N71-28171

NASA CR-118995

MOTION OF A SYMMETRIC RIGID BODY UNDER THE ACTION OF A BODY-FIXED FORCE

by

B. S. Chia and T. R. Kane

Technical Report No. 203

March 1971

Department of Applied Mechanics

STANFORD UNIVERSITY

Department of Applied Mechanics

STANFORD UNIVERSITY

Technical Report No. 203

MOTION OF A SYMMETRIC RIGID BODY UNDER THE ACTION OF A BODY-FIXED FORCE

by

B. S. Chia and T. R. Kane

March 1971

ABSTRACT

Exact dynamical and kinematical equations governing the attitude and translational motions of a six-degree-of-freedom symmetric rigid body under the action of a body-fixed force are formulated. From these equations, an analytical, but approximate, description of the behavior of the system is obtained. The validity of these predictions is established by comparing them with predictions based on digital computer solutions of the exact equations of motion. The solutions are then used to study motions of a symmetric gyrostat.

ACKNOWLEDGEMENT

This work was supported financially by the National Aeronautics and Space Administration under NGR-05-020-209.

TABLE OF CONTENTS

		Page
	ABSTRACT	ii
	ACKNOWLEDGEMENT	iii
	TABLE OF CONTENTS	iv
	LIST OF FIGURES	v
	LIST OF SYMBOLS	vii
PART		
1	INTRODUCTION	1
2	PRELIMINARIES	5
	2.1 System Description . . . 5	
	2.2 Orientation Angles and Direction Cosines . . . 7	
3	ORIENTATION	11
	3.1 Dynamical Equations . . . 11	
	3.2 Kinematical Equations . . . 18	
	3.3 Comparison of Solutions . . . 39	
4	POSITION	50
	4.1 Dynamical Equations . . . 50	
	4.2 Solution of Differential Equations . . . 56	
	4.3 Comparison of Solutions . . . 80	
	4.4 A Special Case . . . 92	
5	SYMMETRIC GYROSTAT	97
	5.1 Introduction . . . 97	
	5.2 System Description . . . 97	
	5.3 Analysis . . . 100	
APPENDIX		
A	Subroutine ARC	117
B	Breakwell's Approximate Solution to the Kinematical Equations	119
	BIBLIOGRAPHY	131

LIST OF FIGURES

FIGURE		Page
1	Model of an Axially Symmetric Rigid Body	6
2	Attitude Angles	8
3	Illustration of the Reference Frame C	12
4	Comparison of Solutions for Orientation Angles $\Omega_1 = 1,0$, $\Omega_3 = 0$, $z = 0.01$	41
5	Comparison of Solutions for Orientation Angles $\Omega_1 = 1,0$, $\Omega_3 = 0$, $z = 0.1$	42
6	Comparison of Solutions for Orientation Angles $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 9.0$	43
7	Comparison of Solutions for Orientation Angles $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 1.0$	44
8	Comparison of Solutions for Orientation Angles $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.05$, $L = 9.0$	45
9	Trajectory of the Symmetry Axis $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$	47
10	Trajectory of the Symmetry Axis $\Omega_1 = 1,0$, $\Omega_3 = 0$, $z = 0.1$	48
11	Trajectory of the Symmetry Axis $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 1.0$	49
12	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$, $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	81
13	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.10$, $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	82
14	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	83

FIGURE		Page
15	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 9.0$, $\mathcal{F}_1 = 0.4$, $\mathcal{F}_2 = 0.4$, $V_{i0} = 0$ $(i = 1,2,3)$	84
16	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 1.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	85
17	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.05$, $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	86
18	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0$, $\Omega_3 = 1.0$, $z = 0.01$, $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$.	87
19	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$, $L = 0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	88
20	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = -0.25$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	89
21	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$, $L = -0.25$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	90
22	Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$, $L = 5000.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$	95
23	Model of a Symmetric Gyrostat	98
B-1	Comparison of Solutions for Orientation Angles $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$	129
B-2	Comparison of Solutions for Orientation Angles $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.1$	130

LIST OF SYMBOLS

Numbers refer to the page on which the symbol is defined.

Symbols	page
Chapter 2	
B, B^*	5
m	5
$\underline{X}$	5
O	5
A	5
$\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$	5
$\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$	5
I, J	5
$\underline{F}$	5
Q	5
$\underline{q}$	5
x_1, x_2, x_3	5
F_1, F_2, F_3	5
q_1, q_2, q_3	5
$\underline{T}$, T	6
$\underline{\alpha}_1$, $\underline{\alpha}_2$, $\underline{\alpha}_3$	6
α	6
$\underline{\beta}_1$, $\underline{\beta}_2$, $\underline{\beta}_3$	6
β	6
γ_1, γ_2, γ_3	9
γ_{ij}	9

Symbols	page
Chapter 3	
C	11
$\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$	11
${}^{C}\underline{\omega}^{B}$	11
s	11
t	11
${}^{A}\underline{\omega}^{C}$	11
p_1, p_2, p_3	11
${}^{A}\underline{\omega}^{B}$	13
${}^{A}\underline{\omega}^{C}$	13
$\underline{H}$	13
L	14
λ	14
p_{10}, p_{20}, p_{30}	14
θ	15
ω_1, ω_2, ω_3	16
ω_{10}, ω_{20}, ω_{30}	16
a_{ij}	18
$a_{ij}(0)$	20
δ_{ij}	20
p_0	21
P_1, P_3	21

Symbols	page
$\underline{e}_1, \underline{e}_2, \underline{e}_3$	26
e_{ij}	27
$\tilde{e}$	27
$k_1, \ldots, k_6$	31
m	31
n	31
x	34
z	34
$\theta_1, \theta_2, \theta_3$	36
$\varphi_1, \varphi_2, \varphi_3$	37
Ω_1, Ω_3	38
ω_0	39
Chapter 4	
X_1, X_2, X_3	50
m	50
f_1, f_2	51
ξ	55
$\delta_1, \ldots, \delta_9$	55
r_g	56
D_1, D_2, D_3	56
$\mathcal{F}_1, \mathcal{F}_2$	56
ζ	56
$\beta_1, \ldots, \beta_7$	56
β_8, β_9	57

Symbols	page
$D_i(0)$	60
V_{i0}	60
w	61
$C_2[w], S_2[w]$	61
$C(a,b,x)$	61
$S(a,b,x)$	61
$C_1(a,b,x)$	61
$S_1(a,b,x)$	62
$C^*(a,b,x)$	62
$S^*(a,b,x)$	62
u, v	62
$\bar{\zeta}$	66
$\bar{\beta}_1, \ldots, \bar{\beta}_9$	66
$C^*(\beta_i, x)$	79
$S^*(\beta_i, x)$	79
v	79
n	92
Chapter 5	
A	97
B, B^*	97
W, W^*	97
m	97
G, G^*	97
$\underline{X}$	97

Symbols	page
O	97
$\underline{a}_1, \underline{a}_2, \underline{a}_3$	97
$\underline{b}_1, \underline{b}_2, \underline{b}_3$	97
C	99
$\underline{c}_1, \underline{c}_2, \underline{c}_3$	99
$\underline{I}^G$	99
$\underline{I}^W$	99
I, J	99
K	99
$\underline{F}$	99
$\underline{Q}$	99
$\underline{q}$	99
$\underline{T}_W$	99
x_1, x_2, x_3	99
F_1, F_2	100
q_1, q_2	100
T_W	100
$\underline{T}_B, T_B$	100
${}^B\underline{\omega}^W$	100
r	100
t	100
s	100
${}^C\underline{\omega}^B$	100
${}^A\underline{\omega}^C$	101
p_1, p_2, p_3	101
${}^A\underline{\omega}^B$	101
${}^A\underline{\omega}^W$	101
$\underline{H}^G$	101
$\underline{H}^W$	102
$\underline{M}$	103
M_3	103
λ_G	104
p_{10}, p_{20}, p_{30}	104
L_G	105
θ_G	105
$\omega_1, \omega_2, \omega_3$	105
$\omega_{10}, \omega_{20}, \omega_{30}$	106
s_0	106
r_0	106
ℓ	108
p_0	108
ω_0	108
P_{G1}, P_{G3}	108
Ω_1, Ω_3	108
R	108
z_B, z_W, z_G	108
x	108
ζ_G	111

Symbols	page
$\beta_{G1}, \ldots, \beta_{G4}$	111
$\beta_{G5}, \ldots, \beta_{G9}$	112
k	114
T_e	114
U	115
Γ	115
Appendix B	
ϵ	123
ρ	123
x	126
z	126

1. INTRODUCTION

The simplest conceivable scheme for providing an astronaut with the capability to perform controlled translational motions in "free" space is to attach a single thruster to a part of his body. Such a thruster would generate pure translational motion if the line of action of the thrust could be made to pass through the center of mass of the system. However, in any real system, it is practically impossible to make the line of action of a thrust pass through a preassigned point, and the use of a single thruster must, therefore, be expected to give rise to both rotation and translation. This fact provided the motivation for the present investigation.

The ideal analytical model for the purpose at hand would be an arbitrary rigid body under the action of a misaligned (not passing through the mass center), body-fixed force. A number of authors [1,2,3]* have dealt with this problem, but a complete, analytical description of the motion does not appear to have been given to date. The complexity of the problem is indicated by the fact that no one has yet been able to obtain a closed-form solution to even the Euler dynamical equations, let alone the kinematical equations for this problem. Leimanis and Lee [1,2] developed expressions for the angular velocity of the body for the special case of a body-fixed force whose moment about the mass center is parallel to one of the principal axes of inertia of the body for the mass

* Numbers in brackets designate references in the Bibliography.

center; however, these expressions involved integrals which, in general, cannot be evaluated by quadrature. Grammel [3] constructed approximate solutions of Euler's dynamical equations, by first considering the case of a torque vector parallel to the axis of minimum or maximum principal moment of inertia for the mass center, letting the initial angular velocity vector have but a small component perpendicular to the chosen axis, and using an iterative method. Next, he turned to the case of a torque vector that deviates only a little from one of the above mentioned axes while the initial angular velocity vector is parallel to the chosen axis. By combining these two solutions, he then obtained an approximate solution of Euler's equations for torques fixed in the vicinity of the axes of largest or smallest principal moment of inertia.

A major step in the direction of decreased analytical complexity can be taken in a physically meaningful way by taking advantage of the fact that the inertia ellipsoid for the mass center of a man in the posture of "attention" is nearly axisymmetric, with the symmetry axis parallel to the man's spine. (The two transverse principal moments of inertia for the mass center of an average man have values of approximately 7.4 and 7.9 slug-ft^2. Furthermore, if a back-pack is mounted on the man, the two transverse principal moments of inertia can be made even more nearly equal to each other by designing the back-pack suitably.) Thus it appears that one may expect to derive useful information from the study of a symmetric rigid body. A further simplification results from restricting the analysis to body-fixed forces whose lines of action lie in the plane containing the mass center of the system and normal to the symmetry axis.

One of the first authors to deal with the problem of a symmetric body subjected to the action of a body-fixed torque was Bödewadt [4], who showed that the solution of Euler's dynamical equations can be expressed in terms of integrals of the Fresnel type. Unfortunately, the next step of the solution, that of integrating a set of kinematical equations, was based on a false premise, because, as may be verified by substitution from Eq. (49) into Eq. (42) of [4], the method proposed by Bödewadt is valid only when the matrices U and W' commute, which is not the case for the problem at hand. Consequently, the orientation of the body remains to be determined as a function of time. (An English translation of Bödewadt's work appears in Chapter 10 of Leimanis' book [1].)

The problem of the symmetric body has also been studied by Auelmann [5], who attempted to find a closed-form solution to the kinematical equations. While falling short of this goal, he succeeded in obtaining an expression for the angle between the symmetry axis and the initial direction of the symmetry axis for the special case when initial angular velocity is normal to the symmetry axis.

A very elegant treatment of the problem of a symmetric rigid body under the action of a constant body-fixed force whose moment about the mass center is parallel to the symmetry axis was presented by Valeev [6], who used continued fractions to obtain a description of the motion of a space-fixed unit vector relative to three body-fixed unit vectors parallel to principal axes of inertia for the mass center.

Finally, the problem of planar motions of a human being under the action of a body-fixed force was attacked by Yatteau and Kane [7], who developed closed-form solutions to the two dimensional attitude and translational equations. The present investigation may thus be regarded as an extension of the work of Yatteau and Kane to three dimensions.

The analysis which follows is divided into four major parts. The first of these, Chapter 2, contains a detailed description of the system to be studied, and the orientation angles to be used in describing the attitude of the body are defined. Next, in Chapter 3, exact dynamical and kinematical equations governing attitude motions are formulated; the dynamical equations are solved exactly; and an approximate solution to the kinematical equations is obtained. The validity of the approximate solution is then tested by comparison with digital computer integrations of the exact equations. In Chapter 4, the motion of the system mass center is studied in terms of an approximate analytical solution, and results are once again tested by means of comparisons with solutions obtained by numerical integrations. Finally, Chapter 5 deals with the application of the results given in Chapters 3 and 4 to symmetric gyrostats subjected to body-fixed forces and torques.

2. PRELIMINARIES

2.1 System Description

The system to be analyzed is shown in Figure 1, where B represents an axially symmetric rigid body of mass m . The center of mass of B , designated B* , is located by a position vector $\underline{X}$ relative to a point O that is fixed in an inertial reference frame A. Mutually perpendicular unit vectors $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3 = \underline{a}_1 \times \underline{a}_2$ are fixed in A , and mutually perpendicular unit vectors $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3 = \underline{b}_1 \times \underline{b}_2$ are fixed in B parallel to principal axes of inertia of B for B* , with $\underline{b}_3$ parallel to the symmetry axis of B .

The moments of inertia of B with respect to lines passing through B* and parallel to $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ have the values I, I, and J, respectively.

It is presumed that B is subjected to the action of a force $\underline{F}$ of constant magnitude and fixed direction in B. This force is applied to B at a fixed point Q which is located relative to B* by a vector $\underline{q}$. Furthermore, it is assumed that both $\underline{F}$ and $\underline{q}$ are perpendicular to $\underline{b}_3$. Hence, if the nine scalar quantities x_i, F_i, and q_i $(i = 1,2,3)$ are defined as

$$x_i \triangleq \underline{X} \cdot \underline{a}_i \tag{2.1}$$

$$F_i \triangleq \underline{F} \cdot \underline{b}_i \qquad (i = 1,2,3) \tag{2.2}$$

$$q_i \triangleq \underline{q} \cdot \underline{b}_i \tag{2.3}$$

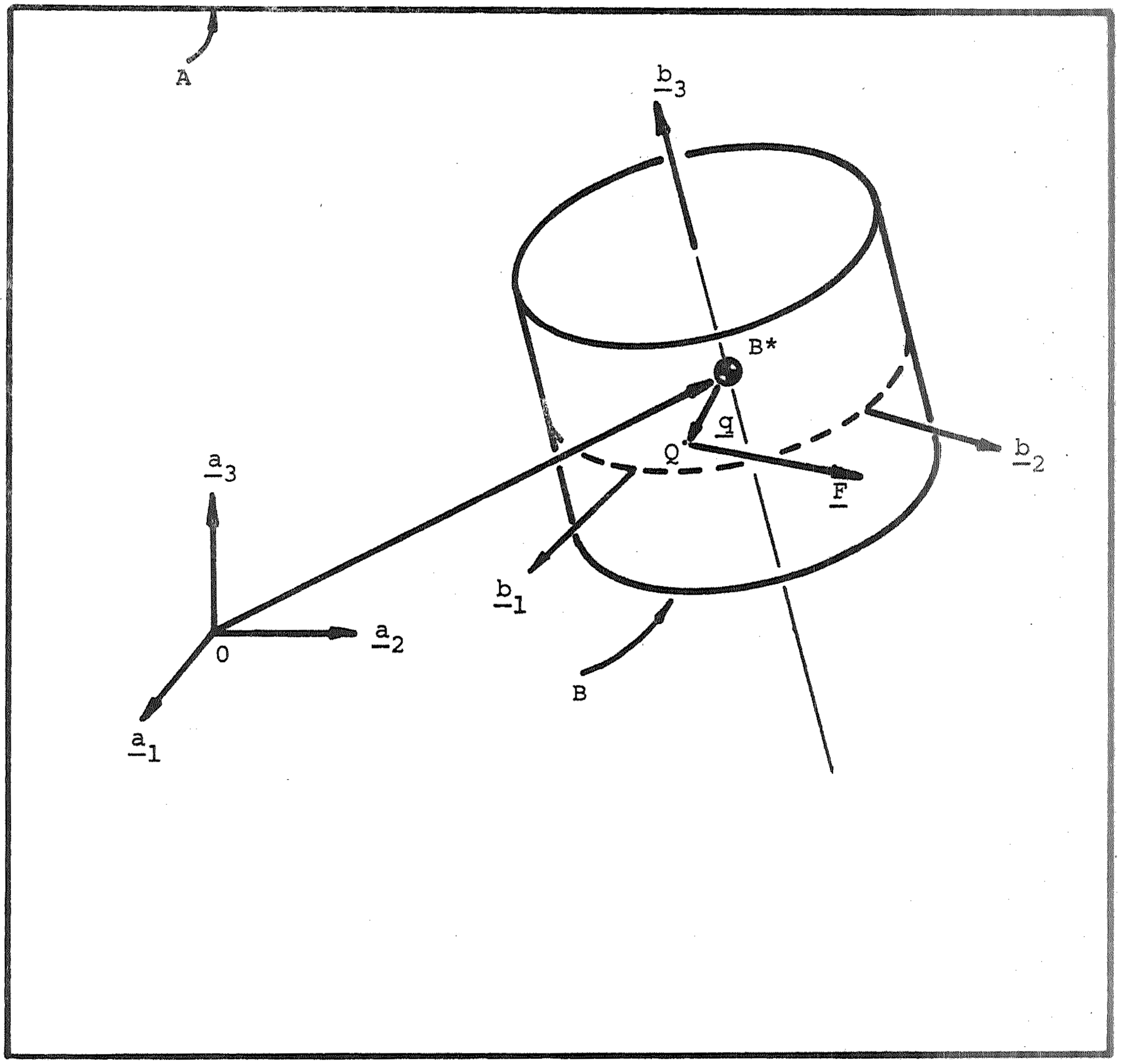

Figure 1. Model of an Axially Symmetric Rigid Body

then

$$F_3 = 0 \qquad (2.4)$$

and

$$q_3 = 0 \qquad (2.5)$$

When the force $\underline{F}$ exerted on B at Q is replaced with a force applied at B* together with a couple of torque $\underline{T}$, then

$$\underline{T} = \underline{q} \times \underline{F} \qquad (2.6)$$

and, in view of Eqs. (2.4) and (2.5),

$$\underline{T} = (q_1F_2 - q_2F_1)\ \underline{b}_3 \qquad (2.7)$$

Hence, if a constant T is defined as

$$T \triangleq q_1F_2 - q_2F_1 \qquad (2.8)$$

then

$$\underline{T} = T\ \underline{b}_3 \qquad (2.9)$$

2.2 Orientation Angles and Direction Cosines

In the sequel, it will be necessary to make use of relationships between orientation angles and direction cosines. This topic is, therefore, discussed briefly in general terms.

If $\underline{\alpha}_1$, $\underline{\alpha}_2$, and $\underline{\alpha}_3$ form a dextral set of orthogonal unit vectors fixed in a reference frame α (see Figure 2), and $\underline{\beta}_1$, $\underline{\beta}_2$, $\underline{\beta}_3$ form a similar set of unit vectors fixed in a rigid body β, then β can be brought into any desired orientation relative to α by aligning $\underline{\beta}_i$

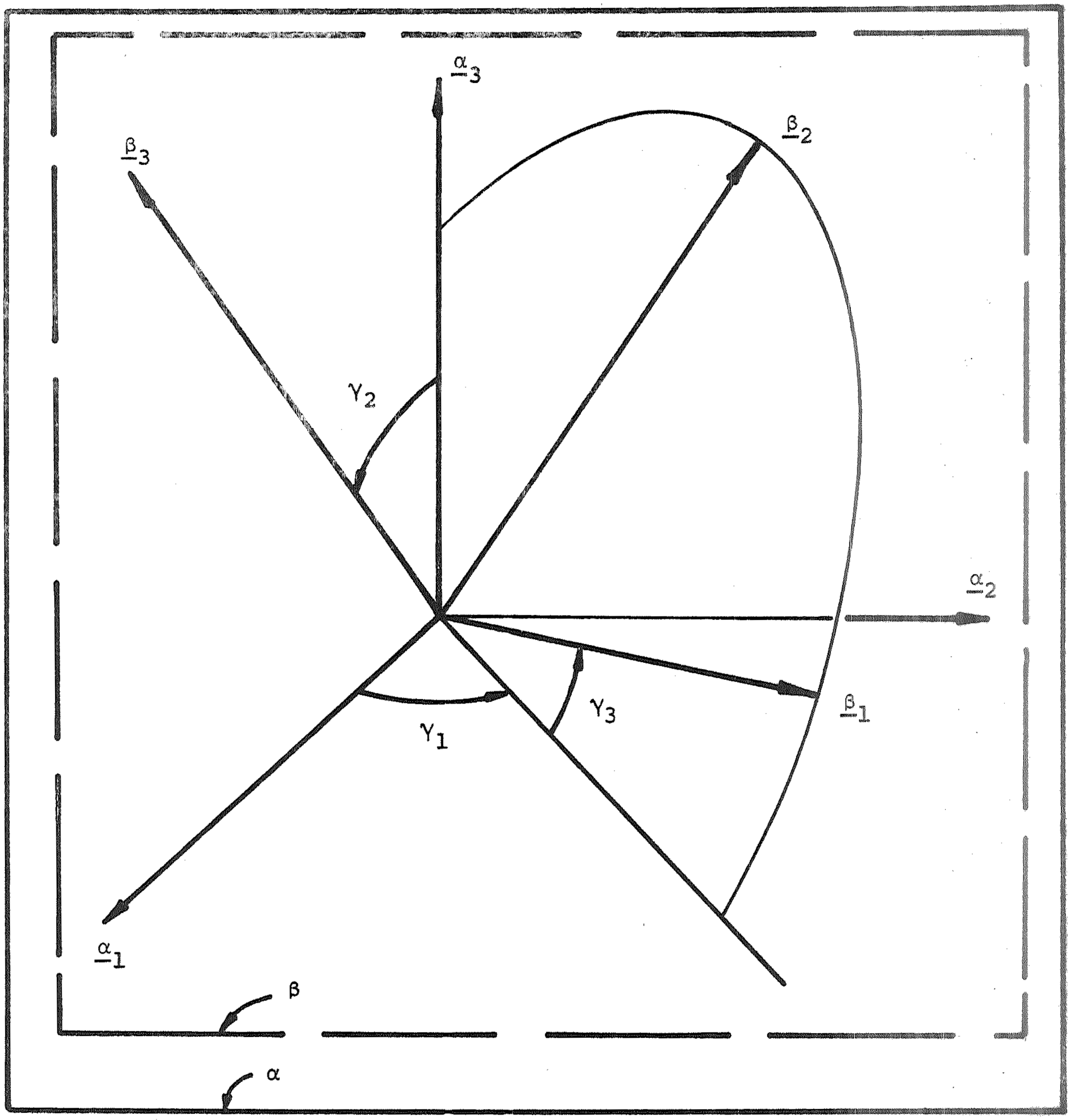

Figure 2. Attitude Angles

with $\underline{\alpha}_i$ (i = 1, 2, 3), and then performing successive rotations of β of amounts γ_1, γ_2, and γ_3, these rotations being characterized by the vectors $\gamma_1\underline{\beta}_3$, $\gamma_2\underline{\beta}_1$, $\gamma_3\underline{\beta}_3$, respectively. Furthermore, if direction cosines γ_{ij} (i, j = 1, 2, 3) are defined as

$$\gamma_{ij} \triangleq \underline{\alpha}_i \cdot \underline{\beta}_j \tag{2.10}$$

these are related to the angles γ_1, γ_2, and γ_3 as follows:

$$\gamma_{11} = \cos\gamma_1 \cos\gamma_3 - \cos\gamma_2 \sin\gamma_1 \sin\gamma_3 \tag{2.11}$$

$$\gamma_{12} = -\cos\gamma_1 \sin\gamma_3 - \cos\gamma_2 \sin\gamma_1 \cos\gamma_3 \tag{2.12}$$

$$\gamma_{13} = \sin\gamma_2 \sin\gamma_1 \tag{2.13}$$

$$\gamma_{21} = \sin\gamma_1 \cos\gamma_3 + \cos\gamma_2 \cos\gamma_1 \sin\gamma_3 \tag{2.14}$$

$$\gamma_{22} = -\sin\gamma_1 \sin\gamma_3 + \cos\gamma_2 \cos\gamma_1 \cos\gamma_3 \tag{2.15}$$

$$\gamma_{23} = -\sin\gamma_2 \cos\gamma_1 \tag{2.16}$$

$$\gamma_{31} = \sin\gamma_2 \sin\gamma_3 \tag{2.17}$$

$$\gamma_{32} = \sin\gamma_2 \cos\gamma_3 \tag{2.18}$$

$$\gamma_{33} = \cos\gamma_2 \tag{2.19}$$

Given the direction cosines, one can find unique values of the angles γ_1, γ_2, and γ_3 in the range from zero to 2π by taking the inverses of the following pairs of trigonometric functions:

$$\left.\begin{aligned}\cos\gamma_1 &= \frac{-\gamma_{23}}{\sqrt{\gamma_{13}^2+\gamma_{23}^2}}\\ \sin\gamma_1 &= \frac{\gamma_{13}}{\sqrt{\gamma_{13}^2+\gamma_{23}^2}}\end{aligned}\right\} \quad (2.20)$$

$$\left.\begin{aligned}\cos\gamma_2 &= \gamma_{33}\\ \sin\gamma_2 &= \frac{-1}{\gamma_{22}\sqrt{\gamma_{13}^2+\gamma_{23}^2}}\left(\gamma_{13}\gamma_{31}+\gamma_{23}\gamma_{32}\gamma_{33}\right)\end{aligned}\right\} \quad (2.21)$$

$$\left.\begin{aligned}\cos\gamma_3 &= \frac{\gamma_{31}}{\sqrt{\gamma_{31}^2+\gamma_{32}^2}}\\ \sin\gamma_3 &= \frac{\gamma_{32}}{\sqrt{\gamma_{31}^2+\gamma_{32}^2}}\end{aligned}\right\} \quad (2.22)$$

The necessity for using pairs of trigonometric functions arises from the fact that the inverse trigonometric functions are multiple-valued. Appendix A contains a Fortran IV subroutine intended to facilitate such inversions of trigonometric functions.

3. ORIENTATION

3.1 Dynamical Equations

The analysis of motions of a rigid body B possessing rotational symmetry is facilitated by introducing a reference frame, C, (see Figure 3) which is fixed neither in the body B nor in the inertial reference frame A, but is constrained to move in such a way that a unit vector, $\underline{c}_3$, fixed in C, remains at all times equal to the unit vector $\underline{b}_3$, which is fixed in B. The angular velocity of B relative to C, to be denoted by ${}^{C}\underline{\omega}^{B}$, is then necessarily parallel to $\underline{c}_3$ and can be expressed as

$$ {}^{C}\underline{\omega}^{B} = s\,\underline{c}_3 \tag{3.1} $$

where s is a function of time t. As will be seen presently, it is the choice of s which furnishes the key to the solution of the problem at hand. However, no matter how s is chosen, the angular velocity of C relative to A, ${}^{A}\underline{\omega}^{C}$, can be expressed as

$$ {}^{A}\underline{\omega}^{C} = p_1\,\underline{c}_1 + p_2\,\underline{c}_2 + p_3\,\underline{c}_3 \tag{3.2} $$

where p_i $(i = 1,2,3)$ are functions of t, and $\underline{c}_1$ and $\underline{c}_2$ are unit vectors fixed in C, perpendicular to each other, and such that $\underline{c}_1 \times \underline{c}_2 = \underline{c}_3$. It then follows that the angular velocity of B relative to A is given by[1]

---

[1] Numbers beneath equal signs are intended to direct attention to equations numbered correspondingly.

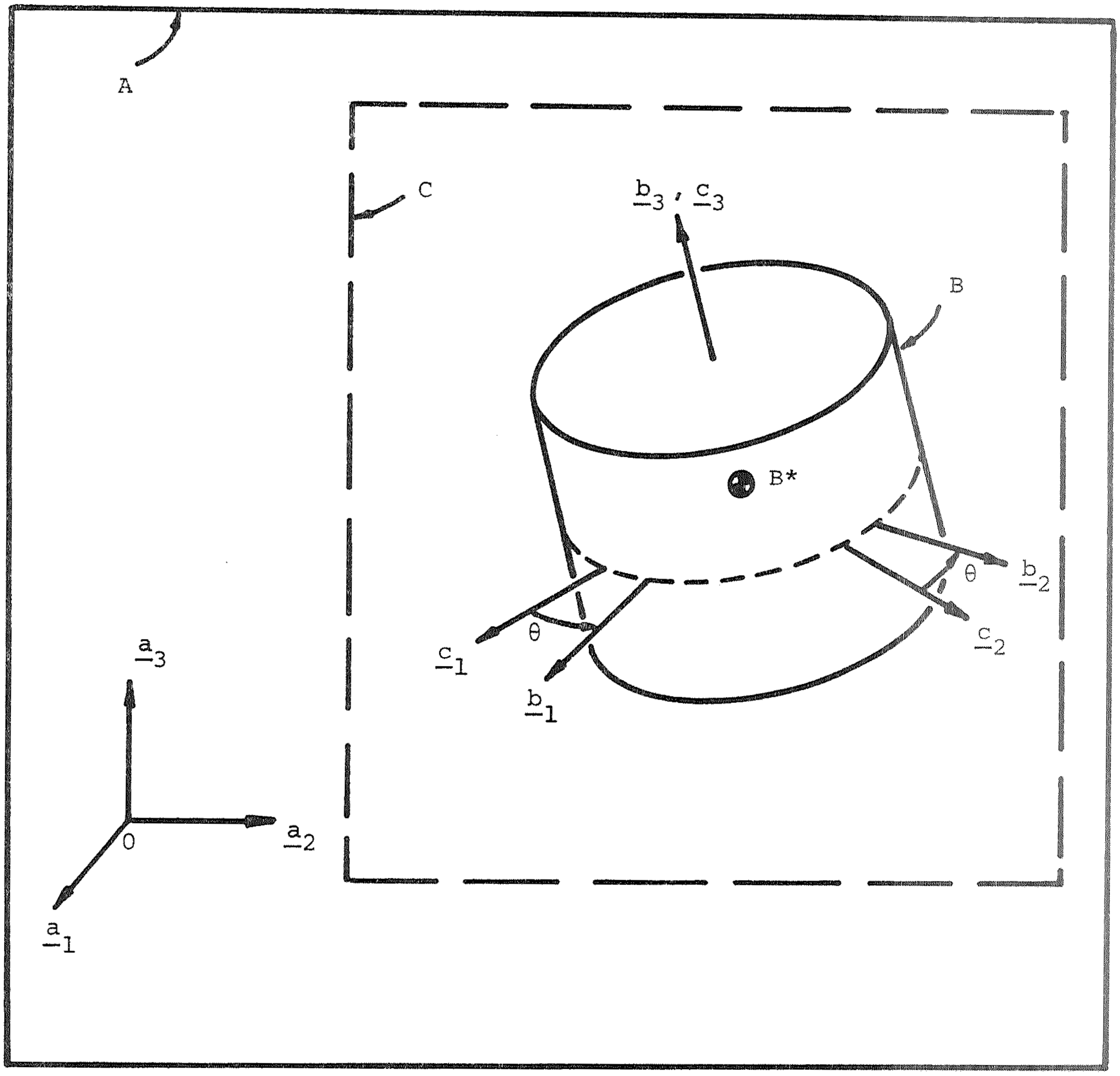

Figure 3. Illustration of the Reference Frame C

$$ {}^{A}\underline{\omega}^{B} = {}^{A}\underline{\omega}^{C} + {}^{C}\underline{\omega}^{B} $$

$$ \underset{(3.1,3.2)}{=} p_1 \underline{c}_1 + p_2 \underline{c}_2 + (p_3 + s) \underline{c}_3 \tag{3.3} $$

It may be verified that the angular momentum $\underline{H}$ of B with respect to B* in A is given by

$$ \underline{H} = I p_1 \underline{c}_1 + I p_2 \underline{c}_2 + J(p_3 + s) \underline{c}_3 \tag{3.4} $$

Hence, if upper left superscripts are used to identify the reference frame in which a time-differentiation of a vector is performed, then

$$ \frac{{}^{A}d\underline{H}}{dt} = \frac{{}^{C}d\underline{H}}{dt} + {}^{A}\underline{\omega}^{C} \times \underline{H} $$

$$ \underset{(3.4,3.2)}{=} \{I \dot{p}_1 + p_2[Js - (I-J)p_3]\} \underline{c}_1 $$

$$ + \{I \dot{p}_2 - p_1[Js - (I-J)p_3]\} \underline{c}_2 $$

$$ + [J(\dot{p}_3 + \dot{s})] \underline{c}_3 \tag{3.5} $$

In accordance with the angular momentum principle, the time rate of change of $\underline{H}$ in A is equal to the total moment about B* of the force(s) acting on B. Thus

$$ \frac{{}^{A}d\underline{H}}{dt} = \underline{q} \times \underline{F} \underset{(2.6)}{=} \underline{T} $$

$$ \underset{(2.9)}{=} T \underline{b}_3 = T \underline{c}_3 \tag{3.6} $$

and substitution into Eq. (3.5) leads to the scalar equations

$$I \dot{p}_1 + p_2[Js - (I-J)p_3] = 0 \tag{3.7}$$

$$I \dot{p}_2 - p_1[Js - (I-J)p_3] = 0 \tag{3.8}$$

$$J(\dot{p}_3 + \dot{s}) = T \tag{3.9}$$

Eqs. (3.7), (3.8), and (3.9) contain four unknown quantities, p_1, p_2, p_3, and s. However, s was introduced solely for the purpose of facilitating the analysis and may, therefore, be chosen at will; and a choice which, indeed, simplifies the subsequent analysis is

$$s = L\, p_3 \tag{3.10}$$

where L is defined as

$$L \triangleq \frac{I-J}{J} \tag{3.11}$$

This permits one to replace Eqs. (3.7)-(3.9) with

$$\dot{p}_1 = 0 \tag{3.12}$$

$$\dot{p}_2 = 0 \tag{3.13}$$

$$\dot{p}_3 = \lambda \tag{3.14}$$

where λ is defined as

$$\lambda \triangleq \frac{T}{I} \tag{3.15}$$

If p_{i0} denotes the initial value of p_i $(i = 1,2,3)$, Eqs. (3.12)-(3.14) lead immediately to

$$p_1 = p_{10} \tag{3.16}$$

$$p_2 = p_{20} \tag{3.17}$$

$$p_3 = \lambda t + p_{30} \tag{3.18}$$

Furthermore,

$$s \underset{(3.10,3.18)}{=} L(\lambda t + p_{30}) \tag{3.19}$$

The angular velocity of B in A can now be found by substitution into Eq. (3.3). However, as it is advantageous to express ${}^{A}\underline{\omega}^{B}$ in terms of $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ rather than $\underline{c}_1$, $\underline{c}_2$, and $\underline{c}_3$, a brief digression will prove useful.

If $\underline{c}_1$ is taken to be equal to $\underline{b}_1$ at $t = 0$, and θ is defined as

$$\theta \triangleq \int_0^t s \, dt$$

$$\underset{(3.19)}{=} L(\frac{1}{2} \lambda t^2 + p_{30} t) \tag{3.20}$$

then θ is the radian measure of the angle between $\underline{b}_1$ and $\underline{c}_1$, and the relationship between the two sets of unit vectors $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ and $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$ can be expressed as

$$\begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{b}_3 \end{bmatrix} \tag{3.21}$$

If ω_i $(i = 1, 2, 3)$ is now defined as

$$\omega_i \triangleq {}^A\underline{\omega}^B \cdot \underline{b}_i \qquad (i = 1, 2, 3) \tag{3.22}$$

it follows from Eq. (3.3) that

$$\omega_1 = p_1 \cos\theta + p_2 \sin\theta \tag{3.23}$$

$$\omega_2 = -p_1 \sin\theta + p_2 \cos\theta \tag{3.24}$$

$$\omega_3 = p_3 + s \tag{3.25}$$

Hence, if ω_{i0} denotes the initial value of ω_i $(i = 1, 2, 3)$, then

$$\omega_{10} \underset{(3.23)}{=} p_{10} \tag{3.26}$$

$$\omega_{20} \underset{(3.24)}{=} p_{20} \tag{3.27}$$

$$\omega_{30} \underset{(3.25,\, 3.19)}{=} (1 + L)p_{30} \tag{3.28}$$

and

$$p_1 \underset{(3.16,\, 3.26)}{=} \omega_{10} \tag{3.29}$$

$$p_2 \underset{(3.17,\, 3.27)}{=} \omega_{20} \tag{3.30}$$

$$p_3 \underset{(3.18,\, 3.28)}{=} \lambda t + \frac{\omega_{30}}{1 + L} \tag{3.31}$$

$$s \underset{(3.19,\, 3.28)}{=} L(\lambda t + \frac{\omega_{30}}{1 + L}) \tag{3.32}$$

and

$$\theta \underset{(3.20,3.28)}{=} L\left(\frac{1}{2}\lambda t^2 + \frac{\omega_{30}}{1+L}t\right) \qquad (3.33)$$

so that, substituting into Eqs. (3.23)-(3.25), one finds

$$\omega_1 = \omega_{10}\cos\theta + \omega_{20}\sin\theta$$

$$= \sqrt{\omega_{10}^2 + \omega_{20}^2}\,\cos\left[\theta + \tan^{-1}\left(-\frac{\omega_{20}}{\omega_{10}}\right)\right] \qquad (3.34)$$

$$\omega_2 = -\omega_{10}\sin\theta + \omega_{20}\cos\theta$$

$$= \sqrt{\omega_{10}^2 + \omega_{20}^2}\,\sin\left[\theta + \tan^{-1}\left(-\frac{\omega_{20}}{\omega_{10}}\right)\right] \qquad (3.35)$$

$$\omega_3 = (1 + L)\lambda t + \omega_{30} \qquad (3.36)$$

Eqs. (3.33)-(3.36) constitute a complete solution of the dynamical equations of rotational motion for the problem at hand. The first two of these equations may be regarded as parametric equations of a circle of radius $\sqrt{\omega_{10}^2 + \omega_{20}^2}$ in the plane normal to $\underline{b}_3$, a plane which we shall call the <u>equatorial plane</u>. The equations show, furthermore, that the orthogonal projection of the vector ${}^A\underline{\omega}^B$ in this plane rotates about the symmetry axis with an angular speed s given by Eq. (3.32). Considering the expression for ω_3 (see Eq. (3.36)), one can thus conclude that the end point of ${}^A\underline{\omega}^B$ describes a helical curve on the surface of a cylinder of radius $\sqrt{\omega_{10}^2 + \omega_{20}^2}$; that ${}^A\underline{\omega}^B$ tends toward parallelism with the symmetry axis of B; and that the symmetry axis approaches some line fixed in reference frame A.

To describe the instantaneous orientation of B in A, one must integrate yet one more set of differential equations, the so-called kinematical equations.

3.2 Kinematical Equations

Conceptually, the simplest way to obtain a complete description of the attitude motion of B in A would be to solve the nine first-order differential equations governing the elements of the direction cosine matrix relating the body-fixed unit vectors $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ to the space-fixed unit vectors $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$. However, since these equations are strongly coupled and involve the time-dependent functions ω_i $(i = 1,2,3)$ (see Eqs. (3.34)-(3.36)), there appears to be little hope of carrying such a solution to a successful conclusion. Alternatively, one may take advantage of the relative simplicity of the differential equations governing the elements of the direction cosine matrix relating $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$ to $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and of the fact that, once the orientation of C in A is known, the orientation of B in A can be found easily, because B performs a simple rotational motion in C, and the time-dependence of the angle θ associated with this rotational motion is already known (see Eq. (3.33)).

Accordingly, one may begin by defining a_{ij} as

$$a_{ij} \triangleq \underline{a}_i \cdot \underline{c}_j \qquad (i,j = 1,2,3) \tag{3.37}$$

and by observing that the first time-derivative of $\underline{a}_i$ in A can be expressed as

$$\frac{{}^{A}d\underline{a}_i}{dt} = \frac{{}^{C}d\underline{a}_i}{dt} + {}^{A}\underline{\omega}^{C} \times \underline{a}_i \qquad (i = 1,2,3) \tag{3.38}$$

Since $\underline{a}_i$ is fixed in A, this derivative is equal to zero. Using Eq. (3.2), one is thus led to

$$\dot{a}_{i1}\,\underline{c}_1 + \dot{a}_{i2}\,\underline{c}_2 + \dot{a}_{i3}\,\underline{c}_3 = (p_3 a_{i2} - p_2 a_{i3})\,\underline{c}_1 + (p_1 a_{i3} - p_3 a_{i1})\,\underline{c}_2 + (p_2 a_{i1} - p_1 a_{i2})\,\underline{c}_3$$

$$(i = 1,2,3) \tag{3.40}$$

The quantities p_i $(i = 1,2,3)$ appearing in Eq. (3.40) are known functions of t (see Eqs. (3.29)-(3.31)); furthermore, p_2 may be set equal to zero, for this implies only that

$$\omega_{20} = 0 \tag{3.41}$$

that is, that ${}^{A}\underline{\omega}^{B}$ is initially perpendicular to $\underline{b}_2$, which can always be arranged by choosing the orientations of $\underline{b}_1$ and $\underline{b}_2$ in B suitably. Consequently, one can replace Eqs. (3.40) with the nine equations

$$\dot{a}_{i1} = p_3 a_{i2} \tag{3.42}$$

$$\dot{a}_{i2} = p_1 a_{i3} - p_3 a_{i1} \qquad (i = 1,2,3) \tag{3.43}$$

$$\dot{a}_{i3} = -p_1 a_{i2} \tag{3.44}$$

or, after using Eqs. (3.16)-(3.18),

$$\dot{a}_{i1} - p_{30} a_{i2} = \lambda t a_{i2} \tag{3.45}$$

$$\dot{a}_{i2} - p_{10}a_{i3} + p_{30}a_{i1} = -\lambda t a_{i1} \qquad (i = 1,2,3) \tag{3.46}$$

$$\dot{a}_{i3} + p_{10}a_{i2} = 0 \tag{3.47}$$

The initial value of a_{ij}, to be denoted by $a_{ij}(0)$, depends on the initial orientation of $\underline{c}_1$, $\underline{c}_2$, and $\underline{c}_3$ relative to $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$. If $\underline{c}_i = \underline{a}_i$ at $t = 0$, which can always be arranged by choosing $\underline{a}_i$ $(i = 1,2,3)$ properly, then (see Eqs. (3.37))

$$a_{ij}(0) = \delta_{ij} \qquad (i,j = 1,2,3) \tag{3.48}$$

where

$$\delta_{ij} \triangleq 1 \quad (i = j) \quad , \quad \delta_{ij} \triangleq 0 \quad (i \neq j) \tag{3.49}$$

These initial conditions also imply that $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ are respectively parallel to $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ at $t = 0$, since $\underline{c}_1$, $\underline{c}_2$, and $\underline{c}_3$ initially coincide with $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$, respectively.

Despite their relative simplicity, Eqs. (3.45)-(3.47) do not appear to admit of an exact solution. Hence we shall resort to a method of successive approximations.

Temporarily confining attention to $i = 1$, we have

$$\dot{a}_{11} - p_{30}a_{12} \underset{(3.45)}{=} \lambda t a_{12} \tag{3.50}$$

$$\dot{a}_{12} - p_{10}a_{13} + p_{30}a_{11} \underset{(3.46)}{=} -\lambda t a_{11} \tag{3.51}$$

$$\dot{a}_{13} + p_{10}a_{12} \underset{(3.47)}{=} 0 \tag{3.52}$$

and the associated initial conditions become (see Eq. (3.48))

$$a_{11}(0) = 1 \quad , \qquad a_{12}(0) = a_{13}(0) = 0 \tag{3.53}$$

We now integrate Eqs. (3.50)-(3.52) with $\lambda = 0$ and require the resulting solution to satisfy Eqs. (3.53). After defining parameters p_0, P_1, and P_3 as

$$p_0 \triangleq \sqrt{p_{10}^2 + p_{30}^2} \tag{3.54}$$

$$P_1 \triangleq \frac{p_{10}}{p_0} \qquad (p_0 \neq 0) \tag{3.55}$$

$$P_3 \triangleq \frac{p_{30}}{p_0} \qquad (p_0 \neq 0) \tag{3.56}$$

so that

$$P_1^2 + P_3^2 = 1 \tag{3.57}$$

we thus arrive at the first approximation,

$$a_{11} \approx P_1^2 + P_3^2 \cos p_0 t \tag{3.58}$$

$$a_{12} \approx -P_3 \sin p_0 t \tag{3.59}$$

$$a_{13} \approx P_1 P_3 (1-\cos p_0 t) \tag{3.60}$$

The physical significance of these results is that they correspond to the torque-free motion of an axially symmetrical rigid body.

By substituting from Eqs. (3.58)-(3.60) into the right-hand members of Eqs. (3.50)-(3.52), we next obtain a set of differential equations whose solution, again required to satisfy Eqs. (3.53), yields the second approximation,

$$a_{11} \approx P_1^2 + P_3^2 \cos p_0 t - P_1^2 P_3 \left(\frac{\lambda t}{p_0}\right)(1-\cos p_0 t)$$

$$- \frac{1}{2} P_3^3 \lambda t^2 \sin p_0 t \tag{3.61}$$

$$a_{12} \approx -P_1^2 \left(\frac{\lambda}{p_0^2}\right) (1-\cos p_0 t) - P_3 \sin p_0 t$$

$$- \frac{1}{2} P_3^2 \lambda t^2 \cos p_0 t \tag{3.62}$$

$$a_{13} \approx P_1 P_3 (1-\cos p_0 t) - P_1 \left(\frac{\lambda}{p_0^2}\right) \sin p_0 t + P_1^3 \left(\frac{\lambda t}{p_0}\right)$$

$$+ P_1 P_3^2 \left(\frac{\lambda t}{p_0}\right) \cos p_0 t + \frac{1}{2} P_1 P_3^2 \lambda t^2 \sin p_0 t \tag{3.63}$$

and, repeating this process once more, we obtain as the third approximation for a_{1j} $(j = 1,2,3)$

$$a_{11} \approx 1 - \left[P_3^2 + \left(\frac{\lambda}{p_0^2}\right)^2 P_1^2 \left(1 + 4P_3^2 + P_1 P_3 - 3P_1 P_3^3\right) \right] (1-\cos p_0 t)$$

$$+ \left(\frac{\lambda}{p_0^2}\right) \left(1 + \frac{3}{2} P_3^2 - \frac{5}{2} P_3^4\right) \left(\frac{\lambda t}{p_0}\right) \sin p_0 t - P_1^2 P_3 \left(\frac{\lambda t}{p_0}\right) (1-\cos p_0 t)$$

$$+ \frac{1}{2} P_1^2 \left(3P_3^2 - 1\right) \left(\frac{\lambda t}{p_0}\right)^2 - \frac{3}{2} P_1^2 P_3^2 \left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t$$

$$- \frac{2}{3} P_1^2 P_3^2 \left(\frac{p_0^2}{\lambda}\right) \left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t - \frac{1}{8} P_3^4 \left(\frac{p_0^2}{\lambda}\right)^2 \left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t$$

$$- \frac{1}{2} P_3^3 \left(\frac{p_0^2}{\lambda}\right) \left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t \tag{3.64}$$

$$a_{12} \approx -P_1^2\left(\frac{\lambda}{p_0^2}\right)(1-\cos p_0 t) - \left[P_3 + P_1^2\left(\frac{\lambda}{p_0^2}\right)^2\left(\frac{3}{2}P_3 + P_1 - 3P_1P_3^2\right)\right]\sin p_0 t$$

$$+ 3P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) - \frac{1}{2}P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)\cos p_0 t - \frac{1}{2}P_1^2P_3\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t$$

$$- \frac{1}{6}P_1^2P_3\left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t - \frac{1}{2}P_3^2\left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t$$

$$+ \frac{1}{8}P_3^3\left(\frac{p_0^2}{\lambda}\right)^2\left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t \qquad (3.65)$$

$$a_{13} \approx \left[P_1P_3 + \left(\frac{\lambda}{p_0^2}\right)^2 \left(4P_1^3P_3 + P_1^4 - 3P_1^4P_3^2 - 3P_1P_3\right)\right](1-\cos p_0 t)$$

$$- P_1\left(\frac{\lambda}{p_0^2}\right)\sin p_0 t + P_1^3\left(\frac{\lambda t}{p_0}\right) - \frac{3}{2}P_1^3P_3\left(\frac{\lambda t}{p_0}\right)^2 + P_1P_3^2\left(\frac{\lambda t}{p_0}\right)\cos p_0 t$$

$$+ P_1P_3\left(\frac{1}{2}P_1^2 + 3P_3^2\right)\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)\sin p_0 t - \frac{3}{2}P_1P_3^3\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t$$

$$+ \frac{1}{2}P_1P_3\left(\frac{1}{3}P_1^2 - P_3^2\right)\left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t$$

$$+ \frac{1}{8}P_1P_3^3\left(\frac{p_0^2}{\lambda}\right)^2\left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t + \frac{1}{2}P_1P_3^2\left(\frac{\lambda t}{p_0}\right)^2\left(\frac{p_o^2}{\lambda}\right)\sin p_0 t \qquad (3.66)$$

For $i = 2$, the same technique leads to the following third approximation for a_{2j} $(j = 1,2,3)$:

$$a_{21} \approx -P_1^2\left(\frac{\lambda}{p_0^2}\right)(1-\cos p_0 t) + P_3\left[1 + \frac{5}{2}P_1^2\left(\frac{\lambda}{p_0^2}\right)^2\right]\sin p_0 t$$

$$+ P_1^2\left(\frac{\lambda t}{p_0}\right)\sin p_0 t - \frac{7}{2}P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)\cos p_0 t - 2P_1^2P_3\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t$$

$$+ \frac{2}{3} P_1^2 P_3 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t + P_1^2 P_3 \left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)$$

$$+ \frac{1}{2} P_3^2 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t - \frac{1}{8} P_3^3 \left(\frac{p_0^2}{\lambda}\right)^2 \left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t \qquad (3.67)$$

$$a_{22} \approx P_1^2 \left(\frac{\lambda}{p_0^2}\right)^2 + \left[1 - P_1^2 \left(\frac{\lambda}{p_0^2}\right)^2\right] \cos p_0 t - \frac{1}{2} P_1^2 \left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \sin p_0 t$$

$$- \frac{1}{6} P_1^2 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t - \frac{1}{8} P_3^2 \left(\frac{p_0^2}{\lambda}\right)^2 \left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t$$

$$- \frac{1}{2} P_3 \left(\frac{p_0^2}{\lambda}\right) \left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t \qquad (3.68)$$

$$a_{23} \approx -P_1 P_3 \left(\frac{\lambda}{p_0^2}\right) (1 - \cos p_0 t) - P_1 \left[1 + \frac{5}{2} P_1^2 \left(\frac{\lambda}{p_0^2}\right)^2 - 3\left(\frac{\lambda}{p_0^2}\right)^2\right] \sin p_0 t$$

$$-P_1^3 \left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) + \frac{1}{2} P_1 \left(\frac{\lambda}{p_0^2}\right)(1 - 7P_3^2)\left(\frac{\lambda t}{p_0}\right) \cos p_0 t$$

$$+ P_1 P_3 \left(\frac{\lambda t}{p_0}\right) \sin p_0 t + \frac{1}{2} P_1 (1 - 4P_3^2)\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t$$

$$- \frac{1}{6} P_1 (1 - 4P_3^2)\left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t - \frac{1}{2} P_1 P_3 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t$$

$$+ \frac{1}{8} P_1 P_3^2 \left(\frac{p_0^2}{\lambda}\right)^2 \left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t \qquad (3.69)$$

Finally, for $i = 3$, one obtains as the third approximation for a_{3j} $(j = 1, 2, 3)$

$$a_{31} \approx P_1 P_3 \left[1 + \left(\frac{\lambda}{p_0^2}\right)^2 (2 - 5P_3^2)\right](1 - \cos p_0 t) + P_1 \left(\frac{\lambda}{p_0^2}\right) \sin p_0 t$$

$$- \frac{5}{2} P_1^3 P_3 \left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \sin p_0 t - P_1^3 \left(\frac{\lambda t}{p_0}\right) \cos p_0 t$$

$$+ P_1 P_3 \left(2 - \frac{3}{2} P_3^2\right)\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t - \frac{1}{2} P_1 P_3 (1 - 3P_3^2)\left(\frac{\lambda t}{p_0}\right)^2$$

$$- P_1 P_3^2 \left(\frac{\lambda t}{p_0}\right) + \frac{1}{8} P_1 P_3^3 \left(\frac{p_0^2}{\lambda}\right)^2 \left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t$$

$$+ \frac{1}{2} P_1 P_3^2 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t + \frac{2}{3} P_1^3 P_3 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t \quad (3.70)$$

$$a_{32} \approx -P_1 P_3 \left(\frac{\lambda}{p_0^2}\right)(1 - \cos p_0 t) + P_1 \left[1 + \left(\frac{\lambda}{p_0^2}\right)^2 \left(2 - 5P_3^2 - \frac{5}{2} P_1^2\right)\right] \sin p_0 t$$

$$+ 3P_1 P_3^2 \left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) + \frac{1}{2} P_1^3 \left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \cos p_0 t - \frac{1}{2} P_1 P_3^2 \left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t$$

$$+ \frac{1}{6} P_1^3 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t + \frac{1}{2} P_1 P_3 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t$$

$$- \frac{1}{8} P_1 P_3^2 \left(\frac{p_0^2}{\lambda}\right)^2 \left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t \quad (3.71)$$

$$a_{33} \approx P_3^2 + P_1^2 \cos p_0 t + 5P_1^2 P_3^2 \left(\frac{\lambda}{p_0^2}\right)^2 (1 - \cos p_0 t) + P_1^2 P_3 \left(\frac{\lambda t}{p_0}\right)$$

$$- \frac{3}{2} P_1^2 P_3^2 \left(\frac{\lambda t}{p_0}\right)^2 - P_1^2 P_3 \left(\frac{\lambda t}{p_0}\right) \cos p_0 t$$

$$+ \frac{1}{2} P_1^2 (1 - 5P_3^2)\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \sin p_0 t - \frac{1}{2} P_1^2 (1 - 3P_3^2)\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t$$

$$- \frac{1}{6} P_1^2 (1 - 4P_3^2)\left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t - \frac{1}{8} P_1^2 P_3^2 \left(\frac{p_0^2}{\lambda}\right)^2 \left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t$$

$$- \frac{1}{2} P_1^2 P_3 \left(\frac{p_0^2}{\lambda}\right)\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t \quad (3.72)$$

An approximate description of the attitude motion of C in A which is equivalent to the one furnished by Eqs. (3.64)-(3.72), but is simpler in form, may be obtained by introducing a dextral set of orthogonal unit vectors, $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$, fixed in reference frame A in such a way that $\underline{e}_1$ has the same directions as the initial angular velocity of C in A, $\underline{e}_2 = \underline{a}_2$, and $\underline{e}_3 = \underline{e}_1 \times \underline{e}_2$. Keeping in mind that $\underline{c}_i = \underline{a}_i$ at $t = 0$, one can then express $\underline{e}_1$ as

$$\underline{e}_1 \underset{(3.2)}{=} \frac{p_{10}\,\underline{a}_1 + p_{30}\,\underline{a}_3}{\left(p_{10}^2 + p_{30}^2\right)^{1/2}} \tag{3.73}$$

or, after using Eqs. (3.54)-(3.56), as

$$\underline{e}_1 = P_1\,\underline{a}_1 + P_3\,\underline{a}_3 \tag{3.74}$$

so that

$$\underline{e}_3 = \underline{e}_1 \times \underline{e}_2 = \underline{e}_1 \times \underline{a}_2 \underset{(3.74)}{=} -P_3\,\underline{a}_1 + P_1\,\underline{a}_3 \tag{3.75}$$

and, in matrix notation,

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} P_1 & 0 & P_3 \\ 0 & 1 & 0 \\ -P_3 & 0 & P_1 \end{bmatrix} \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} \tag{3.76}$$

Now, Eqs. (3.37) are equivalent to the matrix equation

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \end{bmatrix} \tag{3.77}$$

Consequently

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} P_1 & 0 & P_3 \\ 0 & 1 & 0 \\ -P_3 & 0 & P_1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \end{bmatrix} \tag{3.78}$$

and, if e_{ij} is defined as

$$e_{ij} \triangleq \underline{e}_i \cdot \underline{c}_j \qquad (i,j = 1,2,3) \tag{3.79}$$

then the matrix $\tilde{e}$ having e_{ij} as the element in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column is given by

$$\tilde{e} \underset{(3.78)}{=} \begin{bmatrix} P_1a_{11} + P_3a_{31} & P_1a_{12} + P_3a_{32} & P_1a_{13} + P_3a_{33} \\ a_{21} & a_{22} & a_{23} \\ -P_3a_{11} + P_1a_{31} & -P_3a_{12} + P_1a_{32} & -P_3a_{13} + P_1a_{33} \end{bmatrix} \tag{3.80}$$

and substitution from Eqs. (3.64)-(3.72), into Eq. (3.80) yields

$$e_{11} \approx P_1 - P_1P_3\left(\frac{\lambda t}{p_0}\right) - \frac{1}{2} P_1(1 - 4P_3^2)\left(\frac{\lambda t}{p_0}\right)^2 - \frac{1}{2} P_1P_3\left(\frac{\lambda t}{p_0}\right)^2 (1 - \cos p_0 t)$$

$$+ P_1^3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \sin p_0 t + P_1P_3\left(\frac{\lambda}{p_0^2}\right) \sin p_0 t$$

$$- P_1\left(\frac{\lambda}{p_0^2}\right)^2 (P_3^4 + P_1^5P_3 - 2P_1^3P_3^3 + 1 + P_3^2)(1 - \cos p_0 t) \qquad (3.81)$$

$$e_{12} \approx - \frac{1}{2} P_1P_3\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t - P_1(2P_3 + P_1^3 + P_3^3 - 3P_1^3P_3^2)\left(\frac{\lambda}{p_0^2}\right)^2 \sin p_0 t$$

$$- P_1\left(\frac{\lambda}{p_0^2}\right)(1 - \cos p_0 t) + 3P_1P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \qquad (3.82)$$

$$e_{13} \approx P_3 + P_1^2\left(\frac{\lambda t}{p_0}\right) - 2P_1^2P_3\left(\frac{\lambda t}{p_0}\right)^2 + \frac{1}{2} P_1^2P_3\left(\frac{\lambda t}{p_0}\right)^2 (1 - \cos p_0 t)$$

$$- P_1^2\left(\frac{\lambda}{p_0^2}\right) \sin p_0 t + P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \sin p_0 t$$

$$+ P_1^2\left(\frac{\lambda}{p_0^2}\right)^2 (P_1^3 + P_3^3 + P_3 - 3P_1^3P_3^2)\ (1 - \cos p_0 t) \qquad (3.83)$$

$$e_{21} \approx P_3 \sin p_0 t + P_1^2\left(\frac{\lambda t}{p_0}\right) \sin p_0 t - 2P_1^2P_3\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t$$

$$+ \frac{1}{2} P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t + \frac{2}{3} P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t$$

$$- \frac{1}{8} P_3^3\left(\frac{\lambda}{p_0^2}\right)^{-2}\left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t - P_1^2\left(\frac{\lambda}{p_0^2}\right)(1 - \cos p_o t)$$

$$+ \frac{5}{2} P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)^2 \sin p_0 t - \frac{7}{2} P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \cos p_0 t$$

$$+ P_1^2P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \qquad (3.84)$$

$$e_{22} \approx \cos p_0 t - \frac{1}{2} P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t - \frac{1}{6} P_1^2\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t$$

$$- \frac{1}{8} P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-2}\left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t + P_1^2\left(\frac{\lambda}{p_0^2}\right)^2 - P_1^2\left(\frac{\lambda}{p_0^2}\right)^2 \cos p_0 t$$

$$- \frac{1}{2} P_1^2\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \sin p_0 t \tag{3.85}$$

$$e_{23} \approx -P_1 \sin p_0 t + P_1 P_3\left(\frac{\lambda t}{p_0}\right) \sin p_0 t + \frac{1}{2} P_1(1 - 4P_3^2)\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t$$

$$- \frac{1}{2} P_1 P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t - \frac{1}{6} P_1(1 - 4P_3^2)\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t$$

$$+ \frac{1}{8} P_1 P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-2}\left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t - P_1 P_3\left(\frac{\lambda}{p_0^2}\right)(1 - \cos p_0 t)$$

$$- \frac{5}{2} P_1^3\left(\frac{\lambda}{p_0^2}\right)^2 \sin p_0 t + 3P_1\left(\frac{\lambda}{p_0^2}\right)^2 \sin p_0 t - P_1^3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right)$$

$$+ \frac{1}{2} P_1(1 - 7P_3^2)\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \cos p_0 t \tag{3.86}$$

$$e_{31} \approx -P_3 \cos p_0 t - P_1^2\left(\frac{\lambda t}{p_0}\right) \cos p_0 t + 2P_1^2 P_3\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t$$

$$+ \frac{1}{2} P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t + \frac{2}{3} P_1^2 P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t$$

$$+ \frac{1}{8} P_3^3\left(\frac{\lambda}{p_0^2}\right)^{-2}\left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t$$

$$+ P_1^2 P_3\left(\frac{\lambda}{p_0^2}\right)^2 (2 + P_1^2 + P_1 P_3 - 3P_1 P_3^3)(1 - \cos p_0 t)$$

$$- \frac{7}{2} P_1^2 P_3\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \sin p_0 t + P_1^2\left(\frac{\lambda}{p_0^2}\right) \sin p_0 t \tag{3.87}$$

$$e_{32} \approx \sin p_0 t + \frac{1}{2} P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t + \frac{1}{6} P_1^2\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^3 \cos p_0 t$$

$$- \frac{1}{8} P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-2}\left(\frac{\lambda t}{p_0}\right)^4 \sin p_0 t + \frac{1}{2} P_1^2\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \cos p_0 t$$

$$+ P_1^2\left(\frac{\lambda}{p_0^2}\right)^2\left(- \frac{3}{2} + P_1^2 + P_1 P_3 - 3P_1 P_3^3\right) \sin p_0 t \qquad (3.88)$$

$$e_{33} \approx P_1 \cos p_0 t - P_1 P_3\left(\frac{\lambda t}{p_0}\right) \cos p_0 t - \frac{1}{2} P_1(1 - 4P_3^2)\left(\frac{\lambda t}{p_0}\right)^2 \cos p_0 t$$

$$- \frac{1}{2} P_1 P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t - \frac{1}{6} P_1(1 - 4P_3^2)\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^3 \sin p_0 t$$

$$- \frac{1}{8} P_1 P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-2}\left(\frac{\lambda t}{p_0}\right)^4 \cos p_0 t + P_1 P_3\left(\frac{\lambda}{p_0^2}\right) \sin p_0 t$$

$$+ \frac{1}{2} P_1(1 - 7P_3^2)\left(\frac{\lambda}{p_0^2}\right)\left(\frac{\lambda t}{p_0}\right) \sin p_0 t$$

$$+ P_1\left(\frac{\lambda}{p_0^2}\right)^2 (P_1^2 P_3^2 - P_1^3 P_3 + 3P_1^3 P_3^3 + 3P_3^2)\ (1 - \cos p_0 t) \qquad (3.89)$$

If $p_{30} = 0$, so that $P_1 = 1$ and $P_3 = 0$, then e_{ij} is equal to a_{ij} $(i, j = 1, 2, 3)$ because $\underline{e}_i$ is then equal to $\underline{a}_i$ $(i = 1, 2, 3)$.

Eqs. (3.81)-(3.89) have been written in such a way that t is always multiplied by λ and divided by p_0 when it appears outside of the argument of a trigonometric function, and λ is divided by p_0^2 when it is not multiplied by t. This representation of e_{ij} $(i, j = 1, 2, 3)$ permits one to carry out a systematic simplification of the equations and thus to arrive at approximate results which, as will be seen in the sequel, can furnish an entirely satisfactory description of the motion under consideration. Specifically, dropping from Eqs. (3.81)-(3.89) all terms involving $(\lambda/p_0^2)^n$ with $n \geqq 1$ and all terms involving $(\lambda t/p_0)^m$ with $m \geqq 3$, and defining $k_1, \ldots, k_6$ as

$$k_1 \triangleq P_1 - P_1 P_3\left(\frac{\lambda t}{p_0}\right) - \frac{1}{2} P_1(1-4P_3^2)\left(\frac{\lambda t}{p_0}\right)^2 \tag{3.90}$$

$$k_2 \triangleq \frac{1}{2} P_1 P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \tag{3.91}$$

$$k_3 \triangleq P_3 + P_1^2\left(\frac{\lambda t}{p_0}\right) - 2P_1^2 P_3\left(\frac{\lambda t}{p_0}\right)^2 \tag{3.92}$$

$$k_4 \triangleq \frac{1}{2} P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \tag{3.93}$$

$$k_5 \triangleq \frac{1}{2} P_3\left(\frac{\lambda}{p_0^2}\right)^{-1}\left(\frac{\lambda t}{p_0}\right)^2 \tag{3.94}$$

$$k_6 \triangleq 1 \tag{3.95}$$

one arrives at

$$e_{11} \approx k_1 - \frac{1}{2} P_1 P_3\left(\frac{\lambda t}{p_0}\right)^2 (1-\cos p_0 t) \tag{3.96}$$

$$e_{12} \approx -\frac{1}{2} P_1 P_3\left(\frac{\lambda t}{p_0}\right)^2 \sin p_0 t \tag{3.97}$$

$$e_{13} \approx k_3 + \frac{1}{2} P_1^2 P_3\left(\frac{\lambda t}{p_0}\right)^2 (1-\cos p_0 t) \tag{3.98}$$

$$e_{21} \approx k_3 \sin p_0 t + k_4 \cos p_0 t$$

$$= \sqrt{k_3^2 + k_4^2}\cos\left[p_0 t + \tan^{-1}\left(-\frac{k_3}{k_4}\right)\right] \tag{3.99}$$

$$e_{22} \approx - k_5 \sin p_0 t + k_6 \cos p_0 t$$

$$= \sqrt{k_5^2 + k_6^2} \cos\left[p_0 t + \tan^{-1}\left(\frac{k_5}{k_6}\right)\right] \qquad (3.100)$$

$$e_{23} \approx - k_1 \sin p_0 t - k_2 \cos p_0 t$$

$$= \sqrt{k_1^2 + k_2^2} \cos\left[p_0 t + \tan^{-1}\left(-\frac{k_1}{k_2}\right)\right] \qquad (3.101)$$

$$e_{31} \approx - k_3 \cos p_0 t + k_4 \sin p_0 t$$

$$= \sqrt{k_3^2 + k_4^2} \sin\left[p_0 t + \tan^{-1}\left(-\frac{k_3}{k_4}\right)\right] \qquad (3.102)$$

$$e_{32} \approx k_5 \cos p_0 t + k_6 \sin p_0 t$$

$$= \sqrt{k_5^2 + k_6^2} \sin\left[p_0 t + \tan^{-1}\left(\frac{k_5}{k_6}\right)\right] \qquad (3.103)$$

$$e_{33} \approx k_1 \cos p_0 t - k_2 \sin p_0 t$$

$$= \sqrt{k_1^2 + k_2^2} \sin\left[p_0 t + \tan^{-1}\left(-\frac{k_1}{k_2}\right)\right] \qquad (3.104)$$

Finally, if one replaces $\left(\frac{\lambda t}{p_0}\right)$ and $\left(\frac{\lambda t}{p_0}\right)^2$ in Eqs. (3.90)-(3.104) with $\sin\left(\frac{\lambda t}{p_0}\right)$ and $2\left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right]$, respectively, one obtains

$$k_1 \approx P_1 - P_1 P_3 \sin\left(\frac{\lambda t}{p_0}\right) - P_1\left(1 - 4P_3^2\right)\left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right] \qquad (3.105)$$

$$k_2 \approx P_1 P_3 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right] \tag{3.106}$$

$$k_3 \approx P_3 + P_1^2 \sin\left(\frac{\lambda t}{p_0}\right) - 4P_1^2 P_3 \left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right] \tag{3.107}$$

$$k_4 \approx P_3^2 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right] \tag{3.108}$$

$$k_5 \approx P_3 \left(\frac{\lambda}{p_0^2}\right)^{-1} \left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right] \tag{3.109}$$

$$k_6 \approx 1 \tag{3.110}$$

and

$$e_{11} \approx k_1 - P_1 P_3 \left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right] (1 - \cos p_0 t) \tag{3.111}$$

$$e_{12} \approx - P_1 P_3 \left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right] \sin p_0 t \tag{3.112}$$

$$e_{13} \approx k_3 + P_1^2 P_3 \left[1 - \cos\left(\frac{\lambda t}{p_0}\right)\right] (1 - \cos p_0 t) \tag{3.113}$$

$$e_{21} \approx k_3 \sin p_0 t + k_4 \cos p_0 t$$

$$= \sqrt{k_3^2 + k_4^2} \cos\left[p_0 t + \tan^{-1}\left(-\frac{k_3}{k_4}\right)\right] \tag{3.114}$$

$$e_{22} \approx - k_5 \sin p_0 t + k_6 \cos p_0 t$$

$$= \sqrt{k_5^2 + k_6^2} \cos\left[p_0 t + \tan^{-1}\left(\frac{k_5}{k_6}\right)\right] \tag{3.115}$$

$$e_{23} \approx - k_1 \sin p_0 t - k_2 \cos p_0 t$$

$$= \sqrt{k_1^2 + k_2^2} \cos\left[p_0 t + \tan^{-1}\left(-\frac{k_1}{k_2}\right)\right] \qquad (3.116)$$

$$e_{31} \approx - k_3 \cos p_0 t + k_4 \sin p_0 t$$

$$= \sqrt{k_3^2 + k_4^2} \sin\left[p_0 t + \tan^{-1}\left(-\frac{k_3}{k_4}\right)\right] \qquad (3.117)$$

$$e_{32} \approx k_5 \cos p_0 t + k_6 \sin p_0 t$$

$$= \sqrt{k_5^2 + k_6^2} \sin\left[p_0 t + \tan^{-1}\left(\frac{k_5}{k_6}\right)\right] \qquad (3.118)$$

$$e_{33} \approx k_1 \cos p_0 t - k_2 \sin p_0 t$$

$$= \sqrt{k_1^2 + k_2^2} \sin\left[p_0 t + \tan^{-1}\left(-\frac{k_1}{k_2}\right)\right] \qquad (3.119)$$

Considering the process by means of which these equations were obtained, one may expect that they will yield ever better results as λ/p_0^2 and $\lambda t/p_0$ become smaller and smaller.

For purposes of numerical work a form of the above equations involving only dimensionless quantities is convenient and may be obtained by introducing x and z as

$$x \triangleq p_0 t \qquad (3.120)$$

$$z \triangleq \frac{\lambda}{p_0^2} \underset{(3.15)}{=} \frac{T}{I p_0^2} \qquad (3.121)$$

In terms of these quantities, Eqs. (3.105)-(3.119) can be rewritten as

$$k_1 \approx P_1 - P_1P_3 \sin zx - P_1(1-4P_3^2)(1-\cos zx) \quad (3.122)$$

$$k_2 \approx P_1P_3 z^{-1}(1-\cos zx) \quad (3.123)$$

$$k_3 \approx P_3 + P_1^2 \sin zx - 4P_1^2P_3(1-\cos zx) \quad (3.124)$$

$$k_4 \approx P_3^2 z^{-1}(1-\cos zx) \quad (3.125)$$

$$k_5 \approx P_3 z^{-1}(1-\cos zx) \quad (3.126)$$

$$k_6 \approx 1 \quad (3.127)$$

$$e_{11} \approx k_1 - P_1P_3(1-\cos zx)(1-\cos x) \quad (3.128)$$

$$e_{12} \approx - P_1P_3(1-\cos zx) \sin x \quad (3.129)$$

$$e_{13} \approx k_3 + P_1^2P_3(1-\cos zx)(1-\cos x) \quad (3.130)$$

$$e_{21} \approx k_3 \sin x + k_4 \cos x$$

$$= \sqrt{k_3^2 + k_4^2} \cos\left[x + \tan^{-1}\left(-\frac{k_3}{k_4}\right)\right] \quad (3.131)$$

$$e_{22} \approx - k_5 \sin x + k_6 \cos x$$

$$= \sqrt{k_5^2 + k_6^2} \cos\left[x + \tan^{-1}\left(\frac{k_5}{k_6}\right)\right] \quad (3.132)$$

$$e_{23} \approx - k_1 \sin x - k_2 \cos x$$

$$= \sqrt{k_1^2 + k_2^2} \cos\left[x + \tan^{-1}\left(-\frac{k_1}{k_2}\right)\right] \quad (3.133)$$

$$e_{31} \approx - k_3 \cos x + k_4 \sin x$$

$$= \sqrt{k_3^2 + k_4^2}\, \sin\left[x + \tan^{-1}\left(-\frac{k_3}{k_4}\right)\right] \tag{3.134}$$

$$e_{32} \approx k_5 \cos x + k_6 \sin x$$

$$= \sqrt{k_5^2 + k_6^2}\, \sin\left[x + \tan^{-1}\left(\frac{k_5}{k_6}\right)\right] \tag{3.135}$$

$$e_{33} \approx k_1 \cos x - k_2 \sin x$$

$$= \sqrt{k_1^2 + k_2^2}\, \sin\left[x + \tan^{-1}\left(-\frac{k_1}{k_2}\right)\right] \tag{3.136}$$

and results may be expected to improve as z and x become smaller.

Before leaving Eqs. (3.128)-(3.136), it should be mentioned that alternative and, it turns out, more accurate approximations to e_{ij} have been developed by Professor J. V. Breakwell for the special case $p_{30} = 0$. A detailed description of this work appears in Appendix B.

Now that approximate expressions for the elements of the matrix $\tilde{e}$ are at hand, approximate values of three orientation angles, say θ_1, θ_2, and θ_3, describing the orientation of the unit vectors $\underline{c}_1$, $\underline{c}_2$, and $\underline{c}_3$ relative to $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ can be found by employing the procedure described in Section 2.2; that is, replacing γ_i and γ_{ij} in Eqs. (2.20)-(2.22) with θ_i and e_{ij}, respectively, one obtains

$$\left.\begin{aligned} \sin\theta_1 &= \frac{e_{13}}{\sqrt{e_{13}^2 + e_{23}^2}} \\ \cos\theta_1 &= \frac{-e_{23}}{\sqrt{e_{13}^2 + e_{23}^2}} \end{aligned}\right\} \quad (3.137)$$

$$\left.\begin{aligned} \sin\theta_2 &= \frac{-1}{e_{22}\sqrt{e_{13}^2 + e_{23}^2}} (e_{13}e_{31} + e_{23}e_{32}e_{33}) \\ \cos\theta_2 &= e_{33} \end{aligned}\right\} \quad (3.138)$$

$$\left.\begin{aligned} \sin\theta_3 &= \frac{e_{31}}{\sqrt{e_{31}^2 + e_{32}^2}} \\ \cos\theta_3 &= \frac{e_{32}}{\sqrt{e_{31}^2 + e_{32}^2}} \end{aligned}\right\} \quad (3.139)$$

and these equations can be solved for θ_1, θ_2, and θ_3. Next, to determine the motion of the body-fixed unit vectors $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ relative to $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$, it is only necessary to observe that Eqs. (3.20), (3.120), and (3.121) permit one to express θ, the angle between $\underline{b}_1$ and $\underline{c}_1$, as

$$\theta = L(\frac{1}{2} zx^2 + P_3 x) \quad (3.140)$$

and that three orientation angles, say φ_1, φ_2, and φ_3, analogous to θ_1, θ_2, and θ_3, but governing the attitude of $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ relative to $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$, are given by

$$\varphi_1 = \theta_1 \tag{3.141}$$

$$\varphi_2 = \theta_2 \tag{3.142}$$

$$\varphi_3 = \theta_3 + \theta \tag{3.143}$$

The parameters P_1 and P_3, introduced in Eqs. (3.55) and (3.56) for purposes of analytical convenience, can be expressed in a physically more meaningful form after introducing Ω_1 and Ω_3 as

$$\Omega_1 \triangleq \frac{\omega_{10}}{\sqrt{\omega_{10}^2 + \omega_{30}^2}} \tag{3.144}$$

$$\Omega_3 \triangleq \frac{\omega_{30}}{\sqrt{\omega_{10}^2 + \omega_{30}^2}} \tag{3.145}$$

from which it follows that

$$\Omega_1^2 + \Omega_3^2 = 1 \tag{3.146}$$

Referring to Eqs. (3.26), (3.28), (3.54)-(3.56), (3.144), and (3.145), one then obtains

$$P_1 = \frac{\Omega_1}{\sqrt{\Omega_1^2 + \dfrac{\Omega_3^2}{(1+L)^2}}} \tag{3.147}$$

$$P_3 = \frac{\Omega_3 / (1+L)}{\sqrt{\Omega_1^2 + \dfrac{\Omega_3^2}{(1+L)^2}}} \tag{3.148}$$

In addition, defining ω_0 as

$$\omega_0 \triangleq \sqrt{\omega_{10}^2 + \omega_{30}^2} \qquad (3.149)$$

one can express p_0 as (see Eqs. (3.26), (3.28), (3.54), (3.144), and (3.145))

$$p_0 = \omega_0\sqrt{\Omega_1^2 + \frac{\Omega_3^2}{(1+L)^2}} \qquad (3.150)$$

and this relationship will be found useful in the sequel.

Returning now to Eqs. (3.147) and (3.148), it is evident that, if either $\Omega_1 = 1$ (so that $\Omega_3 = 0$) or $\Omega_1 = 0$ (so that $\Omega_3 = 1$), P_1 and P_3 are equal to Ω_1 and Ω_3, respectively, regardless of the value of L. This means that the initial angular velocity of C is equal to that of B if B initially has a pure tumbling or pure spinning motion.

In summary, given the dimensionless parameters Ω_1, Ω_3, L, z, and x, one may proceed as follows to evaluate φ_1, φ_2, and φ_3:

(1) Compute P_1 and P_3 from Eqs. (3.147) and (3.148).

(2) Use Eqs. (3.122)-(3.127) to evaluate $k_1, \ldots, k_6$.

(3) Find e_{ij} $(i,j = 1,2,3)$ from Eqs. (3.128)-(3.136).

(4) Evaluate θ_1, θ_2, and θ_3 by reference to Eqs. (3.137)-(3.139).

(5) Find θ from Eq. (3.140).

(6) Determine φ_1, φ_2, and φ_3 by using Eqs. (3.141)-(3.143).

3.3 Comparison of Solutions

A measure of the utility of the procedure described in the preceding section may be obtained by comparing values of θ_1, θ_2, and θ_3 found by means of this procedure with values resulting from a numerical integration of Eqs. (3.45)-(3.47), subsequent evaluation of e_{ij} $(i, j = 1, 2, 3)$ by reference to Eq. (3.80), and determination of θ_1, θ_2, and θ_3 by inversion of Eqs. (3.137)-(3.139). Figures 4-8 each show θ_1, θ_2, and θ_3 as functions of x, the values corresponding to the digital computer solution of the exact equations being represented by solid curves, whereas the approximate solution is represented by crosses; and each figure applies to a different combination of values of Ω_1, Ω_3, L, and z.

The accuracy of the approximate solutions is seen to be relatively insensitive to the numerical values of Ω_1, Ω_3, (see Figures 4, 6, and 7) and L (see Figures 6 and 7), but to decrease noticeably both when z increases (compare Figure 4 with Figure 5 or Figure 6 with Figure 8) and when x increases (see Figures 4-8). We note for future reference that for $z = 0.01$ (see Figures 4, 6, and 7) the exact and the approximate value of θ_i $(i = 1, 2, 3)$ are nearly indistinguishable from each other so long as x remains smaller than 15.00, and that similarly good agreement obtains for $z = 0.05$ and $z = 0.10$ (see Figures 5 and 8) for $x < 2.5$.

While the approximate solution can furnish considerable quantitative information, it does not lead directly to a clear qualitative description of the motion under consideration. This aspect of the subject can be discussed most effectively in terms of the behavior of the unit vector $\underline{c}_3$, which, it will be recalled, is parallel to the symmetry axis of B. For example, one can plot the trajectory of the tip of $\underline{c}_3$, as has

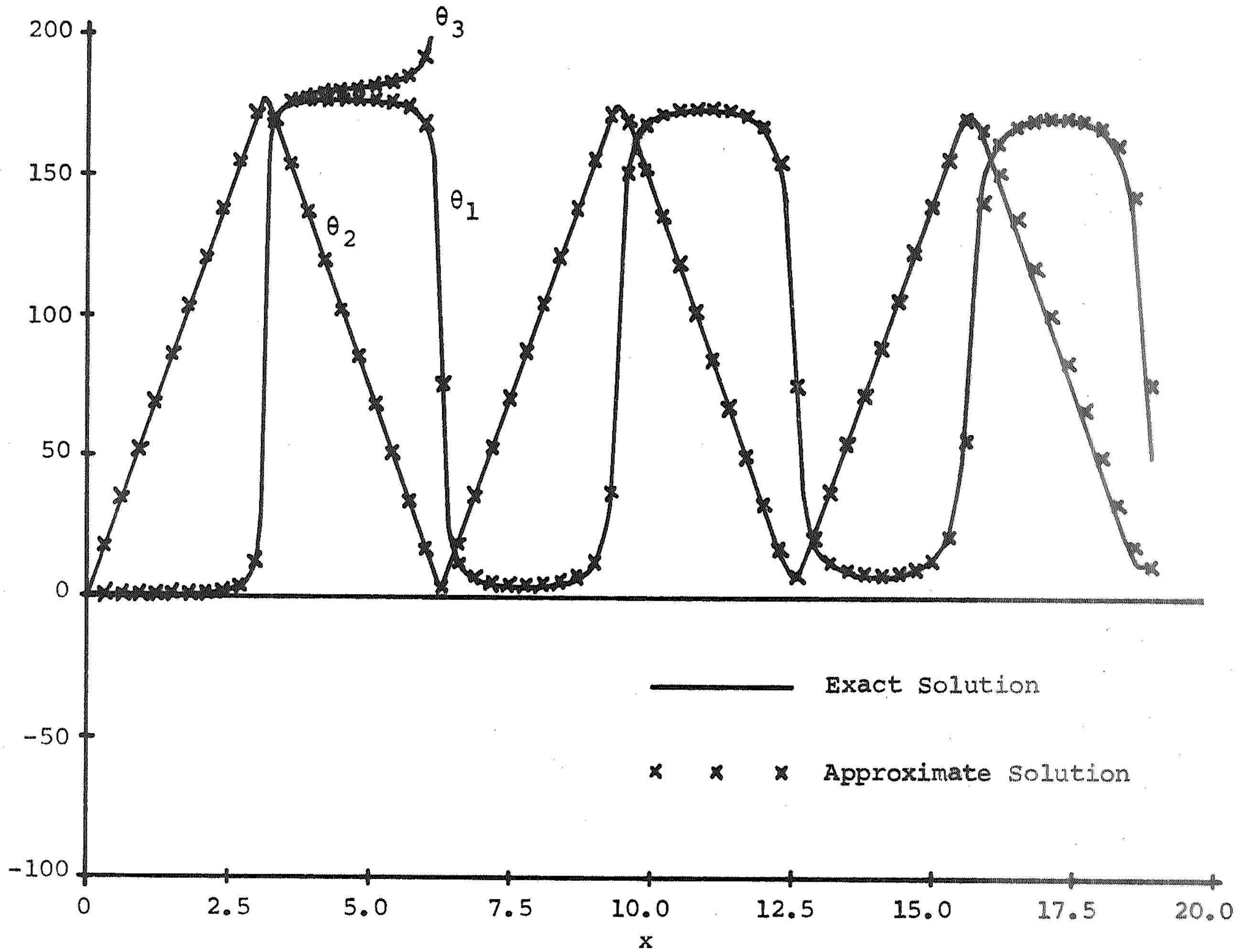

Figure 4. Comparison of Solutions for Orientation Angles

$\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$

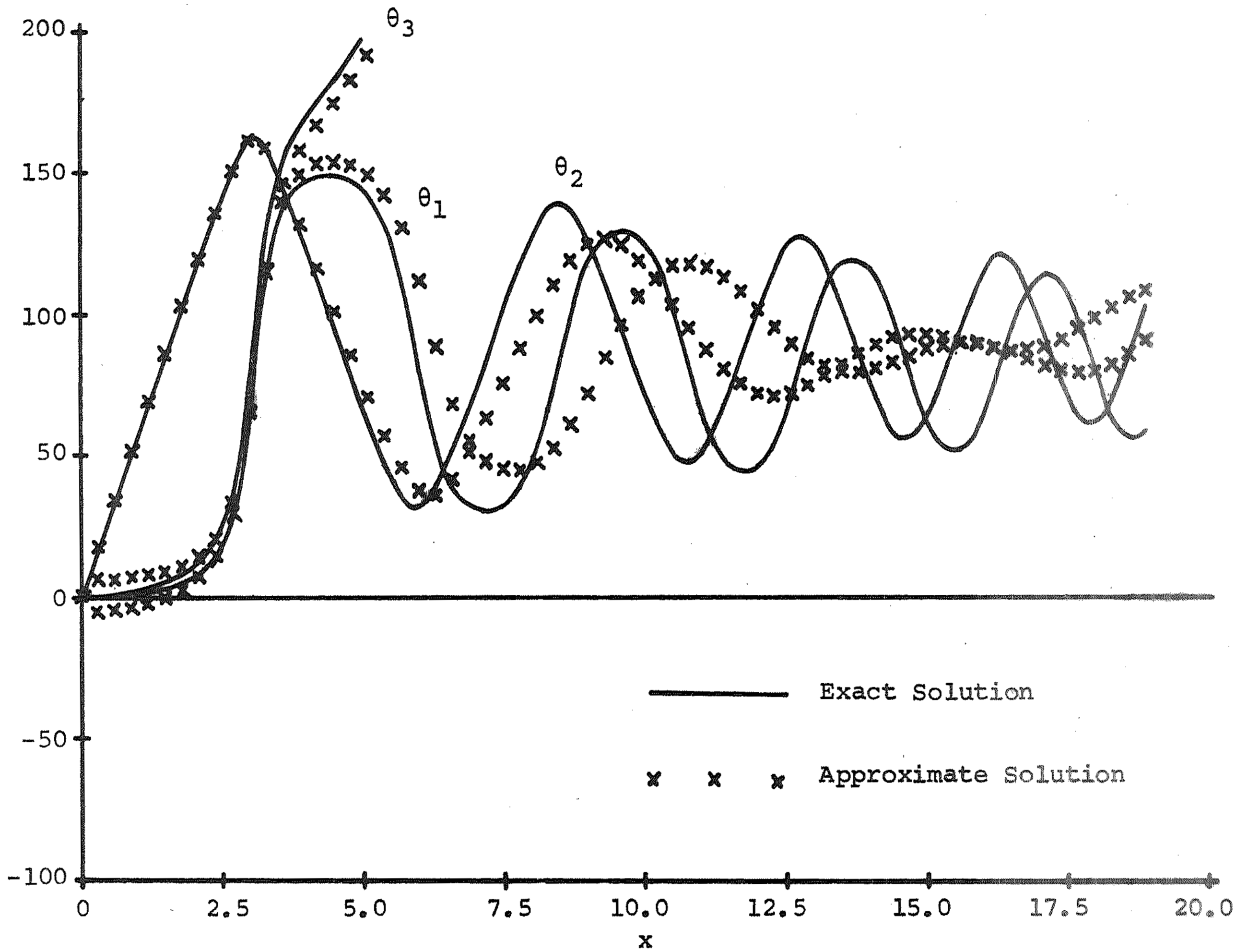

Figure 5. Comparison of Solutions for Orientation Angles
$\Omega_1 = 1.0, \ \Omega_3 = 0, \ z = 0.1$

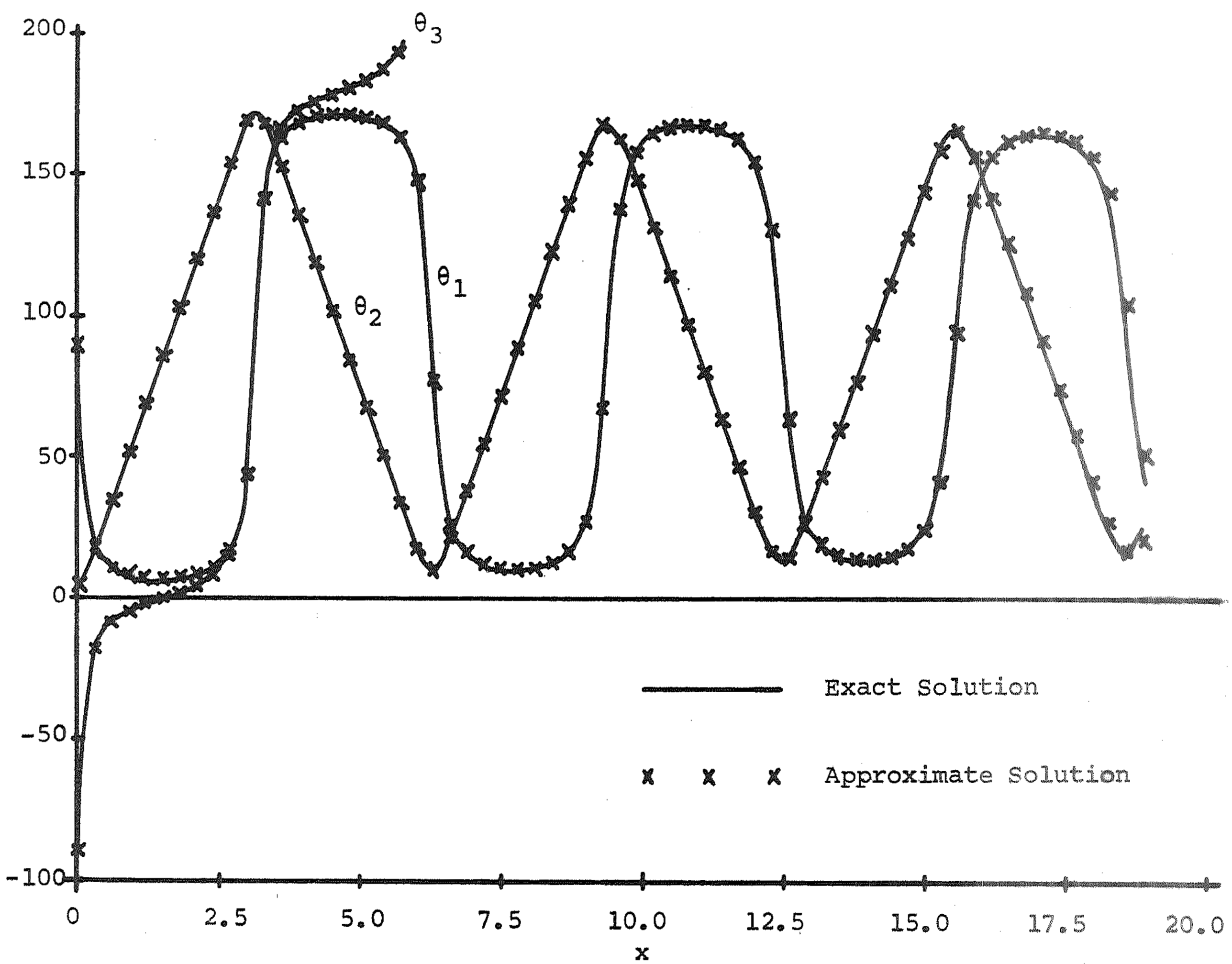

Figure 6. Comparison of Solutions for Orientation Angles
$\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 9.0$

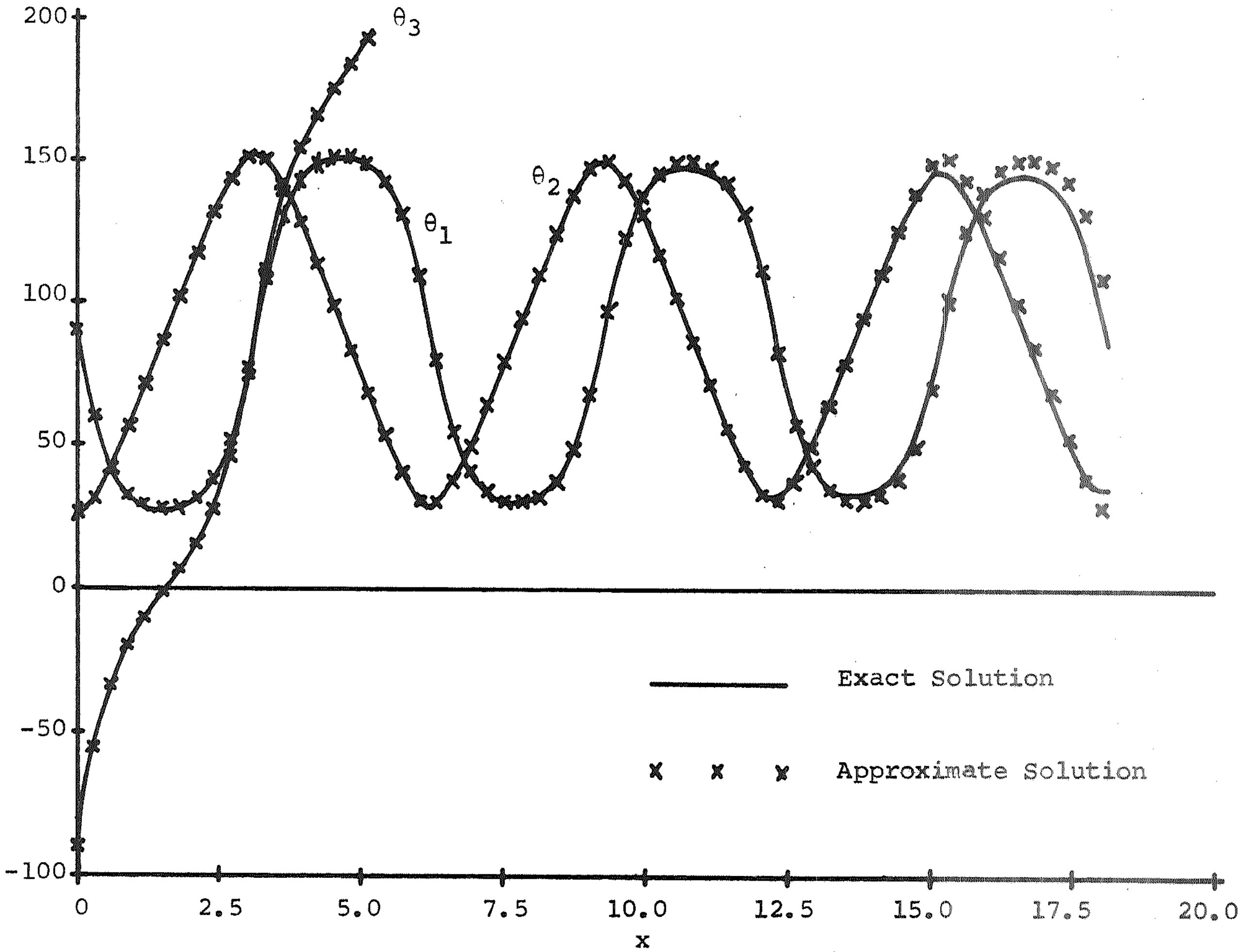

Figure 7. Comparison of Solutions for Orientation Angles
$\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 1.0$

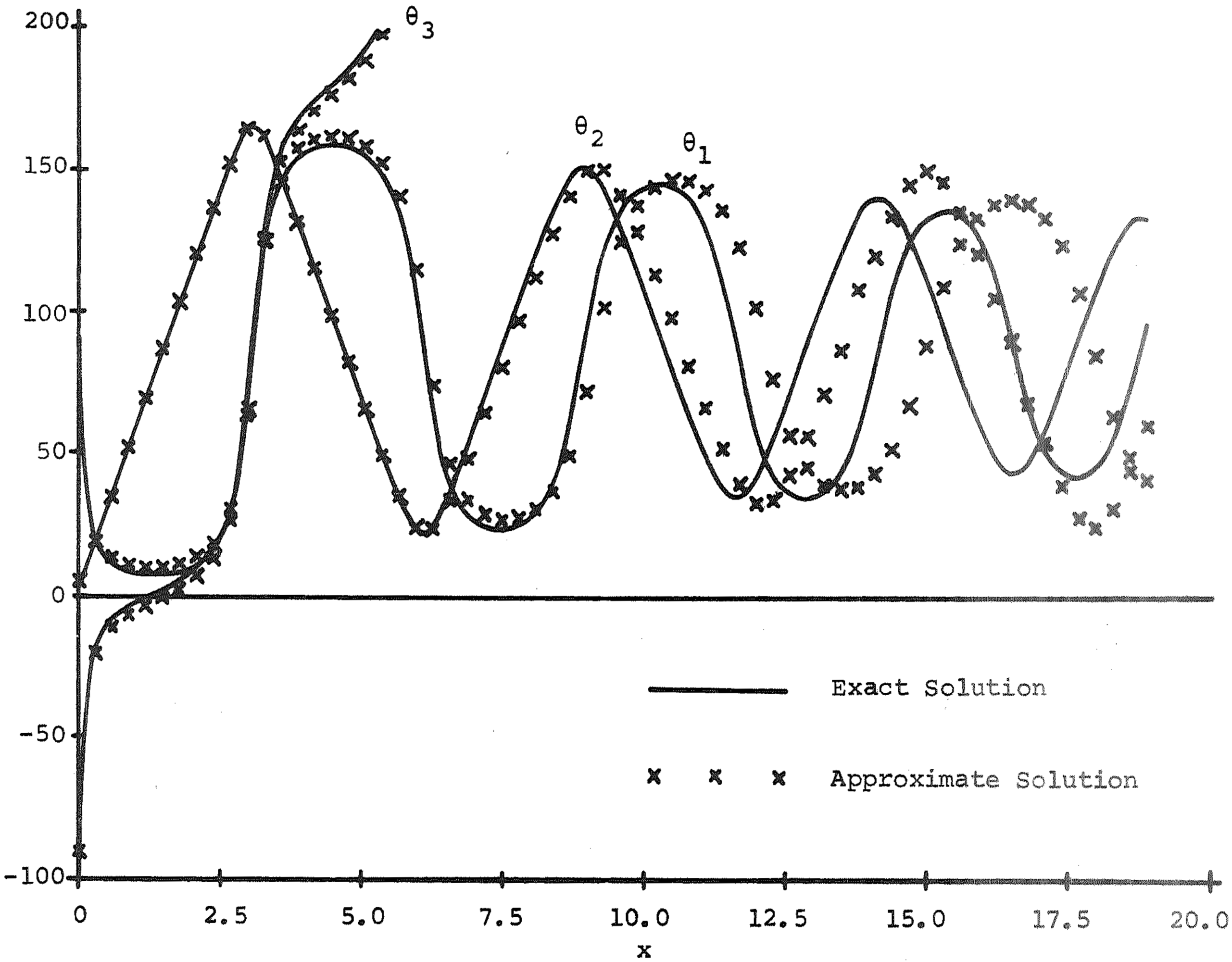

Figure 8. Comparison of Solutions for Orientation Angles
$\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.05$, $L = 9.0$

been done in Figures 9-11 for the parameter values used previously to generate Figures 4, 5, and 7. These plots show that the symmetry axis of B performs a "coning" motion, around the initial angular velocity vector, ${}^{A}\underline{\omega}^{C}(0)$, approaching this vector as t approaches infinity.

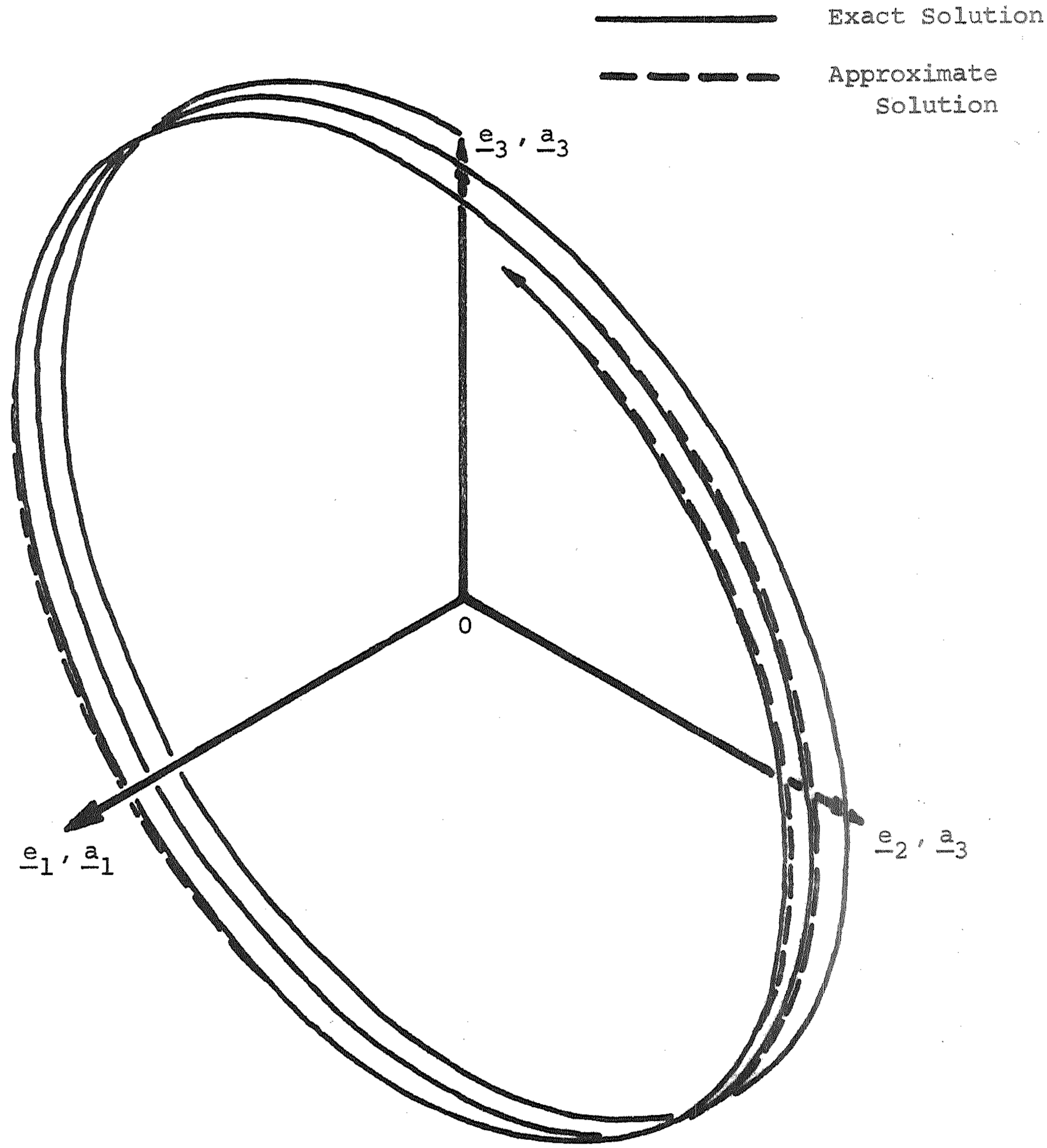

Figure 9. Trajectory of the Symmetry Axis
$\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$

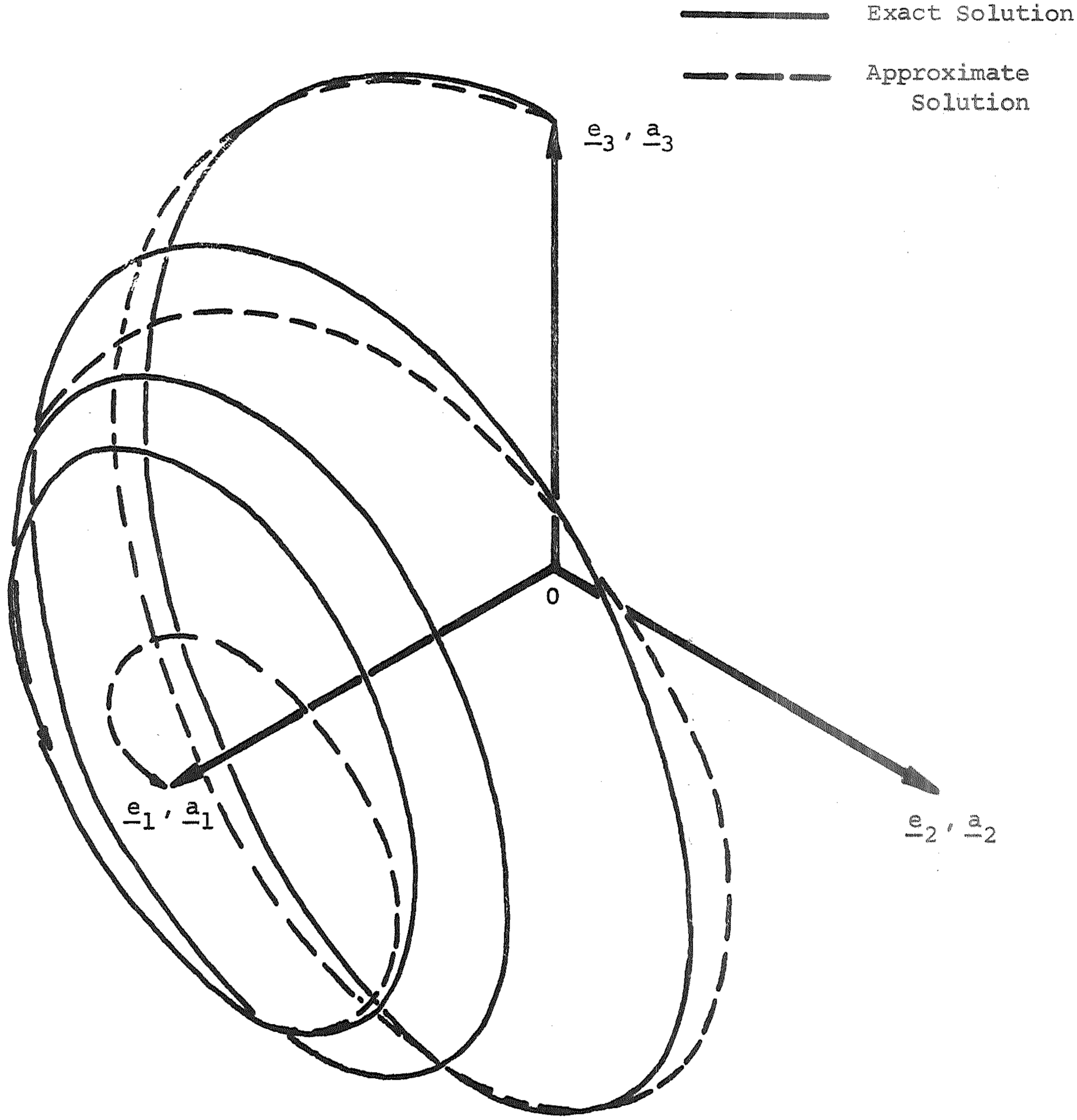

Figure 10. Trajectory of the Symmetry Axis
$\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.1$

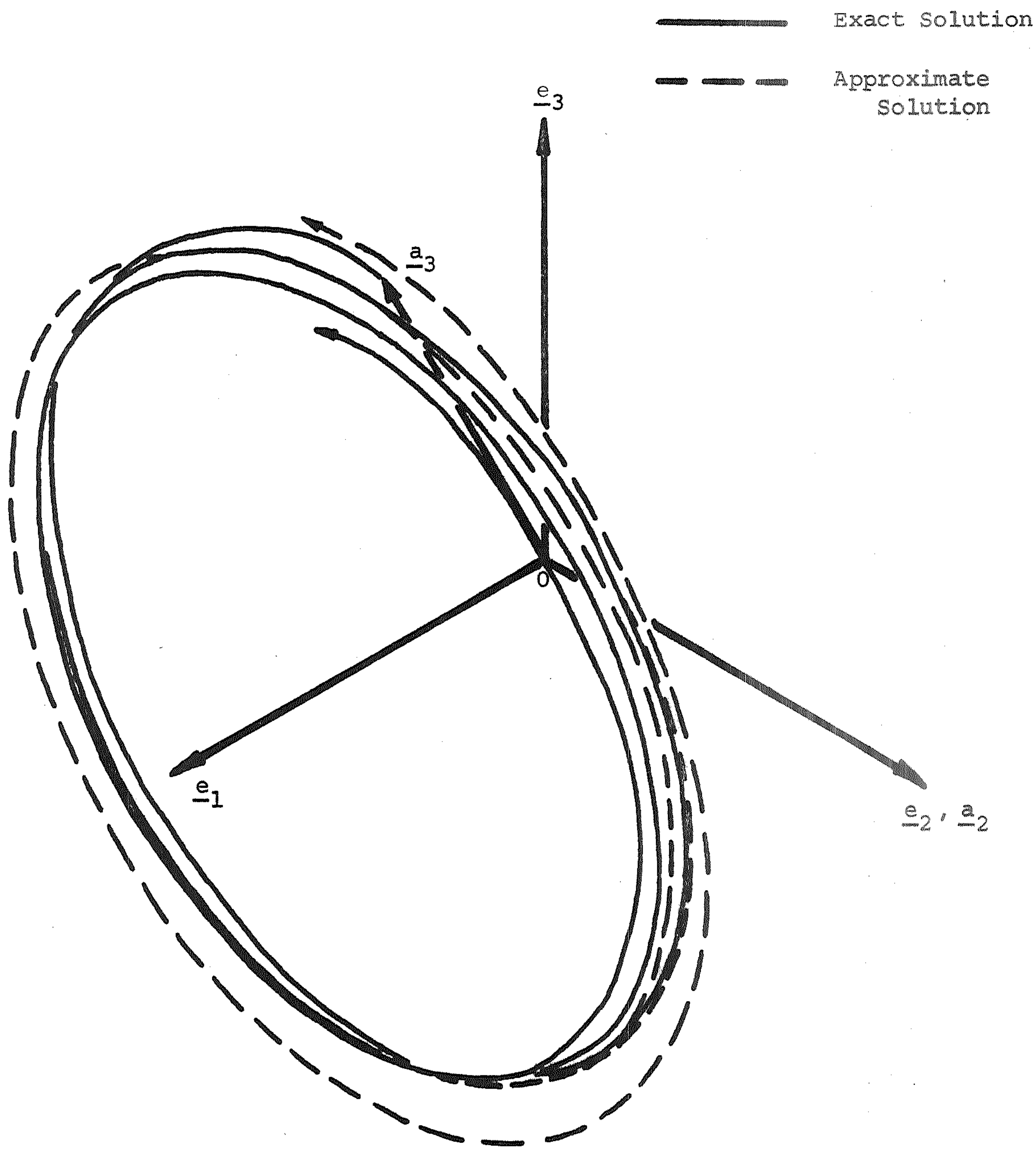

Figure 11. Trajectory of the Symmetry Axis
$\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$,
$z = 0.01$, $L = 1.0$

4. POSITION

4.1 Dynamical Equations

To study the translational motion of B, that is, the motion of the mass center B^* of B, we recall the definitions of m, $\underline{X}$, and $\underline{F}$ (see Chapter 2) and, after defining X_i as

$$X_i \triangleq \underline{X} \cdot \underline{e}_i \qquad (i = 1,2,3) \tag{4.1}$$

express Newton's Second Law of Motion in the form

$$m\,\ddot{\underline{X}} \cdot \underline{e}_i \underset{(4.1)}{=} m\,\ddot{X}_i = \underline{F} \cdot \underline{e}_i \qquad (i = 1,2,3) \tag{4.2}$$

From Eqs. (2.2) and (2.4),

$$\begin{aligned}
\underline{F} &= F_1\,\underline{b}_1 + F_2\,\underline{b}_2 \\
&\underset{(3.21)}{=} (F_1 \cos\theta - F_2 \sin\theta)\,\underline{c}_1 + (F_1 \sin\theta + F_2 \cos\theta)\,\underline{c}_2 \\
&\underset{(3.79)}{=} [e_{11}(F_1 \cos\theta - F_2 \sin\theta) + e_{12}(F_1 \sin\theta + F_2 \cos\theta)]\,\underline{e}_1 \\
&\quad + [e_{21}(F_1 \cos\theta - F_2 \sin\theta) + e_{22}(F_1 \sin\theta + F_2 \cos\theta)]\,\underline{e}_2 \\
&\quad + [e_{31}(F_1 \cos\theta - F_2 \sin\theta) + e_{32}(F_1 \sin\theta + F_2 \cos\theta)]\,\underline{e}_3
\end{aligned} \tag{4.3}$$

Substitution from Eq. (4.3) into Eq. (4.2) thus leads to

$$\ddot{X}_1 = [e_{11}(f_1 \cos\theta - f_2 \sin\theta) + e_{12}(f_1 \sin\theta + f_2 \cos\theta)] \tag{4.4}$$

$$\ddot{X}_2 = [e_{21}(f_1 \cos\theta - f_2 \sin\theta) + e_{22}(f_1 \sin\theta + f_2 \cos\theta)] \tag{4.5}$$

$$\ddot{X}_3 = [e_{31}(f_1 \cos\theta - f_2 \sin\theta) + e_{32}(f_1 \sin\theta + f_2 \cos\theta)] \quad (4.6)$$

Where f_1 and f_2 are defined as

$$f_1 \triangleq \frac{F_1}{m} \quad (4.7)$$

$$f_2 \triangleq \frac{F_2}{m} \quad (4.8)$$

Equations (4.4)-(4.6) are seen to be coupled to the rotational motion of B by the quantities θ and e_{ij} $(i, j = 1, 2, 3)$; and, if the expressions for θ and e_{ij} $(i, j = 1, 2, 3)$ given in Eqs. (3.20) and (3.111)-(3.119) are substituted into Eqs. (4.4)-(4.6), the following approximate expressions for $\ddot{X}_i$ $(i = 1, 2, 3)$ are obtained:

$$\begin{aligned}
\ddot{X}_1 \approx\ & f_1 P_1 \{ (4P_3^2 - P_3) \cos(\xi t^2 + 2\delta_1 t) \\
& - \frac{P_3}{2}[\sin(\xi t^2 + 2\delta_2 t) - \sin(\xi t^2 + 2\delta_3 t)] \\
& + \frac{1}{2}(1 - 4P_3^2 + P_3)[\cos(\xi t^2 + 2\delta_3 t) + \cos(\xi t^2 + 2\delta_2 t)] \\
& + P_3 \cos(\xi t^2 + 2\delta_5 t) - \frac{P_3}{2}[\cos(\xi t^2 + 2\delta_7 t) + \cos(\xi t^2 + 2\delta_9 t)]\} \\
& - f_2 P_1 \{ (4P_3^2 - P_3) \sin(\xi t^2 + 2\delta_1 t) \\
& - \frac{P_3}{2}[\cos(\xi t^2 + 2\delta_3 t) - \cos(\xi t^2 + 2\delta_2 t)] \\
& + \frac{1}{2}(1 - 4P_3^2 + P_3)[\sin(\xi t^2 + 2\delta_2 t) + \sin(\xi t^2 + 2\delta_3 t)] \\
& + P_3 \sin(\xi t^2 + 2\delta_5 t) - \frac{P_3}{2}[\sin(\xi t^2 + 2\delta_7 t) + \sin(\xi t^2 + 2\delta_9 t)]\}
\end{aligned} \quad (4.9)$$

$$\ddot{X}_2 \approx f_1\{\tfrac{1}{2}(P_3 - 4P_1^2P_3)[\sin(\xi t^2 + 2\delta_5 t) - \sin(\xi t^2 + 2\delta_4 t)]$$

$$+ \frac{P_1^2}{4}[\cos(\xi t^2 + 2\delta_8 t) - \cos(\xi t^2 + 2\delta_9 t) - \cos(\xi t^2 + 2\delta_6 t)$$

$$+ \cos(\xi t^2 + 2\delta_7 t)] + P_1^2P_3[\sin(\xi t^2 + 2\delta_7 t) - \sin(\xi t^2 + 2\delta_6 t)$$

$$+ \sin(\xi t^2 + 2\delta_9 t) - \sin(\xi t^2 + 2\delta_8 t)]\}$$

$$+ f_1P_3^2\left(\frac{\lambda}{P_0^2}\right)^{-1}\{\tfrac{1}{2}[\cos(\xi t^2 + 2\delta_4 t) + \cos(\xi t^2 + 2\delta_5 t)]$$

$$- \tfrac{1}{4}[\cos(\xi t^2 + 2\delta_6 t) + \cos(\xi t^2 + 2\delta_7 t) + \cos(\xi t^2 + 2\delta_8 t)$$

$$+ \cos(\xi t^2 + 2\delta_9 t)]\} - f_2\{\tfrac{1}{2}(P_3 - 4P_1^2P_3)[\cos(\xi t^2 + 2\delta_4 t)$$

$$- \cos(\xi t^2 + 2\delta_5 t)] + \frac{P_1^2}{4}[\sin(\xi t^2 + 2\delta_7 t) - \sin(\xi t^2 + 2\delta_6 t)$$

$$- \sin(\xi t^2 + 2\delta_9 t) + \sin(\xi t^2 + 2\delta_8 t)] + P_1^2P_3[\cos(\xi t^2 + 2\delta_8 t)$$

$$- \cos(\xi t^2 + 2\delta_9 t) + \cos(\xi t^2 + 2\delta_6 t) - \cos(\xi t^2 + 2\delta_7 t)]\}$$

$$- f_2P_3^2\left(\frac{\lambda}{P_0^2}\right)^{-1}\{\tfrac{1}{2}[\sin(\xi t^2 + 2\delta_4 t) + \sin(\xi t^2 + 2\delta_5 t)]$$

$$- \tfrac{1}{4}[\sin(\xi t^2 + 2\delta_9 t) + \sin(\xi t^2 + 2\delta_8 t) + \sin(\xi t^2 + 2\delta_7 t)$$

$$+ \sin(\xi t^2 + 2\delta_6 t)]\} + f_1\{\tfrac{1}{2}[\sin(\xi t^2 + 2\delta_4 t) + \sin(\xi t^2 + 2\delta_5 t)]$$

$$- \frac{P_3}{2}\left(\frac{\lambda}{p_0^2}\right)^{-1}\left[\cos(\xi t^2 + 2\delta_4 t) - \cos(\xi t^2 + 2\delta_5 t)\right]$$

$$+ \frac{P_3}{4}\left(\frac{\lambda}{p_0^2}\right)^{-1}\left[\cos(\xi t^2 + 2\delta_8 t) - \cos(\xi t^2 + 2\delta_9 t)\right.$$

$$\left. + \cos(\xi t^2 + 2\delta_6 t) - \cos(\xi t^2 + 2\delta_7 t)\right]\Big\}$$

$$+ f_2\Big\{\frac{1}{2}\left[\cos(\xi t^2 + 2\delta_4 t) + \cos(\xi t^2 + 2\delta_5 t)\right]$$

$$- \frac{P_3}{2}\left(\frac{\lambda}{p_0^2}\right)^{-1}\left[\sin(\xi t^2 + 2\delta_5 t) = \sin(\xi t^2 + 2\delta_4 t)\right]$$

$$+ \frac{P_3}{4}\left(\frac{\lambda}{p_0^2}\right)^{-1}\left[\sin(\xi t^2 + 2\delta_7 t) - \sin(\xi t^2 + 2\delta_6 t)\right.$$

$$\left. + \sin(\xi t^2 + 2\delta_9 t) - \sin(\xi t^2 + 2\delta_8 t)\right]\Big\} \qquad \left(\frac{\lambda}{p_0^2}\right) \neq 0 \tag{4.10}$$

$$\ddot{X}_3 \approx f_1\Big\{\frac{P_3}{2}(-1 + 4P_1^2)\left[\cos(\xi t^2 + 2\delta_4 t) + \cos(\xi t^2 + 2\delta_5 t)\right]$$

$$- \frac{P_1^2}{4}\left[\sin(\xi t^2 + 2\delta_9 t) + \sin(\xi t^2 + 2\delta_8 t) - \sin(\xi t^2 + 2\delta_7 t)\right.$$

$$\left. - \sin(\xi t^2 + 2\delta_6 t)\right] - P_1^2 P_3\left[\cos(\xi t^2 + 2\delta_6 t) + \cos(\xi t^2 + 2\delta_7 t)\right.$$

$$\left. + \cos(\xi t^2 + 2\delta_8 t) + \cos(\xi t^2 + 2\delta_9 t)\right]\Big\}$$

$$+ f_1 P_3^2\left(\frac{\lambda}{p_0^2}\right)^{-1}\Big\{\frac{1}{2}\left[\sin(\xi t^2 + 2\delta_5 t) - \sin(\xi t^2 + 2\delta_4 t)\right]$$

$$- \frac{1}{4}[\sin(\xi t^2 + 2\delta_7 t) - \sin(\xi t^2 + 2\delta_6 t) + \sin(\xi t^2 + 2\delta_9 t)$$

$$- \sin(\xi t^2 + 2\delta_8 t)]\} - f_2\{\frac{P_3}{2}(-1 + 4P_1^2)[\sin(\xi t^2 + 2\delta_4 t)$$

$$+ \sin(\xi t^2 + 2\delta_5 t)] - \frac{P_1^2}{4}[\cos(\xi t^2 + 2\delta_6 t) + \cos(\xi t^2 + 2\delta_7 t)$$

$$- \cos(\xi t^2 + 2\delta_8 t) - \cos(\xi t^2 + 2\delta_9 t)] - P_1^2 P_3[\sin(\xi t^2 + 2\delta_9 t)$$

$$+ \sin(\xi t^2 + 2\delta_8 t) + \sin(\xi t^2 + 2\delta_7 t) + \sin(\xi t^2 + 2\delta_6 t)]\}$$

$$- f_2 P_3^2 \left(\frac{\lambda}{P_0^2}\right)^{-1} \{\frac{1}{2}[\cos(\xi t^2 + 2\delta_4 t) - \cos(\xi t^2 + 2\delta_5 t)]$$

$$- \frac{1}{4}[\cos(\xi t^2 + 2\delta_8 t) - \cos(\xi t^2 + 2\delta_9 t) + \cos(\xi t^2 + 2\delta_6 t)$$

$$- \cos(\xi t^2 + 2\delta_7 t)]\} + f_1\{\frac{P_3}{2}\left(\frac{\lambda}{P_0^2}\right)^{-1} [\sin(\xi t^2 + 2\delta_4 t)$$

$$+ \sin(\xi t^2 + 2\delta_5 t)] + \frac{1}{2}[\cos(\xi t^2 + 2\delta_4 t) - \cos(\xi t^2 + 2\delta_5 t)]$$

$$- \frac{P_3}{4}\left(\frac{\lambda}{P_0^2}\right)^{-1} [\sin(\xi t^2 + 2\delta_9 t) + \sin(\xi t^2 + 2\delta_8 t)$$

$$+ \sin(\xi t^2 + 2\delta_7 t) + \sin(\xi t^2 + 2\delta_6 t)]\}$$

$$+ f_2\{\frac{P_3}{2}\left(\frac{\lambda}{P_0^2}\right)^{-1} [\cos(\xi t^2 + 2\delta_4 t) + \cos(\xi t^2 + 2\delta_5 t)]$$

$$+ \frac{1}{2}[\sin(\xi t^2 + 2\delta_5 t) - \sin(\xi t^2 + 2\delta_4 t)]$$

$$- \frac{p_3}{4}\left(\frac{\lambda}{p_0^2}\right)^{-1} [\cos(\xi t^2 + 2\delta_6 t) + \cos(\xi t^2 + 2\delta_7 t)$$

$$+ \cos(\xi t^2 + 2\delta_8 t) + \cos(\xi t^2 + 2\delta_9 t)]\} \qquad \left(\frac{\lambda}{p_0^2}\right) \neq 0 \tag{4.11}$$

where ξ and δ_i $(i = 1,\ldots,9)$ are constants defined as follows:

$$\xi \triangleq \frac{1}{2} L\lambda \tag{4.12}$$

$$\delta_1 \triangleq (L\, p_{30})/2 \tag{4.13}$$

$$\delta_2 \triangleq (L\, p_{30} + \frac{\lambda}{p_0})/2 \tag{4.14}$$

$$\delta_3 \triangleq (L\, p_{30} - \frac{\lambda}{p_0})/2 \tag{4.15}$$

$$\delta_4 \triangleq (L\, p_{30} - p_0)/2 \tag{4.16}$$

$$\delta_5 \triangleq (L\, p_{30} + p_0)/2 \tag{4.17}$$

$$\delta_6 \triangleq (L\, p_{30} - \frac{\lambda}{p_0} - p_0)/2 \tag{4.18}$$

$$\delta_7 \triangleq (L\, p_{30} - \frac{\lambda}{p_0} + p_0)/2 \tag{4.19}$$

$$\delta_8 \triangleq (L\, p_{30} + \frac{\lambda}{p_0} - p_0)/2 \tag{4.20}$$

$$\delta_9 \triangleq (L\, p_{30} + \frac{\lambda}{p_0} + p_0)/2 \tag{4.21}$$

4.2 Solution of Differential Equations

Before proceeding to the solution of Eqs. (4.9)-(4.11), it is convenient to introduce the following quantities:

$$r_g \triangleq (2J/m)^{1/2} \tag{4.22}$$

$$D_i \triangleq \frac{X_i}{r_g} \qquad (i = 1, 2, 3) \tag{4.23}$$

$$\mathcal{F}_i \triangleq \frac{f_i}{r_g p_0^2} \overset{(4.7, 4.8)}{=} \frac{F_i}{m r_g p_0^2} \qquad (i = 1, 2) \tag{4.24}$$

$$\zeta \triangleq \frac{\xi}{p_0^2} \overset{(4.12, 3.121)}{=} \frac{1}{2} L z \tag{4.25}$$

$$\beta_1 \triangleq \frac{\delta_1}{p_0} \overset{(4.13)}{=} \frac{1}{2} L P_3 \tag{4.26}$$

$$\beta_2 \triangleq \frac{\delta_2}{p_0} \overset{(4.14)}{=} (L P_3 + z)/2 \tag{4.27}$$

$$\beta_3 \triangleq \frac{\delta_3}{p_0} \overset{(4.15)}{=} (L P_3 - z)/2 \tag{4.28}$$

$$\beta_4 \triangleq \frac{\delta_4}{p_0} \overset{(4.16)}{=} (L P_3 - 1)/2 \tag{4.29}$$

$$\beta_5 \triangleq \frac{\delta_5}{p_0} \overset{(4.17)}{=} (L P_3 + 1)/2 \tag{4.30}$$

$$\beta_6 \triangleq \frac{\delta_6}{p_0} \overset{(4.18)}{=} (L P_3 - z - 1)/2 \tag{4.31}$$

$$\beta_7 \triangleq \frac{\delta_7}{p_0} \overset{(4.19)}{=} (L P_3 - z + 1)/2 \tag{4.32}$$

$$\beta_8 \triangleq \frac{\delta_8}{p_0} \overset{=}{(4.20)} (L P_3 + z - 1)/2 \tag{4.33}$$

$$\beta_9 \triangleq \frac{\delta_9}{p_0} \overset{=}{(4.21)} (L P_3 + z + 1)/2 \tag{4.34}$$

Substitution from Eqs. (4.22)-(4.34), (3.120), and (3.121) into Eqs. (4.9)-(4.11) gives

$$D_1'' \approx \mathcal{F}_1 P_1 \{ (4P_3^2 - P_3)\cos(\zeta x^2 + 2\beta_1 x) - \frac{P_3}{2}[\sin(\zeta x^2 + 2\beta_2 x)$$

$$- \sin(\zeta x^2 + 2\beta_3 x) + \cos(\zeta x^2 + 2\beta_7 x) + \cos(\zeta x^2 + 2\beta_9 x)]$$

$$+ P_3\cos(\zeta x^2 + 2\beta_5 x) + \frac{1}{2}(1 - 4P_3^2 + P_3)[\cos(\zeta x^2 + 2\beta_3 x)$$

$$+ \cos(\zeta x^2 + 2\beta_2 x)]\} - \mathcal{F}_2 P_1 \{ (4P_3^2 - P_3)\sin(\zeta x^2 + 2\beta_1 x)$$

$$- \frac{P_3}{2}[\cos(\zeta x^2 + 2\beta_3 x) - \cos(\zeta x^2 + 2\beta_2 x) + \sin(\zeta x^2 + 2\beta_7 x)$$

$$+ \sin(\zeta x^2 + 2\beta_9 x)] + P_3 \sin(\zeta x^2 + 2\beta_5 x) + \frac{1}{2}(1 - 4P_3^2 + P_3)$$

$$[\sin(\zeta x^2 + 2\beta_2 x) + \sin(\zeta x^2 + 2\beta_3 x)]\} \tag{4.35}$$

$$D_2'' \approx \mathcal{F}_1 \{ \frac{1}{2}(P_3 - 4P_1^2 P_3)[\sin(\zeta x^2 + 2\beta_5 x) - \sin(\zeta x^2 + 2\beta_4 x)]$$

$$+ \frac{P_1^2}{4}[\cos(\zeta x^2 + 2\beta_8 x) - \cos(\zeta x^2 + 2\beta_9 x) - \cos(\zeta x^2 + 2\beta_6 x)$$

$$+ \cos(\zeta x^2 + 2\beta_7 x)] + P_1^2 P_3[\sin(\zeta x^2 + 2\beta_7 x) - \sin(\zeta x^2 + 2\beta_6 x)$$

$$+ \sin(\zeta x^2 + 2\beta_9 x) - \sin(\zeta x^2 + 2\beta_8 x)]\}$$

$$+ \mathcal{F}_1 P_3^2 \; z^{-1}\{\tfrac{1}{2}[\cos(\zeta x^2 + 2\beta_4 x) + \cos(\zeta x^2 + 2\beta_5 x)]$$

$$- \tfrac{1}{4}[\cos(\zeta x^2 + 2\beta_6 x) + \cos(\zeta x^2 + 2\beta_7 x) + \cos(\zeta x^2 + 2\beta_8 x)$$

$$+ \cos(\zeta x^2 + 2\beta_9 x)]\} - \mathcal{F}_2\{\tfrac{1}{2}(P_3 - 4P_1^2 P_3)[\cos(\zeta x^2 + 2\beta_4 x)$$

$$- \cos(\zeta x^2 + 2\beta_5 x)] + \frac{P_1^2}{4}[\sin(\zeta x^2 + 2\beta_7 x) - \sin(\zeta x^2 + 2\beta_6 x)$$

$$- \sin(\zeta x^2 + 2\beta_9 x) + \sin(\zeta x^2 + 2\beta_8 x)] + P_1^2 P_3[\cos(\zeta x^2 + 2\beta_8 x)$$

$$- \cos(\zeta x^2 + 2\beta_9 x) + \cos(\zeta x^2 + 2\beta_6 x) - \cos(\zeta x^2 + 2\beta_7 x)]\}$$

$$- \mathcal{F}_2 P_3^2 \; z^{-1}\{\tfrac{1}{2}[\sin(\zeta x^2 + 2\beta_4 x) + \sin(\zeta x^2 + 2\beta_5 x)]$$

$$- \tfrac{1}{4}[\sin(\zeta x^2 + 2\beta_9 x) + \sin(\zeta x^2 + 2\beta_8 x) + \sin(\zeta x^2 + 2\beta_7 x)$$

$$+ \sin(\zeta x^2 + 2\beta_6 x)]\} + \mathcal{F}_1\{\tfrac{1}{2}[\sin(\zeta x^2 + 2\beta_4 x) + \sin(\zeta x^2 + 2\beta_5 x)]$$

$$- \frac{P_3}{2} \; z^{-1}[\cos(\zeta x^2 + 2\beta_4 x) - \cos(\zeta x^2 + 2\beta_5 x)]$$

$$+ \frac{P_3}{4} \; z^{-1}[\cos(\zeta x^2 + 2\beta_8 x) - \cos(\zeta x^2 + 2\beta_9 x) + \cos(\zeta x^2 + 2\beta_6 x)$$

$$- \cos(\zeta x^2 + 2\beta_7 x)]\} + \mathcal{F}_2\{\tfrac{1}{2}[\cos(\zeta x^2 + 2\beta_4 x) + \cos(\zeta x^2 + 2\beta_5 x)]$$

$$- \frac{P_3}{2} z^{-1}[\sin(\zeta x^2 + 2\beta_5 x) - \sin(\zeta x^2 + 2\beta_4 x)]$$

$$+ \frac{P_3}{4} z^{-1}[\sin(\zeta x^2 + 2\beta_7 x) - \sin(\zeta x^2 + 2\beta_6 x) + \sin(\zeta x^2 + 2\beta_9 x)$$

$$- \sin(\zeta x^2 + 2\beta_8 x)]\} \qquad z \neq 0 \qquad (4.36)$$

$$D_3'' \approx \mathcal{F}_1\{\frac{P_3}{2}(-1 + 4P_1^2)[\cos(\zeta x^2 + 2\beta_4 x) + \cos(\zeta x^2 + 2\beta_5 x)]$$

$$- \frac{P_1^2}{4}[\sin(\zeta x^2 + 2\beta_9 x) + \sin(\zeta x^2 + 2\beta_8 x) - \sin(\zeta x^2 + 2\beta_7 x)$$

$$- \sin(\zeta x^2 + 2\beta_6 x)] - P_1^2 P_3[\cos(\zeta x^2 + 2\beta_6 x) + \cos(\zeta x^2 + 2\beta_7 x)$$

$$+ \cos(\zeta x^2 + 2\beta_8 x) + \cos(\zeta x^2 + 2\beta_9 x)]\}$$

$$+ \mathcal{F}_1 P_3^2 z^{-1}\{\frac{1}{2}[\sin(\zeta x^2 + 2\beta_5 x) - \sin(\zeta x^2 + 2\beta_4 x)]$$

$$- \frac{1}{4}[\sin(\zeta x^2 + 2\beta_7 x) - \sin(\zeta x^2 + 2\beta_6 x) + \sin(\zeta x^2 + 2\beta_9 x)$$

$$- \sin(\zeta x^2 + 2\beta_8 x)]\} - \mathcal{F}_2\{\frac{P_3}{2}(-1 + 4P_1^2)[\sin(\zeta x^2 + 2\beta_4 x)$$

$$+ \sin(\zeta x^2 + 2\beta_5 x)] - \frac{P_1^2}{4}[\cos(\zeta x^2 + 2\beta_6 x) + \cos(\zeta x^2 + 2\beta_7 x)$$

$$- \cos(\zeta x^2 + 2\beta_8 x) - \cos(\zeta x^2 + 2\beta_9 x)] - P_1^2 P_3[\sin(\zeta x^2 + 2\beta_9 x)$$

$$+ \sin(\zeta x^2 + 2\beta_8 x) + \sin(\zeta x^2 + 2\beta_7 x) + \sin(\zeta x^2 + 2\beta_6 x)]\}$$

$$- \mathcal{F}_2 P_3^2 \ z^{-1}\{\tfrac{1}{2}[\cos(\zeta x^2 + 2\beta_4 x) - \cos(\zeta x^2 + 2\beta_5)]$$

$$- \tfrac{1}{4}[\cos(\zeta x^2 + 2\beta_8 x) - \cos(\zeta x^2 + 2\beta_9 x) + \cos(\zeta x^2 + 2\beta_6 x)$$

$$- \cos(\zeta x^2 + 2\beta_7 x)]\} + \mathcal{F}_1\{\frac{P_3}{2} \ z^{-1}[\sin(\zeta x^2 + 2\beta_4 x)$$

$$+ \sin(\zeta x^2 + 2\beta_5 x)] + \tfrac{1}{2}[\cos(\zeta x^2 + 2\beta_4 x) - \cos(\zeta x^2 + 2\beta_5 x)]$$

$$- \frac{P_3}{4} \ z^{-1}[\sin(\zeta x^2 + 2\beta_9 x) + \sin(\zeta x^2 + 2\beta_8 x) + \sin(\zeta x^2 + 2\beta_7 x)$$

$$+ \sin(\zeta x^2 + 2\beta_6 x)]\} + \mathcal{F}_2\{\frac{P_3}{2} \ z^{-1} \ [\cos(\zeta x^2 + 2\beta_4 x)$$

$$+ \cos(\zeta x^2 + 2\beta_5 x)] + \tfrac{1}{2}[\sin(\zeta x^2 + 2\beta_5 x) - \sin(\zeta x^2 + 2\beta_4 x)]$$

$$- \frac{P_3}{4} \ z^{-1}[\cos(\zeta x^2 + 2\beta_6 x) + \cos(\zeta x^2 + 2\beta_7 x) + \cos(\zeta x^2 + 2\beta_8 x)$$

$$+ \cos(\zeta x^2 + 2\beta_9 x)]\} \qquad z \neq 0 \qquad (4.37)$$

where primes denote differentiation with respect to x .

The following conditions must be satisfied by D_i $(i = 1, 2, 3)$ at $x = 0$:

$$D_i(0) = 0 \qquad (i = 1, 2, 3) \qquad (4.38)$$

and

$$D_i'(0) = V_{i0} \qquad (i = 1, 2, 3) \qquad (4.39)$$

The reason for taking $D_i(0)$ $(i = 1,2,3)$ equal to zero is that no loss of generality results from regarding B* as initially coincident with point 0, (see Figure 1), since the choice of point 0 is arbitrary.

The solution of Eqs. (4.35)-(4.37) can be expressed in terms of certain functions related to Fresnel integrals. The functions to be employed are defined as follows:

$$C_2[w] \triangleq \frac{1}{\sqrt{2\pi}} \int_0^w \frac{\cos u}{\sqrt{u}}\, du \tag{4.40}$$

$$S_2[w] \triangleq \frac{1}{\sqrt{2\pi}} \int_0^w \frac{\sin u}{\sqrt{u}}\, du \tag{4.41}$$

$$C(a,b,x) \triangleq \left[\frac{\pi}{2a}\sqrt{\frac{2}{a\pi}}\,(ax+b)\right]\left\{\cos\left(\frac{b^2}{a}\right)C_2\left[\frac{1}{a}(ax+b)^2\right] + \sin\left(\frac{b^2}{a}\right)S_2\left[\frac{1}{a}(ax+b)^2\right]\right\} - \frac{1}{2a}\sin(ax^2+2bx)$$

$$a > 0 \tag{4.42}$$

$$S(a,b,x) \triangleq \left[\frac{\pi}{2a}\sqrt{\frac{2}{a\pi}}\,(ax+b)\right]\left\{\cos\left(\frac{b^2}{a}\right)S_2\left[\frac{1}{a}(ax+b)^2\right] - \sin\left(\frac{b^2}{a}\right)C_2\left[\frac{1}{a}(ax+b)^2\right]\right\} + \frac{1}{2a}\cos(ax^2+2bx)$$

$$a > 0 \tag{4.43}$$

$$C_1(a,b,x) \triangleq \sqrt{\frac{\pi}{2a}}\left\{\cos\left(\frac{b^2}{a}\right)C_2\left[\frac{1}{a}(ax+b)^2\right]\right.$$

continued

$$+ \sin\left(\frac{b^2}{a}\right) S_2\left[\frac{1}{a}(ax+b)^2\right]\Bigg\}$$

$$a > 0 \qquad (4.44)$$

$$S_1(a,b,x) \triangleq \sqrt{\frac{\pi}{2a}}\left\{\cos\left(\frac{b^2}{a}\right) S_2\left[\frac{1}{a}(ax+b)^2\right]\right.$$

$$\left. - \sin\left(\frac{b^2}{a}\right) C_2\left[\frac{1}{a}(ax+b)^2\right]\right\}$$

$$a > 0 \qquad (4.45)$$

$$C^*(a,b,x) \triangleq \int_0^x \int_0^v \cos(au^2 + 2bu)\,du\,dv$$

$$\underset{(4.42,\,4.44)}{=} C(a,b,x) - C_1(a,b,0)x - C(a,b,0)$$

$$a > 0 \qquad (4.46)$$

$$S^*(a,b,x) \triangleq \int_0^x \int_0^v \sin(au^2 + 2bu)\,du\,dv$$

$$\underset{(4.43,\,4.45)}{=} S(a,b,x) - S_1(a,b,0)x - S(a,b,0)$$

$$a > 0 \qquad (4.47)$$

Two successive integrations of Eqs. (4.35)-(4.37) and use of Eqs. (4.38) and (4.39) now lead to the following expression for D_1, D_2, and D_3:

$$D_1 \approx \mathcal{F}_1 P_1 \Big\{ (4P_3^2 - P_3)\; C^*(\zeta, \beta_1, x) - \frac{P_3}{2}[S^*(\zeta, \beta_2, x)$$

$$- S^*(\zeta, \beta_3, x) + C^*(\zeta, \beta_7, x) + C^*(\zeta, \beta_9, x)]$$

$$+ P_3 \, C^*(\zeta, \beta_5, x) + \frac{1}{2}(1 - 4P_3^2 + P_3)[C^*(\zeta, \beta_3, x)$$

$$+ C^*(\zeta, \beta_2, x)]\} - \mathcal{F}_2 P_1 \{(4P_3^2 - P_3) \, S^*(\zeta, \beta_1, x)$$

$$- \frac{P_3}{2}[C^*(\zeta, \beta_3, x) - C^*(\zeta, \beta_2, x) + S^*(\zeta, \beta_7, x)$$

$$+ S^*(\zeta, \beta_9, x)] + P_3 \, S(\zeta, \beta_5, x)$$

$$+ \frac{1}{2}(1 - 4P_3^2 + P_3)[S^*(\zeta, \beta_2, x) + S^*(\zeta, \beta_3, x)]\} + V_{10} x$$

$$\zeta > 0 \qquad (4.48)$$

$$D_2 \approx \mathcal{F}_1 \{\frac{1}{2}(P_3 - 4P_1^2 P_3)[S^*(\zeta, \beta_5, x) - S^*(\zeta, \beta_4, x)]$$

$$+ \frac{P_1^2}{4}[C^*(\zeta, \beta_8, x) - C^*(\zeta, \beta_9, x) - C^*(\zeta, \beta_6, x)$$

$$+ C^*(\zeta, \beta_7, x)] + P_1^2 P_3 [S^*(\zeta, \beta_7, x) - S^*(\zeta, \beta_6, x)$$

$$+ S^*(\zeta, \beta_9, x) - S^*(\zeta, \beta_8, x)]\} + \mathcal{F}_1 P_3^2 z^{-1} \{\frac{1}{2}[C^*(\zeta, \beta_4, x)$$

$$+ C^*(\zeta, \beta_5, x)] - \frac{1}{4}[C^*(\zeta, \beta_6, x) + C^*(\zeta, \beta_7, x) + C^*(\zeta, \beta_8, x)$$

$$+ C^*(\zeta, \beta_9, x)]\} - \mathcal{F}_2 \{\frac{1}{2}(P_3 - 4P_1^2 P_3)[C^*(\zeta, \beta_4, x) - C^*(\zeta, \beta_5, x)]$$

$$+ \frac{P_1^2}{4}[S^*(\zeta, \beta_7, x) - S^*(\zeta, \beta_6, x) - S^*(\zeta, \beta_9, x) + S^*(\zeta, \beta_8, x)]$$

$$+ P_1^2 P_3 [C^*(\zeta, \beta_8, x) - C^*(\zeta, \beta_9, x) + C^*(\zeta, \beta_6, x)$$

$$- C^*(\zeta, \beta_7, x)]\} - \mathcal{F}_2 P_3^2 z^{-1} \{\frac{1}{2}[S^*(\zeta, \beta_4, x) + S^*(\zeta, \beta_5, x)]$$

$$- \frac{1}{4}[S^*(\zeta, \beta_9, x) + S^*(\zeta, \beta_8, x) + S^*(\zeta, \beta_7, x)$$

$$+ S^*(\zeta, \beta_6, x)]\} + \mathcal{F}_1 \{\frac{1}{2}[S^*(\zeta, \beta_4, x) + S^*(\zeta, \beta_5, x)]$$

$$- \frac{P_3}{2} z^{-1}[C^*(\zeta, \beta_4, x) - C^*(\zeta, \beta_5, x)] + \frac{P_3}{4} z^{-1}[C^*(\zeta, \beta_8, x)$$

$$- C^*(\zeta, \beta_9, x) + C^*(\zeta, \beta_6, x) - C^*(\zeta, \beta_7, x)]\}$$

$$+ \mathcal{F}_2 \{\frac{1}{2}[C^*(\zeta, \beta_4, x) + C^*(\zeta, \beta_5, x)] - \frac{P_3}{2} z^{-1}[S^*(\zeta, \beta_5, x)$$

$$- S^*(\zeta, \beta_4, x)] + \frac{P_3}{4} z^{-1}[S^*(\zeta, \beta_7, x) - S^*(\zeta, \beta_6, x)$$

$$+ S^*(\zeta, \beta_9, x) - S^*(\zeta, \beta_8, x)]\} + V_{20} x$$

$$\zeta > 0 \qquad (4.49)$$

$$D_3 \approx \mathcal{F}_1 \{\frac{P_3}{2}(-1 + 4P_1^2)\ [C^*(\zeta, \beta_4, x) + C^*(\zeta, \beta_5, x)]$$

$$- \frac{P_1^2}{4}[S^*(\zeta, \beta_9, x) + S^*(\zeta, \beta_8, x) - S^*(\zeta, \beta_7, x) - S^*(\zeta, \beta_6, x)]$$

$$- P_1^2 P_3 [C^*(\zeta, \beta_6, x) + C^*(\zeta, \beta_7, x) + C^*(\zeta, \beta_8, x)$$

$$+ C^*(\zeta, \beta_9, x)]\} + \mathcal{F}_1 P_3^2 z^{-1}\{\tfrac{1}{2}[S^*(\zeta, \beta_5, x) - S^*(\zeta, \beta_4, x)]$$

$$- \tfrac{1}{4}[S^*(\zeta, \beta_7, x) - S^*(\zeta, \beta_6, x) + S^*(\zeta, \beta_9, x)$$

$$- S^*(\zeta, \beta_8, x)]\} - \mathcal{F}_2\{\frac{P_3}{2}(-1 + 4P_1^2)[S^*(\zeta, \beta_4, x)$$

$$+ S^*(\zeta, \beta_5, x)] - \frac{P_1^2}{4}[C^*(\zeta, \beta_6, x) + C^*(\zeta, \beta_7, x)$$

$$- C^*(\zeta, \beta_8, x) - C^*(\zeta, \beta_9, x)] - P_1^2 P_3[S^*(\zeta, \beta_9, x)$$

$$+ S^*(\zeta, \beta_8, x) + S^*(\zeta, \beta_7, x) + S^*(\zeta, \beta_6, x)]\}$$

$$- \mathcal{F}_2 P_3^2 z^{-1}\{\tfrac{1}{2}[C^*(\zeta, \beta_4, x) - C^*(\zeta, \beta_5, x)] - \tfrac{1}{4}[C^*(\zeta, \beta_8, x)$$

$$- C^*(\zeta, \beta_9, x) + C^*(\zeta, \beta_6, x) - C^*(\zeta, \beta_7, x)]\}$$

$$+ \mathcal{F}_1\{\frac{P_3}{2} z^{-1}[S^*(\zeta, \beta_4, x) + S^*(\zeta, \beta_5, x)] + \tfrac{1}{2}[C^*(\zeta, \beta_4, x)$$

$$- C^*(\zeta, \beta_5, x)] - \frac{P_3}{4} z^{-1}[S^*(\zeta, \beta_9, x) + S^*(\zeta, \beta_8, x)$$

$$+ S^*(\zeta, \beta_7, x) + S^*(\zeta, \beta_6, x)]\} + \mathcal{F}_2\{\frac{P_3}{2} z^{-1}[C^*(\zeta, \beta_4, x)$$

$$+ C^*(\zeta, \beta_5, x)] + \tfrac{1}{2}[S^*(\zeta, \beta_5, x) - S^*(\zeta, \beta_4, x)]$$

$$- \frac{P_3}{4} z^{-1}[C^*(\zeta, \beta_6, x) + C^*(\zeta, \beta_7, x) + C^*(\zeta, \beta_8, x)$$

$$C^*(\zeta, \beta_9, x)]\} + V_{30}x \qquad \zeta > 0 \tag{4.50}$$

Eqs. (4.48)-(4.50) are valid only for $\zeta > 0$. To deal with $\zeta < 0$, parameters $\bar{\zeta}$ and $\bar{\beta}_i$ $(i = 1,\ldots,9)$ are introduced as

$$\bar{\zeta} \triangleq -\zeta \tag{4.51}$$

and

$$\bar{\beta}_i \triangleq -\beta_i \qquad (i = 1,\ldots,9) \tag{4.52}$$

which permits one to rewrite Eqs. (4.35)-(4.37) as follows:

$$\begin{aligned}
D_1'' \underset{(4.35)}{\approx} & \mathscr{F}_1 P_1 \{(4P_3^2 - P_3)\cos(\bar{\zeta}x^2 + 2\bar{\beta}_1 x) + \frac{P_3}{2}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_2 x) \\
& - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_3 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)] \\
& + P_3\cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x) + \frac{1}{2}(1 - 4P_3^2 + P_3)[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_3 x) \\
& + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_2 x)]\} - \mathscr{F}_2 P_1 \{-(4P_3^2 - P_3)\ \sin(\bar{\zeta}x^2 + 2\bar{\beta}_1 x) \\
& - \frac{P_3}{2}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_3 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_2 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) \\
& - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)] - P_3\ \sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x) \\
& - \frac{1}{2}(1 - 4P_3^2 + P_3)[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_2 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_3 x)]\}
\end{aligned} \tag{4.53}$$

$$
\begin{aligned}
D_2''\underset{(4.36)}{\approx} & \mathcal{F}_1\{\tfrac{1}{2}(P_3 - 4P_1^2P_3)[-\sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_4 x)] \\
& + \frac{P_1^2}{4}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) \\
& + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x)] - P_1^2P_3[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) \\
& + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_8 x)]\} \\
& + \mathcal{F}_1 P_3^2\ z^{-1}\{\tfrac{1}{2}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)] \\
& - \tfrac{1}{4}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) \\
& + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)]\} - \mathcal{F}_2\{\tfrac{1}{2}(P_3 - 4P_1^2P_3)[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) \\
& - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)] - \frac{P_1^2}{4}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) \\
& - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_8 x)] + P_1^2P_3[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) \\
& - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x)]\} \\
& - \mathcal{F}_2 P_3^2\ z^{-1}\{-\tfrac{1}{2}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)] \\
& - \tfrac{1}{4}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) \\
& + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_6 x)]\} + \mathcal{F}_1\{-\tfrac{1}{2}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)]
\end{aligned}
$$

$$- \frac{P_3}{2} z^{-1}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)]$$

$$+ \frac{P_3}{4} z^{-1}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)$$

$$+ \cos(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x)]\}$$

$$+ \mathcal{F}_2\{\frac{1}{2}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)]$$

$$+ \frac{P_3}{2} z^{-1}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_4 x)]$$

$$- \frac{P_3}{4} z^{-1}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)$$

$$- \sin(\bar{\zeta}x^2 + 2\bar{\beta}_8 x)]\} \qquad z \neq 0 \qquad (4.54)$$

$$D_3'' \underset{(4.37)}{\approx} \mathcal{F}_1\{\frac{P_3}{2}(-1 + 4P_1^2)[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)]$$

$$+ \frac{P_1^2}{4}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x)$$

$$- \sin(\bar{\zeta}x^2 + 2\bar{\beta}_6 x)] - P_1^2 P_3[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x)$$

$$+ \cos(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)]\}$$

$$+ \mathcal{F}_1 P_3^2 z^{-1}\{-\frac{1}{2}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_4 x)]$$

$$+ \frac{1}{4}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)$$

$$- \sin(\bar{\zeta}x^2 + 2\bar{\beta}_8 x)]\} - \mathscr{F}_2\{- \frac{P_3}{2}(-1 + 4P_1^2)[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_4 x)$$

$$+ \sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)] - \frac{P_1^2}{4}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x)$$

$$- \cos(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)] + P_1^2 P_3[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)$$

$$+ \sin(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_6 x)]\}$$

$$- \mathscr{F}_2 P_3^2 z^{-1}\{\frac{1}{2}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)]$$

$$- \frac{1}{4}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_6 x)$$

$$- \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x)]\} + \mathscr{F}_1\{- \frac{P_3}{2} z^{-1}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_4 x)$$

$$+ \sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)] + \frac{1}{2}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_4 x) - \cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)]$$

$$+ \frac{P_3}{4} z^{-1}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_9 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_8 x) + \sin(\bar{\zeta}x^2 + 2\bar{\beta}_7 x)$$

$$+ \sin(\bar{\zeta}x^2 + 2\bar{\beta}_6 x)]\} + \mathscr{F}_2\{\frac{P_3}{2} z^{-1}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_4 x)$$

$$+ \cos(\bar{\zeta}x^2 + 2\bar{\beta}_5 x)] - \frac{1}{2}[\sin(\bar{\zeta}x^2 + 2\bar{\beta}_5 x) - \sin(\bar{\zeta}x^2 + 2\bar{\beta}_4 x)]$$

$$- \frac{P_3}{4} z^{-1}[\cos(\bar{\zeta}x^2 + 2\bar{\beta}_6 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_7 x) + \cos(\bar{\zeta}x^2 + 2\bar{\beta}_8 x)$$

$$+ \cos(\bar{\zeta}x^2 + 2\bar{\beta}_9 x)]\} \qquad z \neq 0 \qquad (4.55)$$

Using the functions defined in Eqs. (4.40)-(4.47), one can express the solution to Eqs. (4.53)-(4.55) satisfying Eqs. (4.38)-(4.39) as

$$D_1 \approx \mathcal{F}_1 P_1 \{(4P_3^2 - P_3) C^*(\bar{\zeta}, \bar{\beta}_1, x) - \frac{P_3}{2}[-S^*(\bar{\zeta}, \bar{\beta}_2, x)$$

$$+ S^*(\bar{\zeta}, \bar{\beta}_3, x) + C^*(\bar{\zeta}, \bar{\beta}_7, x) + C^*(\bar{\zeta}, \bar{\beta}_9, x)]$$

$$+ P_3 \; C^*(\bar{\zeta}, \bar{\beta}_5, x) + \frac{1}{2}(1 - 4P_3^2 + P_3)[C^*(\bar{\zeta}, \bar{\beta}_3, x)$$

$$+ C^*(\bar{\zeta}, \bar{\beta}_2, x)]\} - \mathcal{F}_2 P_1 \{-(4P_3^2 - P_3) S^*(\bar{\zeta}, \bar{\beta}_1, x)$$

$$- \frac{P_3}{2}[C^*(\bar{\zeta}, \bar{\beta}_3, x) - C^*(\bar{\zeta}, \bar{\beta}_2, x) - S^*(\bar{\zeta}, \bar{\beta}_7, x)$$

$$- S^*(\bar{\zeta}, \bar{\beta}_9, x)] - P_3 \; S^*(\bar{\zeta}, \bar{\beta}_5, x)$$

$$- \frac{1}{2}(1 - 4P_3^2 + P_3)[S^*(\bar{\zeta}, \bar{\beta}_2, x) + S^*(\bar{\zeta}, \bar{\beta}_3, x)]\} + V_{10} x$$

$$\zeta < 0 \qquad (4.56)$$

$$D_2 \approx \mathcal{F}_1 \{\frac{1}{2}(P_3 - 4P_1^2 P_3)[-S^*(\bar{\zeta}, \bar{\beta}_5, x) + S^*(\bar{\zeta}, \bar{\beta}_4, x)]$$

$$+ \frac{P_1^2}{4}[C^*(\bar{\zeta}, \bar{\beta}_8, x) - C^*(\bar{\zeta}, \bar{\beta}_9, x) - C^*(\bar{\zeta}, \bar{\beta}_6, x)$$

$$+ C^*(\bar{\zeta}, \bar{\beta}_7, x)] - P_1^2 P_3 [S^*(\bar{\zeta}, \bar{\beta}_7, x) - S^*(\bar{\zeta}, \bar{\beta}_6, x)$$

$$+ S^*(\bar{\zeta}, \bar{\beta}_9, x) - S^*(\bar{\zeta}, \bar{\beta}_8, x)]\} + \mathcal{F}_1 P_3^2 \; z^{-1} \{\frac{1}{2}[C^*(\bar{\zeta}, \bar{\beta}_4, x)$$

$$+ C^*(\bar{\zeta}, \bar{\beta}_5, x)] - \frac{1}{4}[C^*(\bar{\zeta}, \bar{\beta}_6, x) + C^*(\bar{\zeta}, \bar{\beta}_7, x) + C^*(\bar{\zeta}, \bar{\beta}_8, x)$$

$$+ C^*(\bar{\zeta}, \bar{\beta}_9, x)]\} - \mathcal{F}_2\{\frac{1}{2}(P_3 - 4P_1^2P_3)[C^*(\bar{\zeta}, \bar{\beta}_4, x)$$

$$- C^*(\bar{\zeta}, \bar{\beta}_5, x)] - \frac{P_1^2}{4}[S^*(\bar{\zeta}, \bar{\beta}_7, x) - S^*(\bar{\zeta}, \bar{\beta}_6, x)$$

$$- S^*(\bar{\zeta}, \bar{\beta}_9, x) + S^*(\bar{\zeta}, \bar{\beta}_8, x)] + P_1^2P_3[C^*(\bar{\zeta}, \bar{\beta}_8, x)$$

$$- C^*(\bar{\zeta}, \bar{\beta}_9, x) + C^*(\bar{\zeta}, \bar{\beta}_6, x) - C^*(\bar{\zeta}, \bar{\beta}_7, x)]\}$$

$$- \mathcal{F}_2P_3^2\ z^{-1}\{-\frac{1}{2}[S^*(\bar{\zeta}, \bar{\beta}_4, x) + S^*(\bar{\zeta}, \bar{\beta}_5, x)]$$

$$+ \frac{1}{4}[S^*(\bar{\zeta}, \bar{\beta}_9, x) + S^*(\bar{\zeta}, \bar{\beta}_8, x) + S^*(\bar{\zeta}, \bar{\beta}_7, x) + S^*(\bar{\zeta}, \bar{\beta}_6, x)]\}$$

$$+ \mathcal{F}_1\{-\frac{1}{2}[S^*(\bar{\zeta}, \bar{\beta}_4, x) + S^*(\bar{\zeta}, \bar{\beta}_5, x)] - \frac{P_3}{2}\ z^{-1}[C^*(\bar{\zeta}, \bar{\beta}_4, x)$$

$$- C^*(\bar{\zeta}, \bar{\beta}_5, x)] + \frac{P_3}{4}\ z^{-1}[C^*(\bar{\zeta}, \bar{\beta}_8, x) - C^*(\bar{\zeta}, \bar{\beta}_9, x)$$

$$+ C^*(\bar{\zeta}, \bar{\beta}_6, x) - C^*(\bar{\zeta}, \bar{\beta}_7, x)]\} + \mathcal{F}_2\{\frac{1}{2}[C^*(\bar{\zeta}, \bar{\beta}_4, x)$$

$$+ C^*(\bar{\zeta}, \bar{\beta}_5, x)] + \frac{P_3}{2}\ z^{-1}[S^*(\bar{\zeta}, \bar{\beta}_5, x) - S^*(\bar{\zeta}, \bar{\beta}_4, x)]$$

$$- \frac{P_3}{4}\ z^{-1}[S^*(\bar{\zeta}, \bar{\beta}_7, x) - S^*(\bar{\zeta}, \bar{\beta}_6, x) + S^*(\bar{\zeta}, \bar{\beta}_9, x)$$

$$- S^*(\bar{\zeta}, \bar{\beta}_8, x)]\} + V_{20}x \qquad \zeta < 0 \qquad (4.57)$$

$$D_3 \approx \mathcal{F}_1\{\frac{P_3}{2}(-1+4P_1^2)\,[C^*(\bar{\zeta},\bar{\beta}_4,x) + C^*(\bar{\zeta},\bar{\beta}_5,x)]$$

$$+ \frac{P_1^2}{4}[S^*(\bar{\zeta},\bar{\beta}_9,x) + S^*(\bar{\zeta},\bar{\beta}_8,x) - S^*(\bar{\zeta},\bar{\beta}_7,x)$$

$$- S^*(\bar{\zeta},\bar{\beta}_6,x)] - P_1^2P_3[C^*(\bar{\zeta},\bar{\beta}_6,x) + C^*(\bar{\zeta},\bar{\beta}_7,x)$$

$$+ C^*(\bar{\zeta},\bar{\beta}_8,x) + C^*(\bar{\zeta},\bar{\beta}_9,x)]\} + \mathcal{F}_1P_3^2\ z^{-1}\{-\frac{1}{2}[S^*(\bar{\zeta},\bar{\beta}_5,x)$$

$$- S^*(\bar{\zeta},\bar{\beta}_4,x)] + \frac{1}{4}[S^*(\bar{\zeta},\bar{\beta}_7,x) - S^*(\bar{\zeta},\bar{\beta}_6,x)$$

$$+ S^*(\bar{\zeta},\bar{\beta}_9,x) - S^*(\bar{\zeta},\bar{\beta}_8,x)]\} - \mathcal{F}_2\{-\frac{P_3}{2}(-1+4P_1^2)[S^*(\bar{\zeta},\bar{\beta}_4,x)$$

$$+ S^*(\bar{\zeta},\bar{\beta}_5,x)] - \frac{P_1^2}{4}[C^*(\bar{\zeta},\bar{\beta}_6,x) + C^*(\bar{\zeta},\bar{\beta}_7,x)$$

$$- C^*(\bar{\zeta},\bar{\beta}_8,x) - C^*(\bar{\zeta},\bar{\beta}_9,x)] + P_1^2P_3[S^*(\bar{\zeta},\bar{\beta}_9,x)$$

$$+ S^*(\bar{\zeta},\bar{\beta}_8,x) + S^*(\bar{\zeta},\bar{\beta}_7,x) + S^*(\bar{\zeta},\bar{\beta}_6,x)]\}$$

$$- \mathcal{F}_2P_3^2\ z^{-1}\{\frac{1}{2}[C^*(\bar{\zeta},\bar{\beta}_4,x) - C^*(\bar{\zeta},\bar{\beta}_5,x)] - \frac{1}{4}[C^*(\bar{\zeta},\bar{\beta}_8,x)$$

$$- C^*(\bar{\zeta},\bar{\beta}_9,x) + C^*(\bar{\zeta},\bar{\beta}_6,x) - C^*(\bar{\zeta},\bar{\beta}_7,x)]\}$$

$$+ \mathcal{F}_1\{-\frac{P_3}{2}\ z^{-1}[S^*(\bar{\zeta},\bar{\beta}_4,x) + S^*(\bar{\zeta},\bar{\beta}_5,x)] + \frac{1}{2}[C^*(\bar{\zeta},\bar{\beta}_4,x)$$

$$- C^*(\bar{\zeta},\bar{\beta}_5,x)] + \frac{P_3}{4}\ z^{-1}[S^*(\bar{\zeta},\bar{\beta}_9,x) + S^*(\bar{\zeta},\bar{\beta}_8,x)$$

$$+ S^*(\bar{\zeta}, \bar{\beta}_7, x) + S^*(\bar{\zeta}, \bar{\beta}_6, x)]\} + \mathcal{F}_2\{\frac{P_3}{2} z^{-1}[C^*(\bar{\zeta}, \bar{\beta}_4, x)$$

$$+ C^*(\bar{\zeta}, \bar{\beta}_5, x)] - \frac{1}{2}[S^*(\bar{\zeta}, \bar{\beta}_5, x) - S^*(\bar{\zeta}, \bar{\beta}_4, x)]$$

$$- \frac{P_3}{4} z^{-1}[C^*(\bar{\zeta}, \bar{\beta}_6, x) + C^*(\bar{\zeta}, \bar{\beta}_7, x) + C^*(\bar{\zeta}, \bar{\beta}_8, x)$$

$$+ C^*(\bar{\zeta}, \bar{\beta}_9, x)]\} + V_{30}x \qquad \zeta < 0 \qquad (4.58)$$

Finally, for $\zeta = 0$, which is the case when the inertia ellipsoid of B for B* is a sphere (see Eqs. (4.25) and (3.11)), Eqs. (4.35)-(4.37) reduce to

$$D_1'' \approx \mathcal{F}_1 P_1\{4P_3^2 - P_3 - \frac{P_3}{2}[\sin(2\beta_2 x) - \sin(2\beta_3 x) + \cos(2\beta_7 x)$$

$$+ \cos(2\beta_9 x)] + P_3\cos(2\beta_5 x) + \frac{1}{2}(1 - 4P_3^2 + P_3)[\cos(2\beta_3 x)$$

$$+ \cos(2\beta_2 x)]\} - \mathcal{F}_2 P_1\{- \frac{P_3}{2}[\cos(2\beta_3 x) - \cos(2\beta_2 x)$$

$$+ \sin(2\beta_7 x) + \sin(2\beta_9 x)] + P_3\sin(2\beta_5 x)$$

$$+ \frac{1}{2}(1 - 4P_3^2 + P_3)[\sin(2\beta_2 x) + \sin(2\beta_3 x)]\} \qquad (4.59)$$

$$D_2'' \approx \mathcal{F}_1\{\frac{1}{2}(P_3 - 4P_1^2 P_3)[\sin(2\beta_5 x) - \sin(2\beta_4 x)] + \frac{P_1^2}{4}[\cos(2\beta_8 x)$$

$$- \cos(2\beta_9 x) - \cos(2\beta_6 x) + \cos(2\beta_7 x)] + P_1^2 P_3[\sin(2\beta_7 x)$$

$$- \sin(2\beta_6 x) + \sin(2\beta_9 x) - \sin(2\beta_8 x)]\}$$

$$+ \mathcal{F}_1 P_3^2\ z^{-1}\{\tfrac{1}{2}[\cos(2\beta_4 x) + \cos(2\beta_5 x)] - \tfrac{1}{4}[\cos(2\beta_6 x)$$

$$+ \cos(2\beta_7 x) + \cos(2\beta_8 x) + \cos(2\beta_9 x)]\}$$

$$- \mathcal{F}_2\{\tfrac{1}{2}(P_3 - 4P_1^2 P_3)[\cos(2\beta_4 x) - \cos(2\beta_5 x)] + \frac{P_1^2}{4}[\sin(2\beta_7 x)$$

$$- \sin(2\beta_6 x) - \sin(2\beta_9 x) + \sin(2\beta_8 x)] + P_1^2 P_3[\cos(2\beta_8 x)$$

$$- \cos(2\beta_9 x) + \cos(2\beta_6 x) - \cos(2\beta_7 x)]\}$$

$$- \mathcal{F}_2 P_3^2\ z^{-1}\{\tfrac{1}{2}[\sin(2\beta_4 x) + \sin(2\beta_5 x)] - \tfrac{1}{4}[\sin(2\beta_9 x)$$

$$+ \sin(2\beta_8 x) + \sin(2\beta_7 x) + \sin(2\beta_6 x)]\} + \mathcal{F}_1\{\tfrac{1}{2}[\sin(2\beta_4 x)$$

$$+ \sin(2\beta_5 x)] - \frac{P_3}{2}\ z^{-1}[\cos(2\beta_4 x) - \cos(2\beta_5 x)]$$

$$+ \frac{P_3}{4}\ z^{-1}[\cos(2\beta_8 x) - \cos(2\beta_9 x) + \cos(2\beta_6 x) - \cos(2\beta_7 x)]\}$$

$$+ \mathcal{F}_2\{\tfrac{1}{2}[\cos(2\beta_4 x) + \cos(2\beta_5 x)] - \frac{P_3}{2}\ z^{-1}[\sin(2\beta_5 x)$$

$$- \sin(2\beta_4 x)] + \frac{P_3}{4}\ z^{-1}[\sin(2\beta_7 x) - \sin(2\beta_6 x) + \sin(2\beta_9 x)$$

$$- \sin(2\beta_8 x)]\} \qquad z \neq 0 \qquad (4.60)$$

$$D_3'' \approx \mathcal{F}_1\left\{\frac{P_3}{2}(-1+4P_1^2)\,[\cos(2\beta_4 x) + \cos(2\beta_5 x)] - \frac{P_1^2}{4}[\sin(2\beta_9 x)\right.$$

$$+ \sin(2\beta_8 x) - \sin(2\beta_7 x) - \sin(2\beta_6 x)] - P_1^2 P_3[\cos(2\beta_6 x)$$

$$\left. + \cos(2\beta_7 x) + \cos(2\beta_8 x) + \cos(2\beta_9 x)]\right\}$$

$$+ \mathcal{F}_1 P_3^2\, z^{-1}\left\{\frac{1}{2}[\sin(2\beta_5 x) - \sin(2\beta_4 x)] - \frac{1}{4}[\sin(2\beta_7 x)\right.$$

$$\left. - \sin(2\beta_6 x) + \sin(2\beta_9 x) - \sin(2\beta_8 x)]\right\}$$

$$- \mathcal{F}_2\left\{\frac{P_3}{2}(-1+4P_1^2)\,[\sin(2\beta_4 x) + \sin(2\beta_5 x)] - \frac{P_1^2}{4}[\cos(2\beta_6 x)\right.$$

$$+ \cos(2\beta_7 x) - \cos(2\beta_8 x) - \cos(2\beta_9 x)] - P_1^2 P_3[\sin(2\beta_9 x)$$

$$\left. + \sin(2\beta_8 x) + \sin(2\beta_7 x) + \sin(2\beta_6 x)]\right\}$$

$$- \mathcal{F}_2 P_3^2\, z^{-1}\left\{\frac{1}{2}[\cos(2\beta_4 x) - \cos(2\beta_5 x)] - \frac{1}{4}[\cos(2\beta_8 x)\right.$$

$$\left. - \cos(2\beta_9 x) + \cos(2\beta_6 x) - \cos(2\beta_7 x)]\right\}$$

$$+ \mathcal{F}_1\left\{\frac{P_3}{2}\, z^{-1}[\sin(2\beta_4 x) + \sin(2\beta_5 x)] + \frac{1}{2}[\cos(2\beta_4 x)\right.$$

$$- \cos(2\beta_5 x)] - \frac{P_3}{4}\, z^{-1}[\sin(2\beta_9 x) + \sin(2\beta_8 x) + \sin(2\beta_7 x)$$

$$\left. + \sin(2\beta_6 x)]\right\} + \mathcal{F}_2\left\{\frac{P_3}{2}\, z^{-1}[\cos(2\beta_4 x) + \cos(2\beta_5 x)]\right.$$

$$+ \frac{1}{2}[\sin(2\beta_5 x) - \sin(2\beta_4 x)] - \frac{P_3}{4} z^{-1}[\cos(2\beta_6 x) + \cos(2\beta_7 x)$$

$$+ \cos(2\beta_8 x) + \cos(2\beta_9 x)]\} \qquad z \neq 0 \qquad (4.61)$$

The solution to Eqs. (4.59)-(4.61) that satisfies the initial conditions in Eqs. (4.38) and (4.39) is

$$D_1 \approx \mathcal{F}_1 P_1 \{(4P_3^2 - P_3)\left(\frac{x^2}{2}\right) - \frac{P_3}{2}[S^*(\beta_2, x) - S^*(\beta_3, x)$$

$$+ C^*(\beta_7, x) + C^*(\beta_9, x)] + P_3 C^*(\beta_5, x)$$

$$+ \frac{1}{2}(1 - 4P_3^2 + P_3)[C^*(\beta_3, x) + C^*(\beta_2, x)]\}$$

$$- \mathcal{F}_2 P_1 \{- \frac{P_3}{2}[C^*(\beta_3, x) - C^*(\beta_2, x) + S^*(\beta_7, x) + S^*(\beta_9, x)]$$

$$+ P_3 S^*(\beta_5, x) + \frac{1}{2}(1 - 4P_3^2 + P_3)[S^*(\beta_2, x) + S^*(\beta_3, x)]\}$$

$$+ V_{10} x \qquad (4.62)$$

$$D_2 \approx \mathcal{F}_1 \{\frac{1}{2}(P_3 - 4P_1^2 P_3)[S^*(\beta_5, x) - S^*(\beta_4, x)] + \frac{P_1^2}{4}[C^*(\beta_8, x)$$

$$- C^*(\beta_9, x) - C^*(\beta_6, x) + C^*(\beta_7, x)] + P_1^2 P_3[S^*(\beta_7, x)$$

$$- S^*(\beta_6, x) + S^*(\beta_9, x) - S^*(\beta_8, x)]\}$$

$$+ \mathcal{F}_1 P_3^2 z^{-1}\{\frac{1}{2}[C^*(\beta_4, x) + C^*(\beta_5, x)] - \frac{1}{4}[C^*(\beta_6, x)$$

$$+ C^*(\beta_7, x) + C^*(\beta_8, x) + C^*(\beta_9, x)]\}$$

$$- \mathcal{F}_2\{\frac{1}{2}(P_3 - 4P_1^2P_3)[C^*(\beta_4, x) - C^*(\beta_5, x)] + \frac{P_1^2}{4}[S^*(\beta_7, x)$$

$$- S^*(\beta_6, x) - S^*(\beta_9, x) + S^*(\beta_8, x)] + P_1^2P_3[C^*(\beta_8, x)$$

$$- C^*(\beta_9, x) + C^*(\beta_6, x) - C^*(\beta_7, x)]\}$$

$$- \mathcal{F}_2P_3^2\ z^{-1}\{\frac{1}{2}[S^*(\beta_4, x) + S^*(\beta_5, x)] - \frac{1}{4}[S^*(\beta_9, x)$$

$$+ S^*(\beta_8, x) + S^*(\beta_7, x) + S^*(\beta_6, x)]\}$$

$$+ \mathcal{F}_1\{\frac{1}{2}[S^*(\beta_4, x) + S^*(\beta_5, x)] - \frac{P_3}{2}\ z^{-1}[C^*(\beta_4, x)$$

$$- C^*(\beta_5, x)] + \frac{P_3}{4}\ z^{-1}[C^*(\beta_8, x) - C^*(\beta_9, x) + C^*(\beta_6, x)$$

$$- C^*(\beta_7, x)]\} + \mathcal{F}_2\{\frac{1}{2}[C^*(\beta_4, x) + C^*(\beta_5, x)]$$

$$- \frac{P_3}{2}\ z^{-1}[S^*(\beta_5, x) - S^*(\beta_4, x)] + \frac{P_3}{4}\ z^{-1}[S^*(\beta_7, x)$$

$$- S^*(\beta_6, x) + S^*(\beta_9, x) - S^*(\beta_8, x)]\} + V_{20}x$$

$$z \neq 0 \qquad (4.63)$$

$$D_3 \approx \mathcal{F}_1\{\frac{P_3}{2}(-1 + 4P_1^2)[C^*(\beta_4, x) + C^*(\beta_5, x)] - \frac{P_1^2}{4}[S^*(\beta_9, x)$$

$$+ S^*(\beta_8, x) - S^*(\beta_7, x) - S^*(\beta_6, x)] - P_1^2 P_3 [C^*(\beta_6, x)$$

$$+ C^*(\beta_7, x) + C^*(\beta_8, x) + C^*(\beta_9, x)]\}$$

$$+ \mathcal{F}_1 P_3^2 z^{-1}\{\tfrac{1}{2}[S^*(\beta_5, x) - S^*(\beta_4, x)] - \tfrac{1}{4}[S^*(\beta_7, x)$$

$$- S^*(\beta_6, x) + S^*(\beta_9, x) - S^*(\beta_8, x)]\}$$

$$- \mathcal{F}_2\{\frac{P_3}{2}(-1 + 4P_1^2)[S^*(\beta_4, x) + S^*(\beta_5, x)] - \frac{P_1^2}{4}[C^*(\beta_6, x)$$

$$+ C^*(\beta_7, x) - C^*(\beta_8, x) - C^*(\beta_9, x)] - P_1^2 P_3 [S^*(\beta_9, x)$$

$$+ S^*(\beta_8, x) + S^*(\beta_7, x) + S^*(\beta_6, x)]\}$$

$$- \mathcal{F}_2 P_3^2 z^{-1}\{\tfrac{1}{2}[C^*(\beta_4, x) - C^*(\beta_5, x)] - \tfrac{1}{4}[C^*(\beta_8, x)$$

$$- C^*(\beta_9, x) + C^*(\beta_6, x) - C^*(\beta_7, x)]\}$$

$$+ \mathcal{F}_1\{\frac{P_3}{2} z^{-1}[S^*(\beta_4, x) + S^*(\beta_5, x)] + \tfrac{1}{2}[C^*(\beta_4, x)$$

$$- C^*(\beta_5, x)] - \frac{P_3}{4} z^{-1}[S^*(\beta_9, x) + S^*(\beta_8, x) + S^*(\beta_7, x)$$

$$+ S^*(\beta_6, x)]\} + \mathcal{F}_2\{\frac{P_3}{2} z^{-1}[C^*(\beta_4, x) + C^*(\beta_5, x)]$$

$$+ \tfrac{1}{2}[S^*(\beta_5, x) - S^*(\beta_4, x)] - \frac{P_3}{4} z^{-1}[C^*(\beta_6, x) + C^*(\beta_7, x)$$

$$+ C^*(\beta_8, x) + C^*(\beta_9, x)]\} + V_{30}x \qquad z \neq 0$$

(4.64)

where the functions $C^*(\beta_i, x)$ and $S^*(\beta_i, x)$ are defined as

$$C^*(\beta_i, x) \triangleq \int_0^x \int_0^v \cos(2\beta_i u)\,du\,dv$$

$$= \frac{1}{4\beta_i^2}[1 - \cos(2\beta_i x)] \qquad (i = 1, \ldots, 9) \qquad (4.65)$$

and

$$S^*(\beta_i, x) \triangleq \int_0^x \int_0^v \sin(2\beta_i u)\,du\,dv$$

$$= -\frac{1}{4\beta_i^2}\sin(2\beta_i x) + \frac{x}{2\beta_i} \qquad (i = 1, \ldots, 9) \qquad (4.66)$$

In summary, given the dimensionless parameters Ω_1, Ω_3, L, z, x, $\mathcal{F}_1$, $\mathcal{F}_2$, and V_{i0} $(i = 1, 2, 3)$, one may proceed as follows to evaluate D_1, D_2, and D_3:

(1) Determine P_1 and P_3 by reference to Eqs. (3.147)-(3.148).

(2) Use Eqs. (4.25)-(4.34) to evaluate ζ and β_i $(i = 1, \ldots, 9)$.

(3) If $\zeta > 0$, evaluate D_1, D_2, and D_3 by reference to Eqs. (4.48)-(4.50) together with the definitions in Eqs. (4.46) and (4.47).

(4) If $\zeta < 0$, form $\bar{\zeta}$ and $\bar{\beta}_i$ $(i = 1, \ldots, 9)$ by using Eqs. (4.51) and (4.52), then find D_1, D_2, and D_3 from Eqs. (4.56)-(4.58), using the definitions in Eqs. (4.46) and (4.47).

(5) If $\zeta = 0$, determine D_1, D_2, and D_3 by using Eqs. (4.62)-(4.64) together with the definitions in Eqs. (4.65) and (4.66).

4.3 Comparison of Solutions

Once again we turn to the computer to obtain a measure of the accuracy of the solution just obtained. In choosing parameter values for this purpose it is well to keep in mind that the approximate solution was generated by using approximate expressions for e_{ij} $(i, j = 1, 2, 3)$, which suggests that the magnitudes of z and x bear directly on the accuracy of D_1, D_2, and D_3; that is, one must expect the discrepancies between the approximate and the exact solution to grow as z and x increase.

Figures 12-21 each show D_1, D_2, and D_3 plotted as a function of x. Numerical solutions of the exact equations of motion are represented by solid curves, while values obtained by using the approximate solution are represented by crosses. In all cases, B^* is presumed to be at rest initially. The parameter values used to generate each plot are tabulated below.

Table 1

Figure	Ω_1	Ω_3	L	z	$\mathcal{F}_1$	$\mathcal{F}_2$
12	1.0	0	9.0	0.01	0.2	0.2
13	1.0	0	9.0	0.10	0.2	0.2
14	0.70711	0.70711	9.0	0.01	0.2	0.2
15	0.70711	0.70711	9.0	0.01	0.4	0.4
16	0.70711	0.70711	1.0	0.01	0.2	0.2
17	0.70711	0.70711	9.0	0.05	0.2	0.2
18	0	1.0	9.0	0.01	0.2	0.2
19	1.0	0	0	0.01	0.2	0.2
20	0.70711	0.70711	-0.25	0.01	0.2	0.2
21	1.0	0	-0.25	0.01	0.2	0.2

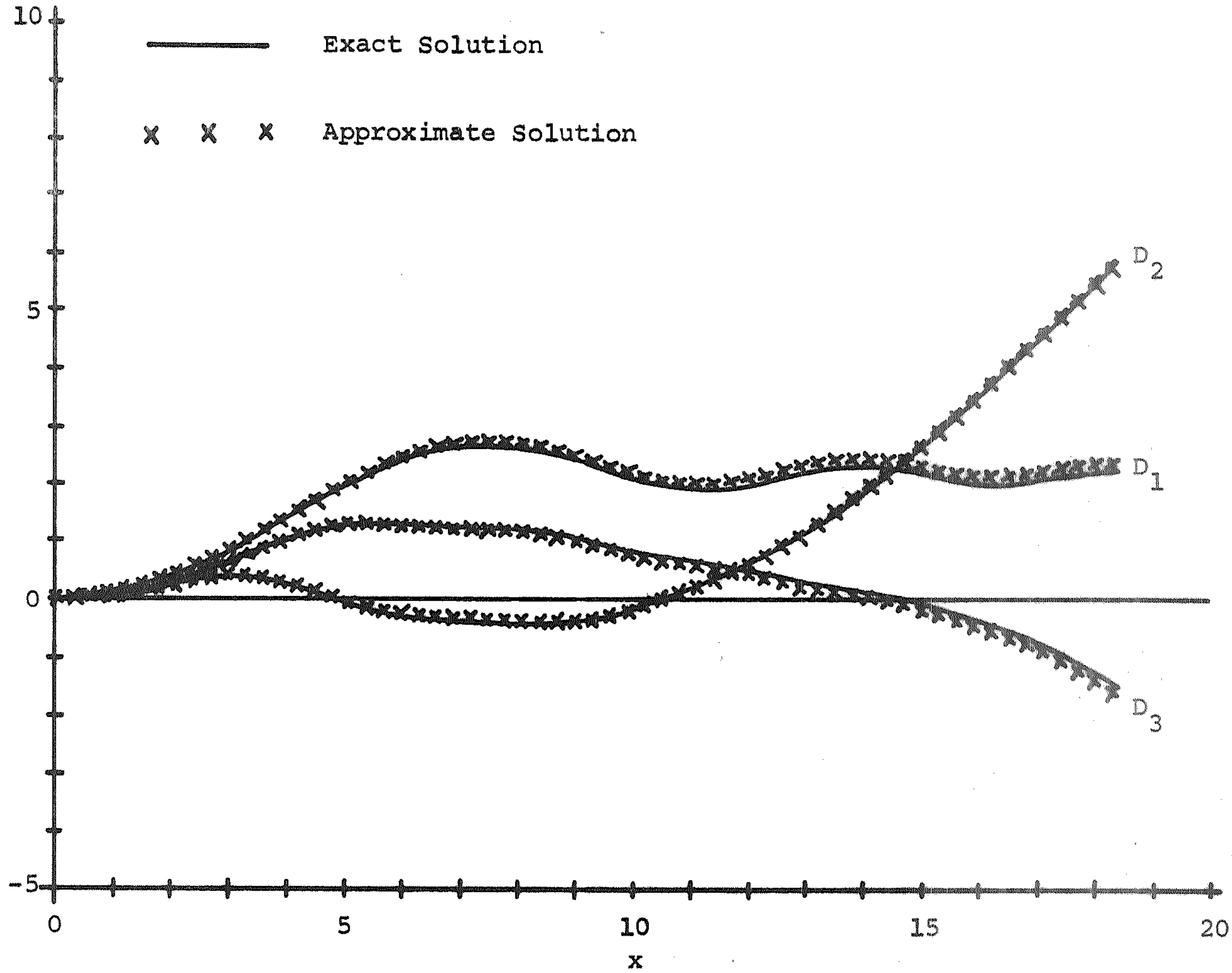

Figure 12. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$, $L = 9.0$ $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

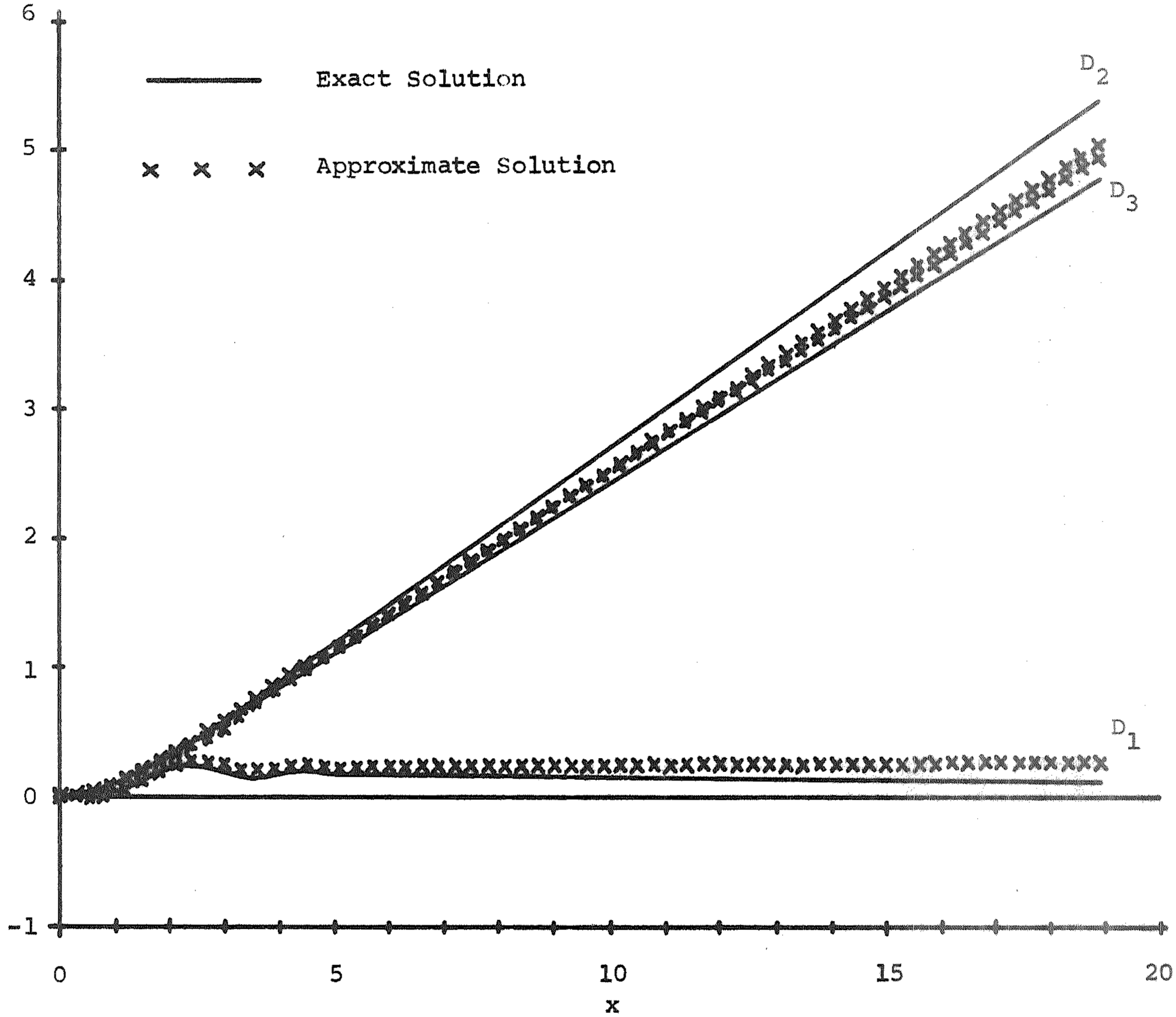

Figure 13. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.10$, $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

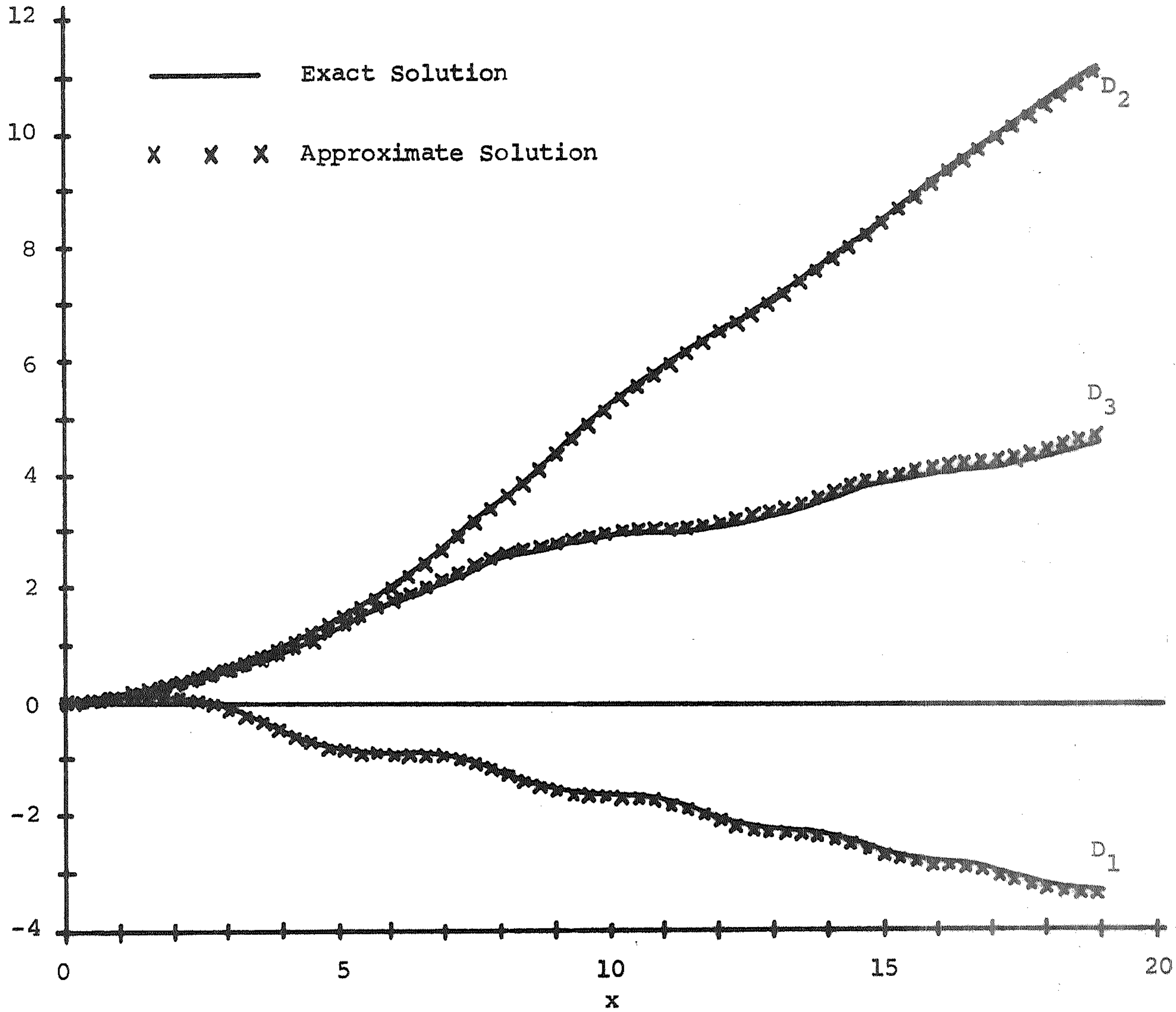

Figure 14. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

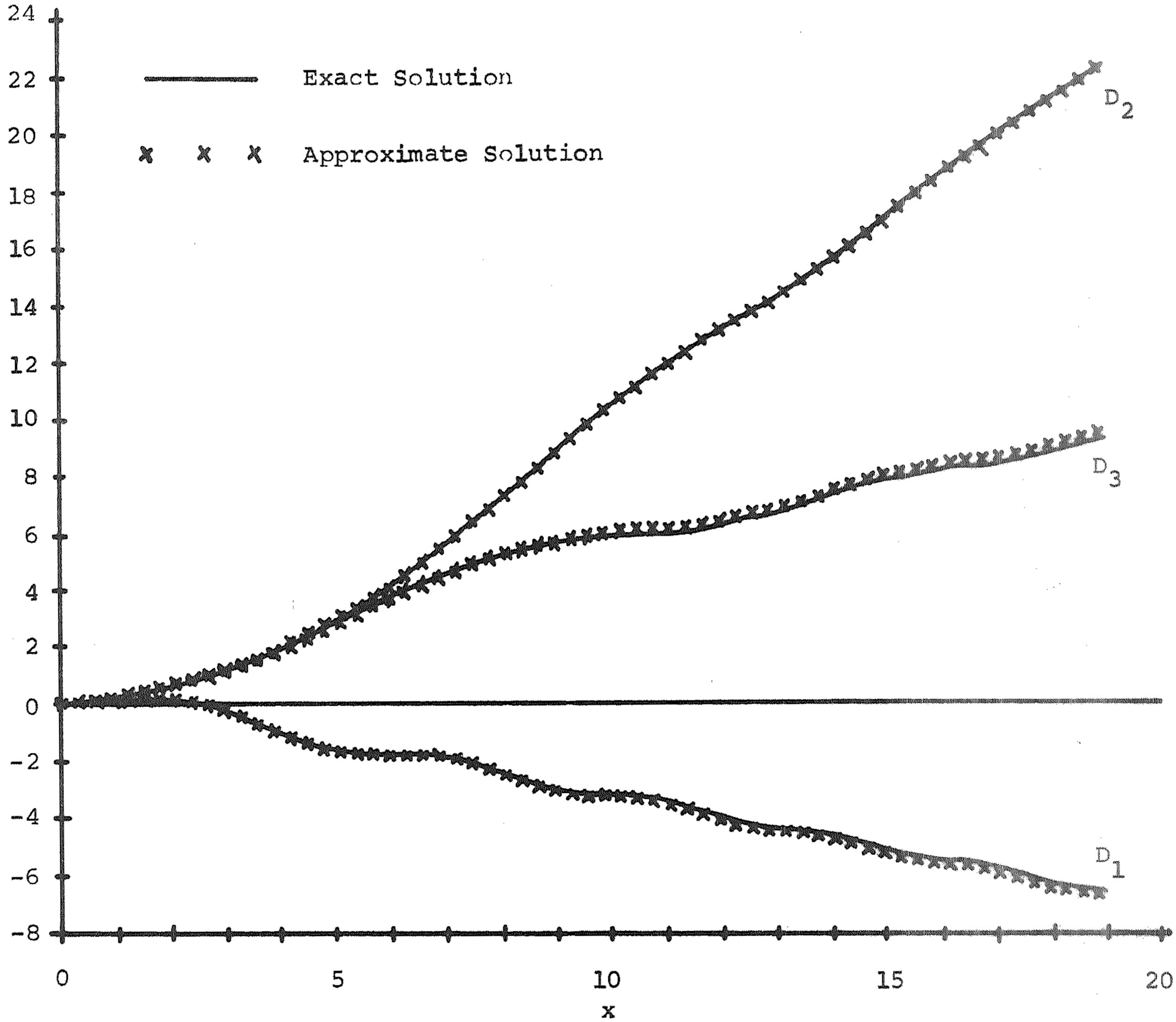

Figure 15. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 9.0$, $\mathcal{F}_1 = 0.4$, $\mathcal{F}_2 = 0.4$, $V_{i0} = 0$ $(i = 1, 2, 3)$

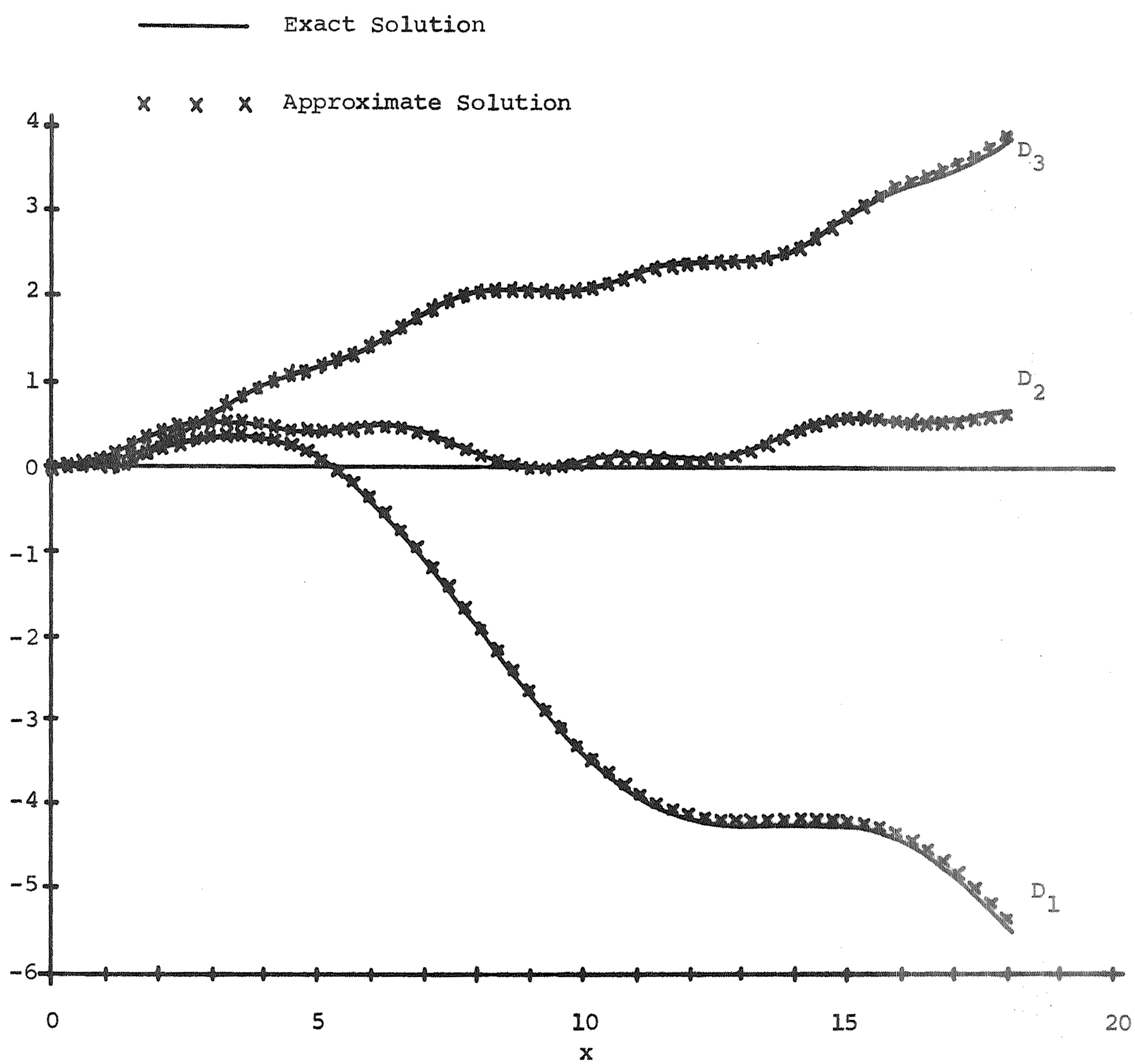

Figure 16. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = 1.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

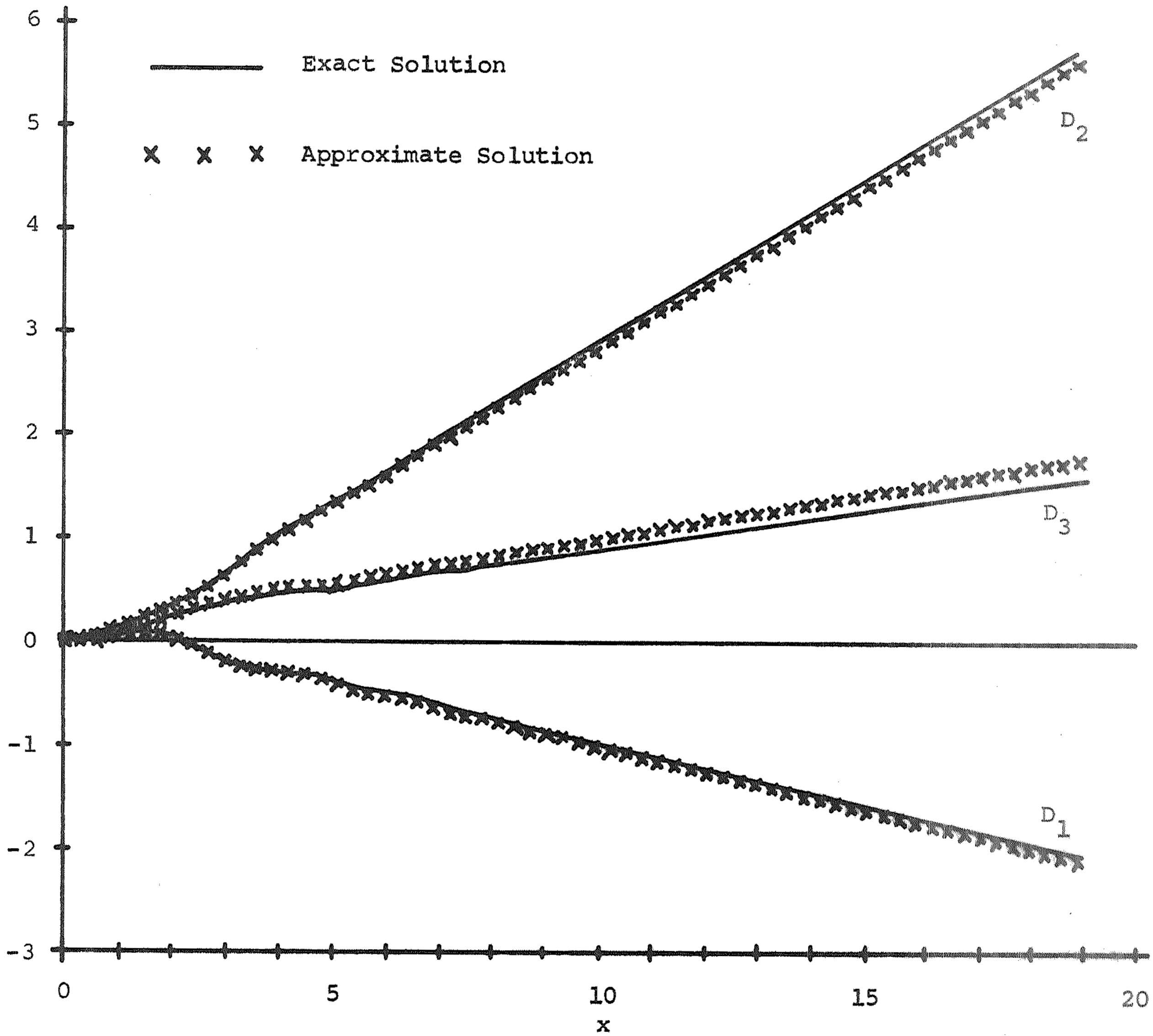

Figure 17. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.05$ $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

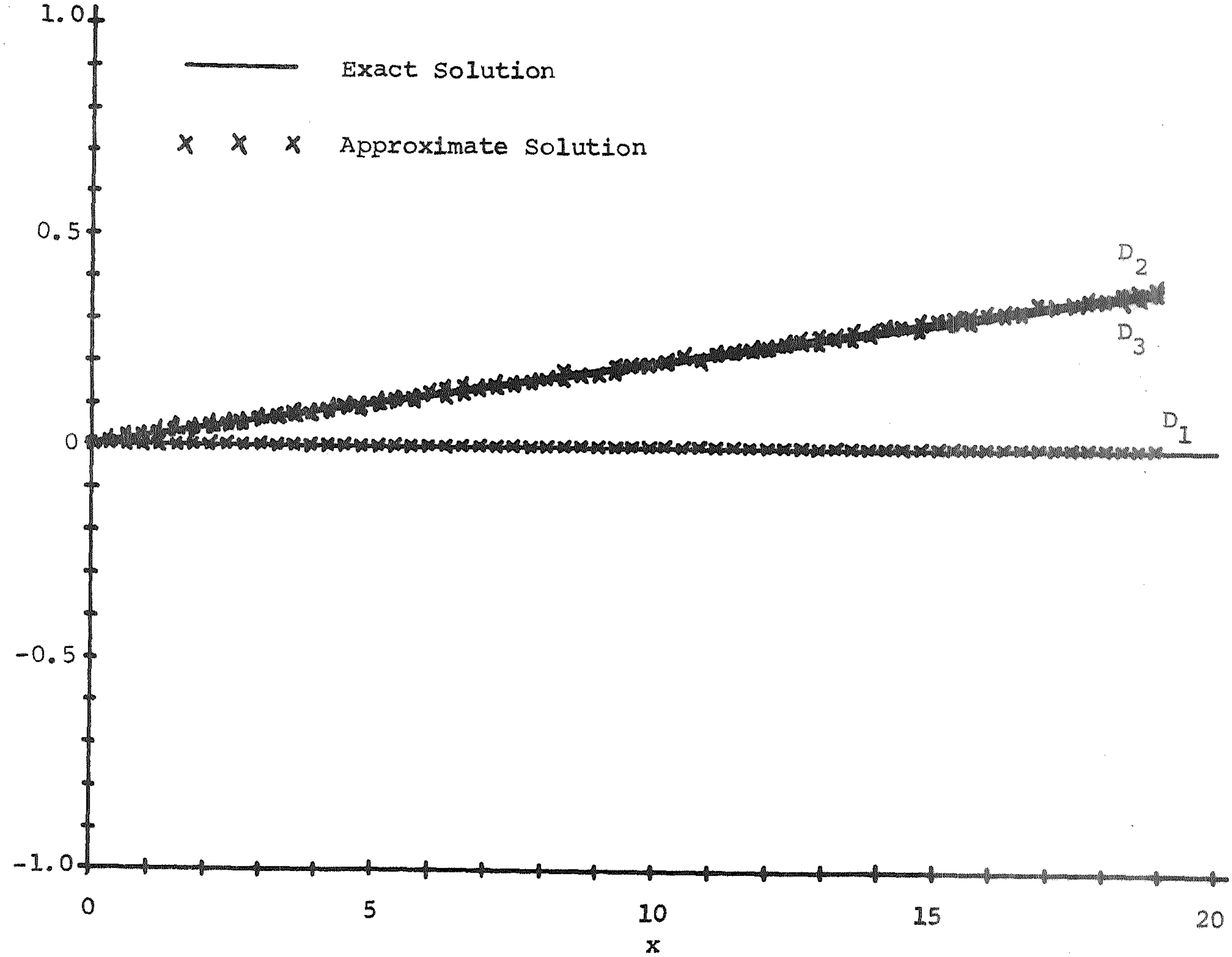

Figure 18. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0$, $\Omega_3 = 1.0$, $z = 0.01$, $L = 9.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

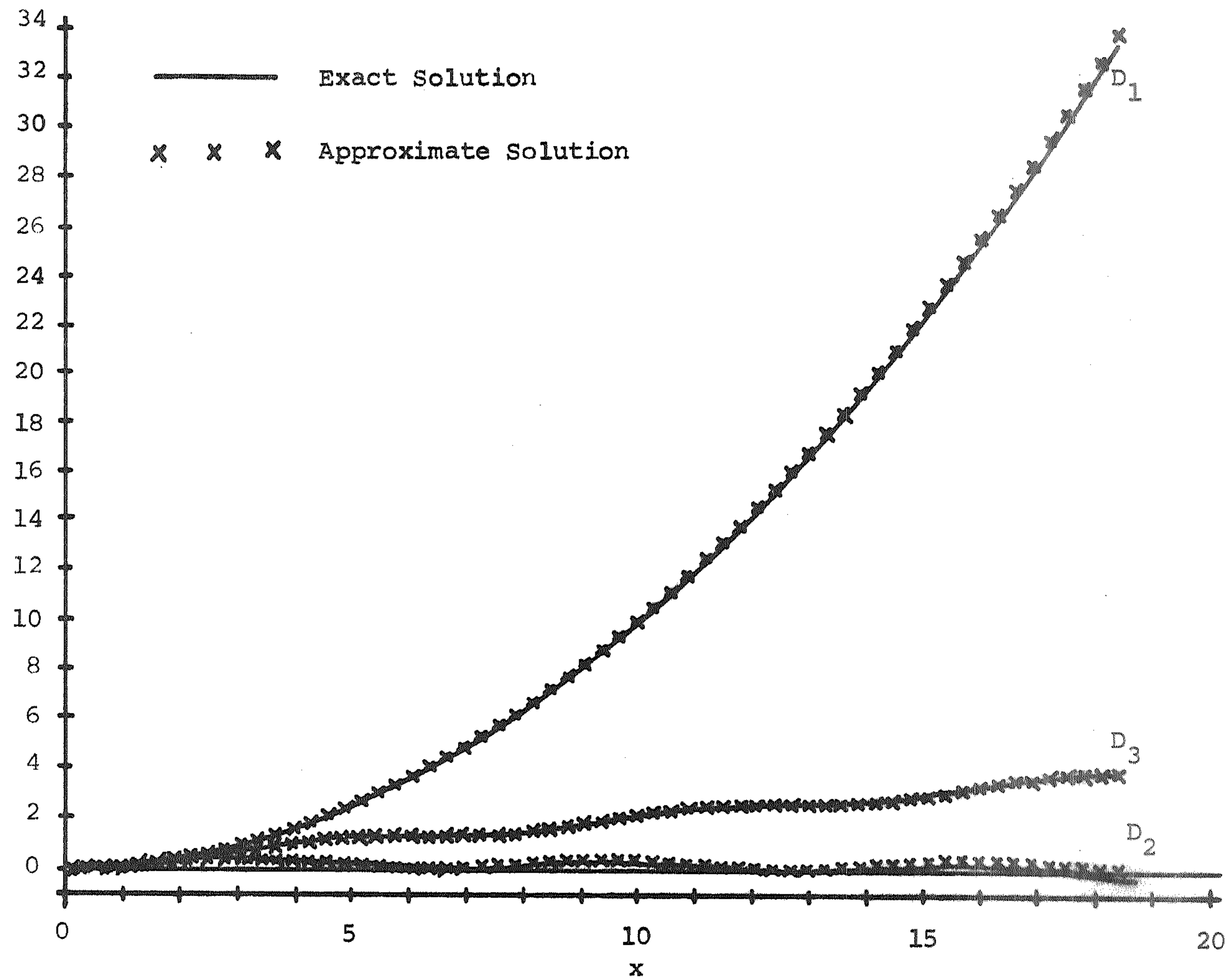

Figure 19. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$, $L = 0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

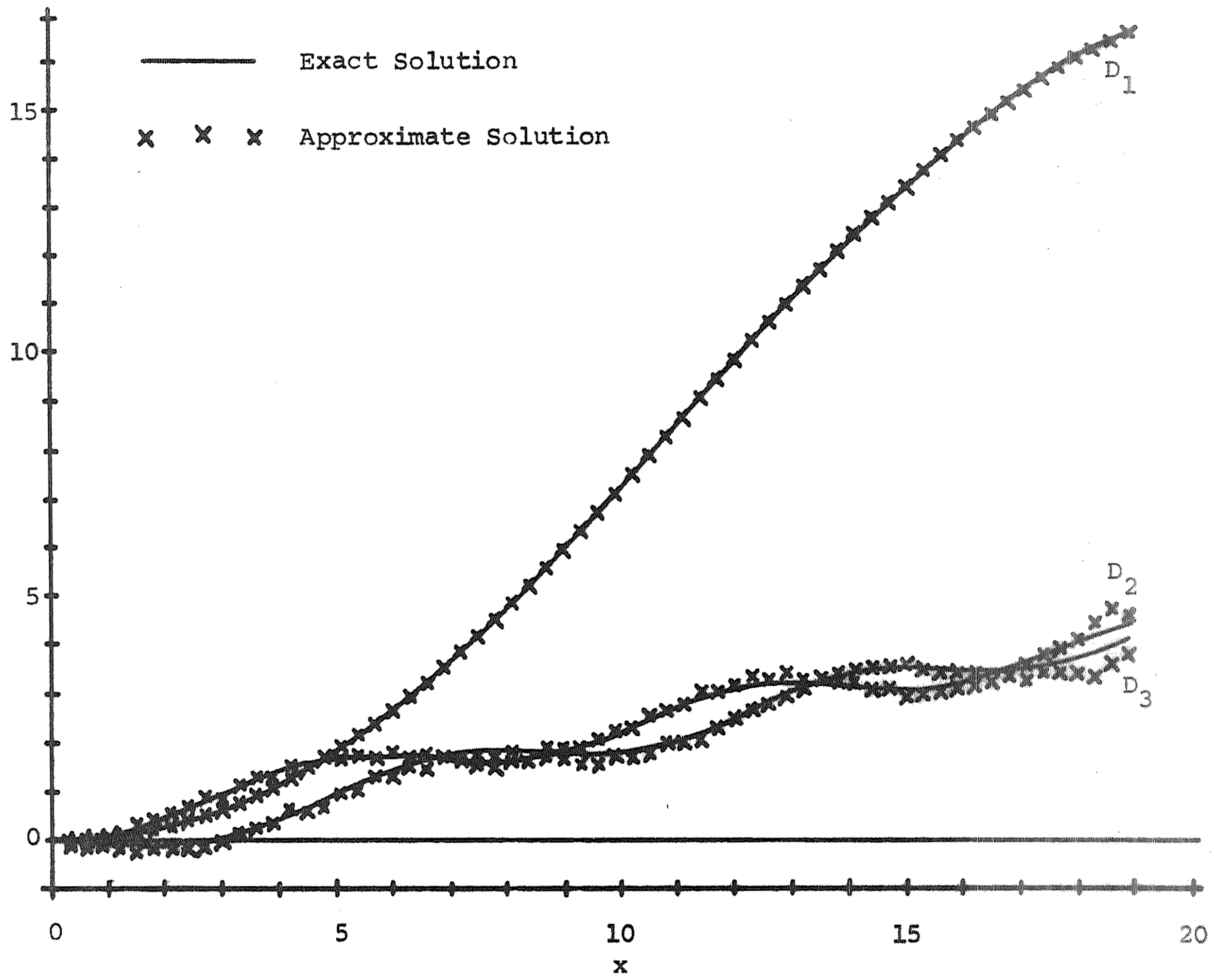

Figure 20. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 0.70711$, $\Omega_3 = 0.70711$, $z = 0.01$, $L = -0.25$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1,2,3)$

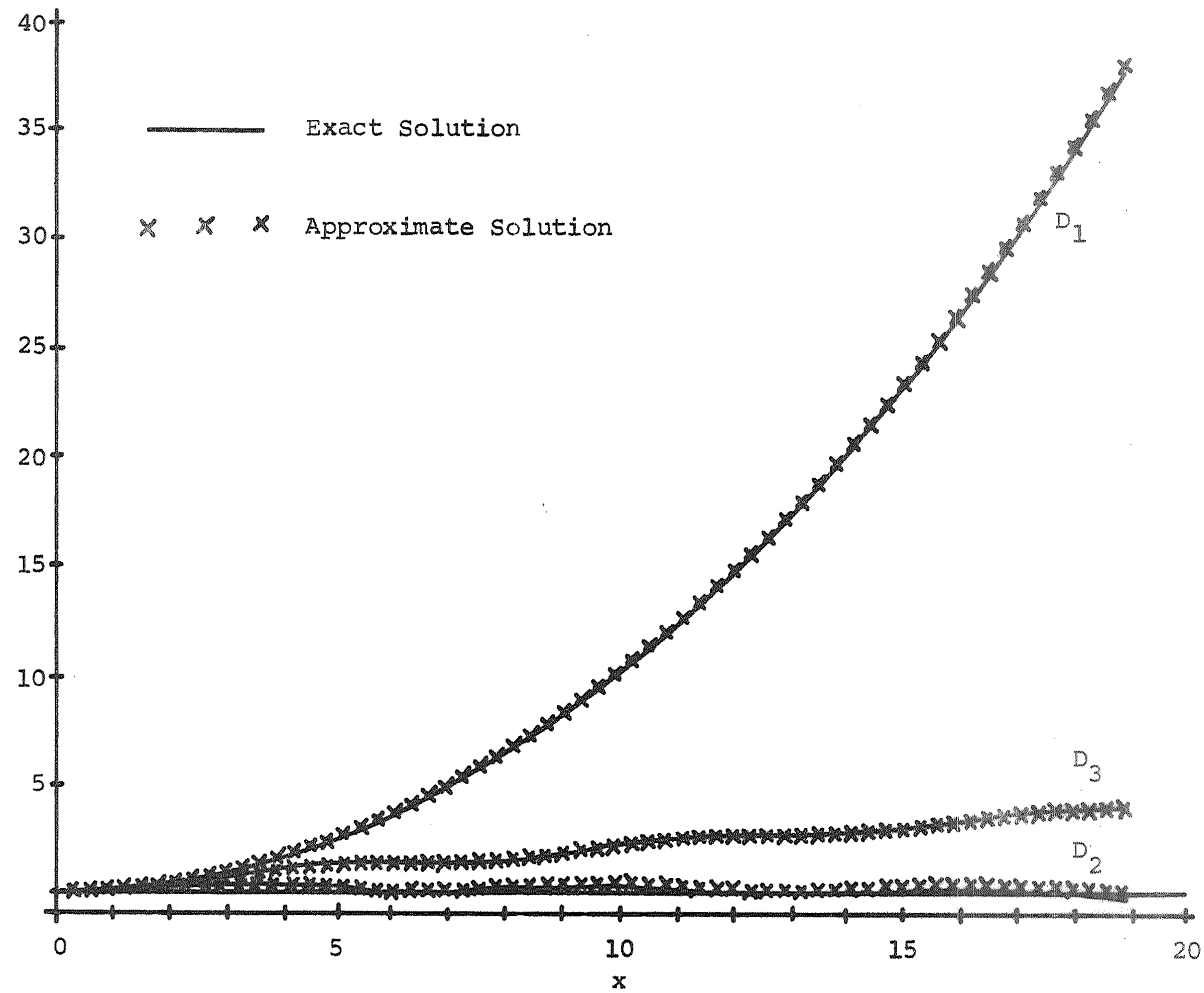

Figure 21. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$, $L = -0.25$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

Cursory examination of Figures 12-21 leads to the immediate conclusion that the approximate solution furnishes qualitatively correct results in all cases and that, as expected, discrepancies increase as x increases. The effect of increasing z can be assessed by comparing Figure 12 with Figure 13 and Figure 14 with Figure 17. In both cases, an increase in z is seen to affect accuracy adversely. As for the effect of other parameters on accuracy, this appears to be negligible; that is, for a given value of z, the accuracy of the approximate solution is independent of the values of Ω_1, Ω_3, $\mathcal{F}_1$, $\mathcal{F}_2$, and L.

Figures 12-21 also permit one to explore the effect of parameter values on the nature of the translational motion. Consider, for example, the parameter L, which characterizes the inertia ellipsoid of the body. Figures 14, 16, and 20 suggest that the mass centers of bodies possessing differently shaped inertia ellipsoids must be expected to follow markedly different trajectories. In particular, the transition from a positive value of L to a negative value can bring about a change in the sign of D_1, and thus alter the direction of motion noticeably. The force level, on the other hand, plays a relatively minor role as regards the shape of the trajectory, as may be seen by reference to Figures 14 and 15, which show that, although the values of $\mathcal{F}_1$ and $\mathcal{F}_2$ used to generate Figure 15 are twice as large as those for Figure 14, the trajectories followed by the mass center are almost identical. This is not to say, however, that the force level is totally irrelevant to the motion. On the contrary, as shown by Eqs. (4.48)-(4.50), (4.56)-(4.58), and (4.62)-(4.64), D_1, D_2, and D_3 are directly proportional to the magnitude of the applied force $\underline{F}$, since $\mathcal{F}_1$ and $\mathcal{F}_2$ are proportional to this quantity; and this proportionality is, in fact, the reason underlying the similarity of Figures 14 and 15.

In conclusion, it can be said that the approximate expressions for D_i $(i = 1, 2, 3)$ describe all essential features of the motion of the mass center and that the agreement between the approximate and the exact solution is such as to justify considerable confidence in the latter.

4.4 A Special Case

Particularly simple and revealing expressions for D_1, D_2, and D_3 can be obtained if B is a "slender" body, if the misalignment of $\underline{F}$ from the mass center is "small," and if B has a pure tumbling motion initially, that is, if

$$L \gg 1 \tag{4.67}$$

$$0 < z \ll 1 \tag{4.68}$$

and

$$\omega_3(0) = 0 \tag{4.69}$$

If, in addition,

$$(Lz)^n \gg 1 \qquad (n = 1/2,\ 1,\ 3/2) \tag{4.70}$$

then Eqs. (4.40)-(4.47) lead to (see Eqs. (4.25)-(4.34))

$$C_2\left[\frac{1}{\zeta}(\zeta x + \beta_i)^2\right] \underset{(4.40)}{\approx} 1/2 \qquad (i = 1, \ldots, 9) \tag{4.71}$$

$$S_2\left[\frac{1}{\zeta}(\zeta x + \beta_i)^2\right] \underset{(4.41)}{\approx} 1/2 \qquad (i = 1,\ldots,9) \tag{4.72}$$

$$C(\zeta, \beta_i, x) \underset{(4.42)}{\approx} \sqrt{\frac{\pi}{2}}\left[\frac{\sqrt{2x}}{\sqrt{Lz}} + 0\right]\left\{1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}\right\}$$

$$= \sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{z} \qquad (i = 1,\ldots,9) \tag{4.73}$$

$$S(\zeta, \beta_i, x) \underset{(4.43)}{\approx} \sqrt{\frac{\pi}{2}}\left[\frac{\sqrt{2x}}{\sqrt{Lz}} + 0\right]\left\{1 \cdot \frac{1}{2} - 0 \cdot \frac{1}{2}\right\}$$

$$= \sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{z} \qquad (i = 1,\ldots,9) \tag{4.74}$$

$$C_1(\zeta, \beta_i, x) \underset{(4.44)}{\approx} 0 \cdot \left\{1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}\right\}$$

$$= 0 \qquad (i = 1,\ldots,9) \tag{4.75}$$

$$S_1(\zeta, \beta_i, x) \underset{(4.45)}{\approx} 0 \left\{1 \cdot \frac{1}{2} - 0 \cdot \frac{1}{2}\right\}$$

$$= 0 \qquad (i = 1,\ldots,9) \tag{4.76}$$

$$C^*(\zeta, \beta_i, x) \underset{(4.46)}{\approx} \sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2} \qquad (i = 1,\ldots,9) \tag{4.77}$$

$$S^*(\zeta, \beta_i, x) \underset{(4.48)}{\approx} \sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2} \qquad (i = 1,\ldots,9) \tag{4.78}$$

Hence, substitution from Eqs. (4.77) and (4.78) into Eqs. (4.48)-(4.50) results in

$$D_1 \underset{(4.48)}{\approx} \mathcal{F}_1\left\{\frac{1}{2}\left[\sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2} + \sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2}\right]\right\}$$

$$- \mathcal{F}_2\left\{\frac{1}{2}\left[\sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2} + \sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2}\right]\right\} + V_{10}x$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{Lz}}\,(\mathcal{F}_1 - \mathcal{F}_2)\,x + V_{10}x \tag{4.79}$$

$$D_2 \underset{(4.49)}{\approx} \mathcal{F}_1\left\{\frac{1}{2}\left[\sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2} + \sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2}\right]\right\}$$

$$+ \mathcal{F}_2\left\{\frac{1}{2}\left[\sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2} + \sqrt{\frac{\pi}{Lz}} \cdot \frac{x}{2}\right]\right\} + V_{20}x$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{Lz}}\,(\mathcal{F}_1 + \mathcal{F}_2)\,x + V_{20}x \tag{4.80}$$

$$D_3 \underset{(4.50)}{\approx} 0 + V_{30}x = V_{30}x \tag{4.81}$$

Since D_i $(i = 1,2,3)$ are linear functions of x, it must be concluded that the mass center B^* of B moves with constant speed on a straight line.

To test the accuracy of Eqs. (4.79)-(4.81), a graphical comparison (see Figure 22) between the results obtained by using a digital computer solution (solid line) of the exact equations of motion, on the one hand, and the analytical description (crosses), on the other hand, was made for

$$\Omega_1 = 1, \qquad \Omega_3 = 0, \qquad z = 0.01$$

$$L = 5000, \qquad \mathcal{F}_1 = 0.2, \qquad \mathcal{F}_2 = 0.2$$

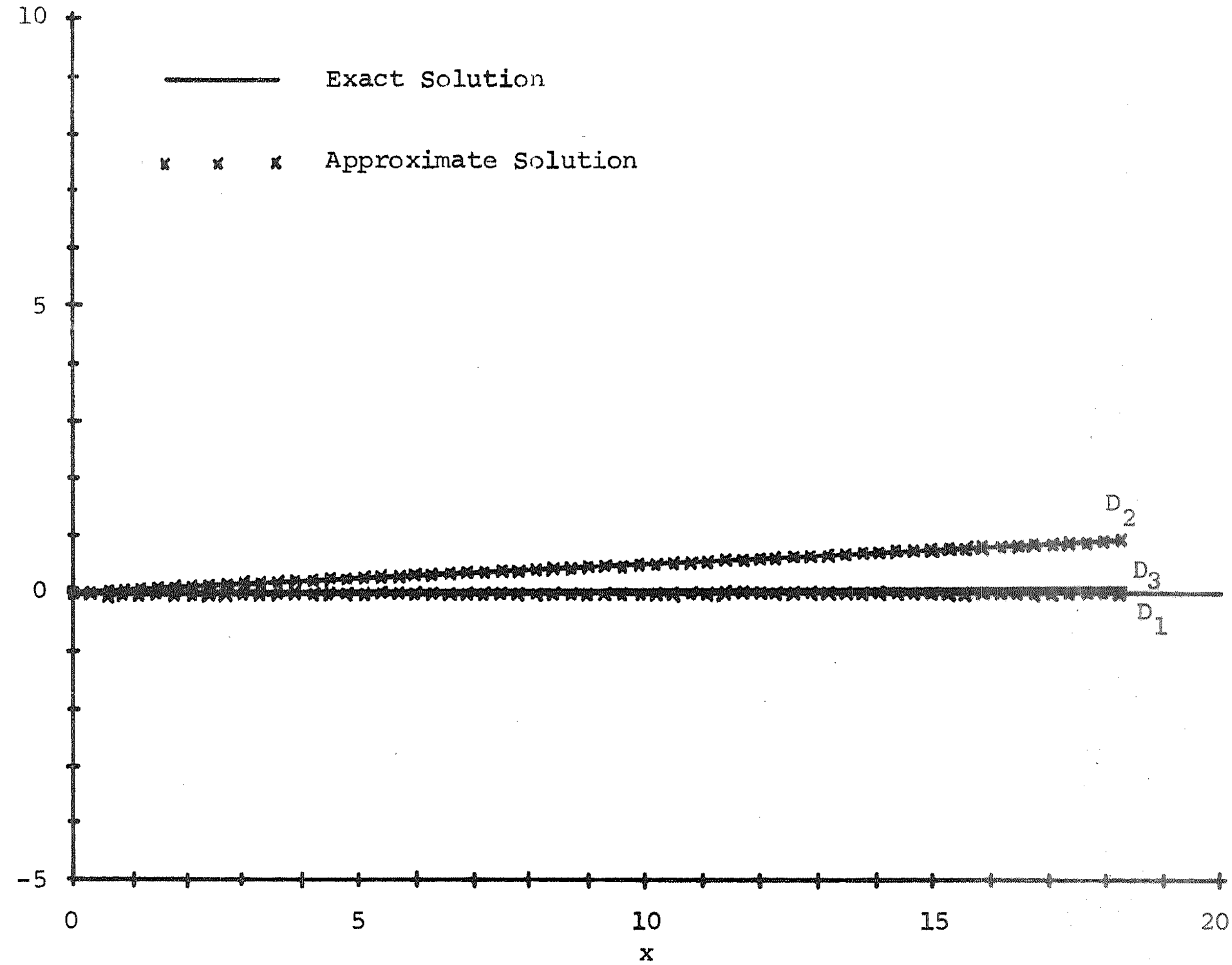

Figure 22. Comparison of Solutions for Position Coordinates of Mass Center. $\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.01$, $L = 5000.0$, $\mathcal{F}_1 = 0.2$, $\mathcal{F}_2 = 0.2$, $V_{i0} = 0$ $(i = 1, 2, 3)$

$$V_{10} = 0 \quad , \qquad V_{20} = 0 \quad , \qquad V_{30} = 0$$

The agreement between the predictions of the exact and of the approximate solutions is seen to be excellent.

5. SYMMETRIC GYROSTAT

5.1 Introduction

It is the purpose of this chapter to show that the approximate solution derived in the preceding chapters is not restricted to the description of motions of a single symmetric rigid body, but can also be applied to a symmetric gyrostat acted upon by a force fixed relative to the main body and by a couple applied to the rotor. This sort of problem is of interest in connection with space vehicles possessing spinning components.

5.2 System Description

The system to be analyzed, shown in Figure 23, consists of a main body B and a rotor W, whose combined mass is m. Together, B and W form a gyrostat G. The inertia ellipsoid of B for the mass center B* of B may have three unequal principal diameters, whereas the inertia ellipsoid of W for the mass center W* of W is presumed to be an ellipsoid of revolution. Furthermore, it is assumed that W is connected to B in such a way that (1) W* and the spin axis of W are fixed in B, but W can rotate relative to B about this axis, and (2) the inertia ellipsoid of G for the mass center G* of G is an ellipsoid of revolution whose axis is parallel to the spin axis of W.

The center of mass of the gyrostat, G*, is located by a position vector $\underline{X}$ relative to a point 0 that is fixed in an inertial reference frame A. Mutually perpendicular unit vectors $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3 = \underline{a}_1 \times \underline{a}_2$ are fixed in A, and mutually perpendicular unit vectors $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3 = \underline{b}_1 \times \underline{b}_2$ are fixed in B parallel to the principal

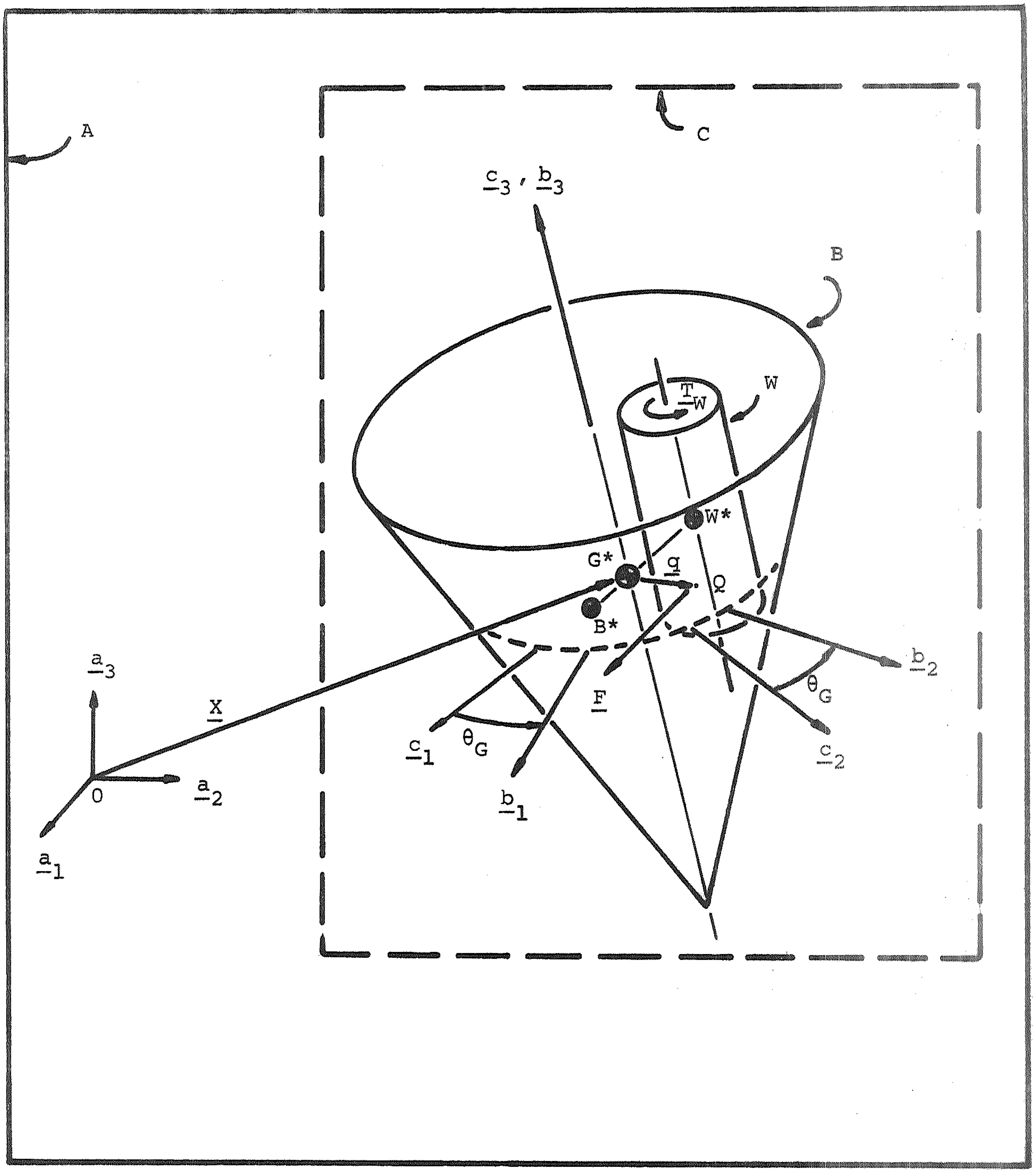

Figure 23. Model of a Symmetric Gyrostat

axes of inertia of G for G*, with $\underline{b}_3$ parallel to the symmetry axis of G. C designates a reference frame fixed neither in the gyrostat G nor in the inertial reference frame A, but constrained to move in such a way that a unit vector $\underline{c}_3$, fixed in C, remains at all times equal to the unit bector $\underline{b}_3$; furthermore, mutually perpendicular unit vectors $\underline{c}_1$ and $\underline{c}_2$ are fixed in C and are normal to $\underline{c}_3$. Symbols denoting moments of inertia of interest can be defined in terms of these unit vectors and the inertia dyadics $\underline{I}^G$ of G for G* and $\underline{I}^W$ of W for W* by letting

$$I \triangleq \underline{c}_1 \cdot \underline{I}^G \cdot \underline{c}_1 = \underline{c}_2 \cdot \underline{I}^G \cdot \underline{c}_2 \tag{5.1}$$

$$J \triangleq \underline{c}_3 \cdot \underline{I}^G \cdot \underline{c}_3 \tag{5.2}$$

and

$$K \triangleq \underline{c}_3 \cdot \underline{I}^W \cdot \underline{c}_3 \tag{5.3}$$

A force $\underline{F}$ of constant magnitude is applied to B at a point Q which is located relative to G* by a vector $\underline{q}$. The orientation of $\underline{F}$ is fixed in B, and both $\underline{F}$ and $\underline{q}$ lie in the plane containing G* and normal to $\underline{b}_3$. Furthermore, W is subjected to the action of a moment $\underline{T}_W$ of constant magnitude and parallel to the spin axis of W.

The following scalar quantities are used in the sequel:

$$x_i \triangleq \underline{X} \cdot \underline{a}_i \qquad (i = 1, 2, 3) \tag{5.4}$$

$$F_i \triangleq \underline{F} \cdot \underline{b}_i \qquad (i = 1, 2) \tag{5.5}$$

$$q_i \triangleq \underline{q} \cdot \underline{b}_i \qquad (i = 1, 2) \tag{5.6}$$

$$T_W \triangleq \underline{T}_W \cdot \underline{c}_3 \tag{5.7}$$

$$T_B \triangleq q_1 F_2 - q_2 F_1 \tag{5.8}$$

When the force $\underline{F}$ exerted on B at Q is replaced with a force applied at G* together with a couple of torque $\underline{T}_B$, then

$$\underline{T}_B \underset{(5.5, 5.6)}{=} T_B \, \underline{c}_3 \tag{5.9}$$

5.3 Analysis

Since W can perform only spinning motion relative to B about a line parallel to $\underline{c}_3$, the angular velocity ${}^B\underline{\omega}^W$ of W relative to B can be expressed as

$${}^B\underline{\omega}^W = r \, \underline{c}_3 \tag{5.10}$$

where r denotes a function of time t; and, as B is constrained to rotate relative to C about a line parallel to $\underline{c}_3$, the angular velocity ${}^C\underline{\omega}^B$ of B relative to C is necessarily parallel to $\underline{c}_3$ and can, therefore, be expressed as

$${}^C\underline{\omega}^B = s \, \underline{c}_3 \tag{5.11}$$

where s is a function of t. Next, the angular velocity ${}^A\underline{\omega}^C$ of C relative to A can always be expressed in terms of functions p_1, p_2, and p_3 of t as

$$ {}^A\underline{\omega}^C = p_1 \underline{c}_1 + p_2 \underline{c}_2 + p_3 \underline{c}_3 \tag{5.12} $$

and the angular velocities, ${}^A\underline{\omega}^B$ and ${}^A\underline{\omega}^W$, of B and W relative to A are then given by

$$ {}^A\underline{\omega}^B = {}^A\underline{\omega}^C + {}^C\underline{\omega}^B = p_1 \underline{c}_1 + p_2 \underline{c}_2 + (p_3 + s)\, \underline{c}_3 \tag{5.13} $$

and

$$ {}^A\underline{\omega}^W = {}^A\underline{\omega}^B + {}^B\underline{\omega}^W = p_1 \underline{c}_1 + p_2 \underline{c}_2 + (p_3 + s + r)\, \underline{c}_3 \tag{5.14} $$

The angular momentum $\underline{H}^G$ of G with respect to G^* in A can be expressed as

$$ \underline{H}^G = I\, p_1 \underline{c}_1 + I\, p_2 \underline{c}_2 + [J(p_3 + s) + Kr]\, \underline{c}_3 \tag{5.15} $$

Hence, if Eq. (5.15) is differentiated with respect to t in the reference frame A, then

$$ \frac{{}^A d\underline{H}^G}{dt} = \frac{{}^C d\underline{H}^G}{dt} + {}^A\underline{\omega}^C \times \underline{H}^G $$

$$ \underset{(5.15)}{=} \{I\, \dot{p}_1 + [(J - I)p_3 + Js + Kr]p_2\}\, \underline{c}_1 $$

$$ + \{I\, \dot{p}_2 - [(J - I)p_3 + Js + Kr]p_1\}\, \underline{c}_2 $$

continued

$$+ [J(\dot{p}_3 + \dot{s}) + K\dot{r}] \underline{c}_3 \tag{5.16}$$

The angular momentum $\underline{H}^W$ of W with respect to W* in A can be expressed as

$$\underline{H}^W = p_1 \underline{\underline{I}}^W \cdot \underline{c}_1 + p_2 \underline{\underline{I}}^W \cdot \underline{c}_2 + (p_3 + s + r) \underline{\underline{I}}^W \cdot \underline{c}_3 \tag{5.17}$$

from which it follows that

$$\underline{H}^W \cdot \underline{c}_3 = (p_3 + s + r)K \tag{5.18}$$

and, differentiating with respect to t, one obtains

$$\frac{d}{dt}(\underline{H}^W \cdot \underline{c}_3) = (\dot{p}_3 + \dot{s} + \dot{r})K \tag{5.19}$$

Now, it can be shown that

$$\frac{d}{dt}(\underline{H}^W \cdot \underline{c}_3) = \frac{{}^A d\underline{H}^W}{dt} \cdot \underline{c}_3 \tag{5.20}$$

Hence, from Eqs. (5.20) and (5.19),

$$\frac{{}^A d\underline{H}^W}{dt} \cdot \underline{c}_3 = (\dot{p}_3 + \dot{s} + \dot{r})K \tag{5.21}$$

In accordance with the angular momentum principle, the time-derivative of $\underline{H}^G$ in A is equal to the total moment about G* of all forces acting on G. Thus

$$\frac{{}^A d\underline{H}^G}{dt} = \underline{T}_W + \underline{T}_B \tag{5.22}$$

and it follows by substitution from Eqs. (5.7), (5.8) and (5.16) into Eq. (5.22) that

$$I\dot{p}_1 + [(J - I)p_3 + Js + Kr]p_2 = 0 \tag{5.23}$$

$$I\dot{p}_2 - [(J - I)p_3 + Js + Kr]p_1 = 0 \tag{5.24}$$

$$J(\dot{p}_3 + \dot{s}) + K\dot{r} = T_W + T_B \tag{5.25}$$

Applying the angular momentum principle to the rigid body W, one has

$$\frac{{}^{A}d\underline{\underline{H}}^{W}}{dt} = \underline{T}_W + \underline{M} \tag{5.26}$$

where $\underline{M}$ denotes the moment about W^* of all forces exerted by B on W. It then follows from Eqs. (5.21) and (5.26) that

$$(\dot{p}_3 + \dot{s} + \dot{r})K = T_W + M_3 \tag{5.27}$$

where

$$M_3 \triangleq \underline{M} \cdot \underline{c}_3 \tag{5.28}$$

Eqs. (5.23)-(5.25) together with Eq. (5.27) constitute the first set of basic dynamical equations for the attitude motion of the symmetric gyrostat G. The excess of unknowns over equations is attributable to the fact that s is a function of t which was introduced solely for the purpose of analytical convenience, and may, therefore, be chosen at will. A choice which indeed simplifies the subsequent analysis is

$$s = \frac{(I - J)p_3 - Kr}{J} \tag{5.29}$$

for this permits one to replace Eqs. (5.23)-(5.25) and (5.27) with

$$\dot{p}_1 = 0 \tag{5.30}$$

$$\dot{p}_2 = 0 \tag{5.31}$$

$$\dot{p}_3 = \lambda_G \tag{5.32}$$

and

$$\dot{r} = \left(\frac{1}{J - K}\right)\left[\frac{J}{K}(M_3 + T_W) - (T_W + T_B)\right] \tag{5.33}$$

where λ_G is defined as

$$\lambda_G \triangleq (T_W + T_B)/I \tag{5.34}$$

(It follows from the definition of J and K that J exceeds K, so that the quantity $(J - K)$ appearing in the denominator of Eq. (5.33) is positive.)

Integration of Eqs. (5.30)-(5.32) immediately yields

$$p_1 = p_{10} \tag{5.35}$$

$$p_2 = p_{20} \tag{5.36}$$

$$p_3 = \lambda_G t + p_{30} \tag{5.37}$$

where the subscript "0" here and in what follows denotes the value of the associated quantity for $t = 0$. Furthermore, from Eqs. (5.37) and (5.29),

$$s = L_G(\lambda_G t + p_{30}) - \frac{K}{J} r \tag{5.38}$$

where

$$L_G \triangleq \frac{I - J}{J} \tag{5.39}$$

At this point it becomes desirable to express the components of angular velocity ${}^A\underline{\omega}^C$ of C in A in terms of the initial values of quantities directly associated with the motion of G. If $\underline{c}_1$ is taken to be equal to $\underline{b}_1$ at $t = 0$, and θ_G is defined as the radian measure of the angle between $\underline{b}_1$ and $\underline{c}_1$, then

$$\theta_G \triangleq \int_0^t s\,dt$$

$$\underset{(5.38)}{=} \int_0^t [L_G(\lambda_G t + p_{30}) - \frac{K}{J} r]dt \tag{5.40}$$

Moreover, if ω_i $(i = 1, 2, 3)$ is now defined as

$$\omega_i \triangleq {}^A\underline{\omega}^B \cdot \underline{b}_i \qquad (i = 1, 2, 3) \tag{5.41}$$

it follows by reference to Eq. (3.21) that

$$\omega_1 = p_1 \cos\theta_G + p_2 \sin\theta_G \tag{5.42}$$

$$\omega_2 = - p_1 \sin\theta_G + p_2 \cos\theta_G \tag{5.43}$$

$$\omega_3 = p_3 + s \tag{5.44}$$

Consequently, at $t = 0$

$$\omega_{10} = p_{10} \tag{5.45}$$

$$\omega_{20} = p_{20} \tag{5.46}$$

$$\omega_{30} = p_{30} + s_0 \tag{5.47}$$

where s_0 is given by

$$s_0 \underset{(5.38)}{=} L_G\, p_{30} - \frac{K}{J}\, r_0 \tag{5.48}$$

from which one can obtain p_{30} as

$$p_{30} \underset{(5.47, 5.48)}{=} \frac{1}{1 + L_G} \left(\omega_{30} + \frac{K}{J}\, r_0\right) \tag{5.49}$$

Eqs. (5.45), (5.46) and (5.49) now permit one to express p_i $(i = 1, 2, 3)$ in terms of ω_{i0} $(i = 1, 2, 3)$ and r_0 as follows:

$$p_1 = \omega_{10} \tag{5.50}$$

$$p_2 = \omega_{20} \tag{5.51}$$

$$p_3 = \lambda_G t + \frac{1}{1 + L_G} \left(\omega_{30} + \frac{K}{J}\, r_0\right) \tag{5.52}$$

Eqs. (5.50)-(5.52) have exactly the same form as Eqs. (3.29)-(3.31). This means that the C-frame for a symmetric gyrostat moves like that for a symmetric rigid body, regardless of the interaction between W and B.

Having determined the angular velocity ${}^{A}\underline{\omega}^{C}$ of C relative to A, we next consider the motion of B relative to C, and of W relative to B. As regards the former, it is only necessary to recall that s denotes the angular speed of B relative to C and to observe that Eqs. (5.38) and (5.49) yield

$$s = L_G\lambda_G t + \frac{L_G}{1+L_G}\left(\omega_{30} + \frac{K}{J}r_0\right) - \frac{K}{J}r \qquad (5.53)$$

from which it is apparent that B spins relative to C with an angular speed s that depends on the motion of W relative to B, that is, on r. The two motions under consideration are thus intimately related to each other, and an assumption regarding the interaction of B and W must be made before one can arrive at final expressions for r, s, and θ_G.

In what follows, three kinds of interaction will be considered: W constrained to remain fixed in B, so that G moves as a rigid body; W permitted to rotate relative to B about its axis, this axis supported by frictionless bearings at both ends; and W subjected to the action of a torque which retards rotation of W relative to B, the magnitude of the torque being taken proportional to the spin rate r, as might be the case if forces are transmitted from B to W through a viscous medium.

Before proceeding with the analysis of these three cases, it is convenient to introduce the following symbols:

$$\ell \triangleq \frac{K}{J} \tag{5.54}$$

$$p_0 \triangleq (p_{10}^2 + p_{30}^2)^{1/2} \tag{5.55}$$

$$\omega_0 \triangleq (\omega_{10}^2 + \omega_{30}^2)^{1/2} \tag{5.56}$$

$$P_{G1} \triangleq p_{10}/p_0 \tag{5.57}$$

$$P_{G3} \triangleq p_{30}/p_0 \tag{5.58}$$

$$\Omega_1 \triangleq \omega_{10}/\omega_0 \tag{5.59}$$

$$\Omega_3 \triangleq \omega_{30}/\omega_0 \tag{5.60}$$

$$R \triangleq r_0/\omega_0 \tag{5.61}$$

$$z_B \triangleq \frac{T_B}{I\ p_0^2} \tag{5.62}$$

$$z_W \triangleq \frac{T_W}{I\ p_0^2} \tag{5.63}$$

$$z_G \triangleq \frac{\lambda_G}{p_0^2} \underset{(5.34)}{=} \frac{T_B}{I\ p_0^2} + \frac{T_W}{I\ p_0^2}$$

$$= z_B + z_W \tag{5.64}$$

and

$$x \triangleq p_0 t \tag{5.65}$$

We note for future reference that (see Eqs. (5.54)-(5.61) and (5.45)-(5.49))

$$p_{G1} = \frac{\Omega_1}{\left[\Omega_1^2 + \left(\frac{1}{1+L_G}\right)^2 (\Omega_3 + \ell R)^2\right]^{1/2}} \tag{5.66}$$

and

$$p_{G3} = \frac{\left(\frac{1}{1+L_G}\right)(\Omega_3 + \ell R)}{\left[\Omega_1^2 + \left(\frac{1}{1+L_G}\right)^2 (\Omega_3 + \ell R)^2\right]^{1/2}} \tag{5.67}$$

From Eq. (5.33), it follows that W remains permanently fixed in B if

$$M_3 = \frac{K}{J}\left[T_B + \left(1 - \frac{J}{K}\right) T_W\right] \tag{5.68}$$

and

$$r_0 = 0 \qquad \text{(or } R = 0\text{)} \tag{5.69}$$

Thus, if W is initially at rest in B, then a constant moment is required to prevent W from rotating relative to B. Expressions for p_i $(i = 1, 2, 3)$ and θ_G for this motion are

$$p_1 \underset{(5.50)}{=} \omega_{10} \tag{5.70}$$

$$p_2 \underset{(5.51)}{=} \omega_{20} \tag{5.71}$$

$$p_3 \underset{(5.52)}{=} \lambda_G t + \frac{\omega_{30}}{1+L_G} \tag{5.72}$$

and

$$\theta_G \underset{(5.40)}{=} L_G\left(\frac{1}{2}\,\lambda_G t^2 + p_{30}t\right)$$

$$= L_G\left(\frac{1}{2}\,z_G x^2 + P_{G3}x\right) \tag{5.73}$$

and the attitude and translational motions of G can now be determined by making use of the solutions in Chapters 3 and 4, since the gyrostat G now behaves like a rigid body. However, before one can apply the previous solutions, the parameters P_1, P_3, L, and z must now be replaced with P_{G1}, P_{G3}, L_G, and z_G, respectively.

If W is allowed to rotate without resistance relative to B, that is, if M_3 is equal to zero, then integration of Eq. (5.33) results in

$$r = \left(\frac{1}{J-K}\right)\left[\left(\frac{J-K}{K}\right)T_W - T_B\right] t + r_0 \tag{5.74}$$

and substitution from Eq. (5.74) into Eq. (5.53) yields

$$s = \left\{L_G\lambda_G - \left(\frac{1}{J-K}\right)\left[\left(\frac{J-K}{J}\right)T_W - \frac{K}{J}\,T_B\right]\right\}t + \left(\frac{1}{1+L_G}\right)\left(L_G\omega_{30} - \frac{K}{J}\,r_0\right) \tag{5.75}$$

so that (see Eqs. (5.40) and (5.75))

$$\theta_G = \frac{1}{2}\left\{L_G\lambda_G - \left(\frac{1}{J-K}\right)\left[\left(\frac{J-K}{J}\right)T_W - \frac{K}{J}\,T_B\right]\right\} t^2 + \left(\frac{1}{1+L_G}\right)\left(L_G\omega_{30} - \frac{K}{J}\,r_0\right)t$$

continued

$$= \zeta_G x^2 + 2\beta_{G1} x \tag{5.76}$$

where

$$\zeta_G \triangleq \frac{1}{2}\left[L_G z_B - z_W + \frac{1 + L_G}{(\ell^{-1} - 1)} z_B\right] \tag{5.77}$$

and

$$\beta_{G1} \triangleq \frac{L_G \Omega_3 - \ell R}{2(1 + L_G)\left[\Omega_1^2 + \left(\frac{1}{1 + L_G}\right)^2 (\Omega_3 + \ell R)^2\right]^{1/2}} \tag{5.78}$$

Eqs. (5.50)-(5.52) and (5.76) are similar in character to Eqs. (3.29)-(3.31) and (3.20). Hence the attitude motion of the symmetric gyrostat with a rotor mounted on a set of frictionless bearings can be determined by simply following the six-step procedure described at the end of Section 3.2, with the exception that the parameters P_1, P_3, L, z, and θ are now replaced with P_{G1}, P_{G3}, L_G, z_G, and θ_G defined in Eqs. (5.66), (5.67), (5.39), (5.64), and (5.76), respectively. Moreover, a description of the motion of G^* can be obtained by utilizing the five-step procedure established at the end of Section 4.2. However, the parameters ζ and β_1 must first be replaced with ζ_G and β_{G1}, respectively; furthermore, β_i $(i = 2, \ldots, 9)$ must be replaced with β_{Gi} $(i = 2, \ldots, 9)$, which are defined as follows:

$$\beta_{G2} \triangleq \beta_{G1} + z_G/2 \tag{5.79}$$

$$\beta_{G3} \triangleq \beta_{G1} - z_G/2 \tag{5.80}$$

$$\beta_{G4} \triangleq \beta_{G1} - 1/2 \tag{5.81}$$

$$\beta_{G5} \triangleq \beta_{G1} + 1/2 \tag{5.82}$$

$$\beta_{G6} \triangleq \beta_{G1} - (z_G + 1)/2 \tag{5.83}$$

$$\beta_{G7} \triangleq \beta_{G1} - (z_G - 1)/2 \tag{5.84}$$

$$\beta_{G8} \triangleq \beta_{G1} + (z_G - 1)/2 \tag{5.85}$$

$$\beta_{G9} \triangleq \beta_{G1} + (z_G + 1)/2 \tag{5.86}$$

A special case of interest arises when the dimensionless parameters z_W and R, which characterize respectively the external moment exerted on W and the initial spin rate of W relative to B, are chosen in such a way that θ_G remains equal to zero at all times. Eqs. (5.76)-(5.78) show that this can be accomplished by taking

$$z_W = [L_G + (1 + L_G)(\ell^{-1} - 1)^{-1}]z_B \tag{5.87}$$

and

$$R = \left(\frac{L_G}{\ell}\right)\Omega_3 \tag{5.88}$$

when Eqs. (5.87) and (5.88) are satisfied, B and C move together as a unit relative to A, and Eqs. (5.66), (5.67), and (5.64) assume the form

$$p_{G1} \underset{(5.66,5.88)}{=} \Omega_1 \tag{5.89}$$

$$p_{G3} \underset{(5.67,5.88)}{=} \Omega_3 \tag{5.90}$$

and

$$z_G \underset{(5.64,\,5.87)}{=} \left(\frac{1+L_G}{1-\ell}\right) z_B \tag{5.91}$$

One may recall from the plots of θ_i $(i = 1, 2, 3)$ vs. x in Chapter 3 that the angle θ_3 governing the "spinning" motion of C in A for a symmetric rigid body is very sensitive to the magnitude of z. This means that the orientation angle θ_3 governing the "spinning" motion of C in A for the symmetric gyrostat G is sensitive to the "effective z", or z_G. Hence, for a symmetric gyrostat with a frictionless rotor, not only can the angle θ_G be kept equal to zero through the proper choice of z_W and R, but the angle θ_3 can be reduced by selecting the shape factors ℓ and L_G in the expression for z_G (see Eq. (5.91)) suitably.

Another interesting motion of G with a frictionless rotor arises when the rotor is forced to spin at a constant rate, say r_0. The externally applied moment T_W which is needed to maintain the constant spin rate r_0 for W is (see Eq. (5.74))

$$T_W = \left(\frac{K}{J-K}\right) T_B \tag{5.92}$$

In terms of the dimensionless parameters, Eq. (5.92) becomes

$$z_W = \frac{z_B}{(\ell^{-1} - 1)} \tag{5.93}$$

and it follows that

$$z_G = z_W + z_B = \frac{z_B}{(1+\ell)} \tag{5.94}$$

Consequently, the attitude and translational motions of G in A can be easily obtained by following the same scheme established in the beginning half of the discussion of this case; however, z_W is now replaced with $z_B/(\ell^{-1} - 1)$.

When the rotation of W relative to B is resisted by an internal moment whose magnitude is proportional to r , M_3 can be expressed as

$$M_3 = -\frac{K}{J}(J-K)kr \tag{5.95}$$

where k is a positive constant. Eq. (5.33) then becomes

$$\dot{r} + kr \underset{(5.33,\,5.95)}{=} \left(\frac{1}{J-K}\right)\left[\left(\frac{J-K}{k}\right)T_W - T_B\right] \tag{5.96}$$

and integration of Eq. (5.96) yields

$$r = \left(r_0 - \frac{T_e}{k}\right)e^{-kt} + \frac{T_e}{k} \tag{5.97}$$

where T_e is defined as

$$T_e \triangleq \left(\frac{1}{J-K}\right)\left[\left(\frac{J-K}{K}\right)T_W - T_B\right] \tag{5.98}$$

Substitution from Eq. (5.97) into Eq. (5.40) leads to the following expression for θ_G:

$$\theta_G = \frac{1}{2}L_G\lambda_G t^2 + \left[\frac{L_G}{1+L_G}\left(\omega_{30} + \frac{K}{J}r_0\right) - \frac{K}{J}\frac{T_e}{k}\right]t$$

$$-\frac{K}{J}\left(r_0 - \frac{T_e}{k}\right)\left(\frac{1-e^{-kt}}{k}\right) \tag{5.99}$$

If a new parameter U, associated with the damping constant k, is now defined as

$$U \triangleq k/p_0 \tag{5.100}$$

then Eq. (5.99) can be expressed in terms of dimensionless parameters as

$$\theta_G = \zeta_G x^2 + 2\beta_{G1} x - \Gamma \tag{5.101}$$

where the constants ζ_G, β_{G1}, and Γ are defined as

$$\zeta_G \triangleq \frac{1}{2} L_G z_G \tag{5.102}$$

$$\beta_{G1} \triangleq \frac{1}{2}\left\{L_G P_{G3} - \left(\frac{1+L_G}{U}\right)\left[z_W - \left(\frac{\ell}{1-\ell}\right) z_B\right]\right\} \tag{5.103}$$

and

$$\Gamma \triangleq \left\{\frac{\ell R}{\left[\Omega_1^2 + \left(\frac{1}{1+L_G}\right)^2 (\Omega_3 + \ell R)^2\right]^{1/2}} - \left(\frac{1+L_G}{U}\right)\left[z_W - \left(\frac{\ell}{1-\ell}\right) z_B\right]\right\}\left(\frac{1-e^{-Ux}}{U}\right) \tag{5.104}$$

Comparison of Eqs. (5.101) and (3.140) shows that the expression for θ_G in Eq. (5.101) contains a constant term, Γ, having no counterpart in Eq. (3.140). Despite this difference between these two equations, the attitude motion of G can be determined by following the six-step procedure outlined at the end of Section 3.2; however, an additional system parameter, U, must now be specified before carrying out the computation of θ_G from Eq. (5.101).

As far as the translational motion of G is concerned, the extra constant term in the expression for θ_G presents a substantial problem, because this term makes it impossible to use the solution in Chapter 4. However, if the system parameters R and z_W are chosen such that

$$R = 0 \tag{5.105}$$

and

$$z_W = \left(\frac{\ell}{1-\ell}\right) z_B \tag{5.106}$$

then $\Gamma = 0$ and θ_G becomes

$$\theta_G = L_G\left(\frac{1}{2} z_G x^2 + P_{G3} x\right) \tag{5.107}$$

which is identical to Eq. (5.73). Under these circumstances, the translational motion of G can thus be determined by reference to the solution in Chapter 4; and it can be seen that a symmetric gyrostat with a viscously damped rotor can perform attitude and translational motions similar to those of a symmetric rigid body, regardless of the damping constant U, so long as U is not equal to zero.

APPENDIX A

Subroutine ARC

```
      SUBROUTINE   ARC    (CA,SA,N,QDOLD,QDNEW,AR,AD)
C***INPUT:     CA=COS(A)           SA=SIN(A)
C***ONE MUST INITIALIZE PARAMETERS:N,QDOLD,& QDNEW IMMEDIATELY
C    FOLLOWING THE READ STATEMENT AT THE BEGINNING OF THE MAIN PROGRAM
C    BY SETTING THEM EQUAL TO ZEROS (REMEMBER: THEY ARE INTEGERS)
C    IF THIS SUBROUTINE IS CALLED "M" TIMES SIMULTANEOUSLY IN THE
C    MAIN PROGRAM TO COMPUTE INVERSES OF THE TRIGONOMETRIC FUNCTIONS
C    OF "M" ANGLES, THEN ONE MUST INITIALIZE N, QDNEW, AND QDOLD "M"
C    TIMES AT THE BEGINNING OF THE MAIN PROGRAM AS FOLLOWS:
C                    N1=N2=.....=NM=0
C                    QDNEW1=QDNEW2=.....=QDNEWM=0
C                    QDOLD1=QDOLD2=.....=QDOLDM=0
C** OUTPUT:   AR=ANGLE-A IN RADIANS    AD=ANGLE-A IN DEGREES

      INTERGER   GDNEW, QDOLD, N
      O=0.0
      PI=4.*ATAN(1.)
      C=180./PI
      If (CA.GT.O.AND.SA.GE.O)    QDNEW=1
      IF (CA.LE.O.AND.SA.GT.O)    QDNEW=2
      IF (CA.LT.O.AND.SA.LE.O)    QDNEW=3
      IF (CA.GE.O.AND.SA.LT.O)    QDNEW=4
      IF  (QDCLD.EQ.QDNEW)  GO TO 135
      IF  ((QDOLD.EQ.1.AND.QDNEW.EQ.2).OR.(QDOLD.EQ.3.AND.QDNEW.EQ.4))
     1GO TO 161
      IF  ((QDOLD.EQ.2.AND.QDNEW.EQ.1).OR.(QDOLD.EQ.4.AND.QDNEW.EQ.3))
     1GO TO 162
      IF (QDOLD.EG.0.AND.QDNEW.EQ.3)  GO TO 177
      IF (QDOLD.EQ.0.AND.QDNEW.EQ.2)  GO TO 178
      IF  ((QDOLD.EQ.4.AND.QDNEW.EQ.1).OR.(QDOLD.EQ.1.AND.QDNEW.EQ.4)
     1 .OR.(QDOLD.EQ.2.AND.QDNEW.EQ.3).OR.(QDOLD.EQ.3.AND.QDNEW.EQ.2)
     2 .OR.(QDOLD.EQ.0.AND QDNEW.EQ.1).OR.(QDOLD.EQ.0.AND.QDNEW.EQ.4))
     3  GO TO 135
      WRITE (6,131)
  131 FORMAT ('C',10X,'*WARNING*  A QUADRANT OR MORE WAS SKIPPED',/)
      GO TO 173
  135 N=N
      GO TO 173
  161 N=N+1
      GO TO 173
  162 N=N-1
      GO TO 173
  177 N=-1
      GO TO 173
```

```
178 N=1
    GO TO 173
173 QDOLD=QDNEW
    IF (CA.EQ.O) GO TO 174
    AR=ATAN(SA/CA)+N*PI
    AD=C*AR
    GO TO 175
174 AR=(-.5+N)*PI
    AD=C*AR
175 RETURN
    END
```

APPENDIX B

Breakwell's Approximate Solution to the Kinematical Equations

Consider a symmetric rigid body under the action of a body-fixed force as described in Section 2.1, and suppose that the angular velocity vector is initially perpendicular to the symmetry axis, that is, $\omega_{30} = p_{30} = 0$. Under these circumstances, the kinematical equations together with the initial conditions as given by Eqs. (3.45)-(3.48) assume the following form:

$$\mathring{a}_{i1} = \lambda t \, a_{i2} \qquad \text{(B.1)}$$

$$\mathring{a}_{i2} - p_{10} a_{i3} = -\lambda t \, a_{i1} \qquad (i = 1,2,3) \qquad \text{(B.2)}$$

$$\mathring{a}_{i3} + p_{10} a_{i2} = 0 \qquad \text{(B.3)}$$

and

$$a_{ij(0)} = \delta_{ij} \qquad (i,j = 1,2,3) \qquad \text{(B.4)}$$

It is, perhaps, remarkable that this apparently simple third order system of equations is not solvable without recourse to some approximation techniques. The object of this appendix is to present an approximate solution developed by Professor J. V. Breakwell* in the course of our investigation.

---

* Professor of Aeronautics & Astronautics, Stanford University.

The first step is to use the method of successive approximations employed in Section 3.2. After carrying out the iteration process four times, one obtains the following as the fourth approximation to a_{ij} $(i,j = 1,2,3)$:

$$a_{11} \approx 1 - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^2 - \left(\frac{\lambda}{2p_{10}}\right)^2 (1 - \cos p_{10}t) + \left(\frac{\lambda}{2p_{10}}\right)\left(\frac{\lambda t}{p_{10}}\right) \sin p_{10}t \tag{B.5}$$

$$\begin{aligned} a_{12} \approx & - \left[\left(\frac{\lambda}{2p_{10}}\right) + 2\left(\frac{\lambda}{2p_{10}}\right)^3\right](1 - \cos p_{10}t) + \frac{3}{2}\left(\frac{\lambda}{2p_{10}}\right)\left(\frac{\lambda t}{p_{10}}\right)^2 \\ & - \frac{1}{2}\left(\frac{\lambda}{2p_{10}}\right)^2\left(\frac{\lambda t}{p_{10}}\right) \sin p_{10}t - \frac{1}{6}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t \end{aligned} \tag{B.6}$$

$$\begin{aligned} a_{13} \approx & - \left[\left(\frac{\lambda}{2p_{10}}\right) + \frac{5}{2}\left(\frac{\lambda}{2p_{10}}\right)^3\right] \sin p_{10}t + \left(\frac{\lambda t}{p_{10}}\right) + 2\left(\frac{\lambda}{2p_{10}}\right)^2\left(\frac{\lambda t}{p_{10}}\right) \\ & - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 + \frac{1}{2}\left(\frac{\lambda}{2p_{10}}\right)^2\left(\frac{\lambda t}{p_{10}}\right) \cos p_{10}t + \frac{1}{2}\left(\frac{\lambda}{2p_{10}}\right)\left(\frac{\lambda t}{p_{10}}\right)^2 \sin p_{10}t \\ & - \frac{1}{6}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t \end{aligned} \tag{B.7}$$

$$\begin{aligned} a_{21} \approx & - \left[\left(\frac{\lambda}{2p_{10}}\right) + 2\left(\frac{\lambda}{2p_{10}}\right)^3\right](1 - \cos p_{10}t) + \frac{1}{2}\left(\frac{\lambda}{2p_{10}}\right)\left(\frac{\lambda t}{p_{10}}\right)^2 \\ & + \left(\frac{\lambda t}{p_{10}}\right) \sin p_{10}t + 2\left(\frac{\lambda}{2p_{10}}\right)^2\left(\frac{\lambda t}{p_{10}}\right) \sin p_{10}t \\ & - \frac{3}{2}\left(\frac{\lambda}{2p_{10}}\right)\left(\frac{\lambda t}{p_{10}}\right)^2 \cos p_{10}t - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t \end{aligned}$$

$$+ \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^4 \cos p_{10}t - \frac{1}{6}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t \tag{B.8}$$

$$a_{22} \approx \cos p_{10}t - \left(\frac{\lambda}{p_{10}^2}\right)^2 (1 - \cos p_{10}t) - \frac{1}{2}\left(\frac{\lambda}{p_{10}^2}\right)\left(\frac{\lambda t}{p_{10}}\right) \sin p_{10}t$$

$$- \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t \tag{B.9}$$

$$a_{23} \approx - \sin p_{10}t - \left(\frac{\lambda}{p_{10}^2}\right)\left(\frac{\lambda t}{p_{10}}\right) + \frac{1}{2}\left[\left(\frac{\lambda}{p_{10}^2}\right)^2 + \left(\frac{\lambda t}{p_{10}}\right)^2\right]\sin p_{10}t$$

$$- \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t + \frac{1}{2}\left(\frac{\lambda}{p_{10}^2}\right)\left(\frac{\lambda t}{p_{10}}\right) \cos p_{10}t \tag{B.10}$$

$$a_{31} \approx \left[\left(\frac{\lambda}{p_{10}^2}\right) + \frac{5}{2}\left(\frac{\lambda}{p_{10}^2}\right)^3\right]\sin p_{10}t - \left(\frac{\lambda t}{p_{10}}\right) \cos p_{10}t$$

$$- \frac{5}{2}\left(\frac{\lambda}{p_{10}^2}\right)^2\left(\frac{\lambda t}{p_{10}}\right) \cos p_{10}t - \frac{3}{2}\left(\frac{\lambda}{p_{10}^2}\right)\left(\frac{\lambda t}{p_{10}}\right)^2 \sin p_{10}t$$

$$+ \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t + \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^4 \sin p_{10}t$$

$$+ \frac{1}{6}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t \tag{B.11}$$

$$a_{32} \approx \sin p_{10}t - \frac{1}{2}\left(\frac{\lambda}{p_{10}^2}\right)^2 \sin p_{10}t + \frac{1}{2}\left(\frac{\lambda}{p_{10}^2}\right)\left(\frac{\lambda t}{p_{10}}\right) \cos p_{10}t$$

$$+ \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t \tag{B.12}$$

$$a_{33} \approx \cos p_{10}t - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^2 \cos p_{10}t + \frac{1}{2}\left(\frac{\lambda}{p_{10}^2}\right)\left(\frac{\lambda t}{p_{10}}\right) \sin p_{10}t$$

$$- \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t \qquad \text{(B.13)}$$

As in Section 3.2, Eqs. (B.5)-(B.13) have been written in such a way that t is always multiplied by λ and divided by p_{10} when it appears outside of the argument of a trigonometric function, and λ is divided by p_{10}^2 when it is not multiplied by t. Confining attention to situations in which (λ/p_{10}^2) is relatively small, so that one may drop all terms containing $(\lambda/p_{10}^2)^n$ with $n \geqq 1$, and omitting terms containing only the product of $\frac{1}{6}\left(\frac{\lambda t}{p_{10}^2}\right)^3$ and a trigonometric function, one arrives at

$$a_{11} \approx 1 - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^2 \qquad \text{(B.14)}$$

$$a_{12} \approx 0 \qquad \text{(B.15)}$$

$$a_{13} \approx \left(\frac{\lambda t}{p_{10}}\right) - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 \qquad \text{(B.16)}$$

$$a_{21} \approx \left(\frac{\lambda t}{p_{10}}\right) \sin p_{10}t - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t$$

$$+ \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^4 \cos p_{10}t \qquad \text{(B.17)}$$

$$a_{22} \approx \cos p_{10}t - \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t \qquad \text{(B.18)}$$

$$a_{23} \approx - \sin p_{10}t + \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^2 \sin p_{10}t$$

$$- \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t \qquad \text{(B.19)}$$

$$a_{31} \approx - \left(\frac{\lambda t}{p_{10}}\right) \cos p_{10}t + \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t$$

$$+ \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^4 \sin p_{10}t \qquad \text{(B.20)}$$

$$a_{32} \approx \sin p_{10}t + \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t \qquad \text{(B.21)}$$

$$a_{33} \approx \cos p_{10}t - \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^2 \cos p_{10}t$$

(B.22)

Next, if one defines ϵ and ρ as

$$\epsilon \triangleq (p_{10}^2 + \lambda^2 t^2)^{1/2} = p_{10} + \frac{1}{2}\frac{\lambda^2 t^2}{p_{10}} - \frac{1}{8}\frac{\lambda^4 t^4}{p_{10}^3} + \ldots \qquad (p_{10}^2 > \lambda^2 t^2)$$

(B.23)

and

$$\rho \triangleq \int_0^t (p_{10}^2 + \lambda^2 \tau^2)^{1/2} d\tau$$

$$= \frac{1}{2}\left(\frac{\lambda}{p_{10}^2}\right)\left[p_{10}t\sqrt{p_{10}^2 t^2 + \left(\frac{\lambda}{p_{10}^2}\right)^{-2}}\right.$$

continued

$$+ \left(\frac{\lambda}{p_{10}^2}\right)^{-2} \log\left(p_{10}t + \sqrt{p_{10}^2 t^2 + \left(\frac{\lambda}{p_{10}^2}\right)^{-2}}\right)\Bigg]$$

$$+ \frac{1}{2}\left(\frac{\lambda}{p_{10}^2}\right)^{-1} \log\left(\frac{\lambda}{p_{10}^2}\right)$$

$$= p_{10}t + \frac{1}{6}\frac{\lambda^2 t^3}{p_{10}} - \frac{1}{40}\frac{\lambda^4 t^5}{p_{10}^3} + \cdots \qquad (p_{10}^2 > \lambda^2 t^2) \tag{B.24}$$

then

$$\frac{p_{10}}{\epsilon} \approx 1 - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^2 + \frac{3}{8}\left(\frac{\lambda t}{p_{10}}\right)^4 \tag{B.25}$$

$$\frac{\lambda t}{\epsilon} \approx \left(\frac{\lambda t}{p_{10}}\right) - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 + \frac{3}{8}\left(\frac{\lambda t}{p_{10}}\right)^5 \tag{B.26}$$

$$\sin\rho \approx \sin p_{10}t + \frac{\lambda^2 t^3}{6p_{10}} \cos p_{10}t \tag{B.27}$$

$$\cos\rho \approx \cos p_{10}t - \frac{\lambda^2 t^3}{6p_{10}} \sin p_{10}t \tag{B.28}$$

$$\frac{\lambda t}{\epsilon}\sin\rho \approx \left(\frac{\lambda t}{p_{10}}\right)\sin p_{10}t - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t$$

$$+ \frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^4 \cos p_{10}t$$

$$- \frac{1}{12}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^6 \cos p_{10}t \tag{B.29}$$

$$-\frac{p_{10}}{\epsilon}\sin\rho \approx -\sin p_{10}t + \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^2 \sin p_{10}t$$

$$-\frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t$$

$$+\frac{1}{12}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^5 \cos p_{10}t \qquad (B.30)$$

$$-\frac{\lambda t}{\epsilon}\cos\rho \approx -\left(\frac{\lambda t}{p_{10}}\right)\cos p_{10}t + \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^3 \cos p_{10}t$$

$$+\frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^4 \sin p_{10}t$$

$$-\frac{1}{12}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^6 \sin p_{10}t \qquad (B.31)$$

$$\frac{p_{10}}{\epsilon}\cos\rho \approx \cos p_{10}t - \frac{1}{2}\left(\frac{\lambda t}{p_{10}}\right)^2 \cos p_{10}t$$

$$-\frac{1}{6}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^3 \sin p_{10}t$$

$$+\frac{1}{12}\left(\frac{\lambda}{p_{10}^2}\right)^{-1}\left(\frac{\lambda t}{p_{10}}\right)^5 \sin p_{10}t \qquad (B.32)$$

and comparison of these expressions with those in Eqs. (B.14)-(B.22) suggests that the following may be good approximations to a_{ij} $(i, j = 1, 2, 3)$:

$$a_{11} \approx p_{10}/\epsilon \qquad (B.33)$$

$$a_{12} \approx 0 \tag{B.34}$$

$$a_{13} \approx \lambda t/\epsilon \tag{B.35}$$

$$a_{21} \approx \frac{\lambda t}{\epsilon} \sin \rho \tag{B.36}$$

$$a_{22} \approx \cos \rho \tag{B.37}$$

$$a_{23} \approx - \frac{p_{10}}{\epsilon} \sin \rho \tag{B.38}$$

$$a_{31} \approx - \frac{\lambda t}{\epsilon} \cos \rho \tag{B.39}$$

$$a_{32} \approx \sin \rho \tag{B.40}$$

$$a_{33} \approx \frac{p_{10}}{\epsilon} \cos \rho \tag{B.41}$$

As in Section 3.2, a dimensionless form of the above expressions for a_{ij} $(i,j = 1,2,3)$ can be obtained by introducing x and z as follows:

$$x \triangleq p_{10}t \tag{B.42}$$

$$z \triangleq \lambda/p_{10}^2 \tag{B.43}$$

In terms of these quantities, Eqs. (B.33)-(B.41) become

$$a_{11} \approx (1 + z^2x^2)^{-1/2} \tag{B.44}$$

$$a_{12} \approx 0 \tag{B.45}$$

$$a_{13} \approx zx(1+z^2x^2)^{-1/2} \tag{B.46}$$

$$a_{21} \approx zx(1+z^2x^2)^{-1/2} \sin\rho \tag{B.47}$$

$$a_{22} \approx \cos\rho \tag{B.48}$$

$$a_{23} \approx -(1+z^2x^2)^{-1/2} \sin\rho \tag{B.49}$$

$$a_{31} \approx -zx(1+z^2x^2)^{-1/2} \cos\rho \tag{B.50}$$

$$a_{32} \approx \sin\rho \tag{B.51}$$

$$a_{33} \approx (1+z^2x^2)^{-1/2} \cos\rho \tag{B.52}$$

where the function ρ as defined in Eq. (B.24) can now be written in terms of z and x by reference to Eqs. (B.42) and (B.43), with the result

$$\rho = \frac{z}{2}\left[x\sqrt{x^2+z^{-2}} + z^{-2}\log\left(x+\sqrt{x^2+z^{-2}}\right)\right] + \frac{1}{2}z^{-1}\log z \tag{B.53}$$

Now that the approximate expressions for the direction cosines a_{ij} $(i,j = 1,2,3)$ are at hand, approximate values of the orientation angles θ_1, θ_2, and θ_3, describing the orientation of the unit vectors $\underline{c}_1$, $\underline{c}_2$, and $\underline{c}_3$ relative to $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$, can be found by making use of Eqs. (2.20)-(2.22) with γ_{ij} and γ_i in place of a_{ij} and θ_i, respectively.

In order to obtain a measure of the accuracy of the solution under consideration, we shall resort to a graphical comparison of the orientation angles θ_1, θ_2, and θ_3 obtained by various means.

Figures B-1 and B-2 each show the results obtained by using a digital computer solution (solid lines) of the exact equations of motion, the approximate solution (crosses) described in Section 3.2, and Breakwell's solution (circles). In both figures, the agreement between the circles and solid lines is seen to be better than that between the crosses and solid lines. The superiority of Breakwell's solution over the one developed in Section 3.2 becomes more pronounced as z and x increase. This is not surprising since Breakwell's solution is based on the fourth approximation to a_{ij} while the approximate solution derived in Section 3.2 is based on the third approximation to a_{ij}.

While Breakwell's approximate solution is more accurate than the one developed in Section 3.2, it has two drawbacks: It is restricted to situations in which the angular velocity vector is initially perpendicular to the symmetry axis (i.e., $\omega_{30} = 0$), and the difficulties associated with integrating Eqs. (4.4)-(4.6) when e_{ij} $(i, j = 1, 2, 3)$ are given by expressions such as those in Eqs. (B.33)-(B.41) are such as to preclude solution of the differential equations governing the motion of the mass center.

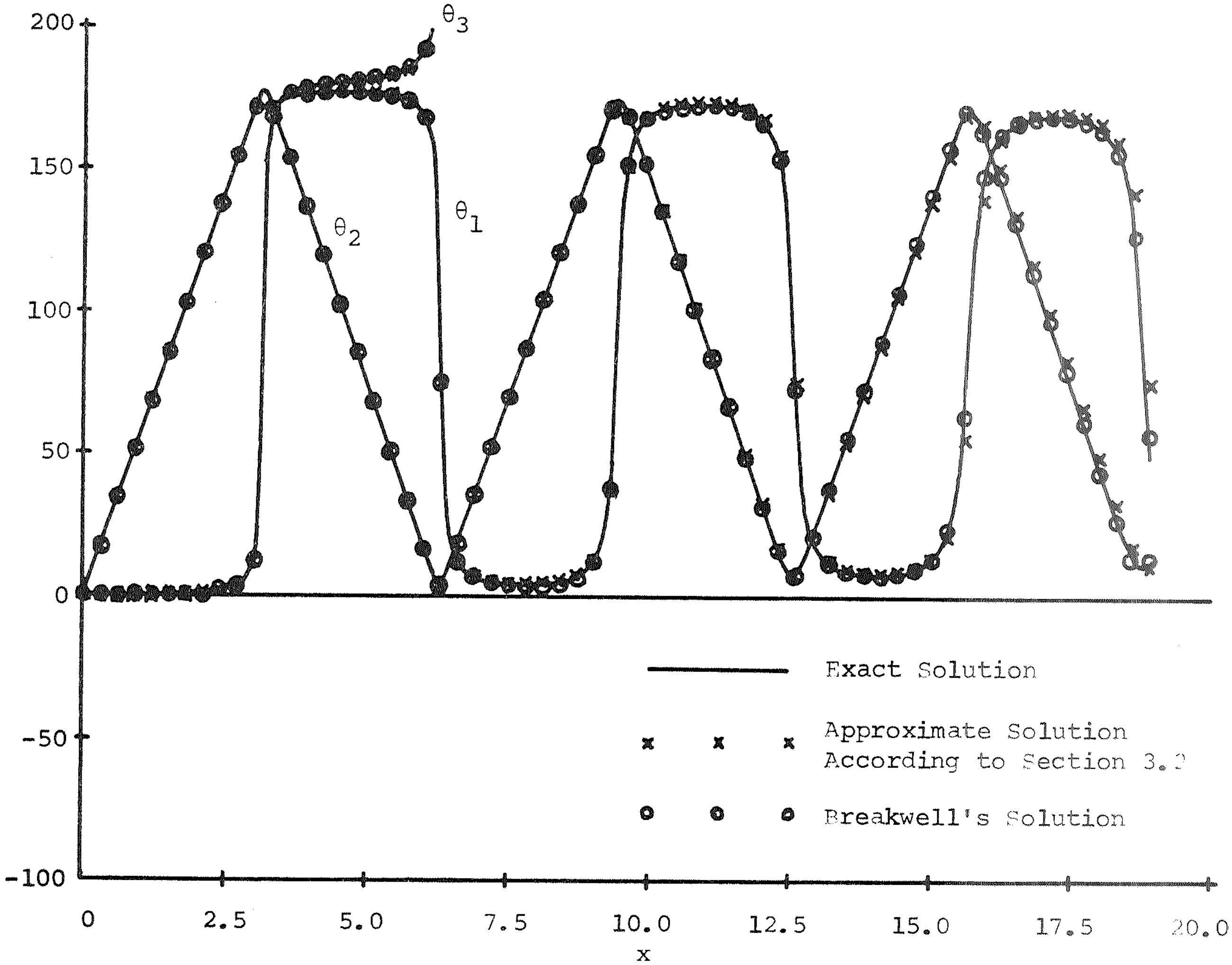

Figure B-1. Comparison of Solutions for Orientation Angles.

$\Omega_1 = 1.0, \ \Omega_3 = 0, \ z = 0.01$

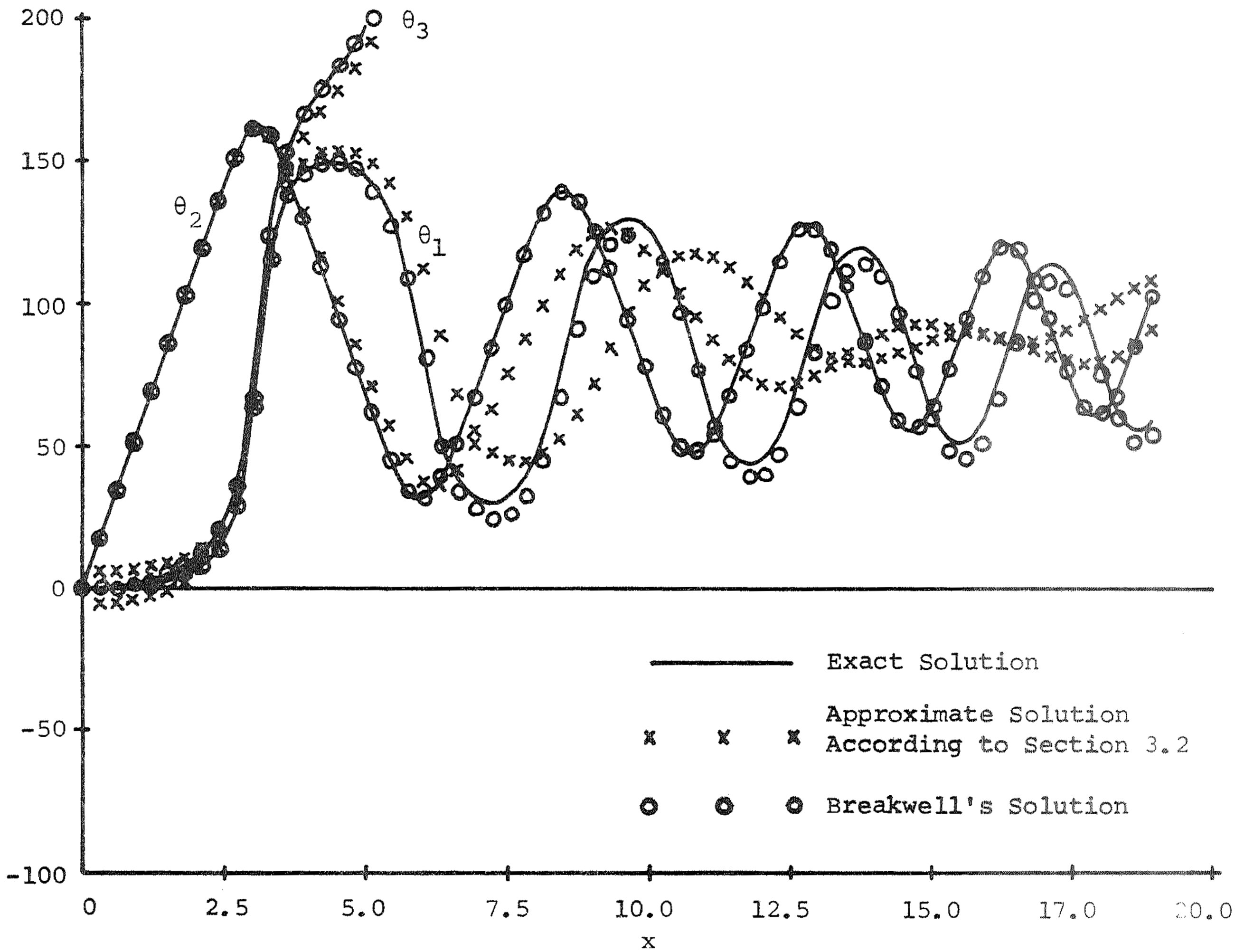

Figure B-2. Comparison of Solutions for Orientation Angles.

$\Omega_1 = 1.0$, $\Omega_3 = 0$, $z = 0.1$

BIBLIOGRAPHY

[1] Leimanis, E., The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point, (Springer-Verlag, New York, 1965).

[2] Lee, R., "The Motion of a Self-Excited Asymmetric Rigid Body," Archive for Rotational Mechanics and Analysis, Vol. 22, No. 2, (1966), pp. 129-142.

[3] Grammel, R., "Der Selbsterregte Unsymmetrische Kreisel," Journal of Applied Mathematics and Mechanics, Vol. 18, (1954), pp. 73-97.

[4] Bödewadt, U.T., "Der Symmetrische Kreisel bei Zeitfester Drehkraft," Mathematische Zeitschrift, Vol. 55, (1952), pp. 310-320.

[5] Auelmann, R.R., "On the Motion of a Symmetrical Rigid Body with Constant Body-Fixed Torques," (private communication).

[6] Valeev, K.G., "On the Problem of the Motion of an Axially Symmetrical Body Under the Action of a Constant Moment," Journal of Applied Mathematics and Mechanics, Vol. 29, (1965), pp. 162-165.

[7] Yatteau, J.D., and Kane, T.R., "Planar Motion of a Human Being Under the Action of a Body-Fixed Thrust," Technical Report No. 199, (Dept. of Applied Mechanics, Stanford University, Sept. 1969).

[8] Smith, G.W., Tewell, J.R., and Murrish, C.H., "A New Simulation Technique for Extravehicular Activities," Report R-66-2, (Martin Marietta Corp., Denver, May 1966).

[9] Tewell, J.R., and Johnson, C.H., "EVA/IVA Simulation Dynamics," Report R-67-8, (Martin Marietta Corp., Denver, Feb. 1967).

[10] Kane, T.R., "Solution of the Equations of Rotational Motion for a Class of Torque-Free Gyrostats," AIAA Journal, Vol. 8, No. 6, (June 1970), pp. 1141-1143.

[11] Scher, M.P., and Kane, T.R., "Alteration of the State of Motion of a Human Being in Free Fall," Technical Report No. 198, (Dept. of Applied Mechanics, Stanford University, July 1969).

[12] Johansen, K.F., Dynamics of the Spherical Pendulum, Engineering Degree Thesis, (Dept. of Applied Mechanics, Stanford University, Oct. 1968).

[13] Leimanis, E., "Some Recent Advances in the Dynamics of Rigid Bodies and Celestrial Mechanics," Surveys in Applied Mathematics, Vol. 2, (John Wiley and Sons, Inc., 1958), pp. 1-108.

[14] Whittaker, E.T., Analytical Dynamics of Particles and Rigid Bodies, 4th Ed., (Cambridge University Press, 1964).

[15] Grammel, R., "Drehzahlabhängige Selbsterregung des Unsymmetrischen Kreisels," Ingenieur-Archiv, Vol. 29, (1969), pp. 153-159.

[16] Grammel, R., "Ein Problem des Selbsterregten Unsymmetrischen Kreisels," Annali di Matematica Pura ed., Series 4, Vol. 50, (1960), pp. 187-196.

[17] Wenglarz, R.A., and Kane, T.R., "Gyroscopic Drifts Associated with Rotational Vehicle Vibrations," Paper No. 68-APM-V, Journal of Applied Mechanics, (May 1968).

[18] Hildebrand, F.B., Advanced Calculas for Applications, (Prentice Hall, Inc., 1962).

[19] Hildebrand, F.B., Methods of Applied Mathematics, 2nd Ed., (Prentice Hall, Inc., 1965).

[20] Kane, T.R., Dynamics, (Holt, Rinehart, and Winston, Inc., 1968).

[21] Goldstein, H., Classical Mechanics, (Addison-Wesley, Inc., 1959).

[22] Routh, E.J., Advanced Dynamics of a System of Rigid Bodies, 6th Ed., (Dover Publications, 1955).

[23] MacMillan, W.D., Dynamics of Rigid Bodies, (Dover Publications, 1960).

[24] Abramowitz, M., and Stegun, I.A., Handbook of Mathematical Functions, National Bureau of Standards, AMS. 55, (U.S. Government Printing Office, Washington, D.C., June 1964).