for determining the second derivatives with respect to ε. The equation obtained is

$$6\ c(U^*_{k\varepsilon} + U^*_{tk}\ m_k)\quad \frac{d}{d\varepsilon} \log \frac{m_k}{\bar{m}_k}$$

$$+ 3\ \Delta V_k [U^*_{k\varepsilon\varepsilon} + 2\ U^*_{tk\varepsilon}\ m_k + 2\ U^*_{tk}\ m_{k\varepsilon} + U^*_{ttk}\ m_k^{\ 2}$$

$$+ \frac{c}{2}\ (L^{*T}_k\ Q^*_{yk}\ L^*_k) m_k^{\ 2}] + 6\ c\ \Delta m_k\ U^*_{tk\varepsilon} + 6\ c\ \bar{m}_k\ \Delta U^*_{tk\varepsilon}$$

$$+ 6\ c\ \Delta m_{k\varepsilon}\ U^*_{tk} + \frac{9}{2}\ c(\bar{m}_k^{\ 2} - m_k^{\ 2}) U^*_{ttk} + \frac{3}{2}\ c\ \bar{m}_k^{\ 2}\ \Delta U^*_{ttk}$$

$$+ 3\ c^2 (\bar{m}_k^{\ 2} - m_k^{\ 2}) L^{*T}_k\ Q^*_{yk}\ L^*_k = 0$$

i.e.,

$$6\ c(U^*_{k\varepsilon} + U^*_{tk} m_k) \left(\frac{\bar{m}_k m_{k\varepsilon} - m_k \bar{m}_{k\varepsilon}}{m_k\ \bar{m}_k} \right)$$

$$+ \frac{3\Delta V_k}{|\lambda_k|}\ \dot{\lambda}^T_k \lambda_{k\varepsilon\varepsilon} + 3\Delta V_k L^{*T}_k \dot{\lambda}_{k\varepsilon\varepsilon}$$

$$+ \frac{3\Delta V_k}{|\lambda_k|}\ [2\dot{\lambda}^T_{k\varepsilon} \lambda_{k\varepsilon} - \frac{2}{|\lambda_k|}\ (L^{*T}_k \lambda_{k\varepsilon})(\dot{\lambda}^T_k \lambda_{k\varepsilon} + \lambda^T_k \dot{\lambda}_{k\varepsilon})]$$

$$+ 3\Delta V_k (2U^*_{tk\varepsilon} m_k + 2U^*_{tk} m_{k\varepsilon} + U^*_{ttk} m_k^{\ 2} + \frac{c}{2}\ m_k^{\ 2} L^{*T}_k Q^*_{yk} L^*_k)$$

$$+ 6\ c\Delta m_k U^*_{tk\varepsilon} + 6\ c\bar{m}_k [-a_{2k} L^{*T}_k Q^*_{yk} L^*_k - \Delta m_k L^{*T}_k Q^*_{tk} - \frac{3\Delta m_k}{|\lambda_k|}\ \dot{\lambda}^T_k Q^*_k]$$

$$+ 6\ c\Delta m_{k\varepsilon} U^*_{tk} + \frac{9}{2}\ c(\bar{m}_k^{\ 2} - m_k^{\ 2}) U^*_{ttk}$$

$$+ \frac{3}{2}\ c\ \bar{m}_k^{\ 2}\ \Delta V_k L^{*T}_k Q^*_{yk} L^*_k + 3\ c^2 (\bar{m}_k^{\ 2} - m_k^{\ 2}) L^{*T}_k Q^*_{yk} L^*_k = 0$$

i.e.,

$$\frac{\Delta V_k}{|\lambda_k|}\dot{\lambda}^T{}_k\lambda_{k\varepsilon\varepsilon} + \Delta V_k L^*{}_k{}^T\dot{\lambda}_{k\varepsilon\varepsilon} = 2cU^*{}_{k\varepsilon}\left(\frac{\bar{m}_k m_{k\varepsilon} - m_k\bar{m}_{k\varepsilon}}{m_k\bar{m}_k}\right)$$

$$- 2\left(c\ \frac{\bar{m}_k m_{k\varepsilon} - m_k m_{k\varepsilon}}{\bar{m}_k} + \Delta V_k m_{k\varepsilon} + c\Delta m_{k\varepsilon}\right) U^*{}_{tk}$$

$$- \frac{\Delta V_k}{|\lambda_k|}[2\dot{\lambda}^T{}_{k\varepsilon}\lambda_{k\varepsilon} - \frac{2}{|\lambda_k|}(L^*{}_k{}^T\lambda_{k\varepsilon})(\dot{\lambda}^T{}_k\lambda_{k\varepsilon} + \lambda^T{}_k\dot{\lambda}_{k\varepsilon})]$$

$$- (\Delta V_k m_k + 2c\Delta m_k)U^*{}_{tk\varepsilon} - [\Delta V_k m_k{}^2 + \frac{3}{2}c\ (\bar{m}_k{}^2 - m_k{}^2)]U^*{}_{ttk}$$

$$- \frac{1}{2}[c\Delta V_k m_k{}^2 - 4c\bar{m}_k a_{2k} + \Delta V_k c\bar{m}_k{}^2 + 2\ c^2(\bar{m}_k{}^2 - m_k{}^2)]L^*{}_k{}^T Q^*{}_{yk}L^*{}_k$$

$$+ 2c\bar{m}_k\Delta m_k\ (L^*{}_k{}^T Q^*{}_{tk} + \frac{3}{|\lambda_k|}\dot{\lambda}^T{}_k Q^*{}_k)$$

i.e.,

$$\frac{\Delta V_k}{|\lambda_k|}\dot{\lambda}^T{}_k\lambda_{k\varepsilon\varepsilon} + \Delta V_k L^*{}_k\dot{\lambda}_{k\varepsilon\varepsilon} = \mu_k$$

where

$$\mu_k = -\ 2cU^*{}_{k\varepsilon}\left(\frac{\bar{m}_k m_{k\varepsilon} - m_k\bar{m}_{k\varepsilon}}{m_k\bar{m}_k}\right) - \frac{2}{\bar{m}_k}(c\Delta m_k\bar{m}_{k\varepsilon} + \Delta V_k\bar{m}_k m_{k\varepsilon})U^*{}_{tk}$$

$$- \frac{\Delta V_k}{|\lambda_k|}\left[2\dot{\lambda}^T{}_{k\varepsilon}\lambda_{k\varepsilon} - \frac{2}{|\lambda_k|}(L^*{}_k{}^T\lambda_{k\varepsilon})(\dot{\lambda}^T{}_k\lambda_{k\varepsilon} + \lambda^T{}_k\dot{\lambda}_{k\varepsilon})\right]$$

$$- (\Delta V_k m_k + 2c\Delta m_k)U^*{}_{tk\varepsilon} - [\Delta V_k m_k{}^2 + \frac{3}{2}c(\bar{m}_k{}^2 - m_k{}^2)]U^*{}_{ttk}$$

$$- \frac{1}{2}[c\Delta V_k(\bar{m}_k{}^2 + m_k{}^2) - 4c\bar{m}_k a_{2k} + 2c^2(\bar{m}_k{}^2 - m_k{}^2)]L^*{}_k{}^T Q^*{}_{yk}L^*{}_k$$

$$+ 2c\bar{m}_k\ \Delta m_k(L^*{}_k{}^T Q^*{}_{tk} + \frac{3}{|\lambda_k|}\dot{\lambda}^T{}_k Q^*{}_k)$$

In summary,

$$\Delta\dot{y}_{k\varepsilon\varepsilon}(0) = \frac{c}{m_k \bar{m}_k}(\Delta m_k m_{k\varepsilon\varepsilon} - m_k \Delta m_{k\varepsilon\varepsilon})L^*_k + \frac{\Delta V_k}{|\lambda_k|}(I - L^*_k L^{*T}_k)\lambda_{k\varepsilon\varepsilon} + \eta_k$$

$$\Delta y_{k\varepsilon\varepsilon}(0) = \xi_k$$

$$\Delta\dot{\lambda}_{k\varepsilon\varepsilon}(0) = \zeta_k \qquad (9\text{-}3)$$

$$\Delta\lambda_{k\varepsilon\varepsilon}(0) = \rho_k$$

$$\Delta K_{k\varepsilon\varepsilon}(0) = \nu_k$$

where η_k, ξ_k, ζ_k, ρ_k, and ν_k contain no second derivatives with respect to ε. The necessary conditions for the second derivatives at $\varepsilon = 0$, applying at the k^{th} thrust are,

$$K_{k\varepsilon\varepsilon} = 0 \qquad \text{(or perhaps } \bar{K}_{k\varepsilon\varepsilon} = 0)$$

$$(9\text{-}4)$$

$$\frac{\Delta V_k}{|\lambda_k|}(\dot{\lambda}_k \lambda_{k\varepsilon\varepsilon} + \lambda^T_k \dot{\lambda}_{k\varepsilon\varepsilon}) = \mu_k \qquad \text{(does not apply if } U^*_k(0) \neq 0)$$

Observe that equations (9-3) and (9-4) have the same form as the equations (9-1) and (9-2) corresponding to the first derivatives.

Section X

EXAMPLE 1: BURN-COAST INTERCEPT

As an example consider a thrust-coast intercept problem in which the vehicle moves from a given point in state space to a point with given position components. The time t_F of intercept is also specified. In this problem the initial time t_I must be identified with t_1, and t_2 with t_F.

We will impose the scaling condition $\lambda_1^T \lambda_1 = 1$ upon the initial values of the Lagrange multipliers. Therefore,

$$\lambda_1^T \lambda_{1\varepsilon} = 0 \tag{10-1}$$

The condition $\bar{K}_1 = 0$ implies $\bar{K}_{1\varepsilon} = 0$. The last of equations (9-1) gives $K_{I\varepsilon} = -\Delta V_1 U^*_1$.

Since y_2 is fixed, we have

$$y_{2\varepsilon} = 0.$$

From equations (6-8) it follows that

$$\left[\frac{\partial y_2}{\partial \bar{y}_1}\right] (\bar{y}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] (\dot{\bar{y}}_{1\varepsilon} - \bar{t}_{1\varepsilon}\bar{G}^*_1) = 0$$

Since $y_{1\varepsilon} = \dot{y}_{1\varepsilon} = 0$,

$$\left[\frac{\partial y_2}{\partial \bar{y}_1}\right] (\Delta y_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] (\Delta\dot{y}_{1\varepsilon} - \bar{t}_{1\varepsilon}\bar{G}^*_1) = 0$$

From equations (9-1) it follows that, for $\varepsilon = 0$, we have

$$\left[\frac{\partial y_2}{\partial \bar{y}_1}\right] (-a_{21}\lambda_1 - \Delta m_1\dot{y}_1 - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] [-c \frac{\Delta m_{1\varepsilon}}{\bar{m}_1} \lambda_1 + \Delta V_1 \lambda_{1\varepsilon}$$

$$+ a_{11}L^*_{t1} - \Delta m_1 G^*_1 - \bar{t}_{1\varepsilon}\bar{G}^*_1] = 0$$

where $\Delta V_1 = |\Delta \dot{y}_1|$.

Since $\Delta m_1 = -\Delta t_{1\varepsilon} = -\bar{t}_{1\varepsilon}$, $G^*_1 = \bar{G}^*_1$, $\Delta \dot{y}_1 = \Delta V_1 \lambda_1$, and $L^*_{t1} = (I - \lambda_1 \lambda_1^T)\dot{\lambda}_1$ at $\varepsilon = 0$, the equation reduces to

$$-a_{11} \left[\frac{\partial y_2}{\partial \bar{y}_1}\right] \lambda_1 + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] [-c \frac{\Delta m_{1\varepsilon}}{\bar{m}_1} \lambda_1 + \Delta V_1 \lambda_{1\varepsilon} + a_{11}(I - \lambda_1 \lambda_1^T)\dot{\lambda}_1] = 0 \quad (10\text{-}2)$$

Equations (10-1) and (10-2) may be expressed as

$$\begin{bmatrix} -\frac{c}{\bar{m}_1} \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] \lambda_1 & \Delta V_1 \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] \\ 0 & \lambda_1^T \end{bmatrix} \begin{bmatrix} \Delta m_{1\varepsilon} \\ \lambda_{1\varepsilon} \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}$$

where

$$\gamma = a_{11} \left[\frac{\partial y_2}{\partial \bar{y}_1}\right] \lambda_1 - a_{11} \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] (I - \lambda_1 \lambda_1^T)\dot{\lambda}_1$$

Inversion of the linear system yields

$$\begin{bmatrix} \Delta m_{1\varepsilon} \\ \lambda_{1\varepsilon} \end{bmatrix} = \begin{bmatrix} -\frac{\bar{m}_1}{c} \lambda_1^T \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} & \frac{\bar{m}_1}{c} \Delta V_1 \\ \frac{1}{\Delta V_1} (I - \lambda_1 \lambda_1^T) \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} & \lambda_1 \end{bmatrix} \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \quad (10\text{-}3)$$

In general (see Appendix A),

$$\lambda_2 = \left[\frac{\partial y_2}{\partial \bar{y}_1}\right] \lambda_1 + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] \dot{\lambda}_1$$

But for an intercept problem the general transversality condition of the calculus of variations implies that $\lambda_2 = 0$. Therefore,

$$\dot{\lambda}_1 = -\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_1 \tag{10-4}$$

Utilizing the latter expression to simplify equation (10-3), we have

$$\Delta m_{1\varepsilon} = \frac{\bar{m}_1}{c}\left[a_{11}\lambda_1^T\dot{\lambda}_1 + a_{11}\lambda_1^T(I - \lambda_1\lambda_1^T)\dot{\lambda}_1\right]$$

$$\Delta m_{1\varepsilon} = \frac{\bar{m}_1 a_{11}}{c}\lambda_1^T\dot{\lambda}_1 \tag{10-5}$$

$$\lambda_{1\varepsilon} = \frac{1}{\Delta V_1}(I - \lambda_1\lambda_1^T)\left[-a_{11}\dot{\lambda}_1 - a_{11}(I - \lambda_1\lambda_1^T)\dot{\lambda}_1\right]$$

$$\lambda_{1\varepsilon} = \frac{2a_{11}}{\Delta V_1}\left[(\lambda_1^T\dot{\lambda}_1)\lambda_1 - \dot{\lambda}_1\right] \tag{10-6}$$

where it is understood that λ_1, $\dot{\lambda}_1$, etc., are the impulsive values. If one were to derive expressions for $\Delta m_{1\varepsilon\varepsilon}$ and $\lambda_{1\varepsilon\varepsilon}$, the equations would have the same coefficient matrix as equations (10-3).

An expression for $\dot{\lambda}_1$ can be obtained from the transversality condition $\lambda_2 = 0$. Thus,

$$\lambda_{2\varepsilon} = 0$$

and, from equations (6-8), we have

$$\left[\frac{\partial \lambda_2}{\partial \bar{y}_1}\right](\bar{y}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + \left[\frac{\partial \lambda_2}{\partial \dot{\bar{y}}_1}\right](\dot{\bar{y}}_{1\varepsilon} - \bar{t}_{1\varepsilon}\bar{G}^*{}_1)$$

$$+\left[\frac{\partial y_2}{\partial \bar{y}_1}\right](\bar{\lambda}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{\lambda}}_1) + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right](\dot{\bar{\lambda}}_{1\varepsilon} - \bar{t}_{1\varepsilon}\bar{Q}^*{}_1) = 0$$

since $\left[\frac{\partial\lambda_2}{\partial\bar{\lambda}_1}\right] = \left[\frac{\partial y_2}{\partial\bar{y}_1}\right]$ and $\left[\frac{\partial\lambda_2}{\partial\dot{\bar{\lambda}}_1}\right] = \left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]$ (see Appendix A).

Employing equations (9-1), one obtains

$$\left[\frac{\partial\lambda_2}{\partial\bar{y}_1}\right](-a_{21}\lambda_1 - \Delta m_1\dot{y}_1 - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + \left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right]\left[-\frac{c}{m_1}\Delta m_{1\varepsilon}\lambda_1 + \Delta V_1(I-\lambda_1\lambda_1^T)\lambda_{1\varepsilon} + a_{11}L^*{}_{t1}\right]$$

$$+\left[\frac{\partial y_2}{\partial\bar{y}_1}\right]\lambda_{1\varepsilon} + \left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]\dot{\lambda}_{1\varepsilon} = 0$$

Therefore, we have

$$\dot{\lambda}_{1\varepsilon} = \left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]^{-1}\left\{a_{11}\left[\frac{\partial\lambda_2}{\partial\bar{y}_1}\right]\lambda_1 - \left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right]\left[-\frac{c}{m_1}\Delta m_{1\varepsilon}\lambda_1 + (I-\lambda_1\lambda_1^T)(\Delta V_1\lambda_{1\varepsilon} + a_{11}\dot{\lambda}_1)\right]\right.$$

$$\left. - \left[\frac{\partial y_2}{\partial\bar{y}_1}\right]\lambda_{1\varepsilon}\right\}$$

$$\dot{\lambda}_{1\varepsilon} = \left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]^{-1}\left(a_{11}\left[\frac{\partial\lambda_2}{\partial\bar{y}_1}\right]\lambda_1 + \left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right]\left\{\frac{c}{m_1}\Delta m_{1\varepsilon}\lambda_1 + a_{11}[\dot{\lambda}_1 - (\lambda_1^T\dot{\lambda}_1)\lambda_1]\right\}\right.$$

$$\left. - \left[\frac{\partial y_2}{\partial\bar{y}_1}\right]\lambda_{1\varepsilon}\right) \qquad (10\text{-}7)$$

It is noteworthy that equations (10-5) and (10-6) have very simple forms which do not involve any transition matrices. However, equation (10-7) does require evaluation of the transition matrices.

In a constant gravitational field, $\lambda_2 = \lambda_1 + (t_2 - t_1)\dot{\lambda}_1$ at ε
$\lambda_2 = 0$, $\dot{\lambda}_1 = -\frac{1}{t_2 - t_1}\lambda_1$. Therefore, equations (10-6) and (10-7) im
$\lambda_{1\varepsilon} = \dot{\lambda}_{1\varepsilon} = 0$ when G is constant. This is not surprising, because
6 it has been shown that the impulsive solution provides exact valu
Lagrange multipliers in the burn-coast intercept problem with const
fixed t_F. The latter observation would seem to imply that, if the
problem is roughly equivalent to a constant G problem, then the lin
tions to the impulsive solution should provide accurate formulas.

The Taylor series expansions about ε = 0, truncated after firs
second-order terms, yield the approximate formulas

$$\bar{t}_1 = t_1 + \frac{\varepsilon}{1!}\bar{t}_{1\varepsilon} + \frac{\varepsilon^2}{2!}\bar{t}_{1\varepsilon\varepsilon} = t_1 - \frac{\Delta m_1}{\beta} - \frac{\Delta m_1}{\beta^2}$$

$$\lambda_{1\varepsilon} = \lambda_1 + \frac{1}{\beta}\lambda_{1\varepsilon} \quad , \quad \dot{\lambda}_{1\varepsilon} = \dot{\lambda}_1 + \frac{1}{\beta}\dot{\lambda}_{1\varepsilon}$$

Before considering a numerical example, we will briefly consid
determination of the impulsive solution. Given y_I, y_F, and $t_F - t_I$
conic through y_I and y_F may be easily determined by means of a Lamb
solution (refs. 2 and 3). There are many computer program subrouti
obtaining such solutions. The output data will ordinarily include
Then $\lambda_1 = L^*_1 = \frac{1}{\Delta V_1}\Delta\dot{y}_1$, $\dot{\lambda}_1$ can be computed from equation (10-4), a
$\bar{m}_1 = m_1 e^{-\Delta V_1/c}$.

Consider a burn-coast intercept problem in which

$$t_1 = 0,\ t_F = 380 \text{ sec},\ m_1 = 16892.0 \text{ kg-sec}^2/\text{m}$$

$$y_1^T = (2872.5 \text{ km},\ 5907.8 \text{ km},\ 77.7 \text{ km})$$

$$\dot{y}_1^T = (-7.33 \text{ km/sec},\ 3.22 \text{ km/sec},\ -.47 \text{ km/sec})$$

$$y_2^T = (0 \text{ km},\ 6556.3 \text{ km},\ 0 \text{ km})$$

$$c = 4100 \text{ m/sec},\ \beta = 22 \text{ kg-sec/m}$$

$\mu = .388 \times 10^{15} \ m^3/sec.$

The vehicle (a Saturn S-IVB) is initially 100 n mi above the earth's surface. It must intercept a point a few miles lower in altitude.

An IBM 1130 data processing system was employed in the solution. The results are as follows:

$\bar{t}_{1 \ TRUE} = 86.0$ sec

$\bar{t}_{1 \ IMP} = 76.6$ (10.9 percent error) $\left(\bar{t}_{1 \ IMP} = t_1 - \frac{\Delta m_1}{\beta}\right)$

$\bar{t}_{1 \ COR} = 84.0$ (2.3 percent error)

$m_{2 \ TRUE} = 15{,}000 \ kg\text{-}sec^2/m$

$m_{2 \ IMP} = 15{,}207$ (1.4 percent error)

$m_{2 \ COR} = 15{,}044$ (.3 percent error)

$\lambda_1{}^T{}_{TRUE} = (.6618, .3727, .6506)$

$\lambda_1{}^T{}_{IMP} = (.6631, .3557, .6587)$
ERROR = (.2 percent, 4.6 percent, 1.3 percent)

$\lambda_1{}^T{}_{COR} = (.6623, .3726, .6503)$
ERROR = (.08 percent, .003 percent, .003 percent)

$\dot{\lambda}_1{}^T{}_{TRUE} = (-.1725 \times 10^{-2}, -.1198 \times 10^{-2}, -.1596 \times 10^{-2})$

$\dot{\lambda}_1{}^T{}_{IMP} = (-.1726 \times 10^{-2}, -.1149 \times 10^{-2}, -.1617 \times 10^{-2})$
ERROR = (.06 percent, 4.3 percent, 1.3 percent)

$\dot{\lambda}_1{}^T{}_{COR} = (-.1726 \times 10^{-2}, -.1198 \times 10^{-2}, -.1596 \times 10^{-2})$
ERROR = (.06 percent, 0.00 percent, 0.00 percent)

The subscript IMP signifies values obtained from the impulsive solution. In the case of $\bar{t}_{1\ IMP}$, this means from the first-order term in equations (10-8). The designation COR is for values obtained from the first order corrections in the cases of λ_1 and $\dot{\lambda}_1$, and from the first- and second-order terms in the case of $\bar{t}_1$.

A FORTRAN listing of subroutine OPINT (optimum intercept) is given in Appendix C along with the listings of subroutines which it employs, except for widely available subroutines and LAM which is described in Appendix A. The call statement for OPINT is CALL OPINT(Y1, YD1, Y2, T2, M1, C, GM, B, TB1C, L1C, LD1C) where

Y1 = y_1, YD1 = $\dot{y}_1$, Y2 = y_2, T2 = $t_2 - t_1$,

M1 = m_1, C = c, GM = μ, B = β,

TB1C = $\bar{t}_{1\ COR}$, L1C = $\lambda_{1\ COR}$, LD1C = $\dot{\lambda}_{1\ COR}$

The following subroutines are employed by OPINT or its subroutines:

OPIMP	determines optimum impulsive solution
LAM	determines transition matrices
MINV	obtains matrix inverse (listing not included)
FORC	obtains corrected values of $\bar{t}_1$, λ_1, and $\dot{\lambda}_1$
WR2	data print routine
DYST	obtains Lambert solution (listing not included)
VMAG	obtains vector magnitude (listing not included)
MPRD	multiplies matrices (listing not included)
COF1	computes $a_{11} = c\Delta m_1 + m_1 \Delta V_1$
DOTN	computes dot product of two vectors (listing not included)

Section XI

EXAMPLE 2: CONSTANT BURN INTERCEPT

A constant burn intercept problem will now be discussed to illustrate how the impulsive theory may sometimes be extended to problems with no coast phases. This problem has already been discussed in reference 6. However, for the sake of completeness, a discussion is also given herein.

The physical problem is the same as that discussed in Section X with the exceptions that there is no coast phase and the final time t_F is not specified. In order that the solution can be related to an impulsive solution, a boundary condition problem is defined for any value of $\varepsilon = 1/\beta$ such that, when $\varepsilon = 0$, the impulsive solution is the solution to the problem and, when $\varepsilon = \varepsilon_{TRUE} = 1/\beta_{TRUE}$, one has the solution to the constant thrust problem. In so doing the boundary condition problem is set up such that there will be a thrust phase in the time interval $[t_1, \bar{t}_1]$ and a coast phase in the interval $[\bar{t}_1, t_2]$. However, the time duration of the latter phase shrinks to zero as $\varepsilon \rightarrow \varepsilon_{TRUE}$.

First we define a function

$$T(\varepsilon) = (1 - \frac{\varepsilon}{\varepsilon_{TRUE}})\Delta T(0)$$

which signifies the coast time, where $\Delta T(0)$ is a specified number, approximately equal to $t_2(\varepsilon_{TRUE}) - t_1$ (i.e., the time duration of the constant thrust trajectory).

In reference 6 it is shown that if $\Delta T(0)$ is exactly equal to $t_2(\varepsilon_{TRUE}) - t_1$ and G is constant, then the single impulse solution (i.e., the $\varepsilon = 0$ solution) provides exact values of the Lagrange multipliers. Therefore, one would expect the model to provide an accurate solution to many problems in which the gravitational field is an inverse square field. Although, the constant burn intercept problem seems to be particularly amenable to approximation by impulsive methods, other problems - such as the constant burn rendezvous problem - are not (ref. 6).

The following boundary conditions are imposed:

$$\lambda_1^T(\varepsilon)\lambda_1(\varepsilon) = 1 \quad \text{(scaling)}$$

$$t_2(\varepsilon) - \bar{t}_1(\varepsilon) = \Delta T(\varepsilon)$$

$$y_2(\varepsilon) = a \qquad (11\text{-}1)$$

$$\lambda_2(\varepsilon) = 0 \quad \text{(transversality)}$$

where a is a constant vector.

The scaling condition implies

$$\lambda_1^T \lambda_{1\varepsilon} = 0 \qquad (11\text{-}2)$$

Furthermore,

$$t_{2\varepsilon} - \bar{t}_{1\varepsilon} = \Delta T_\varepsilon \qquad (11\text{-}3)$$

and

$$y_{2\varepsilon} = 0.$$

As in Section X, the latter equation yields

$$t_{2\varepsilon}\dot{y}_2 - a_{11}\left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_1 + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]\left[-c\,\frac{\Delta m_{1\varepsilon}}{\bar{m}_1}\,\lambda_1 + \Delta V_1\lambda_{1\varepsilon} + a_{11}(I-\lambda_1\lambda_1^T)\dot{\lambda}_1\right] = 0 \qquad (11\text{-}4)$$

The term $t_{2\varepsilon}\dot{y}_2$ is new, since $t_{2\varepsilon}$ is not necessarily zero in the problem now under consideration. But equation (11-3) signifies that, at $\varepsilon = 0$,

$$t_{2\varepsilon} = \bar{t}_{1\varepsilon} + \Delta T_\varepsilon = \Delta t_{1\varepsilon} + \Delta T_\varepsilon = -\Delta m_1 + \Delta T_\varepsilon .$$

Therefore, equations (11-2) and (11-4) can be written as

$$\begin{bmatrix} -\dfrac{c}{\bar{m}_1}\left[\dfrac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]\lambda_1 & \Delta V_1\left[\dfrac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] \\ 0 & \lambda_1^T \end{bmatrix} \begin{bmatrix} \Delta m_{1\varepsilon} \\ \lambda_{1\varepsilon} \end{bmatrix} = \begin{bmatrix} \gamma' \\ 0 \end{bmatrix} \qquad (11\text{-}5)$$

where

$$\gamma' = \gamma + (\Delta m_1 - \Delta T_\varepsilon)\dot{y}_2$$

and γ was defined in Section X. Equations (11-5) have the same coefficient matrix as that obtained for the burn-coast intercept problem in Section X. As in the latter section,

$$\begin{bmatrix} \Delta m_{1\varepsilon} \\ \\ \lambda_{1\varepsilon} \end{bmatrix} = \begin{bmatrix} -\frac{\bar{m}_1}{c}\lambda_1^T\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} & \frac{\bar{m}_1}{c}\Delta V_1 \\ \\ \frac{1}{\Delta V_1}(I - \lambda_1\lambda_1^T)\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} & \lambda_1 \end{bmatrix} \begin{bmatrix} \gamma' \\ \\ 0 \end{bmatrix}$$

Therefore, at $\varepsilon = 0$

$$\Delta m_{1\varepsilon} = -\frac{\bar{m}_1}{c}\lambda_1^T\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\gamma'$$

$$\lambda_{1\varepsilon} = \frac{1}{\Delta V_1}(I - \lambda_1\lambda_1^T)\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\gamma'$$

Furthermore, from equation (11-3) it follows that

$$t_{2\varepsilon}(0) = \bar{t}_{1\varepsilon} + \Delta T_\varepsilon = \Delta t_{1\varepsilon} - \frac{\Delta T(0)}{\varepsilon_{TRUE}}$$

$$= -\Delta m_1 + \frac{\Delta T(0)}{\varepsilon_{TRUE}}$$

$$t_{2\varepsilon\varepsilon}(0) = \bar{t}_{1\varepsilon\varepsilon} = \Delta t_{1\varepsilon\varepsilon} = -2\Delta m_{1\varepsilon}.$$

As in Section X, an expression for $\dot{\lambda}_{1\varepsilon}$ can be obtained from the last condition, $\lambda_2 = 0$, of equations (11-1). Hence,

$$\lambda_{2\varepsilon} = 0$$

which implies (Section X) that

$$\dot{\lambda}_{1\varepsilon} = \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} \left(t_{2\varepsilon}\dot{\lambda}_2 + a_{11}\left[\frac{\partial \lambda_2}{\partial \bar{y}_1}\right] + \left[\frac{\partial \lambda_2}{\partial \dot{\bar{y}}_1}\right] \left\{ \frac{c}{\bar{m}_1} \Delta m_{1\varepsilon}\, \lambda_1 + a_{11}\left[\dot{\lambda}_1 - (\lambda_1{}^T \dot{\lambda}_1)\lambda_1\right] \right\} - \left[\frac{\partial y_2}{\partial \bar{y}_1}\right] \lambda_{1\varepsilon} \right)$$

No numerical studies, employing the equations of this section, have been conducted.

Section XII

EXAMPLE 3: BURN–COAST–BURN RENDEZVOUS

As another example consider a thrust-coast-thrust rendezvous problem in which the vehicle moves from a given point in state space to a point with given position and velocity components. The time t_F of rendezvous is also specified. In this problem,

t_1 = initial time = t_I

$\bar{t}_1$ = first engine cut-off time

t_2 = reignition time

$\bar{t}_2$ = final time = t_F.

The boundary conditions are as follows:

$\lambda_1^T\lambda_1 = 1$ (scaling condition)

$\bar{K}_1 = 0$, $K_2 = 0$ (switching conditions)

$\bar{y}_2 = a$, $\dot{\bar{y}}_2 = b$ (rendezvous conditions)

(Furthermore, since $K_2 = \bar{K}_1 = 0$, $\lambda_2^T(0)\lambda_2(0) = 1$.)

The scaling condition implies

$$\lambda_1^T\lambda_{1\varepsilon} = 0 \tag{12-1}$$

The first switching condition implies $\bar{K}_{1\varepsilon} = 0$. Then from the last of equations (9-1) we have the trival condition $K_{1\varepsilon} = - V_1U^*_1$ at $\varepsilon = 0$. The second switching condition gives $K_2 = 0$. Then the last of equations (6-8) yields

$$\frac{c}{\bar{m}_1}(t_{2\varepsilon}U^*_2 - \bar{t}_{1\varepsilon}U^*_1) + \frac{c}{\bar{m}_1}[\lambda_2^T(\lambda_{2\varepsilon} - t_{2\varepsilon}\dot{\lambda}_2) - \lambda_1^T(\bar{\lambda}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\lambda}_1)] = 0$$

for $\varepsilon = 0$. Since $\lambda_{1\varepsilon} = \bar{\lambda}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\lambda}_1$ and $\lambda_1^T\lambda_{1\varepsilon} = 0$, we have

$$\lambda_2^T\lambda_{2\varepsilon} = \Delta m_2\lambda_2^T\dot{\lambda}_2 - \Delta m_1U^*_1 - \Delta m_2U^*_2$$

From equations (6-8) it follows that

$$\lambda_2^T\left\{t_{2\varepsilon}\dot{\lambda}_2 + \left[\frac{\partial\lambda_2}{\partial\bar{y}_1}\right](\bar{y}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + \left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right](\dot{\bar{y}}_{1\varepsilon} - \bar{t}_{1\varepsilon}G^*_1) + \left[\frac{\partial y_2}{\partial\bar{y}_1}\right](\bar{\lambda}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{\lambda}}_1) + \left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right](\dot{\bar{\lambda}}_{1\varepsilon} - \bar{t}_{1\varepsilon}Q^*_1)\right\}$$

$$= \Delta m_2\lambda_2^T\dot{\lambda}_2 - \Delta m_1U^*_1 - \Delta m_2U^*_2$$

It follows from equations (9-1) that

$$\lambda_2^T\left\{\Delta m_2\dot{\lambda}_2 - a_{11}\left[\frac{\partial\lambda_2}{\partial\bar{y}_1}\right]\lambda_1 + \left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right]\left[-\frac{c}{\bar{m}_1}\Delta m_{1\varepsilon}\lambda_1 + \Delta V_1\lambda_{1\varepsilon} + a_{11}(I-\lambda_1\lambda_1^T)\dot{\lambda}_1\right] + \left[\frac{\partial y_2}{\partial\bar{y}_1}\right]\lambda_{1\varepsilon} + \left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]\dot{\lambda}_{1\varepsilon}\right\} = \Delta m_2\lambda_2^T\dot{\lambda}_2 - (\Delta m_1U^*_1 + \Delta m_2U^*_2)$$

Rearranging, we obtain

$$-\Delta m_{1\varepsilon}\frac{c}{\bar{m}_1}\lambda_2^T\left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right]\lambda_1 + \lambda_2^T\left(\Delta V_1\left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right] + \left[\frac{\partial y_2}{\partial\bar{y}_1}\right]\right)\lambda_{1\varepsilon}$$

$$+ \lambda_2^T\left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]\dot{\lambda}_{1\varepsilon} = a_{11}\lambda_2^T\left\{\left[\frac{\partial\lambda_2}{\partial\bar{y}_1}\right]\lambda_1 - \left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right](I-\lambda_1\lambda_1^T)\dot{\lambda}_1\right\}$$

$$- (\Delta m_1U^*_1 + \Delta m_2U^*_2) \qquad (12\text{-}2)$$

The equation $\bar{y}_2 = a$ yields $\bar{y}_{2\varepsilon} = 0$, and from equations (9-1) it follows that

$$y_{2\varepsilon} = a_{22}\lambda_2 + \Delta m_2\dot{y}_2$$

From equations (6-8), we have

$$\left[\frac{\partial y_2}{\partial\bar{y}_1}\right](\bar{y}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + \left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right](\dot{\bar{y}}_{1\varepsilon} - \bar{t}_{1\varepsilon}G^*_1) = a_{22}\lambda_2$$

Employing equations (9-1) once more, we obtain

$$-a_{11}\left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_1 + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]\left[-\frac{c}{\bar{m}_1}\Delta m_{1\varepsilon}\lambda_1 + \Delta V_1\lambda_{1\varepsilon} + a_{11}(I-\lambda_1\lambda_1^T)\dot{\lambda}_1\right] = a_{22}\lambda_2$$

Therefore,

$$-\Delta m_{1\varepsilon}\frac{c}{\bar{m}_1}\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]\lambda_1 + \Delta V_1\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]\lambda_{1\varepsilon} = a_{11}\left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_1$$

$$- a_{11}\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right](I-\lambda_1\lambda_1^T)\dot{\lambda}_1 + a_{22}\lambda_2 \qquad (12\text{-}3)$$

The equation $\dot{\bar{y}}_2 = b$ yields $\dot{\bar{y}}_{2\varepsilon} = 0$, and from equations (9-1) it follows that

$$\dot{y}_{2\varepsilon} + \frac{c}{m_2\bar{m}_2}(\Delta m_2 m_{2\varepsilon} - m_2\Delta m_{2\varepsilon})\lambda_2 + \Delta V_2(I-\lambda_2\lambda_2^T)\lambda_{2\varepsilon}$$

$$= - a_{12}(I-\lambda_2\lambda_2^T)\dot{\lambda}_2 + \Delta m_2 G^*_2$$

From equations (6-8) it follows that

$$\left[\frac{\partial \dot{y}_2}{\partial \bar{y}_1}\right](\bar{y}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + \left[\frac{\partial \dot{y}_2}{\partial \dot{\bar{y}}_1}\right](\dot{\bar{y}}_{1\varepsilon} - \bar{t}_{1\varepsilon}G^*_1) + \frac{c}{\bar{m}_1\,\bar{m}_2}(\Delta m_2\Delta m_{1\varepsilon} - \bar{m}_1\Delta m_{2\varepsilon})\lambda_2$$

$$+ \Delta V_2(I-\lambda_2\lambda_2^T)\left\{t_{2\varepsilon}\dot{\lambda}_2 + \left[\frac{\partial\lambda_2}{\partial\bar{y}_1}\right](\bar{y}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1)\right.$$

$$+ \left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right](\dot{\bar{y}}_{1\varepsilon} - \bar{t}_{1\varepsilon}G^*_1) + \left[\frac{\partial y_2}{\partial\bar{y}_1}\right](\bar{\lambda}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{\lambda}}_1)$$

$$\left. + \left[\frac{y_2}{\partial\dot{\bar{y}}_1}\right](\dot{\bar{\lambda}}_{1\varepsilon} - \bar{t}_{1\varepsilon}Q^*_1)\right\} = -a_{12}(I-\lambda_2\lambda_2^T)\dot{\lambda}_2$$

Hence,

$$B(\bar{y}_{1\varepsilon} - \bar{t}_{1\varepsilon}\dot{\bar{y}}_1) + C(\dot{\bar{y}}_{1\varepsilon} - \bar{t}_{1\varepsilon}G^*_1) + \Delta V_2(I-\lambda_2\lambda_2^T)\left(\left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_{1\varepsilon} + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]\dot{\lambda}_{1\varepsilon}\right)$$

$$+ \Delta m_{1\varepsilon}\frac{c\Delta m_2}{\bar{m}_1\bar{m}_2}\lambda_2 - \Delta m_{2\varepsilon}\frac{c}{\bar{m}_2}\lambda_2 = -(a_{12} + \Delta V_2\Delta m_2)(I-\lambda_2\lambda_2^T)\dot{\lambda}_2$$

where

$$B = \left[\frac{\partial \dot{y}_2}{\partial \bar{y}_1}\right] + \Delta V_2(I-\lambda_2\lambda_2^T)\left[\frac{\partial \lambda_2}{\partial \bar{y}_1}\right]$$

and

$$C = \left[\frac{\partial \dot{y}_2}{\partial \dot{\bar{y}}_1}\right] + \Delta V_2(I-\lambda_2\lambda_2^T)\left[\frac{\partial \lambda_2}{\partial \dot{\bar{y}}_1}\right]$$

Therefore,

$$-a_{11}B\lambda_1 + C[-\frac{c}{\bar{m}_1}\Delta m_{1\varepsilon}\lambda_1 + \Delta V_1\lambda_{1\varepsilon} + a_{11}(I-\lambda_1\lambda_1^T)\dot{\lambda}_1]$$

$$+ \Delta V_2(I-\lambda_2\lambda_2^T)\left(\left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_{1\varepsilon} + \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]\dot{\lambda}_{1\varepsilon}\right)$$

$$+ \Delta m_{1\varepsilon}\frac{c\Delta m_2}{\bar{m}_1\bar{m}_2}\lambda_2 - \Delta m_{2\varepsilon}\frac{c}{\bar{m}_2}\lambda_2 = -a_{22}(I-\lambda_2\lambda_2^T)\dot{\lambda}_2$$

$$\Delta m_{1\varepsilon}\frac{c}{\bar{m}_1}(-C\lambda_1 + \frac{\Delta m_2}{\bar{m}_2}\lambda_2) - \Delta m_{2\varepsilon}\frac{c}{\bar{m}_2}\lambda_2 + \left\{\Delta V_1 C + \Delta V_2(I-\lambda_2\lambda_2^T)\left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\right\}\lambda_{1\varepsilon}$$

$$+ \Delta V_2(I-\lambda_2\lambda_2^T)\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]\dot{\lambda}_{1\varepsilon}$$

$$= a_{11}B\lambda_1 - a_{11}C(I-\lambda_1\lambda_1^T)\dot{\lambda}_1 - a_{22}(I-\lambda_2\lambda_2^T)\dot{\lambda}_2 \qquad (12\text{-}4)$$

Writing equations (12-1) through (12-4) in matrix form, we have

$$
\begin{bmatrix}
0 & 0 & \lambda_1^T & 0 \\[2ex]
-\frac{c}{\bar{m}_1}\lambda_2^T\left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right]\lambda_1 & 0 & \lambda_2^T\left(\Delta V_1\left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right]+\left[\frac{\partial y_2}{\partial\bar{y}_1}\right]\right) & \lambda_2^T\left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right] \\[2ex]
-\frac{c}{\bar{m}_1}\left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]\lambda_1 & 0 & \Delta V_1\left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right] & 0 \\[2ex]
\frac{c}{\bar{m}_1}\left(-C\lambda_1+\frac{\Delta m_2}{\bar{m}_2}\lambda_2\right) & -\frac{c}{\bar{m}_2}\lambda_2 & \Delta V_1 C+\Delta V_2(I-\lambda_2\lambda_2^T)\left[\frac{\partial y_2}{\partial\bar{y}_1}\right] & \Delta V_2(I-\lambda_2\lambda_2^T)\left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]
\end{bmatrix}
\begin{bmatrix}
\Delta m_{1\varepsilon} \\[2ex] \Delta m_{2\varepsilon} \\[2ex] \lambda_{1\varepsilon} \\[2ex] \dot{\lambda}_{1\varepsilon}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 \\[2ex]
a_{11}\lambda_2^T\left\{\left[\frac{\partial\lambda_2}{\partial\bar{y}_1}\right]\lambda_1-\left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right](I-\lambda_1\lambda_1^T)\dot{\lambda}_1\right\}-(\Delta m_1 U^*_1+\Delta m_2 U^*_2) \\[2ex]
a_{11}\left[\frac{\partial y_2}{\partial\bar{y}_1}\right]\lambda_1-a_{11}\left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right](I-\lambda_1\lambda_1^T)\dot{\lambda}_1+a_{22}\lambda_2 \\[2ex]
a_{11}B\lambda_1-a_{11}C(I-\lambda_1\lambda_1^T)\dot{\lambda}_1-a_{22}(I-\lambda_2\lambda_2^T)\dot{\lambda}_2
\end{bmatrix}
$$

The following row operations will be performed:

- The third row will be multiplied by $\left[\frac{\partial y_2}{\partial\dot{\bar{y}}_1}\right]^{-1}$.
- Then $-C$ times the third row will be added to the fourth row.
- Then $-\lambda_2^T\left[\frac{\partial\lambda_2}{\partial\dot{\bar{y}}_1}\right]$ times the third row will be added to the second row.

The result is

$$
\begin{bmatrix}
0 & 0 & \lambda_1^T & 0 \\
0 & 0 & \lambda_2^T\left[\dfrac{\partial y_2}{\partial \bar{y}_1}\right] & \lambda_2^T\left[\dfrac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] \\
-\dfrac{c}{\bar{m}_1}\lambda_1 & 0 & \Delta V_1 I & 0 \\
\dfrac{c\Delta m_2}{\bar{m}_1\bar{m}_2}\lambda_2 & -\dfrac{c}{\bar{m}_2}\lambda_2 & \Delta V_2(I-\lambda_2\lambda_2^T)\left[\dfrac{\partial y_2}{\partial \bar{y}_1}\right] & \Delta V_2(I-\lambda_2\lambda_2^T)\left[\dfrac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]
\end{bmatrix}
\begin{bmatrix}
\Delta m_{1\varepsilon} \\ \Delta m_{2\varepsilon} \\ \lambda_{1\varepsilon} \\ \dot{\lambda}_{1\varepsilon}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 \\
-\lambda_2^T\left[\dfrac{\partial \lambda_2}{\partial \dot{\bar{y}}_1}\right] D + a_{11}\lambda_2^T\left[\dfrac{\partial \lambda_2}{\partial \bar{y}_1}\right]\lambda_1 - (\Delta m_1 U^*_1 + \Delta m_2 U^*_2) \\
D - a_{11}(I-\lambda_1\lambda_1^T)\dot{\lambda}_1 \\
-CD + a_{11}B\lambda_1 - a_{22}(I-\lambda_2\lambda_2^T)\dot{\lambda}_2
\end{bmatrix}
$$

where

$$
D = \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\left(a_{11}\left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_1 + a_{22}\lambda_2\right) = (a_{11} + a_{22})\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\lambda_2 - a_{11}\dot{\lambda}_1
$$

The latter development makes use of equations (A-3) in Appendix A. By inspection, the inverse of the coefficient matrix is

$$
\begin{bmatrix}
\dfrac{\bar{m}_1\Delta V_1}{c} & 0 & -\dfrac{\bar{m}_1}{c}\lambda_1^T & 0 \\
\dfrac{\Delta m_2}{c} & 0 & -\dfrac{\Delta m_2}{c}\lambda_1^T & -\dfrac{\bar{m}_2}{c}\lambda_2^T \\
\lambda_1 & 0 & \dfrac{1}{\Delta V_1}(I-\lambda_1\lambda_1^T) & 0 \\
-\left[\dfrac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\left[\dfrac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_1 & \left[\dfrac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\lambda_2 & -\dfrac{1}{\Delta V_1}\left[\dfrac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\left[\dfrac{\partial y_2}{\partial \bar{y}_1}\right](I-\lambda_1\lambda_1^T) & \dfrac{1}{\Delta V_2}\left[\dfrac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}(I-\lambda_2\lambda_2^T)
\end{bmatrix}
$$

Therefore, multiplying the inverse matrix by the column of constants, we have

$$\Delta m_{1\varepsilon} = -\frac{\bar{m}_1}{c}\lambda_1{}^T D$$

$$\Delta m_{2\varepsilon} = -\frac{\Delta m_2}{c}\lambda_1{}^T D - \frac{\bar{m}_2}{c}\lambda_2{}^T(-CD + a_{11}B\lambda_1)$$

$$\lambda_{1\varepsilon} = \frac{1}{\Delta V_1}(I-\lambda_1\lambda_1{}^T)(D - a_{11}\dot{\lambda}_1)$$

$$\dot{\lambda}_{1\varepsilon} = \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\lambda_2\left\{-\lambda_2{}^T\left[\frac{\partial \lambda_2}{\partial \dot{\bar{y}}_1}\right]D + a_{11}\lambda_2{}^T\left[\frac{\partial \lambda_2}{\partial \bar{y}_1}\right]\lambda_1 - (\Delta m_1 U^*{}_1 + \Delta m_2 U^*{}_2)\right\}$$

$$- \frac{1}{\Delta V_1}\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\left[\frac{\partial y_2}{\partial \bar{y}_1}\right](I-\lambda_1\lambda_1{}^T)(D - a_{11}\dot{\lambda}_1)$$

$$+ \frac{1}{\Delta V_2}\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}(I-\lambda_2\lambda_2{}^T)(-CD + a_{11}B\lambda_1 - a_{22}\dot{\lambda}_2)$$

Substitution of the expressions for C, B, and D yields

$$\Delta m_{1\varepsilon} = \frac{\bar{m}_1}{c}\left\{a_{11}\lambda_1{}^T\dot{\lambda}_1 - (a_{11} + a_{22})\lambda_1{}^T\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\lambda_2\right\} \qquad (12\text{-}5)$$

$$\Delta m_{2\varepsilon} = \frac{\Delta m_2}{\bar{m}_1}\Delta m_{1\varepsilon} + \frac{\bar{m}_2}{c}\lambda_2{}^T\left\{\left[\frac{\partial \dot{y}_2}{\partial \dot{\bar{y}}_1}\right]\left[(a_{11} + a_{22})\left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1}\lambda_2 - a_{11}\dot{\lambda}_1\right]\right.$$

$$\left. - a_{11}\left[\frac{\partial \dot{y}_2}{\partial \bar{y}_1}\right]\lambda_1\right\}$$

In Appendix A it is shown that

$$\dot{\lambda}_2 = \left[\frac{\partial \dot{y}_2}{\partial \bar{y}_1}\right]\lambda_1 + \left[\frac{\partial \dot{y}_2}{\partial \dot{\bar{y}}_1}\right]\dot{\lambda}_1.$$

Therefore,

$$\Delta m_{2\varepsilon} = \frac{\Delta m_2}{\bar{m}_1} \Delta m_{1\varepsilon} + \frac{\bar{m}_2}{c} \left\{ (a_{11} + a_{22}) \lambda_2{}^T \left[\frac{\partial \dot{y}_2}{\partial \dot{\bar{y}}_1}\right] \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} \lambda_2 - a_{11} \lambda_2{}^T \dot{\lambda}_2 \right\} \quad (12\text{-}6)$$

Moreover,

$$\lambda_{1\varepsilon} = \frac{1}{\Delta V_1} (I - \lambda_1 \lambda_1{}^T) \left\{ (a_{11} + a_{22}) \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} \lambda_2 - 2a_{11} \dot{\lambda}_1 \right\} \quad (12\text{-}7)$$

$$\begin{aligned} \dot{\lambda}_{1\varepsilon} = \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} \Bigg\{ & - \left[\lambda_2 \lambda_2{}^T \left[\frac{\partial \lambda_2}{\partial \dot{\bar{y}}_1}\right] + \frac{1}{\Delta V_1} \left[\frac{\partial y_2}{\partial \bar{y}_1}\right] (I - \lambda_1 \lambda_1{}^T) + \frac{1}{\Delta V_2} (I - \lambda_2 \lambda_2{}^T) C \right] D \\ & + a_{11} \lambda_2 \lambda_2{}^T \left[\frac{\partial \lambda_2}{\partial \bar{y}_1}\right] \lambda_1 + \frac{a_{11}}{\Delta V_1} \left[\frac{\partial y_2}{\partial \bar{y}_1}\right] (I - \lambda_1 \lambda_1{}^T) \dot{\lambda}_1 \\ & + \frac{1}{\Delta V_2} (I - \lambda_2 \lambda_2{}^T) (a_{11} B \lambda_1 - a_{22} \dot{\lambda}_2) - (\Delta m_1 \lambda_1{}^T \dot{\lambda}_1 + \Delta m_2 \lambda_2{}^T \dot{\lambda}_2) \lambda_2 \Bigg\} \end{aligned}$$

$$\begin{aligned} \dot{\lambda}_{1\varepsilon} = \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} \Bigg\{ & \Bigg[\frac{1}{\Delta V_1} \left[\frac{\partial y_2}{\partial \bar{y}_1}\right] (I - \lambda_1 \lambda_1{}^T) + \left[\frac{\partial \lambda_2}{\partial \dot{\bar{y}}_1}\right] \\ & + \frac{1}{\Delta V_2} (I - \lambda_2 \lambda_2{}^T) \left[\frac{\partial \dot{y}_2}{\partial \dot{\bar{y}}_1}\right] \Bigg] \left[a_{11} \dot{\lambda}_1 - (a_{11} + a_{22}) \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} \lambda_2 \right] \\ & + \frac{a_{11}}{\Delta V_1} \left[\frac{\partial y_2}{\partial \bar{y}_1}\right] (I - \lambda_1 \lambda_1{}^T) \dot{\lambda}_1 + a_{11} \left[\frac{\partial \lambda_2}{\partial \bar{y}_1}\right] \lambda_1 \\ & + \frac{1}{\Delta V_2} (I - \lambda_2 \lambda_2{}^T) \left(a_{11} \left[\frac{\partial \dot{y}_2}{\partial \bar{y}_1}\right] \lambda_1 - a_{22} \dot{\lambda}_2 \right) - (\Delta m_1 \lambda_1{}^T \dot{\lambda}_1 + \Delta m_2 \lambda_2{}^T \dot{\lambda}_2) \lambda_2 \Bigg\} \quad (12\text{-}8) \end{aligned}$$

Equations (12-5) through (12-8) give the first-order corrections for the burn-coast-burn rendezvous problem with any gravitational acceleration $G(t, y)$. In employing the latter equations, the coefficients a_{11} and a_{22} should be computed by means of the series expansions provided in Appendix B.

In setting up a numerical example to test equations (12-5) through (12-8), a constant gravitational field (i.e., constant G) was chosen. Then $Q = 0$, $\dot{\lambda}_2 = \dot{\lambda}_1 = \text{const.}$, and $\lambda_2(0) = \lambda_1(0) + \Delta T\, \dot{\lambda}_1(0)$, where $\Delta T = t_2(0) - \bar{t}_1(0)$. Therefore, $\dot{\lambda}_1(0) = \frac{1}{\Delta T}[\lambda_2(0) - \lambda_1(0)]$. Furthermore, $y_2(0) = \bar{y}_1(0) + \Delta T\, \dot{\bar{y}}_1(0) + \frac{\Delta T^2}{2} G$, $\dot{y}_2(0) = \dot{\bar{y}}_1(0) + \Delta T G$,

$$\left[\frac{\partial y_2}{\partial \bar{y}_1}\right] = I, \quad \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right] = \Delta T I, \quad \left[\frac{\partial \dot{y}_2}{\partial \bar{y}_1}\right] = 0, \quad \left[\frac{\partial \dot{y}_2}{\partial \dot{\bar{y}}_1}\right] = I, \quad \left[\frac{\partial \lambda_2}{\partial \bar{y}_1}\right] = 0, \text{ etc.}$$

In the constant G case, equations (12-5) through (12-8) reduce to

$$\left.\begin{aligned}
\Delta m_{1\varepsilon} &= -\frac{\bar{m}_1}{c\Delta T}\left(a_{11} + a_{22}\lambda_1^T\lambda_2\right) \\
\Delta m_{2\varepsilon} &= \frac{(a_{11} - a_{22})\bar{m}_2}{c\Delta T}\left(\lambda_1^T\lambda_2 - 1\right) - \Delta m_{1\varepsilon} \\
\lambda_{1\varepsilon} &= \frac{a_{11} - a_{22}}{\Delta T \Delta V_1}\left[(\lambda_1^T\lambda_2)\lambda_1 - \lambda_2\right] \\
\dot{\lambda}_{1\varepsilon} &= -\frac{1}{\Delta T}\lambda_{1\varepsilon} - \frac{1}{\Delta T^2}\left\{\frac{a_{11} - a_{22}}{\Delta V_2}\left[\lambda_1 - (\lambda_1^T\lambda_2)\lambda_2\right]\right. \\
&\qquad \left. + (\bar{m}_2 - 2\bar{m}_1 + m_1)\left[1 - (\lambda_1^T\lambda_2)\right]\lambda_2\right\}
\end{aligned}\right\} \qquad (12\text{-}9)$$

Before presenting numerical results obtained from equations (12-9), we will consider the determination of the impulsive solution for the rendezvous problem. First, the orbit through points y_1 and $\bar{y}_2$, corresponding to a time duration $\bar{t}_2$, is calculated (letting $t_1 = 0$). Then $\dot{\bar{y}}_1$ and $\dot{y}_2$, the velocities at the terminal points of the orbit, are found. Then one determines

$$\lambda_1 = L^*_1 = \frac{1}{\Delta V_1} \Delta \dot{y}_1, \quad \lambda_2 = L^*_2 = \frac{1}{\Delta V_2} \Delta \dot{y}_2$$

where $\Delta V_i = |\Delta \dot{y}_i| = |\dot{\bar{y}}_i - \dot{y}_i|$. As shown in Appendix A,

$$\dot{\lambda}_1 = \left[\frac{\partial y_2}{\partial \dot{\bar{y}}_1}\right]^{-1} \left(\lambda_2 - \left[\frac{\partial y_2}{\partial \bar{y}_1}\right]\lambda_1\right)$$

$$\dot{\lambda}_2 = \left[\frac{\partial \dot{y}_2}{\partial \bar{y}_1}\right]\lambda_1 + \left[\frac{\partial \dot{y}_2}{\partial \dot{\bar{y}}_1}\right]\dot{\lambda}_1$$

Consider a two-dimensional, burn-coast-burn rendezvous problem in which

$t_1 = 0$, $\bar{t}_2 = 380$ sec, $m_1 = .17 \times 10^5$ kg-sec^2/m

$G^T = (4.63 \text{ m/sec}^2, 8.00 \text{ m/sec}^2)$

$y_1^T = (1800 \text{ km}, 6300 \text{ km})$

$\dot{y}_1^T = (6.8 \text{ km/sec}, - 2.0 \text{ km/sec})$

$y_2^T = (4131.02 \text{ km}, 5014.62 \text{ km})$

$\dot{\bar{y}}_2^T = (5.28352 \text{ km/sec}, -5.05836 \text{ km/sec})$

$c = 4100$ m/sec, $\beta = 22$ kg-sec/m

(The vehicle is a Saturn S-IVB.)

An IBM 1130 data processing system was employed in the solution. The results are as follows:

$\bar{t}_{1\ TRUE} = 50.00$ sec

$\bar{t}_{1\ IMP} = 46.47$ (7.1 percent error)

$\bar{t}_{1\ COR} = 49.64$ (.7 percent error)

$t_{2\ TRUE} = 350.00$ sec

$t_{2\ IMP} = 352.47$ (.71 percent error)

$t_{2\ COR} = 350.60$ (.17 percent error)

$\bar{m}_{2\ TRUE} = .15240 \times 10^5$ kg-sec^2/m

$\bar{m}_{2\ IMP} = .15372 \times 10^5$ (.87 percent error)

$\bar{m}_{2\ COR} = .15261 \times 10^5$ (.14 percent error)

$\lambda_1^T{}_{TRUE} = (.8, .6)$

$\lambda_1^T{}_{IMP} = (.8414, .5404)$
ERROR = (5.2 percent, 9.9 percent)

$\lambda_1^T{}_{COR} = (.7992, .6060)$
ERROR = (.1 percent, 1.0 percent)

$\dot{\lambda}_1^T{}_{TRUE} = (-.002, -.004)$

$\dot{\lambda}_1^T{}_{IMP} = (-.001733, -.004009)$
ERROR = (13.3 percent, .2 percent)

$\dot{\lambda}_1^T{}_{COR} = (-.001979, -.004065)$
ERROR = (1.0 percent, 1.4 percent)

A FORTRAN listing of program RENDZ is given in Appendix C. The program determines an impulsive solution and a first-order correction for the three-dimensional, constant G, burn-coast-burn, final time-fixed, rendezvous problem. The input data is C = c, B = β, TBAR2 = $\bar{t}_2$, M1 = m_1, G = G, Y1 = y_1, YD1 = $\dot{y}_1$, YBAR2 = $\bar{y}_2$, YDBAR2 = $\dot{\bar{y}}_2$.

The output data is TB1IM = impulsive value of $\bar{t}_1$, TB1C = corrected value of $\bar{t}_1$, T2IM = impulsive value of t_2, T2C = corrected value of t_2, L1IM = impulsive approximation to λ_1, LD1IM = impulsive approximation to $\dot{\lambda}_1$, L1C = corrected λ_1 vector, LD1C = corrected $\dot{\lambda}_1$ vector.

Section XIII

SUMMARY

A procedure for obtaining first and higher order corrections, to impulsive solutions to space flight optimization problems, has been developed. The steps to be taken in deriving formulas for the first-order corrections (for example) are:

- Derive the transversality conditions for the problem under consideration.
- Differentiate all boundary conditions, including transversality conditions, with respect to $\varepsilon = 1/\beta$.
- Express all derivatives (with respect to ε) in terms of $\lambda_{I\varepsilon}$, $\dot{\lambda}_{I\varepsilon}$, $K_{I\varepsilon}$, $t_{1\varepsilon}$, $t_{2\varepsilon}$, ..., $t_{N\varepsilon}$, $\Delta m_{1\varepsilon}$, $\Delta m_{2\varepsilon}$, ..., $\Delta m_{N\varepsilon}$, $t_{F\varepsilon}$. In so doing one must employ equations (6-8) for the changes in the derivatives over coast arcs and equations (9-1) for the changes over thrust arcs (for $\varepsilon = 0$). The new development in Appendix A may be used to obtain closed expressions for the transition matrices, appearing in equations (6-8), for the case of an inverse square gravitational field.
- Write the resulting systems of equations in matrix form as shown in the example problems of Sections X, XI, and XII.
- Reduce the system of equations by means of elementary row transformations.
- Invert the resulting system of equations, analytically if possible.
- Obtain the first-order corrections; for example, the correction to $\lambda_I(0)$ is simply $\frac{1}{\beta_{TRUE}} \lambda_{I\varepsilon}(0)$.

Section XIV

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Appendix A

FORMULAS FOR ADJOINT VARIABLES AND TRANSITION MATRICES OVER COAST ARCS

Consider a coast arc over the time interval $[\bar{t}_k, t_{k+1}]$ and assume that the state variables are known at times $\bar{t}_k$ and t_{k+1}, and that the Lagrange multipliers are known at time $\bar{t}_k$. This appendix is concerned with the determination of the Lagrange multipliers and the transition matrices (i.e., partial derivative matrices) over the coast arc. It first makes some general observations applicable to any gravitational acceleration function G(t, y) and then restricts the problem to an inverse square gravitational field. Then a concise system of equations for the Lagrange multipliers (adjoint variables) are derived for the case of non-parabolic motion, from which the transition matrices can be obtained in a straightforward manner. Finally, a computer program - based on the aforementioned method for obtaining transition matrices - is briefly discussed, and a FORTRAN listing of the program is provided.

Consider the adjoint differential equations of the form

$$\ddot{p} = G_y \, p \tag{A-1}$$

where p is a 3 x 1 column vector or a 3 x 3 matrix. Since equations (A-1) are linear and homogeneous in p, there exist matrices A(t) and B(t) such that

$$\begin{aligned} p(t) &= A \, \bar{p}_k + B \, \dot{\bar{p}}_k \\ \dot{p}(t) &= \dot{A} \, \bar{p}_k + \dot{B} \, \dot{\bar{p}}_k \end{aligned} \tag{A-2}$$

for $\bar{t}_k \leq t \leq t_{k+1}$. In other words, p(t) and $\dot{p}(t)$ are linearly dependent upon the initial values.

Since equations (A-1) are the equations of variation for the system $\ddot{y} = G(t, y)$, the matrix $\left[\frac{\partial y}{\partial \bar{y}_k}\right]$ is a solution to the system (A-1) with initial conditions

$$\bar{p}_k = \left[\frac{\partial y}{\partial \bar{y}_k}\right]\Bigg|_{\bar{t}_k} = I$$

$$\dot{p}_k = \frac{d}{dt}\left[\frac{\partial y}{\partial \bar{y}_k}\right]\Bigg|_{\bar{t}_k} = 0$$

Therefore, from equations (A-2) it follows that

$$\left[\frac{\partial y}{\partial \bar{y}_k}\right] = A \cdot I + B \cdot 0 = A.$$

Moreover, $\left[\frac{\partial y}{\partial \dot{\bar{y}}_k}\right]$ is a solution to equations (A-1) with initial conditions 0 and I so that

$$\left[\frac{\partial y}{\partial \dot{\bar{y}}_k}\right] = A \cdot 0 + B \cdot I = B.$$

Since the Lagrange multiplier vector λ is a solution to equations (A-1), it follows - from equations (A-2) and the observations we have just made - that

$$\lambda(t) = \left[\frac{\partial y}{\partial \bar{y}_k}\right]\bar{\lambda}_k + \left[\frac{\partial y}{\partial \dot{\bar{y}}_k}\right]\dot{\bar{\lambda}}_k$$

$$\dot{\lambda}(t) = \left[\frac{\partial \dot{y}}{\partial \bar{y}_k}\right]\bar{\lambda}_k + \left[\frac{\partial \dot{y}}{\partial \dot{\bar{y}}_k}\right]\dot{\bar{\lambda}}_k \tag{A-3}$$

It is evident from equations (A-3) that

$$\left[\frac{\partial \lambda}{\partial \bar{\lambda}_k}\right] = \left[\frac{\partial y}{\partial \bar{y}_k}\right], \quad \left[\frac{\partial \lambda}{\partial \dot{\bar{\lambda}}_k}\right] = \left[\frac{\partial y}{\partial \dot{\bar{y}}_k}\right], \quad \left[\frac{\partial \dot{\lambda}}{\partial \bar{\lambda}_k}\right] = \left[\frac{\partial \dot{y}}{\partial \bar{y}_k}\right], \quad \left[\frac{\partial \dot{\lambda}}{\partial \dot{\bar{\lambda}}_k}\right] = \left[\frac{\partial \dot{y}}{\partial \dot{\bar{y}}_k}\right].$$

Furthermore,

$$\left[\frac{\partial \lambda}{\partial \bar{y}_k}\right] = \frac{\partial}{\partial \bar{y}_k}\left\{\left[\frac{\partial y}{\partial \bar{y}_k}\right]\bar{\lambda}_k + \left[\frac{\partial y}{\partial \dot{\bar{y}}_k}\right]\dot{\bar{\lambda}}_k\right\}$$

and similarly for $\left[\frac{\partial \lambda}{\partial \dot{\bar{y}}_k}\right]$, $\left[\frac{\partial \dot{\lambda}}{\partial \bar{y}_k}\right]$, $\left[\frac{\partial \dot{\lambda}}{\partial \dot{\bar{y}}_k}\right]$.

The first and higher order derivatives of y and $\dot{y}$ with respect to λ_k and $\dot{\lambda}_k$ are obviously all zero. It is also evident that

$$\frac{\partial}{\partial \bar{y}_k^{(i)}}\left[\frac{\partial \lambda}{\partial \bar{\lambda}_k}\right] = \frac{\partial}{\partial \bar{y}_k^{(i)}}\left[\frac{\partial y}{\partial \bar{y}_k}\right]$$

$$\frac{\partial}{\partial \bar{y}_k^{(i)}}\left[\frac{\partial \lambda}{\partial \bar{y}_k}\right] = \frac{\partial}{\partial \bar{y}_k^{(i)}} \frac{\partial}{\partial \bar{y}_k}\left\{\left[\frac{\partial y}{\partial \bar{y}_k}\right]\bar{\lambda}_k + \left[\frac{\partial y}{\partial \bar{y}_k}\right]\dot{\bar{\lambda}}_k\right\}$$

and so on, where (i) indicates ith component.

Now we will turn from a general gravitational field to the special case of an inverse square field to derive analytic expressions for λ_{k+1} and $\dot{\lambda}_{k+1}$ in terms of $\bar{\lambda}_k$, $\dot{\bar{\lambda}}_k$, $\bar{y}_k$, $\dot{\bar{y}}_k$, y_{k+1}, and $\dot{y}_{k+1}$.

The desired expressions can be obtained from the following constants of integration for an inverse square field:

$$\bar{L}^*_k = -\dot{y} \times \lambda + y \times \dot{\lambda} \quad \text{(ref. 19)}$$

$$\bar{M}^*_k = [(\dot{y}^T\dot{y} - \frac{\mu}{|y|})I + \frac{\mu}{|y|^3} yy^T]\lambda + [-(\dot{y}^T y)I + 2y\dot{y}^T - \dot{y}y^T]\dot{\lambda} \quad \text{(refs. 6 and 20)}$$

$$\bar{b}_k = \dot{y}^T\lambda + 2y^T\dot{\lambda} + 3\bar{H}_k(t - \bar{t}_k) \quad \text{(ref. 21)}$$

where $\bar{L}^*_k$, $\bar{M}^*_k$, and $\bar{b}_k$ signify the constants evaluated at time $\bar{t}_k$, and $\bar{H}_k$ is the Hamiltonian $H \equiv -\frac{\mu}{|y|^3}(y^T\lambda) - \dot{y}^T\dot{\lambda}$ (on a coast arc) evaluated at $\bar{t}_k$.
(The Hamiltonian is also a constant.)

Let S(p) denote a matrix

$$\begin{bmatrix} 0 & -p^{(3)} & p^{(2)} \\ p^{(3)} & 0 & -p^{(1)} \\ -p^{(2)} & p^{(1)} & 0 \end{bmatrix}$$

such that S(p)q = p x q for any 3 x 1 vectors p and q. We observe that $S^T(p) = -S(p)$ and $S(p)S(q) = qp^T - (q^Tp)I$.

Then the constants of integration can be written in matrix form as:

$$\begin{bmatrix} -S(\dot{y}) & S(y) \\ (\dot{y}^T\dot{y} - \frac{\mu}{|y|})I + \frac{\mu}{|y|^3} yy^T & -(\dot{y}^Ty)I + 2y\dot{y}^T - \dot{y}y^T \\ \dot{y}^T & 2\bar{y}^T \end{bmatrix} \begin{bmatrix} \lambda \\ \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \bar{L}^*_k \\ \bar{M}^*_k \\ \bar{b}_k - 3\bar{H}_k(t-\bar{t}_k) \end{bmatrix} \qquad \text{(A-4)}$$

Before proceeding, we define the following functions:

$L \triangleq y \times \dot{y}$ (angular momentum)

$M \triangleq \dot{y} \times L - \frac{\mu}{|y|} y$ (perifocus vector)

$E \triangleq \frac{1}{2} (\dot{y}^T\dot{y}) - \frac{\mu}{|y|}$ (energy)

(NOTE: L, M, and E are also constant over a coast arc)

Now we will premultiply both members of the matrix equation (A-4) by the following sequence of 7 x 7 row operation matrices:

$$\begin{bmatrix} I & 0 & 0 \\ \frac{2E}{|L|^2} yL^T & I - \frac{1}{|L|^2} yM^T & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} I & 0 & 0 \\ -S(\dot{y}) & I & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} I & -\frac{|y|}{\mu} S(y) & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \frac{|y|}{\mu} \dot{y}^T & 1 \end{bmatrix},$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & -\frac{1}{2E} \frac{\mu}{|y|} \dot{y} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(assuming } E \neq 0\text{; i.e., a non-parabolic orbit)}$$

The resulting reduced matrix equation is

$$\begin{bmatrix} 0 & S(y) \\ -\frac{\mu}{|y|} I & 0 \\ 0 & -\frac{2|y|}{\mu} \bar{E}_k y^T \end{bmatrix} \begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \bar{L}^*_k - \frac{|y|}{\mu} \dot{y} + \bar{M}^*_k + \frac{|y|}{\mu} \bar{H}_k \bar{L}_k + \frac{|y|}{\mu} \dot{y} \times (\dot{y} \times \bar{L}^*_k) \\ \bar{M}^*_k + \bar{H}_k y - \dot{y} \times L^*_k - \frac{|y|}{\mu} Z\dot{y} \\ \frac{2 \bar{E}_k |y|}{\mu} Z \end{bmatrix} \qquad \text{(A-5)}$$

where

$$Z = \frac{1}{2\bar{E}_k} \{\dot{y}^T \bar{M}^*_k + (y^T \dot{y})\bar{H}_k + \frac{\mu}{|y|} [\bar{b}_k - 3\bar{H}_k (t - \bar{t}_k)]\}.$$

If we multiply the coefficient matrix, of the system (A-5) of matrix equations, by the matrix

$$\begin{bmatrix} 0 & -\frac{|y|}{\mu} I & 0 \\ -\frac{1}{|y|^2} S(y) & 0 & -\frac{\mu}{2\bar{E}_k |y|^3} y \end{bmatrix} \qquad \text{(A-6)}$$

on the lefthand side, we obtain an identity matrix. Therefore, the vector $(\lambda, \dot{\lambda})^T$ is equal to the product of the matrix (A-6) and the column of constants of equation (A-5). Thus,

$$\frac{\mu}{|y|}\lambda = \dot{y} \times \bar{L}^*_k - \bar{M}^*_k - \bar{H}_k y + Z\dot{y}$$

$$|y|^2\dot{\lambda} = - y\,\bar{L}^*_k - y \times (\dot{y} \times \lambda) - Zy \qquad \text{(A-7)}$$

Equations (A-7) may be used to compute λ and $\dot{\lambda}$ at any time during a coasting phase, given the corresponding values of the state, the time t, $\bar{M}^*_k$, $\bar{H}_k$, $\bar{L}^*_k$, $\bar{b}_k$, and $\bar{E}_k$.

From equations (A-3) it is evident that if one sets $\lambda_k = I$ and $\dot{\lambda}_k = 0$ in equations (A-7) (considering λ_k and $\dot{\lambda}_k$ as 3 x 3 matrices), one will obtain the state transition matrices:

$$\left[\frac{\partial y}{\partial \bar{y}_k}\right] \equiv \lambda, \quad \left[\frac{\partial \dot{y}}{\partial \bar{y}_k}\right] \equiv \dot{\lambda}\ .$$

Moreover, if one sets $\lambda_k = 0$ and $\dot{\lambda}_k = I$, one will obtain

$$\left[\frac{\partial y}{\partial \dot{\bar{y}}_k}\right] \equiv \lambda, \quad \left[\frac{\partial \dot{y}}{\partial \dot{\bar{y}}_k}\right] \equiv \dot{\lambda}\ .$$

The mathematical expressions for the latter matrices will not be written out in this report except for $\left[\frac{\partial y}{\partial \dot{\bar{y}}_k}\right]$ as an example. Thus, we write the first of the equations (A-7) as follows:

$$\frac{\mu}{|y|}\lambda = \dot{y} \times \bar{L}^*_k - \bar{M}^*_k - \bar{H}_k y + \frac{1}{2\bar{E}_k}\left\{(\dot{y}\dot{y}^T)\bar{M}^*_k + \bar{H}_K[y^T\dot{y} - \frac{3\mu}{|y|}(t-\bar{t}_k)]\dot{y} + \frac{\mu}{|y|}\,\bar{b}_k\dot{y}\right\}$$

$$= S(\dot{y})\bar{L}^*_k + \left[\frac{1}{2\bar{E}_k}(\dot{y}\dot{y}^T) - I\right]\bar{M}^*_k + \frac{\mu}{2\bar{E}_k|y|}\,\bar{b}_k\dot{y}$$

$$+ \bar{H}_k\{-y + \frac{1}{2\bar{E}_k}\,[y^T\dot{y} - \frac{3\mu}{|y|}\,(t-\bar{t}_k)]\dot{y}\}$$

$$= S(\dot{y})[-S(\dot{\bar{y}}_k)\bar{\lambda}_k + S(\bar{y}_k)\dot{\bar{\lambda}}_k] + \frac{\mu}{2\bar{E}_k|y|}\left[\left(\dot{y}\dot{\bar{y}}^T{}_k\right)\bar{\lambda}_k + 2(\dot{y}\bar{y}^T{}_k)\dot{\bar{\lambda}}_k\right]$$

$$+ \left[\frac{1}{2\bar{E}_k}(\dot{y}\dot{y}^T) - I\right]\left\{\left[\left(\dot{\bar{y}}^T{}_k\dot{\bar{y}}_k - \frac{\mu}{|\bar{y}_k|}\right)I + \frac{\mu}{|\bar{y}_k|^3}\bar{y}_k\bar{y}^T{}_k\right]\bar{\lambda}_k\right.$$

$$\left. + \left[-(\dot{\bar{y}}^T{}_k\bar{y}_k)I + 2\,\bar{y}_k\dot{\bar{y}}^T{}_k - \dot{\bar{y}}_k\bar{y}^T{}_k\right]\dot{\bar{\lambda}}_k\right\}$$

$$- \left\{-y + \frac{1}{2\bar{E}_k}[y^T\dot{y} - \frac{3\mu}{|y|}(t-\bar{t}_k)]\dot{y}\right\}\left[\frac{\mu}{|\bar{y}_k|^3}\bar{y}^T{}_k\ \bar{\lambda}_k + \dot{\bar{y}}^T{}_k\dot{\bar{\lambda}}_k\right]$$

Setting $\bar{\lambda}_k = 0$ and $\dot{\bar{\lambda}}_k = I$, we obtain

$$\left[\frac{\partial y}{\partial \dot{\bar{y}}_k}\right] = \frac{|y|}{\mu}\left\{S(\dot{y})S(\bar{y}_k) + \frac{\mu}{\bar{E}_k|y|}\dot{y}\bar{y}^T{}_k\right.$$

$$- \left(\frac{1}{2\bar{E}_k}\dot{y}\dot{y}^T - I\right)\left[(\dot{\bar{y}}^T{}_k\bar{y}_k)I - 2\bar{y}_k\dot{\bar{y}}^T{}_k + \dot{\bar{y}}_k\bar{y}^T{}_k\right]$$

$$\left. + y\dot{\bar{y}}^T{}_k - \frac{1}{2\bar{E}_k}[y^T\dot{y} - \frac{3\mu}{|y|}(t-\bar{t}_k)](\dot{y}\dot{\bar{y}}^T{}_k)\right\}$$

A FORTRAN subroutine, based upon the techniques just presented, has been developed for obtaining the first order transition matrices. The call statement for the subroutine is CALL LAM(XO, XDO, X, XD, AALO, AALDO, GM, DT, IOPT, AAL, AALD, PXX, PLX) where XO = $\bar{y}_k$, XDO = $\dot{\bar{y}}_k$ X = y, XD = $\dot{y}$, AALO = $\bar{\lambda}_k$, AALDO = $\dot{\bar{\lambda}}_k$, GM = μ, DT = $t-\bar{t}_k$, IOPT = control const., AAL = λ, AALD = $\dot{\lambda}$,

$$\mathrm{PXX} = \begin{bmatrix} \left[\dfrac{\partial y}{\partial \bar{y}_k}\right] & \left[\dfrac{\partial y}{\partial \dot{\bar{y}}_k}\right] \\ \left[\dfrac{\partial \dot{y}}{\partial \bar{y}_k}\right] & \left[\dfrac{\partial \dot{y}}{\partial \dot{\bar{y}}_k}\right] \end{bmatrix}, \quad \mathrm{PLX} = \begin{bmatrix} \left[\dfrac{\partial \lambda}{\partial \bar{y}_k}\right] & \left[\dfrac{\partial \lambda}{\partial \dot{\bar{y}}_k}\right] \\ \left[\dfrac{\partial \dot{\lambda}}{\partial \bar{y}_k}\right] & \left[\dfrac{\partial \dot{\lambda}}{\partial \dot{\bar{y}}_k}\right] \end{bmatrix}$$

If the matrices PXX and PLX are desired, they must be set equal to the 6 x 6 identity matrix before calling LAM. There are three options available:

IOPT = 1 → only AAL and AALD are computed

IOPT = 2 → PXX is also computed

IOPT = 3 → PLX is also computed.

A listing of LAM follows.

```
      SUBROUTINE LAM(XO,XDO,X,XD,AALO,AALDO,
     *               GM,DT,IOPT,
     *               AAL,AALD,PXX,PLX)
      DIMENSION XO(3),XDO(3),X(3),XD(3),AALO(3),AALDO(3)
     *          ,AAL(3),AALD(3),PXX(6,6),PLX(6,6),
     *            ALO(3),ALDO(3),AL(3),ALD(3)
     *          ,ALS(3),ALSS(3),AMS(3),AMSS(3)
     *          ,ALSP(3),AMSP(3)
      DIMENSION SL(3),SLD(3)
      EPS=1.0E-5
      R2=DOTN(X,X,3)
      R=SQRT(R2)
      R2O=DOTN(XO,XO,3)
      RO=SQRT(R2O)
      GMR=GM/R
      GMR3=GMR/R2
      GMRO=GM/RO
      GMR3O=GMRO/R2O
      V2=DOTN(XD,XD,3)
      F=.5*V2-GMR
      ALPHA=V2-GMR
      BETA=DOTN(X,XD,3)
      DO 1900 K=1,IOPT
      DO 1800 J=1,6
      GO TO (100,200,300),K
  100 DO 110 I=1,3
      ALO(I)=AALO(I)
  110 ALDO(I)=AALDO(I)
      GO TO 700
  200 DO 210 I=1,3
      ALO(I)=PXX(I,J)
  210 ALDO(I)=PXX(I+3,J)
      GO TO 800
  300 DO 310 I=1,3
      ALO(I)=PLX(I,J)
  310 ALDO(I)=PLX(I+3,J)
      GO TO 800
700   CONTINUE
      Q2O=DOTN(ALO,XO,3)
      Q3O=DOTN(ALDO,XO,3)
      CALL CROS(ALO,XDO,AMS)
      CALL CROS(XO,ALDO,ALS)
      DO 710 I=1,3
  710 ALS(I)=ALS(I)+AMS(I)
      H=-DOTN(ALDO,XDO,3)-GMR3O*Q2O
      CALL MS(XDO,ALS,H,XO,GMRO,ALO,Q3O,AMS)
      C=DOTN(ALO,XDO,3)+2.0*Q3O-3.0*H*DT
      AK=(DOTN(XD,AMS,3)+BETA*H+GMR*C)/(2.0*F)
      CALL MS(XD,ALS,1.0,AMS,H,X,-AK,AL)
      DO 720 I=1,3
  720 AL(I)=AL(I)/GMR
      Q2=DOTN(X,AL,3)
      CALL MS(X,ALS,-Q2,XD,BETA,AL,-AK,ALD)
      DO 730 I=1,3
  730 ALD(I)=-ALD(I)/R2
```

```
      GO TO 900
  800 CONTINUE
      Q2PO=DOTN(ALO,XO,3)
      Q3PO=DOTN(ALDO,XO,3)
      CALL CROS(ALO,XDO,AMSP)
      CALL CROS(XO,ALDO,ALSP)
      DO 810 I=1,3
  810 ALSP(I)=ALSP(I)+AMSP(I)
      HP=-DOTN(ALDO,XDO,3)-GMR3O*Q2PO
      CALL MS(XDO,ALSP,HP,XO,GMRO,ALO,Q3PO,AMSP)
      CP=DOTN(ALO,XDO,3)+2.0*Q3PO-3.0*HP*DT
      AKP=(DOTN(XD,AMSP,3)+BETA*HP+GMR*CP)/(2.0*E)
      CALL MS(XD,ALSP,1.0,AMSP,HP,X,-AKP,AL)
      DO 820 I=1,3
  820 AL(I)=AL(I)/GMR
      Q2P=DOTN(X,AL,3)
      CALL MS(X,ALSP,-Q2P,XD,BETA,AL,-AKP,ALD)
      DO 830 I=1,3
  830 ALD(I)=-ALD(I)/R2
  900 CONTINUE
      GO TO (1000,1200,1300),K
 1000 DO 1010 I=1,3
      AAL(I)=AL(I)
 1010 AALD(I)=ALD(I)
      GO TO 1900
 1200 DO 1210 I=1,3
      PXX(I,J)=AL(I)
 1210 PXX(I+3,J)=ALD(I)
      GO TO 1800
 1300 CONTINUE
      CALL CROS(AALO,ALDO,AMSS)
      CALL CROS(ALO,AALDO,ALSS)
      DO 1310 I=1,3
 1310 ALSS(I)=ALSS(I)+AMSS(I)
      HS=-DOTN(AALDO,ALDO,3)-GMR3O*(DOTN(AALO,ALO,3)-3.0/R2O
     *     *Q2O*Q2PO)
      Q=-GMR3O*Q2PO
      CALL MS(ALDO,ALS,H,ALO,Q,AALO,Q3O,AMSP)
      Q=DOTN(AALDO,ALO,3)
      CALL MS(XDO,ALSS,HS,XO,0.0,XO,Q,AMSS)
      DO 1320 I=1,3
 1320 AMSS(I)=AMSS(I)+AMSP(I)
      CS=DOTN(AALO,ALDO,3)+2.0*Q-3.0*HS*DT
      AKS=(DOTN(ALD,AMS,3)+(DOTN(ALD,X,3)+DOTN(AL,XD,3))*H
     *     -GMR3*Q2P*C+DOTN(XD,AMSS,3)+BETA*HS+GMR*CS
     *     +2.0*HP*AK)/(2.0*E)
      CALL MS(XD,ALSS,1.0,AMSS,HS,X,-AKS,AMSP)
      Q=-GMR3*Q2P
      CALL MS(ALD,ALS,H,AL,Q,AAL,-AK,SL)
      DO 1330 I=1,3
 1330 SL(I)=(SL(I)+AMSP(I))/GMR
      Q1P=DOTN(AL,XD,3)
      Q=Q1P-AKP
      QS=DOTN(AL,AAL,3)+DOTN(X,SL,3)
      CALL MS(X,ALSS,-QS,XD,Q,AAL,-AKS,AMSP)
      CALL MS(AL,ALS,-Q2,ALD,BETA,SL,-AK,SLD)
      DO 1340 I=1,3
```

```
1340 SLD(I)=(-SLD(I)-AMSP(I)-2.0*Q2P*AALD(I))/R2
     DO 1350 I=1,3
     PLX(I,J)=SL(I)
1350 PLX(I+3,J)=SLD(I)
1800 CONTINUE
1900 CONTINUE
     RETURN
     END
```

```
      SUBROUTINE MS(XD,ALS,H,X,A1,AL,A2,AMS)
      DIMENSION X(3),XD(3),AL(3),ALS(3),AMS(3)
      CALL CROS(XD,ALS,AMS)
      DO 10 I=1,3
   10 AMS(I)=AMS(I)-H*X(I)-A1*AL(I)-A2*XD(I)
      RETURN
      END
```

Appendix B
FUNCTIONAL DEVELOPMENT

In this appendix we will develop functions appearing in the first- and/or second-order corrections.

Development of G*

In an inverse square gravitational field,

$$G^* = -\frac{\mu}{|y|^3}\, y$$

where μ is the Gaussian gravitational constant. Since y is continuous over an impulse, in general

$$\bar{G}^*_k \to G^*_k$$

as $\varepsilon \to 0$.

Development of G^*_t

In an inverse square gravitational field

$$G^*_t = -\frac{\mu}{|y|^3}\,\dot{y} + 3\mu\,\frac{y^T\dot{y}}{|y|^5}\, y$$

$$= \frac{\mu}{|y|^3}\left(-\dot{y} + \frac{3y^T\dot{y}}{|y|^2}\, y\right)$$

In general as $\varepsilon \to 0$,

$$\bar{G}^*_{tk} = (\bar{G}_t)^*_k + G^*_{yk}\bar{\dot{y}}_k \to (G_t)^*_k + G^*_{yk}(\dot{y}_k + \Delta\dot{y}_k) \to G^*_{tk} + G^*_{yk}\,\Delta\dot{y}_k$$

Development of G^*_y

In an inverse square gravitational field,

$$G^*_y = \frac{\mu}{|y|^3}\left(-I + \frac{3}{|y|^2}\, yy^T\right)$$

where I is the identity matrix. In general, since y is continuous over an impulse,

$$\bar{G}^*_{yk} \rightarrow G^{*-}_{yk}$$

as $\varepsilon \rightarrow 0$.

Development of $G^*_{k\varepsilon}$

In an inverse square field,

$$G^*_{k\varepsilon} = \frac{\mu}{|y_k|^3} \left(-y_{k\varepsilon} + \frac{3y^T_k y_{k\varepsilon}}{|y_k|^2} y_k\right)$$

In general as $\varepsilon \rightarrow 0$,

$$\bar{G}^*_{k\varepsilon} = \bar{G}^*_{yk} \, \bar{y}_{k\varepsilon} + \bar{t}_{k\varepsilon}(\bar{G}_t)^*_k$$

$$\rightarrow G^*_{yk}[y_{k\varepsilon} - \bar{m}_k \, \Delta\dot{y}_k - \Delta m_k(\dot{y}_k + cL^*_k)] + \Delta t_{k\varepsilon}(G_t)^*_k + t_{k\varepsilon}(G_t)^*_k$$

$$\rightarrow G^*_{k\varepsilon} - G^*_{yk}[\bar{m}_k \, \Delta\dot{y}_k + \Delta m_k(\dot{y}_k + cL^*_k)] - \Delta m_k(G_t)^*_k$$

$$\rightarrow G^*_{k\varepsilon} - (\Delta m_k c + \bar{m}_k \Delta V_k) \, G^*_{yk} L^*_k - \Delta m_k \; G^*_{tk}$$

Development of L*

Since $L^* = \frac{1}{|\lambda|} \lambda$ and λ is continuous over an impulse,

$$\bar{L}^*_k \rightarrow L^*_k$$

as $\varepsilon \rightarrow 0$.

Development of L^*_t

Since $L^*_t = \frac{1}{|\lambda|}\dot{\lambda} - \frac{\dot{\lambda}^T\lambda}{|\lambda|^3}\lambda = \frac{1}{|\lambda|}(I - L^*L^{*T})\dot{\lambda}$ and λ and $\dot{\lambda}$ are continuous over an impulse,

$$\bar{L}^*_{tk} \to L^*_{tk}$$

as $\varepsilon \to 0$.

Development of L^*_{tt}

Since

$$L^*_{tt} = \frac{1}{|\lambda|}Q^* - \frac{2\dot{\lambda}^T\lambda}{|\lambda|^3}\dot{\lambda} + \frac{1}{|\lambda|^3}[-\lambda^TQ^* - \dot{\lambda}^T\dot{\lambda} + \frac{3(\dot{\lambda}^T\lambda)^2}{|\lambda|^2}]\lambda$$

and since Q, λ, and $\dot{\lambda}$ are continuous over an impulse,

$$\bar{L}^*_{ttk} \to L^*_{ttk}$$

as $\varepsilon \to 0$.

Development of L^*_λ

Since

$$L^{(i)}_{\lambda^{(j)}} = \frac{1}{|\lambda|}\delta_{ij} - \frac{\lambda^{(j)}}{|\lambda|^3}\lambda^{(i)},$$

we have

$$L^*_\lambda = \frac{1}{|\lambda|}(I - \frac{1}{|\lambda|^2}\lambda\lambda^T).$$

As $\varepsilon \to 0$,

$$\bar{L}^*_{\lambda k} \to L^*_{\lambda k}$$

Development of $L^*_{k\varepsilon}$

$$L^*_{k\varepsilon} = \frac{1}{|\lambda_k|}\lambda_{k\varepsilon} - \frac{\lambda^T_k\lambda_{k\varepsilon}}{|\lambda_k|^3}\lambda_k = \frac{1}{|\lambda_k|}(I - L^*_kL^{*T}_k)\lambda_{k\varepsilon}$$

As $\varepsilon \to 0$,

$$\bar{L}^*_{k\varepsilon} = \bar{L}^*_{\lambda k}\ \bar{\lambda}_{k\varepsilon} \to L^*_{\lambda k}(\lambda_{k\varepsilon} - \Delta m_k \dot{\lambda}_k)$$

$$\to L^*_{k\varepsilon} - \Delta m_k L^*_{\lambda k}\dot{\lambda}_k = L^*_{k\varepsilon} - \Delta m_k\ L^*_{tk}$$

Development of $L^*_{k\varepsilon\varepsilon}$

$$L^*_{k\varepsilon\varepsilon} = \frac{1}{|\lambda_k|}\ (I - L^*_k\ L^{*T}_k)\lambda_{k\varepsilon\varepsilon} + \frac{1}{|\lambda_k|^2}\ [-2(L^{*T}_k\lambda_{k\varepsilon})\lambda_{k\varepsilon} - (\lambda^T_{k\varepsilon}\lambda_k\ )L^*_k$$

$$+ 3(L^{*T}_k\lambda_{k\varepsilon})^2 L^*_k]$$

Development of $L^*_{tk\varepsilon}$

$$L^*_{tk\varepsilon} = \frac{1}{|\lambda_k|}\ \dot{\lambda}_{k\varepsilon} - \frac{\lambda^T_k\lambda_{k\varepsilon}}{|\lambda_k|^3}\ \dot{\lambda}_k - \frac{\dot{\lambda}^T_k\lambda_k}{|\lambda_k|^3}\ \lambda_{k\varepsilon}$$

$$+ \frac{1}{|\lambda_k|^3}\left[-\lambda^T_{k\varepsilon}\lambda_k - \dot{\lambda}^T_k\lambda_{k\varepsilon} + \frac{3(\dot{\lambda}^T_k\lambda_k)(\lambda^T_k\lambda_{k\varepsilon})}{|\lambda_k|^2}\right]\lambda_k$$

As $\varepsilon \to 0$,

$$\bar{L}^*_{tk\varepsilon} = \frac{\partial}{\partial\varepsilon}\ (\bar{L}^*_{\lambda k}\dot{\bar{\lambda}}_k) = \bar{L}^*_{\lambda k\varepsilon}\dot{\bar{\lambda}}_k + \bar{L}^*_{\lambda k}\dot{\bar{\lambda}}_{k\varepsilon}$$

$$= (\bar{\lambda}_{k\varepsilon}\bar{L}^*_{\lambda\lambda k})\dot{\bar{\lambda}}_k + \bar{L}^*_{\lambda k}\dot{\bar{\lambda}}_{k\varepsilon}$$

$$\to [(\lambda_{k\varepsilon} - \Delta m_k\dot{\lambda}_k)L^*_{\lambda\lambda k}]\dot{\lambda}_k + L^*_{\lambda k}(\dot{\lambda}_{k\varepsilon} - \Delta m_k Q^*_k)$$

$$\to (\lambda_{k\varepsilon}L^*_{\lambda\lambda k})\dot{\lambda}_k + L^*_{\lambda k}\dot{\lambda}_{k\varepsilon} - \Delta m_k[(\dot{\lambda}_k L^*_{\lambda\lambda k})\dot{\lambda}_k + L^*_{\lambda k}Q^*_k]$$

$$\to L^*_{tk\varepsilon} - \Delta m_k[(\dot{\lambda}_k L^*_{\lambda\lambda k})\dot{\lambda}_k + L^*_{\lambda k}Q^*_k]$$

where, for example,

$$(\lambda_{k\varepsilon}L^*_{\lambda\lambda k}) \triangleq \sum_p \lambda^{(p)}_{k\varepsilon}\ L^*_{\lambda\lambda^{(p)}k}$$

and the superscript p indicates p^{th} component.

Development of $(\dot{\lambda}\, L^*_{\lambda\lambda})$

$$(\dot{\lambda}\, L^*_{\lambda\lambda}) \triangleq \sum_p \dot{\lambda}^{(p)} L^*_{\lambda\lambda}(p)$$

$$= \frac{\lambda^T\dot{\lambda}}{|\lambda|^3} \left(-I + \frac{3}{|\lambda|^2}\lambda\lambda^T\right) - \frac{1}{|\lambda|^3}\left(\lambda\dot{\lambda}^T + \dot{\lambda}\lambda^T\right)$$

Development of Q^*

$$Q^* = G^*_y \lambda$$

In an inverse square gravitational field,

$$Q^* = -\frac{\mu}{|y|^3}\left(\lambda - \frac{3y^T\lambda}{|y|^2} y\right)$$

Since y and λ are continuous over an impulse, in general

$$\bar{Q}^*_k \to Q^*_k$$

as $\varepsilon \to 0$.

Development of Q^*_t

In an inverse square field,

$$Q^*_t = \frac{\mu}{|y|^3}\left[-\dot{\lambda} + \frac{3y^T\dot{y}}{|y|^2}\lambda + \frac{3y^T\lambda}{|y|^2}\dot{y} + \frac{3(\dot{y}^T\lambda + y^T\dot{\lambda})}{|y|^2} y - \frac{15(y^T\lambda)(y^T\dot{y})}{|y|^4} y\right]$$

In general as $\varepsilon \to 0$,

$$\bar{Q}^*_{tk} = \bar{Q}^*_{\lambda k}\dot{\bar{\lambda}}_k + \bar{Q}^*_{yk}\dot{\bar{y}}_k + (\bar{Q}_t)^*_k$$

$$\to Q^*_{\lambda k}\dot{\lambda}_k + Q^*_{yk}(\dot{y}_k + \Delta\dot{y}_k) + (Q_t)^*_k$$

$$\to Q^*_{tk} + Q^*_{yk}\,\Delta\dot{y}_k$$

Development of Q^*_y

In an inverse square field,

$$Q^*_y = \frac{3\mu}{|y|^5}\left[\lambda y^T + y\lambda^T - \frac{5y^T\lambda}{|y|^2} yy^T + (y^T\lambda)I\right]$$

In general, as $\varepsilon \to 0$,

$$\bar{Q}^*_{yk} \to Q^*_{yk}$$

Development of $Q^*_{k\varepsilon}$

In an inverse square field

$$Q^*_{k\varepsilon} = \frac{\mu}{|y_k|^3}\left[-\lambda_{k\varepsilon} + \frac{3y^T{}_k y_{k\varepsilon}}{|y_k|^2}\lambda_k + \frac{3y^T{}_k\lambda_k}{|y_k|^2} y_{k\varepsilon}\right.$$

$$\left. + \frac{3(y^T{}_{k\varepsilon}\lambda_k + y^T{}_k\lambda_{k\varepsilon})}{|y_k|^2} y_k - \frac{15(y^T{}_k\lambda_k)(y^T{}_k y_{k\varepsilon})}{|y_k|^4} y_k\right]$$

In general, as $\varepsilon \to 0$,

$$\bar{Q}^*_{k\varepsilon} = \bar{Q}^*_{\lambda k}\bar{\lambda}_{k\varepsilon} + \bar{Q}^*_{yk}\bar{y}_{k\varepsilon} + \bar{t}_{k\varepsilon}(\bar{Q}_t)^*_k$$

$$\to Q^*_{\lambda k}(\lambda_{k\varepsilon} - \Delta m_k\dot{\lambda}_k) + Q^*_{yk}[y_{k\varepsilon} - \bar{m}_k\Delta\dot{y}_k - \Delta m_k(\dot{y}_k + cL^*_k)]$$

$$+ \Delta t_{k\varepsilon}(Q_t)^*_k + t_{k\varepsilon}(Q_t)^*_k$$

$$\to Q^*_{k\varepsilon} - \Delta m_k Q^*_{tk} - (\bar{m}_k\Delta V_k + c\Delta m_k)Q^*_{yk}L^*_k$$

Development of U^*

$$U^* = \frac{1}{|\lambda|}\lambda^T\dot{\lambda}$$

As $\varepsilon \to 0$,

$$\bar{U}^*_k \to U^*_k$$

Development of U^*_t

$$U^*_t = \frac{1}{|\lambda|}(\dot{\lambda}^T\dot{\lambda} + \lambda^T Q^*) - \frac{1}{|\lambda|^3}(\lambda^T\dot{\lambda})^2$$

As $\varepsilon \to 0$,

$$\bar{U}^*_{tk} \to U^*_{tk}.$$

Development of U^*_{tt}

$$U^*_{tt} = \frac{1}{|\lambda|}(3\dot{\lambda}^T Q^* + \lambda^T Q^*_t) + \frac{3}{|\lambda|^3}\left[-(\lambda^T\dot{\lambda})(\dot{\lambda}^T\dot{\lambda} + \lambda^T Q^*) + \frac{(\lambda^T\dot{\lambda})^3}{|\lambda|^2}\right]$$

Therefore, as $\varepsilon \to 0$,

$$\bar{U}^*_{ttk} \to \frac{1}{|\lambda_k|}\left[3\dot{\lambda}^T{}_k Q^*_k + \lambda^T{}_k(Q^*_{tk} + Q^*_{yk}\,\Delta\dot{y}_k)\right]$$

$$+ \frac{3}{|\lambda_k|^3}\left[-(\lambda^T{}_k\dot{\lambda}_k)(\dot{\lambda}^T{}_k\dot{\lambda}_k + \lambda^T{}_k Q^*_k) + \frac{(\lambda^T{}_k\dot{\lambda}_k)^3}{|\lambda_k|^2}\right]$$

$$\to U^*_{ttk} + \frac{1}{|\lambda_k|}\lambda^T{}_k\, Q^*_{yk}\,\Delta\dot{y}_k$$

$$\to U^*_{ttk} + L^{*T}{}_k Q^*_{yk}\,\Delta\dot{y}_k$$

Development of $U^*_{k\varepsilon}$

$$U^*_{k\varepsilon} = \frac{1}{|\lambda_k|}\left[\dot{\lambda}^T{}_k\lambda_{k\varepsilon} + \lambda^T{}_k\dot{\lambda}_{k\varepsilon} - (\dot{\lambda}^T{}_k L^*_k)(L^{*T}{}_k\lambda_{k\varepsilon})\right]$$

Development of $U^*_{k\varepsilon\varepsilon}$

$$U^*_{k\varepsilon\varepsilon} = \frac{1}{|\lambda_k|}\left[\dot{\lambda}^T{}_k - (\dot{\lambda}^T{}_k L^*_k)L^{*T}{}_k\right]\lambda_{k\varepsilon\varepsilon} + L^{*T}{}_k\dot{\lambda}_{k\varepsilon\varepsilon}$$

$$+ \frac{1}{|\lambda_k|}\left[2\dot{\lambda}^T{}_{k\varepsilon}\lambda_{k\varepsilon} - \frac{2}{|\lambda_k|}(L^{*T}{}_k\lambda_{k\varepsilon})(\dot{\lambda}^T{}_k\lambda_{k\varepsilon} + \lambda^T{}_k\dot{\lambda}_{k\varepsilon})\right.$$

$$\left. - \frac{1}{|\lambda_k|}(\dot{\lambda}^T{}_k L^*_k)(\lambda^T{}_{k\varepsilon}\lambda_{k\varepsilon}) + \frac{3}{|\lambda_k|}(\dot{\lambda}^T{}_k L^*_k)(L^{*T}{}_k\lambda_{k\varepsilon})^2\right]$$

Development of $U^*_{tk\varepsilon}$

$$U^*_{tk\varepsilon} = \frac{1}{|\lambda_k|} (2\dot{\lambda}^T{}_k\dot{\lambda}_{k\varepsilon} + \lambda^T{}_{k\varepsilon}Q^*{}_k + \lambda^T{}_kQ^*{}_{k\varepsilon}) - \frac{1}{|\lambda_k|^3} (\lambda^T{}_k\lambda_{k\varepsilon})(\dot{\lambda}^T{}_k\dot{\lambda}_k + \lambda^T{}_kQ^*{}_k)$$

$$- \frac{2}{|\lambda_k|^3} (\lambda^T{}_k\dot{\lambda}_k)(\lambda^T{}_{k\varepsilon}\dot{\lambda}_k + \lambda^T{}_k\dot{\lambda}_{k\varepsilon}) + \frac{3}{|\lambda_k|^5} (\lambda^T{}_k\lambda_{k\varepsilon})(\lambda^T{}_k\dot{\lambda}_k)^2$$

Therefore, as $\varepsilon \to 0$,

$$\bar{U}^*{}_{tk\varepsilon} \to U^*{}_{tk\varepsilon} + \frac{1}{|\lambda_k|} \Big[-2\Delta m_k\dot{\lambda}^T{}_kQ^*{}_k - \Delta m_k\dot{\lambda}^T{}_kQ^*{}_k - \Delta m_k\lambda^T{}_kQ^*{}_{tk}$$

$$- (\bar{m}_k\Delta V_k + c\Delta m_k)\lambda^T{}_kQ^*{}_{yk}L^*{}_k\Big] + \frac{\Delta m_k}{|\lambda_k|^3} (\lambda^T{}_k\dot{\lambda}_k)(\dot{\lambda}^T{}_k\dot{\lambda}_k + \lambda^T{}_kQ^*{}_k)$$

$$+ \frac{2\Delta m_k}{|\lambda_k|^3} (\lambda^T{}_k\dot{\lambda}_k)(\dot{\lambda}^T{}_k\dot{\lambda}_k + \lambda^T{}_kQ^*{}_k) - \frac{3\Delta m_k}{|\lambda_k|^5} (\lambda^T{}_k\dot{\lambda}_k)^3$$

$$\to U^*{}_{tk\varepsilon} - (\bar{m}_k\Delta V_k + c\Delta m_k)L^{*T}{}_kQ^*{}_{yk}L^*{}_k - \Delta m_kL^{*T}{}_kQ^*{}_{tk}$$

$$+ \frac{3\Delta m_k}{|\lambda_k|} \left[-\dot{\lambda}^T{}_kQ^*{}_k + \frac{1}{|\lambda_k|^2} (\lambda^T{}_k\dot{\lambda}_k)(\dot{\lambda}^T{}_k\dot{\lambda}_k + \lambda^T{}_kQ^*{}_k) - \frac{1}{|\lambda_k|^4} (\lambda^T{}_k\dot{\lambda}_k)^3\right]$$

$$\to U^*{}_{tk\varepsilon} - (\bar{m}_k\Delta V_k + c\Delta m_k)L^{*T}{}_kQ^*{}_{yk}L^*{}_k - \Delta m_kL^{*T}{}_kQ^*{}_{tk}$$

$$+ \frac{3\Delta m_k}{|\lambda_k|} (-\dot{\lambda}^T{}_kQ^*{}_k + U^*{}_k\, U^*{}_{tk})$$

The functions

$$a_{1k} \triangleq \Delta V_k m_k + c\Delta m_k$$

$$a_{2k} \triangleq \Delta V_k \bar{m}_k + c\Delta m_k$$

are of frequent occurrence. Furthermore, care must be exercised in their calculation because they are the differences of nearly equal terms. Expansion of a_{1k} gives:

$$a_{1k} = \Delta V_k m_k + cm_k (e^{-\Delta V_k / c} - 1)$$

$$= \Delta V_k m_k + cm_k \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Delta V_k}{c} \right)^n - 1 \right]$$

$$= \Delta V_k m_k - \left[\Delta V_k m_k - cm_k \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Delta V_k}{c} \right)^n \right]$$

$$= cm_k \left(\frac{\Delta V_k}{c} \right)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!} \left(\frac{\Delta V_k}{c} \right)^n$$

The latter series expansion should be employed for the calculation of a_{1k}.

Expansion of a_{2k} yields:

$$a_{2k} = (c + \Delta V_k)\bar{m}_k - cm_k$$

$$= (c + \Delta V_k) m_k e^{-\Delta V_k / c} - cm_k$$

$$= m_k \left[(c + \Delta V_k) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Delta V_k}{c} \right)^n - c \right]$$

$$= m_k \left[c \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Delta V_k}{c} \right)^n + \Delta V_k \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Delta V_k}{c} \right)^n \right]$$

$$= cm_k \sum_{n=0}^{\infty} (-1)^n \left[-\frac{1}{(n+1)!} + \frac{1}{n!} \right] \left(\frac{\Delta V_k}{c} \right)^{n+1}$$

$$= cm_k \sum_{n=1}^{\infty} (-1)^n \frac{n}{(n+1)!} \left(\frac{\Delta V_k}{c} \right)^{n+1}$$

$$= cm_k \left(\frac{\Delta V_k}{c} \right)^2 \sum_{n=0}^{\infty} (-1)^{n+1} \frac{n+1}{(n+2)!} \left(\frac{\Delta V_k}{c} \right)^n$$

Appendix C
PROGRAM LISTINGS

The FORTRAN listings of the burn-coast intercept subroutine and some of its subroutines follow. These listings are followed by a listing of the program for the constant G, burn-coast-burn rendezvous problem.

```
      SUBROUTINE OPINT(Y1,YD1,Y2,T2,M1,C,GM,B,TB1C,L1C,LD1C)
      REAL M1,MB1,MB2
      REAL L1(3),LD1(3),L2(3),LD2(3),L1C(3),LD1C(3),
     1LY(3,3),LYD(3,3)
      DIMENSION Y1(3),YD1(3),Y2(3),YDB1(3),YDB2(3),YD2(3),PXX(6,6),
     1PLX(6,6),YY(3,3),YYD(3,3),P(3),Q(3)
      NIMP=1
      CALL OPIMP(Y1,YD1,Y2,YDB2,T2,M1,C,GM,YDB1,YD2,L1,LD1,L2,LD2,
     1DV1,DV2,MB1,MB2,NIMP)
      DM1=MB1-M1
      A1=COF1(C,DV1,M1,DM1)
      IOPT=3
      DO 105 I=1,6
      DO 105 J=1,6
      IF (I-J) 102,103,102
  102 PXX(I,J)=0.
      PLX(I,J)=0.
      GO TO 105
  103 PXX(I,J)=1.
      PLX(I,J)=1.
  105 CONTINUE
      CALL LAM (Y1,YDB1,Y2,YD2,L1,LD1,GM,T2,IOPT,P,Q,PXX,PLX)
      DO 110 I=1,3
      DO 110 J=1,3
      YY(I,J)=PXX(I,J)
      YYD(I,J)=PXX(I,J+3)
      LY(I,J)=PLX(I,J)
  110 LYD(I,J)=PLX(I,J+3)
      CALL MINV(YYD,3,D,P,Q)
      CALL FORC(YY,YYD,LY,LYD,MB1,A1,C,L1,LD1,DV1,DM1,B,TB1E,TB1EE,
     1TB1C,L1C,LD1C)
      CALL WR2(M1,MB1,MB2,C,GM,B,DV1,DV2,TB1E,TB1EE,TB1C,T2,Y1,YD1,YDB1,
     1Y2,L1,LD1,L2,LD2,L1C,LD1C)
      RETURN
      END


      SUBROUTINE OPIMP(Y1,YD1,Y2,YDB2,T2,M1,C,GM,YDB1,YD2,L1,LD1,L2,LD2,
     1DV1,DV2,MB1,MB2,NIMP)
      REAL M1,MB1,MB2
      REAL L1(3),LD1(3),L2(3),LD2(3)
      DIMENSION Y1(3),Y2(3),YDB1(3),YD2(3),YD1(3),YDB2(3),DYD1(3),
     1DYD2(3),PXX(6,6),PLX(6,6),YY(3,3),YYD(3,3),YDY(3,3),YDYD(3,3),
     2Q1(3),Q2(3),Q3(3),P1(3),RO(6),R(6)
```

```
      DO 20 I=1,3
      RO(I)=Y1(I)
   20 R(I)=Y2(I)
      CALL DYST (RO,R,T2,GM)
      DO 22 I=1,3
      YDB1(I)=RO(I+3)
   22 YD2(I)=R(I+3)
      DO 25 I=1,3
      DYD1(I)=YDB1(I)-YD1(I)
   25 DYD2(I)=YDB2(I)-YD2(I)
      DV1=VMAG(DYD1,3)
      IF (NIMP-1)27,27,29
   27 DV2=0.
      DO 28 I=1,3
   28 L2(I)=0.
      GO TO 35
   29 DV2=VMAG(DYD2,3)
      DO 30 I=1,3
   30 L2(I)=DYD2(I)/DV2
   35 DO 40 I=1,3
   40 L1(I)=DYD1(I)/DV1
      MB1=M1*EXP(-DV1/C)
      MB2=MB1*EXP(-DV2/C)
      IOPT=2
      DO 55 I=1,6
      DO 55 J=1,6
      IF(I-J) 42,43,42
   42 PXX(I,J)=0.
      PLX(I,J)=0.
      GO TO 55
   43 PXX(I,J)=1.
      PLX(I,J)=1.
   55 CONTINUE
      DO 60 I=1,3
   60 LD1(I)=0.
      CALL LAM (Y1,YDB1,Y2,YD2,L1,LD1,GM,T2,IOPT,P1,LD2,PXX,PLX)
      DO 70 I=1,3
      DO 70 J=1,3
      YY(I,J)=PXX(I,J)
      YYD(I,J)=PXX(I,J+3)
      YDY(I,J)=PXX(I+3,J)
   70 YDYD(I,J)=PXX(I+3,J+3)
      CALL MINV(YYD,3,D,Q1,Q2)
      CALL MPRD(YY,L1,Q1,3,3,1)
      CALL MPRD(YDY,L1,Q2,3,3,1)
      DO 75 I=1,3
   75 P1(I)=L2(I)-Q1(I)
      CALL MPRD(YYD,P1,LD1,3,3,1)
      CALL MPRD(YDYD,LD1,Q3,3,3,1)
      DO 76 I=1,3
   76 LD2(I)=Q2(I)+Q3(I)
      RETURN
      END
```

```
      SUBROUTINE FORC(YY,YYD,LY,LYD,MB1 ,A1,C,L1,LD1,DV1,DM1,B,TB1E,
     1TB1EE,TB1C,L1C,LD1C)
      REAL MB1
      REAL LY(3,3),LYD(3,3),L1(3),LD1(3),L1C(3),LD1C(3),L1E(3),LD1E(3)
      DIMENSION YY(3,3),YYD(3,3),P(3),Q(3),DYDE1(3)
      Z=DOTN(L1,LD1,3)
      DM1E= MB1*A1*Z/C
      DO 130 I=1,3
  130 L1E(I)=-2.*A1*(LD1(I)-Z*L1(I))/DV1
      DO 135 I=1,3
  135 DYDE1(I)=-C*DM1E*L1(I)/MB1-A1*(LD1(I)-Z*L1(I))
      CALL MPRD(LY,L1,Q,3,3,1)
      CALL MPRD(LYD,DYDE1,P,3,3,1)
      DO 140 I=1,3
  140 Q(I)=A1*Q(I)-P(I)
      CALL MPRD(YY,L1E,P,3,3,1)
      DO 145 I=1,3
  145 Q(I)=Q(I)-P(I)
      CALL MPRD(YYD,Q,LD1E,3,3,1)
      TB1E=-DM1
      TB1EE=-2.*DM1E
      TB1C=(TB1E+.5*TB1EE/B)/B
      DO 150 I=1,3
      L1C(I)=L1(I)+L1E(I)/B
  150 LD1C(I)=LD1(I)+LD1E(I)/B
      RETURN
      END
```

```
      FUNCTION COF1(C,DV,M,DM)
      REAL M
      X1=DV/C
      X2=X1*X1
      X3=X1*X2
      X4=X2*X2
      COF1=C*M*X2*(.5-X1/6.+X2/24.-X3/120.+X4/720.)
      RETURN
      END
```

```
      SUBROUTINE WR2(M1,MB1,MB2,C,GM,B,DV1,DV2,TB1E,TB1EE,TB1C,T2,Y1,
     1YD1,YDB1,Y2,L1,LD1,L2,LD2,L1C,LD1C)
      REAL M1,MB1,MB2
      REAL L1(3),LD1(3),L2(3),LD2(3),L1C(3),LD1C(3)
      DIMENSION Y1(3),YD1(3),YDB1(3),Y2(3)
      WRITE(3,200) M1,MB1,MB2,C,GM,B
      WRITE(3,201) DV1,DV2,TB1E,TB1EE,TB1C,T2
      WRITE(3,202)(Y1(I),YD1(I),YDB1(I),Y2(I),I=1,3)
      WRITE(3,203)(L1(I),LD1(I),L2(I),LD2(I),L1C(I),LD1C(I),I=1,3)
  200 FORMAT (19X,2HM1,15X,3HMB1,13X,3HMB2,11X,1HC,16X,2HGM,13X,1HB,/,
     112X,6E16.7)
  201 FORMAT (19X,3HDV1,14X,3HDV2,13X,4HTB1E,10X,5HTB1EE,12X,4HTB1C,11X,
     12HT2,/,12X,6E16.7)
  202 FORMAT (18X,2HY1,13X,3HYD1,13X,4HYDB1,12X,2HY2,/,(12X,4E16.7))
  203 FORMAT (18X,2HL1,13X,3HLD1,13X,2HL2,14X,3HLD2,16X,3HL1C,13X,
     14HLD1C,/,(12X,6E16.7))
      RETURN
      END
```

A listing of the burn-coast-burn constant G, rendezvous program follows.

```
      REAL M1,MB1IM,MB2IM
      REAL L1IM(3),L2IM(3),LD1IM(3),L1E(3),LD1E(3),L1C(3),LD1C(3)
      DIMENSION G(3),Y1(3),YD1(3),YBAR2(3),YDBR2(3),YD1IM(3),YD2IM(3),
     1DYD1(3),DYD2(3)
100   FORMAT(5E12.5)
110   FORMAT(10X,'TB1IM=',E15.8,2X,'TB1C=',E15.8,2X,'T2IM=',E15.8,
     12X,'T2C=',E15.8,/)
115   FORMAT(10X,'L1IM(1)=',E15.8,2X,'L1IM(2)=',E15.8,2X,'L1IM(3)=',
     1E15.8,/)
120   FORMAT(10X,'DL1IM(1)=',E15.8,2X,'LD1IM(2)=',E15.8,2X,'LD1IM(3)=',
     1E15.8,/)
125   FORMAT(10X,'L1C(1)=',E15.8,2X,'L1C(2)=',E15.8,2X,'L1C(3)=',E15.8,
     1/)
130   FORMAT(10X,'LD1C(1)=',E15.8,2X,'LD1C(2)=',E15.8,2X,'LD1C(3)=',
     1E15.8,/)
      READ(2,100) C,B,TBAR2,M1
      READ(2,100)(G(I),Y1(I),YD1(I),YBAR2(I),YDBR2(I),I=1,3)
      DO 35 I=1,3
      YD1IM(I)=(YBAR2(I)-Y1(I))/TBAR2+.5*TBAR2*G(I)
      YD2IM(I)=YD1IM(I)-TBAR2*G(I)
      DYD1(I)=YD1IM(I)-YD1(I)
 35   DYD2(I)=YDBR2(I)-YD2IM(I)
      ADYD1=SQRT(DYD1(1)**2+DYD1(2)**2+DYD1(3)**2)
      ADYD2=SQRT(DYD2(1)**2+DYD2(2)**2+DYD2(3)**2)
      MB1IM=M1*EXP(-ADYD1/C)
      MB2IM=MB1IM*EXP(-ADYD2/C)
      DM1=MB1IM-M1
      DM2=MB2IM-MB1IM
      DO 40 I=1,3
      L1IM(I)=DYD1(I)/ADYD1
      L2IM(I)=DYD2(I)/ADYD2
 40   LD1IM(I)=(L2IM(I)-L1IM(I))/TBAR2
      A1TIL=C*M1*(ADYD1/C)**2*(.5-(ADYD1/C)/6.+((ADYD1/C)**2)/24.-
     1((ADYD1/C)**3)/120.+((ADYD1/C)**4)/720.)
      A2TIL=C*MB1IM*(ADYD2/C)**2*(-.5+(ADYD2/C)/3.-((ADYD2/C)**2)/8.
     1+((ADYD2/C)**3)/30.-((ADYD2/C)**4)/144.)
      R=0.
      DO 45 I=1,3
 45   R=R+L1IM(I)*L2IM(I)
      DM1E=-MB1IM/(C*TBAR2)* (A1TIL+A2TIL*R)
      DM2E=(MB2IM*(A1TIL-A2TIL))/(TBAR2*C)*(R-1.)-DM1E
      DO 50 I=1,3
      L1E(I)=(A1TIL-A2TIL)/(TBAR2*ADYD1) * (R*L1IM(I) - L2IM(I))
 50   LD1E(I) =-L1E(I)/TBAR2 +(1./TBAR2**2)*((-1.*(A1TIL-A2TIL)/ADYD2 )*
     1(L1IM(I)-R*L2IM(I)) + (DM2-DM1)*(R-1.) * L2IM(I))
```

```
      B2=B*B
      TB1IM=-DM1/B
      TB1C=TB1IM-DM1E/B2
      T2IM=TBAR2+DM2/B
      T2C=T2IM+DM2E/B2
      DO 55 I=1,3
      L1C(I)=L1IM(I) + L1E(I)/B
   55 LD1C(I)=LD1IM(I)+LD1E(I)/B
      WRITE(3,110)TB1IM,TB1C,T2IM,T2C
      WRITE(3,115)(L1IM(I),I=1,3)
      WRITE(3,120)(LD1IM(I),I=1,3)
      WRITE(3,125)(L1C(I),I=1,3)
      WRITE(3,130)(LD1C(I),I=1,3)
      CALL EXIT
      END
```


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