

NG2-21-002-008

N71-28548

NASA CR-119027

A METHOD FOR APPROXIMATING THE  
EIGENVALUES OF NON SELF-ADJOINT  
ORDINARY DIFFERENTIAL OPERATORS

by

John E. Osborn

TR 71-2

January 28, 1971

CASE FILE  
COPY

Technical Report  
Department of Mathematics  
University of Maryland  
College Park, Maryland  
20742

A METHOD FOR APPROXIMATING THE  
EIGENVALUES OF NON SELF-ADJOINT  
ORDINARY DIFFERENTIAL OPERATORS

by

John E. Osborn

§1. Introduction.

In this paper we develop a method for approximating the eigenvalues of a class of non self-adjoint ordinary differential operators.

There is an extensive literature dealing with the problem of finding upper and lower bounds for large classes of self-adjoint operators on a Hilbert space. This work covers many ordinary and partial differential operators of interest in applications. For the case of self-adjoint operators which are bounded below upper bounds are provided by the well-known Rayleigh-Ritz method. The method of intermediate problems introduced by Weinstein [36,37] provides lower bounds. This method was extended by Weinstein [38,39,40,41], Aronszajn and Weinstein [5], Aronszajn [3,4], Weinberger [34,35], Bazley [7], Bazley and Fox [8,9], and Stenger [28]. For an exposition of these results and further references see Gould [19] and Fox and Rheinboldt [17]. Fichera [15,16] has developed a method for finding lower bounds which is an alternative to the method of intermediate problems.

Throughout the paper we consider an eigenvalue problem of the form

$$\begin{cases} \tilde{L}f = Lf + Af = \lambda f, \\ U_k f = 0, \quad k = 1, \dots, m, \end{cases}$$

where

$$Lf = i^m f^{(m)} + cf, \quad c \text{ real,}$$

the  $m^{\text{th}}$  order boundary conditions  $U_k f = 0$  are self-adjoint relative to  $L$ , and  $A$  is any ordinary differential operator of order less than  $m$ . Thus we consider non self-adjoint operators which are lower order perturbations of the self-adjoint operator  $L$ . Assuming the eigenvalues and eigenvectors of  $L$  are known we consider the problem of approximating the eigenvalues of  $\tilde{L}$ . This problem is completely formulated in Section 2.

The main results of Section 3, Theorems 3.2 and 3.3, locate the eigenvalues of  $\tilde{L}$  in circles centered at the eigenvalues of  $L$ . These basic location results are used in the formulation of the method for obtaining improvable approximations which is developed in Section 4. These results involve a method for estimating the perturbation of eigenvalues which depends on certain estimates for the resolvent operator of  $L$ . Variations of this method have been used by many authors to estimate eigenvalues. The result of Theorem 3.2 is proved by Schwartz [27] for the case where  $A$  is a bounded operator. Agmon [1] used such estimates, especially estimates

similar to that given in Lemma 3.5 which depend on the higher dimensional version of the inequality in Lemma 3.3, to derive results on the angular distribution of eigenvalues of non self-adjoint elliptic partial differential operators. Clark [12] shows that the eigenvalues  $\mu_k$  of  $\tilde{L}$  are related to the eigenvalues  $\lambda_k$  of  $L$  by  $|\lambda_k - \mu_k| = O(|\lambda_k|^{j/m})$  where  $j$  is the order of  $A$ .

The main results of this paper are in Section 4 in which the Galerkin method is applied to  $\tilde{L}$  using the eigenvectors of  $L$  as a basis. We describe the method in the special case where the eigenvalues

$$\lambda_1 < \lambda_2 < \dots$$

of  $L$  are all positive and distinct. For  $n = 1, 2, \dots$  we use the eigenvalues of  $\tilde{L}_n = P_n \tilde{L}|_{S_n}$ , where  $S_n$  is the span of the first  $n$  eigenvectors of  $L$  and  $P_n$  is the projection onto  $S_n$ , as approximations for the eigenvalues of  $\tilde{L}$ . Assuming the circles discussed in the previous paragraph are mutually disjoint, there will be one eigenvalue of  $\tilde{L}$  in each circle: let  $\mu_p$  be the one in the  $p^{\text{th}}$  circle. Also, one of the eigenvalues of  $\tilde{L}_n$  will be in the  $p^{\text{th}}$  circle if  $p \leq n$ ; denote it by  $\eta_p(n)$ . Theorem 4.4 and the lemmas which follow it establish an estimate of the form

$$|\mu_p - \eta_p(n)| \leq \varepsilon_p(n), \quad n \geq p.$$

$\varepsilon_p(n)$  can be explicitly calculated in many cases.

Sections 5 and 6 contain the proof of two results from Section 4.

In Section 7 we present several examples which illustrate the results of the paper. For three of the examples we have computed the Galerkin eigenvalues and corresponding error estimates for  $n = 1, \dots, 20$ . Selected results from these computations are presented. A basic idea in the paper is that explicit estimates for the eigenvalues can be obtained if explicit values can be found for the constants in certain inequalities, such as (3.1), (4.3), and (4.4), which hold for all functions in some function space. For the examples treated in Section 7 values are given for the appropriate constants.

This paper is a continuation of previous work by the author [23,24]. In [23] a method was developed for the case of a bounded perturbation of an unbounded self-adjoint operator and in [24] the method was extended to cover, in its application to ordinary differential operators, perturbations of lower order which were subject to rather restrictive conditions. The purpose of this paper is to extend and improve this method so as to apply to a large class of ordinary differential operators which are lower order perturbations of the self-adjoint operator  $L$ . Example 7.1 was also treated in [24] and the numerical estimates obtained by the methods of this paper are substantially better than those reported there.

In [23] there is a discussion of other methods for approximating the eigenvalues of various types of non self-adjoint operators. In addition to these methods Vainikko [31,32] has obtained the following result. Consider the eigenvalue problem

$$(1.1) \quad \begin{cases} \tilde{L}f = f^{(m)} + \sum_{k=0}^{m-1} a_k(\tau) f^{(k)} = \lambda b(\tau) f, \\ U_k f = 0, \quad k = 1, \dots, m \end{cases}$$

Here  $U_k f = 0$ ,  $k = 1, \dots, m$  are arbitrary boundary conditions and we assume each of the coefficients is in some Sobolev space:  $b \in H_{r_0}$ ,  $a_k \in H_{r_k}$ ,  $k = 0, \dots, m-1$ . (See Section 2 for a definition of these spaces.) Let  $\phi_1, \phi_2, \dots$  be a sequence of polynomials of orders  $m, m+1, \dots$  which satisfy the boundary conditions and for  $n = 1, 2, \dots$  define the  $n$ -dimensional eigenvalue problem

$$(1.2) \quad \det(\tilde{L}\phi_\ell - \lambda M\phi_\ell, L\phi_k)_{\ell, k=1, \dots, n} = 0,$$

where  $Lf = f^{(m)}$  and  $Mf = bf$ . Let an eigenvalue  $\mu^0$  of (1.1) be a pole of order  $h$  of the resolvent of  $\tilde{L}$  with respect to  $M: (\tilde{L} - \lambda M)^{-1}$ . For  $n = 1, 2, \dots$  let  $\mu^n$  be an eigenvalue of (1.2). Then if  $\lim_{n \rightarrow \infty} \mu^n = \mu^0$  we have the asymptotic estimate

$$|\mu^n - \mu^0| = O(n^{-(r+r')/h}),$$

where  $r = \min_{0 \leq k \leq m-1} r_k$  and  $r' = \min_{0 \leq k \leq m-1} [r_k + m - k]$ . See [33]

for similar asymptotic error estimates for a more general class of operators.

The author wishes to thank two of his colleagues, David C. Lay and James M. Ortega, for helpful conversations on several topics in this paper, Messrs. Steven H. Leventhal and John B. Rountree for their significant help with the computations in Example 4.1, and Professor Jacob T. Schwartz for making part of [13] available to him.

This research was partially supported by NASA Grant NsG-398, NSF Grant GP 6291, and Grant DA-HC04-(AROD) 67-C-0062 with the Army Research Office, Durham. The computer time for this project was supported by NASA Grant NsG-398 to the Computer Science Center of the University of Maryland.

§2. Formulation of the Eigenvalue Problem.

For  $m$  a nonnegative integer let  $H_m = H_m[a,b]$  denote the  $m^{\text{th}}$  Sobolev space on the interval  $[a,b]$ , i.e., the set of all complex valued functions  $f$  such that  $f$  has  $m-1$  continuous derivatives,  $f^{(m-1)}$  is absolutely continuous, and  $f^{(m)}$  is square integrable on  $[a,b]$ . For  $k = 0, 1, \dots, m$  let  $|\cdot|_k$  be the seminorm on  $H_m$  defined by

$$|f|_k = \left( \int_a^b |f^{(k)}|^2 dt \right)^{1/2}.$$

Let  $\|\cdot\|_m$  be the norm on  $H_m$  defined by

$$\|f\|_m = \left( \sum_{k=0}^m |f|_k^2 \right)^{1/2}$$

and  $(\cdot, \cdot)_m$  be the inner product defined by

$$(f, g)_m = \sum_{k=0}^m \int_a^b f^{(k)} \overline{g^{(k)}} dt.$$

$H_m$  with this inner product is a Hilbert space and  $H_0 = L_2$ .

Let  $L$  be the  $m^{\text{th}}$  order differential operator defined by

$$Lf = i^m f^{(m)} + cf,$$

where  $c$  is a real constant and let

$$Af = \sum_{k=0}^j a_k(\tau) f^{(k)}$$

be any differential operator of order  $j < m$  with continuous coefficients. Let  $M = (M_{k\ell})$  and  $N = (N_{k\ell})$  be complex  $m \times m$  matrices such that  $\text{rank}(M, N) = m$  and define the



boundary forms  $U_1, \dots, U_m$  by

$$U_k f = \sum_{\ell=1}^m [M_{k\ell} f^{(\ell-1)}(a) + N_{k\ell} f^{(\ell-1)}(b)].$$

We wish to consider the eigenvalue problem determined by  $\tilde{L} = L + A$  and the boundary forms  $U_1, \dots, U_m$ , i.e., the problem of determining those complex values of  $\lambda$  for which there exists a nonzero function  $f \in H_m$  which satisfies

$$\begin{cases} \tilde{L}f = \lambda f, \\ U_k f = 0, \quad k = 1, \dots, m \end{cases}$$

If we define  $V_m$  by

$$V_m = \{f \mid f \in H_m, U_k f = 0, k = 1, \dots, m\}$$

the problem can be restated as that of finding values of  $\lambda$  for which there exist  $f \in V_m$ ,  $f \neq 0$  such that  $\tilde{L}f = \lambda f$ . For the remainder of the paper we will consider  $L$  and  $\tilde{L}$  to be restricted to  $V_m$ . The operator  $L$  is formally self-adjoint. We now assume the boundary conditions are self-adjoint relative to  $L$ , i.e.,

$$\sum_{q=1}^m (-1)^{q-1} M_{\ell q} \bar{M}_{k, m+1-q} = \sum_{q=1}^m (-1)^{q-1} N_{\ell q} \bar{N}_{k, m+1-q},$$

$$\ell, k = 1, \dots, m.$$

Then  $L$  will be a self-adjoint operator in the space  $L_2$  with domain  $V_m$ :

$$(Lf, g)_0 = (f, Lg)_0$$

for all  $f, g \in V_m$ . The spectrum of  $L$  will consist of a countable set of real eigenvalues, each with geometric multiplicity less than or equal to  $m$ , which has no finite limit point. Numbering them in increasing order by magnitude and taking account of geometric multiplicities we have

$$|\lambda_1| \leq |\lambda_2| \leq \dots \uparrow \infty.$$

The eigenvectors  $x_1, x_2, \dots$  can be chosen to be orthonormal in  $L_2$ ; they are complete in  $L_2$ . The spectrum of  $\tilde{L}$  consists of eigenvalues, each with geometric multiplicity less than or equal to  $m$ , with no finite limit point.

Assuming the eigenvalues and eigenvectors of  $L$  are known we consider the problem of approximating the eigenvalues of  $\tilde{L}$ .

Throughout the paper we denote the spectrum, the resolvent set, and the resolvent of an operator  $B$  by  $\sigma(B)$ ,  $\rho(B)$ , and  $R_\lambda(B)$  respectively.

### §3. Basic Location Theorems.

In this section we present a result which gives bounds on the perturbations of the eigenvalues of  $L$  when  $L$  is perturbed by the operator  $A$ . This result depends on certain estimates for the resolvent operator  $R_\lambda(L) = (\lambda - L)^{-1}$ .

Lemma 3.1. If  $\lambda \in \rho(L)$  and  $f \in L_2$ , then

$$|R_\lambda(L)f|_0 \leq |f|_0 (\text{dist}(\lambda, \sigma(L)))^{-1}$$

where  $\text{dist}(\lambda, \sigma(L))$  denotes the distance from  $\lambda$  to  $\sigma(L)$ .

Proof. Since

$$R_\lambda(L)f = \sum_{k=1}^{\infty} (\lambda - \lambda_k)^{-1} (f, x_k) x_k$$

we see that

$$\begin{aligned} |R_\lambda(L)f|_0^2 &\leq \sum_{k=1}^{\infty} |(\lambda - \lambda_k)^{-1} (f, x_k)|^2 \\ &\leq \max_k |(\lambda - \lambda_k)^{-2}| \sum_{k=1}^{\infty} |(f, x_k)|^2. \end{aligned}$$

This gives the result.

Lemma 3.2. If  $\lambda \in \rho(L)$  and  $f \in L_2$ , then

$$|R_\lambda(L)f|_m \leq \sup_k |(\lambda_k - c)(\lambda_k - \lambda)^{-1}| |f|_0.$$

Proof. For  $f \in L_2$  we have

$$\begin{aligned} f &= (\lambda - L)(\lambda - L)^{-1}f \\ &= \lambda(\lambda - L)^{-1}f - L(\lambda - L)^{-1}f \end{aligned}$$

$$= \lambda(\lambda-L)^{-1}f - i^m[(\lambda-L)^{-1}f]^{(m)} - c(\lambda-L)^{-1}f.$$

Hence

$$\begin{aligned} |R_\lambda(L)f|_m^2 &= |(\lambda-c)(\lambda-L)^{-1}f - f|_0^2 \\ &= \sum_{k=1}^{\infty} |(\lambda_k-c)(\lambda_k-\lambda)^{-1}(f, x_k)|^2 \\ &\leq \sup_k |(\lambda_k-c)(\lambda_k-\lambda)^{-1}|^2 |f|_0^2. \end{aligned}$$

We will need the following inequality. Its proof can be found in Agmon [2].

Lemma 3.3. There are constants  $\gamma$  and  $\delta$  depending only on  $a$ ,  $b$ , and  $m$  such that

$$(3.1) \quad |f|_k \leq \gamma |f|_0^{1-\frac{k}{m}} |f|_m^{\frac{k}{m}} + \delta |f|_0$$

for all  $f \in H_m$  and  $k = 0, 1, \dots, m$ .

Lemma 3.4. If  $\lambda \in \rho(L)$  and  $k = 0, 1, \dots, m$ , then

$$\begin{aligned} |R_\lambda(L)f|_k &\leq |f|_0 (\text{dist}(\lambda, \sigma(L)))^{-1} \times \\ &\quad [\delta + \gamma (\text{dist}(\lambda, \sigma(L)) \sup_q |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^{k/m}] \end{aligned}$$

for all  $f \in L_2$  where  $\gamma$  and  $\delta$  are the constants introduced in Lemma 3.3.

Proof. Using Lemmas 3.1, 3.2, and 3.3 we obtain

$$\begin{aligned} |R_\lambda(L)f|_k &\leq \gamma |R_\lambda(L)f|_0^{1-\frac{k}{m}} |R_\lambda(L)f|_m^{\frac{k}{m}} + \delta |(\lambda-L)^{-1}f|_0 \\ &\leq |f|_0 (\text{dist}(\lambda, \sigma(L))) \times \\ &\quad [\delta + \gamma (\text{dist}(\lambda, \sigma(L)) \sup_q |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^{k/m}]. \end{aligned}$$

Lemma 3.5. If  $\lambda \in \rho(L)$  and  $k = 0, \dots, m$  then

$$\|R_\lambda(L)f\|_k \leq \|f\|_0 (\text{dist}(\lambda, \sigma(L)))^{-1} \times \\ \left\{ \sum_{\ell=0}^k [\delta + \gamma(\text{dist}(\lambda, \sigma(L)) \sup_q |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^\ell]^{2/m} \right\}^{1/2}$$

for all  $f \in L_2$ .

Proof. This is an immediate consequence of Lemma 3.4.

Remark. The values of  $\gamma$  and  $\delta$  in Lemma 3.3 are such that (3.1) holds for all  $f \in H_m$ . In Lemmas 3.4 and 3.5 we only used this inequality for functions of the form  $(\lambda - L)^{-1}f$  for  $f \in L_2$ , i.e., for functions in  $V_m$ . Hence when considering a particular set of boundary conditions it may be possible to choose smaller values for  $\gamma$  and  $\delta$ . See Section 7 for further remarks on the choice of  $\gamma$  and  $\delta$ .

Lemma 3.6. If  $f \in H_j$ , where  $j$  is the order of  $A$ , and  $\lambda \in \rho(L)$  then

$$\|R_\lambda(L)Af\|_j \leq \left( \sum_{\ell=0}^j \max_{a \leq t \leq b} |a_\ell|^2 \right)^{1/2} \|f\|_j (\text{dist}(\lambda, \sigma(L)))^{-1} \times \\ \left\{ \sum_{\ell=0}^j [\delta + \gamma(\text{dist}(\lambda, \sigma(L)) \sup_q |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^\ell]^{2/m} \right\}^{1/2}.$$

Proof. This follows directly from Lemma 3.5 and the inequality

$$\|Af\|_0 \leq \left( \sum_{\ell=0}^j \max |a_\ell|^2 \right)^{1/2} \|f\|_j$$

for all  $f \in H_j$ .

Let

$$h_j(\lambda) = (\text{dist}(\lambda, \sigma(L)))^{-1} \times \\ \left\{ \sum_{\ell=0}^j [\delta + \gamma(\text{dist}(\lambda, \sigma(L)) \sup_q |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^{\ell/m}]^2 \right\}^{1/2}$$

and

$$\beta = \left( \sum_{\ell=0}^j \max |a_\ell|^2 \right)^{1/2}.$$

Theorem 3.1. If  $\lambda$  is an eigenvalue of  $\tilde{L}$  then either

$$(3.2) \quad \beta h_j(\lambda) \geq 1$$

or  $\lambda \in \sigma(L)$ .

Proof. Suppose, for the sake of contradiction, that  $\beta h_j(\lambda) < 1$  and  $\lambda \in \rho(L)$ . From Lemma 3.6 we have

$$\|R_\lambda(L)Af\|_j \leq \beta h_j(\lambda) \|f\|_j$$

for all  $f \in H_j$ . This, together with  $\beta h_j(\lambda) < 1$ , implies that  $(\lambda - L)^{-1}A: H_j \rightarrow H_j$  has norm less than 1. Hence  $I_{H_j} - (\lambda - L)^{-1}A$  is invertible. Thus the restriction of this operator to  $V_m$  will be one-to-one. Since

$$\lambda I_{V_m} - \tilde{L} = \lambda I_{V_m} - L - A|_{V_m} \\ = (\lambda I_{V_m} - L)[I_{V_m} - (\lambda I_{V_m} - L)^{-1}A|_{V_m}]$$

this implies that  $\lambda - \tilde{L}$  is one-to-one and hence that  $\lambda$  is not an eigenvalue of  $\tilde{L}$  which contradicts the hypothesis.

This completes the proof.

Inequality (3.2) is the basic result of this section. We will now show more explicitly what it implies about the location of the eigenvalues of  $\tilde{L}$ . Let  $p$  be such that  $\text{dist}(\lambda, \sigma(L)) = |\lambda - \lambda_p|$ . Let  $\lambda_p^+$  be the next element in the sequence  $\{\lambda_p\}$  to the right of  $\lambda_p$  if such an element exists and let  $\lambda_p^-$  be the next element to the left if such an element exists. With  $p$  chosen in this way (3.2) implies

$$(3.3) \quad |\lambda - \lambda_p| \leq \beta \left\{ \sum_{\ell=0}^j [\delta + \gamma (|\lambda - \lambda_p| \sup |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|)^{\ell/m}]^2 \right\}^{1/2}.$$

If

$$(3.4) \quad \sup |(\lambda_q - c)(\lambda_q - \lambda)^{-1}| = |(\lambda_p - c)(\lambda_p - \lambda)^{-1}|$$

this becomes

$$(3.5) \quad |\lambda - \lambda_p| \leq \beta \left\{ \sum_{\ell=0}^j [\delta + \gamma |\lambda_p - c|^{\ell/m}]^2 \right\}^{1/2}.$$

In case (3.4) is not satisfied we derive a different bound for  $\sup |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|$ . It is clearly sufficient to consider those values of  $q$  which belong to

$$A = \{q \mid |\lambda_q - c| > |\lambda_p - c|\}.$$

Let  $B = \{q \mid \lambda_q < \lambda_p\}$  and  $C = \{q \mid \lambda_q > \lambda_p\}$ . Suppose  $q \in A \cap B$ . Then

$$|(\lambda_q - c)(\lambda_q - \lambda)^{-1}| \leq |\lambda_q - c| |\lambda_q - (\lambda_p + \lambda_p^-)/2|^{-1}.$$

A study of the function  $|x - c| |x - (\lambda_p + \lambda_p^-)/2|^{-1}$  for

$x < (\lambda_p^+ + \lambda_p^-)/2$  shows that

$$|(\lambda_q - c)(\lambda_q - \lambda)^{-1}| \leq \max(1, 2(c - \lambda_p^-)(\lambda_p - \lambda_p^-)^{-1}).$$

Similarly we find that

$$|(\lambda_q^+ - c)(\lambda_q - \lambda)^{-1}| \leq \max(1, 2(\lambda_p^+ - c)(\lambda_p^+ - \lambda_p)^{-1})$$

for  $q \in A \cap C$ . Combining these estimates we obtain

$$(3.6) \quad \sup |\lambda_q - c| |\lambda_q - \lambda|^{-1} \leq \begin{cases} \max(1, 2(c - \lambda_p^-)(\lambda_p - \lambda_p^-)^{-1}) \\ \quad \text{if } \lambda_p^+ \text{ does not exist,} \\ \max(1, 2(\lambda_p^+ - c)(\lambda_p^+ - \lambda_p)^{-1}) \\ \quad \text{if } \lambda_p^- \text{ does not exist,} \\ \max(2(c - \lambda_p^-)(\lambda_p - \lambda_p^-)^{-1}, 2(\lambda_p^+ - c)(\lambda_p^+ - \lambda_p)^{-1}) \\ \quad \text{if } \lambda_p^- \text{ and } \lambda_p^+ \text{ both exist.} \end{cases} \\ \equiv g(p).$$

In the frequently occurring special case where

$0 < \lambda_1 < \lambda_2 < \dots$  and  $c \geq 0$  this bound reduces to

$$g(p) = \begin{cases} \max(1, 2(\lambda_2 - c)(\lambda_2 - \lambda_1)^{-1}), & p = 1, \\ \max(2(c - \lambda_{p-1})(\lambda_p - \lambda_{p-1})^{-1}, 2(\lambda_{p+1} - c)(\lambda_{p+1} - \lambda_p)^{-1}), & p \geq 2. \end{cases}$$

If  $c = 0$  we have  $g(p) = 2\lambda_{p+1}(\lambda_{p+1} - \lambda_p)^{-1}$ . Combining (3.3) and (3.6) we get

$$(3.7) \quad |\lambda - \lambda_p| \leq \beta \left\{ \sum_{\ell=0}^j [\delta + \gamma(|\lambda - \lambda_p| g(p))^{\ell/m}]^2 \right\}^{1/2}.$$

We can solve (3.7) for  $|\lambda - \lambda_p|$ , i.e., (3.7) implies



$$|\lambda - \lambda_p| \leq \tau_p$$

where  $\tau_p$  is the unique positive solution of

$$\tau = \beta \left\{ \sum_{\ell=0}^j [\delta + \gamma(\tau g(p))^{\ell/m}]^2 \right\}^{1/2}.$$

Let

$$(3.8) \quad r_p = \max(\beta \left\{ \sum_{\ell=0}^j [\delta + \gamma |\lambda_p - c| \ell/m]^2 \right\}^{1/2}, \tau_p).$$

These results are summarized in

Theorem 3.2. If  $\lambda$  is an eigenvalue of  $\tilde{L}$  then for some  $p$  we have

$$|\lambda - \lambda_p| \leq r_p.$$

Thus we have shown that the eigenvalues of  $\tilde{L}$  are contained in  $\bigcup_{\ell=1}^{\infty} C_{\ell}$ , where

$$C_{\ell} = \{\lambda \mid |\lambda - \lambda_{\ell}| \leq r_{\ell}\}.$$

It will be useful to have an explicit upper bound for  $\tau_p$ . First suppose

$$2g(p)\beta(\delta+\gamma) \leq 1.$$

Then

$$(g(p))^{-1} \geq \beta \left\{ \sum_{\ell=0}^j [\delta + \gamma(\tau g(p))^{\ell/m}]^2 \right\} \Big|_{\tau=(g(p))^{-1}}$$

which implies  $\tau_p \leq (g(p))^{-1}$ . Hence

$$\tau_p \leq \beta(\delta+\gamma)(1+j)^{1/2}.$$

Next suppose

$$2g(p)\beta(\delta+\gamma) > 1.$$

In this case we have  $\tau_p > (g(p))^{-1}$  and hence

$$\tau_p < \beta(\delta+\gamma)(\tau_p g(p))^{j/m} (j+1)^{1/2}$$

which implies

$$\tau_p < [\beta(\delta+\gamma)(j+1)^{1/2}]^{m/(m-j)} (g(p))^{j/(m-j)}.$$

Combining these two estimates we get

(3.9)

$$\tau_p \leq \max(\beta(\delta+\gamma)(1+j)^{1/2}, [\beta(\delta+\gamma)(j+1)^{1/2}]^{m/(m-j)} (g(p))^{j/(m-j)}).$$

Next we wish to count the number of eigenvalues of  $\tilde{L}$  in each circle  $C_\ell$ ; this result will be contained in Theorem 3.3. To this end we will need to consider the family of operators

$$L_t \equiv L + tA, \quad 0 \leq t \leq 1.$$

Clearly  $L_0 = L$  and  $L_1 = \tilde{L}$ . If  $\lambda$  is an eigenvalue of  $L_t$  the null spaces  $\lambda - L_t, (\lambda - L_t)^2, \dots$  form an increasing sequence of linear manifolds. Theorem 3.1, applied to  $L_t$ , shows that not all complex numbers are eigenvalues of  $L_t$ . Thus for some value of  $\mu$ ,  $(\mu - L_t)^{-1}$  is compact. Hence there is an integer  $\alpha$ , called the ascent of  $\lambda - L_t$ , such

that  $(\lambda - L_t)^\alpha$  and  $(\lambda - L_t)^{\alpha+1}$  have the same null space. The dimension of the null space of  $(\lambda - L_t)^\alpha$  is finite and is called the algebraic multiplicity of  $\lambda$ . As is well-known, the algebraic multiplicity of an eigenvalue is at least as great as its geometric multiplicity and the two are equal if  $L_t$  is self-adjoint. If  $\Gamma$  is a simple closed rectifiable curve lying in  $\rho(L_t)$  and containing several eigenvalues of  $L_t$  in its interior, the projection operator associated with  $L_t$  and the part of the spectrum of  $L_t$  within  $\Gamma$  is defined by

$$P(t) = \frac{1}{2\pi i} \int_{\Gamma} R_{\lambda}(L_t) d\lambda ;$$

this is a contour integral of the operator valued analytic function  $R_{\lambda}(L_t)$  defined for  $\lambda \in \rho(L_t)$ . The dimension of the range of  $P(t)$  is equal to the number of eigenvalues of  $L_t$  within  $\Gamma$  counted according to algebraic multiplicities. We will use this characterization of algebraic multiplicity in the proof of Theorem 3.3. A discussion of these ideas can be found in Taylor [30].

Lemma 3.7. Let  $S$  and  $T$  be two linear operators defined on a linear manifold of a Hilbert space. If  $\lambda \in \rho(S) \cap \rho(T)$ , then

$$a) \quad R_{\lambda}(T) - R_{\lambda}(S) = R_{\lambda}(S)(T-S)R_{\lambda}(S)[I + (S-T)R_{\lambda}(S)]^{-1},$$

$$b) \quad R_{\lambda}(T) = [I + R_{\lambda}(S)(S-T)]^{-1}R_{\lambda}(S), \quad \text{and}$$

$$c) \quad R_{\lambda}(T) - R_{\lambda}(S) = R_{\lambda}(S)(T-S)R_{\lambda}(S) \\ + R_{\lambda}(S)(T-S)R_{\lambda}(S)(T-S)R_{\lambda}(S)[I + (S-T)R_{\lambda}(S)]^{-1}.$$

Proof. For  $\lambda \in \rho(S) \cap \rho(T)$  we have

$$(3.10) \quad R_\lambda(T) - R_\lambda(S) = R_\lambda(S)(T-S)R_\lambda(T),$$

$$\lambda - T = [I + (S-T)R_\lambda(S)](\lambda-S),$$

and

$$\lambda - T = (\lambda-S)[I + R_\lambda(S)(S-T)],$$

and hence

$$(3.11) \quad R_\lambda(T) = R_\lambda(S)[I + (S-T)R_\lambda(S)]^{-1},$$

and

$$(3.12) \quad R_\lambda(T) = [I + R_\lambda(S)(S-T)]^{-1}R_\lambda(S).$$

Substitution of (3.11) into (3.10) gives (a), (3.12) is (b), and substitution of (a) in (3.10) gives (c).

Lemma 3.8. Let  $B$  be a Banach space. Suppose  $S$  is a bounded operator defined on  $B$  and suppose  $T$  is a closed operator with domain in  $B$ . If the domain of  $T$  contains the range of  $S$ , then  $TS$  is bounded.

Proof.  $TS$  is defined on all of  $B$ . Suppose  $x_n \rightarrow x$  and  $TSx_n \rightarrow y$ , where convergence is with respect to the norm in the Banach space. Since  $S$  is continuous we have  $Sx_n \rightarrow Sx$ . Since  $T$  is closed we have  $T(Sx) = y$ . This shows that  $TS$  is closed. This, together with the fact that  $TS$  is defined on all of  $B$ , shows that  $TS$  is bounded.

Lemma 3.9. Let  $P$  and  $Q$  be continuous not necessarily orthogonal projections on a Hilbert space such that  $\|P-Q\| < \|P\|^{-1}$ . Then, the dimension of the range of  $P$  is not greater than the dimension of the range of  $Q$ .

Proof. Suppose the range of  $P$  has larger dimension than the range of  $Q$ . Since the dimension of the range of  $PQ$  is less than or equal to the dimension of the range of  $Q$  we see that the range of  $PQ$  is properly contained in the range of  $P$ . The range of  $P$  is closed since  $P$  is a continuous projection and the range of  $PQ$  is closed since the range of  $Q$  and hence that of  $PQ$  is finite dimensional. Thus there is a unit vector  $f$  in the range of  $P$  which is orthogonal to the range of  $PQ$ . In particular  $f$  is orthogonal to  $PQf$  and hence

$$\|f - PQf\|^2 = 1 + \|PQf\|^2 \geq 1.$$

Now  $f = Pf$  and hence  $f = P^2f$ . Thus

$$\begin{aligned} 1 &\leq \|P^2f - PQf\| \\ &= \|P(Pf - Qf)\| \\ &\leq \|P\| \|P - Q\|, \end{aligned}$$

which contradicts the hypothesis. This completes the proof.

Theorem 3.3. Suppose  $q$  of the circles  $C_\ell$  form a connected set  $C$  which does not intersect any of the other circles:

$$C = \bigcup_{k=1}^q C_{\ell_k}.$$

Then, counting according to algebraic multiplicities, there are  $q$  eigenvalues of  $\tilde{L}$  in  $C$ .

Proof. Consider the operator

$$L_t = L + tA$$

for  $0 \leq t \leq 1$ . Theorem 3.2, applied to  $L_t$  instead of  $\tilde{L}$ , shows that the eigenvalues of  $L_t$  are contained in the circles

$$C_\ell^t \equiv \{\lambda \mid |\lambda - \lambda_\ell| \leq \tau_\ell\}.$$

Here we have used the fact that if  $\tau_{p,t}$  is the positive solution of

$$\tau = (t\beta) \left\{ \sum_{\ell=0}^j [\delta + \gamma(\tau g(p))^{\ell/m}]^2 \right\}$$

then  $\tau_{p,t} \leq t\tau_p$  if  $t \leq 1$ .

Let  $C_{\ell_k}^*$  be obtained from  $C_{\ell_k}$  by increasing the radii of the circles by a number  $r$  which is so small that  $C^* = \bigcup_{k=1}^q C_{\ell_k}^*$  does not intersect  $C_j$ ,  $j \neq \ell_1, \dots, \ell_q$ . Now the number of eigenvalues of  $L_t$  in  $C^*$ , counted according to algebraic multiplicities, is equal to the dimension of the range of the projection

$$P(t) = \frac{1}{2\pi i} \int_{\partial C^*} R_\lambda(L_t) d\lambda.$$

For  $0 \leq t, t' \leq 1$  we have

$$P(t) - P(t') = \frac{1}{2\pi i} \int_{\partial C^*} [R_\lambda(L_t) - R_\lambda(L_{t'})] d\lambda.$$

Using formula (a) of Lemma 3.7 we obtain

$$R_\lambda(L_t) - R_\lambda(L_{t'}) = R_\lambda(L_t)(t-t')AR_\lambda(L_t)[I + (t-t')AR_\lambda(L_t)]^{-1}.$$

Since  $A: H_j \rightarrow L_2$  is closed and the range of  $R_\lambda(L_t)$  is contained in  $H_j$  we see, using Lemma 3.8, that  $A(\lambda - L_t)^{-1}$  is bounded. Thus for  $t - t'$  sufficiently small

$$\begin{aligned} \|P(t) - P(t')\| &\leq \frac{1}{2\pi} \text{length}(\partial C^*) \|R_\lambda(L_t)\| |t-t'| \|AR_\lambda(L_t)\| \times \\ &\quad (1 - |t-t'| \|AR_\lambda(L_t)\|)^{-1}, \end{aligned}$$

where  $\|\cdot\|$  denotes the operator norm associated with the  $L_2$  norm. This inequality shows that  $P(t)$  is continuous in  $t$  with respect to the operator norm.

The continuity of  $P(t)$  together with Lemma 3.9 shows that the dimension of the range of  $P(t)$  is continuous in  $t$ . Since the dimension is an integer we see that the dimension of the range of  $P(t)$  is constant. Hence the number of eigenvalues of  $\tilde{L} = L_1$  in  $C^*$  is equal to the number of eigenvalues of  $L = L_0$  in  $C^*$ . Since  $L$  is self-adjoint it has  $q$  eigenvalues in  $C^*$ . Thus, since  $r$  is arbitrary, we have shown that there are  $q$  eigenvalues of  $\tilde{L}$  in  $C^*$ .

Corollary 1. If  $\bigcup_{k=1}^{\infty} C_k$  is not connected, then  $\tilde{L}$  has at least one eigenvalue.

Proof. In this situation there will be an integer  $p$  such that  $C_1 \cup \dots \cup C_p$  is a connected set not intersecting any of the other circles. Theorem 3.3 applied to this set gives the result.

Corollary 2. If  $\tilde{L}$  and the boundary operators  $U_k$  have real coefficients and the circle  $C_{\ell_0}$  does not intersect the other circles, then the eigenvalue of  $\tilde{L}$  in  $C_{\ell_0}$  is real.

Proof. By Theorem 3.3 there is exactly one eigenvalue of  $\tilde{L}$  in  $C_{\ell_0}$ . If this eigenvalue is not real then its complex conjugate will also be an eigenvalue of  $\tilde{L}$ , will be in  $C_{\ell_0}$ , and will be different from it. This contradicts the fact that there is only one eigenvalue in  $C_{\ell_0}$ .



§4. Improvable Approximations to the Initial Eigenvalues of  $\tilde{L}$ .

Let  $S_n = \text{span}(x_1, \dots, x_n)$  where  $x_1, \dots, x_n$  are the first  $n$  eigenvectors of  $L$ . For any operator  $B$  whose domain contains  $S_n$  define  $B_n = (P_n B)|_{S_n}$  where  $P_n$  is the projection onto  $S_n$ . We now give several results which parallel the results of Section 3.

Lemma 4.1. If  $\lambda \in \rho(L_n)$  and  $f \in S_n$ , then

$$|R_\lambda(L_n)f|_0 \leq |f|_0 (\text{dist}(\lambda, \sigma(L_n)))^{-1}.$$

Proof. The proof here is similar to that of Lemma 3.1.

Lemma 4.2. If  $\lambda \in \rho(L_n)$  and  $f \in S_n$ , then

$$|R_\lambda(L_n)f|_m \leq \max_{1 \leq k \leq n} |(\lambda_k - c)(\lambda_k - \lambda)^{-1}| |f|_0.$$

Proof. For  $\lambda \in \rho(L_n)$  and  $f \in S_n$  we have

$$\begin{aligned} f &= (\lambda - L_n)(\lambda - L_n)^{-1}f \\ &= \lambda(\lambda - L_n)^{-1}f - i^m [(\lambda - L_n)^{-1}f]^{(m)} - c(\lambda - L_n)^{-1}f, \end{aligned}$$

and hence

$$\begin{aligned} |R_\lambda(L_n)f|_m^2 &= \sum_{k=1}^n |(\lambda_k - c)(\lambda_k - \lambda)^{-1}(f, x_k)|^2 \\ &\leq \max_k |(\lambda_k - c)(\lambda_k - \lambda)^{-1}|^2 |f|_0^2. \end{aligned}$$

Lemma 4.3. If  $\lambda \in \rho(L_n)$  and  $k = 0, 1, \dots, m$ , then

$$\|R_\lambda(L_n)f\|_k \leq |f|_0(\text{dist}(\lambda, \sigma(L_n)))^{-1} \times \\ [\delta + \gamma(\text{dist}(\lambda, \sigma(L_n))) \max_{1 \leq q \leq n} |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|]^{k/m}$$

for all  $f \in S_n$ .

Proof. This follows immediately from Lemmas 3.3, 4.1, and 4.2.

Lemma 4.4. If  $\lambda \in \rho(L_n)$  and  $k = 0, 1, \dots, m$ , then

$$\|R_\lambda(L_n)f\|_k \leq |f|_0(\text{dist}(\lambda, \sigma(L_n)))^{-1} \times \\ \left\{ \sum_{\ell=0}^k [\delta + \gamma(\text{dist}(\lambda, \sigma(L_n))) \max_{1 \leq q \leq n} |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|]^{2\ell/m} \right\}^{1/2}.$$

Proof. This follows directly from Lemma 4.3.

Lemma 4.5. If  $\lambda \in \rho(L_n)$  and  $f \in S_n$ , then

$$\|R_\lambda(L_n)A_n f\|_j \leq \left( \sum_{\ell=0}^j \max |a_\ell|^2 \right)^{1/2} \|f\|_j (\text{dist}(\lambda, \sigma(L_n)))^{-1} \times \\ \left\{ \sum_{\ell=0}^j [\delta + \gamma(\text{dist}(\lambda, \sigma(L_n))) \max_{1 \leq q \leq n} |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|]^{2\ell/m} \right\}^{1/2},$$

where  $j$  is the order of  $A$ .

Proof. The proof is essentially the same as that of Lemma 3.6.

Let

$$h_{j,n}(\lambda) = (\text{dist}(\lambda, \sigma(L_n)))^{-1} \times \\ \left\{ \sum_{\ell=0}^j [\delta + \gamma(\text{dist}(\lambda, \sigma(L_n))) \max_{1 \leq q \leq n} |(\lambda_q - c)(\lambda_q - \lambda)^{-1}|]^{2\ell/m} \right\}^{1/2}.$$

Theorem 4.1. If  $\lambda$  is an eigenvalue of  $\tilde{L}_n$  then either

$$\beta h_{j,n}(\lambda) \geq 1$$

or  $\lambda \in \sigma(L_n)$ .

Proof. Suppose  $\lambda \in \rho(L_n)$  and  $\beta h_{j,n}(\lambda) < 1$ . This, together with Lemma 4.5, implies that  $(\lambda - L_n)^{-1}A_n$ , considered as an operator on  $S_n$  with the  $j$  norm, has norm less than one. As in the proof of Theorem 3.1 this implies that  $\lambda$  is not an eigenvalue of  $\tilde{L}_n$  which is a contradiction.

Using the same reasoning which led to Theorem 3.2 we obtain

Theorem 4.2. If  $\lambda$  is an eigenvalue of  $\tilde{L}_n$  then for some  $p$  satisfying  $1 \leq p \leq n$  we have

$$|\lambda - \lambda_p| \leq r_p.$$

Theorem 4.3. Suppose  $q$  of the circles  $C_1, \dots, C_n$  form a connected set  $C$  not intersecting the rest of the circles. Then, counting according to algebraic multiplicities, there are  $q$  eigenvalues of  $\tilde{L}_n$  in  $C$ .

Proof. The concept of algebraic multiplicity is defined in this finite dimensional context just as in Section 3. The continuity argument used in the proof of Theorem 3.3 now applies here and gives the desired result.

For the remainder of the paper we assume the circles  $C_\ell$  are mutually disjoint. Theorems 3.2 and 3.3 then assert that  $\tilde{L}$  has a countable set of eigenvalues which are contained in the circles  $C_\ell$ , one eigenvalue to each circle; denote the one in  $C_\ell$  by  $\mu_\ell$ . Also, for  $n$  fixed, Theorem 4.3 asserts that the eigenvalues of  $\tilde{L}_n$  are contained in the circles  $C_1, \dots, C_n$ , one eigenvalue to each circle; denote the one in  $C_\ell$  by  $\eta_\ell(n)$ . The eigenvalues  $\eta_1(n), \dots, \eta_n(n)$  will be called the  $n^{\text{th}}$  stage Galerkin eigenvalues of  $\tilde{L}$ .

We now derive estimates on the error which arises when  $\eta_p(n)$ , the  $p^{\text{th}}$  of the  $n^{\text{th}}$  stage Galerkin eigenvalues of  $\tilde{L}$ , is used as an approximation for  $\mu_p$ , the  $p^{\text{th}}$  eigenvalue of  $\tilde{L}$ ,  $p = 1, \dots, n$ . Let  $y_p$  be a unit eigenvector of  $\tilde{L}$  corresponding to  $\mu_p$  and let  $y_p$  be orthogonally decomposed as follows:  $y_p = y_p(n) + w_p(n)$ ,  $y_p(n) \in S_n$ ,  $w_p(n) \in S_n^\perp$ . Let  $z_k(n)$  be a unit eigenvector of  $\tilde{L}_n$  corresponding to  $\eta_k(n)$ . The vectors  $z_1(n), \dots, z_n(n)$  span  $S_n$ .

Lemma 4.6. Let  $\alpha_{p,1}(n), \dots, \alpha_{p,n}(n)$  be the coefficients in the expansion of  $y_p(n)$  in the basis  $z_1(n), \dots, z_n(n)$ .

Then

$$\left| \sum_{\ell=1}^n \alpha_{p,\ell}(n) z_\ell(n) \right|_0 = (1 - |w_p(n)|_0^2)^{1/2},$$

and

$$\left| \sum_{\ell=1}^n \alpha_{p,\ell}(n) (\mu_p - \eta_\ell(n)) z_\ell(n) \right|_0 = |P_n A w_p(n)|_0.$$

Proof. The first result follows from

$$y_p(n) = \sum_{\ell=1}^n \alpha_{p,\ell}(n) z_\ell(n),$$

and

$$1 = |y_p|_0^2 = |y_p(n)|_0^2 + |w_p(n)|_0^2.$$

The second is a consequence of

$$\begin{aligned} \sum_{\ell=1}^n \alpha_{p,\ell}(n) (\mu_p - \eta_\ell(n)) z_\ell(n) &= \mu_p y_p(n) - \tilde{L}_n y_p(n) \\ &= \mu_p y_p(n) - P_n \tilde{L}(y_p - w_p(n)) \\ &= \mu_p y_p(n) - \mu_p P_n y_p + P_n (L+A) w_p(n) \\ &= P_n A w_p(n). \end{aligned}$$

Lemma 4.7. Let  $\xi_1, \dots, \xi_n$  be a basis for an  $n$ -dimensional inner product space. Let  $\zeta = \sum_{\ell=1}^n \tau_\ell \xi_\ell$ .

Then

$$m(\xi_1, \dots, \xi_n) \sum_{\ell=1}^n |\tau_\ell|^2 \leq \|\zeta\|^2 \leq M(\xi_1, \dots, \xi_n) \sum_{\ell=1}^n |\tau_\ell|^2,$$

where  $M(\xi_1, \dots, \xi_n)$  and  $m(\xi_1, \dots, \xi_n)$  are respectively the greatest and least eigenvalues of the Gram matrix  $((\xi_k, \xi_\ell))$ .

Proof. We have  $\|\zeta\|^2 = (\zeta, \zeta) = \sum_{k,\ell=1}^n \tau_k \bar{\tau}_\ell (\xi_k, \xi_\ell)$ .

Now  $((\xi_k, \xi_\ell))$  is a Hermitian matrix and thus  $\|\zeta\|^2$  is a Hermitian form in  $(\bar{\tau}_1, \dots, \bar{\tau}_n)$ . The result follows from the fact that the value of a Hermitian form lies between the least eigenvalue of its matrix times the square of the Euclidean length of the vector and the greatest eigenvalue

of its matrix times the square of the Euclidean length of the vector.

Lemma 4.8.

$$1 - |w_p(n)|_0^2 \leq M(z_1(n), \dots, z_n(n)) \sum_{\ell=1}^n |\alpha_{p,\ell}(n)|^2,$$

and

$$m(z_1(n), \dots, z_n(n)) \sum_{\ell=1}^n |\alpha_{p,\ell}(n)(\mu_p - \eta_\ell(n))|^2 \leq |P_n A w_p(n)|_0^2.$$

Proof. This follows directly from Lemmas 4.6 and 4.7.

Theorem 4.4. For some  $\ell$ , say  $\ell_0$ , where  $1 \leq \ell_0 \leq n$ , we have

(4.1)

$$|\mu_p - \eta_{\ell_0}(n)| \leq (M(z_1(n), \dots, z_n(n)) / m(z_1(n), \dots, z_n(n)))^{1/2} \times \\ |P_n A w_p(n)|_0 (1 - |w_p(n)|_0^2)^{-1/2}.$$

Proof. We write  $M_n$  and  $m_n$  for  $M(z_1(n), \dots, z_n(n))$  and  $m(z_1(n), \dots, z_n(n))$  for the remainder of the paper.

Suppose

$$|\mu_p - \eta_\ell(n)| > (M_n / m_n)^{1/2} |P_n A w_p(n)|_0 (1 - |w_p(n)|_0^2)^{-1/2},$$

$\ell = 1, \dots, n$ . Then using the first inequality in Lemma 4.8 we have

$$\begin{aligned}
& \sum_{\ell=1}^n |\alpha_{p,\ell}^{(n)}(\mu_p - \eta_{\ell}^{(n)})|^2 \\
& > M_n |P_n A w_p(n)|_0^2 (1 - |w_p(n)|_0^2)^{-1/2} \sum_{\ell=1}^n |\alpha_{p,\ell}^{(n)}|^2 \\
& \geq |P_n A w_p(n)|_0^2
\end{aligned}$$

which contradicts the second inequality in Lemma 4.8. Hence we get the desired result.

Theorem 4.4 provides the basic estimate for  $|\mu_p - \eta_{\ell_0}^{(n)}|$ . In the remainder of this section we will first discuss the convergence to zero of the right hand side of (4.1) and then derive computable bounds for  $|w_p(n)|_0$  and  $|P_n A w_p(n)|_0$ . Since  $w_p(n) = \sum_{\ell=n+1}^{\infty} (y_p, x_{\ell})_0 x_{\ell}$  we see that  $\lim_{n \rightarrow \infty} |w_p(n)|_0 = 0$  for  $p$  fixed. We prove that  $\lim_{n \rightarrow \infty} |P_n A w_p(n)|_0 = 0$  with the help of the following lemma, whose proof can be found in Agmon [2].

Lemma 4.9. There is a constant  $K$  which depends on  $a, b, m, L$ , and  $A$  such that

$$|Af|_0 \leq K(|Lf|_0 + |f|_0)$$

for all  $f \in H_m$ .

Using this lemma we can write

$$\begin{aligned}
|P_n A w_p(n)|_0 & \leq |A w_p(n)|_0 \\
& \leq K(|L w_p(n)|_0 + |w_p(n)|_0).
\end{aligned}$$

This, together with

$$Lw_p(n) = \sum_{\ell=n+1}^{\infty} \lambda_{\ell} (y_p, x_{\ell})_0,$$

shows that  $\lim_{n \rightarrow \infty} |P_n A w_p(n)|_0 = 0$  for  $p$  fixed. Thus the right side of (4.1) converges to zero as  $n \rightarrow \infty$  for  $p$  fixed provided  $M_n/m_n$  is bounded in  $n$ . In Section 6 we give sufficient conditions for  $M_n/m_n$  to be bounded.

Remark. The factor  $(M_n/m_n)^{1/2}$  is present in inequality (4.1) because we are estimating the change in the eigenvalues when the (possibly) non self-adjoint operator  $\tilde{L}_n$  is perturbed to obtain  $\tilde{L}$ .  $(M_n/m_n)^{1/2}$  equals 1 if and only if  $\tilde{L}_n$  is normal. This factor also arises in a fundamental result on the perturbation of eigenvalues of diagonalizable matrices; see Bauer and Fike [6].

Computable bounds on  $|w_p(n)|_0$  and  $|A w_p(n)|_0$  will now be developed.

Lemma 4.10. If  $n \geq p$ , then

$$|w_p(n)|_0 \leq |A y_p|_0 \left( \min_{\ell \geq n+1} |\mu_p - \lambda_{\ell}| \right)^{-1}.$$

Proof. Since

$$w_p(n) = \sum_{\ell=n+1}^{\infty} (y_p, x_{\ell})_0 x_{\ell}$$

we have

$$\begin{aligned} |w_p(n)|_0^2 &= \sum_{\ell=n+1}^{\infty} |(y_p, x_{\ell})_0|^2 \\ &\leq \left( \min_{\ell \geq n+1} |\mu_p - \lambda_{\ell}| \right)^{-1} \sum_{\ell=n+1}^{\infty} |(y_p, x_{\ell})_0 (\mu_p - \lambda_{\ell})|^2. \end{aligned}$$



Now

$$\begin{aligned}
 (\mu_p - \lambda_\ell)(y_p, x_\ell)_0 &= (\mu_p y_p, x_\ell)_0 - (y_p, \lambda_\ell x_\ell)_0 \\
 &= (\tilde{L}y_p, x_\ell)_0 - (y_p, Lx_\ell)_0 \\
 &= (Ay_p, x_\ell)_0
 \end{aligned}$$

and hence

$$\begin{aligned}
 \sum_{\ell=1}^{\infty} |(y_p, x_\ell)_0 (\mu_p - \lambda_\ell)|^2 &\leq \sum_{\ell=1}^{\infty} |(Ay_p, x_\ell)_0|^2 \\
 &= |Ay_p|_0^2.
 \end{aligned}$$

Lemma 4.11.

$$(4.2) \quad |Ay_p|_0 \leq \sum_{\ell=0}^j \max |a_\ell| [\delta + \gamma(|\mu_p - c| + |Ay_p|_0)^{\ell/m}].$$

Proof. Using Lemma 3.3 we have

$$\begin{aligned}
 |Ay_p|_0 &\leq \left( \sum_{\ell=0}^j \max |a_\ell| |y_p|_\ell \right) \\
 &\leq \sum_{\ell=0}^j \max |a_\ell| (\gamma |y_p|_0^{1-\frac{\ell}{m}} |y_p|_0^{\frac{\ell}{m}} + \delta |y_p|_0) \\
 &= \sum_{\ell=0}^j \max |a_\ell| (\gamma |Ly_p - cy_p|_0^{\ell/m} + \delta) \\
 &= \sum_{\ell=0}^j \max |a_\ell| (\gamma |\tilde{L}y_p - Ay_p - cy_p|_0^{\ell/m} + \delta) \\
 &\leq \sum_{\ell=0}^j \max |a_\ell| [\delta + \gamma(|\mu_p - c| + |Ay_p|_0)^{\ell/m}].
 \end{aligned}$$

We can solve (4.2) for  $|Ay_p|_0$ , i.e., (4.2) implies

$$|Ay_p|_0 \leq t_p$$

where  $t_p$  is the unique positive solution of

$$t = \sum_{\ell=0}^j \max |a_{\ell}| [\delta + \gamma(|\mu_p - c| + t)^{\ell/m}].$$

Lemmas 4.10 and 4.11 provide a computable bound for  $|w_p(n)|_0$  since we have a preliminary bound on  $|\mu_p|$ .

We now assume  $m$  is even (let  $\mu = m/2$ ) and that the boundary conditions are of Sturm type, i.e., are of the form

$$\alpha_{10} f(a) + \alpha_{11} f^{(1)}(a) + \cdots + \alpha_{1, \kappa_1 - 1} f^{(\kappa_1 - 1)}(a) + f^{(\kappa_1)}(a) = 0,$$

$$\alpha_{20} f(a) + \alpha_{21} f^{(1)}(a) + \cdots + \alpha_{2, \kappa_2 - 1} f^{(\kappa_2 - 1)}(a) + f^{(\kappa_2)}(a) = 0,$$

$$\vdots$$

$$\alpha_{\mu 0} f(a) + \alpha_{\mu 1} f^{(1)}(a) + \cdots + \alpha_{\mu, \kappa_{\mu} - 1} f^{(\kappa_{\mu} - 1)}(a) + f^{(\kappa_{\mu})}(a) = 0,$$

$$\beta_{10} f(b) + \beta_{11} f^{(1)}(b) + \cdots + \beta_{1, \kappa'_1 - 1} f^{(\kappa'_1 - 1)}(b) + f^{(\kappa'_1)}(b) = 0,$$

$$\vdots$$

$$\beta_{\mu 0} f(b) + \cdots + \beta_{\mu, \kappa'_{\mu} - 1} f^{(\kappa'_{\mu} - 1)}(b) + f^{(\kappa'_{\mu})}(b) = 0,$$

where

$$0 \leq \kappa_{\mu} < \kappa_{\mu-1} < \cdots < \kappa_2 < \kappa_1 \leq m-1,$$

$$0 \leq \kappa'_{\mu} < \kappa'_{\mu-1} < \cdots < \kappa'_2 < \kappa'_1 \leq m-1.$$

Birkhoff [10] and Tamarkin [29] obtained asymptotic formulas for the eigenvalues of a wide class of differential operators and boundary conditions. Under the assumption that the boundary conditions are of Sturm type, they imply that the eigenvalues of

$$(-1)^{\mu} f^{(2\mu)} = \lambda f,$$

$$U_k f = 0, \quad k = 1, \dots, m$$

are bounded below. This implies that  $c$  can be chosen such that  $L$  is positive definite; we assume for the rest of this section that  $c$  is so chosen. There is further explanation and additional applications of these asymptotic formulas in Section 6. For a complete discussion of these formulas see Naimark [22].

A proof of the following lemma can be found in Chapter XIX of Dunford and Schwartz [13].

Lemma 4.12. If  $k = 1, \dots, m-1$ , then the domain of  $L^{k/m}$  is contained in  $H_k$  where  $L^{k/m}$  denotes the  $k/m$  power of  $L$ .

Lemma 4.13. There is a constant  $K$  which is independent of  $n$  such that

$$|Aw_p(n)|_0 \leq K |\mu_p - \lambda_{n+1}|^{(j-m)/m}$$

for  $n \geq p$ .

Proof. From Lemma 4.12 we see that  $H_j$  contains the domain of  $L^{j/m}$ . Hence, using Lemma 3.8, we see that  $AL^{-j/m}$  is bounded as an operator on  $L_2$ . Thus

$$(4.3) \quad \begin{aligned} |Af|_0 &= |AL^{-j/m}L^{j/m}f|_0 \\ &\leq \|AL^{-j/m}\| |L^{j/m}f|_0 \end{aligned}$$

for all  $f$  in the domain of  $L^{j/m}$ ; in particular this holds for all  $f \in V_m$  and we have

$$|Aw_p(n)|_0 \leq \|AL^{-j/m}\| |L^{j/m}w_p(n)|_0.$$

Now

$$\begin{aligned} (L^{j/m} w_p(n), x_k)_0 &= \lambda_k^{j/m} (w_p(n), x_k)_0 \\ &= \begin{cases} 0, & k \leq n, \\ \lambda_k^{j/m} (y_p, x_k)_0, & k > n, \end{cases} \end{aligned}$$

and

$$(\mu_p - \lambda_k)(y_p, x_k)_0 = (Ay_p, x_k)_0$$

and hence

$$\begin{aligned} |L^{j/m} w_p(n)|_a^2 &= \sum_{k=n+1}^{\infty} \lambda_k^{2j/m} |(Ay_p, x_k)_0 (\mu_p - \lambda_k)^{-1}|^2 \\ &\leq \left\{ \max_{k \geq n+1} (\lambda_k |\mu_p - \lambda_k|^{-1})^{2j/m} \right\} |\mu_p - \lambda_{n+1}|^{2(j-m)/m} |Ay_p|_0^2. \end{aligned}$$

Thus

$$\begin{aligned} |Aw_p(n)|_0 &\leq \\ &\|AL^{-j/m}\| \left\{ \max_{k \geq p+1} (\lambda_k |\mu_p - \lambda_k|^{-1})^{j/m} \right\} |Ay_p|_0 |\mu_p - \lambda_{n+1}|^{(j-m)/m}. \end{aligned}$$

Under the assumption that  $j \leq \mu$  we present an additional estimate for  $|Aw_p(n)|_0$  which is in some cases more readily computed than the bound given in Lemma 4.13.

Lemma 4.14. There are constants  $c_1 > 0$ ,  $c_2 \geq 0$  which depend on  $a, b, m, L$ , and the boundary conditions such that

$$(4.4) \quad \|f\|_{\mu}^2 \leq c_1 (Lf, f)_0 + c_2 |f|_0^2$$

for all  $f \in V_m$ .

Inequality (4.4) for functions which vanish near  $a$  and  $b$  is Gårding's inequality. Our concern here is to prove it for functions in  $V_m$  and to show how the constants can be calculated. This is done in Section 5.

Lemma 4.15. If  $j \leq m/2$ , then

$$|Aw_p(n)|_0 \leq \beta \left\{ \max_{k \geq n+1} (c_1 \lambda_k + c_2) \right\}^{1/2} |\mu_p - \lambda_k|^{-1} |Ay_p|_0.$$

Proof. Using Lemma 4.14 we have

$$\begin{aligned} (4.5) \quad |Af|_0^2 &\leq \beta^2 [c_1 (Lf, f)_0 + c_2 |f|_0^2] \\ &= \beta^2 [c_1 |L^{1/2} f|_0^2 + c_2 |f|_0^2] \end{aligned}$$

for any  $f \in V_m$ . If we now estimate  $|Aw_p(n)|_0$ , using (4.5) instead of (4.3) we obtain

$$\begin{aligned} |Aw_p(n)|_0 &\leq \beta \left\{ \sum_{k=n+1}^{\infty} (c_1 \lambda_k + c_2) |(\mu_p - \lambda_k)^{-1} (Ay_p, x_k)_0|^2 \right\}^{1/2} \\ &\leq \beta \left\{ \max_{k \geq n+1} (c_1 \lambda_k + c_2) \right\}^{1/2} |\mu_p - \lambda_k|^{-1} |Ay_p|_0. \end{aligned}$$

Theorem 4.4 and Lemmas 4.10, 4.11, 4.13 and 4.15 give an explicit estimate for  $|\mu_p - \eta_{\ell_0}(n)|$  since  $M_n/m_n$  can be calculated. In many cases we can conclude that  $\ell_0 = p$  and hence we have an estimate for  $|\mu_p - \eta_p(n)|$  for any  $n \geq p$ . See Section 7 for an explanation of the use of these inequalities in numerical computation.

§5. Proof of Lemma 4.14.

In this section we assume that  $m$  is even, the boundary conditions are of Sturm type, and  $j \leq m/2$ . We do not assume that  $c$  is chosen such that  $L$  is positive definite.

Lemma 5.1. Suppose  $\mu \leq i \leq 2\mu-1$ ,  $0 \leq k \leq \mu-1$  and neither  $i$  nor  $k$  is equal to  $\kappa_\ell$  for any  $\ell$ . Let  $D$  be any constant. Then there exists a family of functions  $f_\epsilon(x)$ ,  $0 < \epsilon < 1$ , such that

1)  $f_\epsilon \in V_m$  for all  $\epsilon$ ,

2) 
$$f_\epsilon^{(p)}(a) = \begin{cases} 1, & p = k \\ D\epsilon^{-1}, & p = i \\ 0, & p \neq i, k, \kappa_1, \dots, \kappa_\mu \end{cases}$$

3)  $f_\epsilon^{(p)}(b) = 0$  for all  $p$ ,

4)  $|f_\epsilon|_0$  and  $|f_\epsilon|_\mu$  are bounded in  $\epsilon$ .

Proof. For  $0 \leq p \leq 2\mu-1$  define  $f_p(x) = (x-a)^p/p!$ . Let  $\gamma(x)$  be an infinitely differentiable function satisfying  $0 \leq \gamma(x) \leq 1$ ,  $\gamma(x) = 1$  if  $x \leq a + (b-a)/3$ , and  $\gamma(x) = 0$  if  $x \geq (a+b)/2$ . For  $\mu \leq q \leq 2\mu-1$  define

$$g_{q,\epsilon}(x_q) = \int_a^{x_q} \dots \left( \int_a^{x_1} D\gamma(a + (x_0-a)\epsilon^{-2})\epsilon^{-1} dx_0 \right) \dots dx_q.$$

Then

$$f_p^{(q)}(a) = \begin{cases} 1, & q = p, \\ 0, & q \neq p, \end{cases}$$

and

$$g_{q,\varepsilon}^{(p)}(a) = \begin{cases} D\varepsilon^{-1}, & p = q, \\ 0, & p \neq q. \end{cases}$$

Also

$$|g_{q,\varepsilon}^{(p)}|_0 \leq |D| \left[ \int_a^b \gamma^2(x) dx \right]^{1/2} (b-a)^{q-p} \text{ if } 0 < \varepsilon < 1, \quad p \leq \mu.$$

The boundary conditions at  $a$  can be solved for  $f^{(\kappa_1)}(a)$ ,  $\dots, f^{(\kappa_\mu)}(a)$  in terms of  $f^{(p)}(a)$ ,  $p \neq \kappa_1, \dots, \kappa_\mu$ :

$$f^{(\kappa_\ell)}(a) = \sum_{\substack{p=0 \\ p \neq \kappa_\mu, \dots, \kappa_{\ell+1}}}^{\kappa_\ell-1} c_{\ell p} f^{(p)}(a),$$

where the coefficients  $c_{\ell p}$  depend only on the boundary conditions. If  $k < \kappa_\ell < i$  define

$$h_{\kappa_\ell}(x) = c_{\ell k} f_{\kappa_\ell}(x);$$

if  $i < \kappa_\ell$  define

$$h_{\kappa_\ell}(x) = c_{\ell k} f_{\kappa_\ell}(x) + c_{\ell i} g_{\kappa_\ell, \varepsilon}(x).$$

Now let

$$f_\varepsilon(x) = \gamma(x) \{ g_{i, \varepsilon}(x) + f_k(x) + \sum_{\kappa_\ell > k} h_{\kappa_\ell}(x) \}.$$

$f_\varepsilon$  clearly satisfies conditions 1-4.

The following two lemmas are proved in Agmon [2]. Lemma 5.3 is closely related to Lemma 3.3.

Lemma 5.2. (Sobolev) There is a constant  $\gamma$  depending only on  $a, b,$  and  $\ell$  such that

$$|f^{(k)}(\tau)| \leq \gamma r^{(\ell - \frac{1}{2} - k)} (|f|_{\ell} + r^{-\ell} |f|_0)$$

for all  $f \in H_{\ell}, 0 < r \leq 1, k = 0, \dots, \ell-1, a \leq \tau \leq b.$

Lemma 5.3. There is a constant  $\gamma$  depending only on  $a, b,$  and  $\ell$  such that

$$|f|_k^2 \leq \gamma (r^{\ell-k} |f|_{\ell}^2 + r^{-k} |f|_0^2)$$

for all  $f \in H_{\ell}, 0 < r \leq 1, k = 0, \dots, \ell-1.$

If  $f \in V_m,$  then repeated integration by parts gives

$$\begin{aligned} (5.1) \quad (Lf, f)_0 &= ((-1)^{\mu} f^{(2\mu)} + cf, f)_0 \\ &= |f|_{\mu}^2 + c |f|_0^2 + (-1)^{\mu} \sum_{k=0}^{\mu-1} (-1)^k f^{(2\mu-1-k)} \bar{f}^{(k)} \Big|_a^b. \end{aligned}$$

Since the eigenvalues of  $L$  are bounded below we have

$$(Lf, f)_0 \geq \lambda |f|_0^2$$

for all  $f \in V_m$  where  $\lambda$  is the least eigenvalue of  $L.$

Using this and (5.1) we obtain

$$(5.2) \quad (-1)^{\mu+1} \sum_{k=0}^{\mu} (-1)^k f^{(2\mu-1-k)} \bar{f}^{(k)} \Big|_a^b \leq |f|_{\mu}^2 + (c-\lambda) |f|_0^2$$

for  $f \in V_m.$  As indicated in the proof of Lemma 5.1 the boundary conditions at  $a$  can be solved for  $f^{(\kappa_{\ell})}(a):$

$$f^{(\kappa_{\ell})}(a) = \sum_{\substack{p=0 \\ p \neq \kappa_{\mu}, \dots, \kappa_{\ell+1}}}^{\kappa_{\ell}-1} c_{\ell p} f^{(p)}(a).$$



Similarly the boundary conditions at  $b$  can be solved for  $f^{(\kappa'_\ell)}(b)$ :

$$f^{(\kappa'_\ell)}(b) = \sum_{p=0}^{\kappa'_\ell-1} c'_{\ell p} f^{(p)}(b)$$

Let  $S(x) = (-1)^\mu \sum_{k=0}^{\mu-1} (-1)^k f^{(2\mu-1-k)}(x) \bar{f}^{(k)}(x)$ . In terms of  $S$  inequality (5.2) is

$$(5.3) \quad -S(b) + S(a) \leq |f|_\mu^2 + (c-\lambda)|f|_0^2.$$

If in  $S(a)$  we substitute the above expression for  $f^{(\kappa'_\ell)}(a)$ ,  $\ell = 1, \dots, \mu-1$ , we will get terms of the form

$$d_{ik} f^{(i)}(a) \bar{f}^{(k)}(a)$$

where  $0 \leq i \leq 2\mu-1$ ,  $0 \leq k \leq \mu-1$ ,  $i+k \leq 2\mu-1$ , and neither  $i$  nor  $k$  is equal to  $\kappa'_\ell$  for any  $\ell$ . Likewise for  $S(b)$  we get terms of the form

$$d'_{ik} f^{(i)}(b) \bar{f}^{(k)}(b)$$

where  $0 \leq i \leq 2\mu-1$ ,  $0 \leq k \leq \mu-1$ ,  $i+k \leq 2\mu-1$ , and neither  $i$  nor  $k$  is equal to  $\kappa'_\ell$  for any  $\ell$ . Let  $i, k$  be fixed with  $i \geq \mu$ . Now let  $f_\varepsilon$  be the family of functions shown to exist in Lemma 5.1 where  $D = d_{ik}$ . For these functions (5.3) reduces to

$$|d_{ik}|^2 \varepsilon^{-1} \leq |f_\varepsilon|_\mu^2 + (c-\lambda)|f_\varepsilon|_0^2.$$

Since  $|f_\varepsilon|_\mu$  and  $|f_\varepsilon|_0$  are bounded in  $\varepsilon$  we see that

$d_{ik} = 0$ . Similarly  $d'_{ik} = 0$  if  $i \geq \mu$ . Hence (5.1)

becomes

$$\begin{aligned}
 (5.4) \quad (Lf, f)_0 &= |f|_\mu^2 + c|f|_0^2 + S(b) - S(a) \\
 &= |f|_\mu^2 + c|f|_0^2 + \sum_{\substack{0 \leq i, k \leq \mu-1 \\ i, k \neq \kappa_\ell, \ell=1, \dots, \mu}} d'_{ik} f^{(i)}(b) \bar{f}^{(k)}(b) \\
 &\quad - \sum_{\substack{0 \leq i, k \leq \mu-1 \\ i, k \neq \kappa_\ell, \ell=1, \dots, \mu}} d_{ik} f^{(i)}(a) \bar{f}^{(k)}(b)
 \end{aligned}$$

and thus

$$\begin{aligned}
 (Lf, f)_0 &\geq |f|_\mu^2 + c|f|_0^2 - \sum |d'_{ik}| |f^{(i)}(b)| |\bar{f}^{(k)}(b)| \\
 &\quad - \sum |d_{ik}| |f^{(i)}(a)| |\bar{f}^{(k)}(a)|.
 \end{aligned}$$

Using Lemma 5.2 with  $\ell = \mu$  we have

$$\begin{aligned}
 |f^{(i)}(a)| &\leq \gamma r^{\mu - \frac{1}{2} - i} (|f|_\mu + r^{-\mu} |f|_0) \\
 &\leq \gamma (r^{1/2} |f|_\mu + r^{\frac{1}{2} - \mu} |f|_0), \quad 0 \leq i \leq \mu-1.
 \end{aligned}$$

Using this and a similar estimate for  $|f^{(i)}(b)|$  we obtain

$$\begin{aligned}
 (Lf, f)_0 &\geq |f|_\mu^2 + c|f|_0^2 - \sum |d'_{ik}| \gamma^2 (r^{1/2} |f|_\mu + r^{\frac{1}{2} - \mu} |f|_0)^2 \\
 &\quad - \sum |d_{ik}| \gamma^2 (r^{1/2} |f|_\mu + r^{\frac{1}{2} - \mu} |f|_0)^2 \\
 &\geq |f|_\mu^2 + c|f|_0^2 - 2\gamma^2 (r |f|_\mu^2 + r^{1-2\mu} |f|_0^2) \\
 &\quad + (\sum |d'_{ij}| + \sum |d_{ij}|)
 \end{aligned}$$

$$\begin{aligned}
&= |f|_{\mu}^2 \{1 - 2\gamma^2 r (\sum |d'_{ij}| + \sum |d_{ij}|)\} \\
&\quad + |f|_0^2 \{c - 2\gamma^2 r^{1-2\mu} (\sum |d_{ij}| + \sum |d'_{ij}|)\} \\
&= \frac{1}{2} |f|_{\mu}^2 + \tilde{c} |f|_0^2
\end{aligned}$$

for  $r$  chosen appropriately. Finally, using Lemma 5.3 we have

$$\begin{aligned}
(Lf, f) &\geq \frac{1}{2} \|f\|_{\mu}^2 - \frac{1}{2} \sum_{i=0}^{\mu-1} |f|_i^2 + \tilde{c} |f|_0^2 \\
&\geq \frac{1}{2} \|f\|_{\mu}^2 - \frac{1}{2} \sum_{i=0}^{\mu-1} \gamma (r^{\mu-i} |f|_{\mu}^2 + r^{-i} |f|_0^2) + \tilde{c} |f|_0^2 \\
&\geq \frac{1}{2} \|f\|_{\mu}^2 - \frac{\gamma}{2} (\sum_{i=0}^{\mu-1} r^{\mu-i}) \|f\|_{\mu}^2 + |f|_0^2 (\tilde{c} - \frac{\gamma}{2} \sum_{i=0}^{\mu-1} r^{-i}) \\
&= \|f\|_{\mu}^2 \left\{ \frac{1}{2} - \frac{\gamma}{2} (\sum_{i=0}^{\mu-1} r^{\mu-i}) \right\} + |f|_0^2 (\tilde{c} - \frac{\gamma}{2} \sum_{i=0}^{\mu-1} r^{-i}).
\end{aligned}$$

Now by choosing  $r$  appropriately we arrive at Lemma 4.14.

Remarks. 1. The constants  $c_1$  and  $c_2$  in Lemma 4.14 are computable assuming the constants in Lemmas 5.2 and 5.3 are computable. See Section 7, especially Example 7.4, for further comments on finding explicit values for these constants.

2. Let  $m = 2$  and write the boundary conditions as

$$\begin{aligned}
\alpha_0 f(a) + \alpha_1 f^{(1)}(a) &= 0, \\
\beta_0 f(b) + \beta_1 f^{(1)}(b) &= 0.
\end{aligned}$$

Then, if  $\alpha_0 \alpha_1 \leq 0$  and  $\beta_0 \beta_1 \geq 0$ , (5.4) implies

$$(Lf, f)_0 \geq |f|_1^2 + c |f|_0^2.$$

Hence in this case we obtain (4.4) with  $c_1 = 1$  and

$$c_2 = \max(0, 1-c).$$

§6. Proof that  $M_n/m_n$  is bounded.

We return here to the problem mentioned in Section 4 of showing that

$$M(z_1(n), \dots, z_n(n)) / m(z_1(n), \dots, z_n(n))$$

is bounded in  $n$ . For the remainder of this section assume that  $m = 2\mu$  is even, the boundary conditions are of Sturm type,  $L$  is positive definite, and the circles

$$C_\ell = \{\lambda \mid |\lambda - \lambda_\ell| \leq r_\ell\}$$

are mutually disjoint. We also assume that  $j$ , the order of  $A$ , is less than or equal to  $m-2$ . Let  $d_\ell = \min_{k \neq \ell} |\lambda_k - \lambda_\ell|$ .

Lemma 6.1.  $\sum_{k=1}^{\infty} (\lambda_k^v / d_k)^2 < \infty$  if  $0 < v < 1 - \frac{3}{2m}$ .

Proof. Under the assumptions of this section the asymptotic formula for the eigenvalues of  $L$  which was mentioned in Section 4 is as follows. There is an integer  $p$  and a constant  $\alpha$  such that

$$(6.1) \quad \lambda_k = \left( \frac{\pi(k-p)}{b-a} \right)^m \left\{ 1 + \frac{\alpha}{k} + o\left(\frac{1}{k^2}\right) \right\}$$

for all  $k$ . An immediate consequence of this is an asymptotic formula for  $d_k$ :

$$(6.2) \quad d_k = \left( \frac{\pi}{b-a} \right)^m (k-p)^{m-1} \left\{ 1 + o\left(\frac{1}{k}\right) \right\}.$$

Hence

$$\lambda_k^{2\nu}/d_k^2 = \left(\frac{\pi}{b-a}\right)^{2m(\nu-1)} (k-p)^{2(1+m\nu-m)} O(1)$$

and the result follows.

Lemma 6.2. For sufficiently large  $\ell$ , say  $\ell \geq \ell_1$ ,

$$r_\ell < d_\ell/3.$$

Proof. It follows from (3.6), (3.8), (3.9) and (6.1) that

$$r_\ell \leq d\ell^j, \quad d > 0,$$

and from (6.2) that

$$d_\ell/3 \geq \tilde{d}\ell^{m-1}, \quad \tilde{d} > 0.$$

The result now follows from the fact that  $j < m-1$ .

Lemma 6.3. Let  $\phi_1, \dots, \phi_n$  be any orthonormal basis for  $S_n = \text{span}(x_1, \dots, x_n)$  and let  $T: S_n \rightarrow S_n$  be the linear operator defined by

$$T\phi_k = z_k(n), \quad k = 1, \dots, n.$$

Then

$$(M_n/m_n)^{1/2} = \|T\| \|T^{-1}\|$$

where  $\|\cdot\|$  denotes the operator norm which is associated with the inner product on  $S_n$  which is induced by the  $L_2$  inner product.

Proof.  $\|T\|^2$  is equal to the largest eigenvalue of  $T^*T$  and the matrix of  $T^*T$  with respect to  $\phi_1, \dots, \phi_n$  is the transpose of the Gram matrix  $((z_k(n), z_\ell(n)))_0$ . Hence  $\|T\|^2 = M_n$ . Similar reasoning shows that  $\|T^{-1}\|^2 = m_n$ .

$\tilde{L}_n$  is a diagonalizable operator on  $S_n$ . Let  $Q_1^n, \dots, Q_n^n$  be the projections associated with  $\tilde{L}_n$  and let  $I_n = \{1, 2, \dots, n\}$ . The following lemma is due to Lorch [21].

Lemma 6.4.  $\|T\| \|T^{-1}\| \leq 4 \left\{ \max_{I \subset I_n} \left\| \sum_{k \in I} Q_k^n \right\| \right\}^2$ .

Combining Lemmas 6.3 and 6.4 we see that

$$(6.3) \quad (M_n/m_n)^{1/2} \leq 4 \left\{ \max_{I \subset I_n} \left\| \sum_{k \in I} Q_k^n \right\| \right\}^2.$$

Let  $P_1^n, \dots, P_n^n$  be the projections associated with  $L_n$ .

Then

$$(6.4) \quad \left\| \sum_{k \in I} Q_k^n \right\| = \left\| \sum_{k \in I} P_k^n + \sum_{k \in I} (Q_k^n - P_k^n) \right\| \\ \leq 1 + \left\| \sum_{k \in I} (Q_k^n - P_k^n) \right\|$$

since the  $P_k^n$  are orthogonal. Let  $\Gamma_\ell = \{\lambda \mid |\lambda - \lambda_\ell| \leq d_\ell/3\}$ .

Using Lemma 6.2 and formula (c) of Lemma 3.7 we have

$$(6.5) \quad Q_k^n - P_k^n = \frac{1}{2\pi i} \int_{\partial \Gamma_k} [R_\lambda(\tilde{L}_n) - R_\lambda(L_n)] d\lambda \\ = \frac{1}{2\pi i} \int_{\partial \Gamma_k} R_\lambda(L_n) A_n R_\lambda(L_n) d\lambda \\ + \frac{1}{2\pi i} \int_{\partial \Gamma_k} R_\lambda(L_n) A_n R_\lambda(L_n) A_n R_\lambda(L_n) [I - A_n R_\lambda(L_n)]^{-1} d\lambda$$

if  $k > \ell_1$ . We now estimate the second term in the right side of (6.5).

Lemma 6.5. If  $\lambda \in \rho(L_n)$ , then

$$\|A_n R_\lambda(L_n)\| \leq \|AL^{(2-m)/m}\| \max_{k=1, \dots, n} \lambda_k^{(m-2)/m} |\lambda - \lambda_k|^{-1}.$$

Proof. From Lemma 4.12 we see that the range of  $L^{(2-m)/m}$  is contained in  $H_{m-2}$  and hence it follows from Lemma 3.8 that  $AL^{(2-m)/m}$  is bounded. It follows directly from the definition of a power of an operator that  $L_n^{(2-m)/m} = (L^{(2-m)/m})_n$ . Using this we find that

$$A_n L_n^{(2-m)/m} f = P_n AL^{(2-m)/m} f$$

for all  $f \in S_n$  and hence that

$$\|A_n L_n^{(2-m)/m}\| \leq \|AL^{(2-m)/m}\|.$$

For any  $f \in S_m$

$$L_n^{(m-2)/m} R_\lambda(L_n) f = \sum_{k=1}^n (f, x_k)_0 \lambda_k^{(m-2)/m} (\lambda - \lambda_k)^{-1} x_k,$$

and hence

$$\begin{aligned} \|L_n^{(m-2)/m} R_\lambda(L_n) f\|_0^2 &= \sum_{k=1}^n \lambda_k^{2(m-2)/m} |\lambda - \lambda_k|^{-2} |(f, x_k)_0|^2 \\ &\leq \max_k \lambda_k^{2(m-2)/m} |\lambda - \lambda_k|^{-2} \|f\|_0^2. \end{aligned}$$

Thus

$$\|L_n^{(m-2)/m} R_\lambda(L_n)\| \leq \max_k \lambda_k^{(m-2)/m} |\lambda - \lambda_k|^{-1}.$$

Combining these estimates we have

$$\begin{aligned}
\|A_n R_\lambda(L_n)\| &= \|A_n L_n^{(2-m)/m} L_n^{(m-2)/m} R_\lambda(L_n)\| \\
&\leq \|A_n L_n^{(2-m)/m}\| \|L_n^{(m-2)/m} R_\lambda(L_n)\| \\
&\leq \|A_n L_n^{(2-m)/m}\| \max_k \lambda_k^{(m-2)/m} |\lambda - \lambda_k|^{-1}.
\end{aligned}$$

Lemma 6.6. If  $0 < \nu < 1$  and  $\lambda \in \partial\Gamma_\ell$  then

$$\max_{k=1, \dots, n} \lambda_k^\nu |\lambda - \lambda_k|^{-1} \leq \max(3, 3\lambda_2/2\lambda_1) \lambda_\ell^\nu / d_\ell.$$

Proof. If  $k \neq \ell$ , then

$$\begin{aligned}
\lambda_k^\nu |\lambda - \lambda_k|^{-1} &\leq \lambda_k^\nu |\lambda_k - \lambda_\ell|^{-1} |\lambda - \lambda_\ell| |\lambda_k - \lambda_\ell|^{-1} \\
&\leq (3/2) \lambda_k^\nu |\lambda_k - \lambda_\ell|^{-1} \\
&= (3/2) |\lambda_k - \lambda_\ell|^{\nu-1} (\lambda_k / |\lambda_k - \lambda_\ell|)^\nu \\
&\leq (3/2) |\lambda_k - \lambda_\ell|^{\nu-1} (1 + \lambda_\ell / |\lambda_k - \lambda_\ell|)^\nu \\
&\leq (3/2) (\lambda_\ell + d_\ell)^\nu / d_\ell.
\end{aligned}$$

Since  $\lambda_k \geq d_k$  for  $k \geq 2$  we have

$$(\lambda_\ell + d_\ell)^\nu \leq \max(2, \lambda_2/\lambda_1) \lambda_\ell^\nu.$$

Hence

$$\lambda_k^\nu |\lambda - \lambda_k|^{-1} \leq \max(2, \lambda_2/\lambda_1) \lambda_\ell^\nu / d_\ell$$

if  $k \neq \ell$ . In the case  $k = \ell$  we have



$$\lambda_k^{\nu} |\lambda - \lambda_k|^{-1} = 3 \lambda_{\ell}^{\nu} / d_{\ell}.$$

These two estimates give the result.

Lemma 6.7. There is an integer  $n_1$  such that

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{\partial \Gamma_{\ell}} R_{\lambda}(L_n) A_n R_{\lambda}(L_n) A_n R_{\lambda}(L_n) [I - A_n R_{\lambda}(L_n)]^{-1} d\lambda \right\| \\ \leq 2 \|AL^{(2-m)/m}\|^2 \max(3, 3\lambda_2/2\lambda_1)^2 (\lambda_{\ell}^{(m-2)/m} / d_{\ell})^2 \end{aligned}$$

for  $\ell \geq n_1$ .

Proof. Let

$$d = \max(3, 3\lambda_2/2\lambda_1).$$

Suppose  $\lambda \in \Gamma_{\ell}$ . Lemmas 6.5 and 6.6 imply

$$(6.6) \quad \|A_n R_{\lambda}(L_n)\| \leq \|AL^{(2-m)/m}\| d \lambda_{\ell}^{(m-2)/m} / d_{\ell}$$

if  $\ell > \ell_1$ . Let  $n_1$  be an integer such that  $n_1 > \ell_1$  and

$$\|AL^{(2-m)/m}\| d \lambda_{\ell}^{(m-2)/m} / d_{\ell} < 1/2$$

if  $\ell \geq n_1$ . Then if  $\ell \geq n_1$ ,  $I - A_n R_{\lambda}(L)$  is invertible and

$$(6.7) \quad \|[I - A_n R_{\lambda}(L_n)]^{-1}\| \leq (1 - \|A_n R_{\lambda}(L_n)\|)^{-1} < 2.$$

Now, using (6.6), (6.7) and the fact that  $\|R_{\lambda}(L_n)\| = 3/d_{\ell}$ , we find

$$\begin{aligned}
& \|R_\lambda(L_n)A_nR_\lambda(L_n)A_nR_\lambda(L_n)[I - A_nR_\lambda(L_n)]^{-1}\| \\
& \leq \|R_\lambda(L_n)\| \|A_nR_\lambda(L_n)\|^2 \|[I - A_nR_\lambda(L_n)]^{-1}\| \\
& \leq (6/d_\ell^3) \|AL^{(2-m)/m}\|^2 d^2 \lambda_\ell^{2(m-2)/m}
\end{aligned}$$

if  $n \geq n_1$ .

Next we consider the problem of estimating

$$\left\| \sum_{k \in I} \frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n) d\lambda \right\|,$$

where  $I \subset I_n$ . Let  $1 \leq k \leq n$  and consider the Laurent expansion of  $R_\lambda(L_n)$  about  $\lambda_k$ :

$$R_\lambda(L_n) = \sum_{\ell=-\infty}^{\infty} B_\ell^k (\lambda - \lambda_k)^\ell, \quad \lambda \in \Gamma_k,$$

where

$$B_\ell^k = \frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n) (\lambda - \lambda_k)^{-\ell-1} d\lambda.$$

Since  $L_n$  is compact  $\lambda_k$  is a pole of  $R_\lambda(L_n)$ . The order of  $\lambda_k$  as a pole of  $R_\lambda(L_n)$  is equal to the ascent of  $\lambda_k - L_n$  and since  $\lambda_k$  has algebraic multiplicity one we see that  $\lambda_k$  is a simple pole. Thus the expansion has the form

$$R_\lambda(L_n) = B_{-1}^k (\lambda - \lambda_k)^{-1} + B_0^k + B(\lambda),$$

where  $B(\lambda) = \sum_{\ell=1}^{\infty} B_\ell^k (\lambda - \lambda_k)^\ell$ . Now  $B_{-1}^k = P_k^n$ , the  $k^{\text{th}}$  projection associated with  $L_n$ . Also

$$B_0^k = \frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n) (\lambda - \lambda_k)^{-1} d\lambda.$$

Now

$$\begin{aligned}
R_\lambda(L_n)A_nR_\lambda(A_n) &= P_k^n A_n P_k^n (\lambda - \lambda_k)^{-2} + B_0^k A_n P_k^n (\lambda - \lambda_k)^{-1} \\
&\quad + B(\lambda)A_n P_k^n (\lambda - \lambda_k)^{-1} + P_k^n A_n B_0^k (\lambda - \lambda_k)^{-1} \\
&\quad + B_0^k A_n B_0^k + B(\lambda)A_n B_0^k + P_k^n A_n B(\lambda) (\lambda - \lambda_k)^{-1} \\
&\quad + B_0^k A_n B(\lambda) + B(\lambda)A_n B(\lambda)
\end{aligned}$$

for  $\lambda \in \Gamma_k$  and hence, since

$$\begin{aligned}
\int_{\Gamma_k} (\lambda - \lambda_k)^{-2} d\lambda &= 0, \\
\int_{\Gamma_k} (\lambda - \lambda_k)^{-1} d\lambda &= 2\pi i, \\
\int_{\Gamma_k} B(\lambda) (\lambda - \lambda_k)^{-1} d\lambda &= 2\pi i B(\lambda_k) = 0, \\
\int_{\Gamma_k} d\lambda &= 0, \\
\int_{\Gamma_k} B(\lambda) d\lambda, \\
\int_{\Gamma_k} B^2(\lambda) d\lambda &= 0,
\end{aligned}$$

and

$$\frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n) d\lambda = B_0^k A_n P_k^n + P_k^n A_n B_0^k.$$

Using this formula we obtain

$$\begin{aligned}
(6.8) \quad &\left\| \sum_{k \in I} \frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n)A_nR_\lambda(L_n) d\lambda \right\| \\
&\leq \left\| \sum_{k \in I} B_0^k A_n P_k^n \right\| + \left\| \sum_{k \in I} P_k^n A_n B_0^k \right\|.
\end{aligned}$$

Now, since

$$R_\lambda(L_n) = \sum_{k=1}^n (\lambda - \lambda_k)^{-1} P_k^n,$$

and

$$\frac{1}{2\pi i} \int_{\Gamma_k} (\lambda - \lambda_\ell)^{-1} (\lambda - \lambda_k)^{-1} d\lambda = \begin{cases} 0, & \ell = k \\ (\lambda_k - \lambda_\ell)^{-1}, & \ell \neq k \end{cases},$$

we get

$$\begin{aligned} B_0^k &= \sum_{\substack{\ell=1 \\ \ell \neq k}}^n P_\ell^n \frac{1}{2\pi i} \int_{\Gamma_k} (\lambda - \lambda_\ell)(\lambda - \lambda_k)^{-1} d\lambda \\ &= \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (\lambda_k - \lambda_\ell)^{-1} P_\ell^n. \end{aligned}$$

For  $x \in S_n$  we have

$$\begin{aligned} P_k^n x &= (x, x_k)_0 x_k, \\ A_n P_k^n x &= (x, x_k)_0 P_n A x_k \\ &= (x, x_k)_0 \sum_{q=1}^n (A x_k, x_q)_0 x_q, \end{aligned}$$

and hence

$$\begin{aligned} B_0^k A_n P_k^n x &= \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (\lambda_k - \lambda_\ell)^{-1} P_\ell^n A_n P_k^n x \\ &= \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (\lambda_k - \lambda_\ell)^{-1} P_\ell^n (x, x_k)_0 \sum_{q=1}^n (A x_k, x_q)_0 x_q \\ &= \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (\lambda_k - \lambda_\ell)^{-1} (x, x_k)_0 (A x_k, x_\ell)_0 x_\ell, \end{aligned}$$

and thus

$$\left\| \sum_{k \in I} B_0^k A_n P_k^n x \right\|_0 = \left\| \sum_{k \in I} (x, x_k)_0 \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (\lambda_k - \lambda_\ell)^{-1} (A x_k, x_\ell)_0 x_\ell \right\|_0$$

$$\begin{aligned}
&\leq \sum_{k \in I} \left\| \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (\lambda_k - \lambda_\ell)^{-1} (Ax_k, x_\ell)_0 x_\ell \right\|_0 |(x, x_k)_0| \\
&\leq \left[ \sum_{k \in I} \left\| \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (\lambda_k - \lambda_\ell)^{-1} (Ax_k, x_\ell)_0 x_\ell \right\|_0^2 \right]^{1/2} \left[ \sum_{k \in I} |(x, x_k)_0|^2 \right]^{1/2} \\
&= \left[ \sum_{k \in I} \sum_{\substack{\ell=1 \\ \ell \neq k}}^n |(\lambda_k - \lambda_\ell)^{-1} (Ax_k, x_\ell)_0|^2 \right]^{1/2} \|x\|_0.
\end{aligned}$$

Hence

$$\begin{aligned}
(6.9) \quad &\left\| \sum_{k \in I} B_0^k A_n P_k^n \right\|_0 \\
&\leq \left[ \sum_{k \in I} \sum_{\substack{\ell=1 \\ \ell \neq k}}^n |(\lambda_k - \lambda_\ell)^{-1} (Ax_k, x_\ell)_0|^2 \right]^{1/2} \\
&= \left[ \sum_{k \in I} \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (AL^{(2-m)/m} L^{(m-2)/m} x_k, x_\ell)_0 (\lambda_k - \lambda_\ell)^{-1} |^2 \right]^{1/2} \\
&= \left[ \sum_{k \in I} \sum_{\substack{\ell=1 \\ \ell \neq k}}^n |\lambda_k^{(m-2)/m} (\lambda_k - \lambda_\ell)^{-1} (AL^{(2-m)/m} x_k, x_\ell)_0|^2 \right]^{1/2} \\
&\leq \left[ \sum_{k \in I} \max_{\substack{\ell=1, \dots, n \\ \ell \neq k}} |\lambda_k^{(m-2)/m} (\lambda_k - \lambda_\ell)^{-1}|^2 \sum_{\substack{\ell=1 \\ \ell \neq k}}^n |(AL^{(2-m)/m} x_k, x_\ell)_0|^2 \right]^{1/2} \\
&\leq \left[ \sum_{k \in I} \max_{\ell \neq k} |\lambda_k^{(m-2)/m} (\lambda_k - \lambda_\ell)^{-1}|^2 \|AL^{(2-m)/m} x_k\|_0^2 \right]^{1/2} \\
&\leq \|AL^{(2-m)/m}\| \left[ \sum_{k \in I} \max_{\ell \neq k} |\lambda_k^{(m-2)/m} (\lambda_k - \lambda_\ell)^{-1}|^2 \right]^{1/2} \\
&\leq \|AL^{(2-m)/m}\| \left[ \sum_{k \in I} (\lambda_k^{(m-2)/m} / d_k)^2 \right]^{1/2} \\
&\leq \|AL^{(2-m)/m}\| \left[ \sum_{k=1}^{\infty} (\lambda_k^{(m-2)/m} / d_k)^2 \right]^{1/2}.
\end{aligned}$$

Similarly we find that

$$P_k^n A_n B_0^k x = \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (x, x_\ell)_0 (\lambda_k - \lambda_\ell)^{-1} (Ax_\ell, x_k)_0 x_k$$

for  $x \in S_n$  and thus

$$\begin{aligned}
& \left\| \sum_{k \in I} P_k^n A_n B_0^k x \right\|_0 \\
&= \left\| \sum_{k \in I} \sum_{\substack{\ell=1 \\ \ell \neq k}}^n (x, x_\ell)_0 (\lambda_k - \lambda_\ell)^{-1} (Ax_\ell, x_k)_0 x_k \right\|_0 \\
&= \left\| \sum_{\ell=1}^n \left( \sum_{\substack{k \in I \\ k \neq \ell}} (Ax_\ell, x_k)_0 (\lambda_k - \lambda_\ell)^{-1} x_k \right) (x, x_\ell)_0 \right\|_0 \\
&\leq \sum_{\ell=1}^n \left\| \sum_{\substack{k \in I \\ k \neq \ell}} (Ax_\ell, x_k)_0 (\lambda_k - \lambda_\ell)^{-1} x_k \right\|_0 |(x, x_\ell)_0| \\
&\leq \left[ \sum_{\ell=1}^n \left\| \sum_{\substack{k \in I \\ k \neq \ell}} (Ax_\ell, x_k)_0 (\lambda_k - \lambda_\ell)^{-1} x_k \right\|_0^2 \sum_{\ell=1}^n |(x, x_\ell)_0|^2 \right]^{1/2} \\
&\leq \left[ \sum_{\ell=1}^n \sum_{\substack{k \in I \\ k \neq \ell}} |(Ax_\ell, x_k)_0 (\lambda_k - \lambda_\ell)^{-1}|^2 \right]^{1/2} \|x\|_0.
\end{aligned}$$

Hence

(6.10)

$$\begin{aligned}
\left\| \sum_{k \in I} P_k^n A_n B_0^k \right\| &\leq \left[ \sum_{\ell=1}^n \sum_{\substack{k \in I \\ k \neq \ell}} |(Ax_\ell, x_k)_0 (\lambda_k - \lambda_\ell)^{-1}|^2 \right]^{1/2} \\
&\leq \|AL^{(2-m)/m}\| \left[ \sum_{\ell=1}^n \max_{\substack{k \in I \\ k \neq \ell}} |\lambda_\ell^{(m-2)/m} (\lambda_k - \lambda_\ell)^{-1}|^2 \right]^{1/2} \\
&\leq \|AL^{(2-m)/m}\| \left[ \sum_{\ell=1}^{\infty} (\lambda_\ell^{(m-2)/m} / d_\ell)^2 \right]^{1/2}.
\end{aligned}$$

Combining (6.8), (6.9) and (6.10) gives

Lemma 6.8. For any  $n$  and any  $I \subset I_n$ ,

$$\begin{aligned}
\left\| \sum_{k \in I} \frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda(L_n) A_n R_\lambda(L_n) d\lambda \right\| \\
\leq 2 \|AL^{(2-m)/m}\| \left[ \sum_{k=1}^{\infty} (\lambda_k^{(m-2)/m} / d_k)^2 \right]^{1/2}.
\end{aligned}$$

The method used to prove Lemmas 6.7 and 6.8 is an adaptation of a method used by Kramer [20] to prove a result on perturbation of spectral operators.

Lemma 6.9. Let  $k$  be fixed. Then  $\|Q_k^n\|$  is bounded in  $n$  for  $n \geq k$ .

Proof. Using Lemma 3.5 we find that

$$\begin{aligned} \|R_\lambda(L_n)A_n f\|_j &= \|R_\lambda(L)A_n f\|_j \\ &\leq h_j(\lambda) |A_n f|_0 \\ &\leq \beta h_j(\lambda) \|f\|_j \end{aligned}$$

for  $f \in S_n$ . Let  $C_k^*$  be obtained from  $C_k$  by increasing the radius by such an amount that  $C_k^*$  does not intersect  $C_\ell$ ,  $\ell \neq k$ . If  $\lambda \in \partial C_k^*$ , then

$$\beta h_j(\lambda) < 1.$$

Thus  $R_\lambda(L_n)A_n$  considered as an operator on  $S_n$  with the  $j$  norm has norm less than one. From this it follows that  $I - R_\lambda(L_n)A_n$  is invertible and

$$(6.11) \quad \|[I - R_\lambda(L_n)A_n]^{-1}f\|_j \leq (1 - \beta h_j(\lambda))^{-1} \|f\|_j.$$

Another application of Lemma 3.5 gives

$$(6.12) \quad \|R_\lambda(L_n)f\|_j \leq \beta h_j(\lambda) |f|_0.$$

From (6.11) and (6.12) together with formula (b) of Lemma 3.7 we obtain

$$\begin{aligned}
 \|R_\lambda(\tilde{L}_n)f\|_0 &\leq \|R_\lambda(\tilde{L}_n)f\|_j \\
 &= \|[I - R_\lambda(L_n)A_n]^{-1}R_\lambda(L_n)f\|_j \\
 &\leq (1 - \beta h_j(\lambda))^{-1} \|R_\lambda(L_n)f\|_j \\
 &\leq (1 - \beta h_j(\lambda))^{-1} \beta h_j(\lambda) \|f\|_0.
 \end{aligned}$$

for  $f \in S_n$ ,  $\lambda \in \partial C_k^*$ . Hence

$$\begin{aligned}
 \|Q_k^n\| &= \left\| \frac{1}{2\pi i} \int_{\partial C_k^*} R_\lambda(\tilde{L}_n) d\lambda \right\| \\
 &\leq \text{Radius}(\partial C_k^*) \max_{\lambda \in \partial C_k^*} \beta (1 - \beta h_j(\lambda))^{-1} h_j(\lambda),
 \end{aligned}$$

which shows that  $\|Q_k^n\|$  is bounded in  $n$ .

Now from (6.3) and (6.4) we have

$$(M_n/m_n)^{1/2} \leq 4 \left\{ 1 + \max_{I \subset I_n} \left\| \sum_{k \in I} (Q_k^n - P_k^n) \right\| \right\}^2.$$

Using (6.5) and Lemmas 6.7 and 6.8 we find

$$\begin{aligned}
 &\left\| \sum_{k \in I} (Q_k^n - P_k^n) \right\| \\
 &\leq \left\| \sum_{\substack{k \in I \\ k \geq n_1}} \frac{1}{2\pi i} \int_{\partial \Gamma_k} R_\lambda(L_n) A_n R_\lambda(L_n) d\lambda \right\| \\
 &\quad + \left\| \sum_{\substack{k \in I \\ k \geq n_1}} \frac{1}{2\pi i} \int_{\partial \Gamma_k} R_\lambda(L_n) A_n R_\lambda(L_n) A_n R_\lambda(L_n) [I - A_n R_\lambda(L_n)]^{-1} d\lambda \right\| \\
 &\quad + \sum_{\substack{k \in I \\ k < n_1}} \|Q_k^n\| + \sum_{\substack{k \in I \\ k < n_1}} \|P_k^n\|
 \end{aligned}$$



$$\begin{aligned}
&\leq 2 \|AL^{(2-m)/m}\| \left[ \sum_{k=1}^{\infty} (\lambda_k^{(m-2)/m}/d_k)^2 \right]^{1/2} \\
&\quad + 2 \|AL^{(2-m)/m}\|^2 \max(3, 3\lambda_2/2\lambda_1)^2 \sum_{k=1}^{\infty} (\lambda_k^{(m-2)/m}/d_k)^2 \\
&\quad + \sum_{k=1}^{n_1} \|Q_k^n\| + n_1,
\end{aligned}$$

where  $n_1$  is as in Lemma 6.7. This inequality together with Lemmas 6.1 and 6.9 shows that  $M_n/m_n$  is bounded in  $n$ . This is summarized in

Theorem 6.1. Under the assumptions listed in the first paragraph of this section

$$M(z_1(n), \dots, z_n(n))/m(z_1(n), \dots, z_n(n))$$

is bounded in  $n$ .

§7. Applications.

Example 7.1. Consider the eigenvalue problem

$$\begin{cases} f^{(4)} + 10(\sin \tau)f^{(1)}(\tau) = \lambda f(\tau), & 0 \leq \tau \leq 1, \\ f(0) = f^{(2)}(0) = f(1) = f^{(2)}(1) = 0. \end{cases}$$

This is a problem of the type discussed in Sections 2-6 where

$$Lf = f^{(4)},$$

and

$$Af = 10(\sin \tau)f^{(1)}.$$

The problem determined by  $L$  and the boundary conditions is self-adjoint and  $L$  has eigenvalues and eigenvectors

$$\lambda_k = \pi^4 k^4,$$

$$x_k(\tau) = \sqrt{2} \sin k\pi\tau, \quad k = 1, 2, \dots$$

For these boundary conditions integration by parts shows that we can take  $\gamma = 1$  and  $\delta = 0$  in Lemma 3.3.

We find that  $\beta = 10 \sin 1$ . Thus from (3.8) we have

$$r_p = \max(\beta(1 + p^2\pi^2)^{1/2}, \tau_p)$$

where  $\tau_p$  is the unique positive solution of

$$\tau = \beta \left[ 1 + \left( \frac{2\tau(p+1)^4}{(p+1)^4 - p^4} \right)^{1/2} \right]^{1/2}.$$

An elementary calculation shows that

$$r_p = \beta(1 + p^2\pi^2)^{1/2}.$$

The circles  $C_p = \{\lambda \mid |p^4\pi^4 - \lambda| \leq r_p\}$  are mutually disjoint. Thus from Theorems 3.2 and 3.3 we see that the  $\mu_p$ , the eigenvalues of  $\tilde{L} = L + A$ , satisfy

$$|p^4\pi^4 - \mu_p| \leq 10(\sin 1)(1 + p^2\pi^2)^{1/2}.$$

By Corollary 2 of Theorem 3.3 they are all real.

Using Lemma 4.11 we have

$$|Ay_p|_0 \leq t_p$$

where  $t_p$  is the positive solution of

$$t = \beta(|\mu_p| + t)^{1/4}.$$

Thus

$$|w_p(n)|_0 \leq \frac{t_p}{|\mu_p - \lambda_{n+1}|}, \quad n \geq p,$$

by Lemma 4.10. Since

$$\begin{aligned} |Aw_p(n)|_0 &\leq \beta |w_p(n)^{(1)}|_0 \\ &= \beta \left( \int_0^1 |w_p(n)^{(1)}|^2 d\tau \right)^{1/2} \\ &= \beta \left( \int_0^1 w_p(n)^{(2)} \overline{w_p(n)} d\tau \right)^{1/2} \\ &= \beta (L^{1/2} w_p(n), w_p(n))_0^{1/2} \\ &= \beta |L^{1/4} w_p(n)|_0, \end{aligned}$$

$$|L^{1/4} w_p(n)|_0^2 = \sum_{k=n+1}^{\infty} \lambda_k^{1/2} |(y_p, x_k)_0|^2,$$

and

$$(\mu_p - \lambda_k)(y_p, x_k)_0 = (Ay_p, x_k)_0,$$

we have

$$\begin{aligned} |Aw_p(n)|_0 &\leq \beta \left( \sum_{k=n+1}^{\infty} \frac{\lambda_k^{1/2}}{|\mu_p - \lambda_k|^2} |(Ay_p, x_k)_0|^2 \right)^{1/2} \\ &\leq \beta \max_{k \geq n+1} \frac{\lambda_k^{1/4}}{|\mu_p - \lambda_k|} |Ay_p|_0 \\ &\leq \beta t_p \frac{\lambda_{n+1}^{1/4}}{|\mu_p - \lambda_{n+1}|}. \end{aligned}$$

This estimate for  $|Aw_p(n)|_0$  is essentially that of Lemma 4.13.

Thus for  $p$  fixed and  $n \geq p$  there is by Theorem 4.4 an  $\ell_0$ ,  $1 \leq \ell_0 \leq n$ , such that

$$|\mu_p - \eta_{\ell_0}(n)| \leq \frac{\beta(M_n/m_n)^{1/2} \lambda_{n+1}^{1/4} t_p}{[|\mu_p - \lambda_{n+1}|^2 - t_p^2]^{1/2}} \equiv \tilde{\epsilon}_p(n).$$

We must require that  $|\mu_p - \lambda_{n+1}|^2 - t_p^2 > 0$  for this result to hold. Similar assumptions must be made relative to other error estimates in this paragraph; we will not mention them explicitly. For  $n = p = 1$  we can use  $|\mu_1 - \lambda_2| \geq \lambda_2 - \lambda_1 - r_1$  to obtain

$$|\mu_1 - \eta_1(1)| \leq \frac{\beta(M_1/m_1)^{1/2} \lambda_2^{1/4} t_1}{[|\mu_1 - \lambda_2|^2 - t_1^2]^{1/2}}$$

$$\leq \frac{\beta(M_n/m_n)^{1/2} \lambda_2^{1/4} t_1'}{[(\lambda_2 - \lambda_1 - r_1)^2 - (t_1')^2]^{1/2}} \equiv \varepsilon_1(1),$$

where  $t_1'$  is the positive solution of

$$t = \beta(\lambda_1 + r_1 + t)^{1/4}.$$

Now suppose we have inequalities of the form

$$|\mu_p - \eta_p(n-1)| \leq \varepsilon_p(n-1), \quad p = 1, \dots, n-1.$$

These, together with  $|\mu_n - \lambda_n| \leq r_n$ , yield

$$\frac{\beta(M_n/m_n)^{1/2} \lambda_{n+1} t_p}{[|\mu_p - \lambda_{n+1}|^2 - t_p^2]^{1/2}} \leq \begin{cases} \frac{\beta(M_n/m_n)^{1/2} \lambda_{n+1} t_p''}{[(\lambda_{n+1} - \eta_p(n-1) - \varepsilon_p(n-1))^2 - (t_p'')^2]^{1/2}}, & p < n, \\ \frac{\beta(M_n/m_n)^{1/2} \lambda_{n+1} t_p''}{[(\lambda_{n+1} - \lambda_n - r_n)^2 - (t_p'')^2]^{1/2}}, & p = n, \end{cases}$$

$$\equiv \varepsilon_p(n),$$

where  $t_p''$  is the positive solution of

$$t = \begin{cases} \beta(\eta_p(n-1) + \varepsilon_p(n-1) + t)^{1/4}, & p < n, \\ \beta(\lambda_n + r_n + t)^{1/4}, & p = n. \end{cases}$$

The numbers  $t_1'$ ,  $t_p''$  etc., can be computed numerically. Thus we have

$$|\mu_p - \eta_{\ell_0}(n)| \leq \varepsilon_p(n)$$

for some  $\ell_0$ ,  $1 \leq \ell_0 \leq n$ . We can conclude  $\ell_0 = p$  in this example as follows. From the computed values of

$\eta_1(n), \dots, \eta_n(n)$  and  $\varepsilon_p(n)$  it will be clear that  $|\mu_p - \eta_k(n)| > \varepsilon_p(n)$ ,  $k = 1, \dots, p-1, p+1, \dots, n$ . Hence the only choice for  $\ell_0$  is  $p$ . This will also be true in many other cases and the inequalities  $|\mu_p - \eta_k(n)| > \varepsilon_p(n)$ ,  $k \neq p$ , are simple to verify if  $\varepsilon_p(n)$  is sufficiently small since  $\eta_k(n) \in C_k$ ,  $k = 1, \dots, n$ . Proceeding inductively we define  $\varepsilon_p(n)$ ,  $1 \leq p \leq n$ ,  $n = 1, 2, \dots$  and establish

$$|\mu_p - \eta_p(n)| \leq \varepsilon_p(n).$$

We obtain  $\varepsilon_p(n)$  from  $\tilde{\varepsilon}_p(n)$  by using the estimate for  $\mu_p$  which we had in the  $(n-1)^{\text{st}}$  stage of the estimating procedure. Clearly  $\lim_{n \rightarrow \infty} \varepsilon_p(n) = 0$  for  $p$  fixed. This discussion on the definition of  $\varepsilon_p(n)$  and the proof that  $\ell_0 = p$  pertains to the other examples in this section.

Values for  $\eta_p(n)$ ,  $\varepsilon_p(n)$  were computed for  $1 \leq p \leq n$ ,  $n = 1, 2, \dots, 20$ . Tables 1 and 2 contain the computed values for  $\eta_p(n)$  and  $\varepsilon_p(n)$  for  $p = 1, 2$ , and  $5$ . Table 3 contains the values of  $\eta_p(20)$  and  $\varepsilon_p(20)$  for  $p = 1, \dots, 20$ . The values in Table 3 give the best approximations for  $\mu_p$ ,  $p = 1, \dots, 20$ , that were obtained.  $\eta_p(n)$ ,  $p = 1, \dots, n$ , are the eigenvalues of the matrix  $((\tilde{L}_n x_k, x_\ell)_0)$ ,  $1 \leq \ell, k \leq n$ . We will denote this matrix as well as the corresponding operator by  $\tilde{L}_n$ . This is a real non-symmetric matrix. These eigenvalues were computed as follows. Using the QR method of Francis [18], as implemented in a program

originally written by Professor B.N. Parlett, the eigenvalues were computed; we will call these numbers the first approximations to the eigenvalues of  $\tilde{L}_n$ . Next the eigenvectors were computed. Then a matrix  $P$  was formed with these approximate eigenvectors as columns. Now  $P^{-1}\tilde{L}_nP$  has the same eigenvalues as  $\tilde{L}_n$  and, assuming that the column vectors of  $P$  are close to the eigenvectors of  $\tilde{L}_n$ ,  $P^{-1}\tilde{L}_nP$  should be approximately diagonal and hence the diagonal elements of  $P^{-1}\tilde{L}_nP$  should be approximately equal to the eigenvalues of  $\tilde{L}_n$ . We will call the diagonal elements of  $P^{-1}\tilde{L}_nP$  the 2<sup>nd</sup> approximations to the eigenvalues of  $\tilde{L}_n$ . The difference between the eigenvalues of  $\tilde{L}_n$  and their 2<sup>nd</sup> approximations can be estimated with Gershgorin's theorem. We use these computed diagonal elements of  $P^{-1}\tilde{L}_nP$  for  $\eta_1(n), \dots, \eta_n(n)$ .

In general the off diagonal elements of  $P^{-1}\tilde{L}_nP$  were small. Consider for example  $n = 4$ . The QR method produces

93.2076349,  
1,554.28815,  
7,885.89093

and

24,932.4448.

The diagonal elements of  $P^{-1}\tilde{L}_nP$  are

93.2076495

1,544.28838,

7,885.89194,

and

24,932.4458

and the row sums  $\sum_{l \neq k} |a_{kl}|$  where  $(a_{kl}) = P^{-1} \tilde{L}_n P$  are

$.274 \times 10^{-9}$ ,

$.411 \times 10^{-3}$ ,

$.736 \times 10^{-1}$ ,

and

$.813 \times 10^{-3}$ .

Thus, for instance,

$$|\eta_1(4) - 93.2076495| \leq .274 \times 10^{-9}.$$

These estimates can be refined by scaling in the following way. Suppose we want a better estimate for  $\eta_3(4)$ . The eigenvalues we are seeking are also the eigenvalues of the matrix  $(b_{kl})$  which we obtain from  $(a_{kl})$  by multiplying the 3<sup>rd</sup> row by  $\tau$  and the 3<sup>rd</sup> column by  $\tau^{-1}$ . The diagonal elements of  $(b_{kl})$  and  $(a_{kl})$  are the same but the row sums for  $(b_{kl})$  depend on  $\tau$ . Now choose  $\tau$  as small as possible while keeping the 3<sup>rd</sup> Gershgorin disk separate from the other disks. Here we can choose  $\tau = 10^{-10}$ ; with this choice for  $\tau$  the 3<sup>rd</sup> row sum for  $(b_{kl})$  is  $.736 \times 10^{-11}$ . We can perform



such scaling for each eigenvalue. For  $n = 10$  the radii of the Gershgorin disks before scaling were

.197  $\times 10^{-6}$ ,  
 .881  $\times 10^{-5}$ ,  
 .124  $\times 10^{-3}$ ,  
 .723  $\times 10^{-2}$ ,  
 .253  $\times 10^{-2}$ ,  
 .279,  
 .209  $\times 10^2$ ,  
 .784  $\times 10^2$ ,  
 .789  $\times 10^1$ ,  
 .451  $\times 10^{-4}$ .

The ordering here is such that the 3<sup>rd</sup> radii gives an estimate for  $\eta_3(10)$  for example. After scaling the corresponding radii are

.197  $\times 10^{-6}$ ,  
 .603  $\times 10^{-8}$ ,  
 .196  $\times 10^{-7}$ ,  
 .424  $\times 10^{-6}$ ,  
 .704  $\times 10^{-7}$ ,  
 .427  $\times 10^{-5}$ ,  
 .194  $\times 10^{-3}$ ,  
 .475  $\times 10^{-3}$ ,  
 .329  $\times 10^{-4}$ ,  
 .451  $\times 10^{-4}$ .

These calculations are affected by the round off error which occurs in the inversion of  $P$  and in the calculation of  $P^{-1}\tilde{L}_n P$ .  $P^{-1}$  was computed in single precision; for  $n = 4$  no element of  $P^{-1}P - I$  exceeds in absolute value  $.3 \times 10^{-7}$ . Double precision was used to multiply  $P^{-1}$ ,  $\tilde{L}_n$ , and  $P$ . In addition there is the error which arises when the matrix  $\tilde{L}_n$  is entered in the computer. To reduce this the entries in  $\tilde{L}_n$  were computed in double precision and then truncated to single precision. The remaining calculations were all done in single precision.

For a complete discussion of this method of computing a 2<sup>nd</sup> set of approximations for the eigenvalues of a matrix from an initial set of approximations for the eigenvalues and eigenvectors and estimation of the resulting accuracy using Gershgorin's theorem together with scaling see Wilkenson [42].

In the tables of this section all figures are rounded to the number of digits presented. The notation  $.487|2$  means  $.487 \times 10^2$ . A dash in the table indicates the corresponding number is the same as the number directly above it. All of the computations were performed on the UNIVAC 1108 Computer of the Computer Science Center of the University of Maryland.

Table 1

n/p	$\eta_p(n)$				
	1	2	5	5	15
1	.9309239	2			
2	.9320139	2	.15542023	4	
3	.9320682	2	.15542827	4	
4	.9320765	2	.15542884	4	
5	.9320786	2	.15542895	4	.608764151
6	.9320792	2	.15542898	4	.608764551
7	.9320795	2	.15542899	4	.608764592
8	.9320796	2	.15542900	4	.608764597
9	.9320797	2	—	4	.608764601
10	—	2	—	4	.608764605
11	—	2	—	4	.608764602
12	—	2	—	4	.608764603
13	—	2	—	4	.608764605
14	—	2	—	4	.608764606
15	—	2	—	4	.608764606
16	—	2	—	4	.608764603
17	—	2	—	4	.608764602
18	—	2	—	4	.608764607
19	—	2	—	4	.608764603
20	—	2	—	4	.608764606
					.493133098
					.493133097
					.493133101
					.493133101
					.493133100
					.493133100

Table 2

 $\epsilon_p(n)$ 

n\p	1		2		5		15	
1	.109510	1						
2	.28955	0	.6910	0				
3	.12111	0	.2455	0				
4	.6187	-1	.1210	0				
5	.3577	-1	.691	-1	.3279	0		
6	.2252	-1	.432	-1	.1442	0		
7	.1508	-1	.289	-1	.843	-1		
8	.1059	-1	.203	-1	.555	-1		
9	.772	-2	.148	-1	.390	-1		
10	.580	-2	.111	-1	.287	-1		
11	.447	-2	.85	-2	.218	-1		
12	.351	-2	.67	-2	.170	-1		
13	.281	-2	.54	-2	.135	-1		
14	.229	-2	.44	-2	.110	-1		
15	.188	-2	.36	-2	.90	-2	.12	0
16	.157	-2	.30	-2	.75	-2	.6	-1
17	.132	-2	.25	-2	.63	-2	.4	-1
18	.113	-2	.21	-2	.54	-2	.3	-1
19	.97	-3	.18	-2	.46	-2	.2	-1
20	.83	-3	.16	-2	.40	-2	.2	-1

Table 3

p	$\eta_p(20)$		$\epsilon_p(20)$	
1	.932079727	2	.834	-3
2	.155429005	4	.159	-2
3	.788589868	4	.237	-2
4	.249325000	5	.317	-2
5	.608764606	5	.396	-2
6	.126237964	6	.477	-2
7	.233875011	6	.560	-2
8	.398983423	6	.645	-2
9	.639096829	6	.735	-2
10	.974086689	6	.832	-2
11	.142616228	7	.939	-2
12	.201987069	7	.106	-1
13	.278209682	7	.120	-1
14	.374206342	7	.138	-1
15	.493133100	7	.160	-1
16	.638379791	7	.191	-1
17	.813570038	7	.235	-1
18	.102256125	8	.309	-1
19	.126944459	8	.455	-1
20	.155854502	8	.891	-1

Example 7.2. Consider the eigenvalue problem for the Mathieu equation

$$\begin{cases} -f^{(2)}(\tau) + 2q(\cos 2\tau)f(\tau) = \lambda f(\tau), \\ f(0) = f(\pi) = 0, \end{cases}$$

where  $q$  is a complex constant. The eigenvalues of this problem will be the characteristic values of the Mathieu equation associated with odd periodic solutions. Here we let  $Lf = -f^{(2)}$  and  $Af = 2q(\cos 2\tau)f$ . The eigenvalues and eigenvectors of  $L$  are

$$\begin{aligned} \lambda_k &= k^2, \\ x_k(\tau) &= \sqrt{\frac{2}{\pi}} \sin k\tau, \quad k = 1, 2, \dots \end{aligned}$$

$\tilde{L}$  will be non self-adjoint unless  $q$  is real.

For this example we easily find that

$$C_p = \{\lambda \mid |\lambda - p^2| \leq 2|q|\}.$$

If  $|q| < 3/4$  these circles will be mutually disjoint and there will be one eigenvalue of  $\tilde{L}$  in each  $C_p$ .

Using Theorem 4.4 and Lemma 4.10 we find that for  $p$  fixed and  $n \geq p$  there is an  $\ell_0$ ,  $1 \leq \ell_0 \leq n$ , such that

$$|\mu_p - \eta_{\ell_0}(n)| \leq \frac{(M_n/m_n)^{1/2} 4|q|^2}{[|\mu_p - \lambda_{n+1}|^2 - 4|q|^2]^{1/2}} \equiv \tilde{\varepsilon}_p(n).$$

As in Example 7.1 we can define error estimates  $\varepsilon_p(n)$ , using at the  $n^{\text{th}}$  stage the estimates for  $\mu_p$  from the

$(n-1)^{\text{st}}$  stage in order to estimate  $|\mu_p - \lambda_{n+1}|$ . We can show that  $\ell_0 = p$  as in Example 7.1.

The computations were carried out for  $q = \frac{1+i}{2\sqrt{2}}$  with  $n$  going from 1 to 20. The  $\ell, k$  element of the matrix of  $\tilde{L}_n$  with respect to  $x_1, \dots, x_n$  is

$$\delta_{\ell k} k^2 + \frac{4q}{\pi} \int_0^\pi \cos 2\tau \sin k\tau \sin \ell\tau \, d\tau.$$

The values for  $\eta_1(n), \dots, \eta_n(n)$  were computed using the power method as developed by E.E. Osborne [25]; the computer program was written by Ehrlich [14]. These values were not refined as in Example 7.1. Table 4 gives the real and imaginary parts of  $\eta_p(n)$  and Table 5 gives the values of  $\epsilon_p(n)$  for  $p = 1, 2,$  and  $10$ . Table 6 contains the values of  $\eta_p(20)$  and  $\epsilon_p(20)$  for  $p = 1, \dots, 20$ . The characteristic values for the Mathieu equation for complex values of  $q$  have recently been tabulated by Blanch and Clemm [11].

Table 4

 $\eta_p(n)$ 

n/p	1				2				10				
1	.64645	0	-.35355	0	.40000	1	.00000	0					
2	.64645	0	-.35355	0	.40000	1	.454909	-27					
3	.64508	0	-.38341	0	.399996	1	-.208332	-1					
4	.64508	0	-.38341	0	.399996	1	-.208332	-1					
5	.64511	0	-.38342	0	.399998	1	-.208333	-1					
6	---	0	---	0	---	1	---	-1					
7	---	0	---	0	---	1	---	-1					
8	---	0	---	0	---	1	---	-1					
9	---	0	---	0	---	1	---	-1					
10	---	0	---	0	---	1	---	-1	.10000000	3	.69444442	-2	
11	---	0	---	0	---	1	---	-1	---	3	.69444444	-2	
12	---	0	---	0	---	1	---	-1	---	3	.12626263	-2	
13	---	0	---	0	---	1	---	-1	---	3	.12626262	-2	
14	---	0	---	0	---	1	---	-1	.99999999	2	.12626266	-2	
15	---	0	---	0	---	1	---	-1	---	2	.12626265	-2	
16	---	0	---	0	---	1	---	-1	---	2	.12626265	-2	
17	---	0	---	0	---	1	---	-1	---	2	.12626266	-2	
18	---	0	---	0	---	1	---	-1	---	2	.12626265	-2	71
19	---	0	---	0	---	1	---	-1	---	2	.12626265	-2	
20	---	0	---	0	---	1	---	-1	---	2	.12626266	-2	



Table 5

 $\varepsilon_p(n)$ 

$n \setminus p$	1		2		10	
1	.57735	0				
2	.12955	0	.25820	0		
3	.7132	-1	.9264	-1		
4	.4466	-1	.5190	-1		
5	.3106	-1	.3433	-1		
6	.2269	-1	.2439	-1		
7	.1733	-1	.1830	-1		
8	.1366	-1	.1426	-1		
9	.1105	-1	.1144	-1		
10	.912	-2	.938	-2	.5495	-1
11	.747	-2	.765	-2	.2438	-1
12	.636	-2	.649	-2	.1554	-1
13	.548	-2	.557	-2	.1115	-1
14	.477	-2	.484	-2	.857	-2
15	.419	-2	.425	-2	.686	-2
16	.371	-2	.376	-2	.566	-2
17	.331	-2	.335	-2	.478	-2
18	.297	-2	.300	-2	.410	-2
19	.268	-2	.270	-2	.356	-2
20	.243	-2	.244	-2	.313	-2

Table 6

p	$\eta_p(20)$				$\varepsilon_p(20)$	
1	.64511279	0	-.38341700	0	.24276410	-2
2	.39999774	1	-.20833277	-1	.24462794	-2
3	.90013346	1	.14238454	-1	.24746011	-2
4	.16000023	2	.83332760	-2	.25153524	-2
5	.24999999	2	.52084830	-2	.25697720	-2
6	.36000000	2	.35714295	-2	.26395698	-2
7	.49000001	2	.26041665	-2	.27271085	-2
8	.64000000	2	.19841267	-2	.28356169	-2
9	.81000000	2	.15625002	-2	.29695245	-2
10	.99999999	2	.12626266	-2	.31349869	-2
11	.12100000	3	.10416668	-2	.33407274	-2
12	.14400000	3	.87412561	-3	.35994473	-2
13	.16900000	3	.74404793	-3	.39302948	-2
14	.19600000	3	.64102585	-3	.43634552	-2
15	.22500000	3	.55803573	-3	.49493354	-2
16	.25600000	3	.49019597	-3	.57787746	-2
17	.28899999	3	.43402798	-3	.70335894	-2
18	.32400000	3	.38699679	-3	.91383116	-2
19	.36100000	3	.34722220	-2	.13368434	-1
20	.40000000	3	.32894737	-2	.26733725	-1

Example 7.3. Let  $Lf = f^{(4)}$  and  $Af(\tau) = \tau f^{(2)}(\tau)$  and consider the problem

$$\begin{cases} Lf + Af = \lambda f, \\ f(0) = f^{(1)}(0) = f(1) = f^{(1)}(1) = 0. \end{cases}$$

The eigenvalues of  $L$  are  $\lambda_n = \rho_n^4$  where  $\rho_1, \rho_2, \dots$  are the positive solutions of  $1 = \cosh \rho \cos \rho$ . The unnormalized eigenvectors of  $L$  are

$$\begin{aligned} x_n(\tau) = & (\cos \rho_n - \sin \rho_n - e^{-\rho_n})e^{\rho_n \tau} + (-\cos \rho_n - \sin \rho_n + e^{\rho_n})e^{-\rho_n \tau} \\ & + 2(\cosh \rho_n - \cos \rho_n)\sin \rho_n \tau + 2(\sin \rho_n - \sinh \rho_n)\cos \rho_n \tau. \end{aligned}$$

In (3.1) we can let  $\gamma = 1, \delta = 0$  and we find that

$$r_p = (1 + \lambda_p^{1/2} + \lambda_p)^{1/2}.$$

Thus the eigenvalues of  $\tilde{L}$  occur one each in the circles  $C_1, C_2, \dots$  and they are real.

By Lemma 4.11 we have

$$|Ay_p|_0 \leq t_p$$

where  $t_p$  is the positive solution of  $t = (|\mu_p| + t)^{1/2}$ , i.e.,

$$t_p = \frac{1 + (1 + 4|\mu_p|)^{1/2}}{2}.$$

Using Lemma 4.10 we find

$$|w_p(n)|_0 \leq \tau_p |\mu_p - \lambda_{n+1}|^{-1}.$$

Integration by parts shows that

$$|Aw_p(n)|_0 \leq |Lw_p(n)|_0$$

and, proceeding as in the proof of Lemma 4.13, we obtain

$$|Aw_p(n)|_0 \leq \frac{\lambda_{n+1}^{1/2} \tau_p}{|\mu_p - \lambda_{n+1}|}$$

Combining these results with Theorem 4.4 shows that

$$|\mu_p - \eta_{\ell_0}(n)| \leq \frac{(M_n/m_n)^{1/2} \lambda_{n+1}^{1/2} (1+(1+4|\mu_p|)^{1/2})}{[4|\mu_p - \lambda_{n+1}|^2 - (1+(1+4|\mu_p|)^{1/2})^2]^{1/2}} \equiv \tilde{\varepsilon}_p(n)$$

if  $n \geq p$  for some  $\ell_0 \leq n$ . Just as in Example 7.1 we estimate  $\tilde{\varepsilon}_p(n)$  by  $\varepsilon_p(n)$ , using at the  $n^{\text{th}}$  stage the estimates for  $\mu_p$  obtained at the  $(n-1)^{\text{st}}$  stage, and show that  $\ell_0 = p$ .

The computational procedures used in this example are the same as those used in Example 7.1. Table 7 lists the values of  $\lambda_p$  and  $r_p$  for  $p \leq 10$ , Tables 8 and 9 list the values of  $\eta_p(n)$  and  $\varepsilon_p(n)$  for  $n \leq 20$  and  $p = 1$  and 5, and Table 10 lists the values of  $\eta_p(20)$  and  $\varepsilon_p(20)$  for  $p \leq 20$ .

Table 7

P	$\lambda_p$		$r_p$	
1	.5006	3	.229	2
2	.38035	4	.622	2
3	.146176	5	.1214	3
4	.399438	5	.2004	3
5	.891354	5	.2991	3
6	.1738813	6	.4175	3
7	.3082084	6	.5557	3
8	.5084815	6	.7136	3
9	.7934031	6	.8912	3
10	.11840136	7	.10886	4

Table 8

 $\eta_p(n)$ 

$n \setminus p$	1		5	
1	.4950003	3		
2	.4949835	3		
3	.4949857	3		
4	.4949863	3		
5	.4949868	3	.89007252	5
6	.4949870	3	.89007080	5
7	.4949872	3	.89007077	5
8	.4949872	3	.89007074	5
9	.4949873	3	.89007073	5
10	.4949873	3	.89007073	5
11	.4949874	3	.89007074	5
12	—	3	—	5
13	—	3	—	5
14	—	3	—	5
15	—	3	—	5
16	—	3	—	5
17	.4949875	3	—	5
18	—	3	—	5
19	—	3	—	5
20	—	3	—	5

Table 9

 $\epsilon_p(n)$ 

$n \setminus p$	1		5	
1	.4397	0		
2	.1963	0		
3	.1162	0		
4	.773	-1		
5	.552	-1	.1491	1
6	.414	-1	.763	0
7	.322	-1	.512	0
8	.258	-1	.381	0
9	.211	-1	.299	0
10	.176	-1	.243	0
11	.149	-1	.203	0
12	.128	-1	.172	0
13	.111	-1	.148	0
14	.97	-2	.129	0
15	.85	-2	.114	0
16	.76	-2	.101	0
17	.68	-2	.90	-1
18	.61	-2	.81	-1
19	.55	-2	.73	-1
20	.50	-2	.66	-1

Table 10

p	$\eta_p(20)$		$\epsilon_p(20)$	
1	.494987465	3	.5027	-2
2	.378185872	4	.1370	-1
3	.145702886	5	.2680	-1
4	.398608925	5	.4431	-1
5	.890070740	5	.6631	-1
6	.173697687	6	.9297	-1
7	.307959661	6	.1246	0
8	.508157720	6	.1615	0
9	.792994409	6	.2047	0
10	.118351009	7	.2550	0
11	.170308296	7	.314	0
12	.237742900	7	.385	0
13	.323460204	7	.471	0
14	.430499370	7	.578	0
15	.562133345	7	.718	0
16	.721868858	7	.909	0
17	.913446414	7	.119	1
18	.114084031	8	.165	1
19	.140825862	8	.256	1
20	.172014325	8	.529	1



Example 7.4. Consider an eigenvalue problem of the form

$$\begin{cases} -f^{(2)}(\tau) + a(\tau)f^{(1)}(\tau) = \lambda f(\tau), \\ f^{(1)}(0) + f(0) = 0, \\ f^{(1)}(1) - f(1) = 0. \end{cases}$$

If we let  $Lf = -f^{(2)}$  then  $L$  has one negative eigenvalue. If  $L$  is redefined by  $Lf = -f^{(2)} + 5f$  then  $L$  is positive definite. This will enable us to apply Lemma 4.15. Hence we consider the eigenvalue equation  $-f^{(2)} + af^{(1)} + 5f = \lambda f$ . The eigenvalues of the original problem are found by subtracting 5 from the eigenvalues of this problem. For this example we show how explicit values for the constants in inequalities (3.1) and (4.4) can be obtained.

Redheffer [26] proved a version of the inequality in Lemma 5.3 when  $\ell = 2$  in which the constants are, in a certain sense, best possible. He showed that

$$|f|_1^2 \leq r|f|_2^2 + (12 + \frac{1}{r})|f|_0^2$$

for all  $f \in H_2$  and  $r > 0$ . If  $|f|_2 \leq |f|_0$  we can let  $r = 2$  in this inequality to obtain

$$|f|_1 \leq \sqrt{2} |f|_0^{1/2} |f|_2^{1/2} + \sqrt{12.5} |f|_0.$$

If  $|f|_2 > |f|_0$  we can let  $r = |f|_0 |f|_2^{-1}$  and find

$$|f|_1 \leq \sqrt{2} |f|_0^{1/2} |f|_2^{1/2} + \sqrt{12} |f|_0.$$

Thus in inequality (3.1) we can let  $\gamma = \sqrt{2}$  and  $\delta = \sqrt{12.5}$ .

Now we consider inequality (4.4). Proceeding as in Section 5 we have

$$\begin{aligned} (Lf, f)_0 &= \int_0^1 (-f^{(2)} + 5f)\bar{f} \, d\tau \\ &= |f|_1^2 + 5|f|_0^2 - f^{(1)}\bar{f}|_0^1, \end{aligned}$$

and using the boundary conditions,

$$(Lf, f)_0 = |f|_1^2 + 5|f|_0^2 - |f(1)|^2 - |f(0)|^2.$$

Following the proof of Lemma 5.2 in [2] we find

$$|f(\tau)|^2 \leq h|f|_1^2 + \frac{2}{h}|f|_0^2$$

for  $f \in H_1$  and  $0 < h \leq 1/2$ . Using this to bound  $|f(0)|^2$  and  $|f(1)|^2$  we find

$$\begin{aligned} (Lf, f)_0 &\geq (1-2h)|f|_1^2 + (5 - \frac{4}{h})|f|_0^2 \\ &= (1-2h) \|f\|_1^2 + (4 + 2h - \frac{4}{h})|f|_0^2. \end{aligned}$$

Letting  $h = 1/4$  we obtain

$$\|f\|_1^2 \leq 2(Lf, f)_0 + 23|f|_0^2$$

for all  $f \in H_2$  which satisfy the boundary conditions. Thus in (4.4) we can let  $c_1 = 2$  and  $c_2 = 23$ . These values can then be used in Lemma 4.15.

Example 7.5. The eigenvalue problem.

$$\begin{cases} -f^{(2)} + af^{(1)} = \lambda f, \\ f(0) = f(1) = 0, \end{cases}$$

can be treated by introducing a new independent variable:

$$g(\tau) = f(\tau) \exp\left(-\frac{1}{2} \int_0^\tau a(s) ds\right).$$

This transforms the above problem into

$$\begin{cases} -g^{(2)}(\tau) + \left(\frac{(a(\tau))^2}{4} - \frac{a^{(1)}(\tau)}{4}\right) g(\tau) = \lambda g(\tau), \\ g(0) = g(1) = 0. \end{cases}$$

Hence we can consider the problem as a zero order perturbation of a 2<sup>nd</sup> order problem rather than as a 1<sup>st</sup> order perturbation. This transformation, applied to Example 7.4, would produce non self-adjoint boundary conditions unless  $a(0)$  and  $a(1)$  are real.

## REFERENCES

1. Agmon, S., "On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems," Comm. Pure Appl. Math. 15(1962), 119-147.
2. Agmon, S., Lectures on elliptic boundary value problems, Van Nostrand, Princeton, New Jersey, 1965.
3. Aronszajn, N., "Rayleigh-Ritz and A. Weinstein methods for approximation of eigenvalues, I, II," Proc. Nat. Acad. Sci. U.S.A. 34(1948), 474-480, 594-601.
4. Aronszajn, N., "Approximation methods for eigenvalues of completely continuous symmetric operators," Proceedings of the Symposium on Spectral Theory and Differential Problems, Stillwater, Oklahoma (1951), 179-202.
5. Aronszajn, N. and Weinstein, A., "Existence, convergence, and equivalence in the unified theory of eigenvalues of plates and membranes," Proc. Nat. Acad. Sci. U.S.A., 27 (1941), 188-191.
6. Bauer, F.L. and Fike, C.T., "Norms and exclusion theorems," Numerische Math. 2(1960), 137-141.
7. Bazley, N.W., "Lower bounds for eigenvalues," J. Math. Mech. 10(1961), 289-308.
8. Bazley, N.W. and Fox, D.W., "Truncations in the method of intermediate problems for lower bounds to eigenvalues," J. Res. Nat. Bur. Standards Sect. B, 56(1961).
9. Bazley, N.W. and Fox, D.W., "A procedure for estimating eigenvalues," J. Math. Phys. 3(1962), 469-471.
10. Birkhoff, G.D., "Boundary value and expansion problems of ordinary linear differential equations," Trans. Amer. Math. Soc. 9(1908), 373-395.
11. Blanch, G. and Clemm, D.S., Mathieu's equation for complex parameters, tables of characteristic values, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, 1969.
12. Clark, C., "On relatively bounded perturbations of ordinary differential operators," Pacific J. Math. 25(1968), 59-70.
13. Dunford, N. and Schwartz, J.T., Linear operators, Part III, Interscience, New York, in press.

14. Ehrlich, L.W., "F4 UTEX MATSUB, Eigenvalues and eigenvectors of complex non-Hermitian matrices using the direct and inverse power methods and matrix deflation," Systems Programs for CO-OP, Control Data Corporation.
15. Fichera, G., "Approximation and estimates for eigenvalues," Numerical Solution of Partial Differential Equations, Academic Press, New York (1965), 317-352.
16. Fichera, G., "Further developments in the approximation theory of eigenvalues," Numerical Solution of Partial Differential Equations, II, Academic Press, New York, in press.
17. Fox, D.W. and Rheinboldt, W.C., "Computational methods for determining lower bounds for eigenvalues of operators in Hilbert space," SIAM Rev. 8(1966), 427-462.
18. Francis, J.G.F., "The QR transformation, Parts I and II," Computer J. 4(1961), 265-271, 332-345.
19. Gould, S.H., Variational methods for eigenvalue problems. An introduction to Weinstein's theory of intermediate problems, 2nd edition. University of Toronto Press, Toronto, 1966.
20. Kramer, H.P., "Perturbation of differential operators," Pacific J. Math. 7(1957), 1405-1435.
21. Lorch, E.R., "Bicontinuous linear transformations in certain vector spaces," Bull. Amer. Math. Soc. 45(1939), 564-569.
22. Naimark, M.A., Linear differential operators, Part I, Frederick Ungar, New York, 1967.
23. Osborn, J.E., "Approximation of the eigenvalues of non self-adjoint operators," J. Math. and Phys. 45(1966), 391-401.
24. Osborn, J.E., "Approximation of the eigenvalues of a class of unbounded, non self-adjoint operators," SIAM J. Numer. Anal. 4 (1967), 45-54.
25. Osborne, E.E., "On acceleration and matrix deflation processes used with the power method," J. Soc. Indust. Appl. Math. 6(1958), 279-287.
26. Redheffer, R.M., "Über eine beste Ungleichung zwischen den Normen von  $f$ ,  $f'$ ,  $f''$ ," Math. Z. 80(1962/63), 390-397.

27. Schwartz, J.T., "Perturbations of spectral operators, and applications. I. Bounded perturbations," Pacific J. Math. 4(1954), 415-458.
28. Stenger, W., "On the variational principles for eigenvalues for a class of unbounded operators," J. Math: Mech. 17(1968), 641-648.
29. Tamarkin, Ya. D., "Some points in the theory of differential equations," Rend. di Palermo, 34(1912), 354-382. (French).
30. Taylor, A.E., Introduction to functional analysis, Wiley, New York, 1958.
31. Vainikko, G.M., "Asymptotic error bounds for projection methods in the eigenvalue problem," Ž. Vyčisl. i. Mat. Fiz. 4(1964), 405-425 (Russian).
32. Vainikko, G.M., "On the rate of convergence of certain approximation methods of Galerkin Type in an eigenvalue problem," Izv. Vysš. Učebn. Zaved. Matematika, 1966, No. 2(51), 37-45. Amer. Math. Soc. Transl. (2) Vol. 86, 1970.
33. Vainikko, G.M., "Rapidly of convergence of approximation methods in eigenvalue problems," Ž. Vycisl. i. Mat. Fiz. 7(1967), 997-987. (Russian).
34. Weinberger, H.F., "Error estimation in the Weinstein method for eigenvalues," Proc. Amer. Math. Soc. 3(1952), 643.
35. Weinberger, H.F., "A theory of lower bounds for eigenvalues." (Tech. Note BN-183), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, (1959).
36. Weinstein, A., "Sur la stabilité des plaques encastrées," Compt. Rend. 200(1935), 107-109.
37. Weinstein, A., "Étude des spectres des équations aux dérivées partielles," Mémor. Sci. Math. 88(1937).
38. Weinstein, A., "Bounds for eigenvalues and the method of intermediate problems," Proceedings of the International Conference on Partial Differential Equations and Continuum Mechanics, Univ. of Wisconsin Press, Madison (1961), 39-53.
39. Weinstein, A., A necessary and sufficient condition in the maximum-minimum theory of eigenvalues, studies in mathematical analysis and related topics, Stanford University Press, Stanford, 1962.

40. Weinstein, A., "Intermediate problems and the maximum-minimum theory of eigenvalues," J. Math. Mech. 12(1963), 235-246.
41. Weinstein, A., "An invariant formulation of the new maximum-minimum theory of eigenvalues," J. Math. Mech. 16(1966), 213-218.
42. Wilkinson, J.H., The algebraic eigenvalue problem, Clarendon Press, Oxford, 1965.

Department of Mathematics  
University of Maryland  
College Park, Maryland  
20742