NASA CONTRACTOR
REPORT

NASA CR-1879

NASA CR-1
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SONIC BOOM IN TURBULENCE

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1971

0061127

1. Report No. NASA CR-1879	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Sonic Boom in Turbulence		5. Report Date September 1971
		6. Performing Organization Code
7. Author(s) W. A. Horning		8. Performing Organization Report No.
		10. Work Unit No. 136-80-01-04
9. Performing Organization Name and Address EG&G, Inc Albuquerque, NM		11. Contract or Grant No. NAS1-9284
		13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

Statistics of random overpressure peaks observed in a single sonic boom at ground are derived from first principles. The Fourier transform of the wave equation for sound in weak homogeneous turbulence is the starting point. A temporal propagation equation for the spectral density of scattered waves is derived including multiple scattering effects insofar as they are important in weak turbulence at long sonic pathlengths. Graphical results are compared with the data of Garrick and Maglieri.

17. Key Words (Suggested by Author(s)) Sonic Boom Propagation Shock Wave Turbulence		18. Distribution Statement Unclassified - Unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 82	22. Price* $3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

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SECTION 1

INTRODUCTION

The sonic boom is affected by both large and small scale non-uniformities in the atmosphere. By large scale nonuniformities we mean those which describe the presence of a tropopause or cold front, those whose scale length is large compared to the thickness of the N-wave ($\sim$ 300 ft). E. J. Kane[1] and others have studied large scale nonuniformities by the method of ray acoustics.* Small scale nonuniformities are produced chiefly by typical atmospheric turbulence, which affects the passage of sound by both the temperature and air velocity fluctuations which it contains. S. C. Crow[2] has analyzed the effect of turbulence on sonic bang by a first order scattering theory, and with much agreement with experiment. He suggests that higher order scatterings, the focus of the present study, may be required to fully account for the large pressure spikes which occur in the sonic boom at ground.

V. I. Tatarski's volume[3] (1961) has been a basis for most subsequent theories on the small angle scattering of waves in random media. He shows that wave interference is central in this phenomenon, in results which agree widely with experiment. M. E. Gracheva and A. C. Gurvich[4] (1965) for the case of a light beam in turbulence, measured a wave pathlength L_s beyond which the mean square fluctuations within the beam no longer increase appreciably. This saturation length L_s is from 1 to 2 km in the optical case. It now appears as the pathlength beyond which the earlier results of Tatarski[3] fail, in common with other first order scattering theories. Multiple scattering studies of saturation by Tatarski[5], R. A. Schmeltzer[6], W. P. Brown[7], and D. A. de Wolf[8] have started with the computation of expressions for statistical sample histories of the scattered wave field itself. These expressions, largely based on the perturbation theory of quantum mechanics are series which converge slowly at the saturation length. D.A. de Wolf[9] has nevertheless obtained some good numerical results by their means.

The present study computes no sample histories. Rather, in the spirit of M. T. Beran[10] and V. M. Komissarov[11], it treats statistical averages through their own propagation equations, as is done in the theory of Brownian motion, optical coherence, and thermal noise in electric circuits. We hoped that some field averages might propagate more simply than sample fields, and such is the case in the present problem. The key step in our approach is the termination or closure of the infinite hierarchy of equations which connect cross covariances between random fluctuations of sound and fluctuations of the medium. This step uses factoring of fourth degree covariances and involves an error of third degree in ϵ, the root mean square of the fluctuations in the medium. Such factoring leads to a simple expression for the saturation length, which is found to vary as the inverse of sonic frequency.

*Superscripts refer to the list of references presented on pages 78-80.

The full statistics of the sonic boom at ground are intricate. By largely limiting attention to the probability distribution for the peaks of the larger and more unlikely overpressure spikes, the problem is made somewhat tractable through the use of Poisson statistics. The sonic boom, as a spatially distributed overpressure traveling at a speed of sound, is composed of a wide band of frequencies, because of its sharp leading and trailing edges. The higher of these frequencies are saturated at ground and possess random phases. They comprise Gaussian noise which is statistically almost uniform near each edge of the N-wave and within a distance from it equal to the inverse of the lowest spatial frequency b which is saturated. At frequencies greater than b, we will find the spectral density of this noise to vary as the inverse square of the frequency. That waves propagating through turbulence eventually become random noise has been stated by de Wolf[9]. That this noise is somewhat localized near the sharp edges of the N-wave is shown by data, and has been explained by Crow[2].

The final step of our analysis is a use of an important formula of S. O. Rice for the probability per unit time of unusually high maxima in a random fluctuation with known spectral density. An important input to the calculation is Kolmogorov's spectral density for turbulent flow velocities within the inertial subrange of eddy diameters. It is remarkable that anything about atmospheric flow is as invariable as this seems to be. Nevertheless, it is somewhat uncertain, especially near ground, in part because it omits intermittency in turbulence. Micrometeorological conditions near ground, at the time most of the data[12] were taken, could be only estimated. The above uncertainties, added to theoretical approximations, limit the significance of agreement between theory and experiment. Such agreement, with reasonable inputs, is nevertheless within a factor of 2. This suggests air turbulence as the chief origin of high frequency randomness in sonic boom. This result is of interest in the scattering of microwaves and light by turbulence, phenomena much like the scattering of sound.

The present study is a theoretical interpretation of some of the data on sonic boom given by I. E. Garrick and D. J. Maglieri[12]. The computed results, as reviewed in the summary, are in general agreement with the data.

SECTION 2
SECTION GUIDE

Since the analysis is lengthy, the following outline is given, so that the reader may at any time orient himself relative to the analysis.

Section 3, following previous authors, derives the wave equation for sound in turbulence taking account of both turbulent flow velocity and the associated temperature fluctuations. Section 4 is a reformulation of Section 3 in terms of a special set of Fourier coefficients for the sonic pulse. Sections 3 and 4 are essential to the results on boom fluctuations. Section 5 is a partial introduction to random process theory. Sections 5 and 6 concern basics and approach. The results on statistical symmetry in Section 5 are used continually in what follows. Section 5, and what follows it, presumes as much familiarity with random process theory as is outlined in Appendices A6 and A7.

Sections 6, 7 and 8 contain a treatment of the multiple wave scattering of plane waves incident on weak turbulence for wave pathlengths which may be large compared to the saturation length. The whole analysis applies only when the pathlength is well in excess of all important eddy diameters. The pathlength is treated as large compared to all important wavelengths in the Fourier analysis of the incident wave packet. If the incident plane waves form a disturbance unlimited in their mean direction of propagation, the pathlength in quasi-uniform turbulence must be large compared to the longitudinal correlation length of the incident beam, in order that the above sections apply. Following A. Khinchine[13], Sections 6, 7 and 8 aim at wave statistics valid asymtotically at a pathlength large compared to all characteristic lengths in the phenomenon, where the scattering of any partial wave is by a small percentage over any characteristic length.

Sections 9, 10 and 11, based heavily on Section 8, are chiefly specific to overpressure peaks of unusually large maxima within the sonic boom. Section 12 contains graphical results, while the final Section 13 is a brief critique and summary.

Appendix A1 emphasizes that some mathematical models of nature are non-Markovian, despite the fact that all natural processes are Markovian when viewed from the principle of micro-causality. Appendices A2, A3, A4 and A5 consist of solutions of a few simple random processes by methods of the main text. These illustrations are simple in the sense that they lack heavy formalism, but they show most of the occasional subtleties in continuous random processes. The author finds it saves time to be familiar with some sample problems of the subject and to return to them for guidance.

SECTION 3

THE WAVE EQUATION

The sonic boom in turbulence is a process whose analysis, like that of many other continuous random processes, starts with a temporal propagation equation for sample histories. This, in our case, is a wave equation for sound in a disturbed medium. P. Langevin[14] discussed a random process whose statistics may be found without solving its propagation at all. When the propagation equation for the sample histories of a random process is used only to fix the form of the easier equation for the propagation of the statistics, then the former is often called a Langevin equation.

The technique of solving the wave equation itself is to be sure an attractive branch of analysis, whose field of proper application is large. Its scholars, however, have used detailed solutions of involved wave equations whose role was that of a Langevin equation for a process. Yielding to such a natural compulsion may be retrograde in some cases in proportion to the complexity of the process, and is incidentally outside the spirit of modern random process theory.

The wave equation basic to our problem, while in the literature[4], is derived here in somewhat greater detail. From the first principles of aerodynamics, we quote the equation of mass conservation, namely

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0 \tag{1}$$

where $\rho(t, \vec{r})$ is air density and $\vec{r}(x, y, z)$ is the position in space whose Cartesian coordinates are $x, y,$ and z. Temporal epoch is denoted by t and the local vector velocity by $\vec{q}(t, \vec{r})$. We recall that, in the Eulerian coordinates used in equation (1), $\partial/\partial t$ denotes a partial derivative with $\vec{r}$ held constant, while $\nabla \cdot$ denotes a divergence with t held constant. In the same spirit, we list the isentropic equation of state

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^\gamma \tag{2}$$

where p is local pressure and p_o is the average of p over a sufficiently large volume. ρ_o is a like average of the density, while $\gamma \sim 1.4$ is the well-known ratio of specific heats at constant pressure and at constant volume. The equation of momentum conservation is

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} = \frac{-\nabla p}{\rho} \tag{3}$$

To clarify notation, we list for the vector $(\vec{q} \cdot \nabla)\vec{q}$ its Cartesian coordinate which is parallel to the x axis, namely

$$\left(q_x \frac{\partial}{\partial x} + q_y \frac{\partial}{\partial_y} + q_z \frac{\partial}{\partial_z}\right) q_x \tag{4}$$

For the total velocity $\vec{q}$, we write

$$\vec{q} = \vec{U} + \vec{V} \tag{5}$$

where $\vec{U}$ is the air velocity produced by the sound alone, and $\vec{V}$ the velocity produced by turbulence alone. Such a decomposition is possible when $U \ll c$ and $V \ll c$, where c is the velocity of sound. With these inequalities

$$\rho = \rho_o + R + S \ , \ R \ll \rho_o \ , \ S \ll \rho_o \tag{6}$$

where R is the increment in ρ produced by sound and S the increment produced by turbulence. Similarly

$$p = p_o + P + Q \ , \ P \ll p_o \ , \ Q \ll p_o \tag{7}$$

where P and Q are pressure increments produced by sound and turbulence respectively.

The subscript t will denote $\partial/\partial t$. A substitution in equation (3) from expressions (5), (6), and (7) yields

$$\vec{U}_t + \vec{V}_t + ((\vec{U} + \vec{V})\nabla\cdot)(\vec{U} + \vec{V})$$
$$= (-1/\rho_o + (R + S)/\rho_o{}^2)\nabla(P + Q) \tag{8}$$

except for terms of second degree in R and S. We assume the amplitude of the sound small enough so that

$$\langle U^2 \rangle \ll \langle V^2 \rangle \tag{9}$$

where the brackets $\langle \ldots \rangle$ denote the expected value of their contents. We may ignore U by comparison with V where neither is differentiated. Equation (8) then becomes

$$\vec{U}_t + \vec{V}_t + \vec{V}\cdot\nabla(\vec{U} + \vec{V}) = (-1/\rho_o + (R + S)/\rho_o{}^2)\nabla(P + Q) \tag{10}$$

Since the air velocity produced by typical turbulent flow in the atmosphere is subsonic, we may ignore the air compression produced by such flow, which amounts to putting

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$$\nabla \cdot \vec{V} = 0 \ , \ Q = 0 \ , \ S = 0 \tag{11}$$

Also, we will use the frozen turbulence approximation expressed by

$$\vec{V}_t = Q_t = S_t = 0 \ , \ \vec{V}_{tt} = Q_{tt} = S_{tt} = 0 \tag{12}$$

which is justified by the reasonable assumption that at least the statistics of the turbulent flow remained unchanged during the time of passage of the sound packet over the largest important turbulent eddy. A time derivative of equation (10), with use of relations (11) and (12), yields

$$\vec{U}_{tt} + (\vec{V} \cdot \nabla)\vec{U}_t = -\nabla P_t / \rho_o \tag{13}$$

except for terms of second degree in the small quantities P and R.

Equation (1), by use of (11) and (12), becomes

$$R_t + \rho_o \nabla \cdot \vec{U} + \nabla \cdot (R\vec{V}) = 0 \tag{14}$$

except for a term $\nabla \cdot (R\vec{U})$ which we ignore since it is of second degree in R and $\vec{U}$ both of which are small increments produced by the sound alone. However, because of relation (9) we do not ignore second degree cross products between the velocity produced by the sonic field and that produced by turbulence. From $\nabla \cdot \vec{V} = 0$, we have

$$\nabla \cdot (R\vec{V}) = \vec{V} \cdot \nabla R + R\nabla \cdot \vec{V} = \vec{V} \cdot \nabla R \tag{15}$$

Also, from equation (1) and the inequalities (6) and (7)

$$P/\rho_o \simeq \gamma R/\rho_o \ , \ \nabla R = \nabla P/c^2 \ , \ c^2 \equiv \gamma \rho_o / \rho_o \tag{16}$$

where the symbol c denotes the velocity of sound in the limit U = V = 0. From equation (10)

$$\nabla P = -\rho_o \vec{U}_t \tag{17}$$

except for terms much smaller than $\rho_o \vec{U}_t$. Substituting in equation (14) from (15), (16) and (17) yields

$$P_t / c^2 + \rho_o \nabla \cdot \vec{U} - \rho_o \vec{V} \cdot \vec{U}_t / c^2 = 0 \tag{18}$$

But the irrotational vector field $\vec{U}$ satisfies

$$0 = \nabla \times (\nabla \times \vec{U}) = \nabla \nabla \cdot \vec{U} - \nabla^2 \vec{U} \qquad (19)$$

The gradient of equation (18), with (19), yields

$$\nabla P_t / \rho_0 = -c^2 \nabla^2 \vec{U} + \nabla(\vec{V} \cdot \vec{U}_t) \qquad (20)$$

Substituting in equation (13) from (20) yields

$$\vec{U}_{tt} + (\vec{V} \cdot \nabla)\vec{U}_t + \nabla(\vec{V} \cdot U_t) = c^2 \nabla^2 \vec{U} \qquad (21)$$

A comforting partial check of equation (21) can be obtained by specializing it to the case where $\vec{V}$ is constant and in the direction of propagation of a purely sinusoidal sound wave. Equation (21) then reduces to the familiar

$$V/c = \Delta \omega / \omega \qquad (22)$$

where ω is the sonic frequency and $\Delta \omega$ its Doppler shift. Equation (21) is what the wave equation for a uniform medium becomes after modification to account for a small but otherwise general Doppler effect.

Since $\vec{U}$ is irrotational, we may write

$$\vec{U}(t, \vec{r}) = \nabla \varphi (t, \vec{r}) \qquad (23)$$

where φ is a velocity potential. Consider the contour integral of the second term of equation (21), namely

$$(A \textstyle\int B) \, (\vec{V} \cdot \nabla) \varphi_t \, ds \qquad (24)$$

taken over a spatial path whose increment is ds. The path ends on A, the position of an incremental volume of air prior to the arrival of the sound pulse, and terminates at B, the position of this same small volume at a general time t when the sound pulse overlaps A, a time at which $\vec{V}$ and t are evaluated in expression (21). We assume, in extension of relation (9), that the distance D the air moves as a result of the passage of the sound over point A is small compared to the important scale lengths in the turbulent velocity $\vec{V}(r)$. That is, we assume D is less than a few cm. Then $\vec{V}$ is almost constant over the contour of integral (24), which may then be approximately written as

$$(A \textstyle\int B) \, \nabla(\vec{V} \cdot \nabla) \varphi_t \, ds = [(V \cdot \nabla) \varphi_t]_A^B \qquad (25)$$

A substitution in equation (21) from (23) and (25) followed by integration over the above contour then yields

$$c^2 \nabla^2 \varphi - \varphi_{tt} = 2(\vec{V} \cdot \nabla) \varphi_t \tag{26}$$

Turbulence is commonly associated with air temperature fluctuations which result from the large scale movement of air masses originally at differing temperatures. To include the effect of such temperature variation, we replace c^2 by $c^2(1 + \Theta)$ where $\Theta = T(r)/T_0$ is the ratio of the temperature fluctuation T to the absolute temperature T_0 averaged over many turbulence correlation volumes centered at the point at which T is evaluated. Then equation (25) becomes

$$c^2 \nabla^2 \varphi - \varphi_{tt} = 2(\vec{V} \cdot \nabla) \varphi_t - c^2 \Theta \nabla^2 \varphi \tag{27}$$

This, in summary, is a linearized scalar wave equation for sound in a medium whose material velocity $\vec{V}$ prior to the sound is small compared to the velocity of sound itself, and whose temperature fluctuation Θ is small compared to its mean temperature. Even though there has been some adverse comment[15] on equation (27), this equation is in agreement with Tatarski[3].

From equations (10) and (11), and except for terms of second degree in quantities $\vec{U}$, $\vec{V}$, P, R and φ, we have

$$P = -\rho_o \varphi_t \tag{28}$$

Equations (26) and (27) will be used to find the statistics of fluctuations in overpressure P when its fluctuations reach up to a few times the mean overpressure.

SECTION 4

SPATIAL TRANSFORM OF THE WAVE EQUATION

We proceed to a spatial Fourier transform of the equation
(3-27), equation (27) of Section 3. This tentative step has an initial
motivation not closely reasoned, and which is chiefly justified a pos-
teriori. We do recall, however, that Fourier transforms have been
widely useful in linear partial differential equations, a class which in-
cludes (3-27). Also, such transforms have been so convenient in analy-
sis of continuous random processes that it would be an oversight not to
try them, even if they are not a strict logical necessity.

G. B. Whitham[16] has studied the pressure pulse produced by
a supersonic airplane in a uniform atmosphere. He has shown that after
the pulse has traveled a distance of about 10,000 ft from its source, it
then has the form shown in Figure 1, where z is distance measured out-
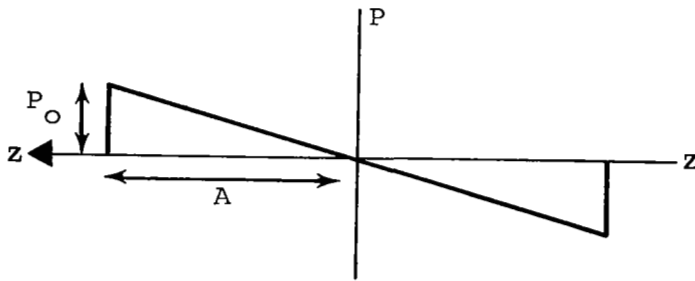ward from the airplane in a direction normal to a surface of constant

Figure 1

pulse overpressure $P = p - p_0$, and where $P_0 \sim$ lb./ft^2, $A \sim 100$ ft.
are typical numerical values. Such an N-wave is largely verified by
experiments, which however show spiky random fluctuations in P. We
will treat surfaces of equal overpressure in the incident N-wave as
plane. Actually, such surfaces are somewhat conical, but their curva-
ture is unimportant for our purposes after they have descended from
cruising altitude ($\sim 60,000$ ft.) to an altitude from below which most
turbulence effects are expected.

The above structure of the N-wave suggests a coordinate
system. We choose right-handed Cartesian coordinates so that a gen-
eral point in space $r(x, y, z)$ is referred to an origin fixed relative to
ground and above the turbulence, as in Figure 2. The direction of in-
creasing z is taken parallel to the mean propagation direction of the
N-wave, not necessarily vertical. The center plane of the N-wave, a
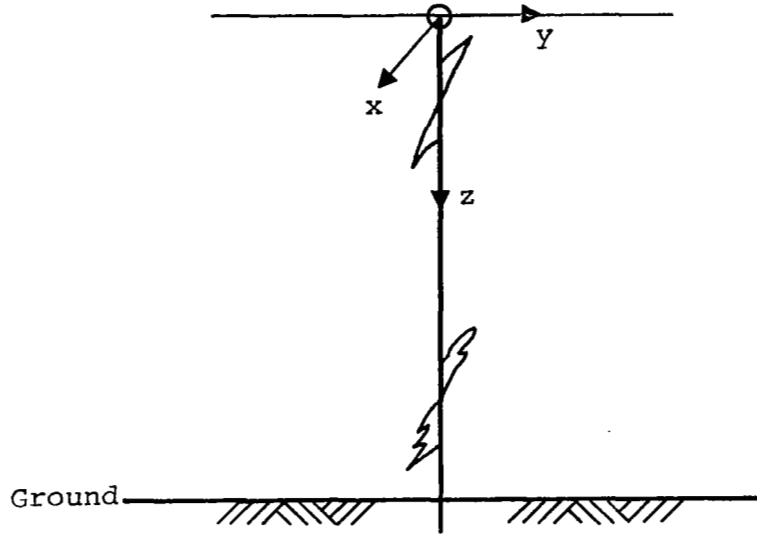plane of zero overpressure, then satisfies $z = ct$, where t is a variable
time.

Figure 2

We write the partial wave expansions

$$\varphi(t, \vec{r}) = \int H(t, \vec{s}) \exp(i\vec{s} \cdot \vec{r} - isct) \, d\vec{s} \tag{1}$$

$$\Theta(\vec{r}) = \int \theta(\vec{s}) \exp(i\vec{s} \cdot \vec{r}) \, d\vec{s} \tag{2}$$

$$\vec{V}(\vec{r}) = \int \vec{v}(\vec{s}) \exp(i\vec{s} \cdot \vec{r}) \, d\vec{s} \tag{3}$$

where

$$\vec{s} = (s_x, s_y, s_z) \ , \quad d\vec{s} = ds_x \, ds_y \, ds_z \tag{4}$$

and where the range of integration is from $-\infty$ to $+\infty$ for each of s_x, s_y and s_z. The vector $\vec{s}$ is called by custom a three dimensional spatial frequency. Equations (1), (2) and (3) define the partial wave coefficients H, θ and $\vec{v}$; and our task is to find some of the statistics of H given the statistics of θ and $\vec{v}$. We will assume that the statistics of turbulence are constant over any plane in air which is parallel to surfaces of equal overpressure in the incident N-wave. This assumption gives slab symmetry to the statistics of the scattered N-wave even

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though its sample histories may lack such symmetry. It suggests the notation

$$\vec{r} = (\vec{\eta}, z) \ , \quad \vec{s} = (\vec{\sigma}, \zeta) \ , \quad \vec{\eta} = (x, y) \ , \quad \vec{\sigma} = (s_x, s_x) \tag{5}$$

Simple consequences of relations (1), (2), (3) are

$$\Theta \nabla^2 \varphi = \int - s'^2 \theta(\vec{s}'') \ H(\vec{s}') \ \exp \left| i(\vec{s}' + \vec{s}'') \cdot \vec{r} - is'ct \right| d\vec{s}' \ d\vec{s}'' \tag{6}$$

$$\vec{V} \cdot \nabla \varphi_t = \int i\vec{s}' \cdot \vec{v}(\vec{s}'')(H_t(\vec{s}') - is'cH(\vec{s}')) \cdot$$

$$\exp[i(\vec{s}' + \vec{s}'') \cdot \vec{r} - is'ct] d\vec{s}' \ d\vec{s}'' \tag{7}$$

or, after a change in the variables of integration

$$\Theta \nabla^2 \varphi = \int - s'^2 \theta(\vec{s} - \vec{s}') \ H(\vec{s}') \ \exp(i\vec{s} \cdot \vec{r} - is'ct) \ d\vec{s}' \ d\vec{s} \tag{8}$$

$$\vec{V} \cdot \nabla \varphi_t = \int i\vec{s}' \cdot \vec{v}(\vec{s} - \vec{s}') \ (H_t(\vec{s}') - is'cH(s')) \cdot$$

$$\exp(i\vec{s} \cdot \vec{r} - is'ct) \ d\vec{s}' \ d\vec{s} \tag{9}$$

In relations (8) and (9), as sometimes in the sequel, we do not write explicitly each dependent variable of a function such as H, θ or $\vec{v}$ when such a variable has the same value throughout an equation. It is important that a sample H changes almost always by only a small percentage over any temporal wave period 1/sc of interest in the problem, because of the weakness of typical atmospheric turbulence; even though H may change by a large percentage over the long spatial wave paths of interest. Thus

$$\left| H_t(\vec{s}) \right| << sc \left| H(\vec{s}) \right| \tag{10}$$

and the first term on the right of equation (9) may be ignored.

We substitute in equation (3-27) from (8), (9) and (10). We recall that the coefficients of

$$\exp(i\vec{s}_1 \cdot \vec{r}) \ , \ \exp(i\vec{s}_2 \cdot \vec{r}), \ \ldots \ \exp(i\vec{s}_i \cdot \vec{r}), \ \ldots \ \exp(i\vec{s}_j \cdot \vec{r}); \ \vec{s}_i \neq \vec{s}_j \tag{11}$$

are all zero in any linear combination of these functions of $\vec{r}$ when this combination is itself zero over any nonzero three dimensional interval of $\vec{r}$. The above substitution thus yields

$$H_{tt} - 2icsH_t = - \int [c^2 s'^2 \theta(\vec{s} - \vec{s}') H(\vec{s}') + 2\vec{s}'\vec{s}' \cdot \vec{v}(\vec{s} - \vec{s}')H(\vec{s}')c] \cdot$$

$$\exp[i(s - s')ct]d\vec{s}' \tag{12}$$

which is the equivalent by Fourier transformation of the original wave equation. Equation (12) in our problem is sensitive to a singularity in the function $H(t, \vec{s})$ because of initial conditions. That is, at $t < 0$, or when the N-wave has not yet entered significant turbulence, we have

$$H(0, \vec{s}) = k(\zeta) \delta(\vec{\sigma}) \tag{13}$$

where

$$\delta(\vec{\sigma}) = \delta(s_x) \delta(s_y) \tag{14}$$

and $\delta(s_x)$ is a one dimensional delta function. Equivalent to equation (13) is the relation

$$\varphi(0, \vec{r}) = \int k(\zeta) \exp(iz\zeta) d\zeta \tag{15}$$

which determines $k(t, \zeta)$ at $t = 0$ in terms of the initial form of the N-wave.

In our problem, the wave detectors are pressure sensors which do not measure the wave angle of arrival α as such. An array of pressure sensors, however, does measure the spatial fluctuation in the total pressure produced by interference of all sinusoidal partial waves which comprise the scattered N-wave. The dominant scale length observed for such fluctuations is much smaller than expected with a non-turbulent atmosphere, which justifies use of the initial condition (13), since the smaller the scale length the larger the range in arrival angle α required to produce this scale length by partial wave interference. Fluctuations in a sonic boom, like those within a laser beam in air, are spatially spiky and neither significant nor easily measureable at scale lengths larger than some value L which for the sonic boom case is about 100 ft. The pressure sensors thus do not distinguish angles of arrival whose difference is less than $\Delta\alpha \sim 1/sL$, where s is some weighted average of the scalar spatial frequency in the N-wave. The detailed structure of $H(t, \vec{s})$ when σ is in the interval

$$0 < \sigma < \sigma_0 \simeq \zeta/sL \tag{16}$$

is hence ignored for all t. Henceforth we may replace the old $H(t, \vec{s})$ of equation (12) by $h(t, \vec{s}) + k(t, \zeta) \delta(\sigma)$ with the understanding that both $k(t, \zeta)$ and $h(t, \vec{s})$ are nonsingular. In the new notation, equation (12)

becomes the pair of coupled equations

$$h_{tt}(\vec{s}) - 2icsh_t(\vec{s}) = - \int [c^2 s'^2 \theta(\vec{s} - \vec{s}') + 2s'\vec{s}' \cdot \vec{v}(\vec{s} - \vec{s}')c] \cdot$$

$$[h(\vec{s}') + k(\zeta') \delta(\vec{\sigma}')] \exp[i(s - s')ct] d\vec{s}' \qquad (17.1)$$

$$[k_{tt}(\zeta) - 2icsk_t(\zeta) \delta(\vec{\sigma}) = - \int [c^2 s'^2 \theta(\vec{\zeta} - \vec{s}') + 2s'\vec{s} \cdot \vec{v}(\vec{\zeta} - \vec{s}')c] \cdot$$

$$[h(\vec{s}') + k(\zeta') \delta(\vec{\sigma}')] \exp[i(s - s')ct] d\vec{s}' \qquad (17.2)$$

Inertial and thermal effects on scattering, represented by $\vec{v}$ and θ, enter unsymmetrically in equation (17). They are symmetrical in the equation of Crow[2] for first order scattering from a weak shock. This makes our analysis less compact than his. Also, we must now seek the temporal dependence of $k(t, \zeta)$, for which there is no analog in the works of Crow[2] or Tatarski[4]. A natural intuition expects a temporal decline in $k(t, \zeta)$ which is somewhat exponential, but details remain to be explored.

The considerable structure of equations (17) makes them unattractive or infeasible to solve as such for plural scattering effects and in the numerical detail needed for comparison with experiment. We attempt no solution of equations (17). Rather, we note a property of the functions

$$h(t, \vec{s}_i) , \quad h_t(t, \vec{s}_i) , \quad k(t, \zeta_i) , \quad k_t(t, \zeta_i) \qquad (18)$$

These functions, with their complex conjugates, comprise a set of eight functions of the single running variable t, for fixed $\vec{s}_1$. For the aggregate of the fixed $\vec{s}_1$ in the $\vec{s}$-plane, these eight functions become replaced by a continuum of functions each dependent on t as the single running variable. Such a continuum comprises precisely the Hamiltonian state variables for a radiation field regarded as a mechanical system. Since $h(t, \vec{s})$ and $h_t(t, \vec{s})$ are uniformly continuous in t and s throughout the full range of st space of interest, and since $k(t, \zeta)$ and $k_t(t, \zeta)$ are similarly continuous in t and ζ; we may now replace the continuum of state variables by a countable discretum of them, one for each of the discrete infinity of small cells which fill $\vec{st}$ space. Equations (17) may now be replaced by quite analogous finite difference equations in $\vec{s}$ and t, although for brevity we will commonly avoid an explicit finite difference notation. Such discretization is realistic because it corresponds to the limited resolution of wave detectors of whatever kind. Such discretization avoids need for the theory of measure, a matter discussed by Gikhman[17].

The above introduction of discrete variables makes N-wave scattering a problem in statistical mechanics[18], which contains several routes for finding the statistics of a system without solving its sample propagation equation. The route we will use goes beyond the usual good man's intuition. Hence we devote the next section to an introduction of the needed statistical concepts.

SECTION 5
STATISTICAL SYMMETRY

Surfaces of equal pressure within the scattered N-wave are randomly corrugated. As such, they possess no strict symmetry relative to simple transformations of the space through which they move. Yet these surfaces, to a good approximation, possess much spatial symmetry of a statistical sort. The present section introduces pertinent statistical symmetries from first principles.

A random function $F(t)$ is called statistically uniform if its ensemble averages or statistics are independent of t. For example, if $F(t)$ is uniform, then

$$\langle F(t) \rangle, \quad \langle F(t)F(t + t_1) \rangle, \quad \langle F(t)F(t + t_1)F(t + t_2) \rangle, \cdots \tag{1}$$

are all independent of t. Take

$$F(t) = \int f(\omega) \exp(i\omega t) \, d\omega \tag{2}$$

which defines the partial wave coefficient $f(\omega)$ of $F(t)$. If $F(t)$ is uniform, then by (2)

$$\langle F(t) \rangle = \int \langle f(\omega) \rangle \exp(i\omega t) \, d\omega \tag{3}$$

is independent of t, which requires

$$\langle f(\omega) \rangle = \Psi(f) \, \delta(\omega) \tag{4}$$

where $\Psi(f)$ is a constant pertaining to f, and $\delta(\omega)$ is the delta function of ω. Substitution in equation (3) from (4) yields

$$\langle F(t) \rangle = \Psi(f) \tag{4.1}$$

Direct extensions of relation (4) are valid for the higher degree covariances of the sequence (1). If $F(t)$ is stationary, then by expression (2)

$$\langle F(t') F(t' + t) \rangle = \int \langle f(\omega') f(\omega'') \rangle \exp[i(\omega't' + \omega''t' + \omega''t)] \, d\omega' d\omega' \tag{5}$$

is independent of t' which requires

$$\langle f(\omega') f(\omega'') \rangle = \Psi(ff; \omega') \, \delta(\omega' + \omega'') \tag{6}$$

where $\Psi(ff; \omega')$ is some function of ω' pertaining to the product $f(\omega')$ $f(\omega'')$. Substitution in equation (5) from (6) yields

$$\langle F(t') \, F(t' + t) \rangle = \int \Psi(ff; \omega) \, \exp(i\omega t) \, d\omega \tag{4.2}$$

Equation (4.2) and its Fourier inverse are together called the Wiener-Khinchin theorem, when taken with appropriate weak restrictions on $F(t)$. Similar to equation (4.2), we have

$$\langle F(t'') \, F(t'' + t') \, F(t'' + t' + t) \rangle$$

$$= \int \Psi(fff; \omega, \omega') \, \exp[i(\omega't' + \omega t' + \omega t)] \, d\omega \, d\omega' \tag{4.3}$$

Let $F^*(t)$ be the complex conjugate of $F(t)$. A slight extension of equation (2) is then

$$F(t) = \int f(\omega) \, \exp(i\omega t) \, d\omega$$

$$F^*(t) = \int f^*(\omega) \, \exp(-i\omega t) \, d\omega \tag{7}$$

from which

$$\langle F^*(t_o) \, F(t_o + t) \rangle = \int \langle f^*(\omega') \, f(\omega'') \rangle \exp[-i(\omega' - \omega'')t_o + i\omega''t]$$

$$d\omega' \, d\omega'' \tag{8}$$

If $F(t)$ is uniform, then by equation (5)

$$\langle F^*(t_o) \, F(t_o + t) \rangle = \int \Psi(f^*f; \omega') \, \exp(i\omega t) \, d\omega \tag{9}$$

Usually, and in the sequel, when we speak of the spectral density of $F(t)$ without further qualification, we always mean $\Psi(f^*f; \omega)$, which is abbreviated as $\Psi(f; \omega)$. Some authors use the average

$$\langle \, | F(t' - t) - F(t') |^2 \rangle \tag{10}$$

and call it the structure function $D(t)$ associated with the random function $F(t)$. It is simply related to the covariance. If F is real and uniform, its structure function is

$$\int \Psi(f; \omega) \, [1 - \exp(i\omega t)] \, d\omega = D(t) \tag{11}$$

The idea of statistical uniformity is easily extended to several dimensions. For example, if $F(\vec{\eta})$ is a random function uniform over the η-plane, where $\vec{\eta}$ equals (x, y), then

$$\langle F(\overline{\eta}') \; F(\overline{\eta}' + \vec{\eta}) \rangle = \int \Psi(ff; \overline{\sigma}) \; \exp(i\overline{\sigma} \cdot \vec{\eta}) \; d\vec{\sigma} \tag{12}$$

If the statistics of $F(\vec{\eta})$ are independent of the direction of the vector $\vec{\eta}$, then $F(\vec{\eta})$ is called statistically isotropic. Then, in equation (12)

$$\Psi(ff; \vec{\sigma}) = \Psi(ff; \sigma) \tag{13}$$

where σ is the length of the vector $\vec{\sigma}$. If F is isotropic, equation (12) becomes

$$\langle F(\overline{\eta}') \; F(\overline{\eta}' + \vec{\eta}) \rangle = \int \Psi(ff; \sigma) \; 2\pi \; J_0(\sigma \eta) \; \sigma d\sigma \tag{14}$$

where

$$J_0(x) = (o \int 2\pi) \; \exp(ix \cos \theta) \; d\theta / 2\pi \tag{15}$$

is the Bessel function of order zero.

The idea of uniformity gains in structure when applied to several distinct random functions. If $F(\vec{\eta})$ and $G(\vec{\eta})$ are two random functions, each uniform over the η-plane, then

$$\langle F(\vec{\eta}') \; G(\overline{\eta}' + \overline{\eta}) \rangle = \int \Psi(fg; \vec{\sigma}) \; \exp(i\vec{\sigma} \cdot \vec{\eta}) \; d\vec{\sigma} \tag{16}$$

where the left member is called the cross covariance of F and G, and where $\Psi(fg, \omega)$ is some function of ω called the cross spectral density. Uniformity and isotropy of F are symmetry properties which express that $F(\vec{\eta})$ is statistically invariant under translations and rotations of the η-plane. If each of $F(\vec{\eta})$, $G(\vec{\eta})$, and $H(\vec{\eta})$ is uniform and isotropic over the η-plane, then as a simple extension of equation (4.3) we can show

$$\langle F(\overline{\eta}'') \; G(\overline{\eta}'' + \vec{\eta}') \; H(\overline{\eta}'' + \overline{\eta}' + \vec{\eta}) \rangle$$

$$= \int \Psi(fgh; \sigma, \overline{\sigma}', \vec{\sigma} \cdot \vec{\sigma}') \exp(i \vec{\eta}' \cdot \vec{\sigma}' + i\vec{\eta} \cdot \sigma + i\vec{\eta} \cdot \vec{\sigma}) \; d\vec{\sigma}' \; d\vec{\sigma} \tag{17}$$

Equation (17) is justified by noting that the provisos on F, G, H require the left member of equation (17) to be a function of η, η', and $\vec{\eta} \cdot \vec{\eta}'$ alone, which in turn require that $\Psi(fgh \ldots)$ be a function of σ, σ' and

$\vec{\sigma} \cdot \vec{\sigma}'$ alone. Expression (17) may have an important symmetry invariance under a group which contains the translations and rotations of the η'' plane as a proper subgroup. For example (17) may be invariant under the group which is the direct product of the group of translations and rotations of the η'' plane alone multiplied by the group of translations and rotations of the η' plane alone. If this higher symmetry is valid, then

$$\langle F(\vec{\eta}'') \, G(\vec{\eta}'' + \vec{\eta}') \, H(\vec{\eta}'' + \vec{\eta}' + \vec{\eta}) \rangle = \langle F(\vec{\eta}) \rangle \langle G(\vec{\eta}') \, H(\vec{\eta}' + \vec{\eta}) \rangle$$

$$= \langle F \rangle \int \Psi(gh;\sigma) \, \exp(i\vec{\eta} \cdot \vec{\sigma}) \, d\vec{\sigma} \tag{18}$$

and then F is statistically independent of G and H. Such higher symmetries are important for covariances of the fourth degree. For example, let $F(\vec{\eta})$, $G(\vec{\eta})$, $H(\vec{\eta})$, $J(\vec{\eta})$ be four uniform and isotropic random functions, where each of F and G is statistically independent of each of H and J. Then a slight extension of (18) is

$$\langle F(\vec{\eta}') \, G(\vec{\eta}' + \vec{\eta}) \, H(\vec{\eta}' + \vec{\eta} + \zeta') \, J(\vec{\eta}' + \vec{\eta} + \zeta' + \zeta) \rangle$$

$$= \int \Psi(fg;\sigma) \, \exp(i\vec{\eta} \cdot \vec{\sigma}) \, d\vec{\sigma} \int \Psi(hj;\sigma) \, \exp(i\vec{\zeta} \cdot \vec{\sigma}) \, d\vec{\sigma} \tag{19}$$

SECTION 6
SECULAR STATE VARIABLES FOR THE SONIC FIELD

The present section derives, from the primary wave equation (4-12), a modification which applies to certain temporal averages of the primary sonic state variables which were introduced in Section 4. In more detail, we consider the average $\underline{H}(t, \vec{s})$ of the random sample function $\underline{H}(t, \vec{s})$ over a time interval γ which is taken as at least a small integral multiple of the associated wave period $1/(cs)$. In a typical atmosphere, for which $\epsilon \ll 1$, the percentage change in almost every sample of $H(t, \vec{s})$ for any s is very small over any interval γ. We hence lose but unimportant detail in the full statistics of the fluctuations of H when we replace the sample functions H, H_t by $\underline{H}$, $\underline{H}_t$; but we thereby gain much in analytic simplicity.

The wave equation (4-12), when v is temporarily ignored, becomes

$$H_{tt}(\vec{s}) - 2ics\, H_t(\vec{s}) = - \int c^2 s'^2 \theta(\vec{s} - \vec{s}')\, H(\vec{s}')\, \exp\left[ic(s - s')t\right]\, d\vec{s}' \quad (1)$$

We will consider only the case, which applies to a turbulent atmosphere, where $\theta(\vec{s})$ is almost always highly peaked very near $s = 0$, even though $\theta(\vec{s})$ may be 0 at precisely $s = 0$. Then the spectral density $\Psi(\theta, s)$ is highly peaked near $s = 0$. In this case, the dominant contribution to the integral of equation (1) comes from values of s' such that $|s - s'| \ll s$ for all values of s of interest. And then $\exp\left[ic(s - s')t\right]$ has only a very small percentage variation during a wave period $1/(cs)$. But equation (1) shows that $H_{tt}(\vec{s})$ and $H_t(\vec{s})$ have large percentage changes during the interval $1/(cs)$. Although H_t itself undergoes large variations during a time interval $1/(cs)$, its contributions to changes in $H(t, \vec{s})$ during such an interval is small compared to H. That is

$$t\int t + 1/cs)\, H_t(t', \vec{s})\, dt' \ll H(t, \vec{s}) \quad (2)$$

is true almost always. The lower and upper limits of integration in equation (2) are indicated as t and $t + [1/cs]$. In summary, we may picture $H(\vec{s})$ as in Figure 3. Here the regular term $h(\vec{s})$ has an instantaneous ensemble average $\langle h(t, \vec{s}) \rangle$ which is 0 for all t and for all those $\vec{s}$ for which σ is greater than in any of the significant partial waves which comprise the well-collimated incident N-wave. The singular term $k(t, \zeta)\, \delta(\vec{\sigma})$ represents any partial wave coefficient for which $k(0, \zeta)$ is significantly nonzero in the incident wave packet. Clearly $\langle k(t, \zeta) \rangle \neq 0$ in general. The time after which the expected energy in the incident partial waves is less than about $1/e$ of its value at $t = 0$ will be called the saturation time, or t_s.
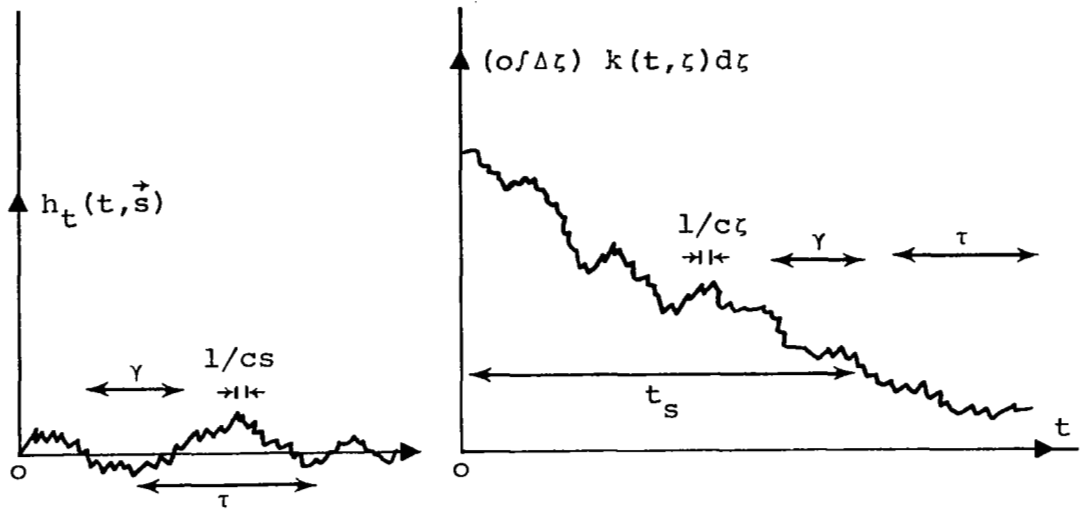
Figure 3

The left member of equation (1) possesses a simple integrating factor $\exp(-2icst)$, which allows a strict formal integration of this equation, namely

$$H_t(t, \vec{s}) = \exp(2icst) \left[-\int c^2 s'^2 \theta(\vec{s} - \vec{s}') (0\int t) H(t'\vec{s}') \exp[-ic(s + s')t'] d\vec{s}' dt' \right.$$

$$\left. + H_t(0, s) \right] \tag{3}$$

where $H_t(0, \vec{s})$ is the possibly nonzero initial value of $H_t(t, \vec{s})$. We will use the integration formula

$$(0\int t) Q(t') \exp(icst') dt' = [Q(t) \exp(icst) - Q(0)]/(ics) \tag{4}$$

which is valid provided the percentage change in the function $Q(t)$ is very small compared to 100 percent over every time interval of duration $1/(cs)$. When formula (4) is used to evaluate the inner integral of equation (3), the latter becomes

$$H_t(t, \vec{s}) = \int c^2 s'^2 \theta(\vec{s} - \vec{s}') H(t, \vec{s}') \exp[ic(s - s')t]/[ic(s + s')]$$

$$+ H_t(0, \vec{s}) \exp(2icst) \tag{5}$$

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The average of equation (5) over a time interval $\gamma \gg 1/cs$, with a center at time t, yields

$$\underline{H}_t(t,\vec{s}) = \int c^2 s'^2\, \theta(\vec{s} - \vec{s}')\, \underline{H}(t,\vec{s}')\, \exp[ic(s-s')t]/[ic(s+s')]\, ds' \quad (6)$$

since $\vec{s} - \vec{s}' \ll s$ as explained above, and since the γ-averaging of the final term of equation (5) yields 0 as γ approaches infinity. Equation (6) is the propagation equation for a random process whose state functions are H(t, s), H*(t, s), a separate pair of such functions of time alone for each of the distinct values of s. It is this random process to which we will apply the concepts of Section 5. Since equation (6), as distinct from (1), does not require rapid oscillations of its state variables; equation (6) is called the secular differential equation for the wave packet, while H and H* are the secular state variables. It is important that the secular equation is of lower differential order than the strict wave equation (1), a matter with close analogies in planetary mechanics.

A formal and strict time integration of equation (3) yields, with use of formula (4)

$$H(t,\vec{s}) = ({}_0\!\!\int^t)\, c^2 s'^2 \theta(\vec{s} - \vec{s}')\, H(t,\vec{s}')\, \exp \big| ic(s - s')t' \big| / \big| ic(s + s') \big|\, dt'\, d\vec{s}'$$

$$+ H_t(0,\vec{s})[\exp(2icst) - 1]/(2ics) + H(0,\vec{s}) \quad (7)$$

where $H(0,\vec{s})$ is the initial value of $H(t,\vec{s})$. The γ-average of equation (7) eliminates the term

$$H_t(0,\vec{s})\, \exp(2icst)/(2ics) \quad (8)$$

and shows that the nonsecular wave equation (1) is consistent with a meaningful γ-average for H in the case of weak turbulence and a form of $\Psi(\theta, s)$ which is very peaked near s = 0.

An an application of the basic propagation equation (6), we proceed to study the variation of $\underline{k}(t,\vec{s})$ with time. We use the brackets $\langle \ldots \rangle$ to denote the ensemble average of their contents. When $\sigma = 0$, the ensemble average of equation (6) becomes

$$\langle \underline{k}(t,\zeta)\rangle_t\, \delta(\vec{\sigma}) = \int cs'^2\, \langle \theta(\vec{\zeta} - \vec{s}')\, \underline{H}(t,\vec{s}')\rangle\, \exp[ic(\zeta - s')t]/$$
$$\big| i(\zeta + s')\big|\, d\vec{s}' \quad (9)$$

Equation (9) requires that the covariance $\langle \theta(\vec{s} - \vec{s}')\, \underline{H}(t,\vec{s})\rangle$ be singular and contain the factor $\delta(\vec{\sigma})$ else its left member is $\vec{0}$, which is against hypothesis. To study this covariance, multiply equation (7) by $\theta(\vec{\zeta} - s)$, average over γ and ensemble average, which all yields

$$\langle \theta(\vec{\zeta} - \vec{s}') \, \underline{H}(t, s') \rangle = (0 \int t) c^2 s''^2 \langle \theta(\vec{\zeta} - \vec{s}') \, \theta(\vec{s}' - \vec{s}'') \, H(t, \vec{s}'') \rangle$$

$$\exp[ic(s' - s'')t'] / [ic(s' + s'')] \, d\vec{s}'' \tag{10}$$

When $\epsilon \ll 1$, then the correlation between any $\theta(\vec{s})$ and any $H(t, \vec{s}')$ is much less than 1 when

$$t \gg t_c \equiv L_\theta / c \tag{11}$$

where L_θ denotes the correlation length of the random function $\theta(\vec{r})$. By definition

$$L_\theta = (-\infty \int \infty) \langle \theta(\vec{r} + \vec{u}z) \, \theta(\vec{r}) \rangle \, dz / \langle \theta(\vec{r}) \theta(\vec{r}) \rangle \tag{12}$$

where $\vec{u}$ is a dimensionless unit vector in any direction.

The above small correlation between θ and H implies

$$\langle \theta(\vec{\zeta} - \vec{s}') \, \theta(\vec{s} - \vec{s}'') \, H(t, \vec{s}'') \rangle = \langle \theta(\vec{\zeta} - \vec{s}') \, \theta(\vec{s}' - \vec{s}'') \rangle \langle \psi(t, \vec{s}'') \rangle + o(\epsilon^3)$$

$$\tag{13}$$

The statistical uniformity of $\theta(\vec{r})$ implies

$$\langle \theta(\vec{\delta} - \vec{s}') \, \theta(\vec{s}' - \vec{s}'') \rangle = \Psi(\theta; \vec{\zeta} - \vec{s}') \, \delta(\vec{\zeta} - \vec{s}'') \tag{14}$$

Substituting in equation (10) from (13) and (14) yields

$$\langle \theta(\vec{\zeta} - \vec{s}') \, \underline{H}(t, \vec{s}') \rangle = c^2 \zeta^2 \Psi(\theta; \vec{\zeta} - \vec{s}') (0 \int t) \langle \underline{k}(t, \zeta) \rangle$$

$$\exp |ic(\vec{s} - \vec{\zeta}) t'| / [ic(\zeta + s')] \, dt' \, \delta(\vec{\sigma}) \tag{15}$$

The limits of integration on the right of equation (15) properly denote integration from the latest instant of the first γ integral following the time origin to the latest instance of the γ interval in which lies the epoch of observation. This is because $\underline{k}$ is itself an average over a γ interval. With this understanding, the trial solution

$$\langle \underline{k}(t, \zeta) \rangle = \langle \underline{k}(0, \zeta) \rangle \exp(-\alpha t) \tag{16}$$

reduces equation (15) to

$$\langle \theta(\zeta - s') \underline{H}(t, s') \rangle = \frac{c\zeta^2 \Psi(\theta; \zeta - s') \exp\left[ic(s' - \zeta)t - at\right]}{i(\zeta + s') \left[ic(s' - \zeta) - a\right]} \cdot \underline{k}(0, \zeta) \delta(\vec{\sigma})$$

(17)

correct in its constant of integration. Substituting in equation (9) from (16) and (17) yields at once

$$a = \int \frac{c^2 s'^2 \zeta \Psi(\theta; \vec{\zeta} - \vec{s'}) \, d\vec{s'}}{(\zeta + s')^2 \left[ic(s' - \zeta) - a\right]}$$

(18)

which is an integral equation for determining the function $a(\zeta)$.

The function $a(\zeta)$ which satisfies equation (18) is imaginary valued when the chief contributions to the integral (18) come from values of s' for which

$$\left| c(s' - \zeta) \right| \gg \left| a \right|$$

(19)

which is the case for values of $\Psi(\theta, \vec{s})$ typical of near ground atmospheric turbulence. However, this does not imply a lack of energy loss in the incident beam.

SECTION 7

PROPAGATION OF THE SONIC SPECTRAL DENSITY

In the present section, we obtain a key expression for the rate of change of the full spectral density of the velocity potential, for the case of a packet of plane waves incident on uniform turbulence. The discussion, based on the secular propagation equation (6-6), leads to moment factorization as a device for closing the hierarchy of moment equations.

We begin by considering the equation

$$\langle \underline{H}^*(t, \vec{s})\ \underline{H}(t, \vec{s})\rangle_t = \langle \underline{H}^*(t, \vec{s})\ \underline{H}_t(t, \vec{s}) + \underline{H}^*(t, \vec{s})\ \underline{H}(t, \vec{s})$$

$$+ \{\underline{H}^*(t, \vec{s})\ \underline{H}(t, \vec{s})\}\rangle \tag{1}$$

where $\underline{H}$ denotes the Fourier coefficient of the velocity potential time averaged over several wave periods. Henceforth, H alone, without the underbar, will denote such a time average. Even after the averaging over a few wave periods, H is still a random quantity. The brackets $\langle ... \rangle$ will denote a further averaging over a time τ which is several eddy traverse times each of duration $1/(cs)$, and also ensemble averaging. The first two terms of equation (1), after substitution from (6-6), become

$$\int c^2 s'^2\ \langle \theta(\vec{s} - \vec{s'})\ H^*(t, \vec{s})\ H(t, \vec{s'})\ \exp[ic(s - s')t]\rangle / [ic(s + s')]\ d\vec{s'} + cc \tag{2}$$

where the symbol cc denotes the complex conjugate of the preceding term. Similarly, the final term of equation (1) becomes by Section A6

$$(t \iint t + \tau) \iint c^2 s'^2 s''^2\ \langle \theta^*(\vec{s} - \vec{s'})\ \theta(\vec{s} - \vec{s''})\ H^*(t', \vec{s'})\ H(t'', \vec{s''})$$

$$\exp[-ic(s - s')\ t' + ic(s - s'')\ t'']\rangle / [(s + s')(s + s'')]\ d\vec{s'}\ d\vec{s''}\ dt'\ dt''/\tau \tag{3}$$

Since $H(t, \vec{s})$ is only weakly correlated to $\theta(\vec{s'})$ when $t \gg \tau$ and $\epsilon \ll 1$ for all t, s and s', the average within expression (3) may be approximately factored as

$$\langle \theta^*(\vec{s} - \vec{s'})\ \theta(\vec{s} - \vec{s''})\ H^*(t', \vec{s'})\ H(t'', \vec{s''})\ \exp[-ic(s - s')t' + ic(s - s'')t'']\rangle$$

$$= \langle \theta^*(\vec{s} - \vec{s'})\ \theta(\vec{s} - \vec{s''})\rangle\ \langle H^*(t', \vec{s'})\ H(t'', \vec{s''})\ \exp[-ic(s - s')t' + ic(s - s'')t'']\rangle$$

$$+ o(\epsilon^3) \tag{4}$$

where ϵ is the root mean square of the largest $\theta(\vec{s})$. Our intent is to keep all terms through $o(\epsilon^2)$ in equation (1), or all terms which arise from first and second order scattering. The neglect of terms $o(\epsilon^3)$ is

justified in evaluating $\langle H*H \rangle_t$ when $\epsilon \ll 1$, since the subsequent time integration of $\langle H*H \rangle$ will then account for all significant contributions to $\langle H*H \rangle$ from all orders of scattering. The neglect of terms $o(\epsilon^3)$, $o(\epsilon^4)$ etc. in evaluating $\langle H*H \rangle$ through ensemble averaging following the computation of sample histories $H(t, \vec{s})$ is not similarly justified. These last two remarks contain the core of our method.

To explore the average within expression (2), we consider its time derivative, namely

$$\langle \theta(\vec{s} - \vec{s}') H_t*(t, \vec{s}) H(t, \vec{s}') \exp |ic(s - s')t| \rangle$$

$$+ \langle \theta(\vec{s} - \vec{s}') H*(t, \vec{s}) H_t(t, \vec{s}') \exp |ic(s - s')t| \rangle$$

$$+ \langle \theta(\vec{s} - \vec{s}') H*(t, \vec{s}) H(t, \vec{s}') ic(s-s') \exp |ic(s - s')t| \rangle$$

$$(5)$$

After substituting from equation (6-6) in the first two terms of (5), they become

$$\int \frac{c s''^2}{i(s' + s'')} \langle \exp [ic(s - s'')t] \theta(\vec{s} - \vec{s}') \theta(\vec{s}' - \vec{s}'') H*(\vec{s}) H(\vec{s}'') \rangle d\vec{s}''$$

$$- \int \frac{c s''^2}{i(s + s'')} \langle \theta*(\vec{s} - \vec{s}'') \theta(\vec{s} - \vec{s}') H*(\vec{s}'') H(\vec{s}') \rangle d\vec{s}''$$

$$(6)$$

Factoring moments as in equation (4) yields

$$\langle \exp [ic(s - s'') t] \theta(\vec{s} - \vec{s}') \theta(\vec{s} - \vec{s}'') H*(\vec{s}) H(\vec{s}'') \rangle$$

$$= \langle \theta(\vec{s} - \vec{s}') \theta(\vec{s} - \vec{s}'') \rangle \langle \exp [ic(s - s'')t] H*(\vec{s}) H(\vec{s}'') \rangle + o(\epsilon^3)$$

$$(7)$$

and

$$\langle \theta*(\vec{s} - \vec{s}'') \theta(\vec{s} - \vec{s}') H*(\vec{s}'') H(\vec{s}') \rangle = \langle \theta*(\vec{s} - \vec{s}'') \theta(\vec{s} - \vec{s}') \rangle \langle H*(\vec{s}'') H(\vec{s}') \rangle$$

$$+ o(\epsilon^3) \qquad (8)$$

By the statistical uniformity and isotropy of turbulence in our model, we have

$$\langle \theta(\vec{s} - \vec{s}') \theta(\vec{s} - \vec{s}'') \rangle = \Psi(\theta; \vec{s} - \vec{s}') \delta(\vec{s} - \vec{s}'') \qquad (9)$$

$$\langle \theta*(\vec{s} - \vec{s}'') \; \theta(\vec{s} - \vec{s}') \rangle = \Psi(\theta; \vec{s} - \vec{s}') \; \delta(\vec{s}' - \vec{s}'') \tag{10}$$

where $\Psi(\theta; \vec{s})$ is the spectral density of the dimensionless temperature fluctuation $\Theta(\vec{r})$.

Substituting in expression (6) from (7) through (10) yields

$$\frac{cs}{2i} \; \Psi(\theta; \vec{s} - \vec{s}') \langle H*(\vec{s}) H(\vec{s}) \rangle - \frac{cs'^2}{i(s + s')} \; \Psi(\theta; \vec{s} - \vec{s}') \langle H*(\vec{s}') H(\vec{s}') \rangle \tag{11}$$

Since $\theta(\vec{s})$ in turbulence is highly peaked near $s = 0$, the dominant contribution to the integral (2) is from values of s' such that $|\vec{s} - \vec{s}'|$ is only a few times $2\pi/L_2$ where L_2 is the largest important eddy diameter. $H(t, \vec{s})$, for all s and $t >> L_2/2\pi c$, varies by a very small percentage over an eddy traverse time when $\epsilon << 1$. Hence the final term of expression (5) is negligible compared to its first two terms, which equal expression (11). Substituting from equation (10) in (4) and then from (4) in (3) shows expression (3) to be $o(\epsilon^2 \tau/t_\bullet)$ where $t_\bullet$ is a time interval over which $H(t, s)$ changes by a large percentage, for any s. Hence we ignore expression (3)

The above analysis of the terms of quantity (1) followed by direct substitution in (1) and time differentiation of (1), both simple steps, yields

$$\langle H*(t, \vec{s}) \; H(t, \vec{s}) \rangle_{tt} = \int \frac{c^2 s'^2}{s - s'} \; \Psi(\theta; \vec{s} - \vec{s}')$$

$$\left[\frac{s'^2}{s + s'} \; \langle H*(t, \vec{s}') H(t, \vec{s}') \rangle - \frac{s}{2} \; \langle H*(t, \vec{s}) H(t, \vec{s}) \rangle \right] d\vec{s}' + cc + o(\epsilon^3) \tag{12}$$

Equation (12) is a basic propagation equation for the second moments of a wave field in weak turbulence. In the sonic boom problem with incident plane waves, the function $H(t, \vec{s})$ is discontinuous in s in the manner discussed in Section 4. We thus need a slight generalization of equation (12), obtained in a manner almost a duplicate of that used to obtain (12). This generalization is

$$\langle H*(\vec{s_1}) \; H(\vec{s_2}) \rangle_{tt}$$

$$= - \int \frac{c^2 s_2^2}{2 s_1 (s_2 + s')} \; \exp[ic(s_1 - s_2)t] \; \Psi(\theta; \vec{s_2} - \vec{s}') \; \langle H*(\vec{s_1}) H(\vec{s_2}) \rangle d\vec{s}'$$

$$+ \int \frac{|\vec{s_1} - \vec{s_2} + \vec{s}'|^2}{(s_1 + s')(s_1 + |\vec{s_1} - \vec{s_2} + \vec{s}'|)} \; \Psi(\theta; \vec{s_2} - \vec{s}') \; \langle H*(\vec{s_1} - \vec{s_2} + \vec{s}') H(\vec{s}') \rangle d\vec{s}'$$

$$\tag{13}$$

$$+ (1*2)$$

where the symbol (1*2) denotes the complex conjugate of the two preceding terms with their subscripts 1 and 2 interchanged. In equation (13) we have chosen $c(s_1 - s_2)$ to be much less than any inverse time of interest in the problem, in preparation for eventually allowing $\vec{s}_1$ to approach $\vec{s}_2$

By Section 4 we have

$$\langle H^*(\vec{s}_1) \, H(\vec{s}_2) \rangle = \langle h^*(\vec{s}_1) \, h(\vec{s}_2) \rangle + \langle k^*(\zeta_1) \, k(\zeta_1) \rangle \, \delta(\vec{\sigma}_1) \, \delta(\vec{\sigma}_2) \qquad (14)$$

where we ignore terms containing factors $\langle hk^* \rangle$ or $\langle h^*k \rangle$, since such factors are 0 by the statistical symmetry of the turbulence and the initial conditions of the problem. The same symmetry requires that the phase of each partial wave coefficient $h(t, \vec{s})$ be between 0 and 2π with uniform probability. That is

$$\langle h^*(\vec{s}_1) \, h(\vec{s}_2) \rangle = \Psi(h; \zeta, \sigma) \, \delta(\vec{s}_1 - \vec{s}_2) \qquad (15)$$

The spectral density $\Psi(h; \zeta, \sigma)$ is independent of the direction of the transverse component σ of $\vec{s} \equiv (\sigma, \zeta)$, but may depend on the direction of $\vec{s}$. Substituting in equation (13) from (14) and (15) yields

$$\Psi_{tt}(h; \zeta_1, \zeta_2, \sigma_1) \, \delta(\vec{\sigma}_1 - \vec{\sigma}_2) + \langle k^*(\zeta_1) \, k(\zeta_2) \rangle_{tt} \, \delta(\vec{\sigma}_1) \, \delta(\vec{\sigma}_2)$$

$$= - \int \frac{c^2 s_2^2 s'^2}{2 s_1 (s_2 + s')} \, \exp[ic(s_1 - s_2)t] \, \Psi(\theta; \vec{s}_2 - \vec{s}') \, \cdot$$

$$[\Psi(h; \zeta_1, \zeta_2, \sigma_1) \, \delta(\vec{\sigma}_1 - \vec{\sigma}_2) + \langle k^*(\zeta_1) k(\zeta_2) \rangle \, \delta(\vec{\sigma}_1) \delta(\vec{\sigma}_2)] \, d\vec{s}'$$

$$+ \int \frac{c^2 |\vec{s}_1 - \vec{s}_2 + \vec{s}'|^2 s'^2}{(s_1 + s')(s_1 + |\vec{s}_1 - \vec{s}_2 + \vec{s}'|)} \, \Psi(\theta; \vec{s}_2 - \vec{s}') \, \cdot$$

$$[\Psi(h; \zeta_1 - \zeta_2 - \zeta', \zeta', \sigma') \, \delta(\vec{\sigma}_1 - \vec{\sigma}_2) +$$

$$\langle k^*(\zeta_1 - \zeta_2 - s') k(\zeta') \rangle \, \delta(\vec{\sigma}_1 - \vec{\sigma}_2 - \vec{\sigma}') \, \delta(\vec{\sigma}')] \, d\vec{s}' + (1*2)$$

$$(16)$$

The compatibility of singularities in equation (16) requires that its coefficients of $\delta(\vec{\sigma}_1) \, \delta(\vec{\sigma}_2)$ in the two members of the equation be equal. Equating such coefficients and then letting $\vec{s}_1$ and $\vec{s}_2$ approach $\vec{s}$ yields

$$\langle k^*(\zeta) k(\zeta) \rangle_{tt} = - \int \frac{c^2 s \, s'^2}{s + s'} \, \Psi(\theta; \vec{s} - \vec{s}') \, d\vec{s}' \, \langle k^*(\zeta) k(\zeta) \rangle \qquad (17)$$

Similarly equating coefficients of $\delta(\vec{\sigma}_1 - \vec{\sigma}_2)$ in equation (16), now with $\vec{s}_1 = (\vec{\sigma}, \zeta_1)$ and $\vec{s}_2 = (\vec{\sigma}, \zeta_2)$, yields

$$\Psi_{tt}(h; \zeta_1, \zeta_2, \sigma) = - \int \frac{c^2 s\, s'^2}{2(s + s')}\ \Psi(\theta; \vec{s}_1 - \vec{s}')\, d\vec{s}'\ \Psi(h; \zeta_1, \zeta_2, \sigma)$$

$$+ \int \frac{c_2\, s'^4}{(s + s')^2}\ \Psi(\theta; \vec{s}_1 - \vec{s}')\ \Psi(h; \zeta_1 - \zeta_2 - \zeta', \zeta', \sigma')\, d\vec{s}'$$

$$+ \int \frac{c^2 |\vec{\zeta}_1 - \vec{\zeta}_2 + \vec{s}'|^2\, s'^2}{(s_1 + s')(s_1 + |\vec{s}_1 - \vec{s}_2 + \vec{s}'|)}\ \Psi(\theta; \vec{s}_2 - \vec{\zeta}')\ \langle k^*(\zeta_1 - \zeta_2 - \zeta') k(\zeta') \rangle\, d\zeta'$$

$$+ (1*2)$$

$$(18)$$

The terms of equation (18) are compatible in their singularities, in consideration that expression (18) has meaning only after integration over a volume of $\vec{\sigma}\, \zeta_1\, \zeta_2$ space. Similar consideration of course does not make all the terms of equation (16) compatible. All contributions to quantity (17) are from double scattering within a few eddy traverse times, as are all contributions to the first term on the right of equation (18). The second term of the right of (18) arises from single scattering from one to another of those partial waves which are 0 at $t = 0$. The third term of equation (18) accounts for single scattering from incident partial waves to others. The first Born approximation, commonly used in problems of the present type, is equivalent to ignoring equation (17), ignoring the first two terms of (18) and their analogues in the three terms of (1*2), and to replacing $\langle k^*(t, \zeta_1 - \zeta_2 - \zeta')\, k(t, \zeta') \rangle$ by $\langle k^*(0, \zeta_1 - \zeta_2 - \zeta')\, k(0, \zeta') \rangle$ in the third term of (18).

SECTION 8

SATURATION OF WAVE FLUCTUATIONS

The present section derives a simple expression for the wave pathlength L_s in turbulence beyond which the mean square of wave fluctuations almost ceases to grow with increases in pathlength.

With a slight abbreviation, equation (7-17) becomes

$$\langle k^*(t, \zeta)\, k(t, \zeta) \rangle_{tt} = - m_\theta^2 \langle k^*(t, \zeta)\, k(t, \zeta) \rangle \qquad (1)$$

where

$$m_\theta^2 = \int \frac{c^2 z\, s'^2}{z + s'} \Psi(\theta; \vec{z} - \vec{s}')\, d\vec{s}' \qquad (2)$$

The solution of equation (1) is

$$\langle k^*(t, \zeta)\, k(t, \zeta) \rangle = \langle k^*(0, \zeta)\, k(0, \zeta) \rangle \cos m_\theta \qquad (3)$$

In physical interpretation, equation (3) introduces a temporary paradox. The left member of (3) is commonly viewed as proportional to the energy per unit range of the spatial frequency ζ. A natural intuition then indicates that this left member should depend on time only through a factor exp $(-\alpha t)$, where α is some real valued quantity independent of time. By this viewpoint, taken literally, there is no understanding of the above time factor cos (m_θ). The resolution of the paradox rests on recognizing that there is no general validity in the idea of a well defined energy density in frequency space, however useful this concept may be in some cases. Strictly, energy is a property of the entire wave packet. The energy of the unscattered portion of the packet is an integral which contains the quantity (3) in its integrand. As time increases, the value of this integral does decline in somewhat the manner indicated by intuition because of the increasingly rapid oscillations of cos (m_θ) relative to changes in ζ. The above paradox occurs also, in slightly different form, in the simpler problem of Appendix A4.

The attenuation from true absorption, of the partial waves within a small frequency range of a beam, is indeed expressed by a factor exp $(-\alpha t)$ with real and positive α. Many analysts will agree that beam attenuation from scattering alone is never strictly expressible in such a manner. However, several authors[19,20,21] have derived approximate formulae for the above real α for the problem of incident partial waves attenuated only by scattering in turbulence. Caution in the use of such alphas has been indicated in the review by J. B. Keller[22].

The effective attenuation time for the partial waves which comprise the incident packet of plane waves may be obtained by considering the function

$$\Delta m = (dm/d\zeta) \, \Delta\zeta \qquad (4)$$

where $\Delta\zeta$ is the bandwidth of the spatial frequencies in the incident packet. When the time t in equation (3) becomes great enough so that $t\Delta m \cong 1$; then the oscillations of cos (mt) relative to ζ reduce the value of the integral of (3) over the range $\Delta\zeta$, and by a factor of about $1/e$ relative to the value of this integral at $t = 0$. Hence $1/\Delta m$ is the attenuation time for the partial waves in a narrow banded incident packet; and $dm/d\zeta$ is the inverse attenuation time per unit bandwidth. The wave pathlength covered during such an attenuation time is

$$L_s \equiv c/\Delta m \qquad (5)$$

There is reason to call L_s the saturation length for fluctuations of the packet produced by its passage through turbulence. For when the pathlength in turbulence is less than L_s, then the partial waves scattered from the incident partial waves grow at the expense of energy in the incident waves. But when this wave path is greater than L_s, then the scattered waves can change chiefly only by interchange of sonic energy among themselves, since their source of energy has then chiefly vanished. We identify L_s with a saturation length measured by Gracheva and Gurvich[5] in optical beams which have traversed a pathlength in turbulence of 1 km or more. The analysis leading to equation (3) applies to these optical beams if we understand $\Theta(r)$ to be twice the fluctuation in the index of refraction of the atmosphere. The above analysis is not clearly valid unless

$$\zeta \gg 2\pi/L_2 \qquad (6)$$

a restriction not satisfied by the lower frequency partial waves which comprise the incident N-wave. However, as S. C. Crow has emphasized, it is only the frequencies for which (6) is satisfied which produce the higher peaks in the fluctuations of boom overpressure at ground level. When the incident wave packet is broad banded, or when $\Delta\zeta$ is not very small compared to a middle frequency for the packet, as in the sonic boom case, then a single saturation length for the entire packet does not exist. In this case, because of the dependence of $m_\theta(\zeta)$ on ζ, small frequency intervals centered at the larger values of ζ become saturated before those centered at lower values. When relation (6) is satisfied, then (2) reduces to

$$m_\theta^2(\zeta) = c^2 \zeta^2 \int \Psi(\theta, \vec{\zeta} - \vec{s}') \, d\vec{s}' \qquad (7)$$

In the high frequency approximation, and when $\Psi(\theta;\vec{s}) = \Psi_0 s^{-11/3}$ as in homogeneous turbulence, then the integral of equation (2) becomes elementary, and this equation reduces to

$$m_\theta^2(\zeta) = 3(2\pi)^{1/3} c^2 \zeta^2 \Psi_0 L_2^{2/3} \, , \quad L_2 \simeq 100 \text{ meter} \tag{8}$$

Here Ψ_0 is a quantity which is slightly weather dependent but independent of s.

A natural hope has been that the chief features of wave fluctuations in turbulence might be independent of the large eddy diameter L_2, which is not the best known parameter of turbulence in the atmosphere. In several important phenomena involving limited wave pathlengths this hope has been realized; although it does not seem justified at the larger pathlengths at which saturation occurs.

SECTION 9

SATURATION FROM RANDOM VELOCITIES

The present section treats velocity effects in about the same way the previous section treated thermal effects. In addition it briefly reviews knowledge of the spectral density of turbulence. It concludes by listing the Fourier transform of the incident N-wave.

If we temporarily set $\theta(\vec{s}) = 0$ for all $\vec{s}$, then equation (4-12) becomes

$$H_{tt} - 2icsH_t = - \int 2cs'\vec{s}' \cdot \vec{v}(\vec{s} - \vec{s}') \, H(\vec{s}') \exp\left| ic(s - s')t \right| d\vec{s}' \tag{1}$$

which is what equation (6-1) becomes after replacement of the first of the two expressions

$$c^2 s'^2 \theta(\vec{s} - \vec{s}') \, , \quad 2cs'\vec{s}' \cdot \vec{v}(\vec{s} - \vec{s}') \tag{2}$$

by the second. With this replacement, the equations of Section 8 apply without further change to the analysis of velocity effects, down to equation (6-14). The quantity (6-14), after the replacement (2), becomes

$$4\langle \vec{s}' \cdot \vec{v}(\vec{s} - \vec{s}') \, \vec{s}'' \cdot \vec{v}(\vec{s}' - \vec{s}'') \rangle / (c^2 s's'') \tag{3}$$

In homogeneous turbulence, components of velocity in perpendicular directions are uncorrelated, which requires

$$\langle v_y v_z \rangle = \langle v_z v_x \rangle = \langle v_x v_y \rangle = 0 \tag{4}$$

And when the turbulence is isotropic

$$\langle v_x v_x \rangle = \langle v_y v_y \rangle = \langle v_z v_z \rangle \tag{5}$$

Thus the average within quantity (3) is

$$\langle s' \cdot v' s'' \cdot v'' \rangle = \langle s'_x s''_x v'_x v''_x + s'_y s''_y v'_y v''_y + s'_z s''_z v'_z v''_z \rangle$$

$$= \vec{s}' \cdot \vec{s}'' \, v_x(\vec{s} - \vec{s}') v_x(\vec{s}' - \vec{s}'')$$

$$= s'^2 \Psi(v; \vec{s} - \vec{s}') \, \delta(\vec{s} - \vec{s}'') \tag{6}$$

where $\Psi(\vec{v}; s)$ is the spatial spectral density of a single Cartesian component of $\vec{v}$.

Substituting from equation (6) in (3) yields for expression (3)

$$4 \Psi(v; \vec{s} - \vec{s}') \, \delta(\vec{s} - \vec{s}'')/c^2 \tag{7}$$

Comparing expressions (6-14) and (7) shows that $\Psi(\theta; \vec{s} - \vec{s}')$ need be replaced simply by $4\Psi(v, s - s')/c^2$ in order for the result (8-1) to apply to velocity effects. In more detail, for the case of wave fluctuations produced only by small random velocities in the medium through which the waves move, we have

$$\langle |k(t, \zeta)|^2 \rangle_{tt} = -m_v^2 \langle |k(t, \zeta)|^2 \rangle \tag{8}$$

with the definition

$$m_v^2(\zeta) = \int \frac{c^2 s'^2 \zeta}{(\zeta + s')} \, \frac{4 \Psi(v, \vec{s} - \vec{s}')}{c^2} \, d\vec{s}' \tag{9}$$

In homogeneous turbulence, temperature and velocity fluctuations have zero cross correlation, or $\langle \theta v \rangle = 0$. Thus, for the case of temperature and velocity fluctuations acting together, we have

$$\langle |k(t, \zeta)|^2 \rangle_{tt} = \langle |k(0, \zeta)|^2 \rangle \cos(mt) \tag{10}$$

where

$$m^2 = m_\theta^2 + m_v^2 \qquad (10.1)$$

When $|\zeta|$ is very large compared to $2\pi/L_2$, then

$$m(\zeta) = n\zeta \qquad (10.2)$$

where

$$n^2 = c^2 \int \Psi(\theta; \vec{s} - \vec{s}') \, d\vec{s}' + 2 \int \Psi(v; \vec{s} - \vec{s}') \, d\vec{s}' \qquad (10.3)$$

We now consider the form of the functions $\Psi(v;\vec{s})$ and $\Psi(\theta;\vec{s})$. A famous dimensional argument by Kolmogorov[3, 23] shows that in homogeneous turbulence

$$\Psi(v;\vec{s}) = 0.041\, C_v^2 s^{-11/3} \qquad (11)$$

where C_v^2 is independent of s and is called the strength of the turbulence. The coefficient C_v^2 is proportional to $\nu v_1^2/L_1$ where ν is viscosity and v_1 the mean scalar velocity in the smallest eddies present, whose diameter is denoted by L_1. Relation (11), now checked by several experimenters, is widely valid provided

$$2\pi/L_2 < s < 2\pi/L_1 \qquad (12)$$

The inertial subrange of eddy diameters L is given by (12) with $s = 2\pi/L$. By an extension of relation (11), it can be shown that

$$\Psi(\theta;\vec{s}) = .033\, C_\theta^2 s^{-11/3} \qquad (13)$$

where C_θ^2 is called the strength of thermal fluctuations, or the strength of thermal turbulence. The numerical constants 0.041 and 0.033 in equations (11) and (13) are in part conventional, but are needed for the above customary definitions of C_θ and C_v. The ratio C_θ/C_v is somewhat weather dependent; but at least not too far from the earth's surface it is typical that

$$C_\theta \simeq C_v/c \qquad (14)$$

a relation which is based on sonic experiments by M. A. Kallistratova[24], and which we use in what follows.

A procedure for finding typical values of C_θ perhaps as good as any now available, involves the relation between C_θ and fluctuations in beams of light. The spectral density for atmospheric fluctuations in the optical index of refraction may be written[4] as

$$\Psi(n;\vec{s}) = 0.033 \; C_n^2 \; s^{-11/3} \tag{15}$$

It can be shown[4] that

$$C_n = 79 \cdot 10^{-6} p \; C_\theta/T \tag{16}$$

where T is temperature in degrees Kelvin and p is total ambient pressure in millibars. Optical measurements[4], show the value of C_n to be in the range

$$4 \cdot 10^{-9} < C_n < 8 \cdot 10^{-9} \; cm^{-1/3} \tag{17}$$

near ground and in about 80 percent or more of nominally unbiased choices of weather conditions. Various atmospheric models allow for a smooth and gradual change in C_n and C_θ over several km of altitude. Such models are useful at times in the case of light. In the case of sonic boom data, the parameters which enter in such atmospheric models have not usually been measured at the time data were obtained. In what follows, we will assume that the sonic boom is scattered by a layer of turbulence of uniform strength between the ground and an altitude of several km.

The log-normal distribution is an almost invariable feature of the fluctuations of waves in turbulence, at least after the pathlength in the atmosphere is 1 km or more, and commonly at much shorter pathlengths. Measurements of the irradiance of light, in a beam which has traversed several km of turbulence, have shown it to be log-normal within one percent almost irrespective of weather conditions. The data of Garrick and Maglieri[12] show a log-normal distribution for the random component of boom overpressure. The Rytov method is a theoretical approach which yields the above log-normal distribution for waves whose path is in homogeneous turbulence with the spectral density (11) for the random flow velocity; and with a uniform strength C_v^2 for the turbulence all along the wave path. This result now appears as an artifact of the approximation. Analyses which include the multiple scattering of waves in uniform turbulence, for example de Wolf's[9], show that the scattered waves are uniformly distributed in phase; and that their irradiance has a Gaussian distribution in vector frequency or a Raleigh distribution in scalar frequency.

How may we explain the everpresence of the above log-normal distribution? We surmise that it is produced by the spatial intermittency of atmospheric turbulence. That is, we consider the coefficient C_v^2 of

expression (11) to be a random function of position along the wave path-length, a function with scale length large compared to L_2. Then the spectral density of the noise comprised by the scattered waves will wax or wain in a random manner as the wave path covers regions of stronger or weaker turbulence. Unfortunately, the parameters of typical intermittency do not seem well enough known to serve as input for a detailed calculation of the pressure peaks in sonic boom. In numerical work, we will stay with the spectral densities (11) and (13). Since values listed for C_n^2 and C_v^2 have been obtained from measured data by a method which ignored intermittency; these values may perhaps compensate for a lack of intermittency in the theory, a matter beyond the present study.

Equation (10) calls for the specification of the spatial transform $k(0, \zeta)$ of the incident wave packet. This transform has been studied by J. R. Young [25], and also by P. B. Oncley and D. G. Dunn [26]. The incident N-wave has the form

$$P = P_o z/A; \quad -A < z < A$$
$$P = 0 \quad ; \quad -\infty < z < -A, \ A < z < \infty \tag{18}$$

where P is overpressure, P_o is the peak value of P, and $2A$ is the height of the region over which the N-wave extends. The Fourier transform $Q(\zeta)$ of the function $P(z)$ is

$$Q(\zeta) = \frac{iP_o}{\pi A} \left(\frac{A}{\zeta} \cos \zeta A - \frac{1}{\zeta^2} \sin \zeta A \right) \tag{19}$$

By Section 4, the overpressure is related to the velocity potential φ by

$$P = \rho_o c \ \partial \varphi / \partial z \tag{20}$$

Since

$$P(z) = \int Q(\zeta) \exp (i \zeta z) \, d\zeta \tag{21}$$

we have

$$\varphi = \int Q(\zeta) \exp (i \zeta z)/(\rho_o c i \zeta) \, d\zeta \tag{22}$$

or, by equation (19), the transform of φ at $t = 0$ is

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$$k(0, \zeta) = \frac{-P_o}{\pi A \rho_o c} \left[\frac{A}{\zeta^2} \cos \zeta A - \frac{1}{\zeta^3} \sin \zeta A \right] \tag{23}$$

which expresses the initial condition for our stochastic boundary value problem.

SECTION 10

ENERGY CONSERVATION

The present section formulates the principle of energy conservation for sonic fluctuations. It then specializes this principle to the high frequency approximation.

The sonic energy E, per unit cross section of a tube with axis parallel to the mean wave propagation direction and of infinite length, is

$$E = \tfrac{1}{2} \rho_o (-\infty \int \infty) \vec{U} * \vec{U} \, dz = \tfrac{1}{2} \rho_o \int \nabla \varphi * \cdot \nabla \varphi \, dz \tag{1}$$

where ρ_o is mean air density, $\vec{U}$ is the sonic contribution to the vector air velocity, and φ is the velocity potential.

The inequalities

$$|U_x| << |U_z| \, , \quad |U_y| << |U_z| \tag{2}$$

are proper for pathlengths of the sound in turbulence which are not so long as to make unrecognizable at ground the original N-shaped form of the overpressure at high altitude. Measurements obtained at the ground are represented in Figure 4. Only the single pressure history at the upper right of the figure suggests violations of relations (2), which we take to be almost always satisfied. In computing statistics at ground, we will hence replace equation (1) by

$$E \simeq \tfrac{1}{2} \rho_o (-\infty \int \infty) \langle \partial \varphi * / \partial z \; \partial \varphi / \partial z \rangle \, dz \tag{3}$$

As in Section 4, we have

$$\varphi(t, \vec{r}) = \int [h(t, \vec{s}) + k(t, \vec{s}) \; \delta(\vec{\sigma})] \; \exp (i\vec{r} \cdot \vec{s} - isct) \, d\vec{s} \tag{4}$$

A substitution in equation (3) from (4), with use of (7-15) yields

$$E(t) = \tfrac{1}{2} \rho_o \int \langle [h^*(t, \vec{s}') \, h(t, \vec{s}'') + k^*(t, \zeta') \, k(t, \zeta'') \, \delta(\vec{\sigma}') \, \delta(\vec{\sigma}'')$$

$$\exp [i \, (s' - s'') \, ct] \rangle \, \exp [i\vec{r} \cdot (\vec{s}'' - \vec{s}')] \, \zeta' \zeta'' \, d\vec{s}' \, d\vec{s}'' \, dz \tag{5}$$

as the total energy in terms of the ensemble average $\langle \ldots \rangle$. We have

$$\int \exp [iz(\zeta'' - \zeta')] \, dz = 2\pi \delta(\zeta'' - \zeta') \tag{6}$$

We may use relation (6) in performing the integration over z indicated in equation (5) so long as the integrand in (5) is not singular at $\zeta' - \zeta'' = 0$. Using (6) in (5) yields

$$E(t) = \tfrac{1}{2} \rho_o \int \langle [h^*(t, \zeta', \vec{\sigma}') \, h(t, \zeta', \sigma'') + k^*(t, \zeta')(kt, \zeta') \, \delta(\vec{\sigma}') \, \delta\vec{\sigma}'']$$

$$\exp [ic(|(\vec{\sigma}', \zeta')| - |(\sigma'', \zeta')|) t] \rangle \exp [i\vec{r} \cdot (\vec{\sigma}'' - \vec{\sigma}')] \, \zeta'^2 d\vec{\sigma}' \, d\vec{\sigma}'' \tag{7}$$

From equation (7-15)

$$\langle h^*(\zeta', \vec{\sigma}) \, h(\zeta', \vec{\sigma}'') \rangle = \Psi(h; \zeta', \zeta', \sigma') \, \delta(\vec{\sigma}' - \vec{\sigma}'') \tag{8}$$

Use of equation (8) in (7) yields

$$E(t) = \tfrac{1}{2} \rho_o \int [\int \Psi(h; t, \zeta', \zeta', \sigma') \, d\vec{\sigma}' + \langle |k(t, \zeta')|^2 \rangle] \, \zeta'^2 d\zeta' \tag{9}$$

which is an expression of energy conservation specialized to the geometry of our problem. When $t = 0$, equation (9) becomes

$$E(0) = \tfrac{1}{2} \rho_o \int |k(0, \zeta')^2| \zeta'^2 d\zeta \tag{10}$$

In our model, true sonic absorption is ignored, or $E(t) = E(0)$. Thus, subtracting equation (10) from (9) yields

$$\int \Psi(h; t, \zeta', \zeta', \sigma') \, d\sigma' \zeta'^2 d\zeta'$$

$$= \int [|k(0, \zeta')|^2 - \langle |k(t, \zeta')|^2 \rangle] \, \zeta'^2 d\zeta' \tag{11}$$

which relates the energy in the scattered waves at any time to that then remaining in the incident beam.

The next paragraphs derive a specialization of equation (11) important in boom analysis. A variant of equation (8-1) is

$$\langle k^*(t, \zeta_a) \, k(t, \zeta_b) \rangle_{tt} = - \underset{\sim}{m}^2 \langle k^*(t, \zeta_a) \, k(t, \zeta_b) \rangle \qquad (12)$$

where

$$\underset{\sim}{m} = \int \frac{c^2 \, \zeta s'^2}{(\underset{\sim}{\zeta} + s')} \, \Psi(\theta; \vec{\underset{\sim}{\zeta}} - \vec{s}') \, d\vec{s}'. \qquad (12.1)$$

where ζ denotes the average of ζ_a and ζ_b. Equation (12) is valid provided

$$\zeta_a \gg 2\pi/L_2, \quad \zeta_b \gg 2\pi/L_2, \quad \zeta_a - \zeta_b \simeq 2\pi/L_2 \qquad (13)$$

and is derived almost the same as equation (8-1). There are significant contributions to the third integral of equation (7-18) only when the factor $\langle k^*(\zeta_a) \, k(\zeta_b) \rangle$ in its integrand satisfies relation (13) since $\Psi(\theta; s)$ is highly peaked near $S = 2\pi/L_2$. Thus, near or after the saturation time for frequency $\underset{\sim}{\zeta} = (\zeta_a + \zeta_b)/2$, the function $\Psi(h; t, \zeta_1, \zeta_2, \sigma)$ will oscillate relative to changes in ζ_1, or ζ_2 with a period $1/(\underset{\sim}{m}t) = 1/(n\zeta t)$; this because of contributions to $\Psi_{tt}(h; \ldots)$ from the third term of equation (7-17)

   We note that the effect of incident frequencies in the boom does not strictly vanish at saturation; but then only produces in $\Psi(h; \ldots)$ strong oscillations relative to changes in frequency or time. The average of $\Psi(h; t, \zeta_1, \zeta_2, \sigma)$ relative to ζ_1, or ζ_2 and over a range $\Delta\zeta > 1/nt$ hence has a dominant peak centered at $|\zeta_1 - \zeta_2| = 1/(nt)$. We may approximate this peak by a delta function, which is the same as writing

$$\Psi(h; t, \zeta_1, \zeta_2, \sigma) = \chi(t, \zeta, \sigma) \, \delta(\zeta_1 - \zeta_2)$$

$$\zeta_1 \gg 2\pi/L_2, \quad \zeta_2 \gg 2\pi/L_2 \qquad (14)$$

where χ is the power spectrum of noise whose frequencies possess no phase correlation.

   Within the wide band of frequencies which comprise the sonic boom at ground, let b denote the lowest which is saturated. Sonic energy may be localized in the frequency domain to within an uncertainty less than $1/(n\zeta t)$. Thus, by equations (5) and (14), the sonic energy within saturated frequencies is

$$E_b(t) = \tfrac{1}{2} \rho_0 \int (b\int \infty) \langle [\chi(t, s') \, \delta(\vec{s}' - \vec{s}'') + k^*(t, \zeta') \, k(t, \zeta'') \, \delta(\vec{\sigma}') \, \delta(\vec{\sigma}'')]$$

$$\exp[ic(s' - s'')t] \rangle \exp[i\vec{r} \cdot (\vec{s}'' - \vec{s}')] \, \zeta' \zeta'' \, d\vec{s}' \, d\vec{s}'' \, dz \qquad (15)$$

We perform first the integration over $\vec{s}''$ in the first term of expression (15), and perform first the integration over z in the second term. The result, in place of equation (9) is formally

$$E(t) = \tfrac{1}{2} \rho_o \int (b \int \infty) \; \chi(t, \zeta'\sigma') \; d\vec{\sigma}' \; \zeta'^2 d\zeta' \; dz$$

$$+ \tfrac{1}{2} \rho_o (b \int \infty) \langle \, |k(t, \zeta')|^2 \rangle \; \zeta'^2 d\zeta' \tag{16}$$

In equation (16), the proper limits for the integration over z have still to be specified. If the random noise with quasi-local properties fully described by χ extended over all values of z, then the limits in question would indeed be infinite. The domain over z of this noise is less than infinity, however, because of its wave interference with frequencies less than b. That is, frequencies ζ for which $\zeta < b$ almost fully determine the form of the N-wave except within an interval $1/b$ of its leading edge, and within a like interval near its trailing edge. We do not here record the somewhat tedious Fourier analysis which confirms this conclusion. With $1/b$ as the range of integration over z in equation (16), after properly ignoring the final term of (16), and after subtracting equation (10) from (16), we obtain

$$\frac{1}{b} (b \int \infty) \; \chi(t, \vec{s}') \; \zeta'^2 d\vec{s}' = (b \int \infty) \; |k(0, \zeta')|^2 \; \zeta'^2 d\zeta' \tag{17}$$

The two integrands in equation (17) may be equated because sonic energy is quasi-localized in frequency when $\zeta > b$. Thus

$$\int \chi(t, \zeta, \sigma) \; d\vec{\sigma}/b = |k(0, \zeta)|^2 \tag{18}$$

where, by relation (9-10) of Section 9

$$b = 1/(n \, t) \tag{19}$$

Equation (18) is the desired specialization of energy conservation; and is an analytic expression of much of the high frequency approximation used previously by S. C. Crow[2] in sonic boom analysis. A virtue of equation (18) is that it transforms the statistics of high overpressure peaks into a problem involving little more than wave saturation, which by equations (8-1) and (8.-8) is one of the rare simple features of multiple scattering.

SECTION 11

POISSON STATISTICS FOR HIGH OVERPRESSURES

The present section compares theoretical and experimental values for the nominal lowest frequency b at which saturation occurs. We then apply a formula of S. O. Rice, which leads to an expression for the probability of pressure peaks above a given value.

A numerical value for b may be obtained by use of the turbulence parameters listed in Section 9.. In a nominally unbiased choice of weather conditions near ground, the parameter C_n for the optical refractive index is below $6 \cdot 10^{-9}$ $cm^{-1/3}$ with a probability of about 10 percent, and above $12 \cdot 10^{-9}$ $cm^{-1/3}$ with a like probability. For a numerical example, we choose $C_n = 10^{-8}$ $cm^{-1/3}$, which suits gusty conditions. To this value corresponds $C_\theta = 14 \cdot 10^{-10}$ $cm^{-2/3}$, by the relations of Section 9. And to this corresponds

$$\Psi(\theta;\vec{s}) = 0.033 \, C_\theta^2 \, s^{-11/3} = 5 \cdot 10^{-11} \, s^{-11/3} \tag{1}$$

as an expression for the spectral density of relative temperature fluctuations in turbulence. The functions $\Psi(\theta;s)$ and $\Psi(v;s)/c^2$ are equal by Kallistratova's relation (9-14). Thus, equation (1), with (8.-11) of Section 8 and with the above choice of C_n, yields

$$m^2(\zeta) = n^2 \zeta^2 , \quad n = 16 \text{ cm/sec.} \tag{2}$$

The above values for C_n came from optical data[3] and a model for the altitude distribution of turbulence of the single step kind, according to which C_n is constant from 0 to 10 km altitude, and zero at higher altitudes. The time t for the traverse of sound through 10 km is $t \simeq 33$ sec. With this value of traverse time, and n = 16 cm/sec. we have

$$b = 2\pi/\lambda_b = 2\pi/(nt) , \quad \text{or} \quad \lambda_b \simeq 5 \text{ meter} \tag{3}$$

where λ_b denotes the largest saturated Fourier wavelength. The value $\lambda_b = 5$ m is obtained from the theory and optical data only. The experimental boom histories shown in Figure 4 column (b), where the duration of each history is about 0.2 sec, show a random disturbance whose length is also about 5 m. Better agreement between an involved theory and a somewhat weather dependent phenomenon would have small significance.

Garrick and Maglieri[12] show data of chief importance as the probability $\Pi(g)$ that the measured overpressure in a boom at ground possess at least once, during its observation by a single microphone, a maximum which exceeds a nominal value P_n by a ratio g. The value P_n represents the single maximum expected without scattering from

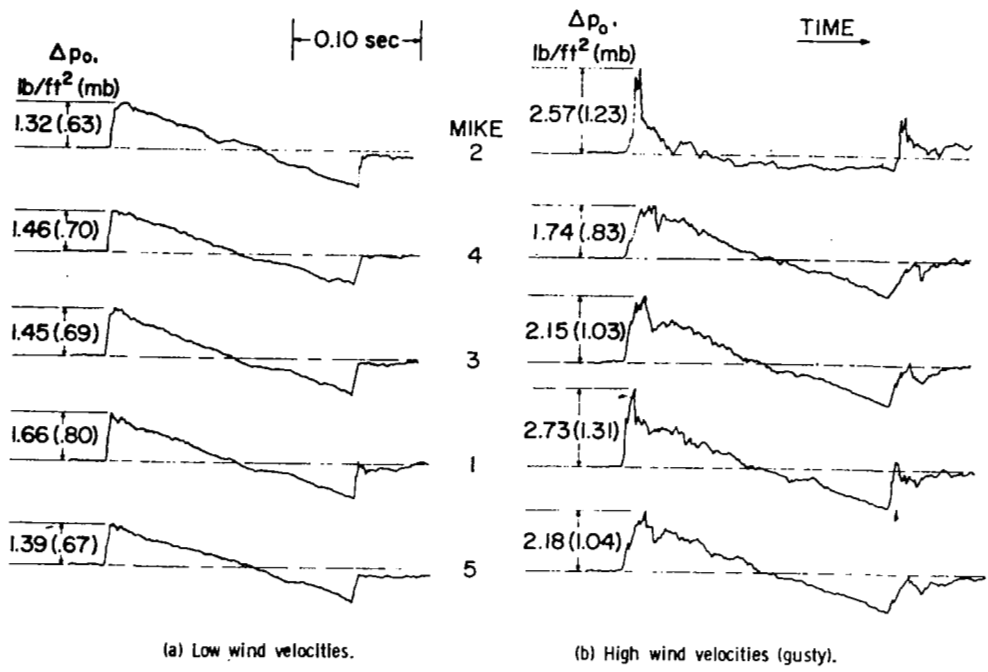(a) Low wind velocities.

(b) High wind velocities (gusty).

Figure 4.   Time histories of sonic-boom overpressure showing
wave-shape variations between microphones for two
flights of a B-58 airplane on different days.
(From ref. 12.)

turbulence. Probabilities of type $\Pi(g)$ have been studied by S. O. Rice[27, 28]. They are in general complicated functionals of the spectral densities with which random functions are usually and most simply described. But when g is large enough so that $\Pi(g) \ll 1$, the case of chief interest for the data, then Poisson statistics apply.

For Poisson statistics there exists a function $M(g, Z)$ as a probability per unit distance Z, measured from the central plane of the boom, that the pressure noise exceed the nonrandom pressure $Q(Z)$ on which it is superposed by a factor g. In the case of a sonic boom

$$Q(Z) \simeq P_o b(A-Z)/2\pi, \qquad A - \frac{2\pi}{b} < Z < A \tag{4}$$

where A is the half distance between leading and trailing edges, and P_o is peak pressure expected without scattering by turbulence. Equation (4) represents a straight line consistent with the lowest graph of Figure 4. A result of Rice[27], for the above function M, is

$$M(g, Z) = (1/2\pi)(D_2/D_1)^{\frac{1}{2}} \exp\left[-(gQ)^2/2D_1\right] \tag{5}$$

where

$$D_1 = (-\infty \int \infty)\ w(\zeta)\ d\zeta \tag{6}$$

and

$$D_2 = (-\infty \int \infty)\ w(\zeta)\ \zeta^2 d\zeta \tag{7}$$

and where $w(\zeta)$ is the spectral density of the noise in the overpressure.

To find $w(\zeta)$ we use its definition[29] adapted to pressure noise of finite duration, namely

$$w(\zeta) = \frac{b}{2\pi^2}(A-2\pi/b \iint A)\ P*(b;t, Z')\ P(b;t, Z'') \tag{8}$$

$$\exp\left[-i\zeta(Z'-Z'')\right]\ dZ'\ dZ''$$

where $P(b; t, Z)$ denotes the contribution to total pressure $P(t, Z)$ from spatial frequencies greater than b. Since the noise of interest consists of a superposition of partial waves whose phases are uncorrelated, the noise is Gaussian, with spatial and ensemble averages assumed equivalent. The ensemble averaging of equation (8) does not change its left member, since $w(\zeta)$ for stochastic noise is an ensemble average as well as a spatial average. But such ensemble averaging replaces P*P on the right of equation (8) by $\langle P*P \rangle$. By equation (3 - 28),

$P = -\rho_0 c \varphi_z$; or in terms of the Fourier transform of φ

$$\langle P*P \rangle = \rho_0^2 c^2 \int \langle [h*h + k*k \ \delta(\vec{\sigma}') \ \delta(\vec{\sigma}'')] \cdot$$

$$\exp [ic(s' - s'')\rangle \ \zeta'\zeta'' \ d\vec{s}' \ d\vec{s}'' \tag{9}$$

Equation (9) is valid in general. For saturated frequencies $\langle P*P \rangle$ becomes $\langle P(b; t, Z) \ P(b; t, Z) \rangle$ where t is the traverse time of the sound in turbulence. In this case the second term of (9) may be ignored, while its first term then becomes the energy of scattered waves at frequencies higher than b, except for a constant. The expressions for this energy derived in the final paragraph of Section 10 transform equation (9) to

$$\langle P*(b;t,Z) \ P(b;t,Z) \rangle = \rho_0^2 c^2 (b\!\int\!\!\int^\infty) \ k*(0, \zeta') \ k(0, \zeta'') \ \zeta'\zeta'' \ d\zeta' d\zeta'' \tag{10}$$

By equation (9 -23), and for $Ab \gg 1$,

$$k(0, \zeta) = \frac{-P_0}{\pi \rho_0 c} \ \frac{e^{i\zeta A} + e^{-i\zeta A}}{2\zeta^2} \tag{11}$$

in which the first term makes the dominant contribution to values of $P(Z)$ near the leading edge of the boom. Substitution from equation (11) in (10), and then from (10) in (8), yields

$$w(\zeta) = \frac{b}{2\pi^2} (A - 2\pi/b \ \int\!\!\int A) \ (b \int\!\!\int^\infty) \ \frac{P_0^2}{4\pi^2 \zeta' \zeta''} \ \cdot$$

$$\exp [-i\zeta (Z' - Z'') - i\zeta'(Z' - A) + i\zeta''(Z'' - A)] \ d\zeta' d \zeta'' \ dZ' \ dZ'' \tag{12}$$

In equation (12), we integrate over Z' and Z'' with the approximation

$$(A - 2\pi/b \ \int A) \ \exp [iZ (\zeta' - \zeta'')] \ dZ \simeq 2\pi \ \delta(\zeta' - \zeta''); \ \zeta' > 1/Z , \ \zeta'' > 1/Z \tag{13}$$

which yields

$$w(\zeta) = P_0^2 \ b/(2\pi^2 \zeta^2) \ ; \ \zeta > b, \ b = 2\pi/nt \tag{14}$$

Definition (6) then becomes, by substitution from (14)

$$D_1 = (b\int^\infty) \ P_0^2 \ b /(2\pi^2 \zeta^2) \ d\zeta = P_0^2/2\pi^2 \tag{15}$$

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Then (7) becomes

$$D_2 = (b \int_o^\infty) P_o^2 b / (2\pi^2) \, d\zeta \tag{16}$$

an integral which diverges at its upper limit.

Taking account of true atmospheric absorption of ultrasonic waves in the atmosphere will change the integral for D_2 so that it converges. A stronger reason for convergence is the cutoff frequency $f \simeq 10^4$ Hz of the modified condenser microphones used in obtaining data. Hence we replace equation (16) by

$$D_2 = P_o^2 \, b(f - b)/(2\pi^2) \tag{17}$$

Relation (5), by use of (15) and (17) becomes

$$M(g', Z) = \frac{1}{2\pi} \left[b(f - b) \right]^{\frac{1}{2}} \exp \left[- \pi g'^2 Q^2 / P_o^2 \right] \tag{18}$$

SECTION 12

GRAPHICAL RESULTS

The present section exhibits significant agreement between data for random peaks in overpressure and the analysis of multiple scattering in typical atmospheric turbulence. Exceptions to such agreement are noted.

In equation (11-18) of Section 11, the function $M(g', Z)$ is a probability per unit length that random pressure rises above its non-random base value Q by a ratio $g'+1$. When g' is such that the total overpressure rises above its single peak P_o expected without turbulence by a factor g, then $g'Q + Q = gP_o$, or

$$g' + 1 = 2\pi g / \left[b(A - Z) \right] \tag{1}$$

A substitution from equation (1) in (11-18), followed by integration over the range of Z within which frequencies in excess of b lack phase correlation, yields

$$\Pi(g) = (bf)^{\frac{1}{2}} (A - 2\pi/b \int A) \exp(-X^2) \, dZ/2\pi \tag{2}$$

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where

$$X = \pi[g - Q/P_o] \tag{3}$$

and where f-b has been replaced by f since $f >> b$. The differential of equation (3), by use of expression (11-4), is

$$dX = bdZ/2 \tag{4}$$

A change in the variable of integration in equation (2) yields

$$\Pi(g) = (f/b)^{\frac{1}{2}} (\pi(g - 1) \int \pi g) \exp(-X^2) \, dX/\pi \tag{5}$$

The error integral, as usually tabulated, is defined as

$$\text{erf}(x) = (2/\pi^{1/2}) (0 \int x) \exp(-x^2) \, dx \tag{6}$$

Substitution in equation (5) from (6) yields

$$\Pi(g) = \tfrac{1}{2}(f/\pi b)^{\frac{1}{2}} [\text{erf}(\pi g) - \text{erf}(\pi (g - 1))] \tag{7}$$

Figure 5 shows plots of equation (7) for three different weather conditions, as shown in the curves drawn solidly. The superposed curves, drawn dashed, summarize extensive experimental data. The dashed curve (2) is consistent with the theory if we suppose that data with the highest overpressure peaks are taken on the average in more gusty weather than those with the lower peaks. Each theoretical curve represents a single value of C_n or a single value of gustiness. The dashed curve (b) is similarly consistent with the theory if we may assume that the root mean square turbulent velocity was about twice as great for curve (b) as it was for curve (a). Curve (c) agrees neither with the other experimental curves (a), (b) nor with the theory; for reasons on which we are unclear. It seems plausible that there was an unusually quiet atmosphere effective at the time the data of curve (c) were taken; and that there was then insufficient air mixing to produce homogeneous turbulence.

The agreement between data and theory, as shown in Figure 5 gives interest to an exploration of the sensitivity of the theory to values of some of the weather or turbulence parameters on which it is based. Figure 6 shows the dependence of $\Pi(g)$ on the thickness of the turbulence of the atmosphere; Figure 7 shows the dependence of $\Pi(g)$ on the cutoff frequency of the sonic detector. Figure 8 shows the dependence of $\Pi(g)$ on L_2, the outer scale of turbulence. $\Pi(g)$ depends on L_2 only through

Figure 5. Measured and Calculated Values of $\Pi(g)$
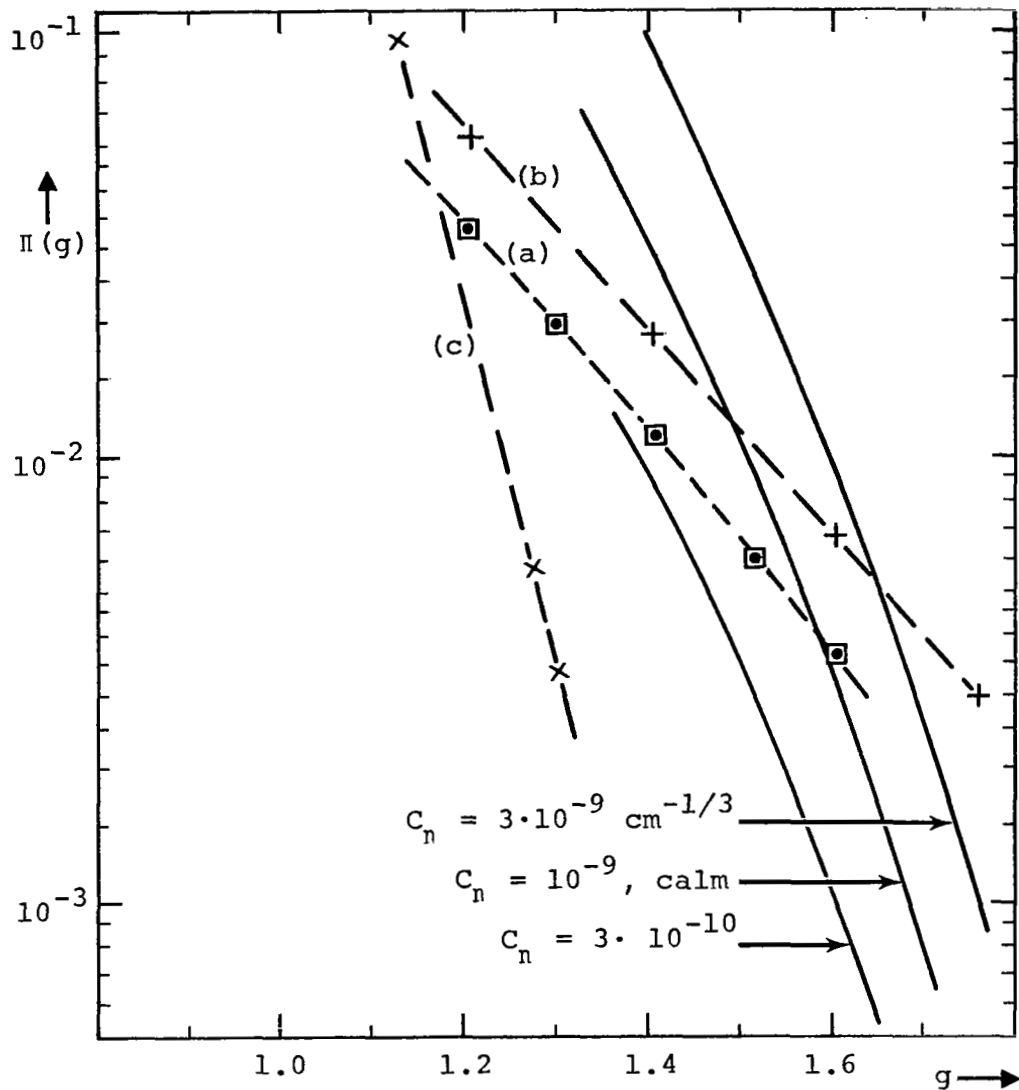
—— Theory

– – Data
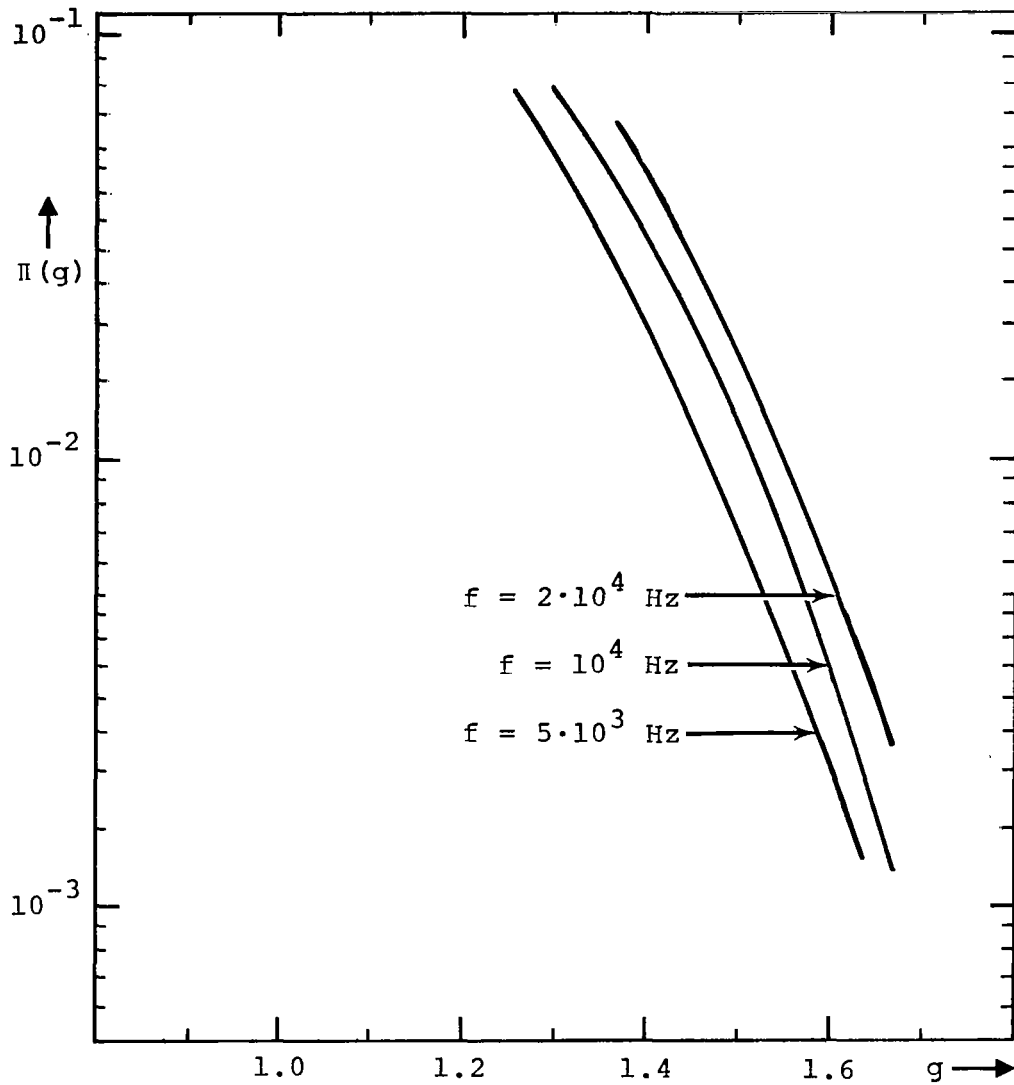
L_2 outer scale of turbulence, 100m

Γ height of effective layer of turbulence, 10 km

f cut off frequency of detectors, 10^4 Hz

+ B-58 flights, 358 data samples

× XB-70 flights, 447 data samples

▣ Composite of XB-70, B-58 and F104 flights, 1597 data samples

Figure 6.   Dependence of $\Pi(g)$ on Γ, the Effective
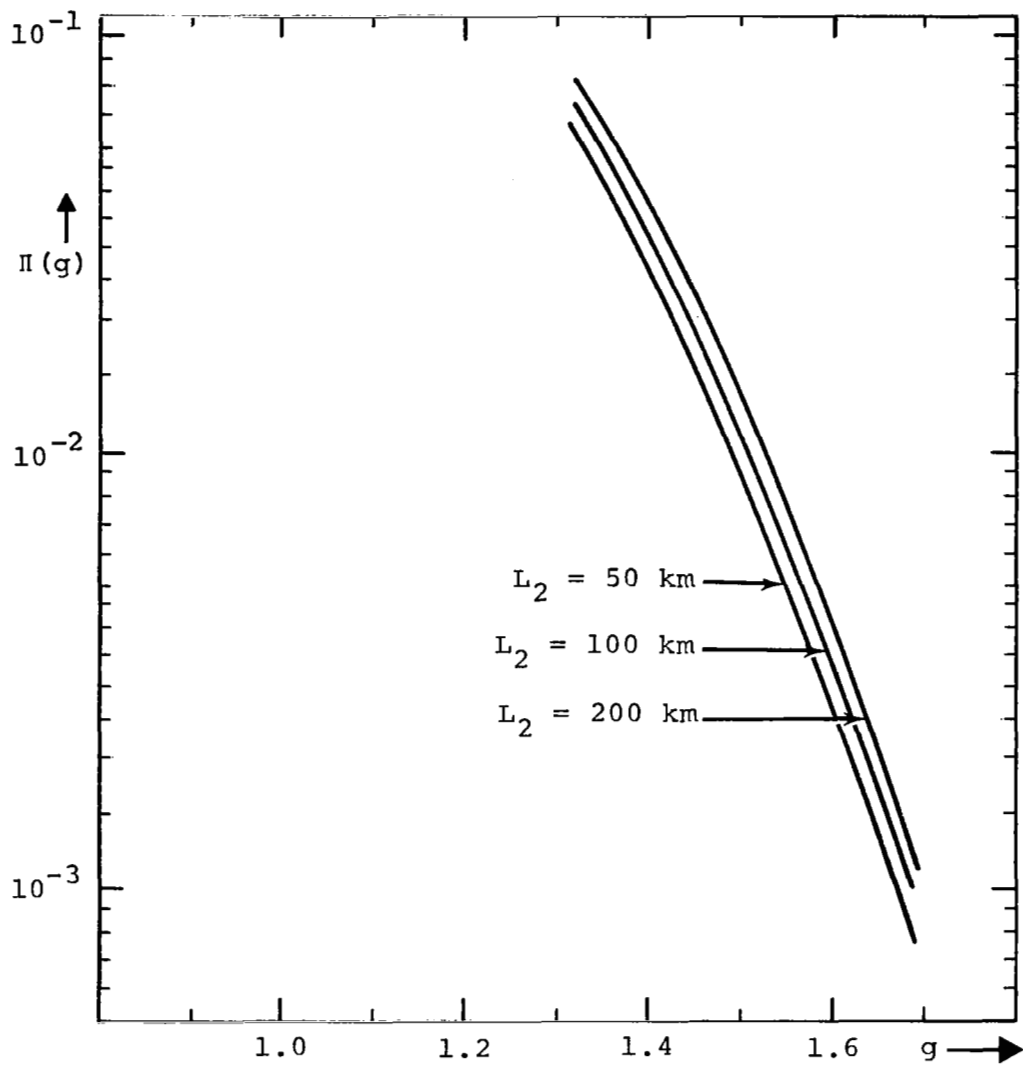Vertical Extent of Turbulence
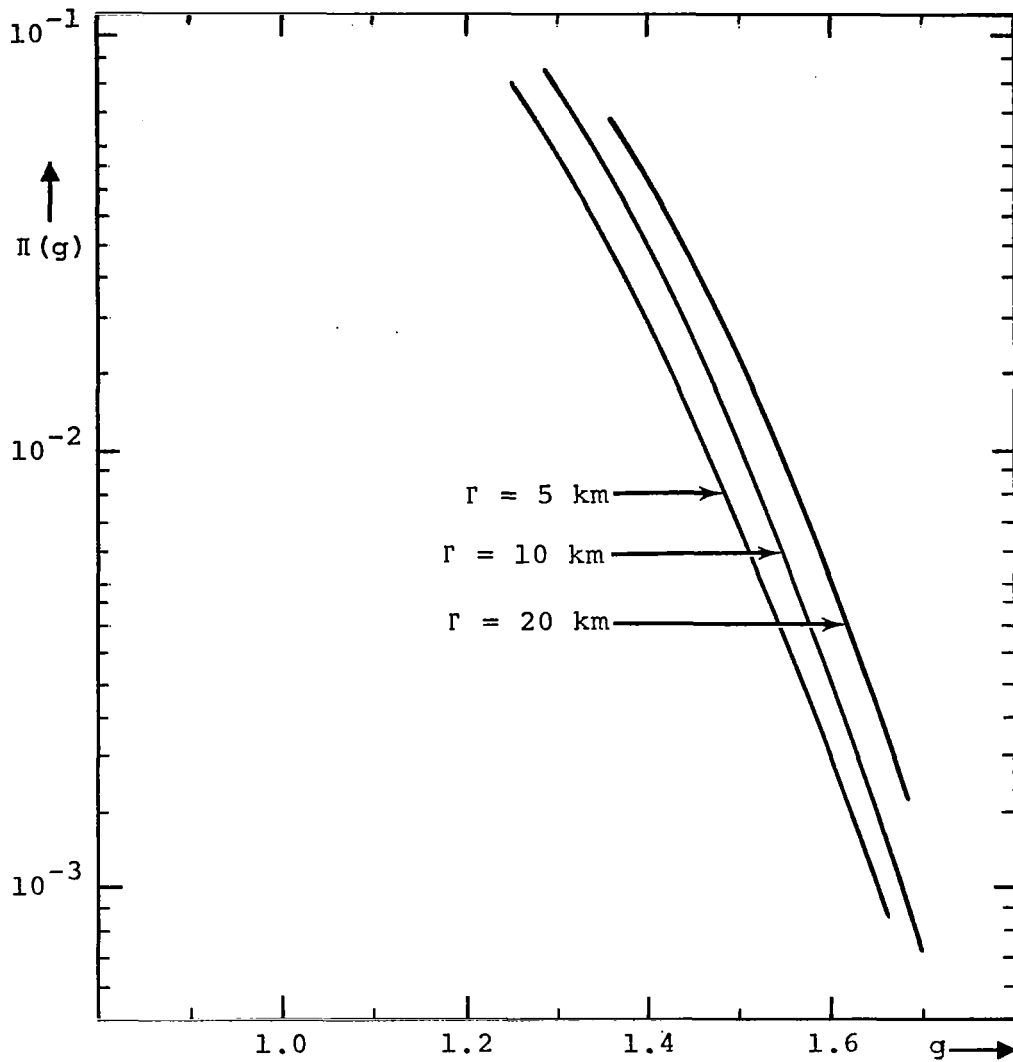
L_2 outer scale of turbulence, 100m

C_n $3 \cdot 10^{-9}$ cm$^{-1/3}$, calm

f detector cut off, 10^4 Hz

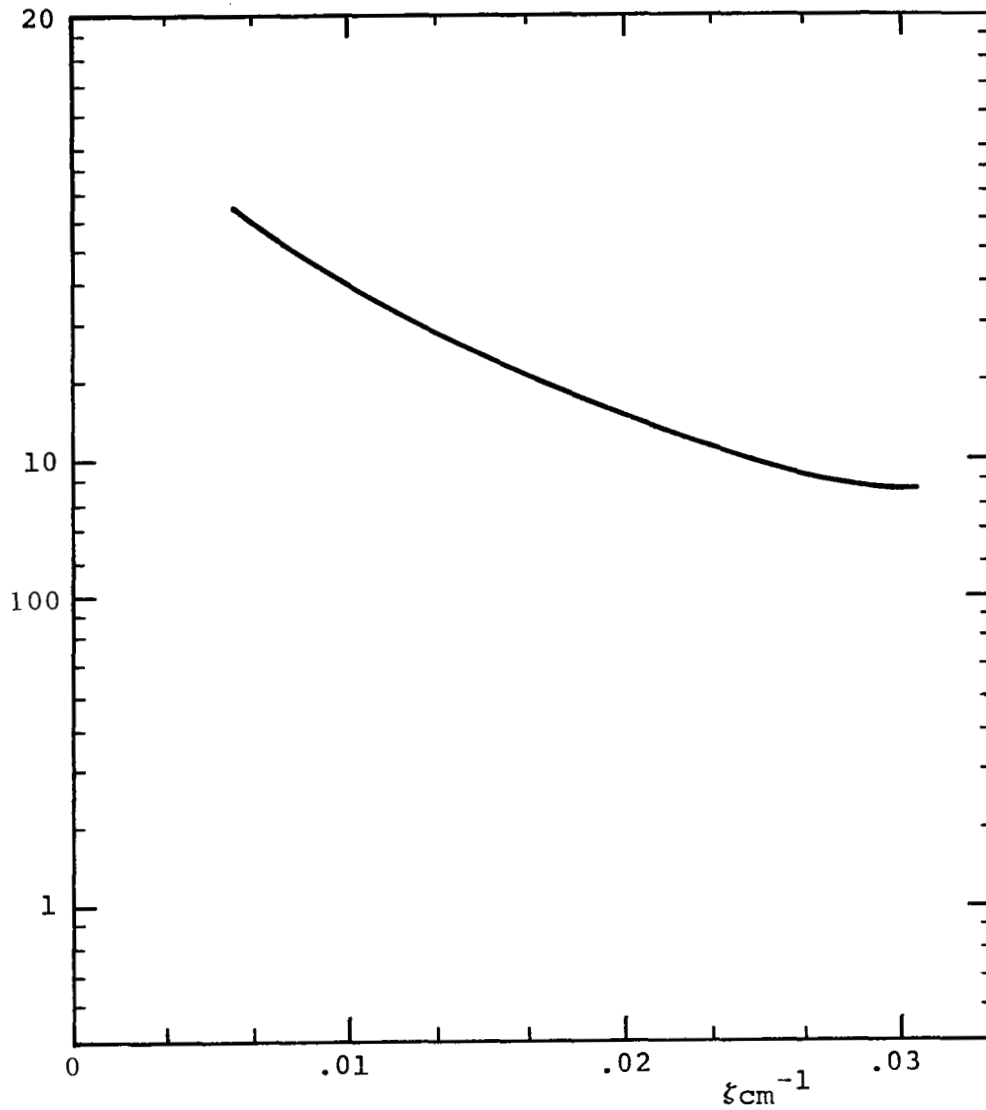L$_2$ outer scale of turbulence, 100 m

C $3 \cdot 10^{-9}$ cm$^{-1/3}$, calm

Γ height of turbulent layer, 10 km

π(g)   probability that overpressure exceed its
         nominal peak value by ratio g

Figure 7. Dependence of π(g) on Detector Cutoff f

Figure 8. Dependence of $\pi(g)$ on Outer Scale L_2

C_n calm, $3 \cdot 10^{-9}$ $cm^{-1/3}$

Γ height of turbulent layer, 10 km

f detector cut off, 10^4 Hz

$\pi(g)$ probability of exceeding nominal peak pressure by a ratio q

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L$_2$ outer scale of turbulence, 100m

C$_n$= $3 \cdot 10^{-9}$ cm$^{-1/3}$

Figure 9.   Dependence of Saturation Length L$_s$
             on Circular Frequency ζ

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the 6th root of the latter; so that plausible uncertainties in L_2 are less important than uncertainties in the general leval of gustiness. Figure 9 shows the dependence of saturation length L_s on frequency ζ, for the case of a single frequency. Broad banded pulses of sound saturate first at their higher frequencies when subject to small angle scattering in turbulence.

The absence in equation (7) of any reference to A, the half length of the sonic boom, need not be taken too literally. If the strength of turbulence in the earth's atmosphere were ten or more times greater than it is in fact; then substantially all frequencies in the sonic boom would be saturated. In such a case the N-shaped form of an overpressure history at ground would be almost fully lost, and the boom would appear as noise whose Fourier components almost all have random and uncorrelated phases. When such atypically strong turbulence occurs, equation (7) does not apply; and then $\Pi(g)$ is significantly dependent on A. That $\Pi(g)$ is in practice almost independent of A is supported by the data of Figure 17 of reference[12], and has been previously discussed by S. C. Crow[2].

The relative simplicity of equation (7) comes chiefly from two reasons, we believe. First, the wave pathlength of the boom in turbulence is large compared to the diameter of a single eddy, so that the random features of the boom at ground result from the combination (not the simple addition) of many small and statistically independent effects. Second, the above wave pathlength is small compared to that required for saturation of the lower Fourier frequencies of the N-wave, so that a high frequency approximation may be used in computing the dominant effect of small angle scattering by turbulence.

SECTION 13

SUMMARY

The extensive data of Garrick and Maglieri[12] have been inter-
preted in large part in the preceding study. The case treated from
first principles is that of random overpressure peaks with maxima
which exceed by an unusual amount the single pressure peak of a sonic
boom in a smoothly varying atmosphere. The random pressure peaks
have been attributed to small temperature and velocity fluctuations in
typical atmospheric turbulence. This attribution is supported by the
agreement between the data and the calculations. Errors in the basic
analysis and numerical work are believed small compared to uncertain-
ties about the turbulence at the times data were measured.

Gustiness near ground, in a layer of air from a few hundred
to a thousand feet thick, may infrequently produce turbulence so strong
that its effect on wave propagation outweighs that of all turbulence
above that ground layer. The statistics of microflow in such unusually
active ground layers is not well understood, nor does the present study
treat their effect on sound. Barring such ground layers, the random
pressure peaks observed in sonic boom are now understood, we believe.

The analysis has been influenced by the complexity of the wave
scattering problem presented by a sonic boom in turbulence. This com-
plexity arises largely because the N-shaped form of the dependence of
overpressure on altitude contains a wide range of important component
frequencies. It is a broad banded signal. Wave scattering by turbu-
lence is a strong function of sonic frequency; with the highest important
frequencies ($\sim 10^4$ Hz) randomized within a 10 meter length of wave
path in typical atmosphere near ground. This small pathlength for
randomization requires multiple scattering in the quantitative analyses
of boom statistics.

The analysis used is a natural extension of the single scattering
theory of S. C. Crow whose conclusions have been verified more quanti-
tatively. The basic theories of Tatarski, on the general problem of
wave propagation in random media, have been somewhat extended in
principle, toward the inclusion of multiple scattering effects. The
analysis used is an outgrowth of the theory of Brownian motion and
other random processes which are subject to relatively simple proba-
bilities asymptotically valid at large times, as explored by A. Khinchine.
Saturation lengths and probabilities for unusually high overpressures
have been given as expressions in closed form. Portions of the analysis,
chiefly Sections 6, 7 and 8 , may be ultimately applicable to the scat-
tering of optical, radio, and plasma waves in random media.

Study of sonic boom supports the hope that additional atmospheric
effects may be subject to more quantitative understanding. -

APPENDIX Al

NON-MARKOVIAN PROCESSES, REMARKS

The present study concerns Markov processes, and of a
special type. In the present section we clarify what a Markov process
is by remarking on what it is not.

A Markov process relative to an interval may be defined as
one with state functions the time averages of whose statistics, over an
observation time interval t_0 centered at a time epoch t, have rates of
change expressible at least in principle as functions of the above aver-
ages over the same time interval t_0. When the interval t_0 shrinks to
a single instant at epoch t, then the above definition specifies a strict
Markov process. When the interval of observation t_0 is large com-
pared to some natural time scale in the process, as is true in many
applications, then the difference between a strict Markov process and
one relative to an interval is important. It is not quite clear that a
process which is strictly Markovian need be Markovian relative to an
interval, while the converse is untrue.

All processes, we may allege, are Markovian from the full
microscopic viewpoint. All the laws of nature in classical physics are
causal; in the sense that rates of change of members of a complete set
of Hamiltonian state variables for any closed mechanical system at an
instant t are expressible as functions of the state variables evaluated
at that same instant. A similar remark holds in quantum theory if the
state variables are taken for example as the spatial Fourier components
of the full wave function for the system. We here ignore the somewhat
speculative exceptions. From the microscopic or maximally detailed
viewpoint, randomness can enter into the motion of a system only as
an expression of our partial ignorance of initial conditions, a matter
much explored in ergodic theory.[17] But such randomness, combined with
causal state propagation equations, leads only to Markov processes.
Just how non-Markovian processes can ever occur is a nontrivial ques-
tion on which we now comment.

An inadequacy in the microscopic viewpoint occurs when, as in
magnetic hysteresis, we are unable to work out even the statistics of a
complete set of state variables for the molecules of an iron bar, but do
know the growth of its magnetization empirically. We then use state
functions to describe the bar which are quite allowable, because the
growth of their statistics is determined in principle with the aid of
empirical knowledge; but which are far from a maximally detailed set
of state functions. It may and sometimes does happen that the changes
in this less than maximal set of state functions is non-Markovian. This
has not been shown to contradict microscopic causality.

A further inadequacy of the microscopic viewpoint occurs when
our knowledge of the coupling of a mechanical system to its surroundings
is chiefly through only a few of their statistics. An example is a chemical

solution in a heat bath; another is the colloidal grain in Brownian motion. In such a case, we naturally model the non-isolated system as a random process with state functions which refer only to the system of chief interest, and the surroundings are represented by the occurrence in the state propagation equations of random functions pertaining to the surroundings. Such a reduction of state functions from the maximum allowable number may or may not lead to a non-Markovian process.

Consider a volume of gas with temperature T, entropy S, heat content Q, and on which the work done is W. By the second law of thermodynamics

$$TdS = dQ + dW \tag{1}$$

Assume, as is very possible, that equation (1) is supplemented by three others, so that the temporal propagation of the joint probability density of T, S, Q and W is in principle determined. Then T(t), S(t), W(t) comprise an allowable set of time dependent statistical state functions for the gas. But, as is well known, expression (1) is not in general a perfect differential, or S(t), as a function of time, in principle cannot be specified in general without reference to past history or to the past path of the system. Here is an example of a class of inherently non-Markovian processes.

Finally, the scattering of waves in weak turbulence is not a strict Markov process in the state functions which describe the waves alone. For as we saw in Sections 5 and 6, it is precisely the instantaneous correlations between the state functions of the waves and the state functions for the random medium which determine the instantaneous rate of change of the state statistics for the waves. Nevertheless, suitable time averaging of the statistics of the state functions of the waves alone makes their time derivatives dependent on only the local time statistics of the waves alone and the constant statistics of the medium alone, to within a practically small time interval. Thus the wave scattering is Markovian to within an interval. A formally enlarged random process whose state functions include those for both the waves and the random medium is of course strictly Markovian, and also possesses the propagation equations (4-17), which must now be considered bilinear in its state variables. But regarding waves in turbulence as a bilinear process seems to change only the viewpoint, not necessarily the details of the analysis through terms o(ϵ^2).

APPENDIX A2

MAXWELL'S DISTRIBUTION

We now give a simple application of the concepts of Section 5. Consider a gas composed of molecules all of the same type and within an enclosure of constant volume. We assume a central force field surrounding each molecule so that the potential energy of the ith molecule is

$$\sum_j V(\,|\vec{q}_i - \vec{q}_j|\,) \tag{1}$$

where $\vec{q}_i$ denotes the vector position of the ith molecule and V is the potential energy of interaction between the ith and j^{th} molecules. The Hamiltonian H for the gas will then be

$$H = \sum_i \vec{p}_i^2/(2m) + \sum_{ij}' V(\,|\vec{q}_i - \vec{q}_j|\,)/2 \tag{2}$$

where the prime on $\sum'$ indicates that terms in the summation for which $i = j$ are omitted. Also $p_i = m\vec{v}_i$ is the momentum of the ith molecule, whose vector velocity is $\vec{v}_i$. Henceforth let p_i and q_i without the arrow, denote any one of the Cartesian components of the momentum and position of a molecule. Then the equations of motion corresponding to the Hamiltonian (2) are

$$\partial H/\partial p_i = q_{it} \quad , \quad -\partial H/\partial q_i = p_{it} \tag{3}$$

where a subscript t denotes differentiation relative to time. Substituting in equations (3) from (2) yields

$$q_{it} = p_i/m \; , \; p_{it} = - \frac{\partial}{2q_i} \sum_j' V(\,|\vec{q}_i - \vec{q}_j|\,) \equiv U_i \tag{4}$$

which incidentally defines the abbreviation U_i.

The aggregate of all the p_i and q_i has properties which allows it to be considered as a set of state functions $\vec{F}(t)$ in a Markov process which models the molecular motion of the gas. That is, if the probability density $W(t, \vec{F})$ of all the p_i and q_i is given at $t = t_o$; then the time derivative of this density is in principle given at later times through equations (4). The Fokker-Planck (FP) equation for the gas, a special case of equation (15) of Appendix A6 is

$$W_t = -\sum_j \frac{\partial}{\partial p_j} (\{p_j\} W) - \sum_j \frac{\partial}{\partial q_j} (\{q_j\} W)$$

$$+ \tfrac{1}{2} \sum_{jh} \frac{\partial^2}{\partial p_j \partial p_h} (\{p_j p_h\} W) + \tfrac{1}{2} \sum_{jh} \frac{\partial^2}{\partial q_j \partial q_h} (\{q_j q_h\} W)$$

$$+ \sum_{jh} \frac{\partial^2}{\partial p_j \partial q_h} (\{p_j q_h\} W) + \ldots \tag{5}$$

The averaging time or observation time interval of equations (5-17), over which we average the instantaneous statistical properties of the gas, is taken small compared to the time during which the gas is in equilibrium, but large compared to the mean time t_c between molecular collision. This same t_c gives an estimate of the correlation time for molecular momenta. We now use equations (A6 - 15) to get

$$\{q_j\} = \langle p \rangle_c / m + o(\tau) \;, \quad \{p_j\} = \langle U_j \rangle_c + o(\tau) \tag{6}$$

$$\{q_j q_h\} = o(\tau) \;, \quad \{q_j p_h\} = o(\tau) \tag{7}$$

$$\{p_j p_h\} = (o \!\int \tau) \langle U_j(t) U_h(t + t') \rangle_c \, dt' + o(\tau) \tag{8}$$

where $\langle \ldots \rangle_c$ denotes a conditional average as defined in Section 5.

We now assume that a time independent statistical state of the gas exists. In more detail, we assume $\partial W / \partial t = o(\tau)$ for this state, that U_j and U_h with j not equal h are uncorrelated, and that the temporal fluctuations in U_j are statistically uniform with a correlation time about equal to t_c. Then, for an equilibrium state, equation (8) becomes

$$\{p_j p_h\} = \sigma \delta_{jh} \quad \text{where} \quad \sigma = (-\infty \!\int \infty) \langle U_j(t) U_j(t + t') \rangle_c \, dt' \tag{9}$$

which defines σ as a constant independent of time and of the p_j and the q_j, and where δ is the Kronecker delta. The occurrence of terms $o(\tau)$ in the above equations is essential in careful analysis as an indication that we are consistently neglecting changes in the statistics over a time interval larger than t_c, although not necessarily over an interval very much larger than t_c. The consequences of this very useful and usual neglect of a physically small but non-zero quantity are easier to follow where the FP equation is taken in the finite difference from (A6-15). Henceforth, however, we omit terms $o(\tau)$. Since $\tau \gg t_c$ we have

$$\langle p_i \rangle_c \simeq 0 \ , \ \langle U_i \rangle_c \simeq 0 \tag{10}$$

We assume the gas to be sufficiently rare so that only binary collisions are important, which means that terms of the series (5) not explicitly written may be ignored.

A substitution in equation (5) from equations (6), (7), (9), (10) yields

$$\sum_j \partial^2 W / \partial p_j^2 = 0 \tag{11}$$

As a trial solution for equation (11) we write

$$W = N \exp (-\beta^2 \sum_j p_j^2) \tag{12}$$

where β is independent of the p_j and q_j and N is here a normalizing constant. Substituting in (11) from (12) yields

$$\sum_{j=1}^n [(2\beta^2 p_j)^2 - 2\beta^2] = 0 \tag{13}$$

or

$$\beta^2 = n/(2 \sum_{j=1} m^2 v_j^2) = 1/(2mkT) \tag{14}$$

where $n/3$ is the number of molecules in the gas. The final equality of equations (14) amounts to the definition of the perfect gas scale for the absolute temperature T, where k is Boltzmann's constant. Substituting in equation (12) from (14), and then integrating over all the v_j except one of them, yields the probability density W_1 for a Cartesian component v of the velocity of a single molecule, namely

$$W_1 = (m/2\pi kT)^{\frac{1}{2}} \exp (-mv^2/2kT) \tag{15}$$

which is the distribution of Maxwell for the molecular velocity in a perfect gas.

A derivation very similar to that just given for equation (15) yields the equilibrium velocity distribution in a Fermi gas or Bose gas. We surmise that most of the statistical results on the transport properties of neutral gases[30] could be obtained starting from equation (A6-15) Varients of the full FP equation (A6-15) have been used in the study of the transport properties of plasmas [31]. The above derivation of the

distribution of Maxwell is about as rigorous as any. It is based on more detailed probability concepts than are some derivations. Its virtue is that it illustrates on familiar ground a procedure for treating more involved problems as Markov processes.

APPENDIX A3

SHOT NOISE

In the present study, the averaging time τ is perhaps the most subtle feature. The physical meaning of τ lies in the limited detail of the data, either through intent or through the limitations of apparatus. In quantum theory as well as in communication engineering, such an averaging time is familiar. The present section contains an analysis of shot noise as an example in which the role of an averaging time appears.

Consider a uniform beam of light incident on a photomultiplier tube. The output current $I(t)$ from the tube is

$$I(t) = e \sum_j F(t - t_j) \tag{1}$$

where e is the electronic charge, where t_j is the time at which the jth photoelectron is released from the photocathode, and where $F(t - t_j)$ is the short pulse of output current produced by this photoelectron. The measured current I_O averaged over a time interval τ is, by equation (1)

$$I_o(t) = \left({}_t\int^{t + \tau} \right) \sum_j F(t' - t_j) \, dt'/\tau \tag{2}$$

Apparatus whose temporal frequency resolution is $1/\tau$ will necessarily average over τ. In a good photomultiplier tube, the halfwidth of the pulse $F(t)$ is about $3 \cdot 10^{-9}$ sec. In the present section, we assume that τ is much greater than this halfwidth, or that the frequency cutoff of the apparatus is less than 10^8 Hz. The more involved case where τ may be less than this halfwidth is discussed by S. O. Rice[28].

Let n denote the number of photoelectrons actually released during the interval τ. Let $\langle n \rangle$ be the expected number released, where the brackets $\langle \ldots \rangle$ denote an ensemble average. Then

$$\langle I_o \rangle = e \left\langle \left({}_t\int^{t + \tau} \right) \sum_j F(t' + t_j) \, dt'/\tau \right\rangle \tag{3}$$

where

$$\left({}_t\int^{t + \tau} \right) F(t' - t_j) \, dt' = 1 \tag{4}$$

provided the epoch t_j lies within the interval $(t, t + \tau)$. Similarly

$$\langle I_o^2(t) \rangle = e^2 \langle (t \int t + \tau) \sum_j F(t' - t_j) \, dt'$$

$$(t \int t + \tau) \sum_h F(t'' - t_h) \, dt'' \rangle / \tau^2$$

$$= e^2 \langle n^2 \rangle / \tau^2 \tag{5}$$

The light beam is considered non-degenerate, or with an average of much less than one photon per normal mode of the radiation field. This assumption suits light from thermal sources of temperature less than 10^4 $^{\circ}$K, as well as laser light after it is sufficiently attenuated. With this assumption, the probability per unit time for release of a photoelectron is constant over τ, and further the probability for release of any other photoelectron. We write

$$n = n_1 + n_2 + \ldots + n_m \tag{6}$$

where n_i is the random number of photoelectrons released during the ith of m equal time intervals τ/m which sum to τ where $m \gg 1$. The mean $\langle n_i \rangle$ and variance σ_i^2 of an n_i satisfy

$$\langle n_i \rangle = \langle n \rangle / m \ll 1 \quad , \quad \langle n_i^2 \rangle = \langle n^2 \rangle / m$$

$$\sigma_i^2 = \langle n_i^2 \rangle - \langle n_i \rangle^2 \simeq \langle n \rangle / m \tag{7}$$

Hence, by the central limit theorem, n satisfies a Gaussian probability density whose variance σ^2 is

$$\sigma^2 = \langle n^2 \rangle - \langle n \rangle^2 = m \sigma_i^2 = \langle n \rangle \tag{8}$$

The variance σ_I of the output current I_o, by equations (3), (5) and (8) satisfies

$$\sigma_I^2 = \langle I_o^2 \rangle - \langle I_o \rangle^2 = e^2 (\langle n^2 \rangle - \langle n \rangle^2) / \tau^2$$

$$= e^2 n / \tau^2 = \langle I_o \rangle e \Delta f \tag{9}$$

where $1/\tau = \Delta f$ is the frequency bandwidth of the apparatus. When positive and negative frequencies are on an equal footing, which is natural when an imaginary valued exponent is used to represent a sinusoid, then equation (9) becomes

$$\sigma_I^2 = 2e \langle I_o \rangle \Delta f \tag{10}$$

which is the standard shot noise formula much used in electronic and optical engineering. The signal-to-noise ratio $\langle I_o \rangle / \sigma_I$, by equations (3) and (9), satisfies

$$(\langle I_o \rangle / \sigma_I)^2 = \langle n \rangle \tag{11}$$

which is simply the expected number of photoelectrons released from the photocathode during an observation time interval.

APPENDIX A4

WAVE SMOOTHING IN A UNIFORM MEDIUM

When a wave packet or wave train traverses a random medium, one effect is a growing irregularity in the wavefronts as a result of their passage through the nonuniformities of the medium. As a second effect, the wavefronts are continually made more regular during their propagation, by the eventual overlap of rays from neighboring hillocks in a wavefront and by diffraction. It is the partial cancelation between these two effects which produces the net wavefront distortion. In the present section, we treat only the second of these effects. The analysis is related to optical coherence, discussed at length by Born and Wolf[32] and by L. Mandel[33].

A purely sinusoidal scalar wave train which moves in the positive z direction of a Cartesian xyz space is represented by

$$E = E_o \exp\left[i(kz - \omega t)\right] \tag{1}$$

where E is wave displacement, E_0 is wave amplitude, z is distance measured along the z-axis, t is time, $\lambda = 2\pi/k$ is wavelength, and ω is the circular frequency. We consider the case where the wave train differs slightly from the idealized from (1) and in which

$$E = U(\vec{r}) \exp\left[i(kz - \omega t)\right] \tag{2}$$

where $U(\vec{r})$ is a complex valued function of the three-dimensional position $r = (x, y, z)$. The wave (2) is assumed to move in a uniform medium for which the wave equation is

$$E_{tt} - c^2 \nabla^2 E = 0 \tag{3}$$

where a subscript z or t will denote partial differentiation. We make the partial wave expansion

$$U(\vec{r}) = \int u(z, \vec{\sigma}) \exp(i\vec{\eta} \cdot \vec{\sigma}) \, d\vec{\sigma} \tag{4}$$

where $\vec{\eta} = (x, y)$ is vector position measured from the z-axis, where $\sigma = (\sigma_x, \sigma_y)$ is a two-dimensional spatial frequency, and u is a partial wave coefficient. Substitution in equation (3) from (2) and (4) yields

$$u_{zz} + 2iku_z - \sigma^2 u = 0 \tag{5}$$

We assume $U(x, y, 0)$ to be a random function of x and y. Then $U(\vec{\sigma}, z)$ is also random, and enters in a random process whose state

functions, with v as a new notation for u_z, are

$$u(z,\vec{\sigma}), \quad v(z,\vec{\sigma}), \quad u*(z,\vec{\sigma}), \quad v*(z,\vec{\sigma}) \qquad (6)$$

where * denotes a complex conjugate. That is, z is a time-like variable in the process, and there is a distinct quadruplet of state functions (6) for each distinct quadruplet of state functions (6) for each distinct value of $\vec{\sigma}$. We assume the scale length of $U(x, y, 0)$ to be large compared to a wavelength λ or

$$\sigma \ll k \qquad (7)$$

or the partial waves are almost collimated. Then the change in u and v will approach 100 percent only over a wave pathlength Λ which is very large compared to λ. In equations (A6-16) we choose τ to satisfy

$$\lambda \ll \tau \ll \Lambda \qquad (8)$$

Then by equations (A6-16) and (5)

$$\{u\} = v + 9(\tau) \quad , \quad \{v\} = -2ikv + \sigma^2 u + o(\tau) \qquad (9)$$

$$\{uu\} = \{vv\} = o(\tau)$$

We will use the abbreviation

$$u_i = u(z,\vec{\sigma}_i) \quad , \quad v_i = v(z,\vec{\sigma}_i) \qquad (10)$$

By equation (A7-3) and (9)

$$\langle u_i^* u_j \rangle_z = \langle u_i^* \{u_j\} + u_j \{u_i\} \rangle = \langle u_i^* v_j \rangle + \langle u_j v_i^* \rangle \qquad (11)$$

where, as henceforth, we omit terms of order τ. Similarly

$$\langle u_i^* v_j \rangle_z = \langle u_i \{v_j\} + v_j \{u_i^*\} \rangle$$
$$= -2ik \langle u_i^* v_j \rangle + \sigma_j^2 \langle u_i^* u_j \rangle + \langle v_i^* v_j \rangle$$

$$(12)$$

$$\langle u_j v_i^* \rangle_z = \langle u_j \{v_i^*\} + v_i^* \{v_j\} \rangle$$
$$= 2ik \langle u_j v_i^* \rangle + \sigma_i^2 \langle u_j u_i^* \rangle + \langle v_i^* v_j \rangle$$

$$(13)$$

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Again by equations (A7-3) and (9), we have

$$\langle v_i^* v_j \rangle_z = \langle v_i^* \{v_j\} + v_j \{v_i^*\} \rangle = \sigma_i^2 \langle u_i^* v_j \rangle + \sigma_j^2 \langle v_i^* u_j \rangle \tag{14}$$

We have now to integrate the four equations (11), (12), (13), (14).

The z derivative of equation (11) is, by (12) and (3)

$$\langle u^* u \rangle_{zz} = 2ik \langle v^* u - u^* v \rangle + 2\sigma^2 \langle u^* u \rangle + 2 \langle v^* v \rangle \tag{15.1}$$

where we have omitted subscripts i and j. Subtracting equation (13) from (12) yields

$$\langle u^* v - uv^* \rangle_z = - 2ik \langle u^* u \rangle_z \tag{15.2}$$

Equation (14) may be expressed as

$$\langle v^* v \rangle_z = \sigma^2 \langle u^* u \rangle_z \tag{15.3}$$

To solve the three equations (15), consider the trial solution

$$\langle u^* u \rangle = A \exp(az) , \quad \langle v^* v \rangle = B \exp(az) \tag{16}$$

Substituting in equations (15) from (16) yields

$$Aa^2 = - 2ikC + 2\sigma^2 A + 2B$$
$$Ca = - 2ikaA , \quad Ba = \sigma^2 Aa \tag{17}$$

Equations (17) are satisfied if

$$a = 0 , \quad 2ikC = 2\sigma^2 A + 2B \tag{18}$$

They are also satisfied if $a \neq 0$ and

$$a^2 = - 4k^2 + 4\sigma^2 \tag{19}$$

or since $k \gg \sigma$, if

$$a = \pm 2ik \mp \sigma^2/k \tag{20}$$

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Thus, the complete solution for $\langle u^*u \rangle$ is of form

$$\langle u^*u \rangle = A_o + A_1 \exp \left[(2ik - i\sigma^2/k)z \right]$$

$$+ A_2 \exp \left[(-2ik + i\sigma^2/k)z \right] \tag{21}$$

whose secular part is

$$\langle u^*u \rangle = A_o + A_1 \exp (-i\sigma^2 z/k)$$

$$+ A_2 \exp (i\sigma^2 z/k) \tag{22}$$

where A_0, A_1 and A_2 are functions to be determined, dependent on σ but not on z.

The spectral density of U by equations (22) is

$$\Psi(z, \vec{\sigma}) = \Psi(0, \vec{\sigma}) \cos (\sigma^2 z/k) \tag{23}$$

Hence, by the Wiener-Khinchin theorem, the covariance of the wave train (2) is, with $\vec{\eta} = (x, y)$

$$\langle E(r)\, E(\vec{r} + \vec{\eta}) \rangle = \int \Psi(0, \vec{\sigma}) \cos (\sigma^2 z/k) \exp (i\vec{\eta} \cdot \vec{\sigma}) \, d\vec{\sigma} \tag{24}$$

which is the chief result of this section. Equation (24) gives precision to the summary statement: that the random hillocks of scale length $1/\sigma = S$ in an almost collimated wave train will fade by a factor of about e during a wave propagation path z of the value $z = S^2/\lambda$. Here the area S^2 is about equal to the first Fresnel zone of a wavefront when viewed normally from a distance $z = S^2/\lambda$. Otherwise spoken, S^2/λ is the Fresnel propagation path for a wavefront irregularity of area S^2.

Consider an example. Suppose a sonic boom traverses a layer of turbulent tropopause with a typical thickness of 1 or 2 km, and with a typical altitude of 10 km. When the downgoing boom leaves the underside of the tropopause, its wavefront irregularity of scale length $S \sim 100m$ in those of its important partial waves with wavelength $\lambda \sim 1$ m will be largely smoothed away in a pathlength $z \sim S^2/\lambda \sim 10$ km.

APPENDIX A5

BROWNIAN MOTION

A pollen grain or other colloidal particle immersed in a liquid or gas moves randomly. Its motion, called Brownian motion, may be seen through a microscope when the particle is suitably illuminated. This obscure phenomena, following A. Einstein[34], became a cornerstone of the molecular theory of matter, and played a role in the early development of quantum theory[35]. Brownian motion has become a traditional proving ground for viewpoints on random processes, as shown for example by the articles collected by N. Wax[36]. In the present section, we treat Brownian motion as an illustration of the ideas of the main text.

Let q_i and $p_i = mq_{it}$ be the Cartesian coordinates of the position $\vec{q}$ and momentum $m\vec{q}_t$ of the colloidal grain, where m is the mass of the grain, where $i = 1, 2, 3$ and where a subscript t denotes differentiation relative to time. We write

$$p_{it} + \gamma p_i = f_i(t) \ , \ q_{it} = p_i/m \ ; \ i = 1, 2, 3 \tag{1}$$

as the Hamiltonian form of the equations of motion for the particle. The new quantity γ expresses the frictional drag of the ambient fluid on the particle. Stokes law for the slow fall of a sphere through a fluid says γ is independent of time provided m is $\succ 10^8$ times the mass of a fluid molecule. The function $f_i(t)$ is the rate of transfer of momentum to the ith component of momentum of the grain, as a result of the collisions of ambient molecules with the grain. The first of equations (1) is commonly termed Langevin's equation, a title which we prefer to use for the state propagation for any Markov process.

The function $f_i(t)$ is commonly viewed as random, with zero mean, with a correlation time t_c not greatly in excess of the mean time between collisions of molecules with the grain, and with statistics independent of the values of p_i and q_i. Under this viewpoint, which we will use, the six $p_i(t)$ and $q_i(t)$ comprise a complete set of state functions $F_i(t)$ for a Markov process with propagation equations (1). We remark, however, that this viewpoint has a difficulty, since the statistics of these $F_i(t)$, by a simple physical intuition, depend at least slightly on recent past history of the $p_i(t)$, the moreso the greater the ratio of q_{it} to the mean molecular velocity. We surmise that the way through this difficulty is to take as state functions the full aggregate of coordinates and momenta of the colloidal grain and of the ambient molecules also. The statistics of this enlarged set of state functions appears to be strictly Markovian and with propagation equations more involved than (1). We have not yet found an analysis of this more accurate type of model, however.

We write

$$f_i(t) = e\breve{f}_i(t) \tag{2}$$

where $\breve{f}_i(t)$ is by definition a function whose root mean square is about one, and where ϵ is a small constant. Although the six state functions p_i, q_i with $i = 1, 2, 3$ and with the Langevin equations (1) do not comprise a strict Markov process without violence to the laws of molecular motion; we will show that these p_i, q_i do comprise an almost Markov process. The situation is thus very analogous to that of Section 8. With the six state functions p_i, q_i as a Markov process, the FP equation (A6-15) is easily written as

$$W_t = \sum_i \frac{\partial}{2p_i}(\{p_i\}W) - \sum_i \frac{\partial}{\partial p_i}(\{q\}W) + \dots \tag{3}$$

The various Planck coefficients in equation (3), by equations (A6-15) are

$$\{p_i\} = -\gamma p_i + (+\int t + \tau)\langle f_i(t')\rangle_c \, dt'/\tau + o(\tau) \tag{4.1}$$

$$\{q_i\} = p_i/m + o(\tau) \tag{4.2}$$

$$\{p_i p_j\} = (t\iint t + \tau)\langle -\gamma p_i(t')f_j(t'') - \gamma p_j(t')f_i(t'') + f_i(t')f_j(t'')\rangle_c \, dt'dt''/ \tag{4.3}$$

$$\{q_i q_j\} = o(\tau) \tag{4.4}$$

$$\{p_i q_j\} = (t\iint t + \tau)\langle f_i(t') p_j(t'')/m\rangle_c \, dt'dt''/c + o(\tau) \tag{4.5}$$

Einstein[34], in his treatment of Brownian motion, introduces a time interval τ which he states "is to be very small compared with the observed interval of time, but, nevertheless of such magnitude that the movements executed by a particle in two consecutive intervals of time τ are to be considered as mutually independent phenomena." This statement was with some justice found objectionable by various authors[37], who then produced a theory of Brownian motion without use of this statement. The particular τ introduced in Appendix A6 is the same as Einstein's τ if within the above quotation we replace the phrase "movements executed by a particle" by the phrase "small changes in the six state variables p_i, q_i almost always for $t_0 \gg \tau$." The time t_0 here denotes the epoch of observation measured from an origin of time at which the six coordinates are known or at least have known probability densities. The above slight change in Einstein's statement makes it unobjectionable, we believe. This change, which reflects relatively recent random process theory, makes his statement a basis for the viewpoint of the present study.

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We have, for example, by Appendix A6

$$\langle p_i \rangle_t = \langle \{p_i\} \rangle , \quad \langle q_i \rangle_t = \langle \{q_i\} \rangle \tag{5.1}$$

$$\langle p_i p_j \rangle_t = \langle p_i \{p_j\} + p_j \{p_i\} + \{p_i p_j\} \rangle \tag{5.2}$$

$$\langle p_i p_j \rangle_t = \ldots \quad (5.3 \qquad \langle q_i q_j \rangle_t = \ldots \tag{5.4}$$

A substitution in equation (5.1) from (4) yields with $\langle f_i \rangle = 0$,

$$\langle p_i(t) \rangle_t = - \gamma \langle p_i(t) \rangle \tag{6}$$

or with p_{io} as the initial value of $\langle p_i(t) \rangle$

$$\langle p_i(t) \rangle = \langle p_{io} \rangle \exp(-\gamma t) \tag{7}$$

Also from equations (5.1) and (4), we have

$$\langle q_i \rangle_t = - \langle p_i(t) \rangle / m \tag{8}$$

or

$$\langle q_i \rangle = \frac{P_o}{m\gamma} [1 - \exp(-\gamma t)] \tag{9}$$

Similar substitution in equation (5) from (4) yield

$$\langle p_i p_j \rangle_t + 2\gamma \langle p_i p_j \rangle = \langle f_j p_i \rangle + \langle f_i p_j \rangle$$

$$+ (t \iint t + \tau) \langle f_i(t') f_j(t'') \rangle \, dt' dt'' / \tau \tag{10.1}$$

$$\tag{10.2}$$

$$\langle p_i q_j \rangle_t = \langle p_i p_j \rangle / m - \gamma \langle p_i q_j \rangle + (t \int t + \tau) \langle f_i q_j \rangle \, dt' / \tau$$

$$\tag{10.3}$$

$$\langle q_i q_j \rangle_t = [\langle q_i p_j \rangle + \langle q_j p_i \rangle] / m$$

We now attend to the cross moments between the random state functions p_i, q_i and the random forces f_i which appear in equations (10). We have

$$\langle f_i q_j \rangle_t = \langle f_i \{q_j\} \rangle = \langle f_i p_j \rangle / m \tag{11}$$

$$\langle f_i p_j \rangle_t = -\gamma \langle f_i p_j \rangle + (t \int t + \tau) \langle f_i(t) f_j(t) \rangle \, dt'/\tau \tag{12}$$

The final term of equation (12) may be ignored since it is proportional to $\langle t_c/\tau \rangle \ll 1$, where t_c is the mean molecular collision time. Since $\langle f_i q_j \rangle$ and $\langle f_i p_j \rangle$ are zero at $t = 0$, they are thus by (12) ignorable at all times. We write

$$(t \iint t + \tau) \langle f_i(t') f_j(t'') \rangle \, dt' dt''/\tau \simeq (-\infty \int \infty) \langle f_i(t) f_j(t') \rangle \, dt' \, \delta_{ij} \equiv \varphi \, \delta_{ij} \tag{13}$$

as the definition of the integral covariance φ of f_i. Then equation (10.1) becomes

$$\langle p_i^2 \rangle_t + 2\gamma \langle p_i^2 \rangle = \varphi \tag{14}$$

whose solution is

$$\langle p_i^2 \rangle = \frac{\varphi}{2\gamma} [1 - \exp(-2\gamma t)] \tag{15}$$

where we now assume $p_{io} = 0$. Substituting in equation (10.2) from (15) yields, with $p \equiv p_i$ and $q \equiv q_i$

$$\langle pq \rangle_t + \gamma \langle pq \rangle = \frac{\varphi}{2\gamma m} [1 - \exp(-2\gamma t)] \tag{16}$$

whose solution is

$$\langle pq \rangle = \frac{\varphi}{2\gamma^2 m} [1 - 2\exp(-\gamma t) + \exp(-2\gamma t)] \tag{17}$$

where the constant of integration is chosen corresponding to $q = 0$ at $\iota = 0$.

Substituting in equation (10.3) from (17) yields a simple equation whose time integral is

$$\langle q^2 \rangle = \varphi t/(\gamma^2 m^2) - [1 - 2e^{-\gamma t} + e^{2\gamma t}] \varphi/(\gamma^3 m^3) \tag{18}$$

At times of usual experimental interest, the first term on the right of equation (18) dominates. Equation (18) exhibits the diffusion constant $D = \varphi/2\gamma^2 m^2$ whose measurement gives information on the microscopic nature of molecular motion.

APPENDIX A6

THE FOKKER-PLANCK EQUATION

The present section is a brief introduction to random process theory, insofar as it is needed in our sonic problem. The section begins with definitions and ends with the Fokker-Planck equation. More sophisticated introductions are by P. Mandl[38] and I. Gikhman[17].

A random variable f is defined as a quantity, such as the position of a given particle at a given time, which plays an identifiable role in some possibly observable process; and whose value may be any of many values whose aggregate is called the ensemble E_f for f. The quantity $h(t, \vec{s})$ of the previous section is a single random variable for each single set of four real numbers which specify values of t and $\vec{s}$. The rules of choice for an ensemble E_f, which are not expected to become ever fully codified, touch on the growing front of applied probability, and are discussed by I. J. Good[39]. A random variable is connected with output numbers through various ensemble averages. We remark that a random variable is not a function in any simple sense.

A random function F(t) is defined as a function whose dependence on its independent variable t is given only by the requirement that this dependence is the same as that of some single function in a set of functions

$$_1F(t) \ , \ _2F(t) \ , \ _3F(t) \ , \ \ldots \tag{1}$$

called in the aggregate the ensemble E_F for F. A random process $\vec{F}(t)$ is defined as a set of functions $F_i(t)$

$$F_1(t) \ , \ F_2(t) \ , \ \ldots \tag{2}$$

whose time dependence is given only by the membership of $\vec{F}(t)$ in some set

$$_1\vec{F}(t) \ , \ _2\vec{F}(t) \ , \ \ldots \tag{3}$$

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whose members have equal statistical weight, and which comprise the ensemble $E_{\vec{F}}$ for $\vec{F}$. A single set ${}_i F(t)$ is called a sample history of the random process. A state of the random process at time t_o is in general a set of numbers

$$_i F_1(t_o) \ , \ _i F_2(t_o) \ , \ \ldots \tag{4}$$

provided simply that the ensemble (3) propagates in time in a manner determined at least in principle. In our sonic problem in particular, there is a propagation equation for each member ${}_i \vec{F}(t)$ of this ensemble. The Hamiltonian variables for a mechanical system comprise an example of state functions which determine a continuous sequence of statistical states of the system.

We define the master density $W(t, \vec{F})$ for any random process $\vec{F}$ as a function whose running variables are the time like variable t and all the state functions $F_i(t)$ which comprise $\vec{F}$; and such that $W(t, \vec{F})$ is the probability that $\vec{F}$ lie in a range $d\vec{F}$ at time t. By $d\vec{F}$ we denote $dF_1 dF_2 \ldots$. The full solution of a random process centers on finding $W(t, \vec{F})$ explicitly from conditions which determine it only implicitly. Frequently, however, an easier task is adequate, namely finding various moment of W such as

$$\int F_i^2 F_j \, W(t, \vec{F}) \, d\vec{F} \tag{5}$$

an integral taken over all $\vec{F}$ space. The transition function

$$\Omega(t_o, \vec{F}_o; t_1, \vec{F}_1) \tag{6}$$

for a random process $\vec{F}$ is defined by the remark that ΩdF_1 is the probability at time t_1 that $\vec{F}$ lie in range dF_1 containing $\vec{F}_1$, under the proviso that $\vec{F}$ had value F_o at time t_o. The relation

$$W(t + \tau, F) = \int W(t, \vec{F}') \, \Omega(t, \vec{F}'; t + \tau, \vec{F}) \, d\vec{F}' \tag{7}$$

is evident, at least on second thought, from the definitions of W and Ω. Equation (7) is called by various authors the Smoluchowski or the Chapman-Kolmogorov equation. We have

$$\int \Omega(t, \vec{F}'; t + \tau, \vec{F}) \, d\vec{F} = 1 \tag{8}$$

since no matter what the state of the system at time t, it must be in at least some state at time $t + \tau$. Thus an integration of equation (7) over $\vec{F}$ space yields simply $1 = 1$. Equation (7) expresses the mass conservation of a conceptual fluid moving in Euclidean space with one dimension for each F_i, a fluid with mass density W and total mass 1. If initial

conditions require

$$W(t_o, \vec{F}) = \delta(\vec{F} - \vec{F}_o) \tag{9}$$

where δ denotes the many dimensional delta function, then

$$W(t, \vec{F}) = \Omega(t_o, \vec{F}_o; t, \vec{F}) \ , \ \text{all } t > t_o \tag{10}$$

We do not mention here the many recognized genera of random processes. But special interest attaches to the case where $\Omega(t, \vec{F}'; t + \tau, \vec{F})$ is independent of the time epoch t, so that the transition probability may be denoted by $\Omega(t, \vec{F}', \vec{F})$. Then equation (7) becomes

$$W(t + \tau, \vec{F}) = \int W(t, \vec{F}') \ \Omega(\tau, \vec{F}', \vec{F}) \ d\vec{F}' \tag{11}$$

which expresses that the dependence of $W(t, \vec{F})$ on $\vec{F}$ at any time is determined by its dependence on $\vec{F}$ at any earlier time. In this case, $F(t)$ is called a Markov process in standard usage. In a Markov process, the master density itself develops fully deterministically.

The integral equation (11) becomes more tractable in applications after a translation to an equivalent differential equation, which is performed most easily as follows[4C]. Consider the integral I, where

$$I \equiv \int W_t(t, \vec{F}) \ Q(\vec{F}) \ d\vec{F} \ ,$$

$$W_t(t, \vec{F}) \equiv [W(t + \tau, \vec{F}) - W(t, \vec{F})]/\tau \tag{12}$$

where $Q(F)$ is any continuous random function which vanishes at $F_i = \pm\infty$ for each F_i. A substitution in equation (12) from the Taylor series

$$Q(\vec{F}) = Q(\vec{F}') + \sum_i \frac{\partial Q(\vec{F})}{\partial F_i} (F_i - F_i') + \ldots \tag{13}$$

and from equation (11) yields

$$I\tau = \int W(t, \vec{F}) \ \Omega(\tau, \vec{F}', \vec{F}) \cdot$$

$$[Q(\vec{F}') + \sum_i \frac{\partial Q}{\partial F_i} (F_i' - F_i) + \ldots] \ d\vec{F}' \ d\vec{F}$$

$$- \int W(t, \vec{F}) \ Q(\vec{F}) \ d\vec{F} \tag{14}$$

By equation (8), the first and last terms of equation (14) cancel. Integration by parts then yields

$$W_t(t,\vec{F}) = - \sum_i \frac{\partial}{\partial F_i} (\{F_i\}W) + \frac{1}{2} \sum_{ij} \frac{\partial^2(\{F_iF_j\}W)}{\partial F_i \partial F_j} + \ldots \qquad (15)$$

where

$$\{F_i\} \equiv \int (F_i' - F_i) \, \Omega(\tau,\vec{F},\vec{F}') \, d\vec{F}'/\tau \qquad (16.1)$$

$$\{F_iF_j\} \equiv \int (F_i' - F_i)(F_j' - F_j) \, \Omega(\tau,\vec{F},\vec{F}') \, d\vec{F}'/\tau \qquad (16.2)$$

where the integrals are over all $\vec{F}$ space. Equation (15) is called the Fokker-Planck (FP) equation for the Markov process $\vec{F}$ and the conditional averages (16) are its Planck coefficients. Equation (15) is the general FP equation, in distinction to its special cases in current use for example by plasma physicists.

Important intuitions about the FP equation (15) may be built on simple examples of which we now give one. Another is in Appendix A2. Suppose the ensemble (3) for the process $\vec{F}$ consists of a single member, say $G(t)$. Then the transition function becomes

$$\Omega(\tau,\vec{F},\vec{F}') = \delta(\vec{F}' - \vec{G}(t + \tau)) \qquad (17)$$

In the limit of small τ, equation (16) becomes

$$\{F_i\} = \partial G_i/\partial t , \quad \{F_iF_j\} = \{F_iF_jF_k\} = \ldots = 0 \qquad (18)$$

A substitution in the FP equation from (18) yields

$$W_t = - \sum_i \frac{\partial}{\partial F_i} \left(\frac{\partial G_i}{\partial t} W \right) = \frac{\partial G_i}{\partial t} \frac{\partial W}{\partial F_i} \qquad (19)$$

The solution of equation (19) gives

$$W(t,\vec{F}) = \delta(\vec{F} - \vec{G}(t)) \qquad (20)$$

as is easily seen by back substitution in equation (19) from (20). In retrospect, in problems in which the ensemble reduces to a single member, or in which randomness plays no role, the FP equation becomes trivial.

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APPENDIX A7
CLOSURE OF MOMENTS

The above master density $W(t, \vec{F})$ for a physical process is itself sometimes partly measured. An example[29] is a laser beam after a long path in weak turbulence, for which $W(\vec{F})$ has been found to be almost multivariate log-normal in the F_i's. More frequently, only a few moments of W are measured. The general moment of $W(t, \vec{F})$ is defined by

$$\int F_i^a F_j^b \ldots F_h^c \, W(t, \vec{F} \, d\vec{F}) \tag{1}$$

where $a, b, \ldots c$ are integral exponents. In expression (1), and as usual when the limits of integration are not explicitly written, it will be understood that the range for each variable of integration is from $-\infty$ to $+\infty$. Expression (1) is commonly called a moment of order $a + b + \ldots c$; but we will call it of degree $a + b + \ldots c$ since we will need the word "order" for another of its common purposes. The moment (1) is alternately denoted by

$$\langle F_i^a F_j^b \ldots F_k^c \rangle \tag{2}$$

In our sonic problem, as in most applied work, we may treat knowledge of $W(t, \vec{F})$ as equivalent to knowledge of all its moments. This is actually true[41] under involved but weak restrictions on the type of process.

We will throughout assume that $W(t, \vec{F})$ approaches 0 as any F_i approaches $\pm \infty$. We multiply equations (A6-15) by F_i and integrate over all $\vec{F}$ space transforming each term on the right of (A6-15) through integration by parts. The formal result is

$$\langle F_i \rangle_t = \langle \{F_i\} \rangle \tag{3.1}$$

Similarly, multiplying equation (A6-15) by $F_i F_j$ and integrating yields

$$\langle F_i F_j \rangle_t = \langle F_i \{F_j\} + F_j \{F_i\} + \{F_i F_j\} \rangle \tag{3.2}$$

And similarly

$$\langle F_i F_j F_k \rangle_t = \langle F_j F_k \{F_i\} + F_k F_i \{F_j\} + F_i F_j \{F_k\}$$

$$+ F_i \{F_j F_k\} + F_j \{F_k F_i\} + F_k \{F_i F_j\} + \{F_i F_j F_k\} \ldots \tag{3.3}$$

and so on. The heirarchy of moment equations (3) is in general equivalent to the FP equation. We here specialize on Markov processes

$$\partial F_i(t)/\partial t = J_i(t, F_i(t)) \tag{4}$$

whose sample histories satisfy deterministic propagation equations of form where $J_i(\vec{F})$ is some function of all the state functions F_i. In processes of physical interest, barring the simplest, the form of J_i is such that the right member of each equation (3.n), the nth in the sequence (3), contains a moment of integral degree greater than n. The heirarchy (3) is then solvable only by an approximation of some kind, which usually ignores all but the first few of equations (3). Finding and justifying such an approximation is the problem of the closure of moments. Many consider it the chief problem today in the analysis of well posed physical random processes.

We now formulate the above closure problem in more detail. The difference $F_i' - F_i$ in equations (A6-16) is strictly

$$F_i' - F_i = (t \int t + \tau) \quad \partial F_i(t')/\partial t' \, dt' \tag{5}$$

where t is the time at which each of the left sides of equations (A6-16) is evaluated. A substitution in equation (A6-16.1) from (5) and (4) yields

$$\{F_i\} = \int (t \int t + \tau) \, J_i(t', \vec{F}(t')) \frac{dt'}{\tau} \quad \Omega(\tau, \vec{F}, \vec{F}') \, d\vec{F}' + o(\tau) \tag{6.1}$$

where $o(\tau)$ denotes, as usual, a quantity of order τ, that is a quantity which satisfies

$$\lim_{\tau \to 0} \frac{o(\tau)}{\tau} = \text{finite quantity, possibly } 0 \tag{7}$$

Substituting similarly in equation (A6-16.2) yields

$$\{F_i F_j\} = \int (t \int\int t + \tau) \, J_i(t', \vec{F}) \, J_j(t'', \vec{F}) \frac{dt' \, dt''}{\tau} \quad \Omega(\tau, \vec{F}, \vec{F}') \, d\vec{F}' \ldots \tag{6.2}$$

where the limits of integration for each of t' and t'' are from t to t + τ. If the J_i are non-linear in the F_j, then obviously the moment equations (3) may be not strictly closed. We remark that even when the J_i are linear in the F_j, then equations (3) may be not strictly closed, which we soon show, but which may be less obvious.

Divide the interval t to t + τ by cuts, namely

$$t_1 = t + \Delta \, , \quad t_2 = t + 2\Delta \, , \quad \ldots \, , \quad t_{a-1} = t + (a - 1)\Delta \tag{8}$$

where $\Delta = \tau/a$. Then equation (6.1) may be put, as a approaches infinity,

$$\{F_i\} = \int J_i(t_1, \vec{F}(t_1)) \; \Omega(\Delta, \vec{F}, \vec{F}(t_1)) \; d\vec{F}(t_1) \Delta/\tau$$

$$+ \int J_i(t_2, \vec{F}(t_2)) \; \Omega(2\Delta, \vec{F}, \vec{F}(t_2)) \; d\vec{F}(t_2) \Delta/\tau + \ldots \tag{9}$$

For any function $Q(t', t'', \ldots \vec{F})$, let

$$\langle Q(t', t'' \ldots, \vec{F}) \rangle_c \tag{10}$$

denote the ensemble average of Q contingent on the possession by $\vec{F}$ at time t of the same values it has on the left of FP equation (A6-15). This defines the new brackets $\langle .. \rangle_c$. Then equation (6.1) clearly becomes

$$\{F_i\} = (t \int t + \tau) \; \langle J_i(t', \vec{F}(t')) \rangle_c \; dt'/\tau + o(\tau) \tag{11.1}$$

and equation (6.2) becomes similarly

$$\{F_i F_j\} = (t \int \int t + \tau) \; \langle J_i(t', \vec{F}(t')) J_j(t'', \vec{F}(t'')) \rangle_c \; dt' \, dt''/\tau \tag{11.2}$$

Useful methods for the closure of moments have in the past been specific to the type of process of interest, and we surmise will continue to be so. Hence we replace equation (4) by its special case

$$\partial F_i(t)/\partial t = \sum_j K(i - j; t) \; F_j(t) \tag{12}$$

in which the J_i are linear and homogeneous in the F_j. The coefficients K, of Toeplitz[42] form, like the F_j comprise a one-dimensional discrete array of functions of t. Equation (12) is general enough to include equation (4-17) provided that the K are regarded as determined only by membership in the ensemble for the random medium. $\vec{F}(0)$ is a sample initial condition, $F(t)$ a sample history for the state functions or waves, while $\vec{K}(t)$ is a set of the function $K(i - j, t)$, or a sample medium. Fixing $\vec{F}(0)$ and $\vec{K}(t)$ determines the $\vec{F}(t)$, or a sample scattering problem is deterministic through the solution of equation (12). We will assume the inverse, or that the fixing of $\vec{F}(0)$ and $\vec{F}(t)$ determines $\vec{K}(t)$. We have not fully studied the validity of the inverse but surmise it to be true except for weak restrictions. This inverse amounts to writing

$$K(j;t) = \breve{K}(j;\vec{F}(t)) \tag{13}$$

where $\breve{K}_j$ is a function of all the F_i's, or a function of t whose dependence on t is entirely through the functional form of the $F_i(t)$'s within the interval from 0 to t. Substituting in equations (11) from (12) yields

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$$\{F_i\} = (t \int t + \tau) \langle \sum_j K(i - j;t')F_j(t') \rangle_c \, dt'/\tau + o(\tau) \qquad (14.1)$$

$$\{F_iF_j\} = (t \int\int t + \tau) \langle \sum_{hk} K(i - h,t') K(j - k,t'') \cdot$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (14.2)$$

$$F_h(t') F_k(t'') \rangle_c \, dt' \, dt''/\tau + o(\tau)$$

The integrals in the right numbers of equations (14) cannot be carried out in simple fashion because by (13) the K's are functions of the F's of a form as yet unknown. But only the ensemble averages of such integrations are needed, as shown by equation (3), and these averages are easier to evaluate as shown in a later section.

We now pause from the main theme to explain the emphasis of the present section on the small interval τ. Many authors start with their Fokker-Planck equation as the limit of equation (A6-15) as τ approaches 0. Such a procedure, if followed consistently, requires an evaluation of the limiting forms of expressions (14) by intuition, which is not possible in our sonic problem with adequate accuracy we believe.

Our analysis through expressions (14) has been quite rigorous. It is the approximation in the evaluation of these expressions which determines the suitability of any mathematical model of a random process or the physical process itself, as implied by I. Oppenheim[43]. N. Wiener[44] introduces for the physical Brownian motion, and with great intuition, a mathematical model whose analysis he then completes rigorously without reference to expressions corresponding to (14). But the analysis of the slight gap between the physics of Brownian motion and his model requires expressions like (14). His model in effect ignores changes in statistics over times as short as a correlation time, which in Brownian motion is about equal to the mean free collision time of the ambient molecules. Our model also will ignore changes in statistics over a correlation time, which in our problem is the time t_c during which the N-wave moves over a suitably typical turbulent eddy diameter. Wiener's analysis is based on random functions which are everywhere nondifferentiable over a large interval. The present study deals only with continuous and differentiable functions, barring isolated points. We surmise, nevertheless, that all essentials of the present study could be based on compactly nondifferentiable functions, a task not yet completed. Such functions should provide a suitable model for waves in any weakly random medium whose fluctuations have a scale length small compared to wave pathlength, provided the statistics of the medium change slowly compared to the rate of change in the statistics of the waves.

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