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STATISTICAL ENERGY METHODS

Jerome E. Manning
Paul J. Remington

Contract No. NAS5-21020
BBN Report No. 2064

26 March 1971

Submitted to:

National Aeronautics and Space Administration
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PREFACE

During the past ten years a new method of vibration analysis, commonly called Statistical Energy Analysis (SEA), has been developed to study the dynamic behavior of large, complex structures and acoustic spaces. This report presents a review of SEA and a guide for its use. The authors hope that the presentation will eliminate some of the confusion about SEA which has inhibited its worthwhile use in many cases.

LIST OF SYMBOLS

A = door area — rod cross sectional area in Appendix A

A_p = plate area

A_1, A_2 = rod cross sectional areas

$A_R(\omega)$ = admittance of receiving system

$A_{R,inf}(\omega)$ = admittance of infinite beam

a_m = weighting factor

b = damping coefficient

$b_{10}, b_{11}, b_{20}, b_{21}$ = damping coefficients of dampers at ends of rods

c_0 = acoustic wave speed — rod wave speed in Appendix A

c_b = beam bending wave speed

C_1, C_2 = damping coefficients of oscillator.

c_1, c_2 = rod wave speeds

C_{mn} = see Eq. 3.3.1.2-5

E = Young's modulus in beam equations

E = total energy

F = force applied to an oscillator

$F(\omega)$ = complex amplitude of force at driving point of structure

F_{BL} = blocked force

$F_{blocked}(\omega)$ = blocked force

F_c = coupling force

F_n = see Eq. 4.3-2

F_m = see Eq. 3.3.2-6

f = force/unit length

f_1, f_2 = see Eq. 3.3.1.1-3

$f_i(t)$ = plate mode i modal force

g_m = see Eq. 3.2.1-22

G = gyroscopic coupling coefficient

$G(x_\xi, x_f, \omega)$ = rod Green's functions

I = acoustic intensity

I = bending moment of inertia

$I(k_1, k_x)$, $I(k_2, k_y)$ = see Eq. 3.2.2-13

$j_i(\omega)$ = joint acceptance of i^{th} mode

K_1, K_2 = oscillator spring stiffness

K_c = coupling spring stiffness

k = oscillator spring stiffness

k_0 = acoustic wavenumber

k_x, k_y = wavenumbers of simply supported mode in a panel

k_1, k_2 = wavenumber space variables

L_1, L_2 = rod lengths

M = oscillator mass

M = total number of modes in a rod in a given frequency band

M_b = beam mass

M_c = coupling mass

M_p = modal mass of plate

m = oscillator mass

m_1, m_2 = mass of rods

m_ℓ = mass/unit length

N = total number of modes in a rod in a given frequency band

$n_1(\omega), n_2(\omega)$ = rod modal density

P = acoustic pressure

Q = quality factor

$R_{i,RAD}(\omega)$ = radiation resistance of the i^{th} panel mode

S = surface area

$S_F(\omega)$, $S_{F_{source}}(\omega)$ = spectral density of the force applied by an ideal force source

$S_{F_{blocked}}(\omega)$ = spectral density of blocked force

$S_{f_i}(\omega)$ = spectral density of modal force f_i

$S_p(\underline{x}_1, \underline{x}_2, \omega)$ = cross spectral density of pressure

$S_p(\underline{k}, \omega)$ = see Eq. 3.2.2-10

$S_p(\omega)$ = spectral density of pressure

$S_V(\omega)$, $S_{V_{source}}(\omega)$ = spectral density of the velocity of an ideal velocity source

$S_{\pi_{in}}(\omega)$ = power input spectral density

T_{rev} = reverberation time

U_m = rod modal amplitude

V = room volume

$V_{free}(\omega)$ = free velocity

V_n = beam modal amplitude

$V_R(\omega)$ = complex velocity of receiving structure

$V_S(\omega)$ = complex velocity of source

$v(\underline{x},t)$ = plate velocity at time t and position $\underline{x}$

$v(x,t)$ = beam velocity at time t and point χ

v_n = rod modal amplitude

x = oscillator displacement

x_{c1} = location of coupling spring in rod 1

x_{c2} = location of coupling spring in rod 2

x_{f1} = location of applied force in rod 1

x_{f2} = location of applied force in rod 2

Z_R = receiving structure impedance

Z_S = source impedance

Z_1, Z_2 = point impedance of rod

Z_0, Z_1 = rod end impedance in Appendix A

α = absorption coefficient

$\overline{\alpha}$ = see Eq. 2-3

β_p = damping coefficient

γ = see Eq. 4.3-7

Δ_1, Δ_2 = see Eq. 3.3.1.2-5

Δ_m, Δ_n = see Eq. 3.3.1.2-5

$\varepsilon_1, \varepsilon_2, \varepsilon_m, \varepsilon_n$ = energy in single mode or a single oscillator

$\varepsilon_m^{(1)}, \varepsilon_n^{(2)}$ = energy in a mode in rod 1 and rod 2, respectively

η = dissipation loss factor

η_m = modal dissipation loss factor

η_{12} = coupling loss factor

κ = see Eq. 3.3.1.1-3

λ_c = acoustic wavelength

λ = see Eq. 3.3.1.1-3

μ = see Eq. 3.3.1.1-3

ξ_1, ξ_2 = rod displacements

ξ_{ER} = see Eq. 4.3-7

ξ_R = right running travelling wave displacement amplitude

ξ_L = left running travelling wave displacement amplitude

π_{diss} = power dissipated

π_{in} = power injected

π_{trans} = power transmitted

π_{1i} = power transmitted from element 1 to element i

π_{mn} = power transmitted between two oscillators

π_{12} = power transmitted from rod 1 to rod 2

ρs = mass/unit area of panel

ρ_0 = air density - rod density in Appendix A

ρ_1, ρ_2 = rod density

ϕ_{12} = coupling coefficient

$\phi_m(x)$ = rod mode shape

$\tilde{\Psi}_i^{-}(k)$ = see Eq. 3.2.2-9

$\psi_i(\underline{x})$ = plate mode shape

$\psi_n(x)$ = beam and rod mode shape

ω = radian frequency

ω_m = beam natural frequencies

ω_c = critical frequency

ω_1, ω_2 = oscillator natural frequencies

ω_m, ω_n = rod natural frequencies

$\langle\ \rangle_t$ = time average

$\langle\ \rangle_{t,\Delta\omega}$ = time average of quantity in frequency band $\Delta\omega$

$\langle\ \rangle_{ens}$ = average over an ensemble of structures

$\langle\ \rangle_{\Delta\omega}$ = average over frequency band $\Delta\omega$

$\langle\ \rangle_L$ = see Eq. 3.3.1.2-9

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1. INTRODUCTION

Recent years have seen a continuing trend toward higher-performance vehicles and larger, more complex structural assemblies. This trend has brought with it a greater number of problems associated with vibrations at high frequencies well above the fundamental resonance frequency of the structure being studied. As a result, aerospace engineers and research scientists have a great need for methods to predict and understand the high-frequency behavior of large complex structures.

Historically, the vibration engineer has focused his attention on the low-frequency range encompassing the first few resonance frequencies of the structure being studied. Thus, a large number of analytical and numerical techniques have been developed dealing with low-frequency vibration problems. However, none of these techniques can deal simply and effectively with a high-frequency vibration problem, in which a large number of modes of vibration contribute to the overall response of the structure.

The classical techniques of vibration analysis, which have served well in studying low-frequency vibrations, are valid, at least in principle, at all frequencies. However, their use at high-frequencies is almost always impractical, particularly when the excitation is random and distributed over the structure.

The most commonly used classical technique of vibration analysis consists of determining the natural modes of vibration, calculating the responses of these modes to the specified excitation of interest, and superposing these responses to determine the total structural response. Use of this technique to determine the response of a large structure to a high-frequency random excitation require⁻ inordinate amounts of computation because of the large number of modes that must be included in the analysis. The computational requirements can be met using a digital computer in most cases. Thus, although the amount of computation required by this classical technique was once the limiting factor in its use, it is no longer, unless one is concerned with the cost of computer time.

A more inhibiting limitation in the use of the classical technique of vibration analysis arises from our inability to calculate accurately the higher-order resonance frequencies, mode shapes and modal damping coefficients. These parameters are much more sensitive to details of the structure than are the same parameters for the lower order modes. Thus, one must be able to describe the structural and material properties and boundary conditions with great precision in order to be able to perform computations involving the higher-order modes meaningfully. The required precision cannot usually be achieved because of manufacturing tolerances and other uncertainties. Even when the properties of the structure are known almost exactly, they are usually so complex that an exact calculation of resonance frequencies, mode shapes, etc, is impossible.

A similar situation exists in the field of room acoustics. In the audio frequency range, a normal-sized living room or office will have thousands of resonance frequencies — a concert hall may have millions of resonance frequencies. Also, because of uncertainties in the location and amount of acoustical damping in the room and because of the very complex shape of the room due to the location of furniture, accurate prediction of the resonance frequencies, mode shapes and damping parameters is impossible. To solve his problems, the room acoustician has used and helped develop statistical energy methods of vibration analysis. The methods are statistical, not only because the source of excitation is considered to be random, but more importantly, because the systems being analyzed are presumed to be drawn from ensembles of systems with random parameters, i.e., resonance frequencies, mode shapes, etc. The methods identify energy as the primary dynamic variable so that the fundamental dynamic equations are simple. Once steady-state is reached, the acoustical power injected into a room must equal the power dissipated in the room plus the power transmitted to other rooms,

$$\pi_{in} = \pi_{diss} + \pi_{trans} \ . \tag{1-1}$$

Combining the statistical and the energy approaches, we average the terms in Eq. 1-1 over time, over the ensemble of systems and, in the case of random excitation, over bands of frequency,

$$\langle\pi_{in}\rangle_{t,ens,\Delta f} = \langle\pi_{diss}\rangle_{t,ens,\Delta f} + \langle\pi_{trans}\rangle_{t,ens,\Delta f} \ . \tag{1-2}$$

Equation 1-2 is the basis of the statistical energy techniques of vibration analysis.

In the early 1960's, R.H. Lyon and his colleagues at Bolt Beranek and Newman Inc. began using statistical energy techniques to study the interaction of sound fields and large panel structures.[1] Later they expanded their studies to include the interaction of connected structures.[2] It would be incorrect to say that Lyon and his colleagues invented the statistical energy approach, or even were the first to use the approach to study the sound-structure interaction. They were, however, the first to identify the fundamental principles on which a statistical energy analysis could be based and to remove much of the empiricism that existed in the use of statistical energy techniques in room acoustics.

To describe their way of looking at the dynamic interaction of sound fields and structures, Lyon and his colleagues coined the name Statistical Energy Analysis (SEA). A number of papers dealing with SEA have attempted to construct a formal "method" of analysis.[3-5] These attempts are important to the development of SEA as a useful tool in vibration analysis. However, it is important not to discriminate between the concepts of SEA and the various methods of implementation of SEA for practical problems. In this report, an attempt both to review the concepts of SEA and to suggest a formal method of analysis is made. The use of Statistical Energy Analysis (SEA) leads to statistical estimates of the time-average energy in each mode of vibration. In its simplest and most commonly used form, SEA leads to the average model energy – the average being taken over time and over all modes with resonance frequencies in a band Δf. For any continuous homogeneous structure, e.g., panels or shells of constant thickness, beams with uniform cross sections, etc.; the average modal energy can be used to find the spatial-energy mean-square response. Most past uses of SEA have been limited to finding spatial-average response of the structural elements making up the complete assembly. This does not mean that the SEA approach is limited to calculation of spatial-average responses. Techniques to calculate statistical estimates of peak response have been suggested[6] and used on occasion.

Statistical Energy Analysis (SEA) has been used many times to estimate the response of laboratory models of complex structures. For the most part the spatial-average vibration levels predicted by SEA have agreed fairly well, within ±5 dB, with measured data. SEA has been used to predict the response of plates[7] and cylindrical shells[8] to an acoustic field, to predict the sound transmission through walls,[9] to predict the vibration transmission between connected plates,[10] to predict the vibration transmission from a shell to a connected instrument package,[11] to predict the vibration transmission in ribbed plates and shells,[12] and for several other problems.[13-18]

In spite of its reasonable success in predicting the response of laboratory models to a vibratory excitation, SEA is not often used to predict the response of field structures. However, a few applications have been made and are worthy of mention. Franken and Lyon[19] have used SEA with success to predict the response of the Titan launch vehicle to acoustic loads. Sevy and Earls[20] have used SEA to predict the vibration transmission to an internally mounted instrument package. Mansour has used SEA to predict vibration transmission in the Mariner '64.[21] Manning has used SEA to predict the noise reduction of the OGO-NIMBUS shroud.[22] And finally, Sawley has used SEA to predict vibration transmission in a ship.[23]

The most extensive use of SEA has been to predict the vibration transmitted from an external acoustic field to a shroud enclosed spacecraft model.[24] The model was quite simple but maintained the very basic properties of the OGO spacecraft.

In Sec. 2 of this report the concepts of SEA are introduced by considering a simple problem from room acoustics. In solving this problem, many of the empirical observations that initially led to the idea of SEA are called on. Then, in Sec. 3, a complete discussion of the SEA concepts is presented. Section 4 illustrates the use of SEA by studying the dynamic interaction of two beams coupled by a spring. And, finally, in Sec. 5 guidelines for the use of SEA in solving particular problems are presented. References to more advanced uses of SEA are given. An appendix of the report gives a bibliography of reports on SEA.

2. AN INTRODUCTORY EXAMPLE FROM ROOM ACOUSTICS

The use of statistical methods of analysis is common in room acoustics.[25] Indeed many of the concepts presently used to analyze the dynamic behavior of complex structures were originally developed for acoustic spaces.[26] For this reason it is appropriate to introduce the basic ideas behind statistical energy analysis (SEA) with a simple room-acoustics problem, even though the solution of this problem considerably predates SEA development.

The problem which will be considered consists of two adjoining rooms connected by a door as shown in Fig. 1. A small air conditioner in room 1 acts as a source of random acoustic noise. The problem is to determine the sound pressure level (SPL) in the rooms — first, with the door closed, and then with it open.

The rooms are considered to be typical of those found in family living areas. They are rectilinear. But, since they are filled with furniture, their shape as an acoustic space is very complex. They will be assumed to be carpeted and to have a typical amount of the wall area covered by drapery. The effect of the carpet, drapery and upholstered furniture is to add damping in a complex way.

It is assumed that the manufacturer of the air conditioner has measured its radiated acoustic power in octave bands in an anechoic chamber, and has supplied this data.

Clearly, there are a number of approaches which can be used to solve the problem. However, it is necessary to point out that use of the classical normal-mode approach would require that one make a very simplified mathematical model for the problem. The exact solution for this mathematical model could probably be

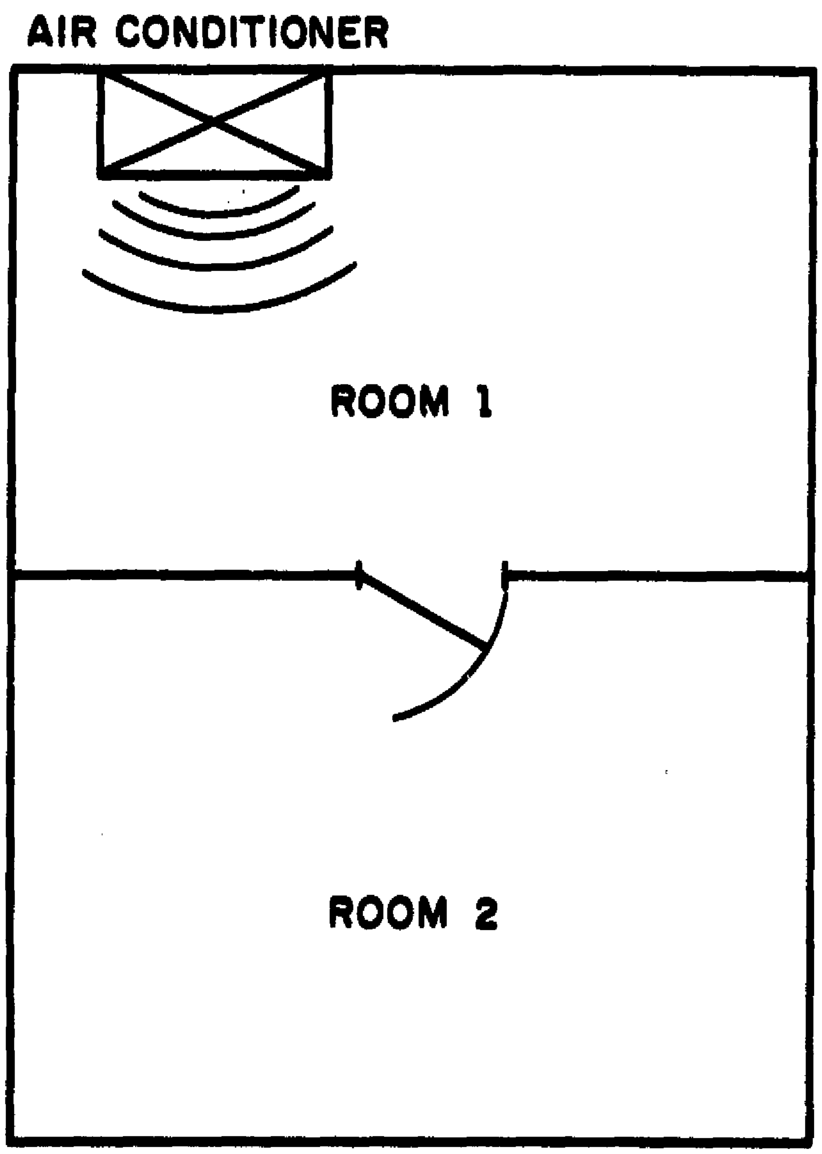

FIG.1 TWO CONNECTED ROOMS EXCITED BY AN AIR CONDITIONER

found by expending a great deal of effort. But then the relevence of this exact solution to the solution of the actual problem would be questionable. The approach most commonly used by acousticians to solve a problem of this type is a statistical energy approach.

Following a statistical energy approach the governing equations for the dynamic behavior of the two rooms are set out in terms of energy and power variables. Each room is treated as an energy storage system as shown in Fig. 2. It will be assumed that the problem is linear so that in each band of frequencies the source injects acoustic power into room 1 which is either dissipated in the room or transmitted to room 2. The power transmitted to room 2 is either dissipated in that room or transmitted back to room 1. Under steady-state conditions the time average energies in the two rooms stay constant. Then, one can write the governing equations for the rooms as a time-average power balance for each band of frequencies. For room 1,

$$\langle \pi_{in,source} \rangle_{t,\Delta\omega} = \langle \pi_{diss,room\ 1} \rangle_{t,\Delta\omega} + \langle \pi_{trans,1\ to\ 2} \rangle_{t,\Delta\omega} \tag{2-1}$$

where $\langle \pi \rangle_{t,\Delta\omega}$ signifies a time-average of the power π in the frequency band $\Delta\omega$ and $\pi_{trans,1\ to\ 2}$ is the net power transmitted from room 1 to room 2. For room 2,

$$\langle \pi_{trans,2\ to\ 1} \rangle_{t,\Delta\omega} = \langle \pi_{diss,room\ 2} \rangle_{t,\Delta\omega} . \tag{2-2}$$

To solve these power balance equations one must first relate the input power in each band of frequencies to known characteristics of the source. Then, the power dissipated in the two rooms and the power transmitted between rooms must be related to the acoustic energy in each room. And, finally, after solving the power balance

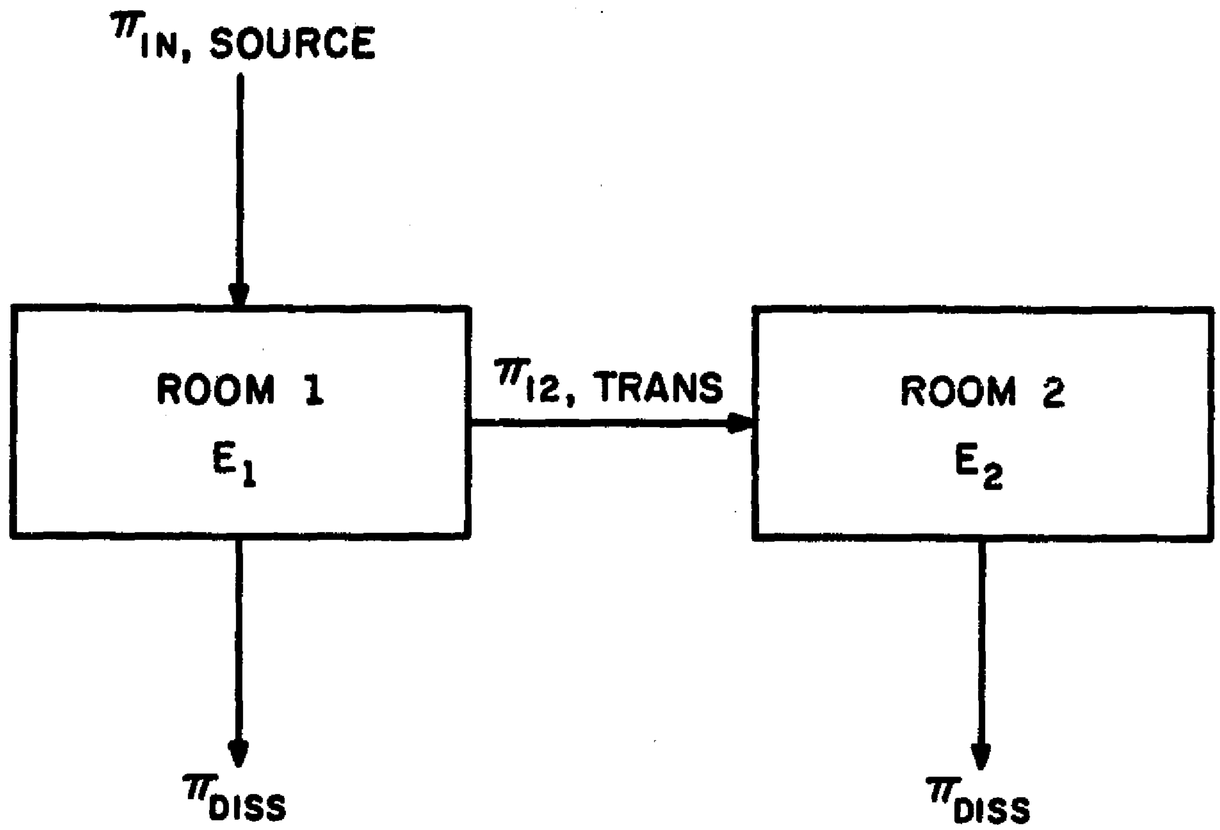

FIG.2 POWER EXCHANGE BETWEEN THE TWO ROOMS

equations for the energy in the two rooms one must relate the sound pressure level in each room to the energy. The power input to room 1 will be found first.

On the basis of simple observation it has long been known that the acoustic power injected into a large room in a band of frequencies by a source of noise does not depend on the characteristics of the room or the location of the source. A room can be considered large as long as its dimensions are much greater than an acoustic wavelength at the center-frequency of the band being considered. Based on the above observation one can assume that the air conditioner injects the same acoustic power into room 1 as into an anechoic chamber. Then, one simply uses the manufacturers radiated power data to give the power input in each octave band of frequencies to room 1.

To continue with the solution of the problem the power dissipated in each room must be related to the acoustic energy. Again, appeal will be made to simple empirical observations which have been made in acoustics. The power dissipated in a room depends on the acoustic energy in the room and the amount of absorptive material, such as carpet, drapery, etc., in the room. When a sound wave impinges on an absorbing surface some of its energy is dissipated and the remainder is reflected back into the room. The ratio of the energy absorbed by a surface to the energy incident on it is defined as the absorption coefficient. In general, absorption coefficients depend on frequency, the angle of incidence of the acoustic waves, the area of material, the way in which the material is mounted and, of course, the properties of the material. In a room with a typical amount of absorptive material the sound waves travel around the room and are reflected many times before their energy is absorbed. In addition, the acoustic waves incident on small reflecting objects are scattered into waves travelling in many different directions. One would expect the sound waves impinging on a particular point in a large irregular room to come from a large number of different directions. As a limiting case the sound field in the room becomes diffuse. In a diffuse field sound waves of equal energy density travel in all possible directions. The absorption coefficient of a material in a diffuse field does not depend on either the exact location or orientation of the material. This result is usually observed in typical rooms, such as the ones being considered, and leads to the common assumption that the sound field in a typical room is diffuse. Techniques to measure diffuse-field absorption coefficients have been established, and data for a number of different materials and mounting configurations are available. One can use the measured absorption coefficients for the materials in the room to obtain an average absorption coefficient for the room. This average absorption coefficient is given by

$$\overline{\alpha} = \frac{1}{s} \Sigma s_i \alpha_i \qquad (2\text{-}3)$$

where $\overline{\alpha}$ is the average absorption coefficient for the room, s is the total surface area of the room, α_i is the absorption coefficient of the *ith* segment of absorptive material, s_i is the surface area of the *ith* segment and the summation is over all absorptive material segments in the room.

In a diffuse field the acoustic energy density is the same at every location in the room. The time-average intensity (power incident per unit area) in the diffuse field is simply given by[27]

$$\langle I \rangle_t = \frac{c_0}{4V} \langle E \rangle_t \qquad (2\text{-}4)$$

where c_0 is the speed of sound, V is the volume of the room and $\langle E\rangle_t$ is the total time-average energy in the room. Combining Eq. 2-4 with the definition of $\bar{\alpha}$ we can relate the power dissipated in the room to its total time-average energy. The result is

$$\langle \pi_{diss}\rangle_{t,\Delta\omega} = \frac{c_0 s}{4V} \bar{\alpha} \langle E\rangle_{t,\Delta\omega} \tag{2-5}$$

where $\bar{\alpha}$ is the value of the room absorption coefficient at the band-center-frequency and $\langle E\rangle_{t,\Delta\omega}$ is the average energy in the room in the frequency band $\Delta\omega$. Equation 2-5 can be applied to either room 1 or 2 by assuming the acoustic fields in these rooms are diffuse and by using the correct values for s, V and $\bar{\alpha}$.

As the third step in the analysis one must find the acoustic power transmitted from rooms 1 to 2. When the door is closed it will be assumed that no power is transmitted between the rooms. When the door is open the sound power incident on the doorway from room 1 will be transmitted to room 2 while the sound power incident on the doorway from room 2 will be transmitted back into room 1. If one assumes that the sound fields in the two rooms are diffuse then the power incident on the doorway from the two rooms can be found from Eq. 2-4. The net time-average power transmitted from room 1 to room 2 is given by

$$\langle \pi_{12}\rangle_{t,\Delta\omega} = \frac{c_0 A}{4} \left[\frac{\langle E_1\rangle_{t,\Delta\omega}}{V_1} - \frac{\langle E_2\rangle_{t,\Delta\omega}}{V_2}\right] \tag{2-6}$$

where A is the area of the door.

Now the power balance equations can be written. With the door closed the result becomes

$$\langle \pi_{in,source} \rangle_{t,\Delta\omega} = \frac{c_0 S_1}{4V_1} \bar{\alpha}_1 \langle E_1 \rangle_{t,\Delta\omega} , \tag{2-7}$$

and with the door open two equations result

$$\langle \pi_{in,source} \rangle_{t,\Delta\omega} = \frac{c_0 S_1}{4V_1} \bar{\alpha}_1 \langle E_1 \rangle_{t,\Delta\omega} + \frac{c_0 A}{4} \left[\frac{\langle E_1 \rangle_{t,\Delta\omega}}{V_1} - \frac{\langle E_2 \rangle_{t,\Delta\omega}}{V_2} \right] \tag{2-8}$$

and

$$\frac{c_0 A}{4} \left[\frac{\langle E_1 \rangle_{t,\Delta\omega}}{V_1} - \frac{\langle E_2 \rangle_{t,\Delta\omega}}{V_2} \right] = \frac{c_0 S_2}{4V} \bar{\alpha}_2 \langle E_2 \rangle_{t,\Delta\omega} . \tag{2-9}$$

Equations 2-7, 2-8 and 2-9 can be solved quite easily for the time-average energies in the two rooms. To complete the solution one must relate the energy in each room to the mean-square sound pressure. Since it has been assumed that the acoustic fields in the two rooms are diffuse, this step is quite simple. The mean-square sound pressure in the diffuse field is independent of location and is related to the total room energy in each frequency band by the equation[27]

$$\langle p^2 \rangle_{t,\Delta\omega} = \frac{\rho_0 c_0^2}{V} \langle E \rangle_t . \tag{2-10}$$

From Eqs. 2-7 and 2-10 the mean-square sound pressure in room 1 with the door closed is given by

$$\langle p_1^2\rangle_{t,\Delta\omega} = \frac{4\rho_0 c_0}{s_1\bar{\alpha}_1}\langle\pi_{in,source}\rangle_{t,\Delta\omega} . \tag{2-11}$$

With the door open solving Eqs. 2-8 and 2-9 leads to

$$\langle p_1^2\rangle_{t,\Delta\omega} = \frac{4\rho_0 c_0}{s_1\bar{\alpha}_1}\langle\pi_{in,source}\rangle_{t,\Delta\omega}\left[\frac{\frac{s_1\bar{\alpha}_1}{A} + \frac{s_1\bar{\alpha}_1}{s_2\bar{\alpha}_2}}{1 + \frac{s_1\bar{\alpha}_1}{A} + \frac{s_1\bar{\alpha}_1}{s_2\bar{\alpha}_2}}\right] . \tag{2-12}$$

Similarly, one finds the mean-square sound pressure in room 2 to be

$$\langle p_2^2\rangle_{t,\Delta\omega} = \frac{1}{1 + \frac{s_2\alpha_2}{A}}\langle p_1^2\rangle_{t,\Delta\omega} . \tag{2-13}$$

Considering the complexity of the problem, the solution is quite simple and requires only the general properties of rooms 1 and 2 such as volume, area of absorption material, etc. By now it is obvious to the reader that this approach should be called an energy approach. However, the reasons for calling it statistical are not yet clear. In carrying out the analysis it was assumed that the input power did not depend on the exact location of the source or on the shape and dissipation in the room. It was also assumed that the sound fields in the two rooms were diffuse. Both of these assumptions are valid in a statistical sense. One defines an ensemble of rooms with the same volume but with random shapes and with the same amount of absorption but with random locations for the absorptive material. Then, the ensemble-average power input will equal the power input to an infinite acoustic field. Similarly, if one averages over the ensemble of rooms he will find sound waves coming from all directions with equal energy density so that the ensemble-average sound field will be diffuse. In future sections of this report the use of statistics will be brought out more explicitly. It is hoped that the simple problem discussed in this section serves to introduce the important ideas behind Statistical Energy Analysis (SEA).

3. FUNDAMENTAL CONCEPTS OF STATISTICAL ENERGY ANALYSIS

In this section the fundamental concepts underlying Statistical Energy Analysis (SEA) will be discussed. The chapter is divided into five subsections. In the first subsection the derivation of a statistical energy model for interconnected complex structures and acoustic spaces is discussed. The model will be an energy model in that each structural element of the complex assembly will be treated as a vibratory energy storage element. The model will be statistical in that the resonance frequencies, mode shapes, connection points, etc. will be treated as random variables. In the second subsection the use of SEA concepts to calculate the power input to a complex structure or acoustic space will be discussed. Then in Subsections 3.3 and 3.4 the use of SEA concepts to calculate the power transmitted between connected structures and acoustic spaces and the power dissipated in a complex structure or acoustic space is covered. Finally, in Sec. 3.5 the power balance equations are written in terms of the energies in each element.

3.1 The Statistical Energy Model

The first step in a Statistical Energy Analysis of the dynamic behavior of a complex structural assembly is the modelling of the assembly by a number of interconnected energy storage elements, as shown in Fig. 3. This step is the most creative and the most difficult part of SEA. Decisions of how to divide up the complete assembly into the energy storage boxes, and which paths of power exchange to include in the analysis require a great deal of insight, which can only be developed by continued use of SEA and comparison of the resulting predictions with data from experiments or exposure of the structure to the actual environment.

A basic assumption in SEA is that the response of a connected structure or acoustic space in a given frequency band can be described by the collective motion of the modes of each isolated structure which are resonant in the band. The dynamic interaction between the structures and acoustic spaces is studied by allowing the modes of the isolated structures to be coupled. Thus, the modes referenced in this report are not modes of the complete structural assembly.

In SEA one commonly assumes that the problems are linear so that vibration in one frequency band cannot be connected into vibration in another band.

On the basis of these combined assumptions a power balance for each frequency band will be formulated and in each band power exchange between modes with resonant frequencies in the band will be studied.

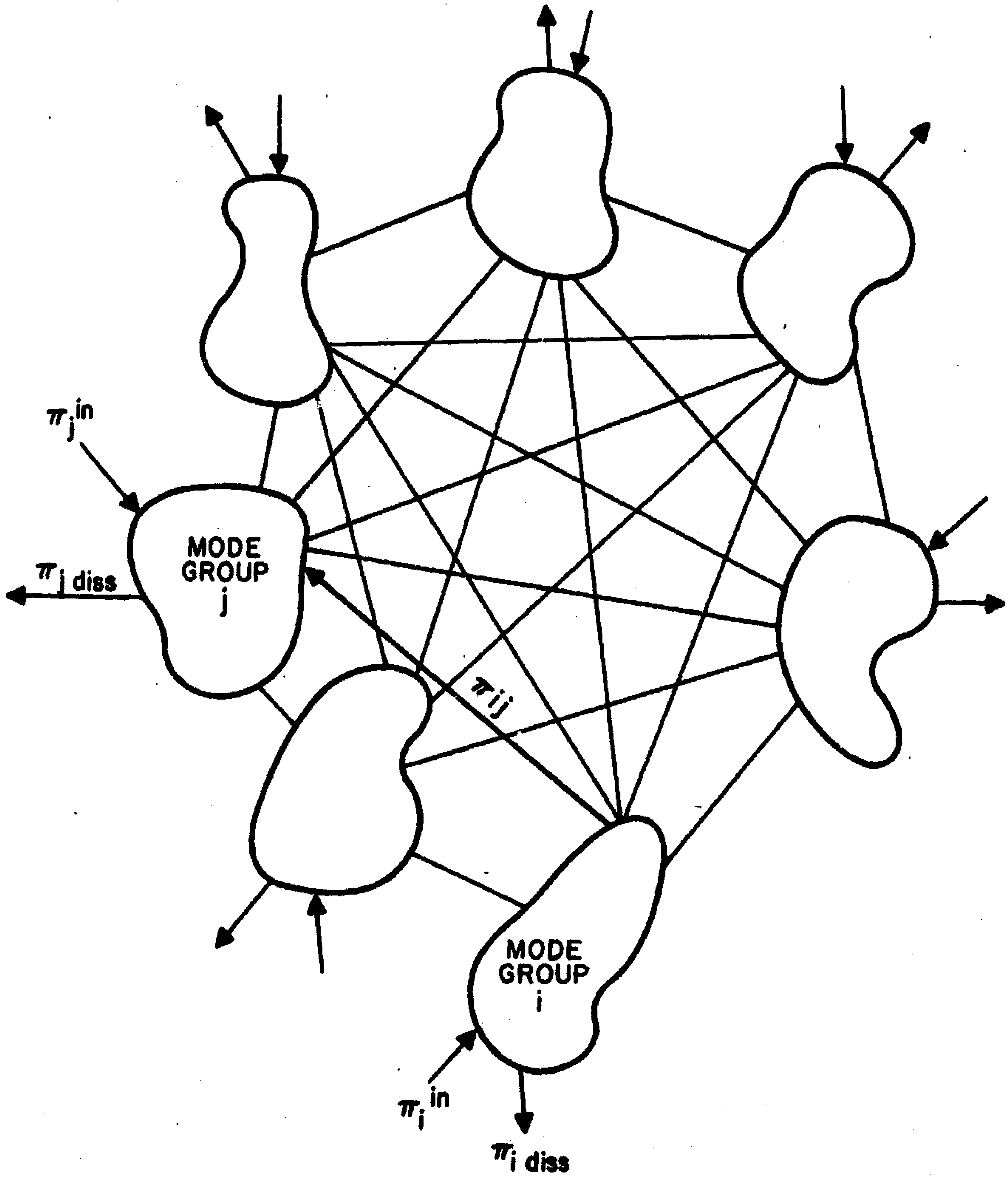

FIG.3 INTER-CONNECTED MODE GROUPS

In many cases the energy storage boxes are identical to the physical elements of the complete assembly, i.e., acoustic spaces, plates, shells, beams, etc. Then, the modeling required for SEA is quite simple. Each energy storage box contains all of the modes of a particular substructure which have resonance frequencies in the band being considered. The modeling is even simpler when the only paths of power exchange are between the modes of touching structural elements.

Two complicating effects are fairly common and must be considered in setting up an SEA model. First, it is possible for a structure or acoustic space to store energy in its resonant modes of vibration and, at the same time, transmit power from one adjoining structure to another through nonresonant motion of the modes with resonance frequencies outside the band being considered. A problem of this type is shown in Fig. 4. In this problem two rooms are separated by a thin partition. An SEA model consists of three energy storage elements for each frequency band of interest, as shown in Fig. 5. Element 1 contains the resonant modes of room 1; element 2 contains the resonant modes of the partition; and element 3 contains the resonant modes of room 2. Two paths of power exchange can be easily identified. Room 1 connects directly to the partition, so that power exchange between the resonant modes of room 1 and the partition clearly takes place. Similarly, power exchange between the resonant modes of the partition and room 2 takes place. A third path of power exchange must also be considered for this problem, as shown in Fig. 5. The resonant modes of room 1 transmit power directly to the resonant modes of room 2 through nonresonant motion of modes in the partition which have resonance

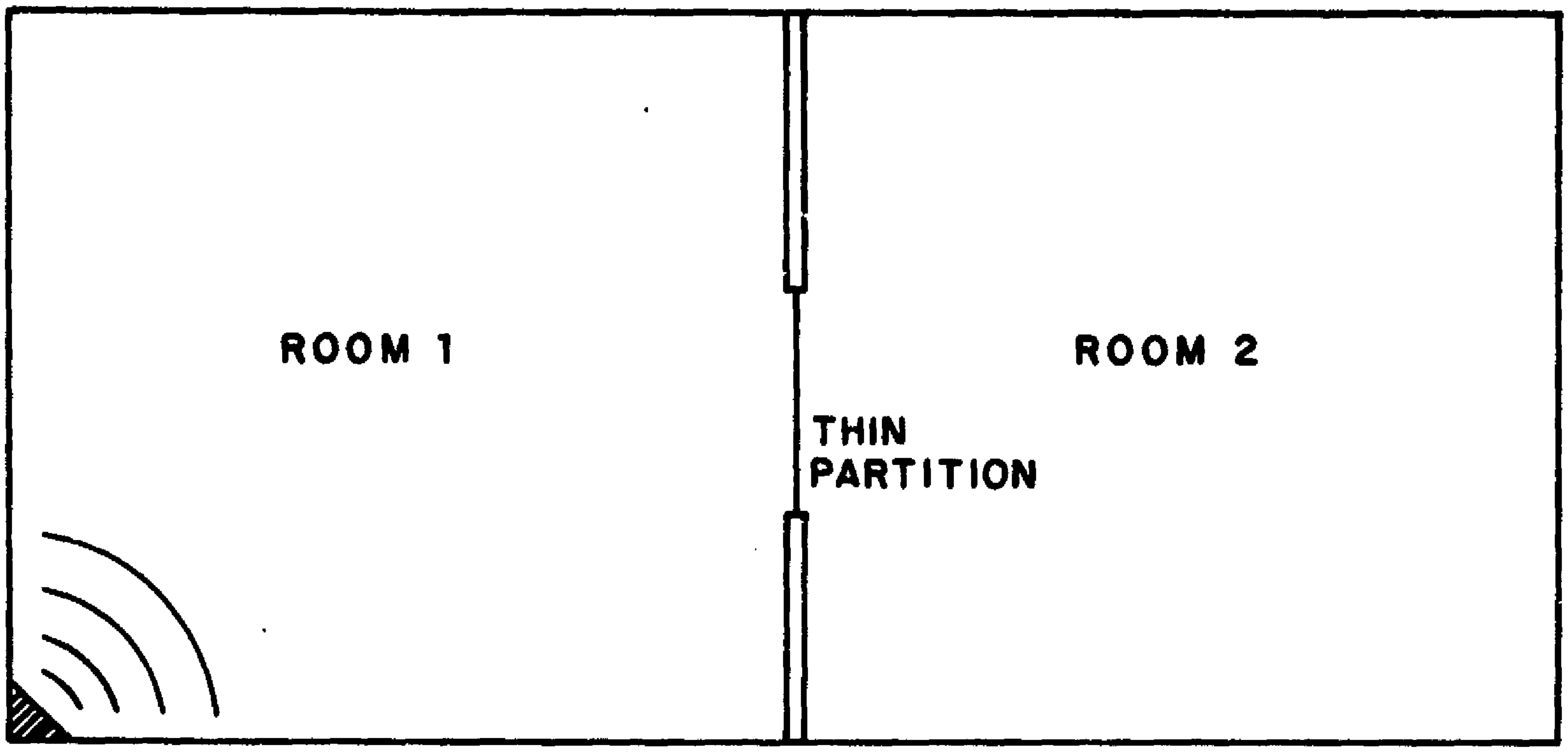

FIG.4 TWO ROOMS SEPARATED BY A PARTITION

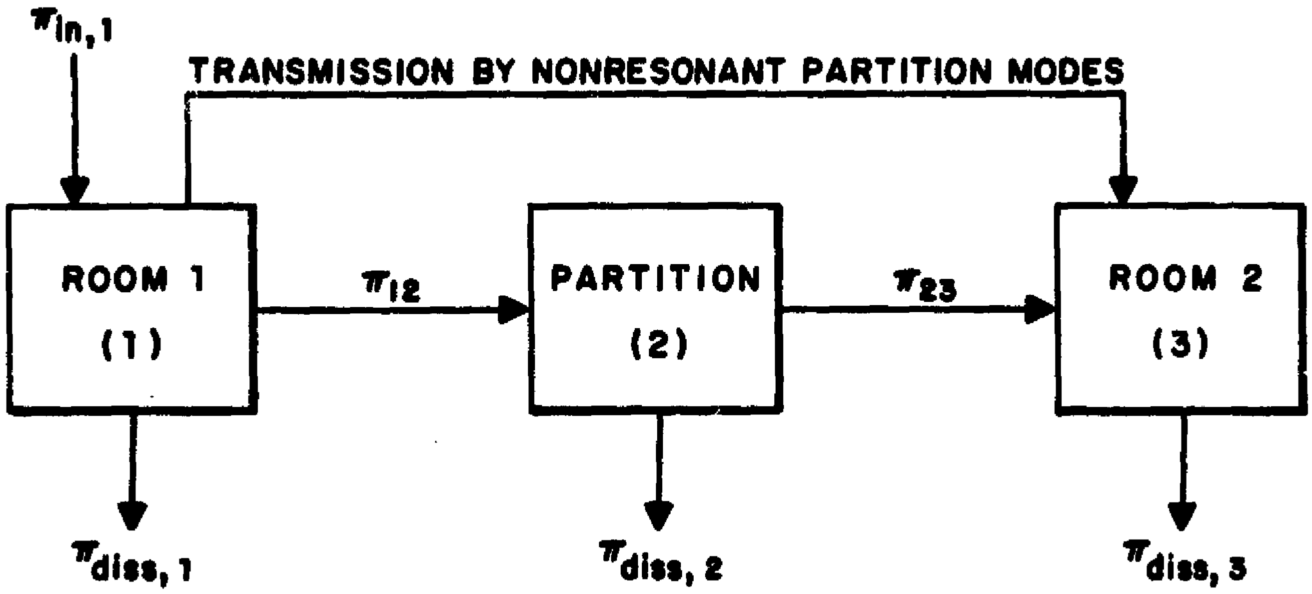

FIG.5 ENERGY STORAGE ELEMENTS AND PATHS OF POWER EXCHANGE FOR TWO ROOMS SEPARATED BY A PARTITION

frequencies below the frequency band being considered. The power transmitted by this third path often exceeds the power transmitted from the resonant modes of room 1 to the resonant modes of the partition and then to the resonant modes of room 2! In considering the power transmitted by nonresonant modes we ignore the small amount of power dissipation due to the motion of these modes.

A second complicating effect is also common. In some cases, the resonant modes of a single structural element must be divided into two or more energy storage elements, because they interact differently with the modes of neighboring elements. For example, if a beam couples two plates, then its torsional and bending resonant modes must be put into separate storage elements It is, of course, possible for a power exchange to also take place between the torsional and bending modes. Other examples of these two complicating effects are in the literature.[9,13,24]

Once the complex structural assembly has been modeled by a group of connected energy storage elements, the basic power balance equations are written. Assuming steady-state conditions the time-average power input to each element must equal the power dissipated plus the power transmitted to other elements. For the *ith* energy storage element, a power balance gives

$$\langle \pi_{in,i} \rangle_{t,\Delta\omega} = \langle \pi_{diss,i} \rangle_{t,\Delta\omega} + \sum_{j} \langle \pi_{ij} \rangle_{t,\Delta\omega} \qquad (3.1\text{-}1)$$

where the summation is over all other elements. In the following sections the use of SEA to compute each of these power quantities will be discussed.

3.2 Power Input

The calculation of vibratory power input from a source to a receiving structure or acoustic space requires that one know both some measure of the source excitation level and output impedance plus the receiving system input impedance. The calculation is made simpler when the actual source can be modelled by an ideal force source (zero output impedance) or an ideal velocity source (infinite output impedance). Simplifications also result when the source can be modelled by a point source or by a line source. However, even when the source can be simply described, for example, as a point force source, the calculation of input power can be complicated because of the need to know the exact input impedance of the receiving system. Unless the receiving system is extremely simple its input impedance will be a complex function of frequency and the parameters of the system-resonance frequencies, mode shapes, surface density, etc. In many cases of practical importance it is impossible to predict or even measure the exact input impedance. However, by using a statistical model of the receiving system, one can avoid this dilemma and obtain quite simple expressions for the input impedance which require only a general description of the receiving system. Randomness is introduced into these statistical models by considering such parameters as resonance frequencies, mode shape, modal damping and excitation point as random variables. Of course, in using a statistical model only statistical measures of the power input can be obtained. For example, the average power input to an ensemble of structures with randomly selected resonance frequencies can be calculated. For any particular member of this ensemble of structures the power input will not equal the calculated average value. However, if the structure is so complex that the resonance frequencies cannot be predicted with any accuracy, the ensemble-average power input will be the best estimate for the particular structure of interest. Confidence in the accuracy of this best estimate can be derived from higher-order statistical measures of the input power, such as the variance around the mean.

So far no specific mention of the temporal variation of the power input has been made. In this report all variations in time will be averaged out by using complex amplitudes for the case of pure-tone excitation and spectral densities for the case of random excitation. This limitation which is imposed on this work should not be taken as a limitation in the use of statistical energy methods of analysis. Indeed, statistical energy methods have been used very successfully in room acoustics to predict the transmission of speech or music – highly transient sounds.[9]

In the sections to follow the power input from both point and distributed-load sources to a variety of different dynamical systems will be calculated. A very simple case will be dealt with at first followed by cases of greater and greater complexity.

3.2.1 Point sources

In many problems of practical interest the source of vibration or sound can be modelled by a point source. As a general rule a source of vibration can be considered to be a point source if the area over which it acts on the receiving system is small compared to λ^2, where λ is the wavelength of the resulting motion in the receiving system. The error introduced into the calculation of input power by modelling a source by a point source will be less than 1 dB (26%) if the largest dimension of the contact area is less than a quarter-wavelength of the resulting motion in the receiving system. Since the wavelength of vibration varies with frequency for most systems, the criteria for treating an actual source as a point source depends on the frequency range of interest. For example, a 6 in. diameter loudspeaker can be treated as a point source in the frequency range below 550 Hz. If one were willing to accept an error of 3 dB or less this frequency range could be extended to 1000 Hz.

Examples of actual sources which can usually be treated as point sources include small sound generators, point-drive mechanical shakers and, in some cases, the mounting supports of vibrating equipment.

General Formulation of Input Power

A formulation of the input power from a point source to any receiving system can be accomplished with complete generality by using mechanical impedance theory. Following this approach, the driving-point impedance of the receiving structure, Z_R, is defined to be the complex force amplitude at the driving point when the velocity is specified to be $V_R = e^{i\omega t}$. Similarly, the source impedance, Z_S, is defined to be the complex force amplitude of the source drive point when its velocity is specified to be $V_S = e^{i\omega t}$ and the source is deactivated. The driving-point impedance of the receiving system and the source will usually be frequency dependent.

With these definitions of impedance, the force generated by a pure-tone source when it is driving the receiving structure can be given by one of two formulas. First, it can be given by

$$F(\omega) = F_{blocked}(\omega) - Z_S(\omega)V_R(\omega) \qquad (3.2.1\text{-}1)$$

where we have assumed $e^{i\omega t}$ dependence, $F(\omega)$ is the complex amplitude of the force acting on the receiving structure, $V_R(\omega)$ is the complex amplitude of the receiving structure and $F_{blocked}(\omega)$ is the complex amplitude of the force generated by the source when its driving point is held motionless. When the blocked force is not known, but the complex amplitude of the driving-point velocity produced by the source when it is generating no force is known, we use the formula,

$$F(\omega) = Z_S(\omega)[V_{free}(\omega) - V_R(\omega)] \qquad (3.2.1\text{-}2)$$

where $V_{free}(\omega)$ is the free velocity. The velocity at the driving point can be related to the force acting on the receiving structure, $F(\omega)$, by the driving-point impedance of the receiving structure,

$$V_R(\omega) = \frac{F(\omega)}{Z_R(\omega)} \quad . \qquad (3.2.1\text{-}3)$$

For a pure tone the time-average power transmitted to the receiving system is given by

$$\langle\pi_{in}\rangle_t = \frac{1}{2} \text{ Re } F(\omega)V_R^*(\omega) \qquad (3.2.1\text{-}4)$$

where Re signifies "Real Part Of". Thus, the time-average power input to the receiving structure becomes

$$\langle\pi_{in}\rangle_t = \frac{1}{2} \left|F_{blocked}\right|^2 \left|\frac{1}{Z_S(\omega)+Z_R(\omega)}\right|^2 \text{ Re } Z_R(\omega) \qquad (3.2.1\text{-}5)$$

or if V_{free} is known,

$$\langle\pi_{in}\rangle_t = \frac{1}{2} \left|V_{free}\right|^2 \left|\frac{Z_S(\omega)}{Z_S(\omega)+Z_R(\omega)}\right|^2 \text{ Re } Z_R(\omega) \qquad (3.2.1\text{-}6)$$

where | | signifies "Magnitude Of".

When the source generates a random force or velocity, spectral densities must be used. Then the time-average power input in a band of frequencies, $\Delta\omega$, is given by

$$\langle \pi_{in} \rangle_{t,\Delta\omega} = \int_{\Delta\omega} d\omega \, S_{\pi_{in}}(\omega) \qquad (3.2.1\text{-}7)$$

where $S_{\pi_{in}}(\omega)$ is the input-power spectral density. From Eq. 3.2.1-5 one can write the input-power spectral density as

$$S_{\pi_{in}}(\omega) = S_{F_{blocked}}(\omega) \left| \frac{1}{Z_S(\omega)+Z_R(\omega)} \right|^2 \text{Re } Z_R(\omega) \qquad (3.2.1\text{-}8)$$

where $S_{F_{blocked}}(\omega)$ is the mean-square blocked-force spectral density. A similar extension of Eq. 3.2.1-6 when the mean-square free-velocity spectral density is known is obvious.

Point Force and Velocity Sources

The source impedance of an ideal force source is zero so that the force generated by the source is not affected by the motion of its driving point (see Eq. 3.2.1-1). It follows that the power input from a random point force source is given by Eq. 3.2.1-8 with $Z_S(\omega) = 0$,

$$S_{\pi_{in}}(\omega) = S_{F_{source}}(\omega) \text{ Re } A_R(\omega) \qquad (3.2.1\text{-}9)$$

where $A_R(\omega)$ is the admittance of the receiving system defined to be $A_R(\omega) = 1/Z_R(\omega)$.

The source impedance of a velocity source is infinite so that the velocity of the drive point is not affected by the force acting on it. The input-power spectral density for the velocity source is given by

$$S_{\pi_{in}}(\omega) = S_{V_{source}}(\omega) \text{ Re } Z_R(\omega) \quad . \qquad (3.2.1\text{-}10)$$

The expressions for power input from point force and velocity sources are much simpler than the general expression for a point source. However, unless the receiving structure is very simple, the required driving-point impedance or admittance is still quite difficult, if not impossible, to predict. In the following sections it will be shown how the use of a statistical model for the receiving structure leads to very simple results. Finally, it will become apparent that the use of a statistical model allows one to replace the driving-point admittance or impedance of the actual receiving system by the driving-point admittance or impedance of an equivalent infinite receiving system. Since the use of infinite system impedances for finite system impedances has not yet been shown for the completely general case, this result will be demonstrated by means of an example. Then a result for the general case will be hypothesized.

Power Input to a Simple Oscillator

The simplest case which can be considered is the power input from a point force source to a simple one-degree-of-freedom oscillator. The input admittance of the simple oscillator is

$$A_R(\omega) = \frac{1}{M} \frac{i\omega}{[\omega_0^2 - \omega^2 + i\eta\omega_0\omega]} \qquad (3.2.1\text{-}11)$$

where M is the mass of the oscillator, ω_0 is its resonance frequency and η is its dissipation loss factor. From Eq. 3.2.1-9 and Eq. 3.2.1-11 one finds the input-power spectral density to be

$$S_{\pi_{in}}(\omega) = S_F(\omega) \frac{1}{M} \frac{\eta\omega^2\omega_0}{(\omega_0^2-\omega^2)^2 + \eta^2\omega^2\omega_0^2} \qquad (3.2.1\text{-}12)$$

As would be expected, the input-power spectral density has a large peak near $\omega - \omega_0$, when the damping loss factor, η, is small.

For the case in which the excitation force spectral density is white — has a constant value S_{F_0} for all frequencies — one can obtain a very simple result for the overall input power. This result is

$$S_{\pi_{in}}^{OA} \triangleq \int_0^\infty d\omega\, S_{\pi_{in}}(\omega) = \frac{\pi}{2} \frac{S_{F_0}}{M} \qquad (3.2.1\text{-}13)$$

where $\triangleq$ signifies a definition. Note that the input power depends only on the level of the excitation and the mass of the oscillator. This simple result will be used often in future sections and allows one to obtain simple results for very complex problems.

The result given above by Eq. 3.2.1-13 tells nothing about the frequency distribution of the input power. However, from Eq. 3.2.1-12 it is apparent that approximately 70% of the overall input power is in the frequency band $\omega_0 - \eta\omega_0$ to $\omega_0 + \eta\omega_0$. Since most of the power input takes place in this frequency band, one commonly uses Eq. 3.2.1-13 to predict the power input from a band-limited random force as long as the bandwidth includes the region $\omega_0 - \eta\omega_0$ to $\omega_0 + \eta\omega_0$.

Power Input to a Beam

The calculation of power input from a point force source to a beam is more complicated than the above calculation. However, the end result will be very simple.

The equation of motion for a beam is given by

$$\frac{EI}{m_\ell}\frac{\partial^4 v}{\partial x^4} + \frac{\partial^2 v}{\partial t^2} + \beta_p \frac{\partial v}{\partial t} = \frac{1}{m_\ell}\frac{\partial f}{\partial t} \qquad (3.2.1\text{-}14)$$

where $v(x,t)$ is the velocity at point x and time t, EI is the bending stiffness, m_ℓ is the mass per unit length, β_p is the damping coefficient and $f(x,t)$ is the force per unit length acting on the beam at point x and time t. To find the driving-point admittance, one assumes that the force per unit length is a pure-tone point force with $e^{i\omega t}$ time dependance so that

$$\frac{\partial f}{\partial t} = i\omega\delta(x-x_0)e^{i\omega t} \qquad (3.2.1\text{-}15)$$

where δ is the Dirac Delta Function and x_0 is the point of application of the point force. Then the beam velocity is expanded in terms of its normal mode shapes,

$$v(x,t) = \sum_n V_n \psi_n(x) e^{i\omega t} \quad , \qquad (3.2.1\text{-}16)$$

where V_n is complex amplitude of the *nth* mode velocity and $\psi_n(x)$ is the normal mode shape. The mode shapes are assumed to be normalized such that

$$\int_L dx\, \psi^2(x) = L \qquad (3.2.1\text{-}17)$$

where L is the length of the beam. If the expansion for v(x,t) is substituted into the equation of motion, multiplied by $\psi_m(x)$ and integrated over the length of the beam, one finds

$$V_m = \frac{1}{M_b} \frac{i\omega\psi_m(x_0)}{\omega_m^2-\omega^2 + i\eta_m\omega_m\omega} \tag{3.2.1-18}$$

where M_b is the total mass of the beam, ω_m is the resonance frequency of the *mth* mode and η_m is the modal dissipation loss factor, $\omega_m\eta_m = \beta_p$. It follows from Eq. 3.2.1-18 that the driving-point admittance can be written as

$$A_R(\omega) = \frac{i\omega}{M_b} \sum_m \frac{\psi_m^2(x_0)}{\omega_m^2 - \omega^2 + i\eta_m\omega_m\omega} . \tag{3.2.1-19}$$

The real part of the admittance, which governs the input power to the beam, can now be written as

$$\text{Re } A_R(\omega) = \frac{1}{M_b} \sum_m \frac{\omega^2\omega_m\eta_m\psi_m^2(x_0)}{(\omega_m^2-\omega^2)^2 + \eta_m^2\omega_m^2\omega^2} \tag{3.2.1-20}$$

It is well-known in classical vibration analysis that the response of a structure can be represented by a superposition of modal responses. Equation 3.2.1-20 shows simply another statement of this result. The real part of the admittance is the weighted sum of the real parts of the admittance for each mode — the weighting factor being the square of the mode shape at the point of application of the force. With this representation in mind one can write Eq. 3.2.1-20 as

$$\text{Re } A_R(\omega) = \sum_m a_m g_m \tag{3.2.1-21}$$

where a_m is the weighting factor, $\psi_m^2(x_0)$, and g_m is the real part of the modal admittance without the weighting factor a_m

$$g_m = \frac{1}{M_b} \frac{\omega^3 \eta_m}{(\omega_m^2 - \omega^2) + \eta_m^2 \omega_m^2 \omega^2} . \qquad (3.2.1\text{-}22)$$

If one were to consider the idealized problem of a beam with simply-supported load conditions and viscous damping the functions a_m and g_m could be computed exactly. The real part of the admittance for this idealized case would appear as in Fig. 6a with peaks occurring at regular intervals of frequency. The height of the peaks would be governed by the values of η_m and a_m with a_m being a periodic function of $\sqrt{\omega}$.

Unfortunately, the regular pattern shown in Fig.6a is rarely observed for an actual beam, particularly at high frequencies, due to boundary conditions which vary with frequency in a seemingly random manner. The resemblance between the plot in Fig. 6b and a sequence of random pulses in time from a random pulse generator is evident. Lyon[28] has used this resemblance as basis for finding statistics of the power input. Following his approach, one defines an ensemble of beams in which the resonance frequencies and the point of application of the source are random variables. The function Re $A_R(\omega)$ for a number of different members of the ensemble is shown in Fig. 6c. The concept of the ensemble average impedance at ω can now be introduced, as well as the variance and other higher order statistics of Re $A_R(\omega)$.

In an actual situation in which the resonance frequencies or the source point is not known the ensemble average value of Re $A_R(\omega)$ is used as a "best" estimate. Confidence in this best estimate for the beam being considered depends on the variance across the ensemble.

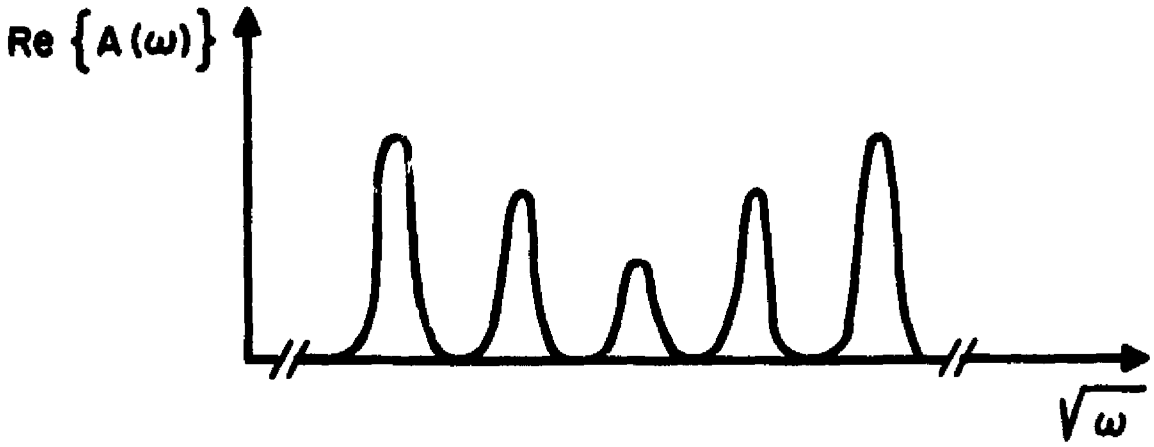

FIG.6a REAL PART OF THE ADMITTANCE FOR A BEAM WITH SIMPLE SUPPORTS

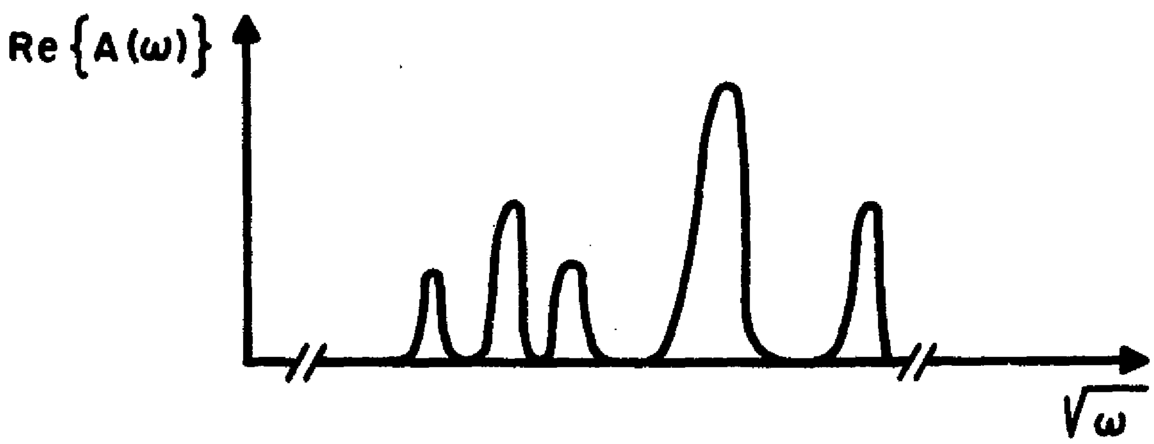

FIG.6b REAL PART OF THE ADMITTANCE FOR A BEAM WITH REALISTIC BOUNDARY CONDITIONS

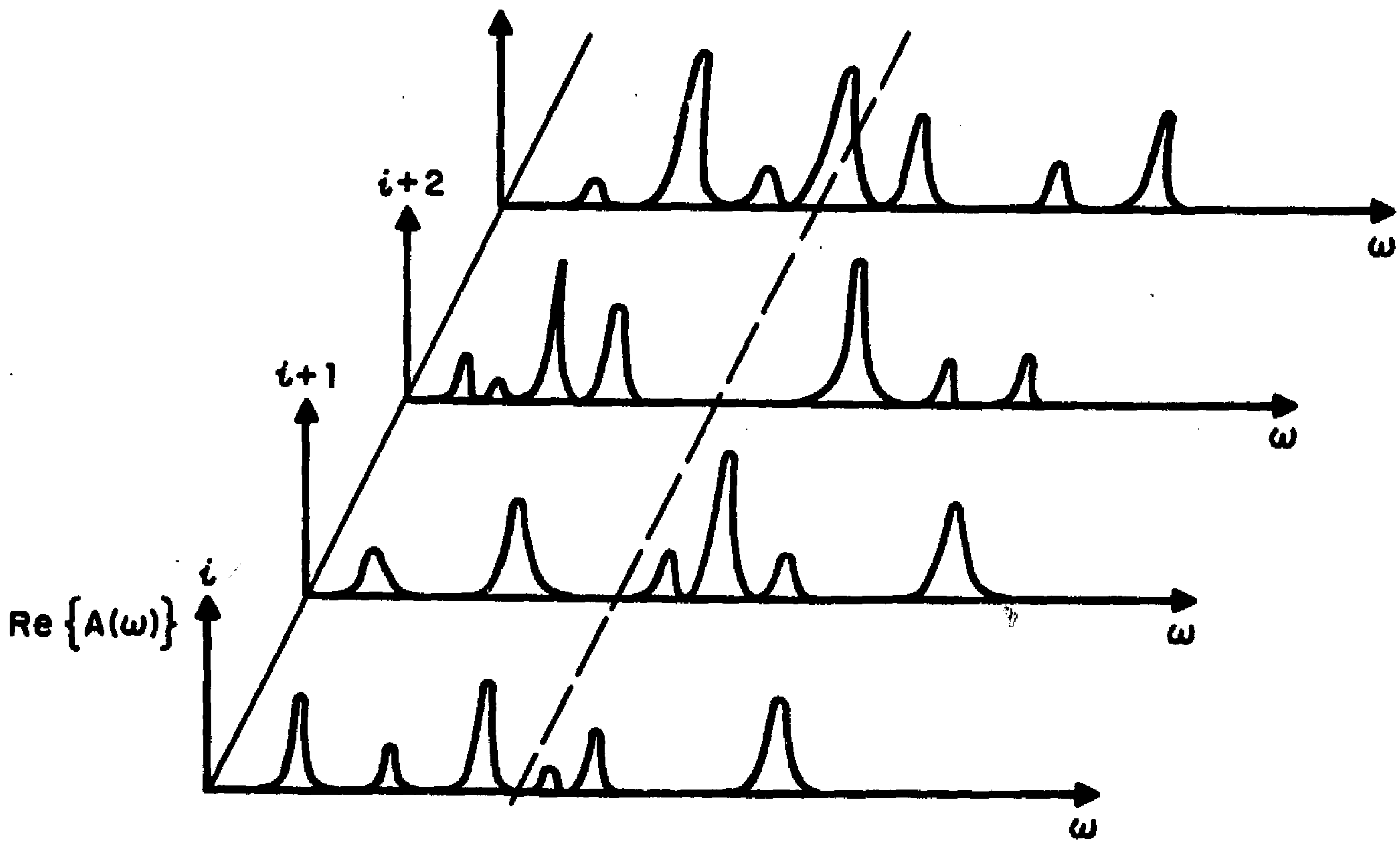

FIG.6c REAL PART OF THE ADMITTANCE FOR AN ENSEMBLE OF BEAMS

Lyon assumed that the separation between resonance frequencies has a Poisson distribution and that all points on the beam are equally probable as points at which the source is connected. Without going into his exact calculations his final results are given below. The ensemble average of the real part of the admittance for light damping is

$$\langle \mathrm{Re}\ A_R(\omega) \rangle_{ens} = \frac{\pi}{2M_b}\, n_b(\omega) \qquad (3.2.1\text{-}23)$$

where $n_b(\omega)$ is the modal density (the ensemble average number of resonance frequencies per unit frequency). The modal density of the beam is

$$n_b(\omega) = \frac{L}{2\pi c_b(\omega)} \qquad (3.2.1\text{-}24)$$

where $c_b(\omega)$ is the bending wavespeed in the beam,

$$c_b^2(\omega) = \omega\sqrt{\frac{EI}{m_\ell}} \qquad (3.2.1\text{-}25)$$

where EI is the bending stiffness of the beam. Combining Eqs. 3.2.1-23 through 3.2.1-25 one finds that

$$\langle \mathrm{Re}\ A_R(\omega) \rangle_{ens} = \mathrm{Re}\ A_{R,inf}(\omega) \qquad (3.2.1\text{-}26)$$

where $A_{R,inf}$ is the real part of the input admittance of an infinite beam,

$$\mathrm{Re}\ A_{R,inf} = \frac{1}{4m_\ell c_b} \quad . \qquad (3.2.1\text{-}27)$$

It is clear from Eq. 3.2.1-26 that the ensemble average input admittance of a finite beam is equal to that of an equivalent infinite beam. This result is very useful since the admittance of an infinite beam depends on parameters typically known with good accuracy.

The variance of the real part of the input admittance has also been found by Lyon.[28] His result shows that this variance depends on the amount of modal overlap, i.e., the number of modal resonances in a modal resonance bandwidth. The larger this number the lower the variance.

When the excitation is band-limited noise one must average the pure-tone result given by Eq. 3.2.1-23 over the band. Since the modal density is not strongly dependent on frequency, one can average the value of the real part of the admittance over the band Δω by

$$\langle \mathrm{Re}\ A_R(\omega)\rangle_{ens,\Delta\omega} = \frac{\pi}{2m_\ell} n_b(\omega_c) \qquad (3.2.1\text{-}28)$$

where ω_c is the band-center frequency.

As for the pure-tone case Lyon[28] has again calculated the variance. For this case the variance is smallest for large numbers of modes in the excitation bandwidth, Δω. This result and that for the pure tone case are consistent since high "modal overlap" implies many modes responding to a pure tone.

When the source is a point velocity source one uses Eq. 3.2.1-10 to predict the time-average power input. This equation requires the real part of the input impedance. To find this impedance the same approach used to find the real part of the admittance is followed. The result is

$$\langle \mathrm{Re}\ Z_R(\omega) \rangle_{ens} = \mathrm{Re}\ Z_{R,inf}(\omega) \qquad (3.2.1\text{-}29)$$

where $Z_{R,inf}$ is the impedance of an infinite beam.

Generalization of the Power Input from a Point Source

The previous section was restructed to the calculation of power input to a beam. This restriction was made in order to simplify the discussion. The method of calculation used can be easily extended to the problem of power input to a generalized multimodal system. In his work Lyon[28] considered the problem of computing power input to plates and acoustic spaces. In the previous section his results were used for computing power input to a beam. Here it is assumed that these results can be used for any multimodal system. Thus, to compute the time-average power input from a point force source the real part of the admittance of the infinite system is used in Eq. 3.2.1-9 to get

$$S_{\pi_{in}}(\omega) = S_F(\omega)\ \mathrm{Re}\ A_{R,inf}(\omega) \qquad (3.2.1\text{-}30)$$

where $A_{R,inf}$ is the admittance of the equivalent infinite system. Similarly, to compute the time-average power input from a point velocity source one uses the infinite system impedance in Eq. 3.2.1-10 to get

$$S_{\pi_{in}}(\omega) = S_V(\omega)\ \mathrm{Re}\ Z_{R,inf}(\omega) \qquad (3.2.1\text{-}31)$$

where $Z_{R,inf}$ is the impedance of the equivalent infinite system. The impedances of a large number of different systems are available in the literature.

When the source is neither a force nor a velocity source but has a finite nonzero source impedance the use of Eqs. 3.2.1-30 and 3.2.1-31 is restricted to cases of modal overlap, i.e., cases in which the average separation between resonance frequencies is less than the damping bandwidth. Scharton[29] has studied the problem of power input from sources of finite impedance. The reader should refer to his work for more detailed information.

3.2.2 Power input from distributed sources

The problem of computing the time-average power input from a distributed source is much more complicated than that from a point source. With a distributed source one must take into account not only the match in frequency between the source and the driven-system modes but also the spatial match between the source and the driven-system mode shapes. Examples of distributed sources include acoustic fields, fluctuating aerodynamic pressure fields and some distributed equipment mounting foundations. As for the point source, simplifications can be made if the source can be treated as either a distributed pressure or velocity source.

To illustrate the computational technique the power input from an acoustic field to a thin plate will be considered. Generalizations to more difficult problems will be clear.

The fluctuating pressure on the surface of a plate in an acoustic field is the sum of a blocked pressure – the pressure when the plate is restrained from moving – plus a radiated pressure – the pressure resulting from the motion of the plate. The radiated pressure can be further divided into a component in phase with the plate velocity — radiation damping — and a component out-of-phase with the plate velocity — fluid loading. For most problems in air the radiation damping will be small compared to the mechanical damping in the plate and the fluid loading will be small compared to the surface density of the plate. Then, the radiated pressure will be negligible and the acoustic field can be considered as a pressure source. Even when radiation damping must be included in the analysis, it is possible to treat the acoustic field as a pressure source and include the radiation damping with the mechanical damping in terms of a total damping loss factor.

The power input from the acoustic field to the plate is given by

$$\langle \pi_{in} \rangle_t = \int_A d\underline{x} \; \langle p(\underline{x},t) v(\underline{x},t) \rangle_t \tag{3.2.2-1}$$

where $p(\underline{x},t)$ is the acoustic pressure on the plate, $v(\underline{x},t)$ is the plate velocity and A is the area of the plate. If one expresses the plate velocity in terms of its normal modes, Eq. 3.2.2-1 becomes

$$\langle \pi_{in} \rangle_t = \sum_i \langle v_i(t) f_i(t) \rangle_t \tag{3.2.2-2}$$

where $v_i(t)$ is the velocity of the *ith* mode and $f_i(t)$ is the modal force given by

$$f_i(t) = \int_A d\underline{x} \; p(\underline{x},t) \; \psi_i(\underline{x}) \tag{3.2.2-3}$$

where $\psi_i(\underline{x})$ is the mode shape. From Eq. 3.2.2-2 it can be seen that the power input is the sum of the power inputs to each mode.

If the acoustic field is random and its spectral density is fairly flat over the band $\Delta\omega$, one can appeal to the power input calculation for a single oscillator to write

$$\langle \pi_{in} \rangle_{t,\Delta\omega} = \sum_{i}^{N_{\Delta\omega}} \frac{\pi}{2} \frac{S_{f_i}}{M_p} (\omega_i) \qquad (3.2.2\text{-}4)$$

where the summation now is over all modes with resonance frequencies, ω_i, in the band $\Delta\omega$, M_p is the modal mass of the plate and $S_{f_i}(\omega_i)$ is the mean-square modal force spectral density for the *ith* mode. The relationship between S_{f_i} and the mean-square pressure spectral density of the exciting pressure field has been defined to be the joint acceptance, $j_i(\omega)$, which is[30]

$$j_i(\omega) = \frac{S_{f_i}(\omega)}{A\ S_p(\omega)} \qquad (3.2.2\text{-}5)$$

In terms of the joint acceptance one can write the input power as

$$\langle \pi_{in} \rangle_{t,\Delta\omega} = \frac{\pi}{2} \frac{1}{M_p} S_p \sum_{i}^{N_{\Delta\omega}} j_i(\omega_i) \qquad (3.2.2\text{-}6)$$

where S_p is assumed to be flat over the band $\Delta\omega$.

The joint acceptance for each mode of an idealized mathematical model of the structure can be calculated. Since the resonance frequencies and shapes of the first few modes of an actual structure can be computed fairly accurately, this mode by mode approach makes good sense at low frequencies. However, at high frequencies confidence in the prediction of resonance frequencies and exact mode shapes falls off rapidly. In addition, the number of modes which must be included in the mode-by-mode calculations becomes very large. For these reasons a statistical approach is called for.

Using a statistical approach, an ensemble of plates is defined in which the resonance frequencies are random variables. For this analysis, however, the mode shapes will be taken as non-random $\sin k_x x \sin k_y y$ functions. In many cases it would also be desirable to make the mode shapes random. But, unfortunately, the calculations for this case have not been worked out. The error introduced by using $\sin k_x x \sin k_y y$ functions for problems in which the boundary conditions are not simple supports is believed to be small, less than ±3 dB.

The ensemble average number of modes with resonant frequencies in a band $\Delta\omega$ is given by the modal density. Thus, the ensemble average power input in the frequency band $\Delta\omega$ can be written

$$\langle \pi_{in} \rangle_{t,\Delta\omega,ens} = \frac{\pi}{2} \frac{1}{M_p} S_p \, n_p(\omega) \, \Delta\omega \langle j_i \rangle_{\Delta\omega,ens} \qquad (3.2.2\text{-}7)$$

where $\langle j_i \rangle_{\Delta\omega,ens}$ is the average joint acceptance — the average being taken both over the band $\Delta\omega$ and over the ensemble. To calculate the average joint acceptance, one calculates the joint acceptance for a $\sin k_x x \sin k_y y$ mode shape and then averages the result over the band $\Delta\omega$ and the ensemble.

The joint acceptance for a particular mode is given by

$$j_i^2(\omega_i) = \frac{1}{S_p A_p^2} \iiiint_{A_p A_p} d\underline{x}_1 \, d\underline{x}_2 \, S_p(\underline{x}_1,\underline{x}_2,\omega_i) \, \psi_i(\underline{x}_1) \, \psi_i(\underline{x}_2) \qquad (3.2.2\text{-}8)$$

where $S_p(\underline{x}_1,\underline{x}_2,\omega_i)$ is the cross-spectral density of the pressures at $\underline{x}_1$ and $\underline{x}_2$. Equation 3.2.2-8 can be put into a form more suitable for the required averaging by Fourier transformation. Defining the following transforms

$$\tilde{\Psi}_i(\underline{k}) \triangleq \frac{1}{2\pi} \iint_{A_p} d\underline{x}\ \psi_i(\underline{x})\ e^{-i(\underline{k}\cdot\underline{x})} \qquad (3.2.2\text{-}9)$$

and

$$S_p(\underline{k},\omega) \triangleq \frac{1}{2\pi} \iint d\underline{\Delta}\ S_p(\underline{x},\underline{x}+\underline{\Delta},\omega)\ e^{+i(\underline{k}\cdot\underline{\Delta})} \qquad (3.2.2\text{-}10)$$

one can write Eq. 3.2.2-8 as

$$j_i^2(\omega_i) = \frac{1}{S_p A_p^2} \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\underline{k}\ S_p(\underline{k},\omega_i)\,|\tilde{\Psi}_i(\underline{k})|^2 \qquad (3.2.2\text{-}11)$$

where $S_p(\underline{k},\omega)$ is the wavenumber-frequency spectrum of the exciting pressure field and $|\tilde{\Psi}_i(\underline{k})|^2$ is the magnitude squared of the transform of the mode shape. The function $|\tilde{\Psi}_i(\underline{k})|^2$ for a sin $k_x y$ sin $k_y y$ mode has the form

$$|\tilde{\Psi}_i(\underline{k})|^2 = \frac{1}{(2\pi)^2} A_p |I(k_1,k_x)|^2\ |I(k_2,k_y)|^2 \qquad (3.2.2\text{-}12)$$

where I is given by

$$I(k_1,k_x) = \frac{1}{L_x} \int_0^{L_x} dx \sin k_x X\ e^{-ik_1 x} \ . \qquad (3.2.2\text{-}13)$$

The function I can be easily evaluated for a number of different values of k_x and k_1.

Smith and Lyon[31] have shown how to average the functions $|I(k_1,k_x)|^2$ and $|I(k_2,k_y)|^2$ for the modes of a plate with resonances in the band $\Delta\omega$. In their work Smith and Lyon compute radiation resistances. The radiation resistance is related to the joint acceptance by the equation

$$R_{i,rad}(\omega_i) = \frac{\rho_0 c_0 k_0^2 A_p^2}{4\pi} j_i(\omega_i) \tag{3.2.2-14}$$

where $R_{i,rad}$ is the radiation resistance, k_0 is the acoustic wavenumber of ω_i and $\rho_0 c_0$ is the characteristic impedance of the acoustic media. Smith and Lyon have found the average value of $R_{i,rad}(\omega_i)$ over the band $\Delta\omega$ and the ensemble for diffuse acoustic excitation. The reader is referred to their work or equivalently to the work by Maidanik[7] for specific results. For plates which are large compared to an acoustic wavelength, we can approximate the ensemble average radiation resistance of modes with resonance frequencies in a given band $\Delta\omega$ by the expressions

$$\langle R_{i,rad}\rangle_{ens,\Delta\omega} = \rho_0 c_0 A \qquad \text{for } \omega > 2\omega_c \tag{3.2.2-15}$$

and

$$\langle R_{i,rad}\rangle_{ens,\Delta\omega} = \frac{1}{\pi^2}\left(\frac{\omega}{\omega_c}\right)^{1/2} \rho_0 c_0 \lambda_c p \qquad \text{for } \omega < \frac{\omega_c}{2} \tag{3.2.2-16}$$

where ω_c is the critical frequency given by

$$\omega_c^4 = \frac{c_0^4 \rho_s}{EI} , \tag{3.2.2-17}$$

λ_c is the acoustic wavelength at ω_c, p is the perimeter of the plate, ρ_s is the mass per unit area of the plate, and EI is the bending stiffness of the plate. The radiation resistance in the region between $\omega_c/2$ and $2\omega_c$ depends in a complex way on the plate parameters. For practical purposes it is reasonable to assume a smooth transition from Eq. 3.2.2-16 to Eq. 3.2.2-15 in this region.

3.3 Power Transmitted

The calculation of the power transmitted between two oscillators has been studied in detail.[1,3,4,5,32] It has been found that if the two oscillators are forced with independent white noise the power flow between them is proportional to the difference in their total energies or

$$\pi_{mn} = \phi_{mn}(\varepsilon_m - \varepsilon_n) \tag{3.3-1}$$

where ϕ_{mn} is, by definition, the factor of proportionality.

The case of the coupling of two multimodal systems is less clearly understood. In this section two general techniques that have been successfully applied to coupled multimodal systems will be presented. To simplify this presentation the two techniques the "mode approach" and the "wave approach" will be used to solve a problem with two rods vibrating longitudinally and coupled together with a spring.

It should be emphasized that the above two techniques are not the only ones applied to coupled multimodal systems. Newland[3,33,34] has developed a technique for measuring the coupling coefficient from the shift in natural frequency that occurs when two systems are coupled together. Gersh[32,35] has applied some techniques from control theory (Liapunov's principle) to deal with the strong coupling case. However, the wave approach and the mode approach have received somewhat more attention in the literature and as a result are, at present, on a somewhat firmer basis.

3.3.1 Mode approach

The mode approach to the calculation of power transmitted between multimodal systems was the first technique applied.[1] It has the advantage that the chain of assumptions involved in the calculations is fairly clear but it has the disadvantage of being cumbersome. The coupled modal equations must be derived and then rather extensive manipulations must be performed on the resulting coupling factor. In the next section the coupling coefficient for two coupled oscillators is derived and in the following section the result is extended to multimodal systems.

3.3.1.1 Two coupled oscillators

If two simple oscillators as shown in Fig. 7 are coupled together and each is driven by a white noise force, the two forces being statistically independent, it can be shown that the time average power flow from the first oscillator to the second is directly proportional to the difference in their time average total energies

$$\pi_{12} = \phi_{12}(\varepsilon_1-\varepsilon_2) \ . \qquad (3.3.1.1\text{-}1)$$

The coupled equations of the two oscillators in Fig. 7 may be written

$$\left(M_1 + \frac{1}{4} M_c\right)\ddot{x}_1 + b_1\dot{x}_1 + (K_1+K_c)x_1 + \frac{1}{4} M_c\ddot{x}_2 - G\dot{x}_2 - K_c x_2 = F_1$$

$$\left(M_2 + \frac{1}{4} M_c\right)\ddot{x}_2 + b_2\dot{x}_2 + (K_2+K_c)x_2 + \frac{1}{4} M_c\ddot{x}_1 - G\dot{x}_1 - K_c x_1 = F_2 \ .$$

(3.3.1.1-2)

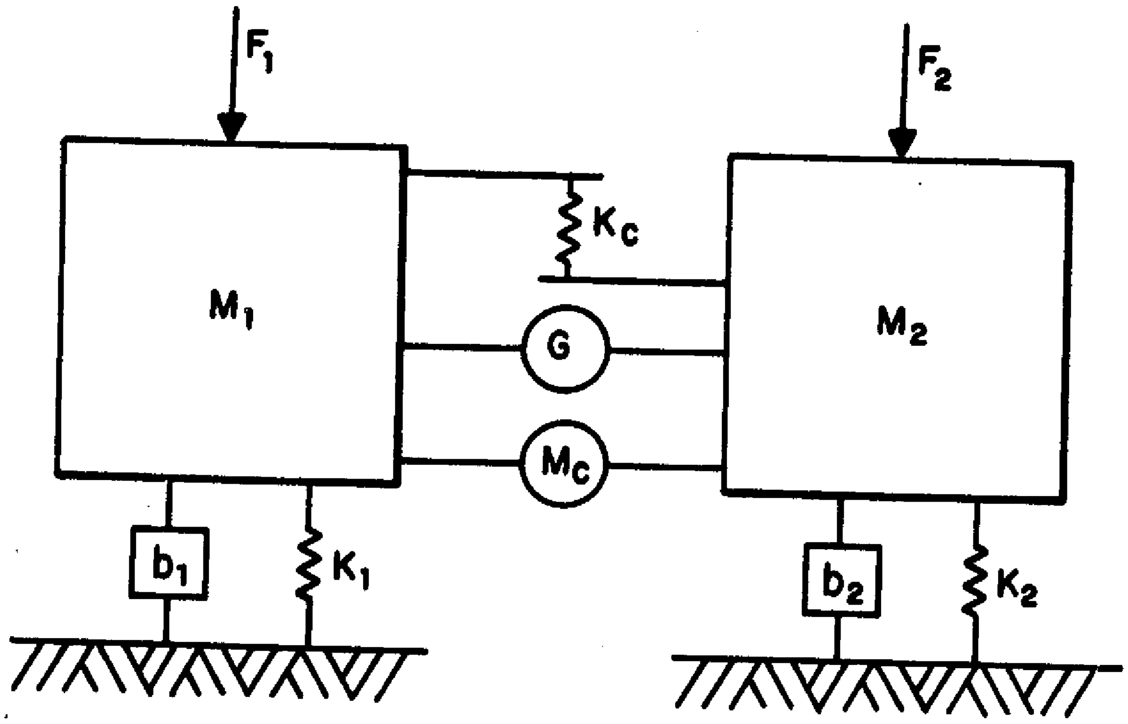

FIG.7 TWO COUPLED OSCILLATORS

Note that there is gyroscopic, inertia and stiffness coupling. The last form of coupling is somewhat uncommon with simple mass-spring oscillators but the coupled equations that result when two continuous systems are coupled together often contain terms of the form $M_c/4\ (\ddot{x}_1+\ddot{x}_2)$ (inertia) and $G\dot{x}$ (gyroscopic) as well as stiffness terms.

We now write Eq. 3.3.1.1-2 with new coefficients

$$\ddot{x}_1 + \Delta_1\dot{x}_1 + \omega_1^2 x_1 + \frac{1}{\lambda}[\mu\ddot{x}_2-\gamma\dot{x}_2-\kappa x_2] = f_1$$

$$\ddot{x}_2 + \Delta_2\dot{x}_2 + \omega_2^2 x_2 + \lambda[\mu\ddot{x}_2-\gamma x_2-\kappa x_2] = f_2 \qquad (3.3.1.1\text{-}3)$$

where

$$\Delta_1 = \frac{b_1}{M_1+M_c/4} \qquad \Delta_2 = \frac{b_2}{M_2+M_c/4}$$

$$\omega_1^2 = \frac{K_1+K_c}{M_1+M_c/4} \qquad \omega_2^2 = \frac{K_2+K_c}{M_1+M_c/4}$$

$$\lambda^2 = \frac{M_1+M_c/4}{M_2+M_c/4} \qquad \mu = \frac{M_c/4}{[(M_1+M_c/4)(M_2+M_c/4)]^{1/2}}$$

$$\gamma = \frac{G}{[(M_1+M_c/4)(M_2+M_c/4)]^{1/2}} \qquad \kappa = \frac{K_c}{[(M_1+M_c/4)(M_2+M_c/4)]^{1/2}}$$

$$f_1 = \frac{F_1}{M_1+M_c/4} \qquad f_2 = \frac{F_2}{M_1+M_c/4} \ .$$

Scharton and Lyon[5] have calculated the coupling coefficient in Eq. 3.3.1.1-1 for the coupled equation of Eq. 3.3.1.1-3 and have found it to be

$$\phi_{12} = \frac{\mu^2[\Delta_1\omega_2^4+\Delta_2\omega_1^4+\Delta_1\Delta_2(\Delta_1\omega_2^2+\Delta_2\omega_1^2)] + (\gamma^2+2\mu\kappa)[\Delta_1\omega_2^2+\Delta_2\omega_1^2] + \kappa^2(\Delta_1+\Delta_2)}{(1-\mu^2)[(\omega_1^2-\omega_2^2)^2 + (\Delta_1+\Delta_2)(\Delta_1\omega_2^2+\Delta_2\omega_1^2)]} \tag{3.3.1.1-4}$$

The result in Eq. 3.3.1.1-4 is always correct independent of the strength of the coupling. The validity of Eq. 3.3.1.1-4 does depend however, on a somewhat special definition of the engeries in Eq. 3.3.1.1-1. For example, ε_1 is the energy that would result if M_2 in Fig. 7 were held stationary and M_1 was allowed to move under the action of the applied force. ε_2 is defined similarly. These energies then contain terms associated with energy in the coupling elements as well as in the oscillators themselves.

It will be found as the coupled modal equations of coupled continuous systems are examined that under suitable assumptions equations of the form of Eq. 3.3.1.1-3 will result. Use of Eq. 3.3.1.1-4 will then yield an equation for the power flow between any two modes of the coupled system.

3.3.1.2 Extension to multimodal systems

For the two oscillator cases it has been shown in the previous section that power transmitted is proportional to the difference in energy of the two oscillators and the proportionality factor was calculated in terms of the properties of the two oscillators and the coupling between them. The somewhat peculiar definition of the total energy of each oscillator (coupling elements are included) used in the previous section can be relaxed somewhat if the coupling is light. In that case oscillator energy can be defined in terms of the oscillator elements not including coupling elements.[1] It remains to apply these results to many oscillators coupled together as in a multimodal system. In order to illustrate this application, the specific example of two rods vibrating longitudinally and coupled together with a spring will be used (see Fig. 8).

The equation of motion for a longitudinally vibrating rod may be written

$$-c^2 \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial t^2} = \frac{f(x,t)}{\rho A} \tag{3.3.1.2-1}$$

where ξ is the displacement in the x direction along the length of the bar, ρ is the density, A is the cross-sectional area, t is the time, c is the wave speed, and $f(x,t)$ is a force per unit length.

Referring to Fig. 8 one may express the resulting equations of motion for the two bars

bar 1

$$\frac{\partial^2 \xi_1}{\partial t^2} - c_1^2 \frac{\partial^2 \xi_1}{\partial x^2} = \frac{F_1}{\rho_1 A_1} \delta(x-x_{f1}) + \frac{K_c}{\rho_1 A} \delta(x-x_{c1}) \left[\xi_2(x_{c2},t) - \xi_1(x_{c1},t)\right]$$

$$- \frac{b_{10}}{\rho_1 A_1} \dot{\xi}_1(o,t)\, \delta(x) - \frac{b_{11}}{\rho_1 A_1} \dot{\xi}(L_1 t)\, \delta(x-L_1) \,. \tag{3.3.1.2-2}$$

bar 2

$$\frac{\partial^2 \xi_2}{\partial t^2} - c_2^2 \frac{\partial^2 \xi_2}{\partial x^2} = \frac{K_c}{\rho_2 A_2} \delta(x-x_{c2}) \left[\xi_2(x_{c2},t) - \xi_1(x_{c1},t)\right] + \frac{F_2}{\rho_2 A_2} \delta(x-x_{f2})$$

$$- \frac{b_{20}}{\rho_2 A_2} \dot{\xi}_2(o,t)\, \delta(x) - \frac{b_{21}}{\rho_2 A_2} \dot{\xi}_2(L_2,t)\, \delta(x-L_2) \,. \tag{3.3.1.2-3}$$

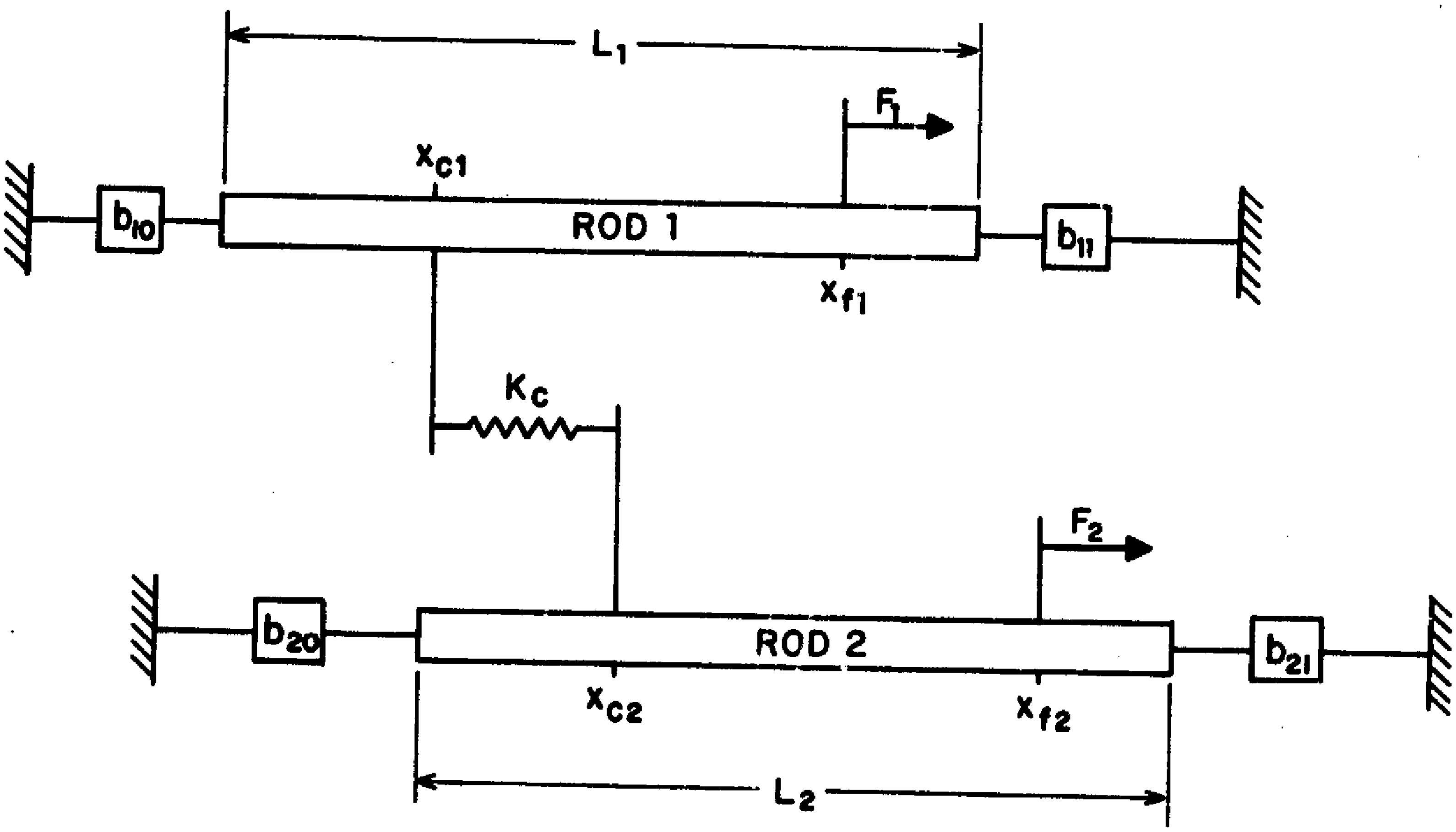

FIG.8 COUPLED RODS IN LONGITUDINAL MOTION

where the effect of the coupling spring and viscous dampers at the end of the beam have been included as external forces, K_c is the coupling spring stiffness, and b_{10}, b_{11}, b_{20}, and b_{21} are the damping coefficients of the viscous dampers.

It is assumed that ξ_1 and ξ_2 can each be expressed as a series of orthogonal functions as follows

$$\xi_1(x,t) = \sum_m u_m(t)\ \phi_m(x)$$

$$\xi_2(x,t) = \sum_n v_n(t)\ \psi_n(x) \qquad (3.3.1.2\text{-}4a)$$

where

$$\frac{\partial^2 \phi_m}{\partial x^2} = -K_m^2\ \phi_m$$

$$\frac{\partial^2 \psi_n}{\partial x^2} = -K_n^2\ \psi_n \qquad (3.3.1.2\text{-}4b)$$

and

$$\int_0^{L_1} \phi_m\ \phi_n\ dx = 0 \text{ if } m \neq n$$

$$= \frac{L_1}{2} \text{ if } m = n$$

$$\int_0^{L_2} \psi_m\ \psi_n\ dx = 0 \text{ if } m \neq n$$

$$= \frac{L_2}{2} \text{ if } m = n \qquad (3.3.1.2\text{-}4c)$$

Because all coupling, forcing, and damping have been included as forcing terms in Eqs. 3.3.1.2-2 and 3.3.1.2-3, the mode shapes required here are specifically the modes shapes for a free-free rod, i.e.,

$$\psi_n(x) = \cos \frac{n\pi x}{L_2} \qquad n = 1, 2, \ldots$$

$$\psi_m(x) = \cos \frac{m\pi x}{L_1} \qquad m = 1, 2, \ldots$$

Thus in terms of the total problem (mode shape and forcing terms) the resulting boundary conditions are the proper ones.

Using Eq. 3.3.1.2-4a in Eq. 3.3.1.2-2 and Eq. 3.3.1.2-3, simplifying the result by the use of Eq. 3.3.1.2-4b, and applying the orthogonality condition of Eq. 3.3.1.2-4c by multiplying the equation by the appropriate orthogonal function and integrating the result over the appropriate length, one obtains

$$\left.\begin{aligned} \ddot{u}_m + \Delta_m \dot{u}_m + \omega_m^2 u_m - \lambda C_{mn} v_n &= F_m \\ \ddot{v}_n + \Delta_n \dot{v}_n + \omega_n^2 v_n - \frac{1}{\lambda} C_{mn} u_m &= G_n \end{aligned}\right\} \qquad (3.3.1.2\text{-}5)$$

where

$$\Delta_m = 2\left\{\frac{b_{10}[\phi_m(0)]^2 + b_{11}[\phi_m(L_1)]^2}{m_1}\right\}$$

$$\Delta_n = 2\left\{\frac{b_{20}[\psi_n^2(0)]^2 + b_{21}[\psi_n(L_2)]^2}{m_2}\right\}$$

$$m_1 = \rho_1 c_1 A_1$$

$$m_2 = \rho_2 c_2 A_2$$

$$\omega_m^2 = c_1^2 K_m^2 + \frac{2K_c}{m_1} [\phi_m(x_{c1})]^2$$

$$\omega_n^2 = c_2^2 K_m^2 + \frac{2K_c}{m_2} [\psi_n(x_{c2})]^2$$

$$C_{mn} = \frac{2K_c \phi_m(x_{c1}) \psi_n(x_{c2})}{(m_1 m_2)^{1/2}}$$

$$\lambda = \left(\frac{m_1}{m_2}\right)^{1/2}$$

$$F_m = \frac{2F_1}{m_1} \phi_m(x_{f1}) + \frac{2K_c}{m_1} \phi_m(x_{c1}) \left\{ \sum_k^{k \neq n} v_k \psi_k(x_{c2}) - \sum_\ell^{\ell \neq m} u_\ell \phi_\ell(x_{c1}) \right\}$$

$$- 2 \sum_k^{k \neq m} \frac{b_{10} [\phi_k(0)]^2 + b_{11} [\phi_k(L_1)]^2}{m_1} \dot{u}_k$$

$$G_n = \frac{2F_2}{m_2} \psi_n(x_{f2}) - \frac{2K_c}{m_2} \psi_n(x_{c2}) \left\{ \sum_k^{k \neq n} v_k \psi_k(x_{c2}) - \sum_\ell^{\ell \neq m} u_\ell \phi_\ell(x_{c1}) \right\}$$

$$- 2 \sum_k^{k \neq n} \frac{b_{20} [\psi_k(0)]^2 + b_{21} [\psi_k(L_2)]^2}{m_2} \dot{v}_k$$

If one assumes that F_1 and F_2 are white, that the u_m's and v_n's are statistically independent and that the summation terms in the F_m and G_n terms each have a spectrum that is flat compared to the admittance spectrum of the *mth* and *nth* modes, respectively, then the results of Scharton and Lyon [5] can be directly applied by substituting the appropriate values from Eq. 3.3.1.2-5 into the expression for the coupling coefficient of Sec. 3.3.1.1

$$\phi_{mn} = \frac{C_{mn}^{\ 2}\ (\Delta_m+\Delta_n)}{(\omega_m^2-\omega_n^2)^2 + (\Delta_m+\Delta_n)\ (\Delta_m\omega_n^2+\Delta_m\omega_n^2)} . \qquad (3.3.1.2\text{-}6)$$

From Eq. 3.3.1.2-6 one can find the power transmitted between any mode in rod 1 and any mode in rod 2

$$\pi_{mn} = \phi_{mn}\left[\varepsilon_m^{(1)} - \varepsilon_n^{(2)}\right] \qquad (3.3.1.2\text{-}7)$$

where $\varepsilon_m^{(1)}$ is the energy of mode m of rod 1 and $\varepsilon_n^{(2)}$ is the energy of mode n of rod 2. To obtain the total power transmitted between the two rods one must sum over all modes in both systems

$$\pi_{12} = \sum_n^N \sum_m^M \phi_{mn}\left[\varepsilon_m^{(1)} - \varepsilon_n^{(2)}\right]. \qquad (3.3.1.2\text{-}8)$$

In general one is dealing with a particular band of frequency and thus the summations refer to all the modes with resonant frequencies in the band of interest, assuming of course that damping is light enough to allow resonant modes to dominate the response.

If one knows the resonant frequencies and mode shapes of both systems well enough, it is possible to apply Eqs. 3.3.1.2-6 and 3.3.1.2-7 directly and sum over the known modes. In general, at nigh frequency one does not know the modal properties that well – not to mention the fact that the numbers of modes involved may be so large that calculations using Eq. 3.3.1.2-8 will be quite tedious. In order to avoid these difficulties, one takes a statistical approach and assumes that the particular systems under consideration are members of an infinite ensemble of systems. It is desired then to find an "average" coupling factor, ϕ, for the ensemble. The definition of the ensemble has purposely been left very loose here. In general the purpose behind averaging ϕ_{mn} of Eq. 3.3.1.2-6 is to end up with a ϕ that is independent of the properties of any particular mode such as natural frequency or mode shape. In order to accomplish this different kinds of averaging will be required for different systems.

For the particular system under discussion here the following form of averaging is appropriate

1. One assumes that there is a uniform probability that the coupling spring is attached at any point along the length of rod 1 and an independent uniform probability that the spring is attached at any point along the length of rod 2. The average value of the coupling coefficient then becomes

$$\langle\phi_{mn}\rangle_L = \frac{1}{L_1 L_2} \int_0^{L_1} dx_{c1} \int_0^{L_2} dx_{c2}\, \phi_{mn} \quad . \qquad (3.3.1.2\text{-}9)$$

2. One assumes that the natural frequencies of the two rods are randomly distributed. To deal with this situation properly two cases must be considered: "well separated modes" in the receiving system (rod 2) and "modal overlap" in the receiving system.

For the case of "well separated modes" one assumes that given the natural frequency, ω_m, of a mode of rod 1, there is a uniform probability that the natural frequency of a mode in rod 2 lies between $\omega_m - 1/2n_2(\omega)$ and $\omega_m + 1/2n_2(\omega)$ [$n_2(\omega) = L_2/\pi C_2$, is the modal density of rod 2]. Because the modes are well separated, i.e., the modal bandwidth Δ is much less than the modal spacing $1/n(\omega)$, it is further assumed that one mode in rod 1 is coupled to *only one mode in rod 2*, or

$$\sum_n \phi_{mn} = n_2(\omega_m) \int_{\omega_m - 1/2n_2(\omega_m)}^{\omega_m + 1/2n_2(\omega_m)} d\omega_n \; \phi_{mn} \qquad (3.3.1.2\text{-}10)$$

where the summation over n refers to summing over all modes in rod 2 in the frequency band of interest.

For the case of "modal overlap" where the modal bandwidth is greater than the spacing between modes one assumes that in the frequency band of interest, $\Delta\omega$, there is a dense array of modes in rod 2 with a uniform probability that a natural frequency lies at any point in $\Delta\omega$. Because of the modal overlap condition one can *no longer assume* that one mode in rod 1 is coupled to only one mode in rod 2. For this reason then, one calculates an average coupling factor for a mode in rod 1 to a single mode in rod 2 and multiplies the result by the number of modes in rod 2 in $\Delta\omega$ or

$$\sum_n \phi_{mn} = [n_2(\omega_m)\Delta\omega] \left[\frac{1}{\Delta\omega} \int_{\omega_m - \Delta\omega/2}^{\omega_m + \Delta\omega/2} d\omega_n \; \phi_{mn} \right] \qquad (3.3.1.2\text{-}11)$$

where the first term in brackets is the number of modes in $\Delta\omega$ in rod 2, the second term in brackets is the average coupling coefficient for a mode in rod 1 to a mode in rod 2 in the band $\Delta\omega$ and ω_m becomes the center frequency of the band.*

If $\Delta\omega$ in Eq. 3.3.1.2-11 is greater than the modal bandwith of a mode in rod 2, then the limits on the integration can be

*To be strictly correct the calculation of the average coupling factor in Eq. 3.3.1.2-11 should also contain an integration over ω_m since the mode of interest in rod 1 could have its natural frequency anywhere in $\Delta\omega$. However, if the modal bandwidths are much less than $\Delta\omega$, the integral over ω_n is approximately the same for all ω_m except near the ends of the band. Thus no averaging in ω_m is required.

changed to $-\infty$ to $+\infty$. The same change in limits on the integration in Eq. 3.3.1.2-10 is possible because of the assumption of well separated modes. As a result, the two results are the same

$$\sum_n \phi_{mn} = \frac{n_2(\omega)}{2} \int_{-\infty}^{\infty} d\omega_n \, \phi_{mn}$$

where the factor of 1/2 is due to the fact that ϕ_{mn} is an even function of ω_n and the limits of integration should actually have been expanded only from 0 to ∞.

Carrying out the above integration using contour integration and assuming that the modal bandwidth Δ is much less than the center frequency ω one obtains

$$\sum_n \phi_{mn} = \frac{1}{2} \frac{K_c^2 L_2}{m_1 m_2 c_2 \omega^2} . \qquad (3.3.1.2\text{-}12)$$

Equation 3.3.1.2-8 then becomes

$$\pi_{12} = \left(\sum_n \phi_{mn}\right) \sum_m^M \varepsilon_m^{(1)} - \sum_m^M \sum_n^N \phi_{mn} \, \varepsilon_n^{(2)} . \qquad (3.3.1.2\text{-}13)$$

Equation 3.3.1.2-13 can be further simplified if one assumes that the summations over $\varepsilon_n^{(2)}$ are negligible compared to the summations over $\varepsilon_m^{(1)}$

$$\pi_{12} = \left(\sum_n \phi_{mn}\right) E_1 \qquad (3.3.1.2\text{-}14)$$

where E_1 is the total energy of rod 1 in the frequency band of interest. Equation 3.3.1.2-14 would be valid if, for example, with a very soft spring coupling the two rods only rod 1 were forced.

If the above is not valid, one often assumes that the coupling between modes within rod 2 is very strong and that each mode, $\varepsilon_n^{(2)}$, has the same energy

$$\varepsilon_n^{(2)} = \frac{E_2}{N} \qquad (3.3.1.2\text{-}15)$$

where E_2 is the total energy of rod 2 in the frequency band of interest and N is the number of modes in rod 2 in that band, Eq. 3.3.1.2-13 then becomes

$$\pi_{12} = \left(\sum_n \phi_{mn}\right)\left(E_1 - \sum_m^M \frac{E_2}{N}\right) = \left(\sum_n \phi_{mn}\right)\left(E_1 - \frac{M}{N} E_2\right) . \qquad (3.3.1.2\text{-}16)$$

For a narrow frequency band $M = n_2(\omega)\Delta\omega$ and $N = n_2(\omega)\Delta\omega$ where $\Delta\omega$ is the bandwidth of interest and n_1 and n_2 are the modal densities of rod 1 and rod 2 respectively, Eq. 3.3.1.2-16 becomes

$$\pi_{12} = (\Sigma\phi_{mn})\; n_1\left(\frac{E_1}{n_1} - \frac{E_2}{n_2}\right) . \qquad (3.3.1.2\text{-}17)$$

Using the appropriate expression for $(\Sigma\phi_{mn})$, Eq. 3.3.1.2-12, in Eq. 3.3.1.2-17 one can calculate the total power flow between rod 1 and rod 2 without specific knowledge of the natural frequencies or mode shapes. Of course this particular problem is simple enough that an exact closed form solution is possible (see Appendix A). Use of this exact solution will be made in a later section.

It is often useful to have an upper limit for the average coupling coefficient. To obtain this limit it is assumed that the modes of the two rods have the same natural frequencies. It is also appropriate to choose the maximum possible value for C_{mn} in Eq. 3.3.1.2-6, i.e., the amplitudes of the two mode shapes at the coupling points are taken to be unity

$$\left(\sum_n \phi_{mn}\right)_{max} = \frac{(C^2_{mn})_{max}}{(\Delta_m+\Delta_n)\omega^2} = \frac{2K^2_c}{m_1 m_2 \omega^2 (b_{10}+b_{11}+b_{20}+b_{21})} \quad . \qquad (3.3.1.2\text{-}18)$$

Use of Eq. 3.3.1.2-18 in Eq. 3.3.1.2-14 or 3.3.1.2-17 will give an upper bound on the power transmitted between the two rods. This result is strictly valid only for the "well separated mode" case. For the "modal overlap" case a possible means of getting at the answer would be simply to multiply Eq. 3.3.1.2-18 by $n_2(\omega)(\Delta_m+\Delta_n)$ or the number of modes in rod 2 contained in the combined modal bandwidths of a mode in rod 1 and a mode in rod 2 or

$$\left(\sum_n \phi_{mn}\right)_{max} = \frac{4K^2_c}{m_1 m_2} \, \frac{L_2}{c_2 \pi \omega^2} \quad . \qquad (3.3.1.2\text{-}19)$$

Although we have formulated the power flow problem in terms of an average coupling coefficient it is more common in SEA to use a coupling loss factor analogous to the damping loss factor. The coupling loss factor is related to the average coupling coefficient by the equation

$$\omega\eta_{12} = \sum_n \phi_{mn} \qquad (3.3.1.2\text{-}20)$$

where η_{12} is the coupling loss factor and ω is the center frequency of the band being considered. Using the coupling loss factor the basic power flow equation becomes

$$\pi_{12} = \omega\eta_{12} n_1 \left(\frac{E_1}{n_1} - \frac{E_2}{n_2} \right) \quad . \qquad (3.3.1.2\text{-}21)$$

Since the power flow from 1 to 2 equals minus the power flow from 2 to 1 we have the basic relationship

$$\eta_{12} n_1 = \eta_{21} n_2 \quad . \qquad (3.3.1.2\text{-}22)$$

This relationship allows some flexibility in using either η_{12} or η_{21}. Note, however, the η_{12} does not equal η_{21}.

3.3.2 Wave approach

In recent years a different approach to the calculation of power transmitted has been developed. This technique, which will be called the "wave approach" in this report, does not depend on knowing the coupled modal equations of the coupled system and, in fact, does not even mention the word mode in the process of calculating the coupling loss factor, η_{12}. It has been demonstrated in many cases[5,3,6] that the coupling factor calculated by the wave approach is the same as that calculated by the mode approach.

The principle outlined in Sec. 3.2 that the impedance of a finite system becomes that of the infinite system given a broad enough frequency band and/or a high enough modal density is the basis of this approach. To illustrate its application the coupling factor for the coupled rod problem of Sec. 3.3.1.2 will be derived here.

Using the total power flow equation of Sec. 3.3.1.2

$$\pi_{12} = \omega\eta_{12} n_1 \left(\frac{E_1}{n_1} - \frac{E_2}{n_2} \right) \qquad (3.3.2\text{-}1)$$

one generally assumes for the *purpose of derivation of* η_{12} that the second term in parentheses is negligible compared to the first. Such an assumption is valid if the modal density n_2 of the second system is much larger than the first or if the coupling is light and system 1 is the only source of excilation for the second. In any event, under this assumption Eq. 3.3.2-1 becomes

$$\pi_{12} = \omega\eta_{12}E_1 \quad . \qquad (3.3.2\text{-}2)$$

Calculating π_{12} and E_1, then, enables one to find η_{12}, the coupling loss factor.

The time-average power flow from rod 1 to rod 2 (see Fig. 8) may be written

$$\pi_{12} = \frac{1}{2}\,\mathrm{Re}\left\{F_c\dot{\xi}_2^*\right\}$$

where F_c is the force amplitude in the coupling spring and $\dot{\xi}_2^*$ is the complex conjugate of the amplitude of the velocity in rod 2 at the point of attachment of the coupling spring. Using the point impedance of rod 2, Z_2, one may write

$$\pi_{12} = \frac{|F_c^2|}{2}\,\mathrm{Re}\left\{\frac{1}{Z_2^*}\right\} . \qquad (3.3.2\text{-}3)$$

It is desirable to express F_c in terms of a quantity easily relateable to the energy in rod 1. For this reacon F_c is expressed as the sum of the force that would be required to hold the point of attachment of the coupling spring rigidly, F_{BL}, and the force due to the motion of that point, F_m

$$F_c = F_{BL} + F_m \; .$$

The force F_m may be written

$$F_m = - Z_1 \dot{\xi}_1 \tag{3.3.2-5}$$

where Z_1 is the point impedance of rod 2 and $\dot{\xi}_1$ is the velocity in rod 1 at the point of attachment of the coupling spring. The total coupling force, F_c, applied to rod 1 is applied equally and oppositely to the coupling spring such that

$$F_c = Z_2' \dot{\xi}_1 \tag{3.3.2-6}$$

where Z_2' is the impedance of the coupling spring attached to rod 2 but separated from rod 1.

Combining Eqs. 3.3.2-3, 3.3.2-4, 3.3.2-5, and 3.3.2-6 leads to an expression for the power flow in terms of F_{BL}

$$\pi_{12} = \frac{|F_{BL}{}^2|}{2} \left| \frac{Z_2'}{Z_1 + Z_2'} \right|^2 \mathrm{Re} \left\{ \frac{1}{Z_2^*} \right\} \quad . \tag{3.3.2-7}$$

Clearly if F_{BL} can be expressed in terms of the energy of rod 1, use of Eq. 3.3.2-1 will lead to an expression for η_{12}.

To this end assume that a right running traveling wave of amplitude $\dot{\xi}_1$ is incident on the coupling point in rod 1 and that that point is held rigidly. It can be easily shown, then, that

$$|F_{BL}{}^2| = |Z_1^2| \; |\dot{\xi}_1^2| \quad .$$

In reality it is equally likely that a left running wave is incident on the junctions from the right. Assuming that this wave has the same amplitude as the right running wave but is uncorrelated with it leads to

$$\left|F_{BL}{}^2\right| = 2|Z_1^2||\dot{\xi}_1|^2 \quad . \qquad (3.3.2\text{-}8)$$

If the coupling is light, the energy contained in rod 1 due to the action of these two uncorrelated traveling waves may be written

$$E_1 = \rho_1 A_1 L_1 |\dot{\xi}_1^2| \qquad (3.3.2\text{-}9)$$

where ρ_1 is the density of rod 1, A its area, and L_1 its length.

Combining Eqs. 3.3.2-1, 3.3.2-7, 3.3.2-8, and 3.3.2-9 and solving for η_{12} leads to

$$\omega\eta_{12} = \frac{|Z_1^2|}{\rho_1 A_1 L_1} \left| \frac{Z_2''}{Z_1 + Z_2'} \right|^2 \mathrm{Re}\, \frac{1}{Z_2{}^*} \quad . \qquad (3.3.2\text{-}10)$$

It can be easily shown that if the two rods are taken to be infinitely long,

$$Z_1 = 2\rho_1 c_1 A_1$$

$$Z_2 = 2\rho_2 c_2 A_2$$

and $$Z_2' = \frac{j\, K_c/\omega\, Z_2}{Z_2 + j\, K_c/\omega}$$

where c_1 is the wave speed in rod 1; ρ_2, c_2, and A_2 are the density, wave speed, and area of rod 2, respectively; and K_c is the stiffness of the coupling spring.

In the case of light coupling ($K_c/\omega << Z_1, Z_2$) Eq. 3.3.2-10 becomes

$$\omega\eta_{12} = \frac{1}{\rho_1 A_1 L_1} \frac{K_c^2}{\omega^2} \frac{1}{2\rho_2 c_2 A_2} = \frac{1}{2} \frac{K_c^2 L_2}{\omega^2 m_1 m_2 c_2} . \qquad (3.3.2\text{-}11)$$

The result in Eq. 3.3.2-11 is exactly the same as that calculated by the mode to mode technique of Sec. 3.3.1.2 for the two cases examined there. Substitution of Eq. 3.3.2-11 into Eq. 3.3.2-1 enables one to calculate the power transmitted between the two rods.

Note that in the calculation of η_{12} it was assumed that in Eq. 3.3.2-1 the second term in parentheses was negligible compared to the first whereas now the entire equation is to be used to calculate power transmitted.

Note that the wave approach as presented here is the approach most commonly found in the literature. It is often much simpler to apply than the mode approach though it is much harder to justify the chain of assumptions yielding the result.

To conclude, the assumptions that are the essence of the wave approach are stated below.

1. All impedances are taken to be those of the infinite system. As in the above example the impedances of the finite rods were taken to be the impedances of infinite rods. However, it should be noted that if the rods had been coupled at the end, the semi infinite rod impedances would have been used.

2. For the *purpose of deriving the coupling coefficient* the power transmitted is taken to be proportional to the energy in the driving system or

$$\pi_{12} = \omega\eta_{12} E_1$$

where 1 denotes the driving system.

3. To calculate the energy in the driving system as well as the blocked force (where force is here meant in the general sense of a moment, stress, pressure, or force) one assumes that the response of the driving system consists of a series of uncorrelated traveling waves incident from all possible directions. For example, in the middle of a rod there are just two possible directions; in a plate waves can be incident on a point from any of 360°; and in an acoustic space one usually assumes a diffuse pressure field (waves incident from all angles), although other fields such as grazing or normally incident should be used when appropriate. If the coupling spring were at the end of the rod, waves could only be incident from one direction; if the coupling point were at the end of a panel, waves could only be incident from any of 180°, etc.

Careful application of the above three assumptions should lead to proper calculation of power transmitted for many different classes of coupled systems.

3.4 Power Dissipated

Definition

In previous sections we have somewhat arbitrarily separated the power leaving a continuous medium (structure, acoustic space, etc.) into two categories, power dissipated and power transmitted. In some cases it is somewhat ambiguous as to how one distinguishes between the two types of power. Clearly the power leaving a continuous medium due to damping (conversion of mechanical to thermal energy) where the damping may be due to interface friction, fluid viscosity, turbulence, mechanical hysteresis, or electromagnetic hysteresis is classified as power dissipated. The ambiguity occurs when one has two continuous media in contact such that power can flow between them, a panel and acoustic space for example. Is such power to be classified as transmitted or dissipated?

In many cases the measurement techniques for damping answer the question for the investigator. For example, in the measurement of panel damping in air one measures both the power lost to internal dissipation as well as the power radiated to the surrounding acoustic medium. Thus in calculating the response of the panel in contact with the same acoustic medium where the damping measurements were taken, to excitation other than that in the medium, one would treat the power radiated to the acoustic medium as added damping.

In other cases the nature of the problem itself makes the classification clear. If three media are in contact such that power can flow between any two, then, to classify the power flow between any two as dissipative may lead to difficulties. For example, if one, a priori, classifies the power flow from medium 3 to medium 1 as dissipative, this inherently assumes that insignificant power is incident on medium 3 from medium 1. If medium 2 is driven and it turns out that more power flows from medium 2 to medium 1 to medium 3 than flow directly from medium 2 to medium 3, then, the above classification of power would be incorrect and would lead to incorrect conclusions. Clearly, then, when the direction of power flow is uncertain, one had best classify the power flow as transmitted. On the other hand if two media are interacting, and the first is the only source of power for the second, then, to classify the power flow between the two as dissipative would cause no difficulties unless, of course, one were interested in the response of the second medium. However, if one measures damping of a medium while it is coupled to a second medium, care must be taken not to extrapolate that measurement to situations in which the first medium is coupled to a different second medium. For example, damping measurements made on a plate in air will be significantly different from measurements made on that same panel in the vacuum of outer space. As a further example, damping measurements made on a component in a structure will be different if made on that same component in a radically different structure. This is *not* to say that damping measurements involving coupling to other media cannot be extrapolated. In actuality as long as the receiving medium is similar to that for which the measurement was performed, extrapolation of the measurement is usually acceptable. For example, the damping measured on a panel in two different rooms would be essentially the same if the rooms were of similar size and had similar absorptive material on the walls.

Analytical Representation

For purposes of analysis it is generally assumed that the power dissipated by a continuous medium, π_{diss}, is proportional to the energy, E, in that medium

$$\pi_{diss} \propto E \tag{3.4-1}$$

a valid assumption for linear systems.

In fact, early measurements of dissipation in materials showed that the coefficient of proportionality in Eq. 3.4-1 varied linearly with frequency. For this reason a loss factor η, a constant, was defined such that

$$\pi_{diss} = \omega\eta E \ .$$

In the case of a damped second order oscillator satisfying the following equation

$$m\ddot{x} + b\dot{x} + kx = F \tag{3.4-2}$$

where m is the mass, b the viscous damping coefficient, k the spring constant, and F a stationary random force applied to the mass, one can multiply both sides of Eq. 3.4-2 by $\dot{x}$ the velocity of the mass, take the time average, and obtain

$$\langle F\dot{x}\rangle = \pi_{diss} = b\langle\dot{x}^2\rangle \ .$$

At resonance, one obtains

$$\pi_{diss} = \omega\eta E \tag{3.4-3}$$

where $\eta = b/\omega_n m$ ($\omega_n = \sqrt{k/m}$ the natural frequency) and E is the total energy of the oscillator which at resonance is twice the kinetic energy since there kinetic and potential energy are equal.

Equation 3.4-3 is commonly used to analytically represent the power dissipated from a continuous medium for any dissipation mechanism. The loss factor η which is generally a function of frequency is one of a great many representations. The damping ratio, ζ, and the amplifications at resonance, Q, are related to η as follows:

$$\eta = 2\zeta = \frac{1}{Q} .$$

Unfortunately, at present, analytical calculation of the loss factor η is not possible except in a few special cases. Radiation damping of panels[1,7] can now be calculated with some degree of confidence and with "applied damping treatments" where sufficient data exists reasonable calculation of η can be made. In most cases, though, one must rely on measurements or educated guesses.

Measurement Technique

In the field of viscoelasticity where damping is modelled by allowing the relevant material moduli to be complex, there are many techniques for measuring damping. While differing in detail all of these techniques use a small material sample with special simple geometry. Such techniques are not particularly applicable to structural damping where friction at mechanical joints may dominate internal material dissipation as the damping mechanism. For this reason two techniques commonly used to measure damping at high frequencies will be briefly mentioned here.

In the following discussion it is assumed that the medium of interest has been isolated from all other interactive media. For example in the case of a panel interacting with an acoustic space one would ideally like to place the panel in a vacuum though this is seldom possible. One can of course leave the medium of interest coupled to other media but it should be realized that the loss factor that one obtains in the case is not the loss factor of the medium of interest but a combined loss factor which is a complicated function of the loss factor of each of the media and the effects of the interaction between the media. This combined loss factor is often called the "total loss factor".

From Eq. 3.4-3 one technique immediately suggests itself. Measure the time average power injected into a medium which by a simple power balance is equal to π_{diss}, measure the time average energy of the medium and use Eq. 3.4-3 to calculate η. The measurement of energy in a resonant structure usually consists of a measurement of the space-time-average velocity squared. The measurement of power injected is somewhat more difficult and in the case of a structure involves time averaging the product of the point force applied and the velocity at the point of application. All these measurements are usually done in frequency bands. In principle this is straightforward but in practice involves the use of some rather sophisticated equipment. A typical setup is shown in Fig. 9a.

From the definition of power input to medium

$$\pi = \frac{\partial E}{\partial t} . \qquad (3.4\text{-}4)$$

A simpler technique is suggested. From Eq. 3.4-4 it is clear that when the excitation to a continuous medium ceases the rate of change of the energy is equal to the power dissipated. Combining Eq. 3.4-4 and Eq. 3.4-3 one obtains

$$\frac{dE}{dt} - \omega\eta E = 0$$

which has a solution

$$E = E_0 e^{-\omega\eta t} . \qquad (3.4\text{-}5)$$

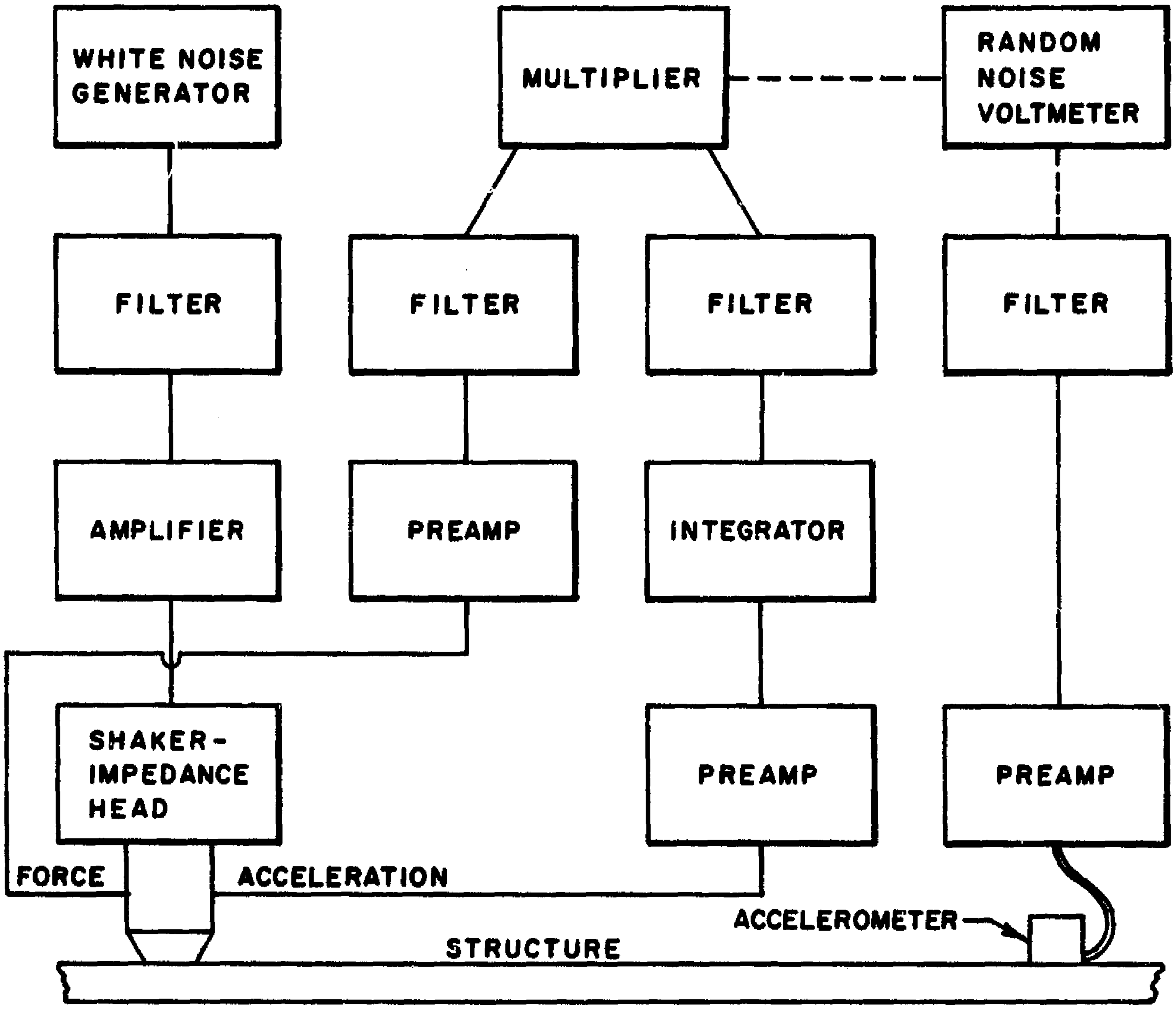

FIG.9a TYPICAL SET UP FOR MEASUTING LOSS FACTOR USING POWER INPUT MEASUREMENT

One simply excites the medium of interest with white noise filtered in a band, turns off the excitation, and measures the rate of decay of energy in the medium in that band. Specialized instruments exist (decay rate meters, graphic level recorders with templates, etc.) that greatly simplify this measurement. Use of Eq. 3.4-5 then yields η, the loss factor.

A commonly defined quantity used with this type of measurement is the reverberation time, T_{REV}, defined as the time for the amplitude E in Eq. 3.4-5 to decay 60 dB or

$$T_{REV} = \frac{\eta f_n}{2.2}$$

where f_n is the center frequency of the band of interest. This type of measurement is used almost exclusively in acoustics due to the difficulty encountered in measuring acoustic power input to an acoustic space. A typical setup for measurement on a structure is shown in Fig. 9b. For an acoustic space one simply uses a microphone in place of the accelerometer and a loudspeaker in place of the shaker.

Rules of Thumb

In the absence of measurements there are rules of thumb for the value of the loss factor. Structures (no damping treatment) have loss factors typically of the order of 10^{-2}. In rooms the reverberation time can be calculated from values of the absorption coefficient of the walls.[25] Values for absorption coefficients for many different wall treatments can also be found in the literature.[37]

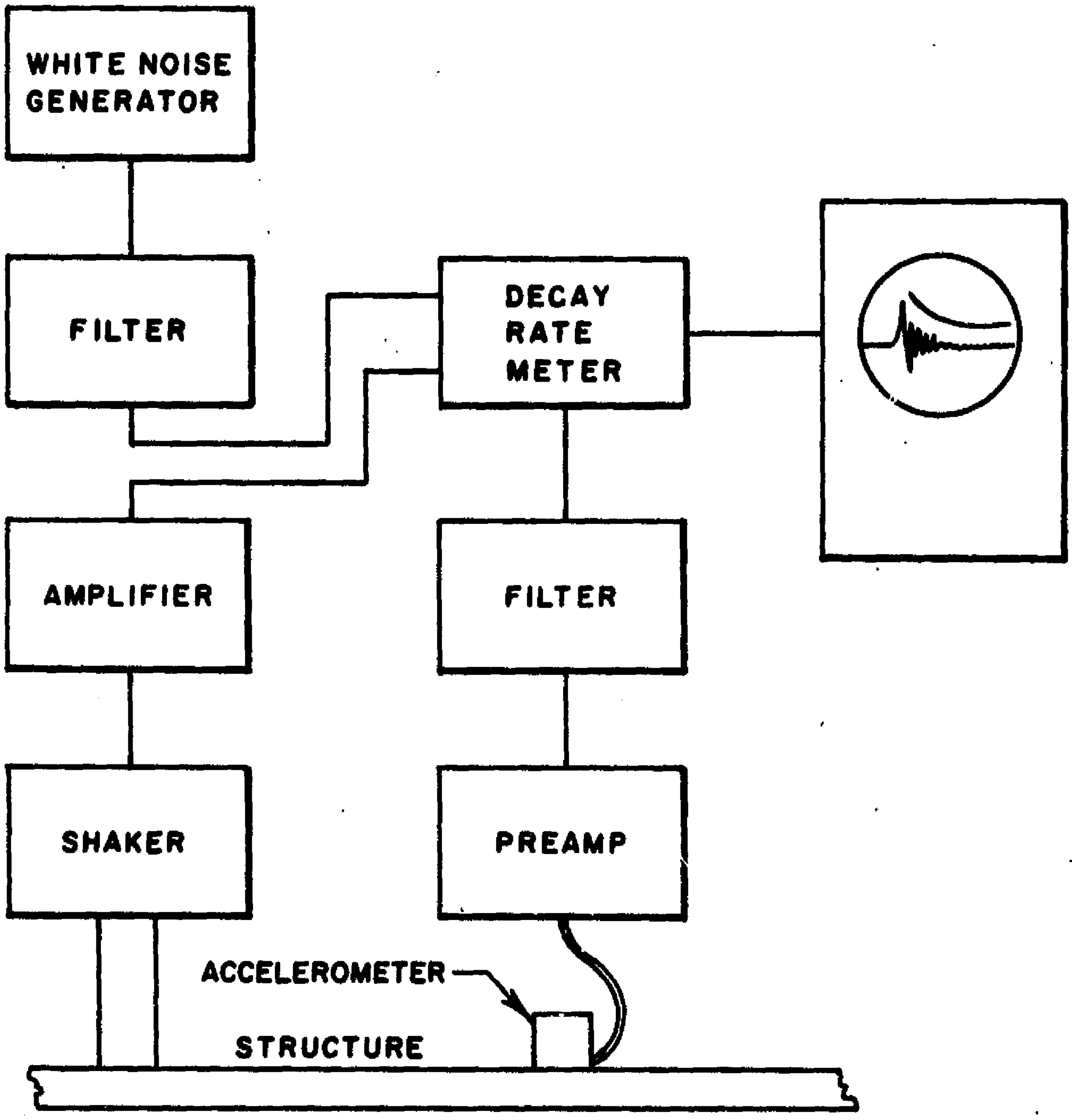

FIG.9b TYPICAL SET UP FOR MEASURING LOSS FACTOR WITH THE DECAY RATE METER

3.5 The Power Balance Equations

In the previous sections the power variables required for the SEA calculations have been related to the energy of the coupled multimodal systems being considered. With these relations one can write the power balance equation for a particular system as

$$<\pi_{in}^{i}>_{t,\Delta\omega,ens} = \sum_{j} \omega\eta_{i,j}\, n_i \left\{ <\frac{E_i}{n_i}>_{t,\Delta\omega,ens} - <\frac{E_j}{n_j}>_{t,\Delta\omega,ens} \right\}$$

$$+ \omega\eta_{i,diss} <E_i>_{t,\Delta\omega,ens} \qquad (3.5\text{-}1)$$

where the summation is over all connected systems. By writing a power balance equation for each system a set of algebraic equations is obtained which can be solved for the energies in each system.

Finally, one must relate the time-average energy of each system to the response variables of interest. Since the time-average kinetic and potential energies are equal for the resonant response of a system, this step is quite simple. The mean-square velocity of the structure averaged over location is given by

$$<\tilde{v}^2>_{t,\underline{x},\Delta\omega} = \frac{<E>_{t,\Delta\omega}}{M} \qquad (3.5\text{-}2)$$

where M is the total mass of the structure. The mean-square pressure in an acoustic space averaged over location is given by

$$<\tilde{p}^2>_{t,\underline{x},\Delta\omega} = \frac{\rho_0 c_0^2}{V} <E>_{t,\Delta\omega} \qquad (3.5\text{-}3)$$

where V is the volume of the space.

4. COMPARISON WITH AN "EXACT" CALCULATION

It will be instructive at this point to compare the approximate calculations in previous sections with an exact calculation. Fortunately, the coupled rod system already studied can be solved exactly in closed form as is demonstrated in Appendix A, using a Green's function approach based on the single rod configuration of Fig. 10. However, some caution must be exercised when making this comparison, for all of the previous approximate calculations are averages over an ensemble of coupled rods. In making an "exact" calculation one must necessarily look at a particular member of that ensemble. As a result, the degree of agreement between the exact and approximate solutions will depend on such things as the standard deviation of the quantity of interest across the ensemble. Lyon[28] has estimated standard deviations of this kind for a few particular cases but such calculations are beyond the scope of the work here. Instead, all that is desired is the development of some confidence in the approximate calculations presented here. This is best accomplished by comparing the exact solution of a particular system with those approximate calculations.

The problem to be examined is shown in Fig. 11 with two rods vibrating longitudinally and coupled with a linear massless spring at the ends. Rod 1 and only rod 1, is forced at the end where it is coupled. The values of the quantities of interest are

$$\rho_1 c_1 A_1 = \rho_2 c_2 A_2 = 590 \text{ lb/sec/ft}$$

$$L_1 = 17 \text{ ft}$$

$$L_2 = 14.2 \text{ ft}$$

$$c_1 = c_2 = 6800 \text{ ft}$$

$$K_c = 59 \text{ lb/ft} .$$

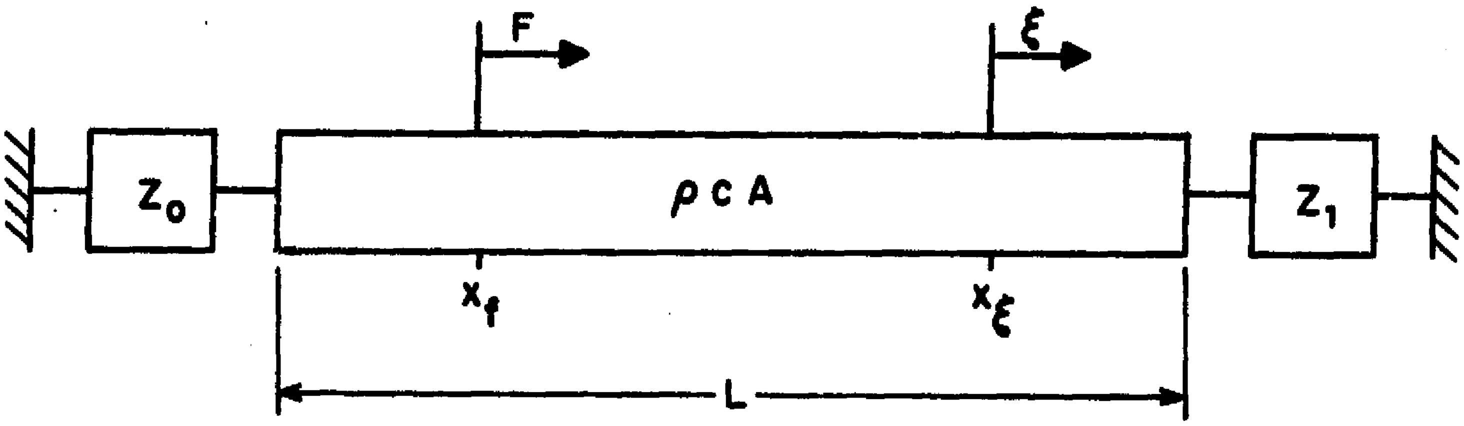

FIG.10 ROD IN LONGITUDINAL MOTION

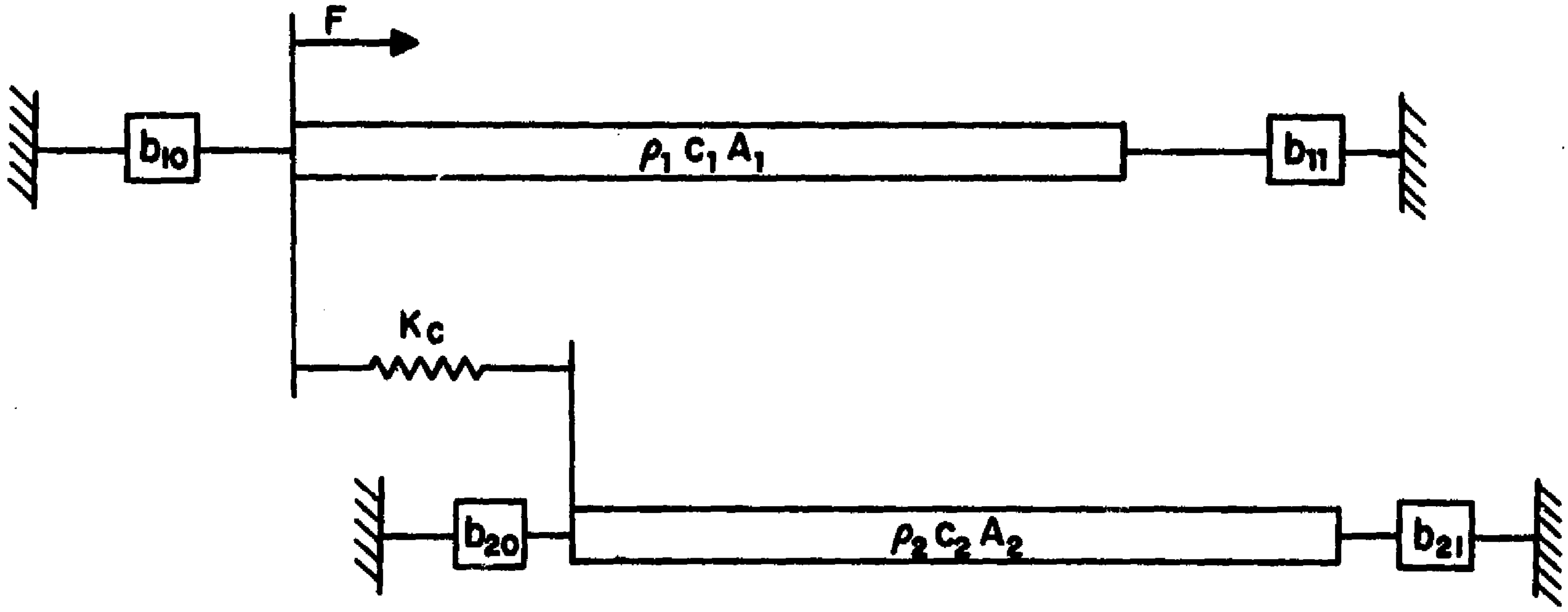

FIG.11 SAMPLE PROBLEM CONFIGURATION

Two values for the damping coefficient of the dampers at the ends of the rods are taken. To simulate the "model overlap" case the damping coefficients are all taken to be the same and equal to 324 lb/sec/ft. The "well-separated mode" case is simulated by again taking all the damping coefficients to be the same but this time equal to 32.4 lb/sec/ft.

4.1 Power Injected

The exact calculation of power injected into rod 1 is shown in Figs. 12 and 13 for the two values of the damping coefficients mentioned above. The locations of the peaks in both figures show the location of the natural frequencies of the modes in rod 1, since because of the light coupling, rod 2 has practically no influence on rod 1.

Using the approximate techniques of Sec. 3.2 yields for the power injected into rod 1

$$\pi_{in} = \frac{|F^2|}{2} \operatorname{Re} \frac{1}{Z_1}$$

Where Z_1 is the impedance of a *semi*-infinite rod. Because of the damper at the end of the rod and because the rod is forced at its end, Z_1 must be modified to include the impedance of the damper. Therefore,

$$\pi_{in} = \frac{|F^2|}{2} \operatorname{Re} \frac{1}{\rho_1 c_1 A_1 + b_{10}} \quad . \qquad (4.1\text{-}1)$$

Eq. 4.1-1 is compared with exact calculations averaged over a 200 Hz band in Figs. 12 and 13.* The agreement is excellent and should give the reader some confidence in the results of Sec. 3.2.

*This averaging in frequency over a 200 Hz band is equivalent to forcing the system with white noise, measuring the quantities of interest in 200 Hz bands, and normalizing the result to 1 Hz bandwidths.

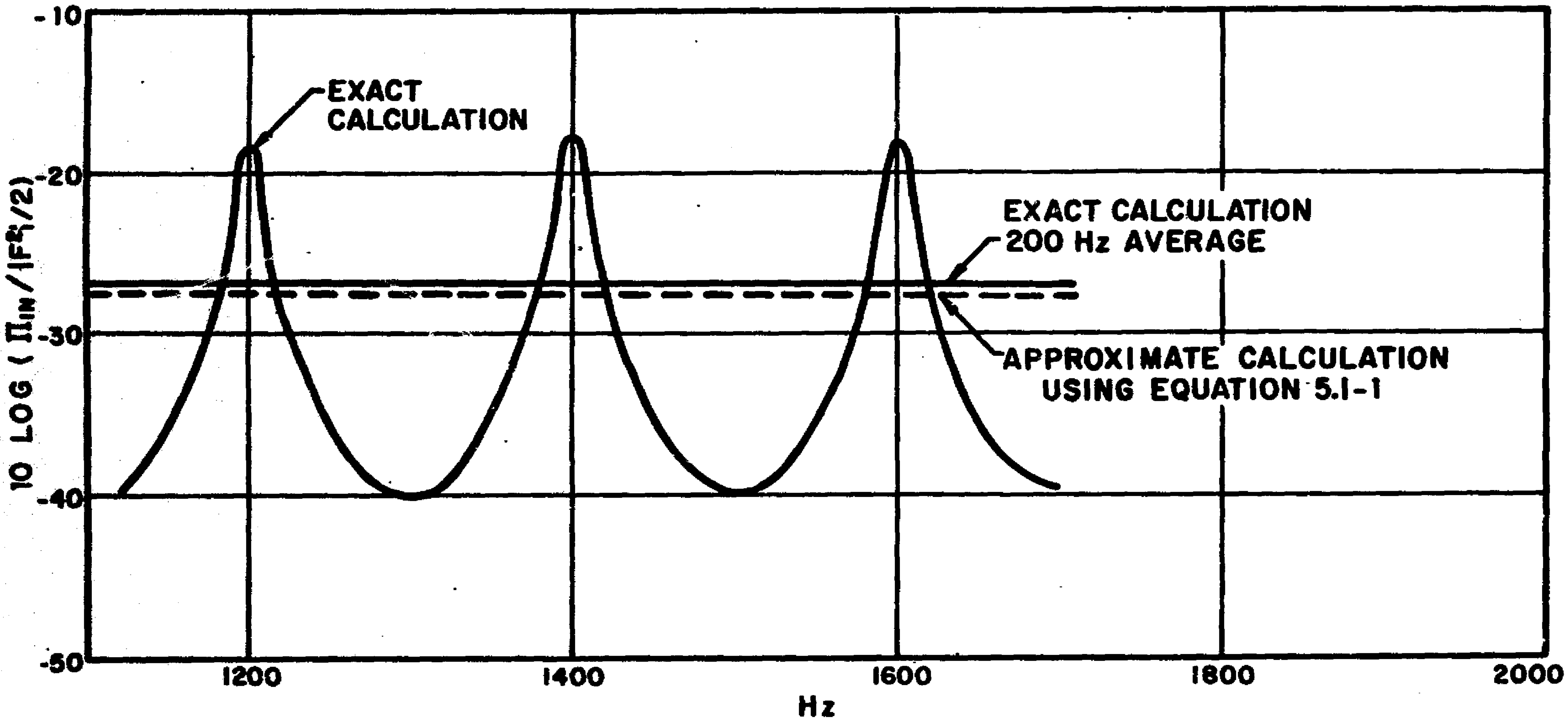

FIG.12 POWER INJECTED INTO
ROD 1
WELL SEPARATED MODE CASE

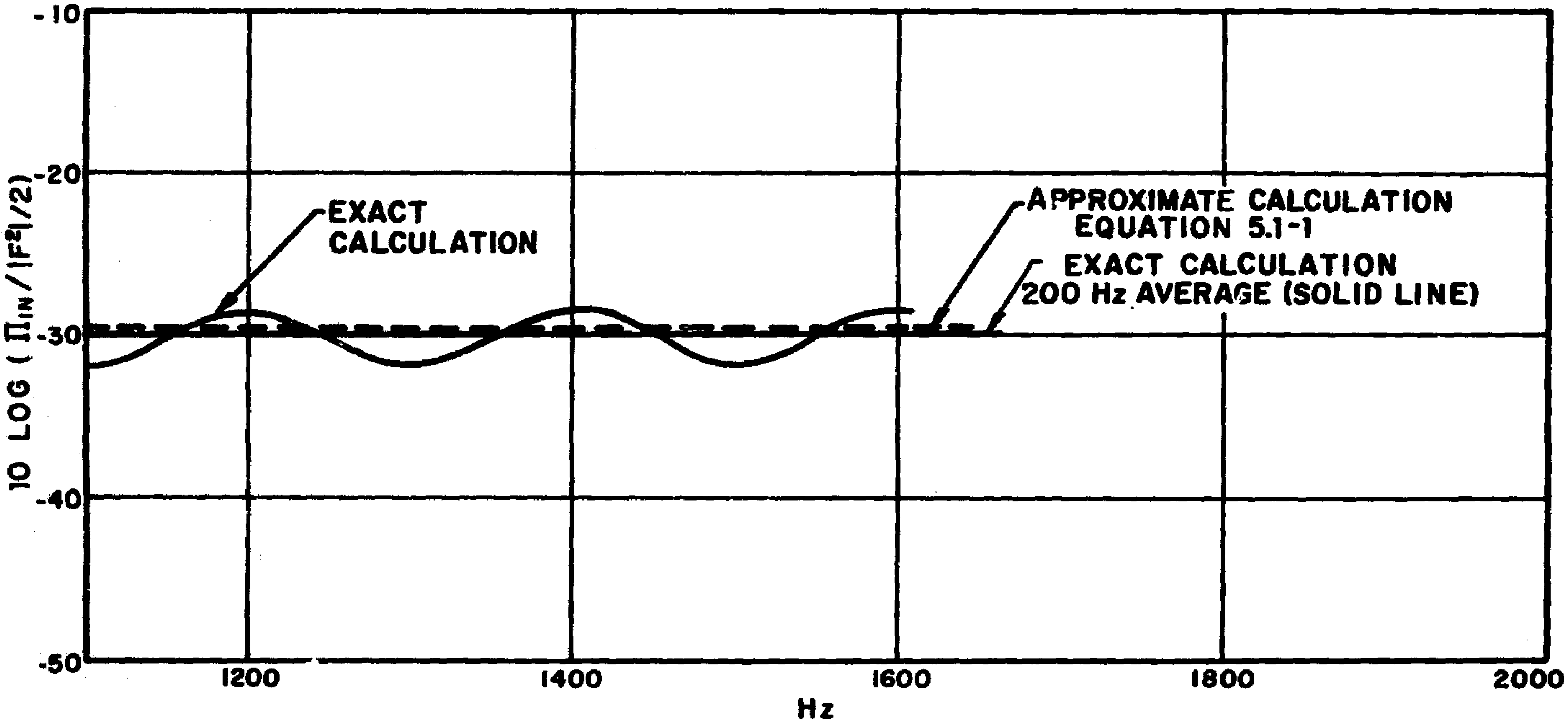

FIG.13 POWER INJECTED INTO ROD 1
MODAL OVERLAP CASE

4.2 Power Transmitted

The power transmitted from rod 1 to rod 2 and the energy in rod 1 (here twice the kinetic energy which is the total energy only at resonance) for the case of well-separated modes are shown in Figs. 14 and 15 and the same quantities for the case of modal overlap are shown in Figs. 16 and 17.

For both cases the energy in rod 2 is negligible (of the order of 100 dB below the energy in rod 1). Figure 14 is of particular interest since it shows clearly the characteristics of the two rods. At 1200 Hz the two rods have the same natural frequency and the power transmitted is high. As the frequency is increased to 1400 and 1600 Hz, the separation between natural frequencies of the two rods increases, as can be seen in the figure, and the power transmitted decreases. The pattern in frequency shown in Fig. 14 will repeat itself and is one reason the particular simple geometry of Fig. 11 was chosen. The same pattern though somewhat less distinct because of the high damping is found in Fig. 16 for the modal overlap case.

In any event the power transmitted from rod 1 to rod 2 can be written

$$\pi_{12} = \omega\eta_{12} E_1 \qquad (4.2\text{-}1)$$

Since the energy in rod 2 is negligible. Using Eqs. 3.3.1.2-12 or 3.3.2-11 one can calculate η_{12} and hence the power transmitted.

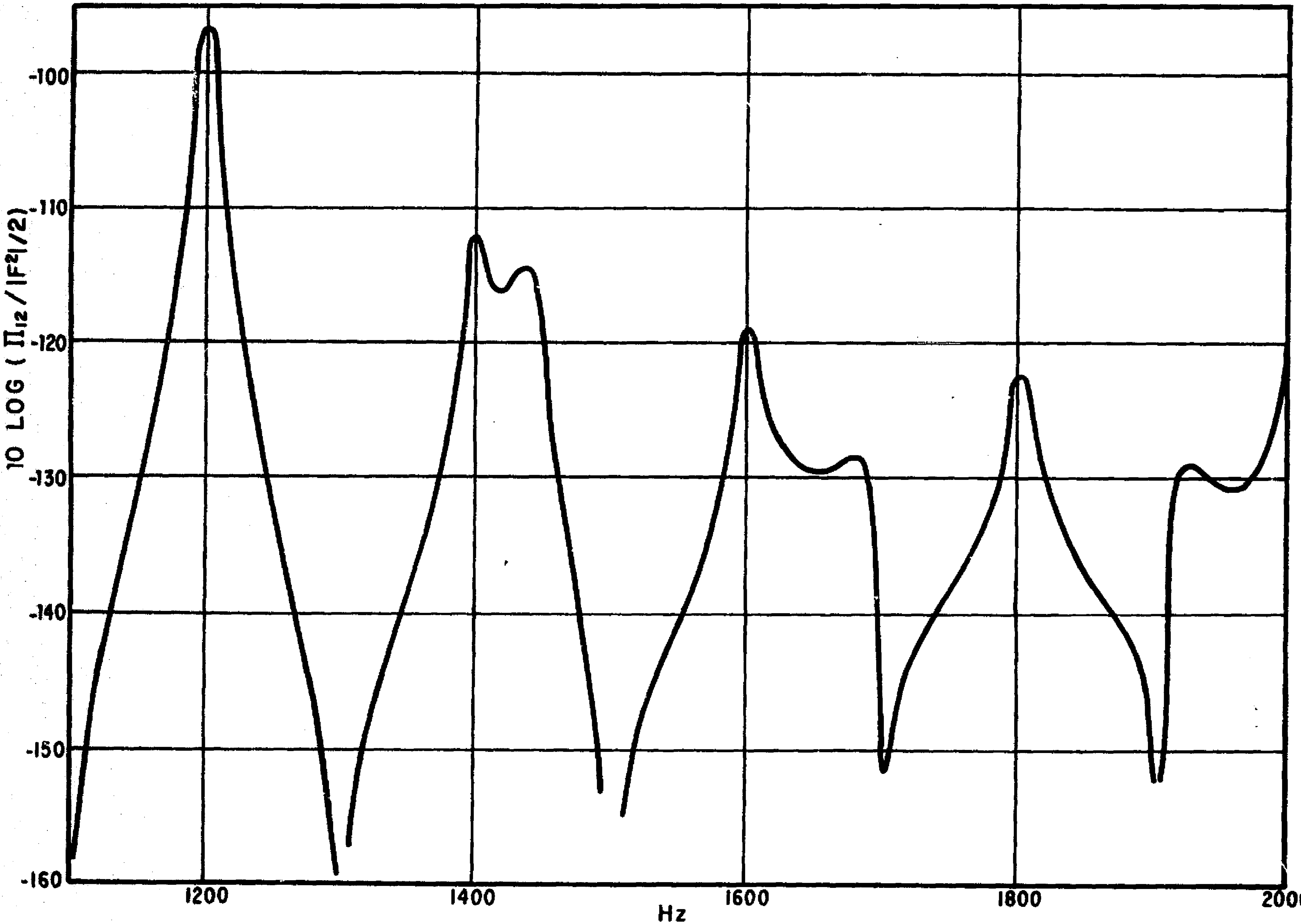

FIG.14 POWER TRANSMITTED FROM
ROD 1 TO ROD 2
WELL SEPARATED MODE CASE

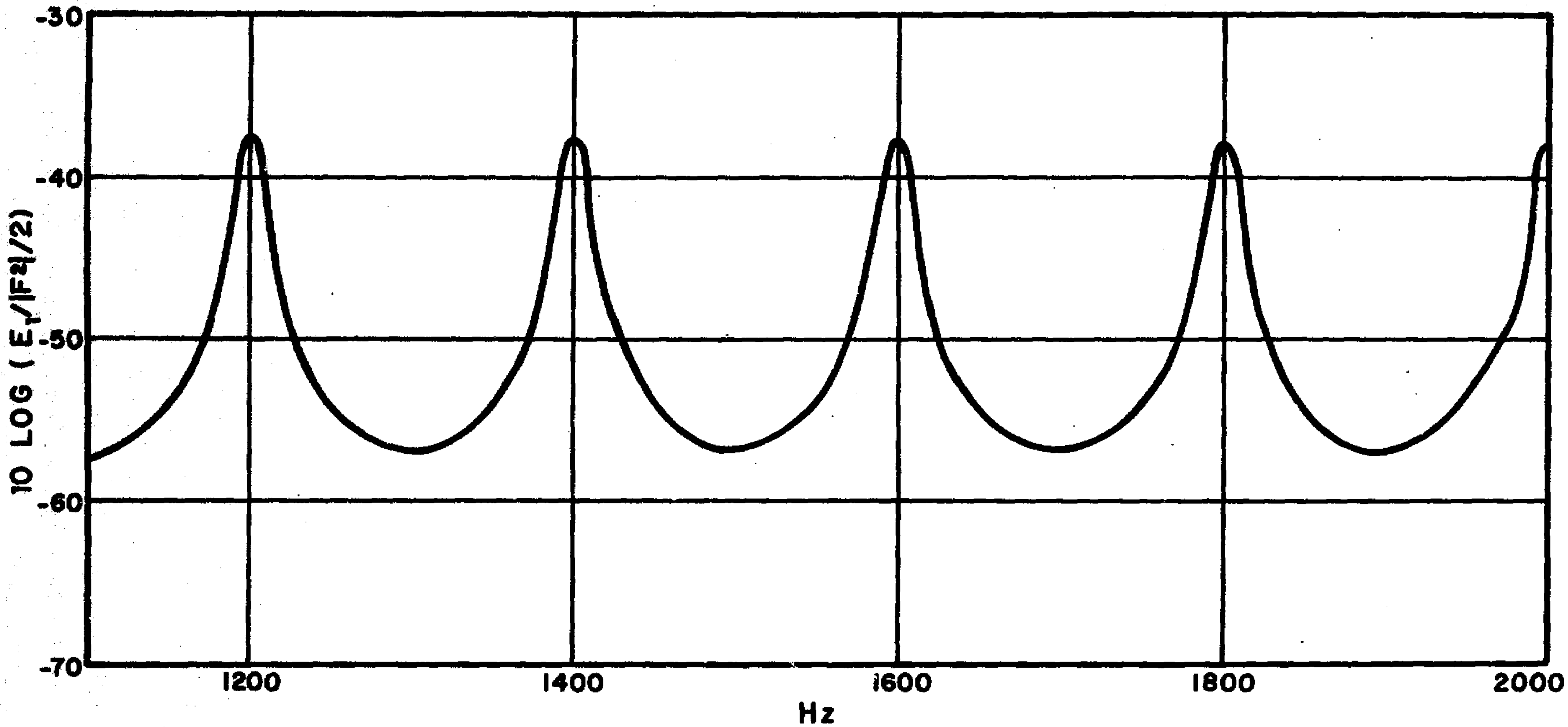

FIG.15 TWICE THE KINETIC ENERGY
OF ROD 1
WELL SEPARATED MODE CASE

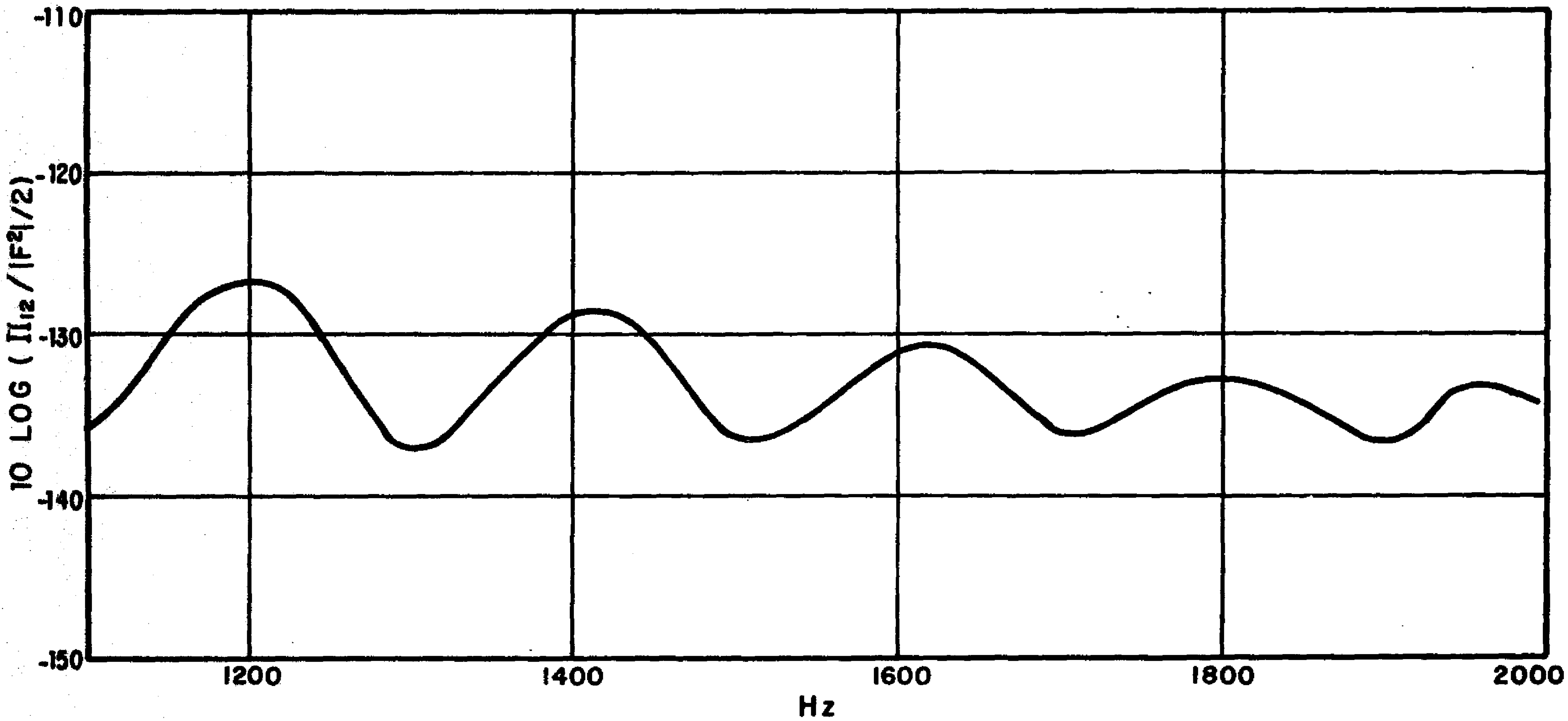

FIG.16 POWER TRANSMITTED FROM
ROD 1 TO ROD 2
MODAL OVERLAP CASE

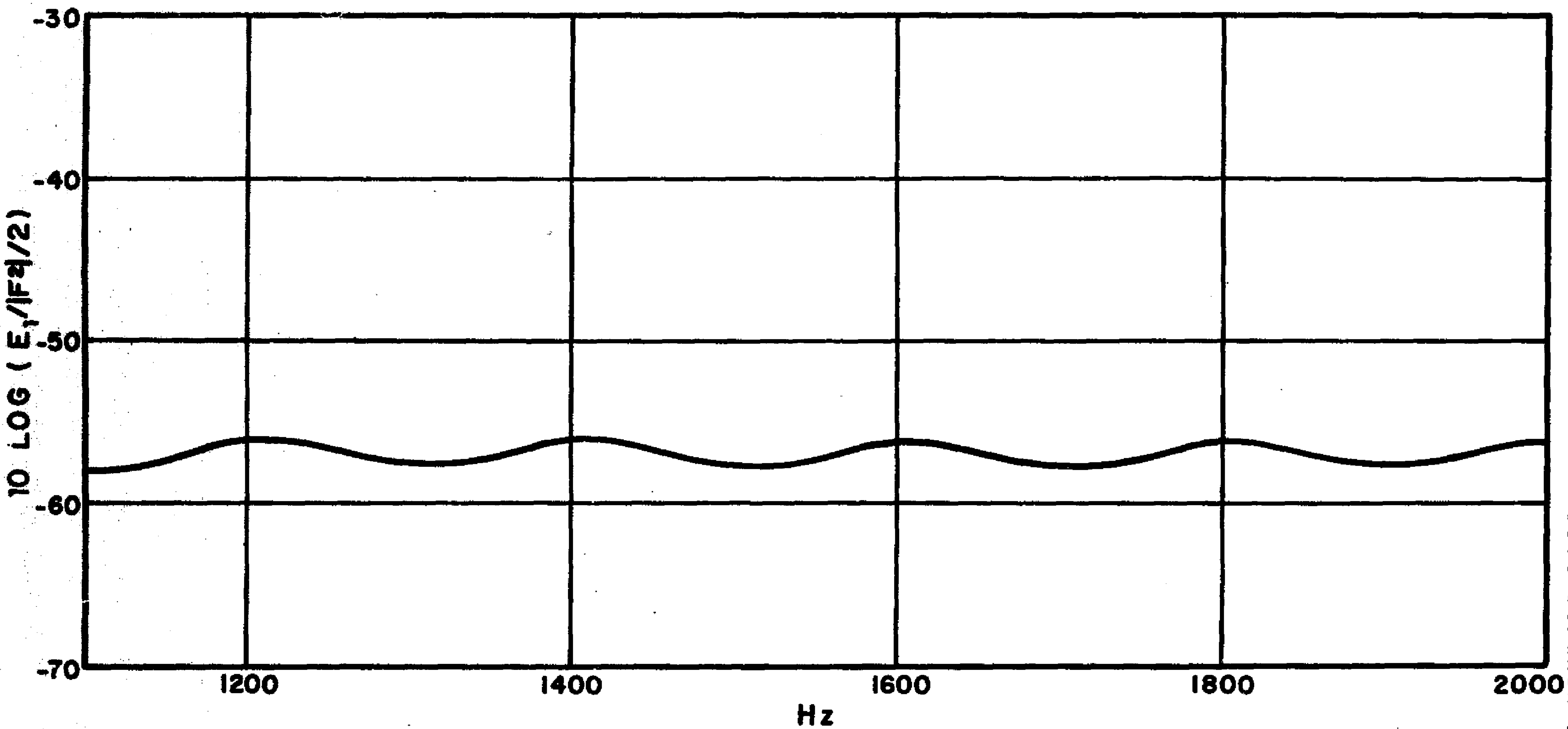

FIG.17 TWICE THE KINETIC ENERGY
IN ROD 1
MODAL OVERLAP CASE

The 200 Hz average of the power flow in Figs. 14 and 16 is compared with the approximate calculation of Eq. 5.2-1 in Figs. 18 and 19 respectively. The energy in E_1 in that approximate calculation has been taken as the 200 Hz average of E_1 in Fig. 15 (-47dB) for Fig. 18 and as the 200 Hz average of E_1 in Fig. 17 (-57dB) for Fig. 19. The approximate calculations are seen to be somewhat low in Fig. 18 for the "well-separated mode" case though a much broader frequency band must be used to average the exact calculation before too much can be said. Taking a 1200 Hz averaging bandwidth the approximate results at 1200 Hz were found to be about 6 dB low. This can be attributed to the fact that the coupling for the example here is at the ends where the amplitude of the mode shapes is always 1. Changing C_{mn} in Eq. 3.3.1.2-12 to reflect this raises the approximate result in Fig. 18 a factor of 4 or 6dB in agreement with the broadband averaged exact result.

The upper bound calculation using Eq. 3.3.1.2-18 for η_{12} is also shown in that figure. Clearly it correctly bounds the power transmitted for the averaging bandwidth. For much narrower averaging bandwidths the bound is found to become about 2 dB low at 1200 Hz where the modes in the two rods coincide in frequency.

The same comparison as above for the "modal overlap" case is shown in Fig. 19. The approximate calculation is seen to be between 1 and 4 dB below the 200 Hz average of the exact calculation. However, the upper bound calculation using η_{12} from Eq. 3.3.1.2-19 and the approximate calculation using η_{12} from Eqs. 3.3.1.2-12 or 3.3.2-11 are found to nicely bracket the exact solution in that figure.

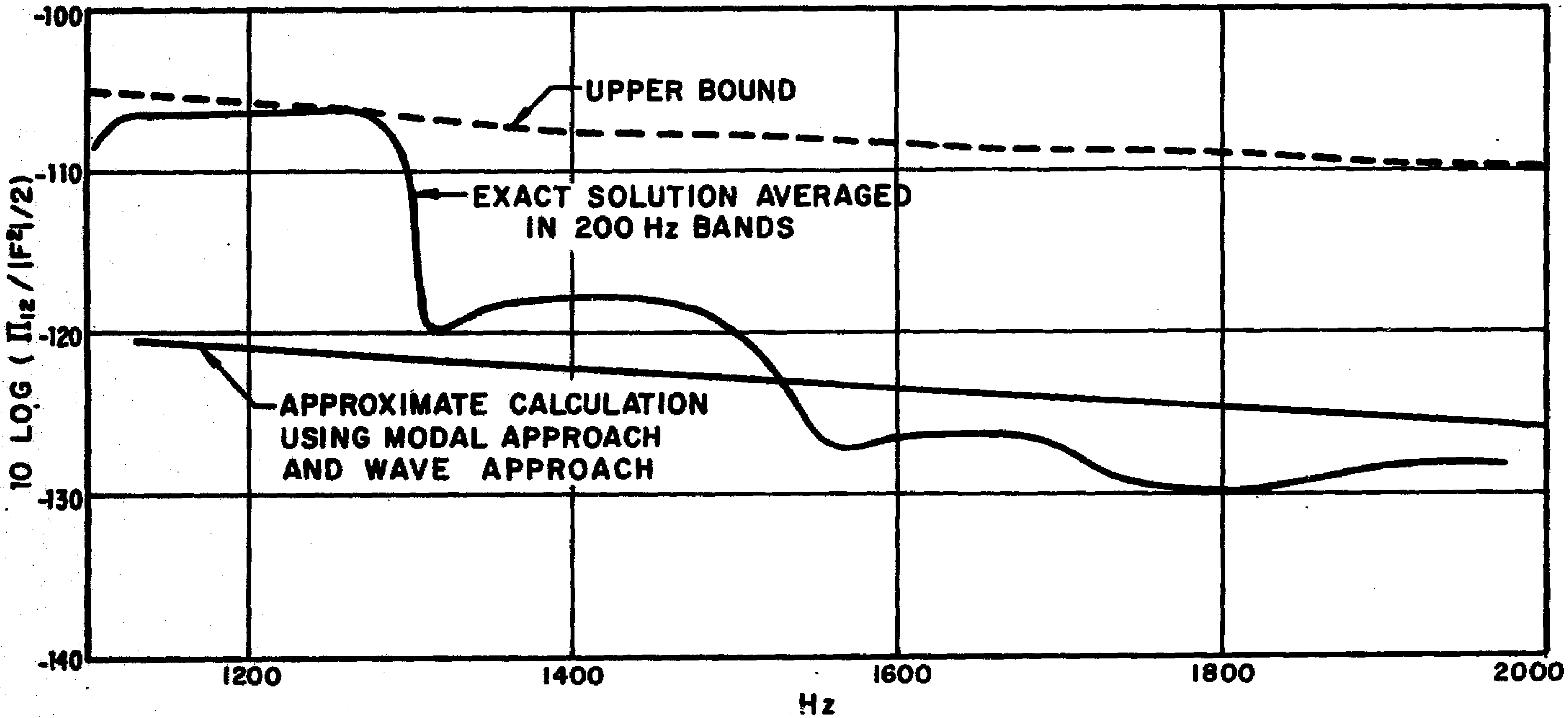

FIG.18 POWER TRANSMITTED FROM ROD 1 TO ROD 2 AVERAGED IN 200 Hz BANDS WELL SEPARATED MODE CASE

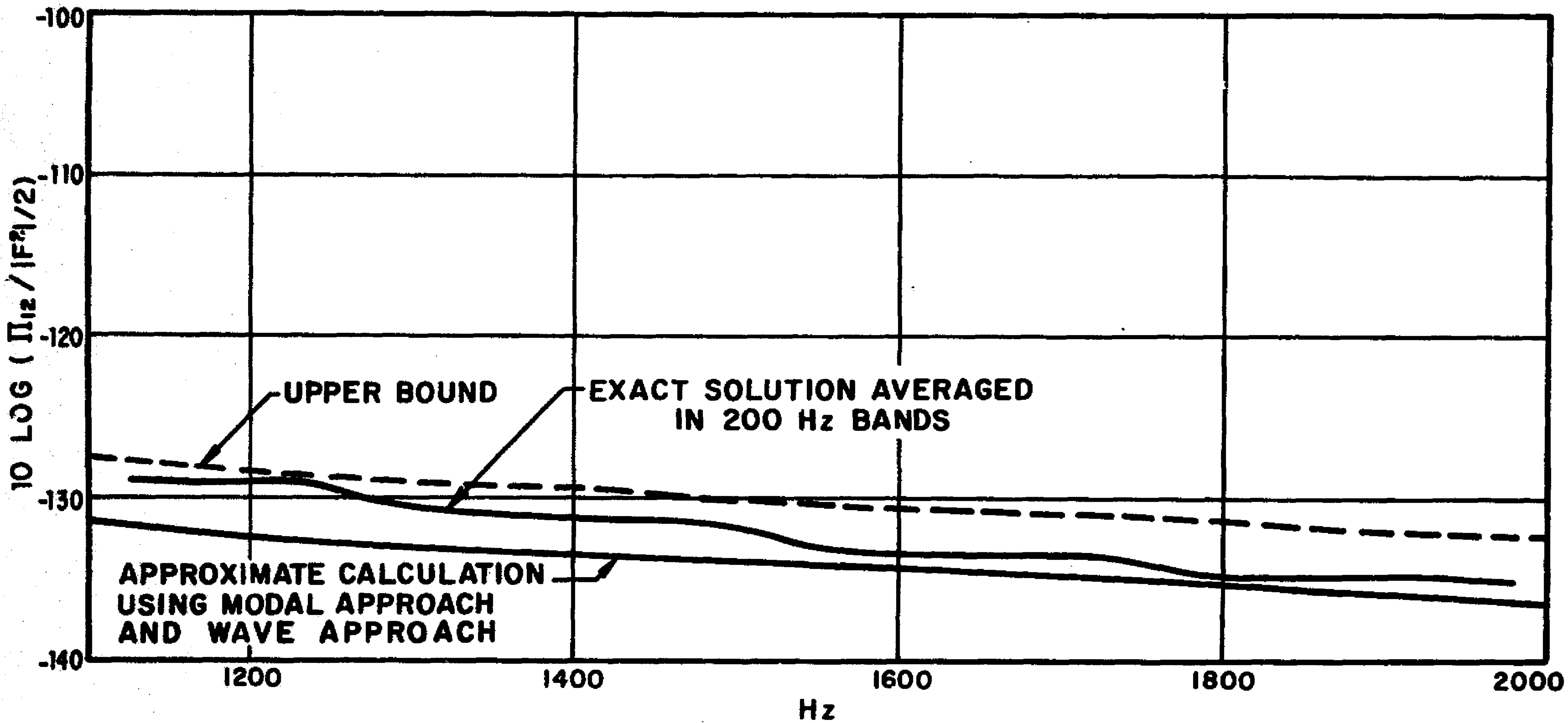

FIG.19 POWER TRANSMITTED FROM ROD 1 TO ROD 2 AVERAGED IN 200 Hz BANDS MODAL OVERLAP CASE

4.3 Power Dissipated

As will be demonstrated in a later section the ratio of energies in the two rods required a knowledge of the power dissipated in rod 2 or equivalently the dissipation loss factor, η_2. The power dissipated in a structure must usually be measured. However, the simplicity of the dissipation mechanism in the example makes an approximate calculation of the dissipated power relatively simple. As a result we will demonstrate how the modal technique and the wave technique can be used to calculate power dissipated in rod 2. From Sec. 3.4 one may write

$$\pi_{diss,2} = \omega\eta_2 E_2 \ . \qquad (4.3\text{-}1)$$

Using the modal technique one requires the modal equations for rod 2 of Sec. 3.3

$$m_n\ddot{v}_n + m_n\Delta_n\dot{v}_n + \omega_n^2 v_n = F_n \qquad (4.3\text{-}2)$$

where $F_n = (G_n + 1/\lambda \ C_{mn}u_m)m_n$

$m_n = m_2/2.$

By multiplying Eq. 4.3-2 through $\dot{v}_n$ and averaging in time one obtains

$$\langle F_n\dot{v}_n\rangle = \Delta_n m_n\langle\dot{v}_n^2\rangle \qquad (4.3\text{-}3)$$

where the left hand side of Eq. 4.3-3 represents the total power injected into mode n of rod 2 and by a simple power balance the right hand side must be the power dissipated. From Sec. 3.3 one can readily see that Δ_n is a constant for all n and that $m_n\dot{v}_n^2$ is the total energy of mode n of rod 2. Summation over all modes in the frequency band of interest in Eq. 4.3-3 yields

$$\pi_{diss,2} = \Delta_n \Sigma m_n \langle \dot{v}_n^2 \rangle = \Delta_n E_2 \tag{4.3-4}$$

where E_2 is the total energy in rod 2 in the frequency band of interest. Combining Eq. 4.3-4 and Eq. 4.3-1 and the expression for Δ_n in Sec. 3.3 one obtains

$$\omega\eta_2 = \Delta_n = \frac{2(b_{20}+b_{21})}{m_2} \quad . \tag{4.3-5}$$

For the wave technique an approximate calculation is also possible. Consider the motion on rod 2 to be made up of a wave traveling to the right and to the left. Considering the interaction of the rod with the dashpot on the right-hand side one can write for the time average dissipated power

$$\pi_{diss,2} = \frac{1}{2} \mathrm{Re}\{F\dot{\xi}_{ER}^{*}\} = \omega^2 b_{21} \frac{|\xi_{ER}^2|}{2} \tag{4.3-6}$$

where ξ_{ER} is the displacement amplitude at the right end of the rod and F is the force applied to the rod by the dashpot with damping coefficient b_{21}. This displacement amplitude can be easily calculated by assuming that the rod is semi-infinite and that there is a single traveling wave of displacement amplitude ξ_R approaching the end of the rod from the left and, then, matching boundary conditions at the end of the rod. After doing this one obtains

$$\xi_{ER} = \xi_R(1+\gamma) \tag{4.3-7}$$

where

$$\gamma = \frac{1-b_{21}/\rho_2 c_2 A_2}{1+b_{21}/\rho_2 c_2 A_2} \ .$$

A similar set of equations may be written for the left-hand side of the rod in terms of the left running wave displacement amplitude ξ_L. Combining Eq. 4.3-6, Eq. 4.3-7 and Eq. 4.3-1 and further assuming that $b_{21} << \rho_2 c_2 A_2$* one may write

$$\pi_{diss,2} = \omega\eta_2 E_2 = 2\omega^2 (b_{21} |\xi_R^2| + b_{20} |\xi_L^2|) \ . \qquad (4.3\text{-}8)$$

Assuming that the two waves are equal in magnitude but uncorrelated the total energy in rod 2 may be written

$$E_2 = \rho_2 A_2 L_2 \omega^2 |\xi_R^2| \qquad (4.3\text{-}9)$$

Substituting Eq. 4.3-9 into Eq. 4.3-8 and solving for $\omega\eta_2$ one obtains

$$\omega\eta_2 = \frac{2(b_{21}+b_{20})}{m_2} \ . \qquad (4.3\text{-}10)$$

Note that the result using the modal technique is the same as that using the wave technique.

It remains to use the result in the power balance equations to calculate the ratio of the energies in the two rods.

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*Note that this assumption is not strictly valid for the strongly damped case; however, the error introduced is small on a decible scale and may be neglected.

4.4 The Power Balance Equations and Energy Ratio

The power balance for the two rods is quite simple

$$\pi_{12} = \pi_{diss,2}$$

or

$$\omega\eta_{12}\left(E_1 - \frac{n_1}{n_2} E_2\right) = \omega\eta_2 E_2 \ .$$

Because E_2 is much less than E_1 this equation may be simplified to

$$\frac{E_2}{E_1} = \frac{\eta_{12}}{\eta_2} \ . \qquad (4.4\text{-}1)$$

Combining the equation with Eq. 3.3.2-11 and Eq. 4.3-10 yields an expression for the ratio of the energy in rod 1 to that in rod 2. These energies are of course directly relatable to the space time average velocity squared in each rod or

$$E_1 = m_1 \langle \dot{\xi}_1^2 \rangle_{x,t}$$

$$E_2 = m_2 \langle \dot{\xi}_2^2 \rangle_{x,t} \ .$$

The ratio of the two energies is plotted in Figs. 20 and 21 for the well separated mode case and the modal overlap case, respectively. For both cases the exact calculation is found from a 200 Hz bandwidth average of twice the time average kinetic energy in rod 2 divided by this same average energy in rod 1. The approximate calculation using Eq. 4.4-1 are also shown in these figures where η_{12} is defined in Eq. 3.3.2-11. The upper

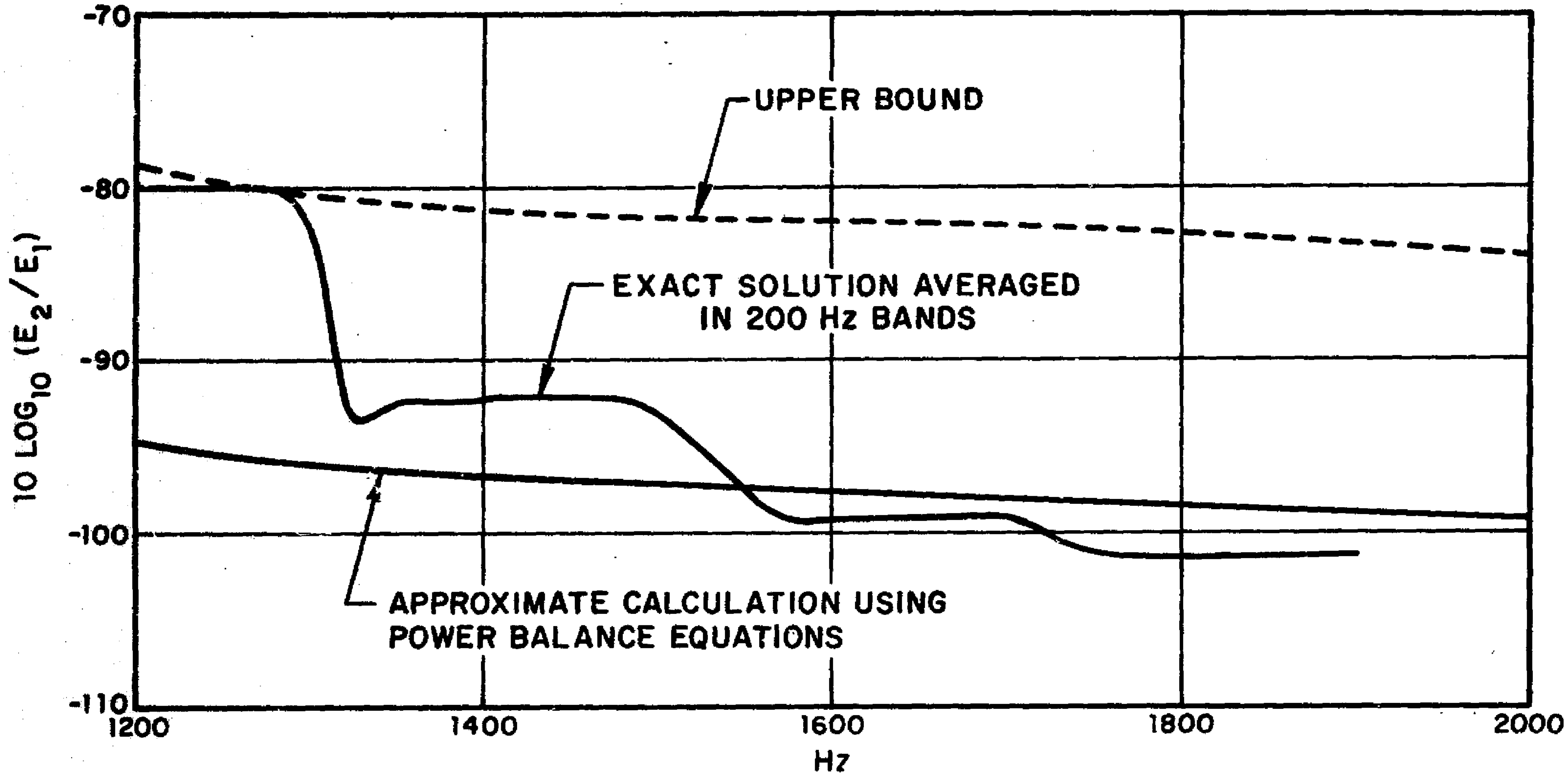

FIG.20 ENERGY RATIO
ROD 2 TO ROD 1
WELL SEPARATED MODE CASE

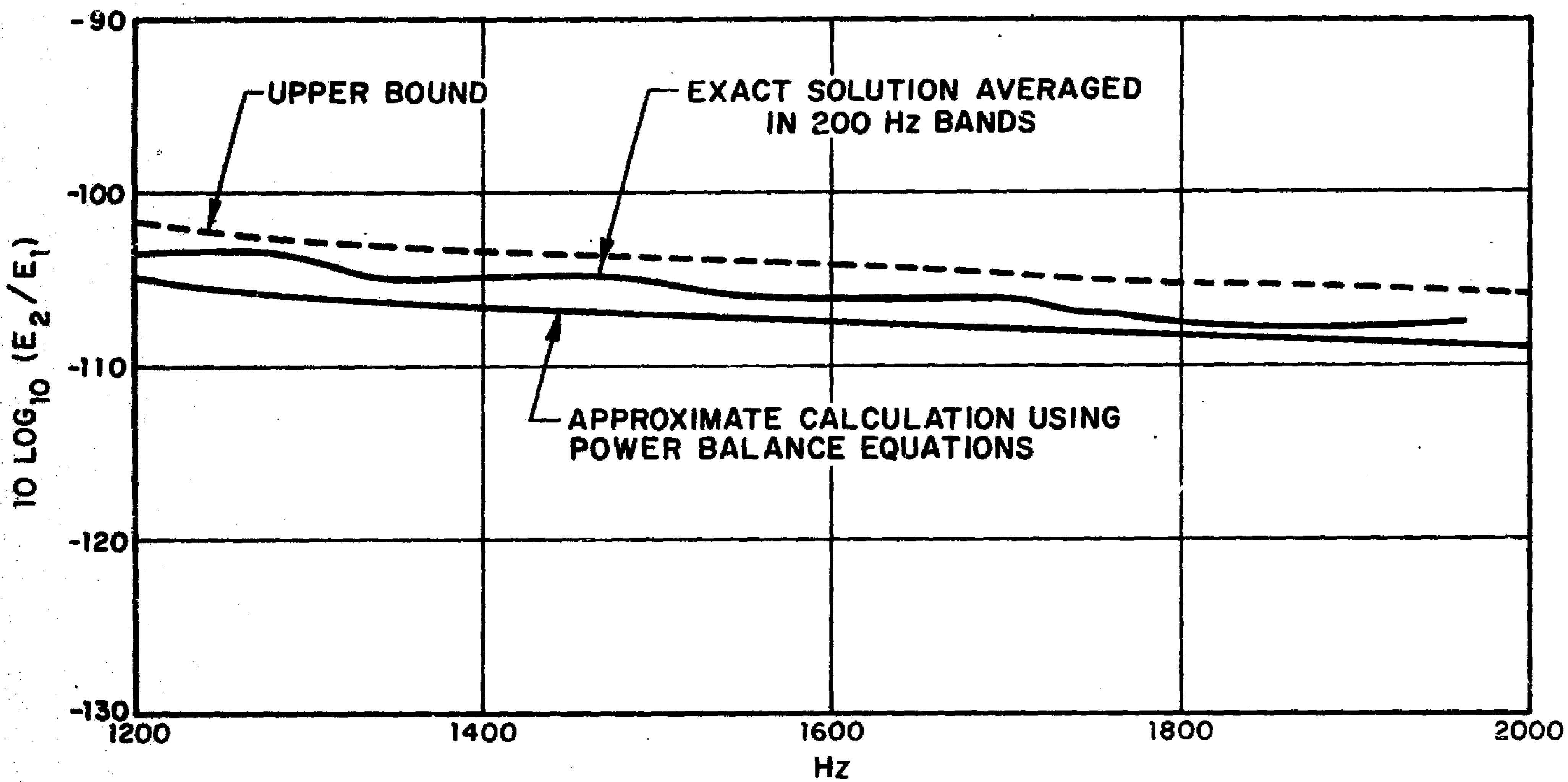

FIG.21 ENERGY RATIO
ROD 2 TO ROD 1
MODAL OVERLAP CASE

bound approximate calculation is formed by using the upper bound calculation for η_{12} in Eq. 3.3.1.2-18 for the well separated mode case and Eq. 3.3.1.2-19 for the modal overlap case. Figures 20 and 21 are seen to be quite similar to Figs. 18 and 19 in which the power transmitted is displayed, and all statements made concerning the agreement of exact and approximate calculations in Figs. 18 and 19 (see Sec. 4.2) apply here. As in Fig. 19 a broader averaging bandwidth in frequency would have improved the agreement in Fig. 20.

These examples clearly point out that given a sufficiently broad measuring bandwidth and sufficient care in applying the techniques, SEA calculations can be applied with confidence. However, it should be borne in mind that SEA results are statistical in nature and as a result any particular member of the ensemble may not agree completely with the SEA result despite a very broad measuring bandwidth. The problem studied here is a clear example of this.

A further caution should be mentioned here, namely, that all of the work presented in this section has been for "*weak*" coupling. This means that the natural frequencies of the two rods do not change significantly upon being coupled together. Figure 14 clearly illustrates this, for each peak in the power flow corresponds to a natural frequency of the uncoupled rods.

5. GUIDELINES FOR THE USE OF STATISTICAL ENERGY ANALYSIS

The potential user of Statistical Energy Analysis (SEA) must understand the basic underlying concepts before he can hope to treat his particular problem of interest. We assume that the reader has gained this basic understanding by reading the first four sections of this report. It is then useful to conclude by giving some guidelines to the use of SEA. These guidelines will help the reader organize his attempt to solve any particular problem using SEA. They will also help the reader decide when to use SEA and what to expect.

Statistical Energy Analysis is most useful in solving vibration and acoustic problems in which many modes of vibration contribute significantly to the response variable of interest. Problems in which displacement is the response variable of interest tend to be less amenable to the SEA approach than problems in which acceleration is of interest, since in most cases many more modes contribute significantly to the acceleration than to the displacement. Also, problems with acoustic or small-scale aerodynamic pressure fluctuations as the dominant source of excitation are more readily treated using SEA because these types of excitation usually excite many modes of vibration quite strongly. Finally, problems involving acoustic spaces and/or large plate and shell structures are more readily treated using SEA than problems involving beams and/or small structures, since the former have many more resonant frequencies in any given band than do the latter.

There are a number of reasons why SEA is more useful for problems involving many modes. SEA is a statistical approach — the statistics being taken over an ensemble of structures and acoustic spaces in which the resonant frequencies and mode shapes are random variables. Like any statistical approach the ensemble average quantities can be used with resonable confidence for one member of the ensemble only if the variance of the quantity across the ensemble is small. Thus, ensemble average response estimates obtained using SEA can be used with confidence for a single member of the ensemble only if the variance of the response across the ensemble is small. Unfortunately, most papers dealing with SEA do not clearly define the ensemble. Furthermore, the calculation of variance has been accomplished only for a few very simple problems. Since a quantitative estimate of variance cannot be made, we must be content, at least for the present, to offer some qualitative guidelines.

The few analytical calculations of variance that have been made, plus a large amount of experimental data, indicate that the variance of the response across the ensemble of structures and acoustic spaces decreases as the number of modes contributing to the response increases. Thus, the average response over an octave band of frequencies for wideband random excitation will have less variance across the ensemble than will the response at a single frequency for pure-tone excitation. Similarly, the average response in a given band of frequencies for a large structure or acoustic space will have less variance across the ensemble than will the response for a small structure or acoustic space which will have fewer resonance frequencies in the band. And, finally, the spatial average response of a structure or acoustic space will have less variance than the response at one point, since many modes can contribute to the spatial average response while only those with antinodes nearby can contribute significantly to the response at one point.

As a practical matter, applications of SEA should be limited to cases in which many modes of vibration contribute to the response. Most uses of SEA have been to calculate spatial-average responses averaged over one-third octave or octave bands of frequencies. Results from numerous experiments show that these spatial average response estimates* usually agree within ±3 dB with the data from a particular structural configuration as long as each energy storage element in the SEA model has at least five (5) modes in each frequency band being considered. This requirement can always be met by averaging the response estimates over sufficiently wide bands of frequency. Narrow band or pure-tone predictions can be obtained using SEA, but they will not be accurate estimates for a particular configuration unless the structures and/or acoustic space are very large. The practical necessity to limit the use of SEA to problems in which many modes contribute to the response means that this method of analysis is not useful in studying the frequency range at and near the first few resonance frequencies of the complete structural assembly (so-called system resonances).

The advantage in using SEA over more classical techniques lies in the fact that exact resonant frequencies and mode shapes are not needed. In many cases of practical interest the boundary conditions and damping mechanisms in a structure or acoustic space are so complex that it is impossible to predict with any accuracy the resonant frequencies and mode shapes. In such a case it does not make sense to use an analysis technique which requires exact information as to these quantities.

Although SEA does not require exact information as to resonant frequencies and mode shapes, it does require the user to define, at least implicitly, an ensemble in which the resonant

*We mean here spatial-average response averaged over the ensemble of structures and acoustic spaces.

frequencies and mode shapes are randomly distributed with known probability distributions. In most applications to date resonant frequencies are randomly distributed so that the spacing between resonant frequencies is a Poisson process. The mode shapes are typically taken as those of a specific structure or acoustic space which is felt to be representative of the ensemble average structure or acoustic space. It is, of course, possible in using SEA to choose any distribution for the resonant frequencies and mode shapes. However, deviating from the common assumptions above would represent an advance in the state-of-the-art.

The advantage described above can become a disadvantage for those problems in which the resonance frequencies and mode shapes are well-known. In this case, it does not make sense to use SEA since more accurate solutions can be obtained using the more classical methods of analysis.

The key steps in solving a vibration problem using Statistical Energy Analysis are:

1. Divide the complex structure into energy storage elements;
2. Determine the paths of energy exchange;
3. Write the power balance equations;
4. Compute the required input powers, coupling loss factors, damping loss factors and modal densities;
5. Solve the power balance equations for the energies of each element;
6. Relate the energy to the response variable of interest.

The most difficult steps above are steps 1, 2 and 4. Each of these steps requires a good understanding of SEA and an intuitive feel for the dynamic behavior of the system being studied.

Although specific rules for accomplishing the above steps cannot be given, some guidelines can be set out.

Completion of the first step requires two successive divisions. First, the complex structural assembly under consideration must be divided into structural subelements, i.e., acoustic spaces, plates, shells, beams, rings, instrument boxes, etc. In most cases this first division is fairly obvious. However, some confusion can occur. Short connecting beams or shells used to join two larger structures may have only a few resonances over the entire frequency range of interest, thereby, invalidating our practical limitation on the use of SEA requiring that each structural element have at least five resonant frequencies in each frequency band of interest. In this case it is best to treat the large structures as energy storage elements and to include the effects of the connecting structure in the calculation of the coupling loss factor between the two storage elements. The connecting structure can in many cases be modeled simply as a spring connection.

A second case in which some confusion can occur is when two structural elements are very intimately coupled together. The question arises as to whether the two subelements should be treated separately or together as one energy storage element. The two elements can always be treated separately and the concept of equipartition of energy, which will be discussed later in this section, used to define the modal energies. However, some care must be used in defining the substructure modes when the coupling is very large. The substructure modes must be selected so that the motion of the coupled substructures in a given band of frequencies can be described accurately by a combination of responses of the substructure modes which have resonant frequencies in the band. For example, consider two beams joined at their ends at right angles. Modes of each beam obtained for simple supports cannot be used to describe the motion of the coupled beams since these modes have no moment at the end of each beam, where they are coupled together. For this problem, it is necessary to use boundary conditions for each beam which allow both a moment and angular velocity. Many other cases of this type exist. Fortunately, however, use of the wave approach (see Sec. 3.3.2) for calculating a coupling loss factor automatically insures that the motion of the junction between two substructures is accurately described.

In many cases, a second division of the total assembly is required. The power transfer calculations of Sec. 3.3 require that each mode in an energy storage element have approximately the same energy. If a group of modes in a particular substructure is much more strongly excited or damped or is more strongly coupled to modes in other energy storage elements, then it will probably have a different energy and should be placed in a separate energy storage element. For example, it is usually necessary to separate bending and torsional modes of a beam into different elements. Similarly, in studying the vibrations of a beam on a plate it may be necessary to divide the plate modes into a set which is well-coupled to the beam modes and a set which is not, depending on whether we can assume that all modes in the plate with resonance frequencies in a given band have the same energy.

Step 2 in SEA is to determine the important paths of energy exchange. As a general rule, the power exchange between any pair of energy storage elements containing modes from substructures which are touching should always be included. In addition, the power exchange in a frequency band between resonant modes of nontouching structures through nonresonant modes of an intervening structure should be considered. None of these indirect paths are the most important paths of energy exchange. For example, the transmission of sound in a frequency band from one acoustic space to another through an intervening structure is usually through nonresonant modes of the structure which have resonant frequencies below the band of interest.

The third step of SEA – writing the power balance equations – is very simple and needs no further discussion.

The fourth step of SEA – calculations of the input powers, coupling loss factors, damping loss factors and modal densities – requires the greatest amount of technical effort. For many problems the SEA user can appeal to results published in the literature. To help the reader toward this end Appendix B of this report lists published papers which give expressions of potential use in SEA. When it is necessary to compute new expressions the following guidelines will be helpful.

Problems in which a power input must be calculated can be divided into two categories. First, when the excitation is distributed over the entire structure it will be necessary to compute the power input to each mode of the structure and then to average the resulting expression over the selected ensemble of structures. The use of joint acceptance expressions to compute the power input from a distributed pressure field to a plate or shell is a good example of such a calculation. Second, when the excitation is localized the impedance approach discussed in Sec. 3.2.1 can be used. The excitation can be considered as localized if it is distributed over at least one less dimension than the structure or acoustic space. For example, a point excitation on a beam, plate, or acoustic space, a line excitation on a plate or acoustic space, and a distributed surface pressure in an acoustic space can all be considered as localized.

When it is necessary to compute a coupling loss factor the authors recommend using the wave approach presented in Sec. 3.3.2, since they have found this approach to be the simplest and most general technique. It is also possible to compute a coupling loss factor directly from the coupled equations of motion if the coupling is small enough that a perturbation technique can be used. Use of the mode to mode calculations for cases of high coupling is not recommended.

In many cases of high coupling we can simplify the problem by appealing to the concept of equipartition of energy. The energy interaction equations show that if the coupling loss factor between two groups of modes is large in comparison to the damping loss factors of each group, then the two groups will have the same average modal energy — equipartion of energy. Since damping loss factors are usually quite small, this concept can often be used to eliminate the need to calculate a coupling loss factor. It is sufficient to say that it is large compared to the damping loss factors:

The calculation of damping loss factors for a new problem is always difficult. Unless a special damping treatment is in use it is impossible to analytically compute the damping loss factor. Empirical prediction techniques can be used with some success but are often not as accurate as one would wish. Measurement is perhaps the best method of calculation, but the structure being analyzed is often not yet constructed. Finally, when it is necessary to compute a modal density, the reader should refer to the references in Appendix B.

Step 5 of SEA — solution of the power balance equations — requires that we solve a number of linear algebraic equations. If the number of energy storage elements in the analysis is small this can be done quite simply. When large numbers of elements are involved it is usually possible to make simplifications so that the solution is still easily accomplished. But even when such simplifications are not possible the linear equations can be easily solved with the help of a digital computer.

The final step of SEA is to relate the modal energies of each energy storage element to the response variable of interest. Assuming that the systems being studied are lightly damped and that the analysis bandwidth is in octaves or narrower, this step is quite simple. In a lightly damped system excited at one or more of its resonances by a band of random excitation, the time-average kinetic energy is equal to the time-average potential energy. Thus, we can obtain either the mean-square velocity of a structure or the mean-square stress through the total time-average energy of the structure. When the band of excitation is narrow (octave band or less) we can find the mean-square acceleration or displacement by multiplying or dividing the mean-square velocity by the band center frequency squared.

The guidelines presented above in conjunction with the first four chapters of the report should help the reader in using SEA. Many specific rules and formulations important to SEA have been left out for the simple reason that they have not yet been completely developed. The authors hope that a continued interest in SEA will someday lead to more explicit formalizations of many of the concepts discussed in this report.

LIST OF REFERENCES

1. R.H. Lyon and G. Maidanik, "Power Flow Between Linearly Coupled Oscillators", *J. Acoust. Soc. Amer.*, **34**, 623–639 (1962).

2. R.H. Lyon and E. Eichler, "Random Vibration of Connected Structures", *J. Acoust. Soc. Amer.*, **36**, 1344–1354 (July 1964).

3. D.E. Newland, "Calculation of Power Flow Between Coupled Oscillators", *J. Sound Vib.*, **3**:3, 262–276 (1966).

4. E.E. Ungar, "Statistical Energy Analysis of Vibrating Systems", *J. Engrg. Ind.*, Paper No. 89 – Vibr-8 (November 1967).

5. T.D. Scharton and R.H. Lyon, "Power Flow and Energy Sharing in Random Vibration", *J. Acoust. Soc. Amer.*, **43**:6, 1332–1343 (June 1968).

6. R.H. Lyon, "Spatial Response Concentrations in Extended Structures", Transactions of the A.S.M.E., *J. Engrg. Ind.*, 754–758 (November 1967).

7. G. Maidanik, "Response of Ribbed Panels to Reverberant Acoustic Fields", *J. Acoust. Soc. Amer.*, **34**:6, 809–826 (1962).

8. J.E. Manning and G. Maidanik, "Radiation Properties of Cylindrical Shells", *J. Acoust. Soc. Amer.*, **36**:9, 1691–1698 (September 1964).

9. A.J. Price and M.J. Crocker, "Sound Transmission Through Double Panels Using Statistical Energy Analysis", *J. Acoust. Soc. Amer.*, **47**:3, 683–693 (March 1970).

10. R.H. Lyon and E. Eichler, "Random Vibration of Connected Structures", *J. Acoust. Soc. Amer.*, **36**:7, 1344–1354 (July 1964).

11. T.D. Scharton and T.M. Yang, "Statistical Energy Analysis of Vibration Transmission into an Instrument Package", Society of Automotive Engineers, Aeronautic and Space Engineering and Manufacturing Meeting, Los Angeles (October 2–6, 1967).

6

12. E.E. Ungar, N. Koronaios, and J. Manning, "Application of Statistical Energy Analysis to Vibration of Multipanel Structure", AFFDL-TR-67-79 (May 1967).

13. R.H. Lyon and T.D. Scharton, "Vibrational — Energy Transmission in a Three Element Structure", *J. Acoust. Soc. Amer.*, **38**, 253–261 (1965).

14. E. Eichler, "Thermal Circuit Approach to Vibrations in Coupled Systems and the Noise Reduction of a Rectangular Box", *J. Acoust. Soc. Amer.*, **37**:6, 995–1007 (June 1965).

15. R.H. Lyon, "Noise Reduction of Rectangular Enclosures with One Flexible Wall", *J. Acoust. Soc. Amer.*, **35**:11, 1791–1797 (November 1963).

16. R.H. Lyon, "An Energy Method for Prediction of Noise and Vibration Transmission", *Shock, Vibration and Associated Environments*, Bulletin 33, Part II, Sec. 1 (February 1964).

17. J.E. Manning and K. Lee, "Predicting Mechanical Shock Transmission", *The Shock and Vibration Bulletin*, No. 37, Part 4 (January 1968).

18. E.E. Ungar and N. Koronaios, "Vibration Distributions in Multipanel Structures: Comparison of Measurements with Statistical Energy Predictions", *The Shock and Vibration Bulletin*, No. 37, Part 2 (January 1968).

19. P.A. Franken and R.H. Lyon, "Energy-Method Estimates of Response to Inflight Dynamic Loads", Institute of Environmental Sciences, 1966 Annual Technical Meeting Proceedings.

20. R.W. Sevy and D.A. Earls, "The Prediction of Internal Vibration Levels of Flight Vehicle Equipments Using Statistical Energy Methods", Technical Report AFFDL-TR-68 (Wright-Patterson) (June 1968).

21. M.N. Mansour, "Applicability of Statistical Energy Concepts to the MM '69 Spacecraft", SAE Paper 700181, (January 1970).

22. See Appendix 1 of Ref. 24.

23. R. Sawley, "The Evaluation of a Shipboard Noise and Vibration Problem Using Statistical Energy Analysis", *Stochastic Processes in Dynamical Problems, ASME*, New York (1969).

24. J.E. Manning, "A Theoretical and Experimental Model Study of the Sound Induced Vibration Transmitted to a Shroud Enclosed Spacecraft", NASA CR-112413 (1970).

25. L.L. Beranek, *Noise Reduction* (McGraw-Hill Book Co., New York, 1960) Chapter 11.

26. P.M. Morse and R.H. Bolt, "Sound Waves in Rooms", *Revs. Mod. Phys.* **16**:2, 69–150 (1944).

27. See, for example, Sec. II.4 of Ref. 25.

28. R.H. Lyon, "Statistical Analysis of Power Injection and Response in Structures and Rooms", *J. Acoust. Soc. Amer.*, **45**:3, 545–565 (1969).

29. T.D. Scharton, "Frequency Average Power Flow into Vibratory Continuous Systems" (to be published in *J. Acoust. Soc. Amer.*).

30. A. Powell, "On the Response of Structures to Random Pressures and to Jet Noise in Particular", Chap. 8 in *Random Vibrations*, S.H. Crandall, ed., Technology Press (1959).

31. P.W. Smith and R.H. Lyon, *Sound and Structural Vibration*, NASA CR-160 (1965).

32. W. Gersch, "Average Power and Power Exchange in Oscillators", *J. Acoust. Soc. Amer.*, **46**:5, 1180–1185 (1969).

33. D.E. Newland, "Energy Sharing in the Random Vibration of Nonlinearly Coupled Modes", *J. Inst. Mathematics and Its Applications*, **1**:3 (September 1965).

34. D.E. Newland, "Power Flow Between a Class of Coupled Oscillators", *J. Acoust. Soc. Amer.*, **43**, 553–559 (1968).

35. W. Gersch, "Mean Square Responses in Structural Systems", *J. Acoust. Soc. Amer.*, **48**:1, 403–413 (1970).

36. S.H. Crandall and R. Lotz, "On the Coupling Loss Factor in Statistical Energy Analysis", *J. Acoust. Soc. Amer.*, **49**:1, 352–356 (1971).

37. Ref. 25, pp. 697–700 and pp. 394–95.

APPENDIX A. GREEN'S FUNCTION SOLUTION TO THE COUPLED ROD PROBLEM

It is possible with the coupled rod problem of Fig. 11 to derive a closed form solution using Green's function techniques. For the problem here, the Green's function $G(x_\xi, x_f, \omega)$ for a rod longitudinally forced and terminated with known impedance Z_0 and Z_1 where G is defined as the complex amplitude of the displacement at x_ξ due to a unit harmonic force of the form $e^{+i\omega t}$ applied at x_f. For the case of Fig. 10 one obtains

$$G(x_\xi, x_f, \omega) = A'\left\{e^{\pm ik(x_\xi - x_f)} + \alpha_1 \cos kx_\xi + \alpha_2 \sin k\, x_\xi\right\} \tag{A-1}$$

where the plus sign is for $x_\xi < x_f$, the minus sign is for $x_\xi > x_f$, k is ω/c_0, and c_0 is the longitudinal wave speed in the rod. The quantities A, α_1, and α_2 are defined as

$$A' = \frac{1}{2i\omega\rho_0 c_0 A}$$

$$\alpha_1 = -\left\{\frac{\left(\frac{Z_1}{\rho_0 c_0 A} - 1\right)e^{-ik(L-x_f)} + \left(\frac{Z_0}{\rho_0 c_0 A} - 1\right)\left(\cos kL + \frac{iZ_1}{\rho_0 c_0 A}\sin kL\right)e^{-ikx_f}}{\frac{Z_0+Z_1}{\rho_0 c_0 A}\cos kL + i\left[\frac{Z_0 Z_1}{(\rho_0 c_0 A)^2} + 1\right]\sin kL}\right.$$

$$\alpha_2 = -\left\{\frac{\frac{iZ_0}{\rho_0 c_0 A}\left(\frac{Z_1}{\rho_0 c_0 A} - 1\right)e^{-ik(L-x_f)} - \left(\frac{Z_0}{\rho_0 c_0 A} - 1\right)\left(\frac{iZ_1}{\rho_0 c_0 A}\cos kL - \sin kL\right)e^{-ikx_f}}{\frac{Z_0+Z_1}{\rho_0 c_0 A}\cos kL + i\left[\frac{Z_0 Z_1}{(\rho_0 c_0 A)^2} + 1\right]\sin kL}\right.$$

(A-2)

where ρ_0 is the rod density, A is the cross sectional area of the rod, and L is the length of the rod.

Using this Green's function one can calculate the relevant characteristics of the coupled rods of Fig. 8. Specifically the power transmitted between rod 1 and rod 2, the power injected into rod 1, the energy in rod 1, and the energy in rod 2, are of interest.

By superposition the displacement in rod 1 at x_{c1} may be written

$$\xi_1(x_{c1}) = F\,G_1(x_{c1},x_f) - F_c G_1(x_{c1},x_{c1}) \quad \text{(A-3)}$$

where G_1 refers to the Green's function of rod 1 and F_c is the force applied by the coupling spring which may be written

$$F_c = k_c[\xi_1(x_{c1}) - \xi_2(x_{c2})] \; . \quad \text{(A-4)}$$

The quantity $\xi_2(x_{c2})$ is the displacement in rod 2 at the point of attachment of the coupling spring which may be written

$$\xi_2(x_{c2}) = F_c G_2(x_{c2},x_{c2}) \; . \quad \text{(A-5)}$$

The time averaged power transmitted from rod 1 to rod 2 can be expressed in terms of F_c and $\dot{\xi}_2(x_{c2})$, the velocity in rod 2 at x_{c2} as follows

$$<\pi_{12}> = \frac{1}{2}\,\mathrm{Re}\{F_c[\dot{\xi}_2(x_{c2})]^*\} \quad \text{(A 6)}$$

where [ ]* means complex conjugate and

$$[\dot{\xi}_2(x_{c2})]^* = -i\omega G_2^*(x_{c2},x_{c2})F_c^* \; . \quad \text{(A-7)}$$

Combining Eqs. A-3 through A-7 one obtains

$$\langle \pi_{12} \rangle_t = \frac{1}{2} |F^2| \left| \frac{G_1(x_{c1}, x_f)}{\frac{1}{k_c} + G_1(x_{c1}, x_{c1}) + G_2(x_{c2}, x_{c2})} \right|^2 \mathrm{Re}\{-i\omega G_2^*(x_{c2}, x_{c2})\} \tag{A-8}$$

The time averaged power injected into rod 1 by the force F can be written

$$\langle \pi_{in} \rangle_t = \frac{1}{2} \mathrm{Re}\, F[\dot{\xi}_1(x_f)]^* \tag{A-9}$$

where $[\dot{\xi}_1(x_f)]^* = -i\omega[F^* G_1^*(x_f, x_f) - F_c^* G_1^*(x_f, x_{c1})]$.

Combining Eqs. A-3 through A-5 with Eq. A-9 the power injected becomes

$$\langle \pi_{in} \rangle_t = \frac{1}{2} |F^2| \, \mathrm{Re}\left\{ -i\omega \left[G_1^*(x_f, x_f) - \frac{G_1^*(x_f, x_{c1}) G_1^*(x_{c1}, x_f)}{\frac{1}{k_c} + G_1^*(x_{c1}, x_{c1}) + G_2^*(x_{c2}, x_{c2})} \right] \right\}. \tag{A-10}$$

By integrating the velocities of the rods along their length the time average kinetic energies of the rods can be derived

$$\langle E_1 \rangle_t = \rho_0 A_1 \int_0^{L_1} \frac{|\dot{\xi}_1(x)|^2}{2} \, dx$$

$$\langle E_2 \rangle_t = \rho_0 A_2 \int_0^{L_2} \frac{|\dot{\xi}_2(x)|^2}{2} \, dx \tag{A-11}$$

where E_1 and E_2 are twice the kinetic energy (the total energy at resonance) for rod 1 and rod 2, respectively. The velocities in

Eq. A-11 can be expressed in terms of the Green's function of the two rods and the applied forces as

$$\dot{\xi}_1(x) = i\omega[FG_1(x,x_f) - F_c G_1(x,x_{c1})]$$

$$\dot{\xi}_2(x) = i\omega\, F_c G_2(x,x_{c2}) \ . \qquad \text{(A-12)}$$

From Eqs. A-3 through A-5 and Eqs. A-12 and A-11 these energies can be expressed

$$\langle E_1 \rangle_t = \frac{|F^2|}{2} \rho_0 A_1 \int_0^{L_1} dx \left| G_1(x,x_f) - \frac{G_1(x,x_{c1})\, G_1(x_{c1},x_f)}{\frac{1}{k_c} + G_1(x_{c1},x_{c1}) + G_2(x_{c2},x_{c2})} \right|^2$$

(A-13)

$$\langle E_2 \rangle_t = \frac{|F^2|}{2} \rho_0 A_1 \int_0^{L_2} dx \left| \frac{G_1(x_{c1},x_f)\, G_2(x,x_{c2})}{\frac{1}{k_c} + G_1(x_{c1},x_{c1}) + G_2(x_{c2},x_{c2})} \right|^2 .$$

(A-14)

Equations A-8, A-10, A-13, and A-14 allow for exact calculation of the relevant characteristics of the coupled system.

APPENDIX B: STATISTICAL ENERGY ANALYSIS – BIBLIOGRAPHY

INTRODUCTORY PAPERS

1. R.H. Lyon and G. Maidanik **(1962)**, "Power Flow Between Linearly Coupled Oscillators", *J. Acoust. Soc. Amer.*, **34**, pp. 623–639.

2. M. Heckl, R.H. Lyon, G. Maidanik, and E.E. Ungar **(1962)**, "New Methods for Understanding and Controlling Vibrations of Complex Structures", Wright-Patterson Technical Note No. ASD-TN-61-122.

3. I. Dyer **(1963)**, "Statistical Vibration Analysis", *International Science and Technology*, pp. 35–41.

4. R.H. Lyon and G. Maidanik **(1964)**, "Statistical Methods in Vibration Analysis", *AIAA Journal*, **2**:6.

5. E. Eichler **(1965)**, "Thermal Circuit Approach to Vibrations in Coupled Systems and the Noise Reduction of a Rectangular Box", *J. Acoust. Soc. Amer.*, **37**:6, 995–1007.

6. F.D. Hart **(1965)**, "Statistical Methods in Sound and Structural Vibration Analysis", NASA Report No. LWP-169.

7. D. Karnopp **(1966)**, "Coupled Vibratory-System Analysis Using the Dual Formulation", *J. Acoust. Soc. Amer.*, **40**:2, pp. 380–384.

8. D.E. Newland **(1966)**, "Calculation of Power Flow Between Coupled Oscillators", *J. Sound Vib.*, **3**:3, pp. 262–276.

9. E.E. Ungar **(1967)**, "Statistical Energy Analysis of Vibrating Systems", *J. Engrg. Ind.*, Paper No. 89 – Vibr-8.

10. D.E. Newland **(1968)**, "Power Flow Between a Class of Coupled Oscillators", *J. Acoust. Soc. Amer.*, **43**, pp. 553–559.

11. T.D. Scharton and R.H. Lyon **(1968)**, "Power Flow and Energy Sharing in Random Vibration", *J. Acoust. Soc. Amer.*, **43**:6, pp. 1332–1343.

12. W. Gersch **(1969)**, "Average Power and Power Exchange in Oscillators", *J. Acoust. Soc. Amer.*, **46**:5, pp. 1180–1185.

13. R.H. Lyon **(1970)**, "What Good is Statistical Energy Analysis Anyway?", *Shock and Vibration Digest*, **2**:6.

PAPERS DESCRIBING APPLICATIONS OF SEA TO PARTICULAR PROBLEMS

1. R.H. Lyon and G. Maidanik **(1962)**, "Power Flow Between Linearly Coupled Oscillators", *J. Acoust. Soc. Amer.*, **34**, pp. 623–639.

2. G. Maidanik **(1962)**, "Response of Ribbed Panels to Reverberant Acoustic Fields", *J. Acoust. Soc. Amer.*, **34**:6, pp. 809–826.

3. R.H. Lyon **(1963)**, "Noise Reduction of Rectangular Enclosures with One Flexible Wall", *J. Acoust. Soc. Amer.*, **35**:11, pp. 1791–1797.

4. R.H. Lyon and E. Eichler **(1964)**, "Random Vibration of Connected Structures", *J. Acoust. Soc. Amer.*, **36**, pp. 1344–1354.

5. R.H. Lyon **(1964)**, "An Energy Method for Prediction of Noise and Vibration Transmission", *Shock, Vibration and Associated Environments*, Bulletin 33, Part II, Sec. 1.

6. J.E. Manning and G. Maidanik **(1964)**, "Radiation Properties of Cylindrical Shells", *J. Acoust. Soc. Amer.*, **36**:9, pp. 1691–1698.

7. E. Eichler **(1965)**, "Thermal Circuit Approach to Vibrations in Coupled Systems and the Noise Reduction of a Rectangular Box", *J. Acoust. Soc. Amer.*, **37**:6, pp. 995–1007.

8. R.H. Lyon and T.D. Scharton **(1965)**, "Vibrational – Energy Transmission in a Three Element Structure", *J. Acoust. Soc. Amer.*, **38**, pp. 253–261.

9. W. Fricke and P. Kehler **(1965)**, "Prediction of Sound Induced Vibrations of VTOL Aircraft", Institute of Environmental Sciences, Annual Technical Meeting Proceedings.

10. P.A. Franken and R.H. Lyon **(1966)**, "Energy-Method Estimates of Response to Inflight Dynamic Loads", Institute of Environmental Sciences, Annual Technical Meeting Proceedings.

11. T.D. Scharton and T.M. Yang **(1967)**, "Statistical Energy Analysis of Vibration Transmission into an Instrument Package", Society of Automotive Engineers, Aeronautic and Space Engineering and Manufacturing Meeting, Los Angeles.

12. E.E. Ungar, N. Koronaios, and J. Manning **(1967)**, "Application of Statistical Energy Analysis to Vibration of Multipanel Structure", AFFDL-TR-67-79.

13. J.E. Manning and K. Lee **(1968)**, "Predicting Mechanical Shock Transmission", *The Shock and Vibration Bulletin,* No. 37, Part 4.

14. R.W. Sevy and D.A. Earls **(1968)**, "The Prediction of Internal Vibration Levels of Flight Vehicle Equipments Using Statistical Energy Methods", Technical Report AFFDL-TR-68 (Wright-Patterson).

15. E.E. Ungar and N. Koronaios **(1968)**, "Vibration Distributions in Multipanel Structures: Comparison of Measurements with Statistical Energy Predictions", *The Shock and Vibration Bulletin,* No. 37, Part 2.

16. F.J. Fahy **(1969)**, "Vibration of Containing Structures Excited by Sound in the Contained Fluid", *J. Sound Vib.,* **10**, No. 3 pp. 490–512.

17. R. Sawley **(1969)**, "The Evaluation of a Shipboard Noise and Vibration Problem Using Statistical Energy Analysis", contribution to *Stochastic Processes in Dynamical Problems, ASME,* New York.

18. J.E. Manning **(1970)**, "A Theoretical and Experimental Model Study of the Sound Induced Vibration Transmitted to a Shroud Enclosed Spacecraft", NASA CR-112413.

19. M.N. Mansour **(1970)**, "Applicability of Statistical Energy Concepts to the MM'69 Spacecraft", SAE Paper No. 700181.

20. A.J. Price and M.J. Crocker **(1970)**, "Sound Transmission Through Double Panels Using Statistical Energy Analysis", *J. Acoust. Soc. Amer.,* **47**:3, pp. 683–693.

21. S.H. Crandall **(1970)**, "Random Vibration of Interconnected Systems", *Developments in Mechanics,* **5**, Proceedings of the 11*th* Midwestern Mechanic's Conference.

PAPERS GIVING CALCULATIONS OF POWER INPUT

1. M. Heckl **(1962)**, "Vibrations of Point-Driven Cylindrical Shells", *J. Acoust. Soc. Amer.*, **34**:10, pp. 1553–1557 (input to cylindrical shells from point sources).

2. P.W. Smith and R.H. Lyon **(1965)**, *Sound and Structural Vibration*, NASA CR-160 (input to beams from point source, input to plates from acoustic fields).

3. L.E. Wittig **(1967)**, "The Measurement of Mechanical Power Flow into a Simple Panel", MIT Acoustics and Vibration Laboratory Report No. DSR 78867-1 (input to plates from point sources).

4. R.H. Lyon **(1969)**, "Statistical Analysis of Power Injection and Response in Structures and Rooms", *J. Acoust. Soc. Amer.*, **45**:3, pp. 545–565 (input to rooms and plates from point source).

5. S.H. Crandall **(1970)**, "Random Vibration of Interconnected Systems", *Developments in Mechanics*, **5**, Proceedings of the 11*th* Midwestern Mechanic's Conference (input to plates from turbulent boundary layer pressure fluctuations).

6. T.D. Scharton **(to be published in** *J. Acoust. Soc. Amer.*), "Frequency Average Power Flow into Vibratory Continuous Systems" (input to acoustic spaces from sources of finite impedance).

PAPERS GIVING CALCULATIONS OF COUPLING LOSS FACTORS

1. R.H. Lyon and G. Maidanik **(1962)**, "Power Flow Between Linearly Coupled Oscillators", *J. Acoust. Soc. Amer.*, **34**, pp. 623–639 (plate to an acoustic space).

2. G. Maidanik **(1962)**, "Response of Ribbed Panels to Reverberant Acoustic Fields", *J. Acoust. Soc. Amer.*, **34**:6, pp. 809–826 (plate to an acoustic space).

3. T.D. Scharton and R.H. Lyon **(1962)**, "Power Flow and Energy Sharing in Random Vibration", *J. Acoust. Soc. Amer.*, **34**, pp. 623–639 (beam to beam).

4. R.H. Lyon **(1963)**, "Noise Reduction of Rectangular Enclosures with One Flexible Wall", *J. Acoust. Soc. Amer.*, **35**:11, pp. 1791–1797 (plate to small acoustic space).

5. R.H. Lyon and E. Eichler **(1964)**, "Random Vibration of Connected Structures", *J. Acoust. Soc. Amer.*, **36**, pp. 1344–1354 (plate to beam connected at a point, plate to plate connected along an edge).

6. J.E. Manning and G. Maidanik **(1964)**, "Radiation Properties of Cylindrical Shells", *J. Acoust. Soc. Amer.*, **36**:9, pp. 1691–1698 (cylindrical shell to an acoustic space).

7. P.W. Smith and R.H. Lyon **(1965)**, *Sound and Structural Vibration,* NASA CR-160 (plate to an acoustic space).

8. R.H. Lyon and T.D. Scharton **(1965)**, "Vibrational – Energy Transmission in a Three Element Structure", *J. Acoust. Soc. Amer.*, **38**, pp. 253–261 (beam to plate connected at a point).

9. E.E. Ungar, N. Koronaios, and J. Manning **(1967)**, "Application of Statistical Energy Analysis to Vibration of Multipanel Structure", AFFDL-TR-67-79 (plate to plate across a beam).

10. F.J. Fahy **(1969)**, "Vibration of Containing Structures Excited by Sound in the Contained Fluid", *J. Sound Vib.*, **10**, No. 3, pp. 490–512 (plate to acoustic space).

11. J.E. Manning **(1970)**, "A Theoretical and Experimental Model Study of the Sound Induced Vibration Transmitted to a Shroud Enclosed Spacecraft", NASA CR-112413 (cylindrical shell to an acoustic space, plate to an acoustic space, beam to beam connected at a point).

12. S.H. Crandall and R. Lotz **(1971)**, "On the Coupling Loss Factor in Statistical Energy Analysis", *J. Acoust. Soc. Amer.*, **49**:1, pp. 352–356 (beam to beam).

PAPERS GIVING CALCULATIONS OF MODAL DENSITY

1. M. Heckl **(1962)**, "Vibrations of Point-Driven Cylindrical Shells", *J. Acoust. Soc. Amer.*, **34**:10, pp. 1553–1557 (cylindrical shells).

2. V.V. Bolotin **(1963)**, "On the Density of the Distribution of Natural Frequencies of Thin Elastic Shells", *J. Appl. Math.*, **27**, No. 2, pp. 538–543 (cylindrical shells, spherical shells).

3. F.D. Hart **(1965)**, "Statistical Methods in Sound and Structural Vibration Analysis", NASA Report No. LWP-169 (beams, plates, acoustic spaces).

4. J.P.D. Wilkinson **(1968)**, "Modal Densities of Certain Shallow Structural Elements", *J. Acoust. Soc. Amer.*, **43**, No. 2, pp. 245–251 (shell segments).

5. H. Runieds **(1969)**, "Modal Densities of Spherical Shells", Paper No. 5138 given at 40*th* Symposium on Shock and Vibration (spherical shells).