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ON THE DETERMINATION OF NEAR-BODY ORBITS USING MASS CONCENTRATION MODELS

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ABSTRACT

The observations of a near-body satellite are used in the determination of certain constants appearing in a mathematical representation of the gravitational field of the central body. This representation is based on the assumption that the mass of the central body can be closely modeled by several concentrated masses located near its geometric center. The determination method employs a perturbation technique, numerical integration of linear and nonlinear differential equations, least-squares fitting criteria, and matrix inversion to determine estimates for the parameters involved.

Several models for the representation of central body gravitational fields are discussed and the numerical techniques for evaluation of the parameters involved are briefly reviewed. The computations performed indicate that the parameters of such models can be estimated numerically. Some of the problems associated with using the method described herein to solve typical trajectory problems are discussed and some actual results are presented. Finally, suggestions for additional study of the theory and its applications are proposed.

NOMENCLATURE

Gm	The product of the Newtonian gravitational constant and the mass of the central body
U	Gravitational potential
g	Acceleration due to gravity
r	Length of the position vector in spherical coordinates
ϕ	Co-latitude in spherical coordinates
θ	Longitude in spherical coordinates
$\bar{\rho}$	Vector appearing in the two-mass model
K	Constant appearing in the two-mass model
M_i, d_i, α	Constants appearing in the various n-mass models
${}^{n}(\ )$	Superscript denoting the known approximation
${}^{n+1}(\ )$	Superscript denoting a newly calculated approximation
$(\ )_n$	Subscript denoting the known approximation
$(\ )_{n+1}$	Subscript denoting a newly calculated approximation

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CHAPTER I

INTRODUCTION

Knowledge of the gravitational field of the earth and the moon are of current interest in a wide range of studies. Pinpoint landings at prescribed points on the lunar surface require a good representation of the gravitational field of that body. The future space stations with long mission times in near-earth orbits must also use mathematical models to correct for the gradual deterioration of these orbits caused in part by the irregular gravitational fields of these bodies.

Much effort has been devoted to the accurate determination of space trajectories. Classical developments have included, most notably, the perturbation theory of Encke and others as described by Ehricke [1] and harmonic series expansions of the geopotential functions done by Kaula [2] and many others. In each case the resulting formulation for the solution of an initial value problem in space orbits results in a complicated evaluation procedure. However, sophisticated computer programs incorporating these models have been generated that yield satisfactory solutions to such problems. Sufficient computer time must be available and a clear knowledge of the perturbing factors must exist.

In any technique for the prediction of long duration orbits, an accurate model of the gravitational field of the earth is required. All the existing models (e.g., Ehricke [1]) assume the earth to be an oblate spheroid, circular in cross section in planes parallel to the equatorial plane, and elliptic in planes taken through the poles. The circumpolar ellipse has its semimajor axis in the equatorial plane and its semiminor axis toward the poles. Quantitatively, the parameters of these models have been estimated, calculated, and predicted by various techniques and methods. One notable effort in this area was the calculation of tesseral and zonal harmonic coefficients by Holloway [3] using perturbation techniques for parameter estimation.

In the process of formulating the solution for the harmonic series coefficients, Holloway [3] proposed a two-mass model for the representation of the gravitational field of the earth. This thesis expands that model and examines further the problem of determining the unknown constants required by that model and others similar to it. Specifically, it will be shown that a knowledge of the total position and velocity of a point mass in orbit about a central force field made up of mass concentrations located near the geometric center of the system is sufficient under certain circumstances to determine the location and mass of each concentration, uniquely, for the models described herein. Simulations which use this type of data for the determination of the gravitational potential of the central body are described.

In the descriptions associated with the simulations the term "convergence" is used to describe the determination of a locally unique numerical result. Convergence will then be taken to mean that all elements of two successive result vectors agree to a given number of significant decimal digits, normally five or six. The terms "close" and "small" are also used to describe results that are convergent (in the sense used here) and whose magnitude are on the order of 10^{-5} or 10^{-6}, respectively.

CHAPTER II

GENERAL THEORY

Basic Assumptions and Formulations

The motion of a space vehicle around the earth is determined by the forces acting upon its mass and by some initial values of its state vector at a given time. Formally, using Newton's law,

$$\bar{F} = m\bar{a}$$

are the differential equations of state. Another familiar form of these differential equations is usually written

$$\ddot{\bar{r}} = f(\bar{r},\dot{\bar{r}},t)$$

where $\bar{r}$ represents the position, $\dot{\bar{r}}$ the velocity, $\ddot{\bar{r}}$ the acceleration, and t the time. Again, an initial value for $\bar{r}$, $\dot{\bar{r}}$, and t must be known, and the forces that affect terms in the differential equations must be described analytically.

The forces that must be considered are listed in Gaposchkin and Lambeck [4] as the gravitational attraction of the earth, sun, moon, and the nongravitational effects of radiation pressure and air drag. Other sources such as Ehricke [1] consider, in addition, gravitational effects of the major planets. The presentation here neglects all nongravitational forces and the gravitational effects of the the sun, moon, and planets. Also, of the complications involved in describing the earth's gravitational field: namely, precession, nutation, polar motion, rotation, and temporal variations in the gravitational field; only the latter two effects are considered here.

In all cases, the central body is considered to be established relative to an inertial reference frame such that the axis of rotation of the central force field is the Z-axis of the system. The X-Y plane is normally called the equatorial plane, and the X-axis normally passes through the prime meridian of the central body. The classical form of spherical coordinates is used, and all differential equations of motion are written assuming a point mass in orbit beyond the radius of the central body. Again the forces described in the equations of motion are only those of a rotating central force field obeying the inverse square laws of gravitational attraction to a point mass in orbit; however, the center of mass of the central body is not restricted to its geometric center.

The Two-Mass Model

The model proposed by Holloway [3] assumes the mass of the earth to be distributed such that, for a dynamic representation of its gravity field, two concentrated masses, Km and (1-K)m, could be located at the ends of a vector $\bar{\rho}$. Fig. 1 illustrates the model and indicates that orbital motion may be calculated as a function of the displaced mass concentrations.

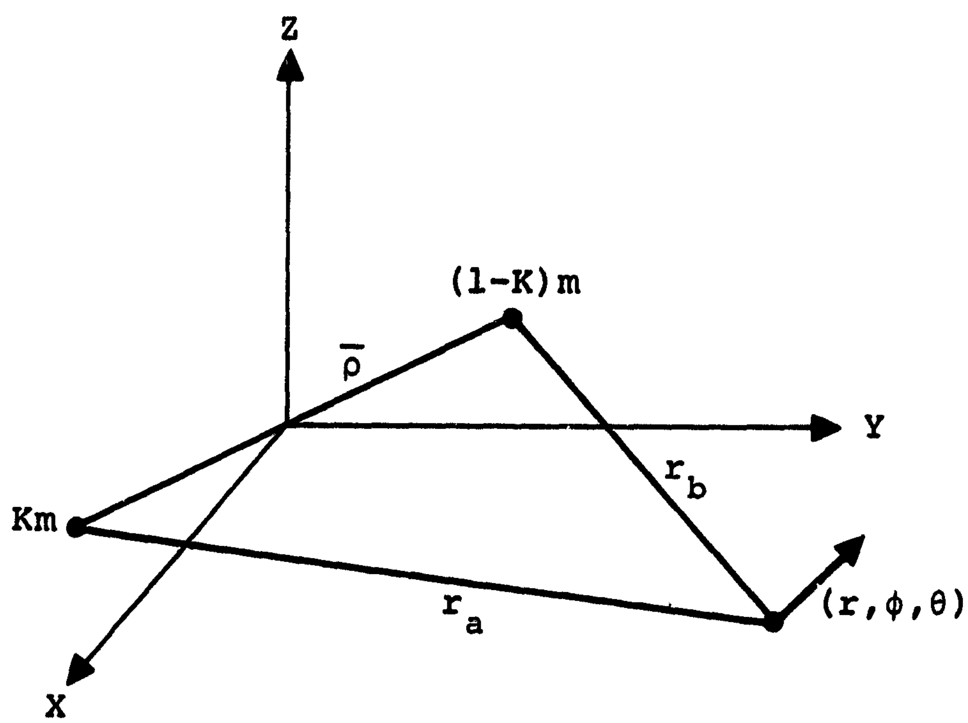

Figure 1. - The two-mass gravitational model.

The gravitational potential function for the two-mass model may be expressed as

$$U = U_a + U_b = -Gm\left(\frac{K}{r_a} + \frac{(1 - K)}{r_b}\right) \qquad (2.1)$$

The distances r_a and r_b are expressed as

$$r_a = \left((1 - K)^2\rho^2 + r^2 + 2(1 - K)\rho r\Phi\right)^{1/2} \qquad (2.2)$$

$$r_b = \left(K^2\rho^2 + r^2 - 2K\rho r\Phi\right)^{1/2} \qquad (2.3)$$

where

$$\bar{\rho} = \rho \cos \phi_0 \cos \theta_0\hat{i} + \rho \cos \phi_0 \sin \theta_0\hat{j} + \rho \sin \phi_0\hat{k} \qquad (2.4)$$

$$\bar{r} = r \cos \phi \cos \theta\hat{i} + r \cos \phi \sin \theta\hat{j} + \rho \sin \phi\hat{k} \qquad (2.5)$$

$$\Phi = \sin \phi \sin \phi_0 + \cos \phi \cos \phi_0 \cos (\theta - \theta_0) \qquad (2.6)$$

A Four-Mass Model

An alternate model is proposed here that is more flexible than the two-mass model representing the gravitational potential of the earth or any other central body. This model assumes that the mass of the central body is concentrated into four masses with two located along the axis of rotation and two located in the equatorial plane. This would seem to follow clearly the standard semimajor and semiminor axes model of the earth for representing the mass distribution. Fig. 2 illustrates the model and indicates that orbital motion may be calculated as a function of the displaced mass concentrations.

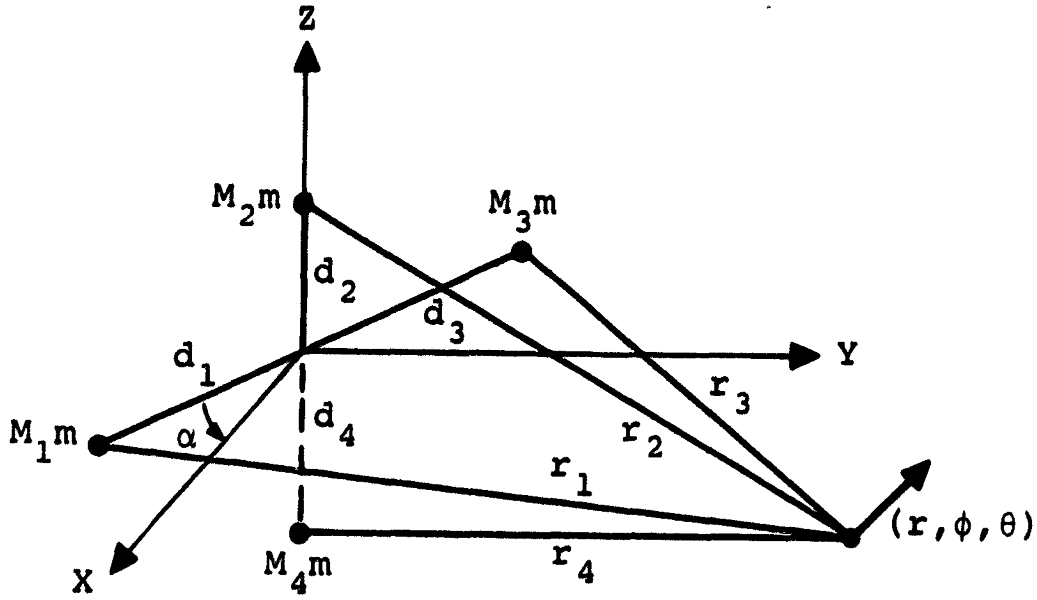

Figure 2. - A four-mass model.

The gravitational potential function for the four-mass model may be expressed as

$$U = U_1 + U_2 + U_3 + U_4 = -Gm\left(\frac{M_1}{r_1} + \frac{M_2}{r_2} + \frac{M_3}{r_3} + \frac{M_4}{r_4}\right) \tag{2.7}$$

The distances r_1, r_2, r_3, and r_4 are expressed as functions of r, ϕ, θ, and α as well as their individual displacements, d_1, d_2, d_3, and d_4 .

It can be illustrated that the four-mass model proposed herein can be reduced to the two-mass model proposed by Holloway [3]. The components of the vector $\bar{\rho}$ indicated by Eq. (2.4) can be shown to be

$$\left.\begin{aligned} \rho_X &= \rho \cos \phi_0 \cos \theta_0 \\ \rho_Y &= \rho \cos \phi_0 \sin \theta_0 \\ \rho_Z &= \rho \sin \phi_0 \end{aligned}\right\} \quad (2.8)$$

with the assumption that $K = 1$. The four-mass model illustrated in Fig. 2 shows that a similar sequence of vectors is

$$\left.\begin{aligned} \sigma_X &= d_1 \cos \alpha \\ \sigma_Y &= d_1 \sin \alpha \\ \sigma_Z &= d_2 \end{aligned}\right\} \quad (2.9)$$

with the assumption that $d_3 = d_4 = 0$. This corresponds to choosing $K = 1$. These components combine to produce the vector

$$\bar{\sigma} = d_1 \cos \alpha\hat{i} + d_1 \sin \alpha\hat{j} + d_2\hat{k} \quad (2.10)$$

which can be seen to resemble the previously defined vector, $\bar{\rho}$. In fact, the two vectors would be the same provided

$$d_1 = \rho \cos \phi_0$$

$$d_2 = \rho \sin \phi_0$$

since it is seen from Figs. 1 and 2 that $\alpha = \theta_0$. A close examination of these figures also shows the similarity of the two models and conveys an appreciation for the flexibility of the four-mass model.

The Parameter Estimation Technique

A system of differential equations whose initial values are all known can form the basis for the estimation of certain parameters through known boundary conditions. As indicated by Doiron [5], Childs [6], Holloway [3], and Bellman and Kalaba [7], this type of problem can be approached by perturbation techniques with considerable success. Consider a vector of first order, normally nonlinear, differential equations

$$\dot{\bar{y}} = \bar{f}(\bar{y},t,\bar{A})$$

where $\bar{y}$ is an $(n\times1)$ vector of dependent variables for the equations, t is the independent variable, and $\bar{A}$ is an $(m\times1)$ vector of parameters (constants) that enter into the calculations of the time-dependent values of the vector $\dot{\bar{y}}$. This system of equations can be written as an $[(n+m)\times1]$ system

$$(\dot{\bar{x}}) = \begin{pmatrix} \dot{\bar{y}} \\ \dot{\bar{A}} \end{pmatrix} = \begin{pmatrix} \bar{f}(\bar{y},t,\bar{A}) \\ 0 \end{pmatrix} \qquad (2.11)$$

Thus the evaluation of $\bar{A}$ at some initial time becomes a problem of estimating initial values for the augmented vector $\begin{pmatrix}\bar{y}\\ \bar{A}\end{pmatrix}$.

As indicated by all the noted references, this system may now be expanded by a Newton-Raphson-Kantorovich expansion to obtain the related set of linear differential equations required in the solution algorithm employed here. The equations ordinarily obtained would be of the form

$$(\dot{\bar{x}})_{n+1} = \bar{f}_n + (\bar{J}_n)(\bar{x}_{n+1} - \bar{x}_n) \qquad (2.12)$$

which would be subject to the initial conditions $\left(\bar{y}(0)\right)_n$. In this equation $(\bar{J}_n)$ is the standard Jacobian matrix of the system evaluated at $\bar{x}_n$; $\bar{x}$, $\bar{y}$, and $\bar{f}_n$ are as before.

The use of the augmented vector and the linearized system of equations for the estimation of parameters required by the definition of mass concentration models of a geopotential field is then an application of the theory outlined by Doiron [5] and explained in detail by Childs, et al. [8]. The variables of orbital motion, namely $\bar{r}$ and $\dot{\bar{r}}$, form the vector $\bar{y}$, and the various parameters of the mass model form the vector $\bar{A}$. Thus the vector $\bar{x}$ of Eq. (2.11) must be such that

$$(\dot{\bar{x}}) = \begin{pmatrix} \dot{\bar{r}} \\ \ddot{\bar{r}} \\ \dot{\bar{A}} \end{pmatrix} = \begin{pmatrix} \bar{f}(\bar{r},t,\bar{A}) \\ \bar{g}(\dot{\bar{r}},\bar{r},t,\bar{A}) \\ 0 \end{pmatrix} . \tag{2.13}$$

and the method of solution requires that values of $\bar{r}$, $\dot{\bar{r}}$ be known at some initial time t_0, and at selected boundary condition times, $t_1, t_2, \ldots, t_n$ for (n) at least as large as the dimension of $\bar{A}$. In addition, the (J_n) appearing in Eq. 2.12 requires analytical expressions for $\frac{\partial \dot{\bar{x}}}{\partial \bar{x}}$. The differential equations for the system, both nonlinear and linearized, are given in the following sections.

The Systems of Equations

A more detailed illustration of the model indicated by Fig. 2 is given in Fig. 3. The elements of geometry required for the model are then derived and are supported by other illustrations. A table of partial derivatives required in the development of the linearized equations is also given. The equations of motion are given with the details of the four-mass geometry included. Finally, the linearized equations indicated by Eq. (2.12) are presented in detail.

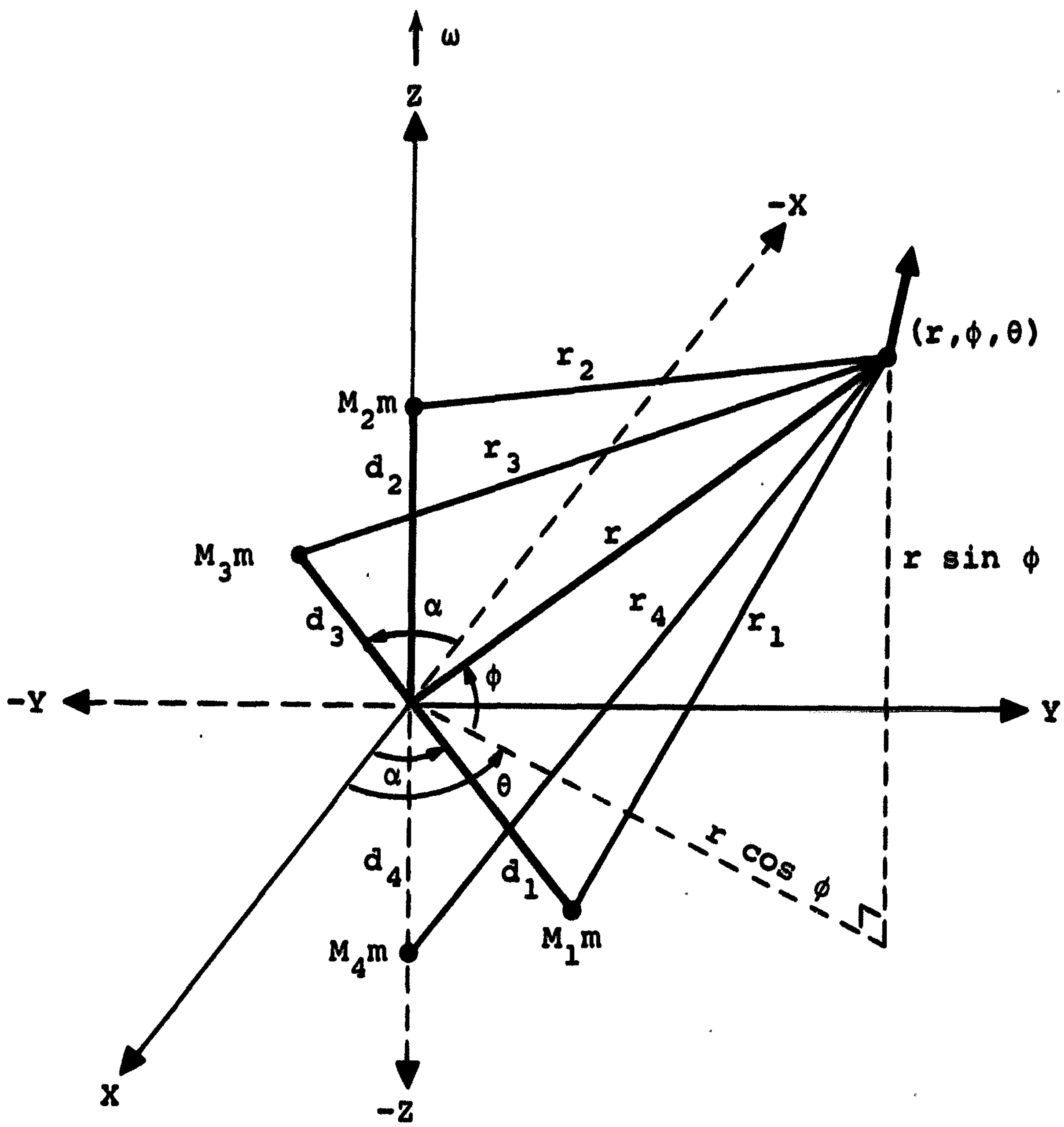

Figure 3. - The detailed four-mass model.

The variables of the system can be used to develop an expression for r_1 . Let $\beta = \theta - \alpha$ and introduce the variable σ_1 as indicated by Fig. 4.

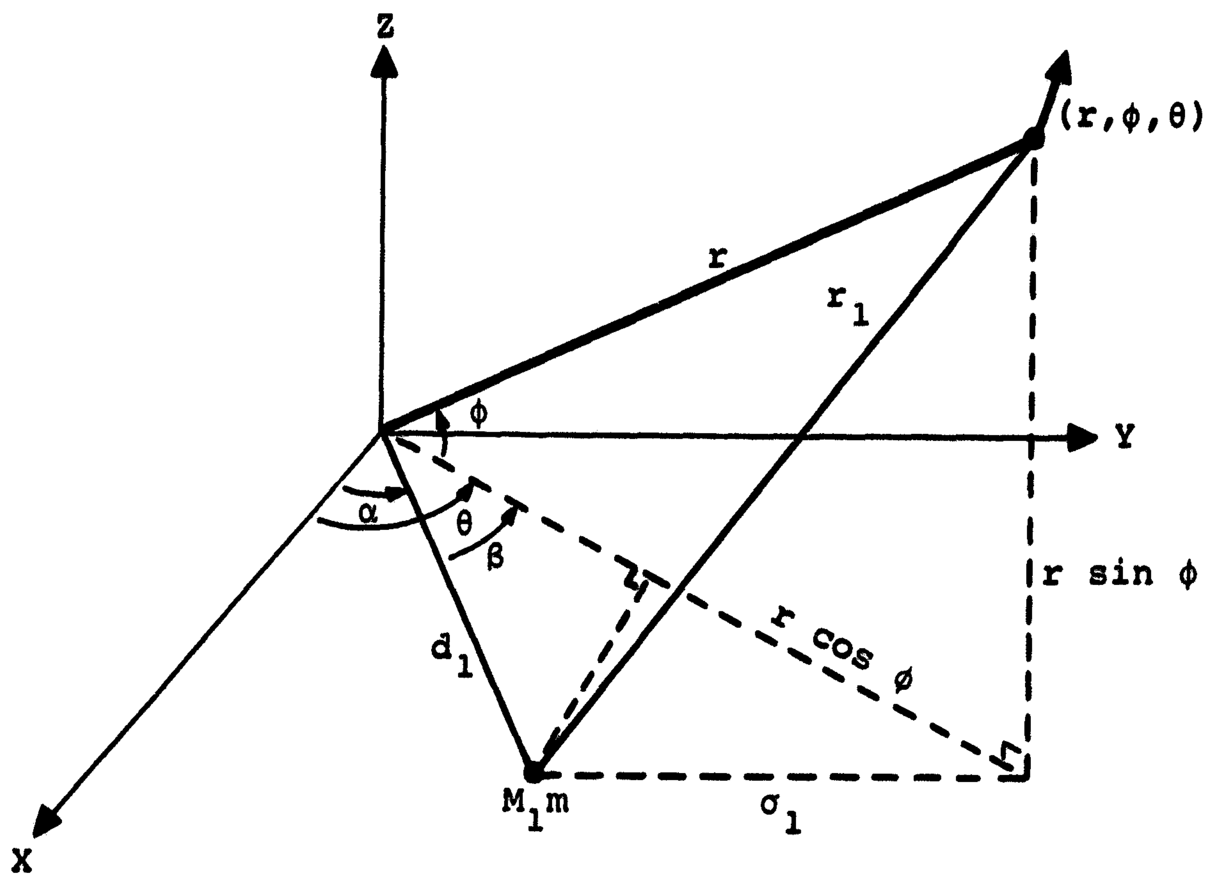

Figure 4. - The detailed geometry of r_1.

Thus σ_1^2 can be determined as

$$\sigma_1^2 = \left(r \cos \phi - d_1 \cos \beta\right)^2 + \left(d_1 \sin \beta\right)^2$$

$$= r^2 \cos^2 \phi - 2rd_1 \cos \phi \cos \beta + d_1^2 \cos^2 \beta + d_1^2 \sin^2 \beta$$

Using $\cos^2 \beta + \sin^2 \beta = 1$, this reduces to

$$\sigma_1^2 = r^2 \cos^2 \phi + d_1^2 - 2rd_1 \cos \phi \cos \beta$$

Now it is seen that

$$r_1^2 = \sigma_1^2 + (r \sin \phi)^2$$

$$= r^2 \cos^2 \phi + r^2 \sin^2 \phi + d_1^2 - 2rd_1 \cos \phi \cos \beta$$

which reduces finally to

$$r_1 = \left(r^2 + d_1^2 - 2rd_1 \cos \phi \cos \beta\right)^{1/2} \qquad (2.14)$$

As illustrated in Fig. 5, the development of expressions for r_2 and r_4 is straightforward.

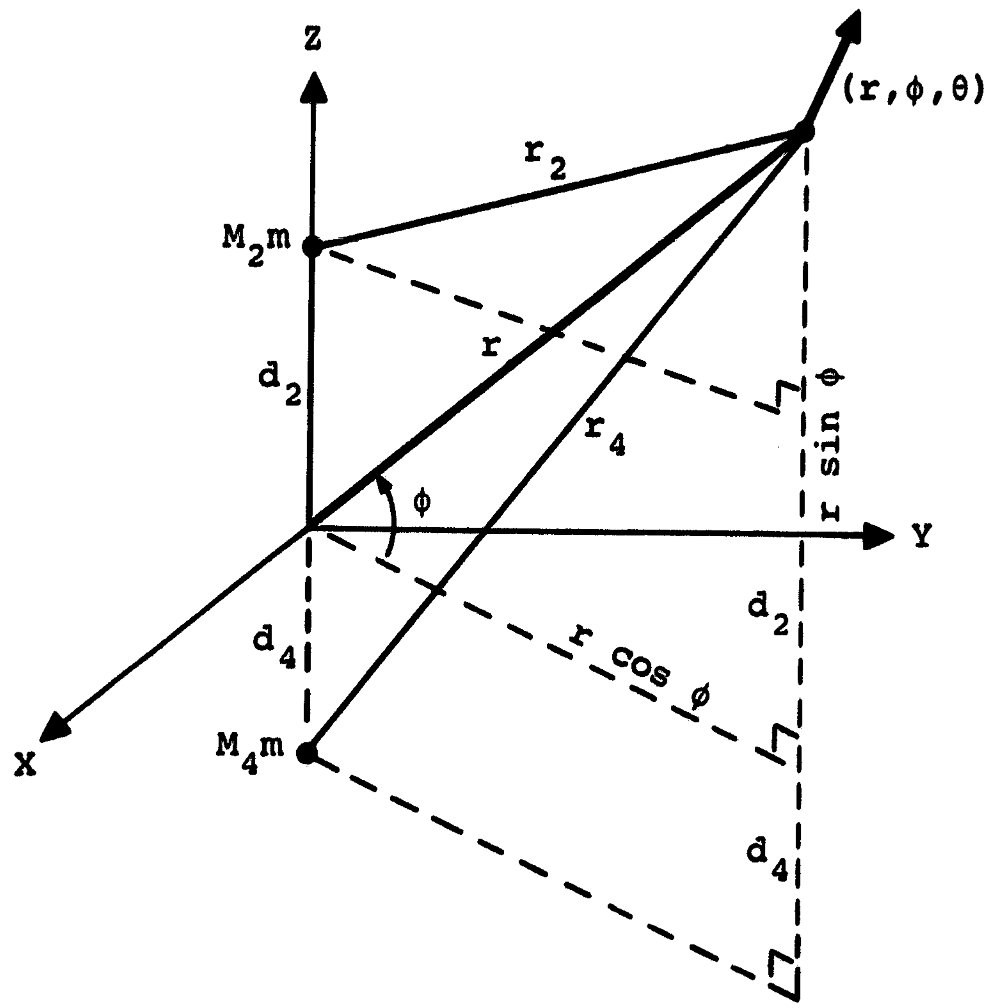

Figure 5. - The detailed geometry of r_2 and r_4 .

Immediately it is seen that

$$r_2^2 = (r\cos\phi)^2 + \left(r\sin\phi - d_2\right)^2$$
$$= r^2\cos^2\phi + r^2\sin^2\phi - 2rd_2\sin\phi + d_2^2$$

which reduces to

$$r_2 = \left(r^2 + d_2^2 - 2rd_2\sin\phi\right)^{1/2} \qquad (2.15)$$

Similarly, it can be determined that

$$r_4 = \left(r^2 + d_4^2 + 2rd_4\sin\phi\right)^{1/2} \qquad (2.16)$$

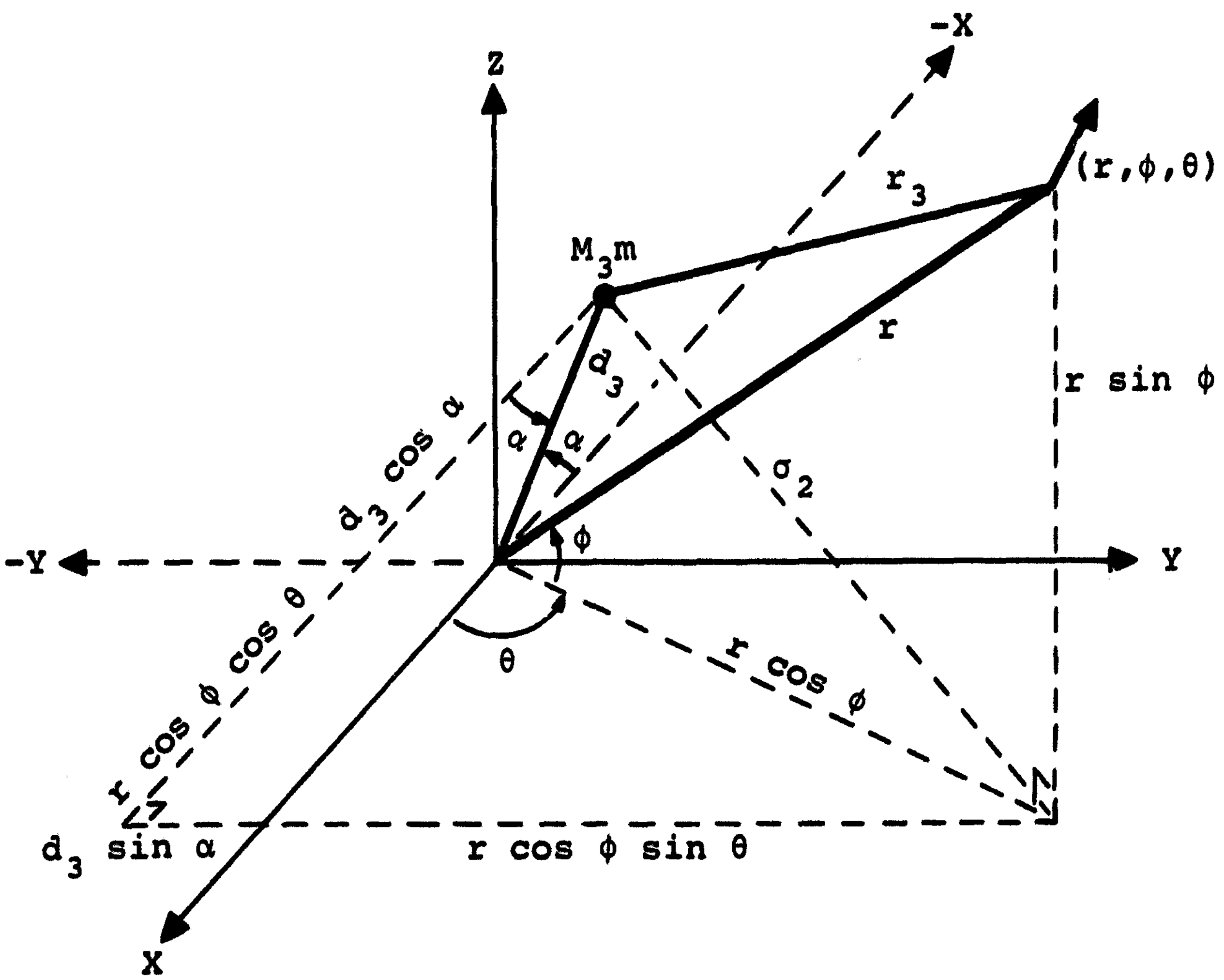

Figure 6. - The detailed geometry of r_3.

There are several ways in which the geometry r_3 can be expressed in terms of the system variables. Fig. 6 presents a straightforward one. Since $\beta = \theta - \alpha$, a trigonometric identity states that

$$\cos \beta = \cos(\theta - \alpha) = \cos \theta \cos \alpha + \sin \alpha \sin \theta$$

The intermediate variable σ_2 is introduced such that

$$\sigma_2^2 = \left(d_3 \cos \alpha + r \cos \phi \cos \theta\right)^2 + \left(d_3 \sin \alpha + r \cos \phi \sin \theta\right)^2$$

$$= d_3^2 + r^2 \cos^2 \phi + 2rd_3 \cos \phi(\cos \theta \cos \alpha + \sin \phi \sin \theta)$$

$$= d_3^2 + r^2 \cos^2 \phi + 2rd_3 \cos \phi \cos \beta$$

Now it is seen that

$$r_3^2 = (r \sin \phi)^2 + \sigma_2^2$$

which reduces to

$$r_3 = \left(r^2 + d_3^2 + 2rd_3^2 \cos \phi \cos \beta\right)^{1/2} \qquad (2.17)$$

<u>The Nonlinear Equations of Motion</u>

The geopotential function was given in Eq. (2.7) as

$$U = -Gm\left(\frac{M_1}{r_1} + \frac{M_2}{r_2} + \frac{M_3}{r_3} + \frac{M_4}{r_4}\right)$$

The equations of motion require expressions for

$$-\frac{\partial U}{\partial r}, \quad -\frac{1}{r^2}\frac{\partial U}{\partial \phi}, \quad -\frac{1}{r^2 \cos^2 \phi}\frac{\partial U}{\partial \theta}$$

These expressions may be derived analytically as follows:

$$-\frac{\partial U}{\partial r} = -Gm\left[M_1 \frac{\partial\left(\frac{1}{r_1}\right)}{\partial r} + M_2 \frac{\partial\left(\frac{1}{r_2}\right)}{\partial r} + M_3 \frac{\partial\left(\frac{1}{r_3}\right)}{\partial r} + M_4 \frac{\partial\left(\frac{1}{r_4}\right)}{\partial r}\right]$$

$$-\frac{1}{r^2}\frac{\partial U}{\partial \phi} = -\frac{Gm}{r^2}\left[M_1 \frac{\partial\left(\frac{1}{r_1}\right)}{\partial \phi} + M_2 \frac{\partial\left(\frac{1}{r_2}\right)}{\partial \phi} + M_3 \frac{\partial\left(\frac{1}{r_3}\right)}{\partial \phi} + M_4 \frac{\partial\left(\frac{1}{r_4}\right)}{\partial \phi}\right]$$

$$-\frac{1}{r^2 \cos^2 \phi}\frac{\partial U}{\partial \theta} = \frac{-Gm}{r^2 \cos^2 \phi}\left[M_1 \frac{\partial\left(\frac{1}{r_1}\right)}{\partial \theta} + M_3 \frac{\partial\left(\frac{1}{r_3}\right)}{\partial \theta}\right]$$

but for q representing any variable from (r,ϕ,θ)

$$\frac{\partial\left(\frac{1}{r_i}\right)}{\partial q} = \frac{\partial r_i^{-1}}{\partial q} = -\frac{1}{r_i^2}\frac{dr_i}{dq}$$

Using the partial derivatives from Table I

$$-\frac{\partial U}{\partial r} = -Gm\left[M_1\frac{(r - d_1\cos\phi\cos\beta)}{r_1^3} + M_2\frac{(r - d_2\sin\phi)}{r_2^3}\right.$$
$$\left. + M_3\frac{(r + d_3\cos\phi\cos\beta)}{r_3^3} + M_4\frac{(r + d_3\sin\phi)}{r_4^3}\right]$$

$$-\frac{1}{r^2}\frac{\partial U}{\partial\phi} = -\frac{Gm}{r^2}\left[M_1\left(\frac{rd_1\sin\phi\cos\beta}{r_1^3}\right) + M_2\left(\frac{-rd_2\cos\phi}{r_2^3}\right)\right.$$
$$\left. + M_3\left(\frac{-rd_3\sin\phi\cos\beta}{r_3^3}\right) + M_4\left(\frac{rd_4\cos\phi}{r_4^3}\right)\right]$$

$$-\frac{1}{r^2\cos^2\phi}\frac{\partial U}{\partial\theta} = \frac{-Gm}{r^2\cos^2\phi}\left[M_1\left(\frac{rd_1\cos\phi\sin\beta}{r_1^3}\right)\right.$$
$$\left. + M_3\left(\frac{-rd_3\cos\phi\sin\beta}{r_3^3}\right)\right]$$

TABLE I

VARIOUS PARTIAL DERIVATIVES OF r_i

	r_1 $\left(r^2 + d_1^2 - 2rd\ \cos\phi\cos\beta\right)^{1/2}$	r_2 $\left(r^2 + d_2^2 - 2rd_2\sin\phi\right)^{1/2}$	r_3 $\left(r^2 + d_3^2 + 2rd_3\cos\phi\cos\beta\right)^{1/2}$	r_4 $\left(r^2 + d_4^2 + 2rd_4\sin\phi\right)^{1/2}$
$\frac{dr_i}{dr}$	$\frac{r - d_1\cos\phi\cos\beta}{r_1}$	$\frac{r - d_2\sin\phi}{r_2}$	$\frac{r + d_3\cos\phi\cos\beta}{r_3}$	$\frac{r + d_4\sin\phi}{r_4}$
$\frac{dr_i}{d\phi}$	$\frac{rd_1\sin\phi\cos\beta}{r_1}$	$\frac{-rd_2\cos\phi}{r_2}$	$\frac{-rd_3\sin\phi\cos\beta}{r_3}$	$\frac{rd_4\cos\phi}{r_4}$
$\frac{dr_i}{d\beta}$	$\frac{rd_1\cos\phi\sin\beta}{r_1}$	0	$\frac{-rd_3\cos\phi\sin\beta}{r_3}$	0
$\frac{dr_i}{d\alpha}$	$\frac{-rd_1\cos\phi\sin\beta}{r_1}$	0	$\frac{rd_3\cos\phi\sin\beta}{r_3}$	0
$\frac{dr_i}{dd_1}$	$\frac{d_1 - r\cos\phi\cos\beta}{r_1}$	0	0	0
$\frac{dr_i}{dd_2}$	0	$\frac{d_2 - r\sin\phi}{r_2}$	0	0
$\frac{dr_i}{dd_3}$	0	0	$\frac{d_3 + r\cos\phi\cos\beta}{r_3}$	0
$\frac{dr_i}{dd_4}$	0	0	0	$\frac{d_4 + r\sin\phi}{r_4}$

The previous terms may be simplified to

$$-\frac{\partial U}{\partial r} = -Gm\left[r\left(\frac{M_1}{r_1^3} + \frac{M_2}{r_2^3} + \frac{M_3}{r_3^3} + \frac{M_4}{r_4^3}\right)\right.$$

$$\left. - \cos\phi \cos\beta\left(\frac{M_1 d_1}{r_1^3} - \frac{M_3 d_3}{r_3^3}\right) - \sin\phi\left(\frac{M_2 d_2}{r_2^3} - \frac{M_4 d_4}{r_4^3}\right)\right]$$

$$\frac{-1}{r^2}\frac{\partial U}{\partial r} = \frac{-Gm}{r}\left[\sin\phi \cos\beta\left(\frac{M_1 d_1}{r_1^3} - \frac{M_3 d_3}{r_3^3}\right)\right.$$

$$\left. - \cos\phi\left(\frac{M_2 d_2}{r_2^3} - \frac{M_4 d_4}{r_4^3}\right)\right]$$

$$-\frac{1}{r^2 \cos^2\phi}\frac{\partial U}{\partial \theta} = \frac{Gm}{r\cos\phi}\left[\sin\beta\left(\frac{M_1 d_1}{r_1^3} - \frac{M_3 d_3}{r_3^3}\right)\right]$$

Arranging terms for convenience only, the final expressions are

$$-\frac{\partial U}{\partial r} = -Gm\left[r\left(\frac{M_1}{r_1^3} + \frac{M_2}{r_2^3} + \frac{M_3}{r_3^3} + \frac{M_4}{r_4^3}\right)\right.$$

$$\left. + \cos\phi \cos\beta\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) + \sin\phi\left(\frac{M_4 d_4}{r_4^3} - \frac{M_2 d_2}{r_2^3}\right)\right] \tag{2.18}$$

$$-\frac{1}{r^2}\frac{\partial U}{\partial \phi} = \frac{Gm}{r}\left[\sin\phi\cos\beta\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right)\right.$$

$$\left. - \cos\phi\left(\frac{M_4 d_4}{r_4^3} - \frac{M_2 d_2}{r_2^3}\right)\right] \qquad (2.19)$$

$$-\frac{1}{r^2\cos^2\phi}\frac{\partial U}{\partial \theta} = \frac{Gm\sin\beta}{r\cos\phi}\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) \qquad (2.20)$$

Thus the equations of motion

$$\dot{r} = v$$

$$\dot{\phi} = \eta$$

$$\dot{\theta} = \gamma$$

$$\dot{v} = r\left(\eta^2 + (\gamma + \omega)^2 \cos^2\phi\right) - \frac{\partial U}{\partial r}$$

$$\dot{\eta} = \frac{-2v}{r}\eta - (\gamma + \omega)^2 \cos\phi \sin\phi - \frac{1}{r^2}\frac{\partial U}{\partial \phi}$$

$$\dot{\gamma} = -2(\gamma + \omega)\left(\frac{v}{r} - \eta\tan\phi\right) - \frac{1}{r^2\cos^2\phi}\frac{\partial U}{\partial \theta}$$

$$\dot{\alpha} = \omega$$

are seen, using the four-mass model, as

$$\dot{r} = v \qquad (2.21)$$

$$\dot{\phi} = \eta \qquad (2.22)$$

$$\dot{\theta} = \gamma \qquad (2.23)$$

$$\dot{v} = r\left(\eta^2 + (\gamma + \omega)^2 \cos^2 \phi\right) - Gm\left[r\left(\frac{M_1}{r_1^3} + \frac{M_2}{r_2^3} + \frac{M_3}{r_3^3} + \frac{M_4}{r_4^3}\right)\right.$$

$$\left. + \cos \phi \cos \beta\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) + \sin \phi \left(\frac{M_4 d_4}{r_4^3} - \frac{M_2 d_2}{r_2^3}\right)\right] \tag{2.24}$$

$$\dot{\eta} = \frac{-2v}{r} \eta - (\gamma + \omega)^2 \cos \phi \sin \phi$$

$$+ \frac{Gm}{r}\left[\sin \phi \cos \beta\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) - \cos \phi\left(\frac{M_4 d_4}{r_4^3} - \frac{M_2 d_2}{r_2^3}\right)\right] \tag{2.25}$$

$$\dot{\gamma} = -2(\gamma + \omega)\left(\frac{v}{r} - \eta \tan \phi\right) + \frac{Gm \sin \beta}{r \cos \phi}\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) \tag{2.26}$$

$$\dot{\alpha} = \omega \tag{2.27}$$

The Linearized Equations of the System

As stated previously, the notations used in deriving linearized equations are

q_n = ${}^{n}(q)$ = present value of q, some variable of the problem.

q_{n+1} = ${}^{n+1}(q)$ = new value of q, some variable of the problem.

The linearized equations then may be written as follows:

$$\dot{r}_{n+1} = v_{n+1}$$

$$\dot{\phi}_{n+1} = \eta_{n+1}$$

$$\dot{\theta}_{n+1} = \gamma_{n+1}$$

$$\begin{aligned}\dot{v}_{n+1} = \; & v_n + {}^{n}\!\left(\frac{\partial \dot{v}}{\partial r}\right)(r_{n+1} - r_n) + {}^{n}\!\left(\frac{\partial \dot{v}}{\partial \phi}\right)(\phi_{n+1} - \phi_n) \\ & + {}^{n}\!\left(\frac{\partial \dot{v}}{\partial \theta}\right)(\theta_{n+1} - \theta_n) + {}^{n}\!\left(\frac{\partial \dot{v}}{\partial v}\right)(\dot{v}_{n+1} - v_n) \\ & + {}^{n}\!\left(\frac{\partial \dot{v}}{\partial \eta}\right)(\eta_{n+1} - \eta_n) + {}^{n}\!\left(\frac{\partial v}{\partial \gamma}\right)(\gamma_{n+1} - \gamma_n) \\ & + {}^{n}\!\left(\frac{\partial \dot{v}}{\partial \alpha}\right)(\alpha_{n+1} - \alpha_n) + \sum_{i=1}^{4}\left[{}^{n}\!\left(\frac{\partial \dot{v}}{\partial M_i}\right)\left({}^{n+1}\!\left(M_i\right) - {}^{n}\!\left(M_i\right)\right)\right] \\ & + \sum_{i=1}^{4}\left[{}^{n}\!\left(\frac{\partial \dot{v}}{\partial d_i}\right)\left({}^{n+1}\!\left(d_i\right) - {}^{n}\!\left(d_i\right)\right)\right]\end{aligned}$$

The ∇ operator of differential calculus reduces the form to

$$\dot{v}_{n+1} = \dot{v}_n + \nabla\dot{v}_n\left[{}^{n+1}\!\left(q_i\right) - {}^{n}\!\left(q_i\right)\right]$$

and the remaining equations appear as

$$\dot{\eta}_{n+1} = \dot{\eta}_n + \nabla\dot{\eta}_n\left[{}^{n+1}\!\left(q_i\right) - {}^{n}\!\left(q_i\right)\right]$$

$$\dot{\gamma}_{n+1} = \dot{\gamma}_n + \nabla\dot{\gamma}_n\left[{}^{n+1}\!\left(q_i\right) - {}^{n}\!\left(q_i\right)\right]$$

$$\dot{\alpha}_{n+1} = \omega$$

Noting that the following relationships hold

$$\frac{\partial\dot{v}}{\partial r_1} = \frac{3GmrM_1}{r_1^4} - \frac{3GmM_1d_1\cos\phi\cos\beta}{r_1^4} = \frac{3GmM_1}{r_1^3}\left(\frac{r - d_1\cos\phi\cos\beta}{r_1}\right) = \frac{3GmM_1}{r_1^3}\frac{\partial r_1}{\partial r}$$

$$\frac{\partial\dot{v}}{\partial r_2} = \frac{3GmrM_2}{r_2^4} - \frac{3GmM_2d_2\sin\phi}{r_2^4} = \frac{3GmM_2}{r_2^3}\left(\frac{r - d_2\sin\phi}{r_2}\right) = \frac{3GmM_2}{r_2^3}\frac{\partial r_2}{\partial r}$$

$$\frac{\partial\dot{v}}{\partial r_3} = \frac{3GmrM_3}{r_3^4} + \frac{3GmM_3d_3\cos\phi\cos\beta}{r_3^4} = \frac{3GmM_3}{r_3^3}\frac{\partial r_3}{\partial r}$$

$$\frac{\partial\dot{v}}{\partial r_4} = \frac{3GmM_4}{r_4^3}\frac{\partial r_4}{\partial r}$$

and, using the relations contained in Table I with repeated application of the chain rule for partial differentiation, the expanded linearized equation for $\dot{v}$ can be seen to be

$$\begin{aligned}
\dot{v}_{n+1} = {}& {}^{n}\left\{r\left(\eta^2 + (\gamma+\omega)^2\cos^2\phi\right) - Gm\left[r\left(\frac{M_1}{r_1^3} + \frac{M_2}{r_2^3} + \frac{M_3}{r_3^3} + \frac{M_4}{r_4^3}\right) + \cos\phi\cos\beta\left(\frac{M_3d_3}{r_3^3} - \frac{M_1d_1}{r_1^3}\right) + \sin\phi\left(\frac{M_4d_4}{r_4^3} - \frac{M_2d_2}{r_2^3}\right)\right]\right\} \\
& + (r_{n+1} - r_n)\,{}^{n}\left\{\eta^2 + (\gamma+\omega)^2\cos^2\phi - Gm\left(\frac{M_1}{r_1^3} + \frac{M_2}{r_2^3} + \frac{M_3}{r_3^3} + \frac{M_4}{r_4^3}\right) + \frac{3GmM_1}{r_1^3}\left(\frac{\partial r_1}{\partial r}\right)^2 + \frac{3GmM_2}{r_2^3}\left(\frac{\partial r_2}{\partial r}\right)^2 + \frac{3GmM_3}{r_3^3}\left(\frac{\partial r_3}{\partial r}\right)^2\right. \\
& \left. + \frac{3GmM_4}{r_4^3}\left(\frac{\partial r_4}{\partial r}\right)^2\right\} + \left(\phi_{n+1} - \phi_n\right)\,{}^{n}\left\{-2r(\gamma+\omega)^2\cos\phi\sin\phi + Gm\sin\phi\cos\beta\left(\frac{M_3d_3}{r_3^3} - \frac{M_1d_1}{r_1^3}\right) + (-Gm\cos\phi)\right. \\
& \left. \cdot\left(\frac{M_4d_4}{r_4^3} - \frac{M_2d_2}{r_2^3}\right) + \frac{3GmM_1}{r_1^3}\frac{\partial r_1}{\partial r}\frac{\partial r_1}{\partial\phi} + \frac{3GmM_2}{r_2^3}\frac{\partial r_2}{\partial r}\frac{\partial r_2}{\partial\phi} + \frac{3GmM_3}{r_3^3}\frac{\partial r_3}{\partial r}\frac{\partial r_3}{\partial\phi} + \frac{3GmM_4}{r_4^3}\frac{\partial r_4}{\partial r}\frac{\partial r_4}{\partial\phi}\right\} + \left(\theta_{n+1} - \theta_n\right) \\
& \cdot\,{}^{n}\left\{Gm\cos\phi\sin\beta\left(\frac{M_3d_3}{r_3^3} - \frac{M_1d_1}{r_1^3}\right) + \frac{3GmM_1}{r_1^3}\frac{\partial r_1}{\partial r}\frac{\partial r_1}{\partial\theta} + \frac{3GmM_3}{r_3^3}\frac{\partial r_3}{\partial r}\frac{\partial r_3}{\partial\theta}\right\} + (v_{n+1} - v_n)(0) + \left(\eta_{n+1} - \eta_n\right)^{n}(2r\eta) \\
& + \left(\gamma_{n+1} - \gamma_n\right)^{n}\left\{2r(\gamma+\omega)\cos^2\phi\right\} + \left(\alpha_{n+1} - \alpha_n\right)^{n}\left\{-Gm\cos\phi\sin\beta\left(\frac{M_3d_3}{r_3^3} - \frac{M_1d_1}{r_1^3}\right) + \frac{3GmM_1}{r_1^3}\frac{\partial r_1}{\partial r}\frac{\partial r_1}{\partial\alpha}\right. \\
& \left. + \frac{3GmM_3}{r_3^3}\frac{\partial r_3}{\partial r}\frac{\partial r_3}{\partial\alpha}\right\} + \left({}^{n+1}(M_1) - {}^{n}(M_2)\right)^{n}\left\{\frac{-Gmr}{r_1^3} + \frac{Gmd_1\cos\phi\cos\beta}{r_1^3}\right\} + \left({}^{n+1}(M_2) - {}^{n}(M_2)\right)^{n}\left\{\frac{-Gmr}{r_2^3} + \frac{Gmd_2\sin\phi}{r_2^3}\right\} \\
& + \left({}^{n+1}(M_3) - {}^{n}(M_3)\right)^{n}\left\{\frac{-Gmr}{r_3^3} - \frac{Gmd_3\cos\phi\cos\beta}{r_3^3}\right\} + \left({}^{n+1}(M_4) - {}^{n}(M_4)\right)^{n}\left\{\frac{-Gmr}{r_4^3} - \frac{Gmd_4\sin\phi}{r_4^3}\right\} + \left({}^{n+1}(d_1) - {}^{n}(d_1)\right) \\
& \cdot\,{}^{n}\left\{\frac{GmM_1\cos\phi\cos\beta}{r_1^3} + \frac{3GmM_1}{r_1^3}\frac{\partial r_1}{\partial r}\frac{\partial r_1}{\partial d_1}\right\} + \left({}^{n+1}(d_2) - {}^{n}(d_2)\right)^{n}\left\{\frac{GmM_2\sin\phi}{r_2^3} + \frac{3GmM_2}{r_2^3}\frac{\partial r_2}{\partial r}\frac{\partial r_2}{\partial d_2}\right\} + \left({}^{n+1}(d_3) - {}^{n}(d_3)\right) \\
& \cdot\,{}^{n}\left\{\frac{-GmM_3\cos\phi\cos\beta}{r_3^3} + \frac{3GmM_3}{r_3^3}\frac{\partial r_3}{\partial r}\frac{\partial r_3}{\partial d_3}\right\} + \left({}^{n+1}(d_4) - {}^{n}(d_4)\right)^{n}\left\{\frac{-GmM_4\sin\phi}{r_4^3} + \frac{3GmM_4}{r_4^3}\frac{\partial r_4}{\partial r}\frac{\partial r_4}{\partial d_4}\right\}
\end{aligned} \qquad 2.28$$

Similarly

$$\frac{\partial \dot{n}}{\partial r_1} = \frac{3GmM_1 d_1 \sin\phi \cos\theta}{r r_1^4} + \frac{3GmM_1}{r^2 r_1^3}\frac{dr_1}{d\phi}$$

$$\frac{\partial \dot{n}}{\partial r_2} = \frac{-3GmM_2 d_2 \cos\phi}{r r_2^4} + \frac{3GmM_2}{r^2 r_2^3}\frac{dr_2}{d\phi}$$

$$\frac{\partial \dot{n}}{\partial r_3} = \frac{-3GmM_3 d_3 \sin\phi \cos\theta}{r r_3^4} + \frac{3GmM_3}{r^2 r_3^3}\frac{dr_3}{d\phi}$$

$$\frac{\partial \dot{n}}{\partial r_4} = \frac{3GmM_4}{r^2 r_4^3}\frac{dr_4}{d\phi}$$

and again, using Table I, the expanded linearised equation for $\dot{n}$ is

$$\dot{n}_{n+1} = {}^{n}\left\{\frac{-2v}{r}\,n - (\gamma + \omega)^2 \cos\phi \sin\phi + \frac{Gm}{r}\left[\sin\phi \cos\theta\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) - \cos\phi\left(\frac{M_4 d_4}{r_4^3} \quad \frac{M_2 d_2}{r_2^3}\right)\right]\right\}$$

$$+ \left(r_{n+1} - r_n\right)^{n}\left\{\frac{2v}{r^2}\,n - \frac{Gm}{r^2}\left[\sin\phi \cos\theta\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) - \cos\phi\left(\frac{M_4 d_4}{r_4^3} - \frac{M_2 d_2}{r_2^3}\right)\right] + \frac{3Gm}{r^2}\right.$$

$$\left.\cdot\left[\frac{M_1}{r_1^3}\frac{dr_1}{d\phi}\frac{dr_1}{dr} + \frac{M_2}{r_2^3}\frac{dr_2}{d\phi}\frac{dr_2}{dr} + \frac{M_3}{r_3^3}\frac{dr_3}{d\phi}\frac{dr_3}{dr} + \frac{M_4}{r_4^3}\frac{dr_4}{d\phi}\frac{dr_4}{dr}\right]\right\} + \left(\phi_{n+1} - \phi_n\right)^{n}\left\{-(\gamma + \omega)^2\right.$$

$$\cdot\left(\cos^2\phi - \sin^2\phi\right) + \frac{Gm}{r}\left[\cos\phi \cos\theta\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) + \sin\phi\left(\frac{M_4 d_4}{r_4^3} - \frac{M_2 d_2}{r_2^3}\right)\right] + \frac{3Gm}{r^2}$$

$$\left.\cdot\left[\frac{M_1}{r_1^3}\left(\frac{dr_1}{d\phi}\right)^2 + \frac{M_2}{r_2^3}\left(\frac{dr_2}{d\phi}\right)^2 + \frac{M_3}{r_3^3}\left(\frac{dr_3}{d\phi}\right)^2 + \frac{M_4}{r_4^3}\left(\frac{dr_4}{d\phi}\right)^2\right]\right\} + \left(\theta_{n+1} - \theta_n\right)^{n}\left\{\frac{-Gm}{r}\sin\phi \sin\theta\right.$$

$$\left.\cdot\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) + \frac{3Gm}{r^2}\left[\frac{M_1}{r_1^3}\frac{dr_1}{d\phi}\frac{dr_1}{d\theta} + \frac{M_3}{r_3^3}\frac{dr_3}{d\phi}\frac{dr_3}{d\theta}\right]\right\} + \left(v_{n+1} - v_n\right)^{n}\left\{\frac{-2}{r}\,n\right\} + \left(n_{n+1} - n_n\right)$$

$$\cdot\,{}^{n}\left\{\frac{-2v}{r}\right\} + \left(\gamma_{n+1} - \gamma_n\right)^{n}\left\{-2(\gamma + \omega)\cos\phi \sin\phi\right\} + \left(\alpha_{n+1} - \alpha_n\right)^{n}\left\{\frac{Gm}{r}\sin\phi \sin\theta\right.$$

$$\left.\cdot\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) + \frac{3Gm}{r^2}\left[\frac{M_1}{r_1^3}\frac{dr_1}{d\phi}\frac{dr_1}{d\alpha} + \frac{M_3}{r_3^3}\frac{dr_3}{d\phi}\frac{dr_3}{d\alpha}\right]\right\} + \left({}^{n+1}(M_1) - {}^{n}(M_1)\right)$$

$$\cdot\,{}^{n}\left\{\frac{-Gm}{r}\sin\phi \cos\theta\;\frac{d_1}{r_1^3}\right\} + \left({}^{n+1}(M_2) - {}^{n}(M_2)\right)^{n}\left\{\frac{Gm}{r}\cos\phi\;\frac{d_2}{r_2^3}\right\} + \left({}^{n+1}(M_3) - {}^{n}(M_3)\right)$$

$$\cdot\,{}^{n}\left\{\frac{Gm}{r}\sin\phi \cos\theta\,\frac{d_3}{r_3^3}\right\} + \left({}^{n+1}(M_4) - {}^{n}(M_4)\right)^{n}\left\{\frac{-Gm}{r}\cos\phi\,\frac{d_4}{r_4^3}\right\} + \left({}^{n+1}(d_1) - {}^{n}(d_1)\right)$$

$$\cdot\,{}^{n}\left\{\frac{-Gm}{r}\sin\phi \cos\theta\,\frac{M_1}{r_1^3} + \frac{3GmM_1}{r^2 r_1^3}\frac{dr_1}{d\phi}\frac{dr_1}{dd_1}\right\} + \left({}^{n+1}(d_2) - {}^{n}(d_2)\right)^{n}\left\{\frac{Gm}{r}\cos\phi\,\frac{M_2}{r_2^3}\right.$$

$$\left.+ \frac{3GmM_2}{r^2 r_2^3}\frac{dr_2}{d\phi}\frac{dr_2}{dd_2}\right\} + \left({}^{n+1}(d_3) - {}^{n}(d_3)\right)^{n}\left\{\frac{Gm}{r}\sin\phi \cos\theta\,\frac{M_3}{r_3^3} + \frac{3GmM_3}{r^2 r_3^3}\frac{dr_3}{d\phi}\frac{dr_3}{dd_3}\right\}$$

$$+ \left({}^{n+1}(d_4) - {}^{n}(d_4)\right)^{n}\left\{\frac{-Gm}{r}\cos\phi\,\frac{M_4}{r_4^3} + \frac{3GmM_4}{r^2 r_4^3}\frac{dr_4}{d\phi}\frac{dr_4}{dd_4}\right\} \qquad 2.29$$

Similarly,

$$\frac{\partial \dot{\gamma}}{\partial r_1} = \frac{3GmM_1 d_1 \sin \beta}{r r_1^4 \cos \phi} = \frac{3GmM_1}{r^2 \cos^2 \phi \, r_1^3} \frac{dr_1}{d\theta}$$

$$\frac{\partial \dot{\gamma}}{\partial r_3} = \frac{-3GmM_3 d_3 \sin \beta}{r r_3^4 \cos \phi} = \frac{3GmM_3}{r^2 \cos^2 \phi \, r_3^3} \frac{dr_3}{d\theta}$$

and again using the contents of Table II, the expanded linearized equation for $\dot{\gamma}$ is seen to be

$$\begin{aligned}
\dot{\gamma}_{n+1} = \; & {}^{n}\left\{-2(\gamma + \omega)\left(\frac{v}{r} - \eta \tan \phi\right) + \frac{Gm \sin \beta}{r \cos \phi}\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right)\right\} + \left(r_{n+1} - r_n\right){}^{n}\left\{2(\gamma + \omega)\right. \\
& \left. \cdot \frac{v}{r^2} - \frac{Gm \sin \beta}{r^2 \cos \phi}\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) + \frac{3GmM_1}{r^2 \cos^2 \phi \, r_1^3} \frac{dr_1}{d\theta} \frac{dr_1}{dr} + \frac{3GmM_3}{r^2 \cos^2 \phi \, r_3^3} \frac{dr_3}{d\theta} \frac{dr_3}{dr}\right\} \\
& + \left(\phi_{n+1} - \phi_n\right){}^{n}\left\{\frac{2(\gamma + \omega)\eta}{\cos^2 \phi} + \frac{Gm \sin \phi \sin \beta}{r \cos^2 \phi}\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) + \frac{3Gm}{r^2 \cos^2 \phi}\right. \\
& \left. \cdot \left(\frac{M_1}{r_1^3} \frac{dr_1}{d\theta} \frac{dr_1}{d\phi} + \frac{M_3}{r_3^3} \frac{dr_3}{d\theta} \frac{dr_3}{d\phi}\right)\right\} + \left(\theta_{n+1} - \theta_n\right){}^{n}\left\{\frac{Gm \cos \beta}{r \cos \phi}\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right)\right. \\
& \left. + \frac{3Gm}{r^2 \cos^2 \phi}\left[\frac{M_1}{r_1^3}\left(\frac{dr_1}{d\theta}\right)^2 + \frac{M_3}{r_3^3}\left(\frac{dr_3}{d\theta}\right)^2\right]\right\} + \left(v_{n+1} - v_n\right){}^{n}\left\{\frac{-2(\gamma + \omega)}{r}\right\} + \left(\eta_{n+1} - \eta_n\right) \\
& \cdot {}^{n}\left\{2(\gamma + \omega) \tan \phi\right\} + \left(\gamma_{n+1} - \gamma_n\right){}^{n}\left\{-2\left(\frac{v}{r} - \eta \tan \phi\right)\right\} + \left(\alpha_{n+1} - \alpha_n\right) \\
& \cdot {}^{n}\left\{\frac{-Gm \cos \beta}{r \cos \phi}\left(\frac{M_3 d_3}{r_3^3} - \frac{M_1 d_1}{r_1^3}\right) + \frac{3Gm}{r^2 \cos^2 \phi}\left(\frac{M_1}{r_1^3} \frac{dr_1}{d\theta} \frac{dr_1}{d\alpha} + \frac{M_3}{r_3^3} \frac{dr_3}{d\theta} \frac{dr_3}{d\alpha}\right)\right\} \\
& + \left({}^{n+1}(M_1) - {}^{n}(M_1)\right){}^{n}\left\{\frac{-Gm \sin \beta}{r \cos \phi} \frac{d_1}{r_1^3}\right\} + \left({}^{n+1}(M_3) - {}^{n}(M_3)\right){}^{n}\left\{\frac{Gm \sin \beta}{r \cos \phi} \frac{d_3}{r_3^3}\right\} \\
& + \left({}^{n+1}(d_1) - {}^{n}(d_1)\right){}^{n}\left\{\frac{-Gm \sin \beta}{r \cos \phi} \frac{M_1}{r_1^3} + \frac{3GmM_1}{r^2 \cos^2 \phi \, r_1^3} \frac{dr_1}{d\theta} \frac{dr_1}{dd_1}\right\} \\
& + \left({}^{n+1}(d_3) - {}^{n}(d_3)\right){}^{n}\left\{\frac{Gm \sin \beta}{r \cos \phi} \frac{M_3}{r_3^3} + \frac{3GmM_3}{r^2 \cos^2 \phi \, r_3^3} \frac{dr_3}{d\theta} \frac{dr_3}{dd_3}\right\} \\
& + \left({}^{n+1}(M_2) - {}^{n}(M_2)\right)(0) + \left({}^{n+1}(M_4) - {}^{n}(M_4)\right)(0) \\
& + \left({}^{n+1}(d_2) - {}^{n}(d_2)\right)(0) + \left({}^{n+1}(d_4) - {}^{n}(d_4)\right)(0)
\end{aligned} \qquad 2.30$$

CHAPTER III

COMPUTATIONAL RESULTS

Several arbitrarily defined models of the earth's geopotential field were examined, two of which are reported here as being generally representative of mass concentration models. The examinations were conducted as follows: (1) a model was proposed, (2) data points (boundary conditions) were calculated from an assumed initial condition using numerical integration, and (3) the parameter estimation program described by Childs et al. [8] was modified and used to regenerate parameters of the model.

A Three-Mass Model of the Earth

In the first model, half the mass is assumed to be at the geometric center of the earth, and the other half is located in the equatorial plane as shown in Fig. 7. Several orbits of a point mass were generated and used to attempt parameter evaluation. Most of the calculations used data obtained from an elliptical orbit approximately 200 km by 250 km altitude in a plane of about 40° inclination to the equatorial plane.

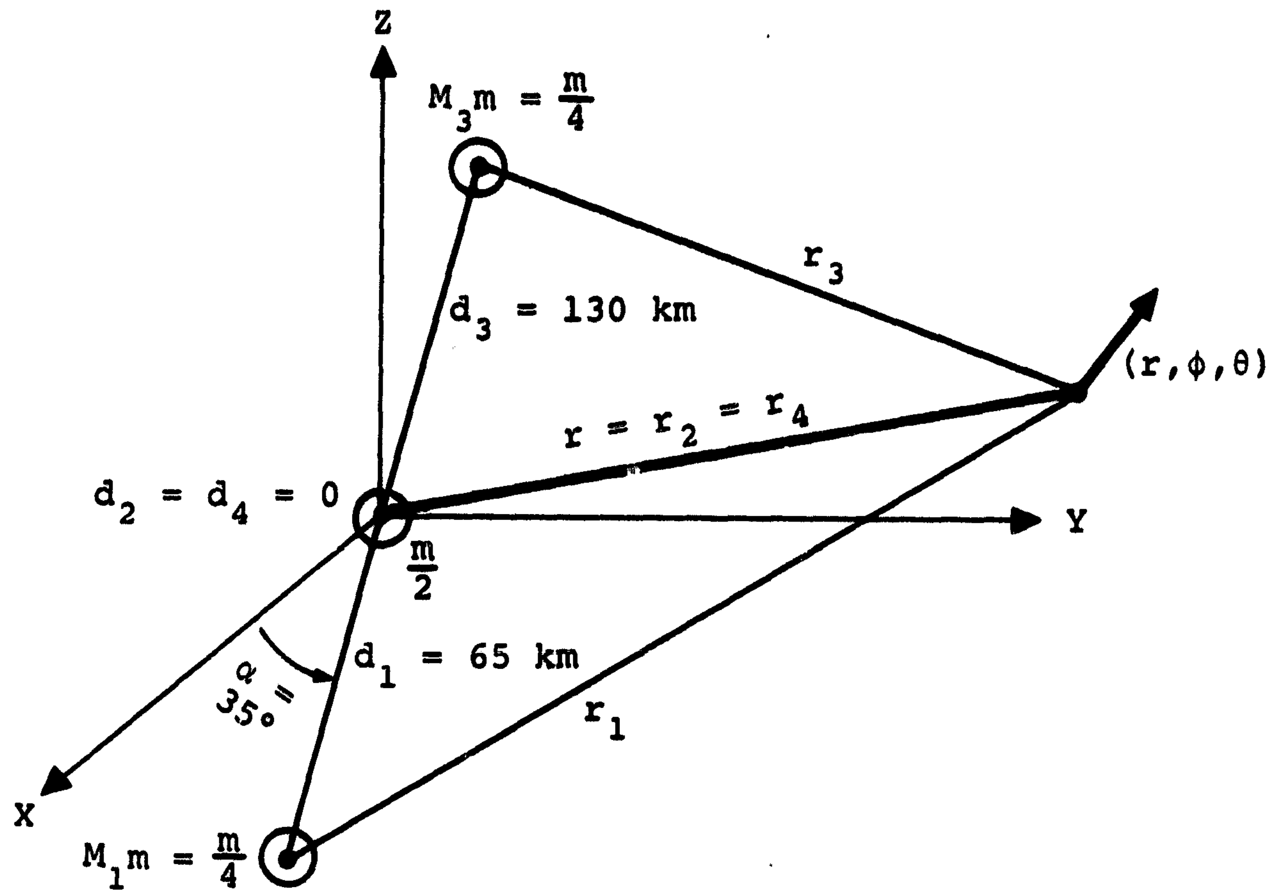

Figure 7. - A specific three-mass model of the earth.

The estimations reported in this paper were limited to five of the nine parameters of the model described previously. It was deemed beyond the scope of this study to attempt evaluation of more parameters. The justification for this was that most models of central gravitational fields assume the primary mass to be at the geometric center of the body and then small perturbations from that basis are computed. This type of model requires only five of the nine parameters be estimated. It was assumed that half of the mass was located at the geometric center (this corresponds to assuming $d_2 = d_4 = 0$ and $M_2 + M_4 = 0.5$), and attempts were made to estimate values of the remaining parameters, namely M_1, M_3, d_1, d_3, and α .

A set of computer runs was made with d_1, d_3, M_1, M_3, and α being estimated from arbitrary guesses at their prescribed values. The results obtained for these cases so resembled the results of the four-mass model study that a separate report of them is not required.

A Four-Mass Model of the Earth

A second model that was given careful consideration is indicated by Fig. 8. In this model, half of the mass was located in the equatorial plane as before, but the other half of the mass was divided and moved out from the

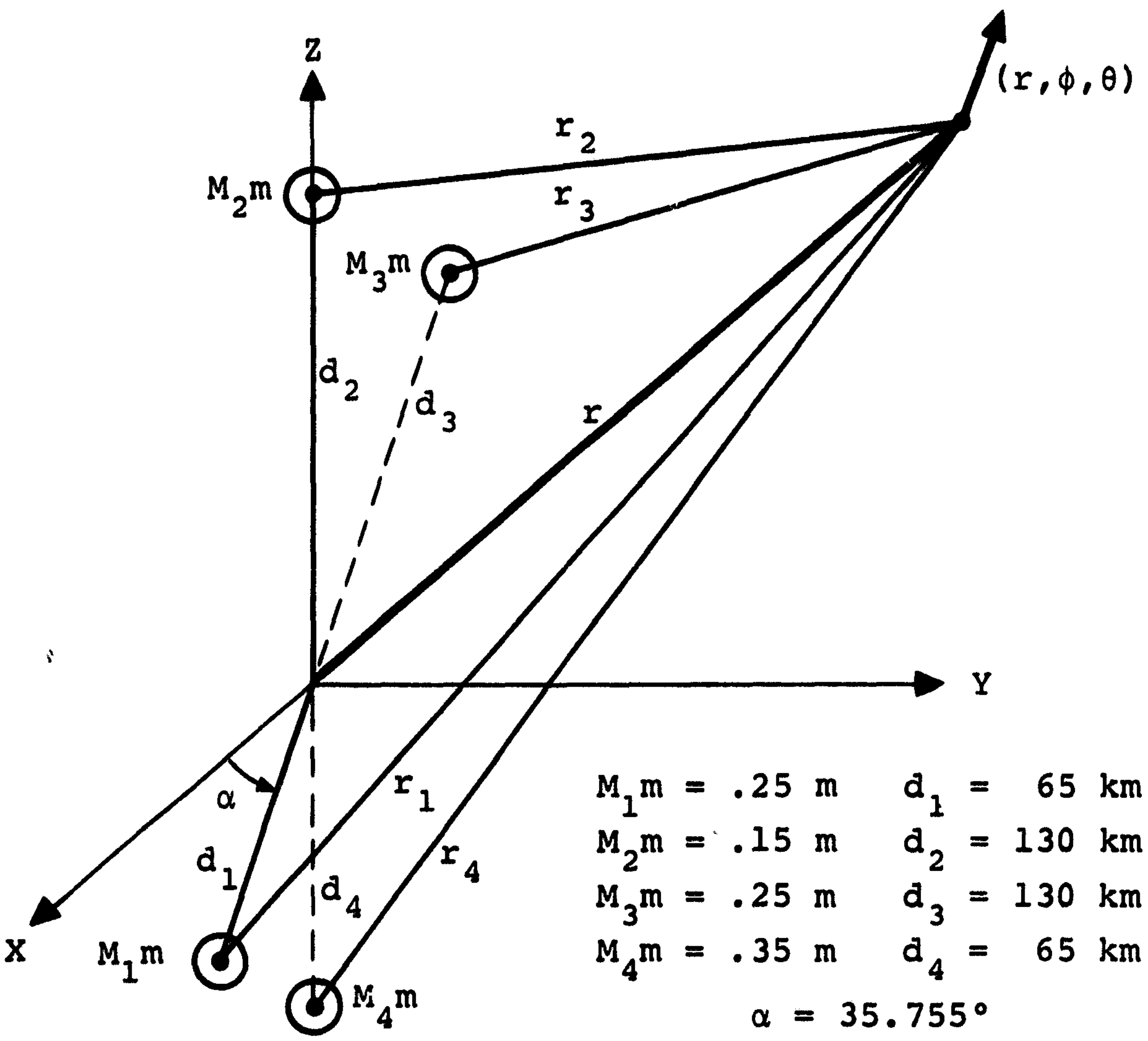

Figure 8. - A specific four-mass model of the earth.

geometric center along the axis of rotation. Again several orbits were generated to serve as boundary conditions for subsequent parameter estimations.

As before, the estimations reported for this paper were limited to five of the nine parameters of the model. The distances, d_2 and d_4, and the masses located at these distances, M_2 and M_4, were held at their prescribed values. The computer runs were made in an effort to estimate M_1, M_3, d_1, d_3, and α from arbitrary guesses at their proper values.

Actual Results of Estimating the Four-Mass Parameters

The process for evaluating M_1, M_3, d_1, d_3, and α assumed $y(t_0)$ to be a point on the orbit exactly above the equator, and boundary conditions were used that spanned about two-thirds of an orbit. The actual boundary conditions used in the computer runs were taken to be linear combinations of r and $\dot{r}$ at about 1,000-second intervals. A scaling multiplier was used on each so that the actual boundary conditions were on the order of 1.0. These boundary conditions proved quite successful compared to other boundary conditions, such as r or $\dot{r}$ alone and the nonlinear boundary condition $\ddot{r}$. All of these three types of boundary conditions were used during this study, but none matched the success of the linearly combined r and $\dot{r}$.

The first computer runs indicated that calculations leading to convergence to the known model required double precision, or about 17 decimal digits, for intermediate calculations. The computer program used for the estimations, called QUASI and described by Childs et al. [8], was modified to do all its calculations in double precision. Even with that modification, convergence to the known values of the gravitational model was not obtained in a reasonable number of iterations unless the initial guesses for d_1 and d_3 were close to their prescribed values. In fact with accurate guesses for d_1 and d_3 , almost any initial guess could be made for M_1, M_3, and α , and the operations of QUASI would result in convergence in about 10 iterations. In addition, if all the M_i and α were supplied, the d_i were successfully estimated from many starting points by QUASI in a reasonable number of iterations.

Estimation of all five of the parameters chosen in this study was not obtained until modifications were made to QUASI. In its natural form, QUASI is strictly a Newton-Raphson iteration procedure using (n) variables and at least (n) boundary conditions. At each iteration a perturbation for each unknown variable of the problem is calculated by application of the classical Newton-Raphson scheme for solving such numerical problems. In the case being described here, the direction of the perturbations seemed to be proper; however, each indicated change seemed to be several orders of magnitude too small for rapid solution of the problem. Thus modifications were made to QUASI to use a larger step in the same direction so that the solution might be obtained in fewer iterations.

Certain variables are used in QUASI in limiting the magnitude of perturbations calculated in the Newton-Raphson scheme. In this case these variables were convenient to develop the algorithm that increased the rate of convergence. The theory involved in applying Newton's method to n-dimensional problems of this type does not provide any definite proof that the perturbations calculated are the best size for obtaining the desired solution rapidly. The scheme used in this case was to let the Newton-Raphson procedure indicate the direction to step in n-space and to determine the size of the step by an arbitrary strategy. This scheme is a gradient method with certain artificial controls.

The controlled gradient scheme was developed by the following considerations. Let $\|\overline{\Delta A}_i\|$ represent the Cartesian norm of the step indicated by the *ith* Newton-Raphson iteration. Program QUASI arbitrarily limits $\|\overline{\Delta A}_i\|$ by a program variable, represented in this discussion by P . Thus the $\|\overline{\Delta A}_i\|$ step is always limited to

$$\frac{P}{\|\overline{\Delta A}_i\|}\,\overline{\Delta A}_i \rightarrow \overline{\Delta A}_i^*$$

where typical values of P as supplied by the program user range from 0.5 to 20.0, depending on the type of problem being solved.

The strategy employed here was somewhat different since the predicted step on a Newton-Raphson iteration had a tendency to be quite small relative to the change needed for convergence. The algorithm is described by the logic flow chart given in Fig. 9. In words, the actual algorithm that was developed can be described by the following steps:

1. Perform sufficient Newton-Raphson steps such that the direction indicated is essentially the same

GIVEN: P, $\|\overline{\Delta A}_i\|$, $\overline{\Delta A}_i$

$D_1 = \overline{\Delta A}_{i-1} \cdot \overline{\Delta A}_i$, $D_2 = \overline{\Delta A}_{i-2} \cdot \overline{\Delta A}_{i-1}$, $D_3 = \overline{\Delta A}_{i-3} \cdot \overline{\Delta A}_{i-2}$

k, $j = 0$ INITIALLY

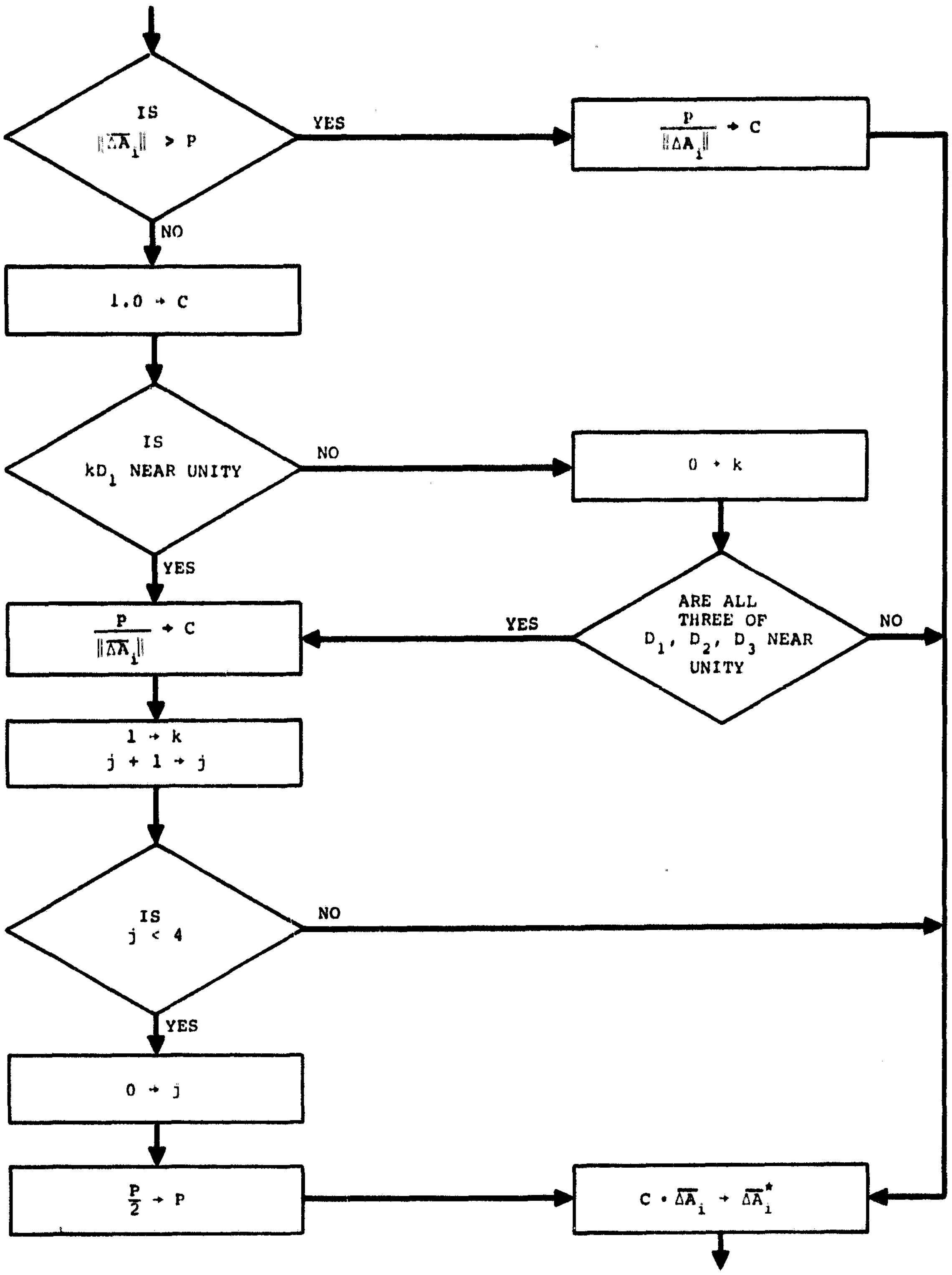

Figure 9. - Logic flow of gradient step-size control.

(in n-space) for at least three consecutive steps. This directional idea is calculated as the vector dot product of three consecutive (unitized) $\overline{\Delta A}_i$ vectors produced by the Newton-Raphson iterations. When this dot product is near unity in two successive vectors, the direction of the two vectors is said to be the same. When four such successive $\overline{\Delta A}_i$ vectors are found, proceed to step 2.

2. Take one step along the indicated gradient, i.e., the gradient of the latest Newton-Raphson step, of an arbitrarily computed size, namely

$$\frac{P}{\|\overline{\Delta A}_i\|}\,\overline{\Delta A}_i \rightarrow \overline{\Delta A}_i^*$$

3. Evaluate the dot product of unitized $\overline{\Delta A}_i$, the last gradient step, and unitized $\overline{\Delta A}_{i-1}$. If this is near unity, repeat step 2. If this dot product is negative, it can be interpreted as indicating the objective has been overstepped. If this dot product is positive but not near unity, it can be interpreted as indicating that the gradient being used is no longer accurate. If either of the last two situations exists, proceed to step 4.

4. Return to step 1 except for every fourth entry of step 4; on those entries, the step control, P, is halved before returning to step 1.

A significant point must be made relative to the ability of QUASI to satisfy the given boundary conditions. Almost any initial guess for M_1, M_3, d_1, d_3, and α would be modified by the *standard operations* of QUASI within seven to 10 iterations to provide a fit of at least three significant decimal digits between the given and calculated boundary conditions. The results indicated for the parameters of the four-mass model would, in some cases, not be accurate to even one decimal digit. In fact, all the cases investigated, with the exception of using the true values for all initial guesses, produced unacceptable results; i.e. at least one parameter was not estimated accurately. It was estimated that QUASI would have required at least 2000 iterations to have obtained an acceptable answer. With the special modifications described above, results that match the known values of the stated parameters to four significant decimal digits were obtained in about 60 iterations. The significant point is that to provide this accuracy in the parameters being estimated, the computer program had to obtain seven and eight significant decimal digits of agreement for the boundary conditions. This certainly verifies both the need for double precision and the need for a specially devised strategy for convergence in the five variables.

A comparison of the data for the standard QUASI and the modified program is given in Table II. These data support the claims of the preceding paragraph. The difference in the rate of convergence of the two schemes becomes evident upon thorough examination of the presented data. It may again be stated that convergence seems assured by the standard Newton-Raphson procedure. However, the calculations of so many consecutive steps of small size, all in the proper direction for convergence, lead to the conclusion that the function being evaluated produces inordinately small changes. This conclusion of a good direction, or gradient, for convergence allows for the significantly increased step size utilized to obtain the rapid convergence of the modified method.

Problems Associated with Using Actual Orbit Data

Actual spacecraft orbital data from the various Apollo missions were available in some limited forms. These data were obtainable in terms of earth-based radar sightings during earth and lunar orbits. These data were examined only to insure that the boundary conditions used in the models presented here represent meaningful boundary conditions in actual orbit data. The utilization of r and $\dot{r}$ in the model cases appears to closely resemble the utilization of range and range rate normally recorded

TABLE II

COMPUTED RESULTS FOR THE FOUR-MASS MODEL

Standard Newton-Raphson	Initial Guess	7 Iterations	60 Iterations	100 Iterations	140 Iterations	Actual Solution
$\bar{A}_1$ = M_1	.30000	.21035	.21125	.21164	.21211	.25000
M_3	.20000	.28965	.28875	.28836	.28789	.25000
d_1	50.000	81.648	81.236	81.097	80.836	65.000
d_3	115.00	115.39	115.69	115.80	116.00	130.00
[illegible]	45.000	35.772	35.758	35.755	35.755	35.755
$\overline{\Delta A}_1$		.000035	.000015	.000015	.000015	
		-.000035	-.000015	-.000015	-.000015	
		-.001605	-.000698	-.000692	-.000682	
		.001212	.000526	.000523	.000518	
		-.000002	-.000000	.000000	.000000	
P	20.0	20.0	20.0	20.0	20.0	
$\Vert\overline{\Delta A}_1\Vert$		.02032	.00874	.00867	.00856	
Boundary Values at						
1000 sec		1.885495	1.885554	1.885554	1.885554	1.885545
5000 sec		.09733961	.09731822	.09731833	.09731846	.09732894
Newton-Raphson Coupled with Gradient Step	**Initial Guess**	**7 Iterations**	**17 Iterations**	**26 Iterations**	**59 Iterations**	**Actual Solution**
$\bar{A}_1$	.30000	.21616	.24221	.25596	.25005	.25000
	.20000	.28384	.25779	.24404	.24995	.25000
	50.000	72.450	67.861	62.725	64.981	65.000
	115.00	116.76	126.85	132.38	130.02	130.00
	45.000	35.273	35.836	35.813	35.755	35.755
$\overline{\Delta A}^*_1$		-.00217	.01432	-.00724	-.000000	
		.00217	-.01432	.00724	.000000	
		8.0000	-6.0831	2.7318	.000007	
		.01308	5.1957	-2.9218	-.000001	
		.00742	-.0001	.00007	.000000	
P	8.0	8.0	8.0	4.0	1.0	
$\Vert\overline{\Delta A}_1\Vert$		7.4703	.00419	.00131	.00001	
Boundary Values at						
1000 sec		2.002146	1.886800	1.885535	1.885545	1.885545
5000 sec		.04227049	.09683903	.09731846	.09732878	.09732894

in radar sightings. No earth orbit data was actually used because of the effort involved in converting the available range and range rate data into r and $\dot{r}$ data. However, the complexities of using real world data were beyond the scope of this work.

For an analysis of the lunar gravitational potential, data were obtained in a form reduced from earth-based sightings by a computer program developed by Clark et al. [9] for one lunar orbit during the Apollo 11 mission. The actual mission data had been manipulated considerably and had been fit, through orbital prediction schemes, to a formal model of the lunar potential field. Some computer runs were made in an attempt to estimate a three-mass model that would duplicate the orbital data. The complexities of the coordinate system and scaling differences were sufficient to preclude any definite conclusion to this work.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The results of the computer runs made with boundary conditions taken from orbit points around the three- and four-mass models indicate the adequacy of the method to solve the boundary value problem. As indicated previously, agreement of seven significant decimal digits between the given and calculated boundary conditions was obtained. In addition, five of the nine parameters defining the gravitational potential model were estimated to four significant decimal digits via the solution of the boundary value problem.

A special strategy was required for convergence to the desired results in a reasonable number of iterations. A standard Newton-Raphson technique was coupled with a gradient search algorithm. The gain in convergence was remarkable (60 iterations versus an estimated 2000), particularly considering that the method developed should be applicable to any system. It was concluded that the controlled gradient-stepping technique should in general compare favorably with a standard Newton-Raphson procedure in solving n-dimensional problems.

Finally it was concluded that mass concentration models can be used to determine near-body orbits for certain central force fields. The results of estimating the numerical descriptions of the models examined herein indicate that such models can be determined from observations of a combination of r (range) and $\dot{r}$ (range rate) of the orbiting body. Although actual earth or moon orbit data have not yet been used successfully, the results of the work performed so far indicate that the gravitational potential of such bodies can be represented by mass concentration models.

Recommended Additional Work

Several items must be mentioned as incomplete in the investigations reported here. These may be described as follows:

Rate of Convergence. — The rate of convergence of the standard process is slow when the initial guesses are not near the true values of the parameters being estimated. Employing second derivatives in the linearized equations is at least one idea that should be given proper consideration.

Changes in Scaling. — The convergence characteristics observed in the computer runs reported here may have been caused by scaling problems. A kilogram-kilometer-second measuring system was used. A possible improvement might be a slug-ton-earth radii-hour measuring system.

Application to Real Orbital Data. – The measurement of gravity forces over the surface of the earth has led to the definition of a geoid describing the gravity of the earth as an oblate spheroid with undulations in its surface. Uotila [10], Hirvonen [11], and others have presented various charts and supporting equations that describe the undulations of surface gravity forces. Under close examination, each chart reinforces the idea of an n-mass model of the earth's geopotential. Considerable effort should be expended in continuing such an investigation.

In addition the observation of lunar orbits leads to the conclusion that the potential function of the moon is varying with time in some way not yet properly understood. Further investigation into the application of n-mass models of the lunar potential are also proposed.

It must be remarked that the perturbations of orbits about the earth and the moon caused by irregularities in the gravity potential function of those bodies are considered of the same order of magnitude as those caused by other forces present in the system. To properly study the perturbations caused by irregularities in the potential field, all these other extraneous forces must be included in the model before actual spacecraft data can be used.

SELECTED BIBLIOGRAPHY

1. Ehricke, K. A., "Environment and Celestial Mechanics," *Spaceflight*, Vol. I, D. Van Nostrand, Princeton, New Jersey, 1960.

2. Kaula, W. M., *Theory of Satellite Geodesy*, Blaisdell, Waltham, Massachusetts, 1966.

3. Holloway, C. C., "Identification of the Earth's Geopotential," University of Houston, Ph.D. Thesis, Houston, Texas, 1968.

4. Gaposchkin, E. M. and Lambeck, K., "1969 Smithsonian Standard Earth (II)," Smithsonian Astrophysical Observatory, Special Report 315, Cambridge, Massachusetts, 1969.

5. Doiron, H. H., "An Indirect Optimization Method with Improved Convergence Characteristics," University of Houston, Ph.D. Thesis, Houston, Texas, 1970.

6. Childs, S. B., "Identification of Systems," Report No. RE 2-68 Project THEMIS Information Processing Systems, ONR Contract No. N00014-68-A-0151, August 1968.

7. Bellman, R. E. and Kalaba, R. E., *Quasilinearization and Nonlinear Boundary-Value Problems*, American Elsevier Pub. Co., New York, 1965.

8. Childs, S. B., et al.; "QUASI, Solution of Multipoint Boundary Value Problems of Quasilinear Differential Equations," Project Themis Report RE7-69, ONR Contract N00014-68-A-0151, September 1969.

9. Clark, L., Johnson, G., Lenzo, D., Gaffey, E., and Contreras, H. "Apollo Orbit Determination Program," National Aeronautics & Space Admin., Manned Spacecraft Center, Computation and Analysis Division Computer Program Documentation, Program E184, Project 1260, Vol. 1, June 1970.

10. Uotila, U. A., "Investigations on the Gravity Field and Shape of the Earth," Isostatic Institute of the International Association of Geodesy, No. 33, Helsinki, Finland, 1960.

11. Hirvonen, R. A., "New Theory of the Gravemetric Geodesy," Isostatic Institute of the International Associations of Geodesy, No. 32, Helsinki, Finland, 1960.

12. Garland, G. D., *The Earth's Shape and Gravity*, Permagon Press, New York, 1965.

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13. ABSTRACT

The observations of a near-body satellite are used in the determination of certain constants appearing in a mathematical representation of the gravitational field of the central body. This representation is based on the assumption that the mass of the central body can be closely modeled by several concentrated masses located near its geometric center. The determination method employs a perturbation technique, numerical integration of linear and nonlinear differential equations, least-squares fitting criteria, and matrix inversion to determine estimates for the parameters involved.

Several models for the representation of central body gravitational fields are discussed and the numerical techniques for evaluation of the parameters involved are briefly reviewed. The computations performed indicate that the parameters of such models can be estimated numerically. Some of the problems associated with using the method described herein to solve typical trajectory problems are discussed and some actual results are presented. Finally, suggestions for additional study of the theory and its applications are proposed.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Non-linear equations Multipoint boundary value problem Orbit determination Geopotential Perturbation method Lumped mass models						