NASA CR-119015

INSTITUTE
for
FLUID DYNAMICS
and
APPLIED MATHEMATICS

Technical Note BN 686 March 1971

The Stabilizing Effect of Boundaries
on Thermal Convection

by

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Paper presented at the Twenty-Third Annual Meeting, Division of Fluid Dynamics, American Physical Society, Charlottesville, Virginia, 23-25 November 1970

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The computations reported in this paper were supported by the National Aeronautics and Space Administration Grant ~~(NSG-398)~~ NGL-21-002-008 to the Computer Science Center of the University of Maryland. The author gratefully acknowledges the partial support of the National Science Foundation through Grant GZ-761.

ABSTRACT

The classical theory for Benard Convection of a viscous, incompressible fluid layer confined between horizontal surfaces is generalized to semi-bounded and unbounded fluids. Two and three-layer models are solved numerically to give curves of neutral stability in the Rayleigh number wavenumber plane. For each model a family of neutral curves is given which is parameterized by the static stability of adjacent layers of fluid. The results demonstrate a destabilization of the classical problem which is most pronounced at small wavenumbers when the static stability of adjacent layers is neutral.

1. INTRODUCTION

It has been almost a century since Count Rumford discovered the phenomenon of thermal convection. Numerous researchers have since contributed to our present understanding of the subject. The classical laboratory experiments were performed by Benard and the classical theory is due to Lord Rayleigh.

Thermal convection has long been recognized as having an important dynamical influence in the atmosphere. In this connection Satellite photography has provided us with evidence of thermally generated cells[1,2]. More recently sensitive radar has been used as a probe to reveal patterns of cellular motion in clear air[3].

The stability theory for parallel shear flow has had a rather long and controversial history. Only recently have some of the fundamental difficulties been resolved. Instead of the cellular motions of thermal convection the most unstable disturbances to parallel shear flow are two dimensional. When thermal instability occurs in the presence of a "subcritical" parallel shear flow an interaction of mechanisms results in the generation of longitudinal rolls with axes oriented in the direction of flow.

In their study of the stability of plane Poiseuille flow of a thermally stratified fluid, Gage and Reid[4] show that the stability boundary in the R, Re-plane recovers the classical results for Benard convection with rigid-rigid boundaries as $Re \rightarrow 0$ and the basic flow is removed from the problem. The reason that this classical result should be recovered is perfectly clear since the boundary conditions employed in the characteristic value problem for the stability of the stratified channel flow were consistent with the rigid-rigid boundary conditions of the classical thermal problem.

If one considers, alternatively, the stability of a thermally stratified boundary layer profile it is not clear whether any classically known result will be recovered by the stability boundary in the R - Re plane as $Re \to 0$. In fact, when we recall that the upper "boundary condition" in the stability theory for parallel shear flows of the boundary-layer type is really a boundedness condition, we notice a basic inconsistency between the formulation of the characteristic value problem for shear flow instability and the classical formulation of the characteristic value problem for thermal instability. This observation motivates the reexamination of thermal convection theory with the goal of developing a more general theory for semibounded and unbounded models. In order to accomplish this objective we shall replace some of the classical boundary conditions with matching conditions and the boundedness conditions. It is anticipated that the results of this study will provide the proper limits for further studies on thermally stratified boundary layer and unbounded flows.

2. THE MATHEMATICAL MODELS

The mathematical formulation used in the models described below is similar to that used by Currie[5] to study the effect of the heating rate on thermal instability. The novelty here is to allow the fluid to extend to infinity replacing classical boundary conditions by matching conditions at an interface and boundedness conditions at infinity.

The Two-Layer Model

Consider a statically unstable layer of viscous, heat-conducting, incompressible fluid bounded below by a free or rigid horizontal surface and above by an infinitely deep layer of the same fluid. The vertical temperature gradient will be specified to be constant in each layer and all properties of the fluid will be assumed horizontally uniform; the fluid extending to infinity in the horizontal directions. Our goal will be to determine the curve of neutral stability and to see how it depends on the static stability R_U of the upper layer.

The eigenfunctions $W_\ell(Z)$ satisfy the usual sixth-order equation

$$(D^2-a^2)^3 W_\ell + a^2 R_\ell W_\ell = 0 \tag{2.1}$$

in each layer. In equation (2.1) W_ℓ is the vertical perturbation velocity, a is the horizontal wavenumber, D is the derivative with respect to the vertical coordinate and R_ℓ is the Rayleigh number defined by

$$R_\ell = \frac{- g \gamma \left.\frac{\partial T}{\partial z}\right|_\ell H^4}{\kappa \nu}$$

where $\left.\frac{\partial T}{\partial z}\right|_\ell$ is the temperature gradient of each layer and H is the depth of the statically unstable layer. For the two layer models, then, there are two Rayleigh numbers of interest: the positive Rayleigh number, R, of the statically unstable layer and the negative Rayleigh number R_U of the infinitely deep statically stable layer of fluid.

When $R_\ell > 0$ the solutions to equations (2.1) can be written

$$W_\ell = \sum_{i=1}^{6} C_\ell^{(i)} \exp \{ r_\ell^{(i)} z \} \tag{2.2}$$

where $r_\ell^{(i)}$ are the roots of

$$(r_\ell^2 - a^2)^3 + a^2 R_\ell = 0 \tag{2.3}$$

The roots , of course, differ for each layer and the eigenfunctions and their derivatives must be matched at the interface of the two layers.

We obtain the characteristic determinant for the two layer model by requiring simultaneous satisfaction of the lower boundary conditions, the matching conditions at the interface, and the boundedness condition at infinity. The lower boundary conditions are

$$W = DW = (D^2-a^2)^2\, W=0 \text{ and } W = D^2W = (D^2-a^2)^2\, W=0 \qquad (2.4)$$

for rigid and free boundaries respectively. The boundedness condition at infinity requires that we reject the exponentially large eigensolutions in the upper layer. If we denote the eigensolutions W_U, W_L, in the upper and lower layers then the matching conditions are

$$W_L^{(n)}\,(+1) = W_U^{(n)}\,(+1) \qquad (n=0,1,2,..5) \qquad (2.5)$$

where n represents the order of the derivative. The physical conditions consistent with the matching of the eigenfunction and its first five derivatives are the requirements of continuity of the vertical perturbation velocity, horizontal perturbation velocities, stress, pressure, perturbation temperature, and vertical derivative of perturbation temperature.

The infinitely deep upper layer can also be considered neutrally stable. If this is done, the solution to eq. (2.1) becomes

$$\begin{aligned} W_U = {} & C_U^{(1)} e^{-az} + C_U^{(2)} z e^{-az} + C_U^{(3)} z^2 e^{-az} \\ & + C_U^{(4)} e^{+az} + C_U^{(5)} z e^{+az} + C_U^{(6)} z^2 e^{+az} \end{aligned} \qquad (2.6)$$

The characteristic determinant is obtained as before by employing the solutions of (2.6) instead of (2.2).

Another special case is obtained in the limit $R_u \to -\infty$, when the eigensolutions of the upper layer vanish identically. It would then appear reasonable to replace the matching conditions (2.5) by the boundary conditions

$$W(+1) = W^{(1)}(+1) = W^{(2)}(+1) = 0 \qquad (2.7)$$

With the boundary conditions of (2.7) the characteristic value problem reduces to the solution of a 6x6 determinant.

The Three-Layer Models

We now turn our attention to the stability of three-layer unbounded models in which an infinitely deep, neutrally or stably stratified layer of fluid also bounds the statically unstable layer from below. The lower boundary conditions are then replaced by matching conditions at the lower interface and the solutions of the governing equation are required to satisfy a boundedness condition at minus infinity.

Consistent with the above model we have three bounded solutions in the top and bottom layers and six matching conditions to satisfy at each interface. As before, we wish to determine the curves of neutral stability in the wavenumber-Rayleigh number plane. These curves will be parameterized by the negative Rayleigh numbers R_T and R_B of the top and bottom layers.

3. RESULTS OF THE COMPUTATIONS

Curves of neutral stability were obtained by searching for zeroes in the characteristics determinants for the two and three layer models. The determinants were evaluated numerically with the aid of a "Math-Pack" subroutine supplied with the Univac 1108.

The Two-Layer Models

The results for the computations on the two-layer model are presented in Figures 1 through 5 and Table 1. Figure 1 contains the results for the stability of the two-layer model bounded above by a neutrally stable layer. The curve of neutral stability for the rigidly bounded case is considerably less stable than the classical rigid-free model. Destabilization is most pronounced at small wavenumbers. In fact the minimum Rayleigh number of 32 is approached asymptotically as the wavenumber approaches zero. The neutral curve for the freely bounded case is still less stable with apparently no critical Rayleigh number being achieved. Both curves show greatest instability at the smallest wavenumber.

Figures 2 and 3 show the stabilizing effect of an infinitely deep statically stable layer above the unstable layer. Slight static stability has the same effects on both freely and rigidly bounded models. With stratification there exist critical wavenumbers and Rayleigh numbers which are shown in Table 1 and Figures 4 and 5 to increase monotonically with increasing static stability of the upper layer. These critical values are consistently lower with the free lower surface.

With a rigid lower boundary and statically stable fluid above the unstable layer our two-layer model is somewhat similar to the models employed to investigate penetrative thermal convection. Provided the upper layer is sufficiently stable the presence of an upper boundary will not appreciably effect the stability. For this situation we have demonstrated agreement between our results and those of Faller for penetrative convection[6].

The Three-Layer Models

Figures 6 through 12 and Table 2 contain the results of the computations for the three layer model. The general behavior of these models is the same as the two layer models. When the top and bottom layers are neutrally stable, there exists no critical wavenumber or Rayleigh number and the curve of neutral stability lies below that of the bounded models discussed above.

The results of Figures 7 through 12 document the stabilizing effects of the static stability of upper and lower layers. Table 2 and Figures 11 and 12 show the montonic increase in critical values of Rayleigh numbers and wavenumbers with increasingly stable stratification of the top and bottom layers.

CONCLUDING REMARKS

Part of the motivation for this work came from the desire to understand the proper limiting situation for thermal instability in the presence of a shear flow as the Reynolds number of the flow approaches zero. Further motivation for pursuing these models in detail was derived from the realization that the boundedness conditions usually applied in stability investigations of boundary layer profiles of parallel shear flow might provide a more realistic model for applications to geophysical fluid dynamics. The use of statically stable layers in place of boundaries make these models even more realistic for modeling local thermal instability. The results of these computations would suggest that the static stability of layers adjacent to an unstable layer could fundamentally alter the stability of the unstable layer. Further computations have suggested that the depths of these adjacent layers will also be important parameters.

Several authors have previously noted the destabilizing effect of altering boundary conditions on the onset of thermal instability of the classical problem. Sasaki[7], in particular, has examined the effect of partial or complete insulation of a boundary with qualitatively similar dynamical effects.

The stabilizing effect of boundaries on thermal instability is the most basic result to come from the investigation of these simple models. The lack of curvature in the neutral curve for the unstable layer bounded above and below by infinitely deep layers of unstratified fluid suggests that the stabilization of small wavenumber disturbances is due to the influence of boundaries. Furthermore, the existence of a critical wavenumber at which the Rayleigh number achieves a minimum on the neutral curve is due to simultaneous stabilizing effects of boundaries, most pronounced for small wavenumbers, and viscosity most pronounced for large wavenumbers. That boundaries are responsible for stabilizing small wavenumber disturbances is also consistent with the results of stability investigations of unbounded parallel shear flow. The work of Esch[8], Tatsumi and Kakutani[9], Tatsumi and Gotoh[10], and Clenshaw and Elliot[11] demonstrate instability at low Reynolds numbers and small wavenumbers for unbounded flows.

TABLE 1

THE VALUES OF THE CRITICAL WAVENUMBERS AND RAYLEIGH NUMBERS ALONG THE CURVES OF NEUTRAL STABILITY FOR THE TWO-LAYER MODELS

	Rigid lower surface		Free lower surface	
R_S	a_c	R_c	a_c	R_c
-10^{-2}	0.347	46.54	.215	8.49
-1	0.800	82.12	.605	32.1
-10^{2}	1.65	275.8	1.40	170
-10^{4}	2.73	1080	2.38	731
-10^{6}	3.44	2100	3.01	1438
-10^{8}	3.78	2766	3.33	1912

TABLE 2

THE VALUES OF THE CRITICAL WAVENUMBERS, a_c AND CRITICAL RAYLEIGH NUMBERS R_c, ALONG THE CURVES OF NEUTRAL STABILITY FOR THE THREE-LAYER MODELS

	$R_L = 0.0$		$R_L = -1$		$R_L = -10^4$	
R_U	a_c	R_c	a_c	R_c	a_c	R_c
-10^{-2}	-	-	-	-	.801	92.57
-1	-	-	-	-	1.00	105.5
-10^{2}	.378	16.67	.681	25.26	1.67	247.3
-10^{4}	.795	92.32	1.00	105.5	2.52	781.3
-10^{6}	1.15	207.7	1.27	219.4	3.02	1343
-10^{8}	1.37	300.6	1.45	311.0	3.27	1684

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11. C.W. Clenshaw and D. Elliot, Q.J. Mech. and Applied Math. 13, 300 (1960).

Captions for the Figures

Figure 1. The curves of neutral stability for a two layer model bounded above by an infinitely deep layer of unstratified fluid and below by rigid and free surfaces.

Figure 2. The dependence of the curves of neutral stability of the two layer model bounded below by a rigid surface on the Rayleigh number, R_U, of the upper layer.

(a) R_U small

(b) R_U large

Figure 3. The dependence of the curves of neutral stability of the two layer model bounded below by a free surface on the Rayleigh number R_U of the upper layer.

(a) R_U small

(b) R_U large

Figure 4. The variation of the critical wavenumber, a_c, on the Rayleigh number R_U for the two layer model with rigid and free lower boundaries.

Figure 5. The variation of the critical Rayleigh number, R_c, on the Rayleigh number R_U for the two-layer model with rigid and free lower boundaries.

Figure 6. The curve of neutral stability for the three layer model with unstratified layers above and below. For comparison the results of figure 1 are reproduced.

Figure 7. The dependence of the curves of neutral stability for the three layer model with R_B=0 upon the Rayleigh number R_T of the top layer.

Figure 8. The dependence of the curves of neutral stability for the three layer model with $R_B = -1$ upon the Rayleigh number R_T of the top layer.

Figure 9. The dependence of the curves of neutral stability for the three layer model with $R_B = -10^4$ upon the Rayleigh number R_T of the top layer.

Figure 10. The dependence of the curves of neutral stability for the three layer model with lower boundary conditions $W = DW = D^2W = 0$ upon the Rauleigh number R_T of the top layer.

Figure 11. The variation of the critical wavenumber a_c for the three layer model for several values of the Rayleigh number R_B of the bottom layer with the Rayleigh number R_T of the top layer.

Figure 12. The variation of the critical Rayleigh number R_c for the three layer model for several values of the Rayleigh number R_B of the bottom layer upon the Rayleigh number R_T of the top layer.

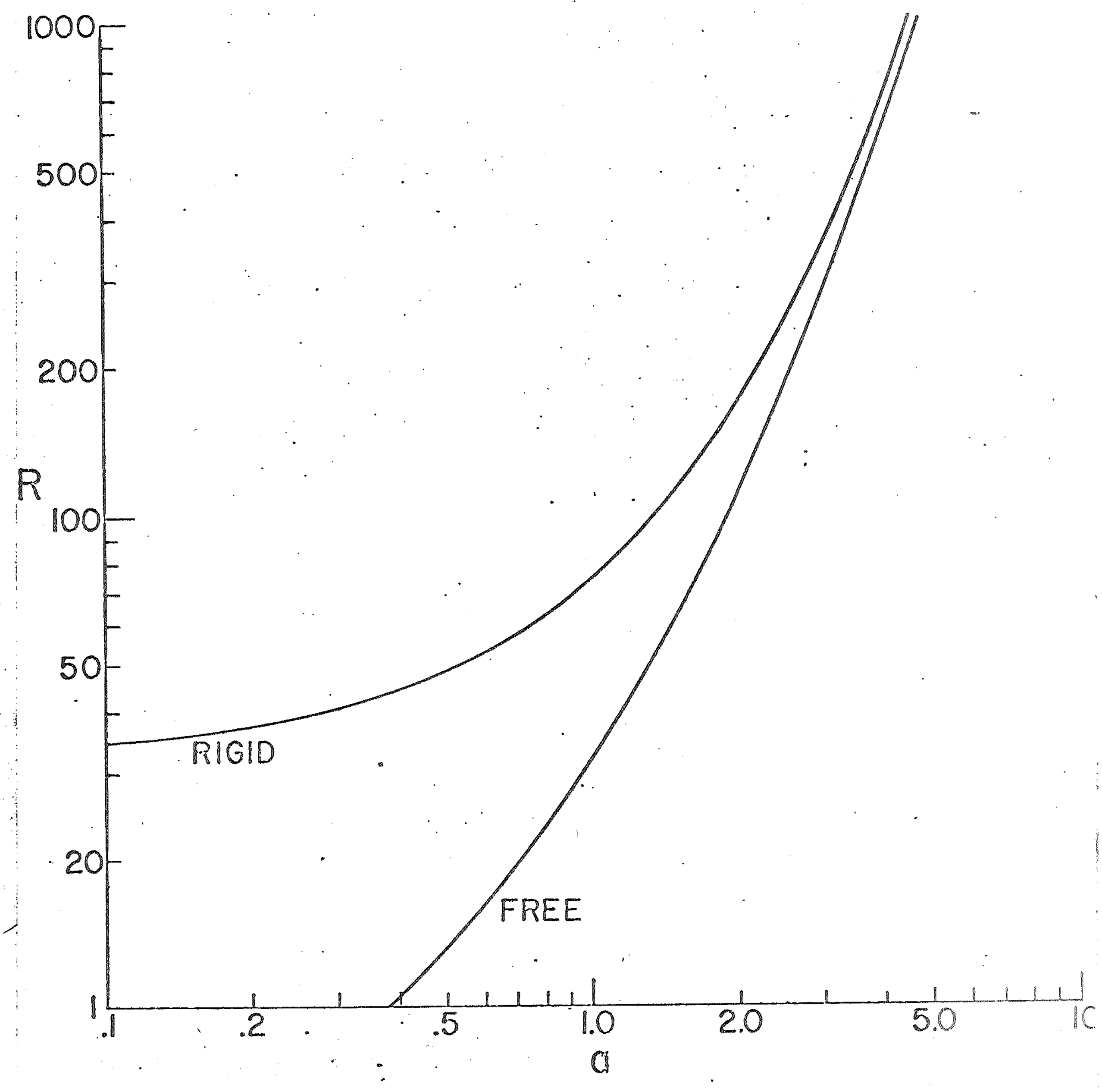


Figure 1

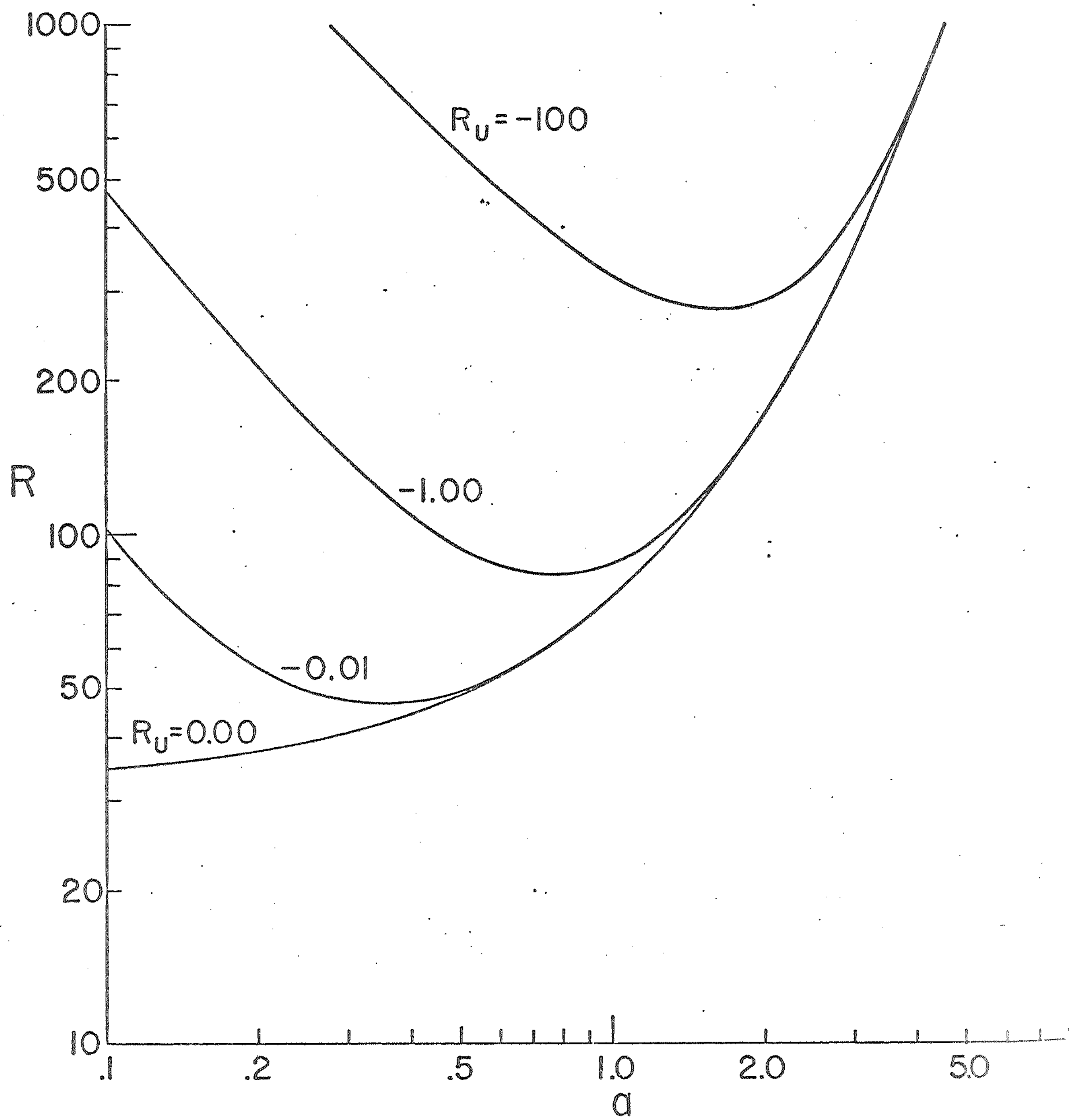


Figure 2a

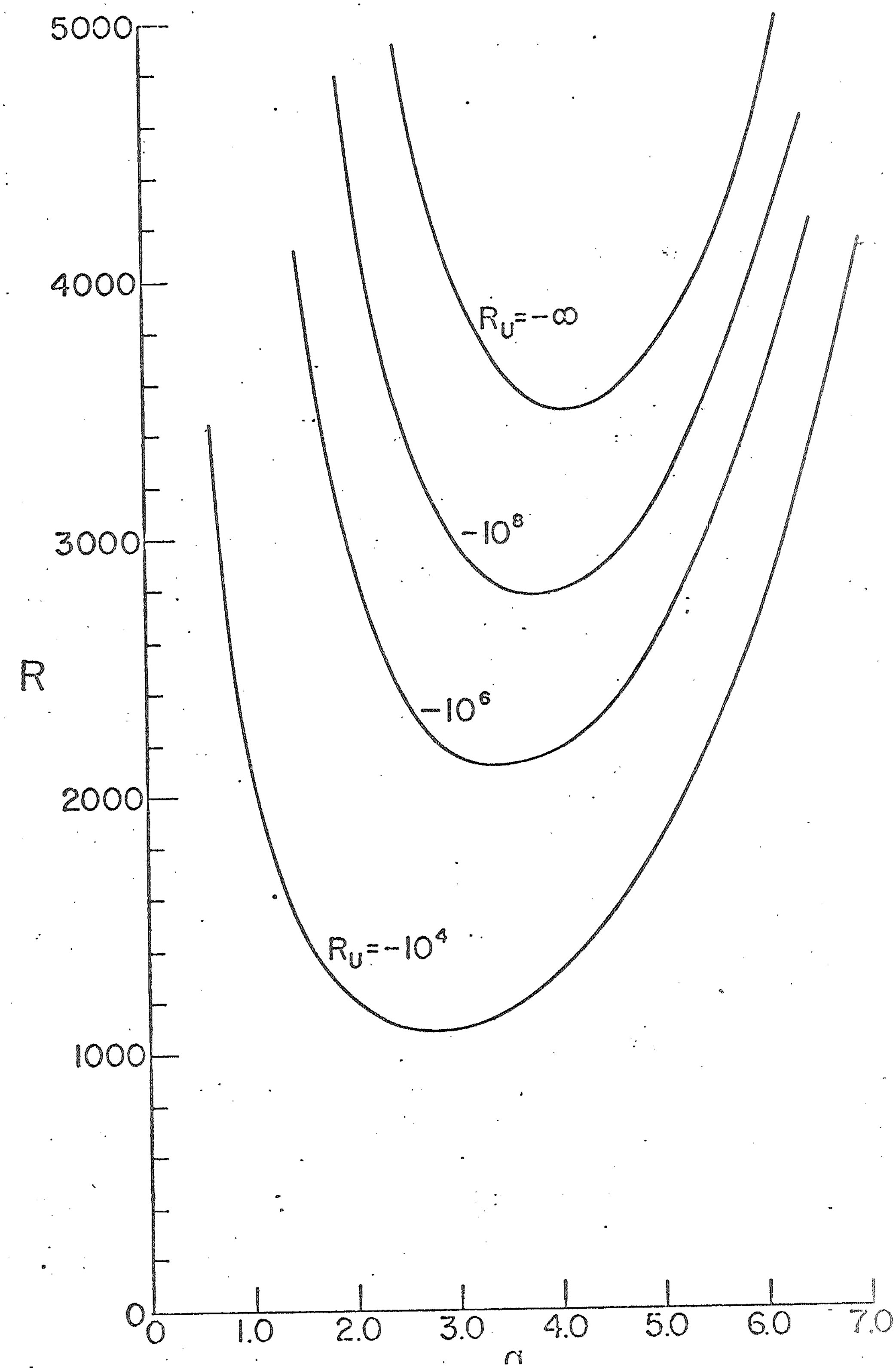

5000
4000
3000
2000
1000
0
R
R_U=-∞
-10^8
-10^6
R_U=-10^4
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
α

Figure 2b

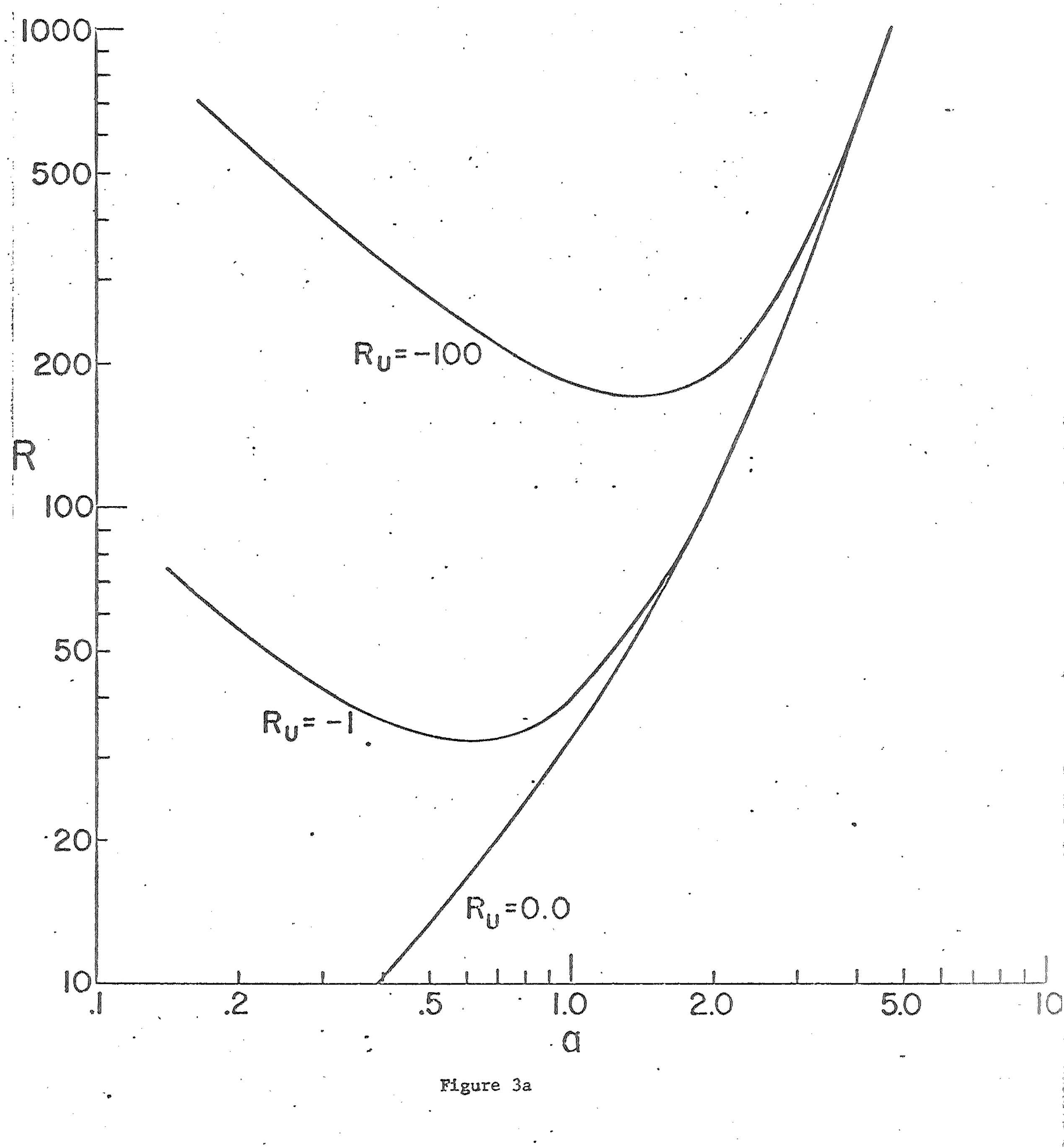
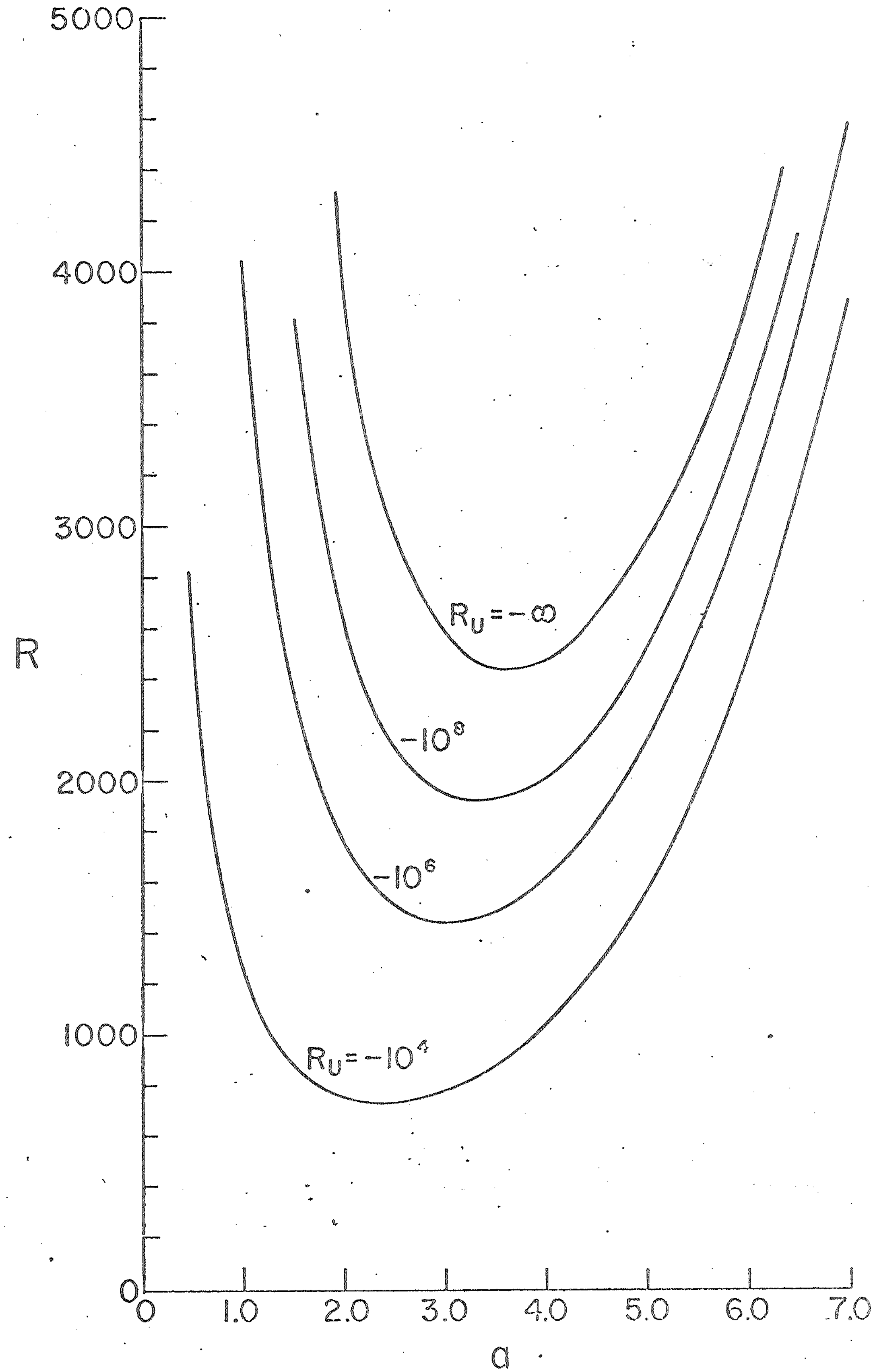


Figure 3a



5000
4000
3000
2000
1000
0
R
$R_U=-\infty$
-10^8
-10^6
$R_U=-10^4$
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
a

Figure 3b

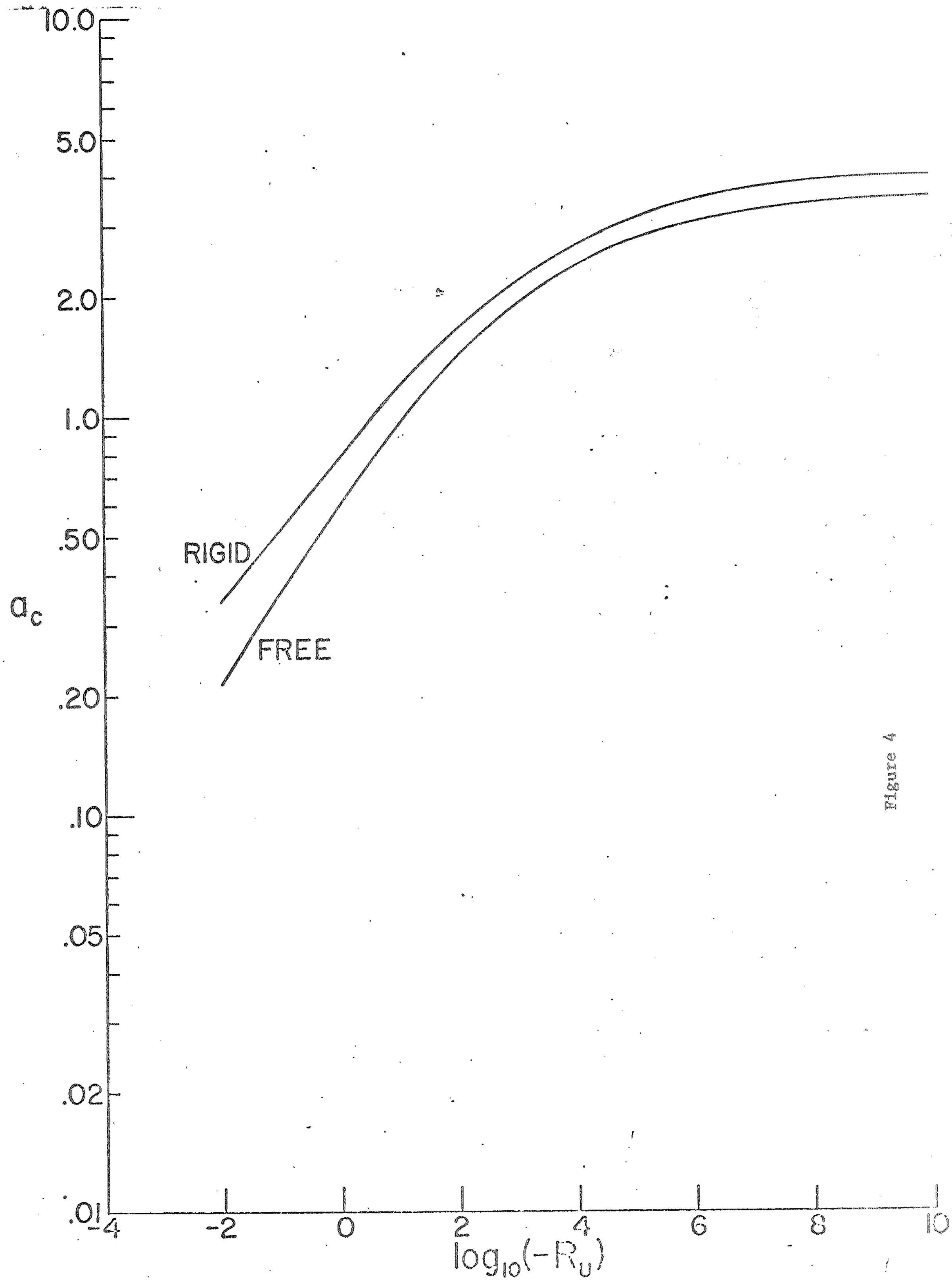


Figure 4

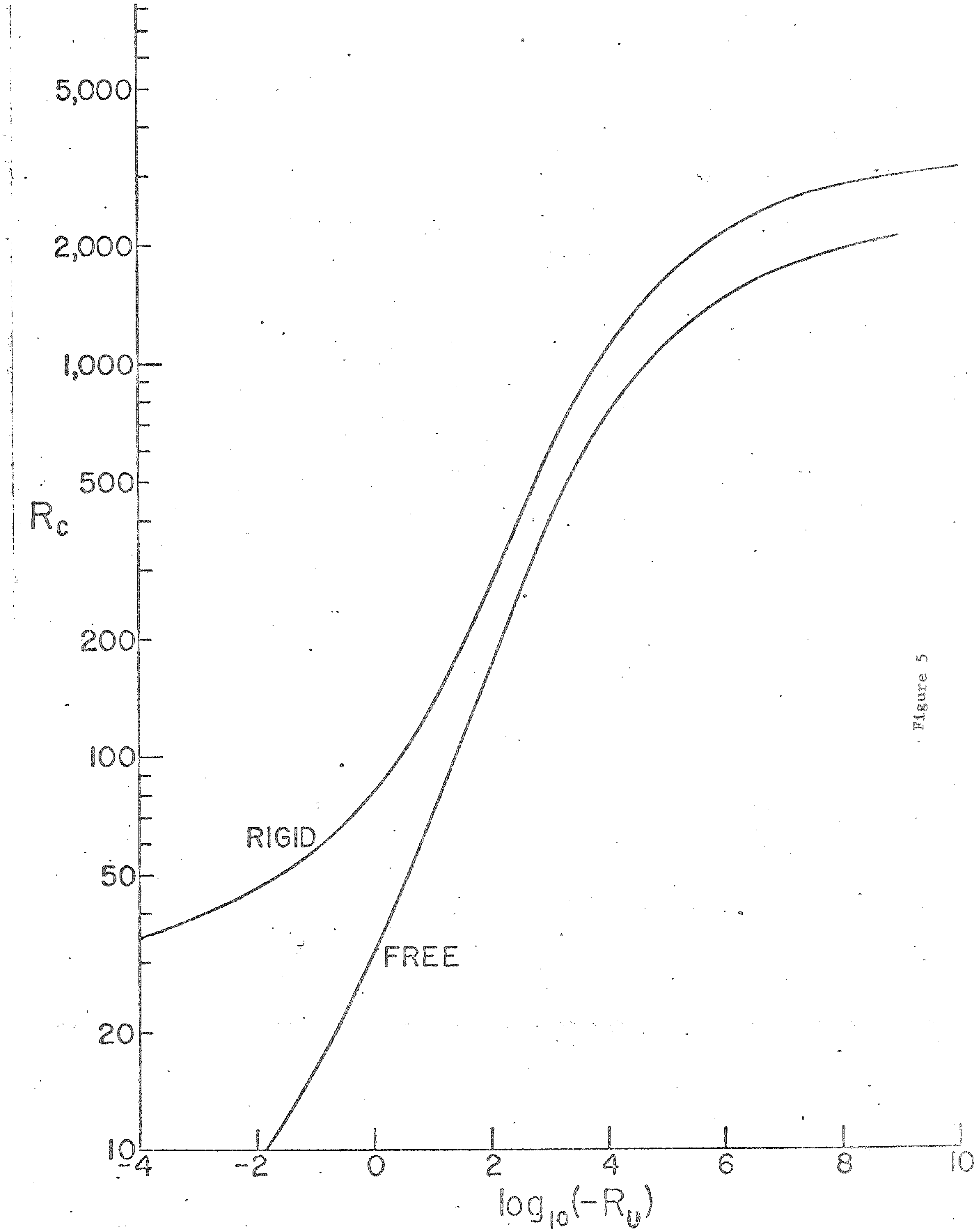

5,000
2,000
1,000
500
200
100
50
20
10
R_c
RIGID
FREE
-4
-2
0
2
4
6
8
10
$\log_{10}(-R_y)$

Figure 5

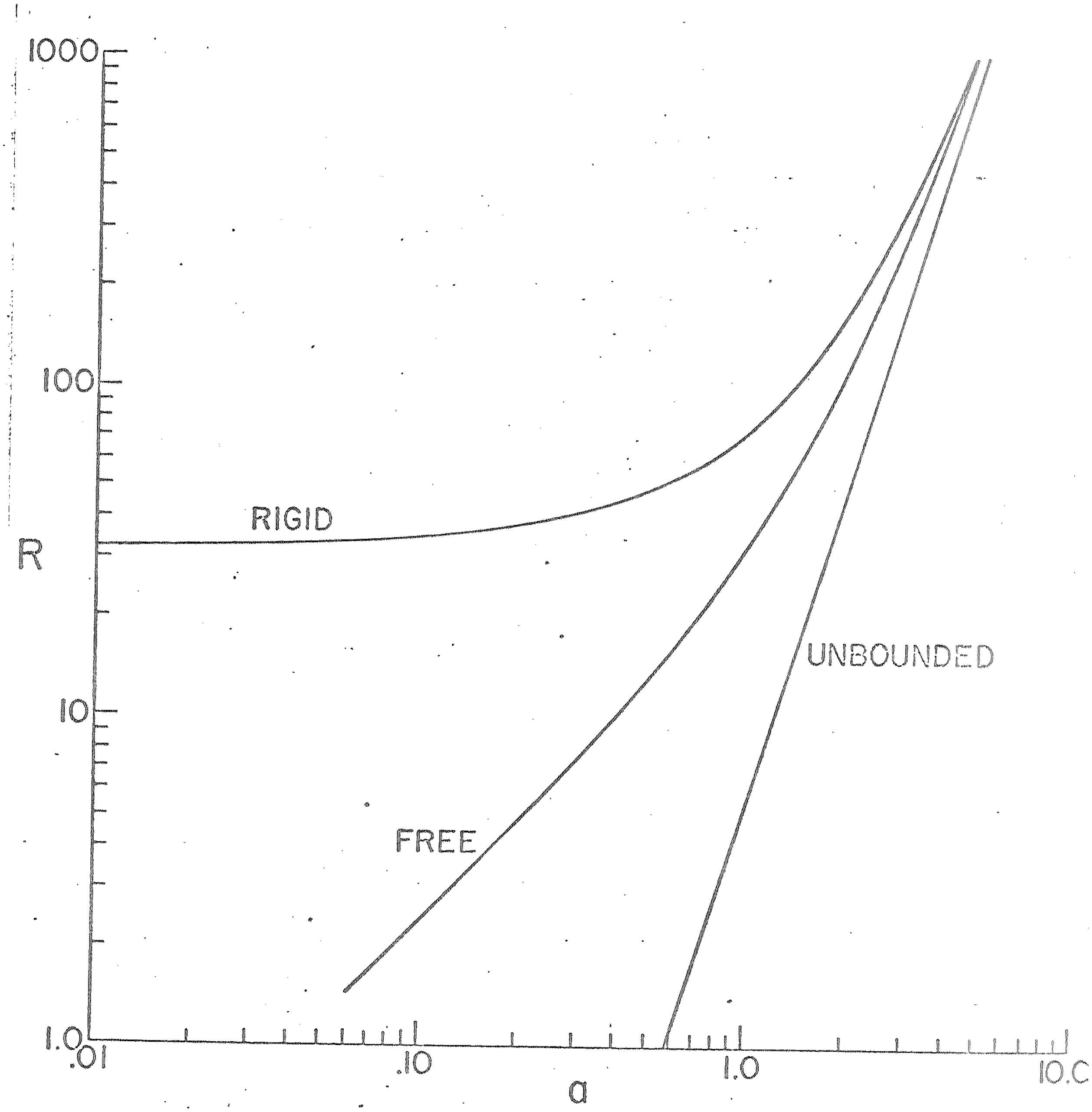


Figure 6

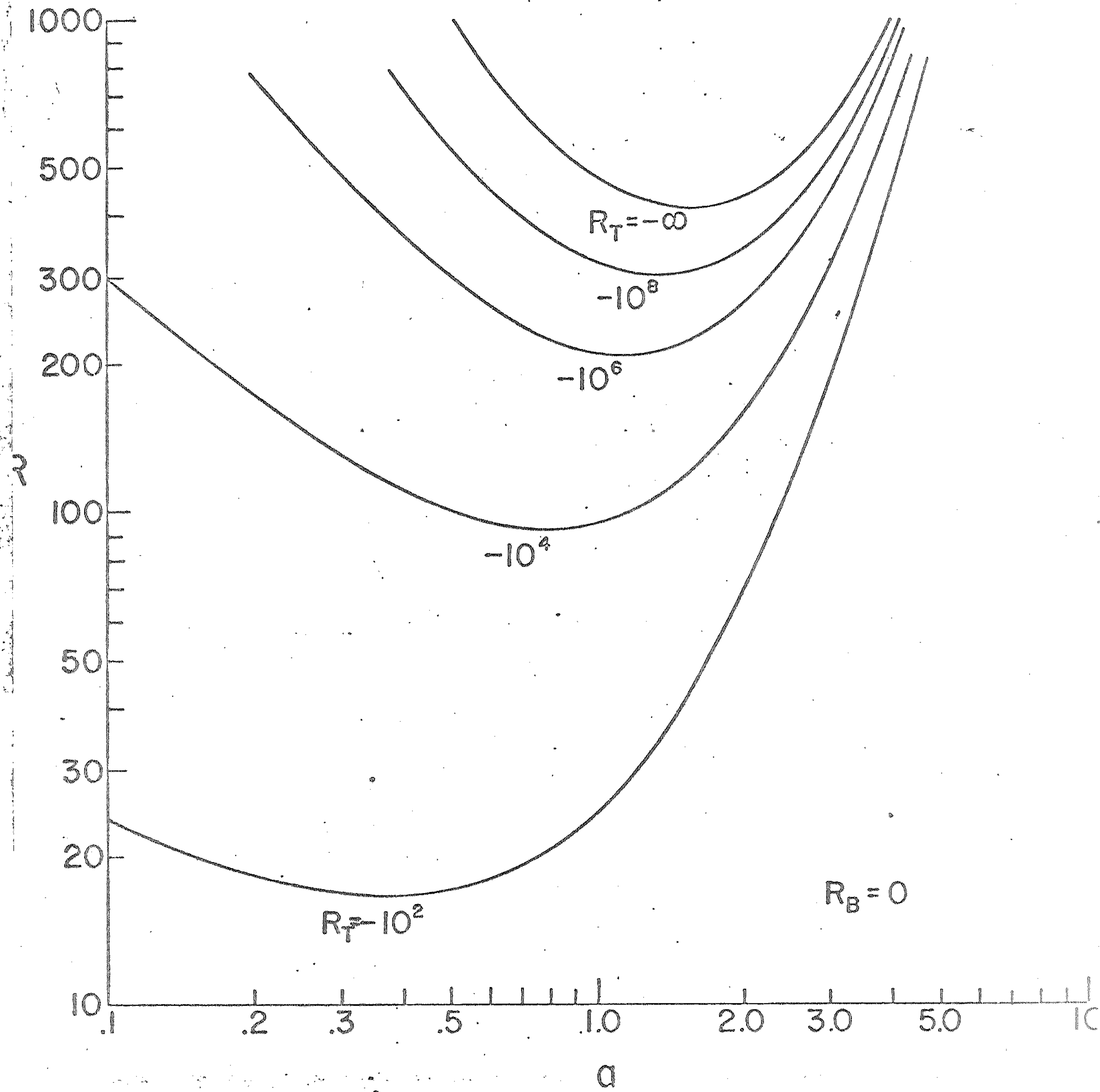
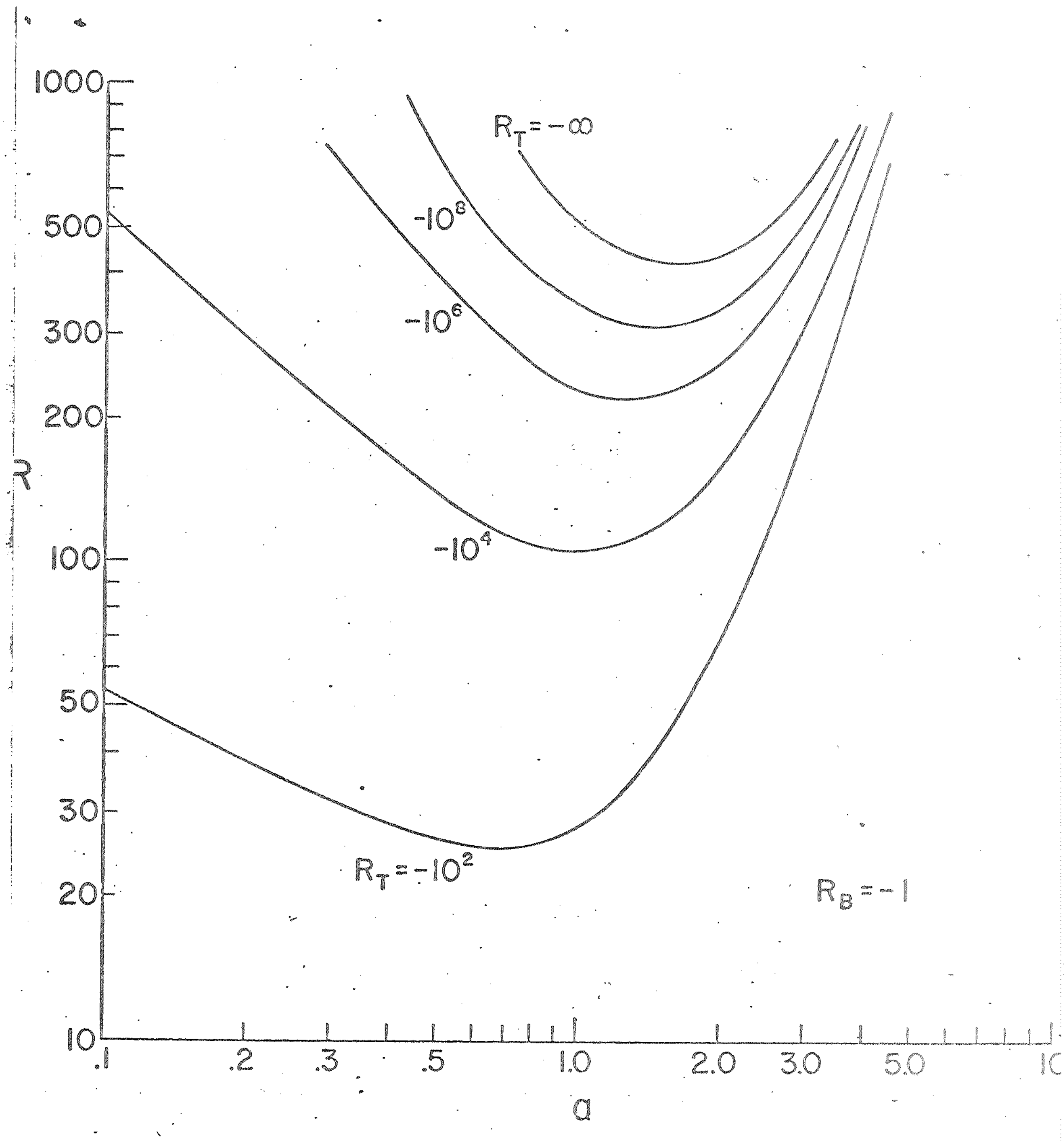
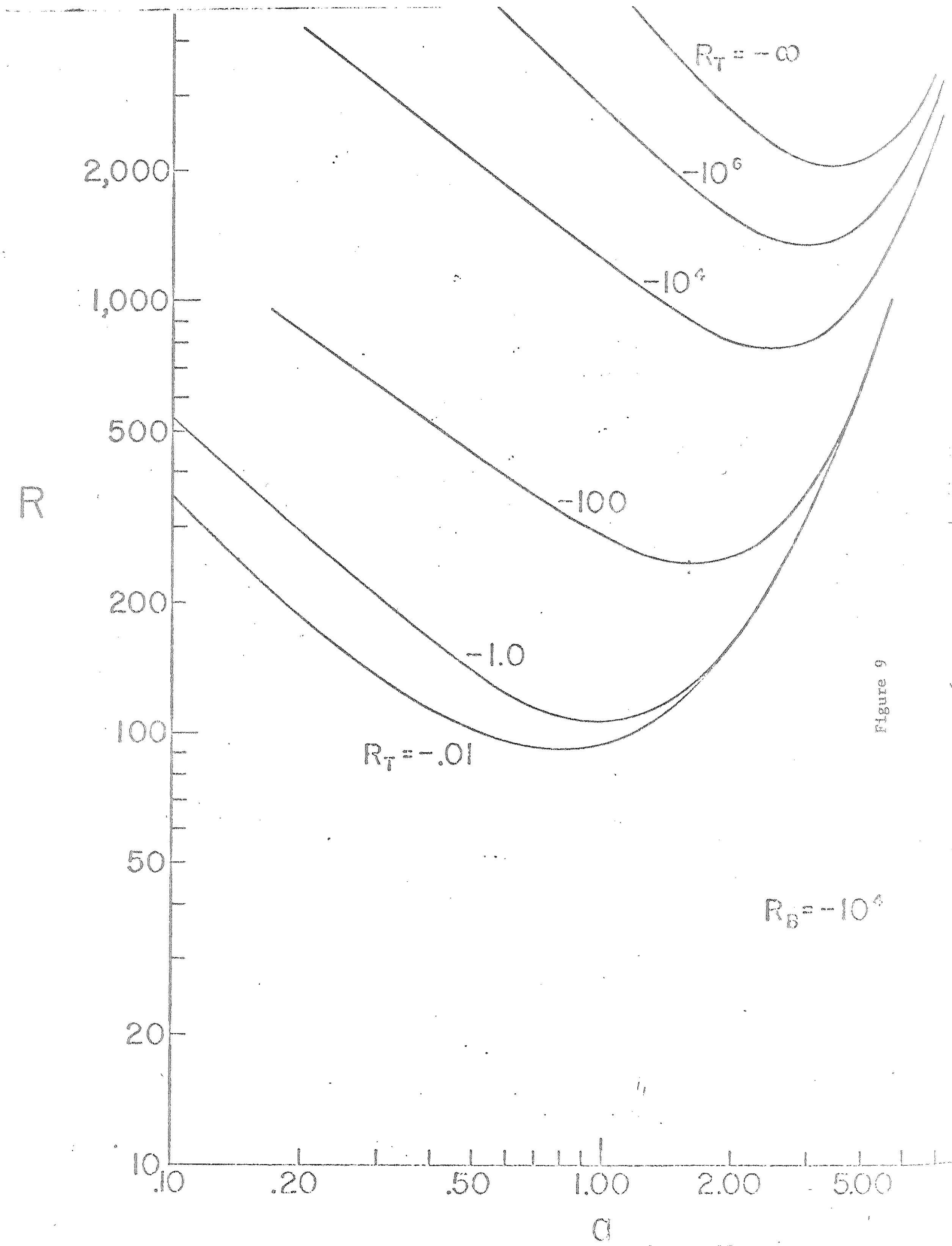

1000
500
300
200
100
50
30
20
10
R
$R_T = -\infty$
-10^8
-10^6
-10^4
$R_T = -10^2$
$R_B = 0$
.1
.2
.3
.5
1.0
2.0
3.0
5.0
a

Figure 7



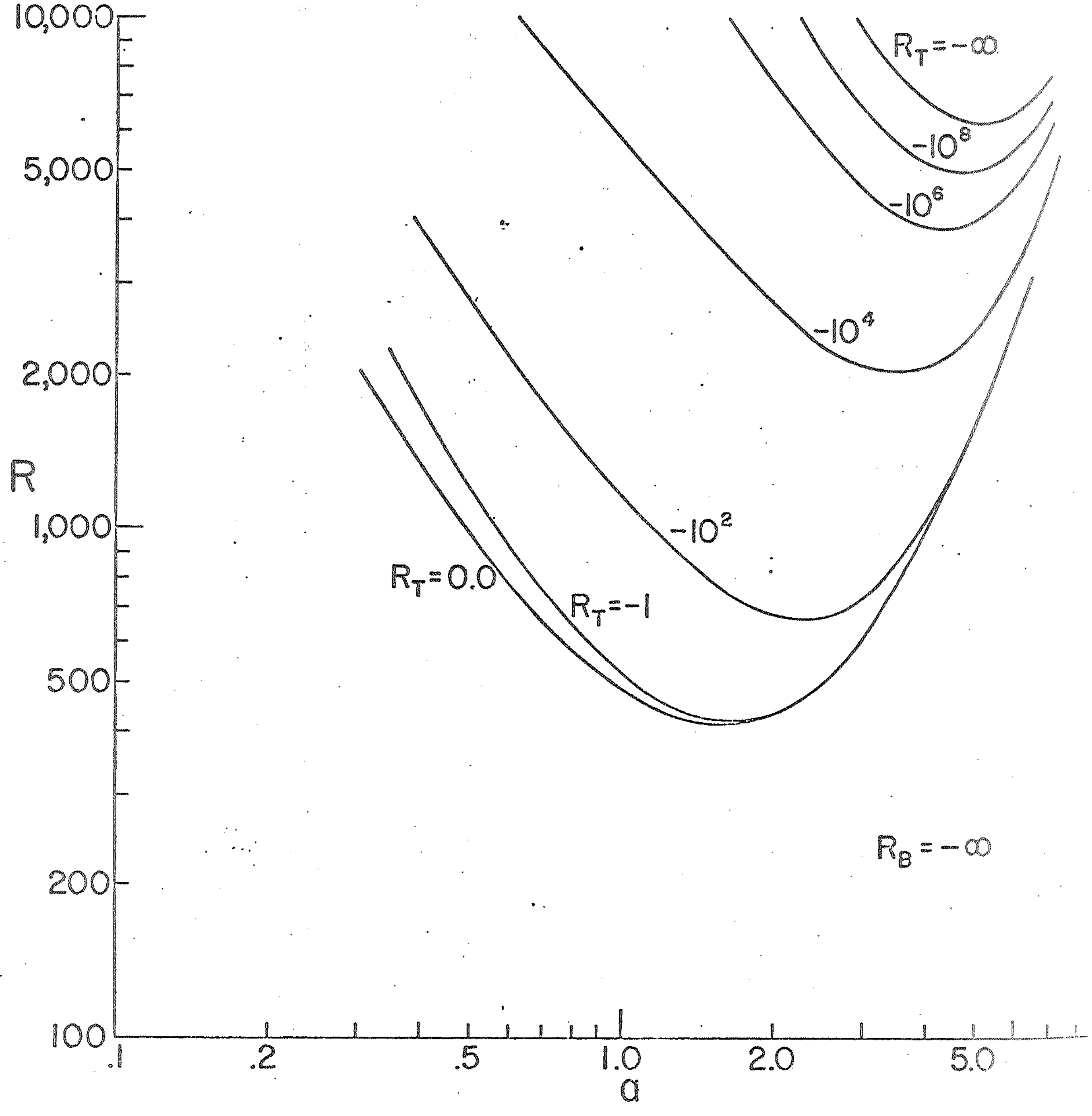
1000
500
300
200
R
100
50
30
20
10
.1
.2
.3
.5
1.0
2.0
3.0
5.0
10
a
$R_T = -\infty$
-10^8
-10^6
-10^4
$R_T = -10^2$
$R_B = -1$

Figure 8



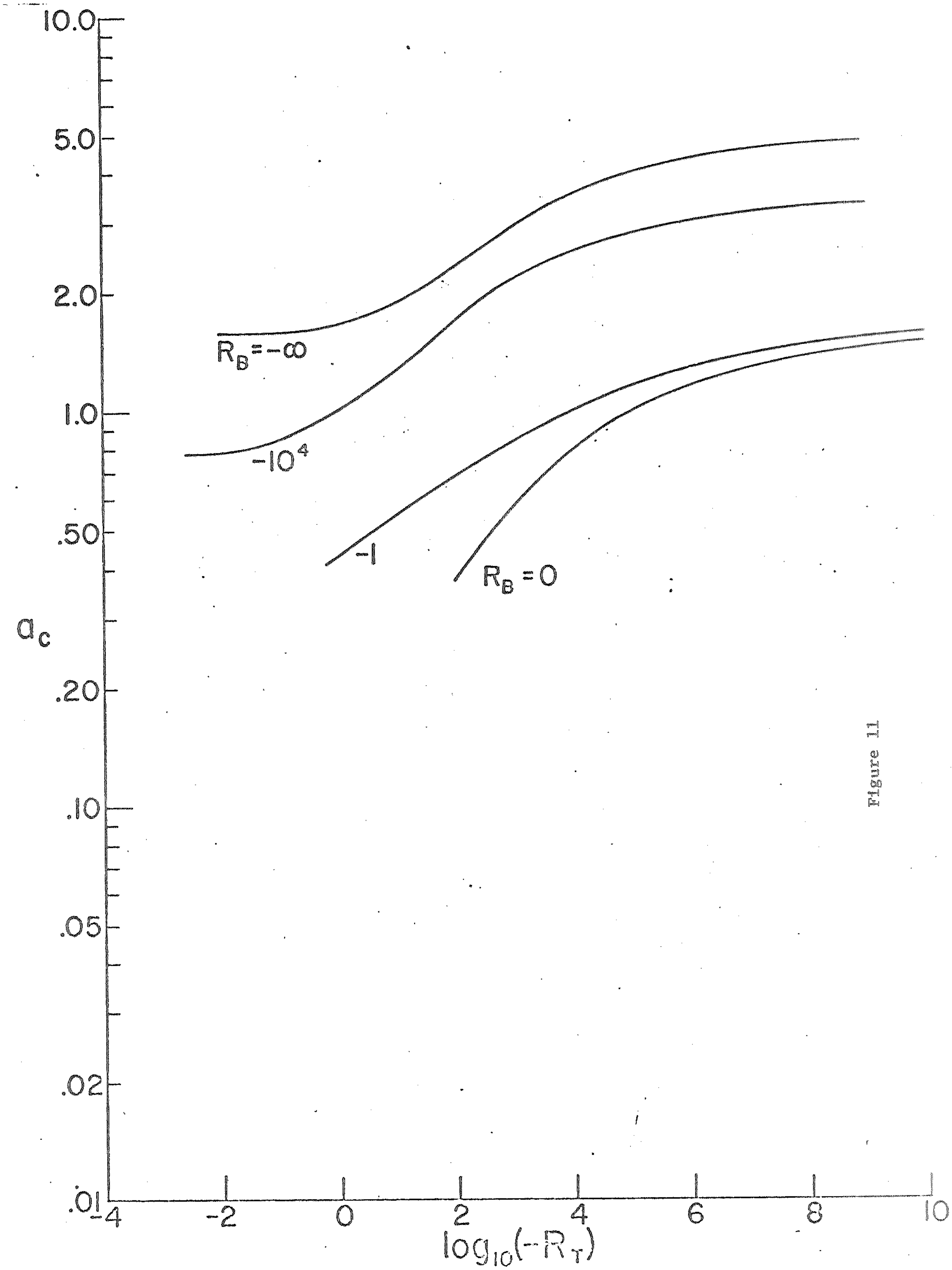
$R_T = -\infty$
-10^6
-10^4
-100
-1.0
$R_T = -.01$
$R_B = -10^4$
R
2,000
1,000
500
200
100
50
20
10
.10
.20
.50
1.00
2.00
5.00
a

Figure 9



10,000
5,000
2,000
R
1,000
500
200
100
.1
.2
.5
1.0
2.0
5.0
a
$R_T = -\infty$
-10^8
-10^6
-10^4
-10^2
$R_T = 0.0$
$R_T = -1$
$R_B = -\infty$

Figure 10



10.0
5.0
2.0
1.0
.50
.20
.10
.05
.02
.01
a_c
$R_B = -\infty$
-10^4
-1
$R_B = 0$
-4
-2
0
2
4
6
8
10
$\log_{10}(-R_T)$

Figure 11

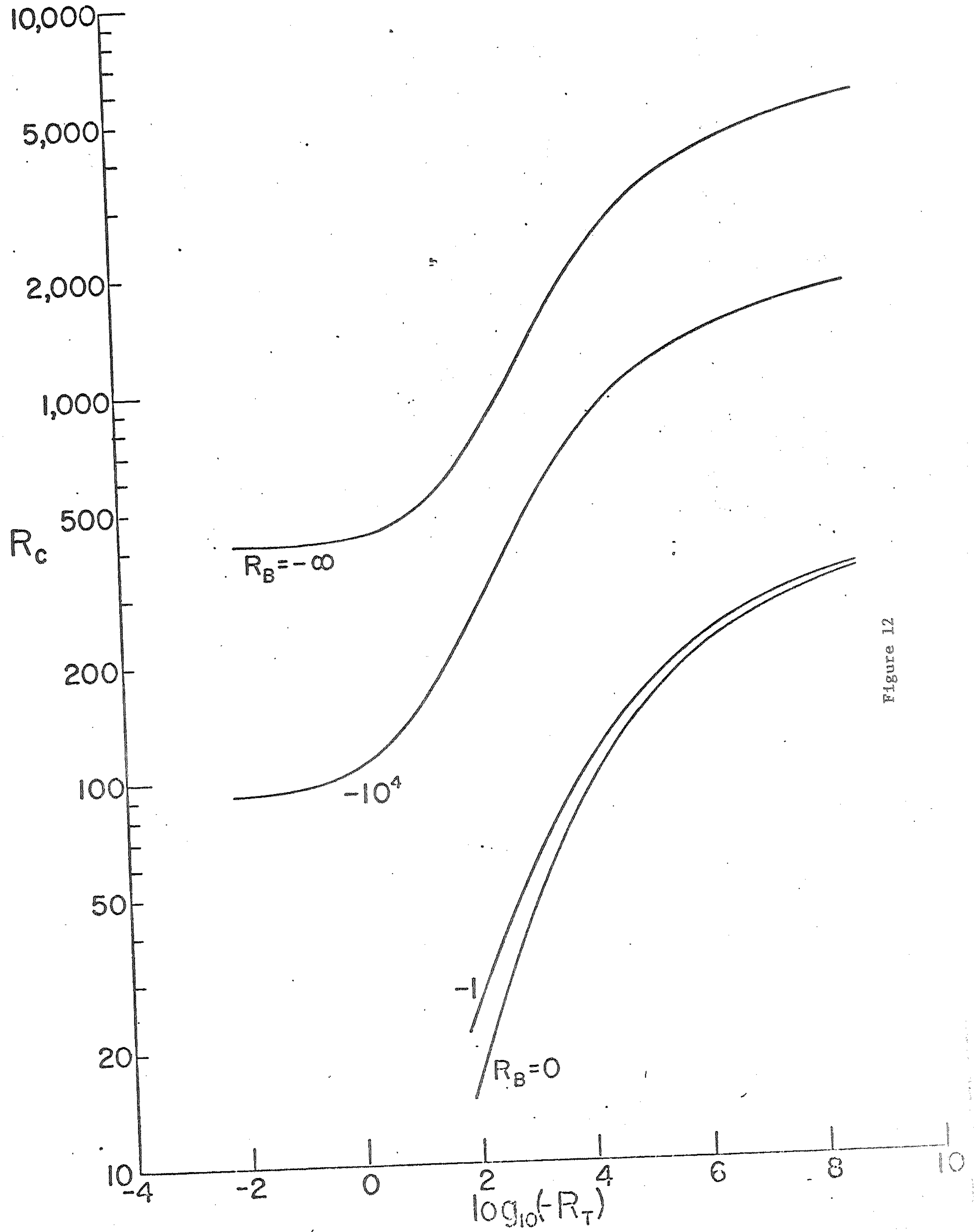

10,000
5,000
2,000
1,000
500
200
100
50
20
10
R_c
$R_B = -\infty$
-10^4
-1
$R_B = 0$
-4
-2
0
2
4
6
8
10
$\log_{10}(-R_T)$

Figure 12

The computations reported in this technical note were performed during the Summer of 1969. At that time the author was unaware that similar work was being pursued elsewhere. In particular the results reported here are in general agreement with results reported in:

Whitehead, J.A. and M.M. Chen, 1970: Thermal Stability and Convection of a thin fluid layer bounded by a Stably-Stratified region, J. Fluid Mech., 40, 549-576.

and

Oguva, Y. and H. Kondo, 1970: A linear stability of convective motion in a thermally unstable layer below a stable region. J. Meteor. Soc. Japan, 48, 204-215.