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AN APPLICATION OF CONTROL THEORY METHODS TO THE OPTIMIZATION OF STRUCTURES HAVING DYNAMIC OR AEROLASTIC CONSTRAINTS

OCTOBER
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This investigation described in this report was supported by NASA Grant NGL-05-020-243 and by the Air Force Office of Scientific Research under Contract No. F44620-68-C-0036.

SUDAAR
NO. 412

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Department of Aeronautics and Astronautics
Stanford University
Stanford, California

AN APPLICATION OF CONTROL THEORY METHODS TO THE OPTIMIZATION OF STRUCTURES HAVING DYNAMIC OR AEROELASTIC CONSTRAINTS

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ABSTRACT

A great deal of interest and attention has recently been focused on the optimal design of structures. By optimal design it is meant that a structure performs the same function as another similar structure while minimizing some performance index, usually the weight of the structure. This study investigates some simple structures whose weights are minimized subject to several types of constraints involving fixed eigenvalues. These eigenvalues may be related to free vibration, in which case a least weight structure is determined while holding one or more natural frequencies constant. Similarly, the eigenvalues may be related to aeroelastic instabilities where a least weight structure is found while holding the flutter speed constant.

With one exception, the models are idealized one-dimensional structures with fixed geometry and spatial dimensions. These models are adequately described by a set of N simultaneous first-order ordinary differential equations which come from the general Nth order equilibrium equation. Methods adapted from optimal control theory are used to develop differential equations and boundary conditions which are necessary to ensure optimality. This optimization problem then becomes a two-point boundary value problem with $2N$ simultaneous non-linear differential equations.

The solution method used to solve these equations is an adaptation of a numerical technique used in optimal control theory which is referred to as the "transition matrix" procedure. This method involves perturbing the optimality equations and boundary conditions to find successive neighboring extremal solutions until the optimum design is reached.

Solutions presented include optimum weight configurations for beams and thin-walled cylinders whose bending or torsional vibration frequencies are held fixed and which may or may not have minimum thickness constraints. The problem of finding an optimum panel whose aerodynamic flutter parameter in high supersonic flow is specified is also studied. Finally a simple study is presented

which provides insight into the accuracy and usefulness of discrete or finite element methods when they are used to generate the structural model for an optimization search involving discrete parameters.

ACKNOWLEDGMENTS

The author wishes to express his appreciation to: Professor Holt Ashley who acted as principal advisor for this thesis and whose encouragement, judgment, and engineering sixth sense made this task easier; Professor Sam McIntosh who acted as "junior" advisor and whose excellent technical advice and embarrassing questions helped immensely; his fellow students in Professor Ashley's research group at Stanford who provided details for computer programming and acted as sounding boards for ideas; and last but never least, his wife Mary who somehow found time from her other duties to type this thesis.

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NOMENCLATURE

A	Arbitrary modal constant
$\tilde{e}$	Error vector for transition matrix
EI	Bending stiffness
GJ	Torsional rigidity
H	Hamiltonian
J	Performance index or merit function
L	Dimensional length
L( )	Integrand for J
MR	Mass ratio
M	Integration iteration number
m	Mass per unit length
N	Constraint equation order
p	Nondimensional deflection slope
q	Nondimensional bending moment
r	Nondimensional shear
s	Nondimensional torque
t(x)	Nondimensional thickness parameter
t_i	Design variable parameter
T	Dimensional thickness
$\overline{T}, T_{ij}$	Transition matrix
v	State variable
w	Nondimensional deflection
W	Dimensional deflection
X	Dimensional spacial coordinate
x	Nondimensional spacial coordinate (= X/L)
y	System variable, may be either state variable, adjoint variable, or combination
z_o	Frequency parameter for flutter

Z	Number of finite elements in beam model
α	Frequency parameter $(= \omega^2 \delta_1)$
β	Frequency parameter $(= \omega^2 (1 - \delta_1))$
$\tilde{\beta}_c$	Calculated vector of end conditions with specified values
$\tilde{\beta}_e$	Exact values of end conditions
δ_1	Ratio of initial structural mass to initial total mass
δ_2	Ratio of nonstructural mass to initial mass
$\delta(\ )$	Variation (in calculus of variations) or perturbation (in differential calculus)
$\tilde{\mu}$	Vector of control constants
κ	Step size for transition matrix
$\tilde{\epsilon}$	Error tolerance (greater than zero)
θ	Nondimensional rotation
ω	Frequency
ω_o	Reference frequency
λ_z	Lagrange multiplier or adjoint variable for state variable z
λ_o	Aerodynamic parameter for panel flutter
λ_o^*, λ^*	Critical aerodynamic parameter for flutter

Subscripts and Symbols

$(\ )_o$	Initial or reference value
$(\ )_f$	Final value
$(\ )'$	Differentiation with respect to x
$(\ )_i$	ith element of a vector
$(\ )(\ )$	Multiplication
$(\ )/(\ )$	Division

exp( )	Exponentiation
	Row matrix
	Column matrix
$(\ )^T$	Matrix Transpose
$(\ )^{-1}$	Matrix Inverse
$(\ )_{final}$	Final value
$(\ )_{int}$	Initial value

N. B. The above nomenclature is followed in the majority of the thesis. However, because the problems involve several areas of study there may be an overlapping of terminology. Where this is the case, this fact will be pointed out.

1. INTRODUCTION

1.1 Background

The optimization of structural configurations to achieve least weight designs, while preserving certain design requirements such as maximum strength, has received a great deal of attention over the last few years. This field of design optimization is still so new that many seemingly simple problems remain to be solved. The term "structural optimization" usually refers to the search for a structure which is similar to a reference or initial design structure, but which weighs less and fulfills the original design requirements and purpose of the initial structure. These design requirements or objectives may be viewed as constraints on the optimization search since the optimization search for a minimum weight structure must always be guided or constrained such that the original design objectives are preserved. The problem can then be said to be that of finding a minimum weight structure subject to a given set of constraints.

Design constraints usually fall into one of two categories, or they may be a combination of both. The first category is usually referred to as the static constraint such as would occur if the design structure is required to support a static load of a given magnitude. The second constraint category is the dynamic constraint, which will occur if an eigenvalue such as natural frequency of vibration is held fixed during the least weight search.

The field of aeroelastic optimization belongs in the category of dynamic constraints. Aeroelastic problems involve aerodynamically induced static or dynamic instability, however, in either case the problems involve the solution of an eigenvalue problem. Therefore, all aeroelastic optimization problems properly belong under the heading of "dynamic constraints."

Once a constraint criterion has been established, the actual choice of a solution or optimization method is rather broad. The optimization technique depends, first of all, on the choice of structural models. Structures, in the simplest cases, may be sufficiently described by differential equations of equilibrium or, if they are extremely complicated, they may be described by a discrete parameter technique which expresses the equilibrium equations as a matrix equation. Once the structural modeling method has been established, an optimization technique must be found which will reduce the weight while still satisfying the constraints placed on the problem. No matter what the structural modeling technique may be, optimization techniques are many and varied, with each researcher having his own favorites. In addition, each problem has its own peculiarities which may make one method nonapplicable and another perfectly adaptable. It has been noted many times that optimization is partly art, partly science.

Methods of solution for optimization problems are determined both by the structural model and the constraints. In the case of a differential equation structural model, one is led to the calculus of variations and a set of equations known as the Euler-Lagrange equations. Many problems in various scientific disciplines can be described in this way, the most notable being the field of optimal control theory. In the case of a discrete element model there are many techniques which use optimization theory to search for the optimum value of a finite set of design parameters such as material thicknesses or areas. For this reason, this area is commonly referred to as "parameter optimization."

1.2 Purpose

In the field or aircraft or missile engineering, optimum weight structures are of prime importance. Because of the so-called "growth factor," the addition of a pound of structural weight to an aircraft wing may cause the increase of the gross weight by several additional pounds. Conversely, the removal of an unnecessary pound may result in fuel savings which in turn lead to cost savings. For these and other reasons, the purpose of this thesis is not entirely academic, but also has practical engineering objectives as well.

This investigation has several purposes. First of all, using one-dimensional differential equation structural models, it seeks to find solutions to some previously unsolved problems having dynamic or fixed eigenvalue constraints. These solutions will be obtained using numerical techniques adapted from optimal control theory. The numerical techniques are designed for use on a digital computer and will be studied to determine relative advantages and disadvantages of each method. In addition, estimation techniques will be suggested and discussed together with interesting properties of the optimal solutions obtained. The study will also determine the relative weight savings of the same structural configuration with several different boundary conditions. In doing so, it is hoped that one may extrapolate from these results to formulate guidelines which may be used in "real world" engineering problems.

In addition to using the differential equation approach to structural modeling, a discrete or finite element model will also be studied. A gradient technique will be used to compare some "parameter optimization" solutions to those "exact" solutions obtained using differential equation techniques.

1.3 Scope

All of the structural models used in this investigation are idealized one-dimensional configurations such as beams in bending or thin-walled cylinders in torsion. By one-dimensional, we mean linear elastic systems whose deformation state can be adequately specified by a set of functions of a single spatial variable. The constraints imposed upon these models will be in the form of eigenvalue equations in which one or more of the eigenvalues are held fixed. These eigenvalue equations will describe conservative system motion such as free vibration and also nonconservative motion such as is encountered in aeroelastic problems.

Among the problems treated will be beam flexural vibration with multiple dynamic constraints and panel flutter in supersonic flow. As used in this study, the term "dynamic constraint" will refer to a situation where a structure has one or more of its natural frequencies fixed. The term "aeroelastic constraint" will refer to a structure which has a fixed aeroelastic eigenvalue such as flutter speed or critical dynamic pressure.

The structural models will include those whose characteristic stiffness per unit length is a linear function of its mass per unit length and those whose stiffness is a polynomial function of the mass per unit length. In all cases, nondimensional quantities will be used so that the optimum configuration is always referred to a base or reference structure with uniform properties.

1.4 Previous Investigations

If one were to list all papers involving some type of optimization, the list would be quite long. However, if only those papers dealing with structural optimization were retained, the list would be considerably shortened. If one went further and eliminated all those papers dealing with static constraints the resulting list would contain probably no more than thirty or forty references.

Ashley (Ref. 1) has covered the early history of dynamic and aeroelastic constraints and cites what he believes to be the earliest work in the field. Significantly, this paper is by Turner (Ref. 2) and was published as an internal report at Vought-Sikorsky Aircraft in 1942. In the field of optimization of one-dimensional structures, a study of beam flexural vibration by Niordson (Ref. 3) appears to be the first published. This paper was closely followed with articles by Turner (Ref. 4), who had first worked in the area twenty-five years earlier, Taylor (Ref. 5), and Prager and Taylor (Ref. 6). In the latter paper, a proof was given for the uniqueness of the solution to several problems involving conservative vibration problems.

The use of control theory techniques to solve simple aeroelastic optimization problems was first suggested by Ashley and McIntosh (Ref. 7). A following study by Armand and Vitte (Ref. 8) formulated some basic problems in more precise control theory terminology and described a perturbation matrix method used for the numerical solution of many aeroelastic and dynamic optimization problems. This suggestion by Ashley was a highly significant contribution and considerably advanced the state-of-the-art.

The primary numerical solution techniques used in both references (7) and (8) above were all adapted from those described by Bryson and Ho (Ref. 9). These methods, with some modification, are also used in this thesis. Further use will be made to some of the above references in later chapters of this investigation.

The field of weight optimization of actual engineering structures is more widely discussed in the literature. A paper by MacDonough (Ref. 10) in 1953 is regarded as the first real attempt at the design of a structure to satisfy flutter requirements. More recently, Schmit (Ref. 11) and several authors including Turner (Ref. 12) have dealt with optimization of discrete parameter systems. One particularly interesting paper by Rubin (Ref. 13) details a gradient method which is used to find minimum weight structures whose frequency may be specified or held within some tolerance. An adaptation of this technique is used in Chapter 7 of this thesis.

Although a flurry of optimization papers has appeared from time to time, the literature dealing with dynamic or aeroelastic optimization is surprisingly sparse. This thesis is intended not only to fill several gaps left by earlier papers, but to extend the solution methods to more complicated problems and, where necessary, to develop new techniques for solving aeroelastic optimization problems.

1.5 Methods of Approach

The general method of approach to both types of problems covered in this thesis will now be briefly outlined. First of all, the equations necessary to define the so-called "optimality conditions" for systems with differential equation constraints will be discussed. This theory, which is adapted from the field of optimal control is well developed and will be applied to some one-dimensional structural optimization problems in which the total mass of the structure is to be minimized. In all cases, the important variable will be a nondimensional thickness distribution parameter $t(x)$, which may be related to the stiffness and inertial properties of the structure. This nondimensional thickness parameter will be referenced to a uniformly dimensioned structure having the specified dynamic or aeroelastic eigenvalues. The overall geometry, such as the length, will be assumed fixed. The constraints which are imposed upon the optimal structure will be eigenvalue differential equations which will involve a fixed parameter, e.g., free vibration equations with a frequency held fixed. These eigenvalue constraint equations will be expressed as a set of N first-order, ordinary differential equations.

Given these N equations, the use of optimal control theory to generate necessary conditions for an optimal thickness distribution always yields a set of 2N first-order, nonlinear differential equations. Since the problem is one-dimensional, boundary conditions are specified at only two points, leaving one with the task of solving a nonlinear, two-point boundary value problem.

A numerical solution technique, commonly referred to as the "transition matrix" solution method, has been adapted from control theory to solve this nonlinear two-point boundary problem. This solution method involves the initial estimation of the so-called "natural boundary conditions" and a cyclic or iterative integration of the differential equations to arrive at a final solution.

The latter part of this thesis will briefly study two cases involving parameter optimization problems. The first of these two cases is the weight optimization of a finite-element structural model for a beam on simple supports with the requirement that the lowest frequency be held fixed. The second case studied is the optimization of a finite-element model for a simply supported panel of infinite aspect whose aerodynamic flutter parameter is held fixed. In both cases, an elementary gradient technique similar to that used by Rubin (Ref. 13) will be used, together with an eigenvalue perturbation method.

2. OPTIMIZATION THEORY FOR SYSTEMS HAVING LINEAR DIFFERENTIAL EQUATION CONSTRAINTS

2.1 Introduction

The development of the differential equations and boundary conditions which are necessary to find the optimal thickness parameter distribution for a minimum weight structure with eigenvalue constraints is, in general, straightforward and simple. It is the actual solution of the differential equation - boundary value system which poses the difficulty in defining the optimal system. In this section a review of the theory necessary to formulate the governing equations of the optimal system is given together with an example of a simple structural model involving torsional vibration at a given frequency.

The theory and nomenclature are taken from the theory of optimal control, as first suggested by Ashley and McIntosh (Ref. 7). The nomenclature and method of expressing the constraint equations are also taken from the same reference.

2.2 An Example of Constraint Equations — A Thin Walled Cylinder in Torsion

Consider (Figure 2.1) a thin-walled cylinder with the end $X = 0$ built in and the end $X = L$ free. The equation of free torsional vibration is given by

$$\frac{d}{dX}\left(GJ\frac{d\theta}{dX}\right) - m\ddot{\theta} = 0 \qquad 0 \le X \le L \tag{2.2.1}$$

where $GJ(X)$ is the torsional stiffness per unit length at station X and $m(X)$ is the moment of inertia per unit length at station X. If we assume harmonic motion i.e. $\theta(X,\tau) = \theta(X)e^{i\omega\tau}$ then equation (2.1.1) reduces to

$$\frac{d}{dX}\left(GJ\frac{d\theta}{dX}\right) + m\omega^2\theta(X) = 0 \tag{2.2.2}$$

If we were to solve the eigenvalue problem for uniform $GJ(X) = GJ_o$ and $m(X) = m_o$ we would find (Ref. 14) the nth eigenvalue equal to

$$\omega_n = \left(\frac{2n+1}{2}\right)\pi\sqrt{\frac{GJ_o}{m_o L^2}} \tag{2.2.3a}$$

with the nth eigenfunction

$$\theta_n(X) = A_n \sin(\omega_n \frac{X}{L}) \qquad (2.2.3b)$$

$n = 0, 1, ----\infty$

We may nondimensionalize equation (2.2.2) to the following equation

$$\frac{d}{dx}\left(\frac{GJ(x)}{GJ_o}\frac{d\theta}{dx}\right) + \left(\frac{m}{m_o}\right)\left(\frac{m_o L^2}{GJ_o}\right)\omega^2\theta(x) = 0 \qquad 0 \le x \le 1 \qquad (2.2.4)$$

where

$$x = \frac{X}{L}$$

If we let $\omega = \omega_o$ and note that, for a thin-walled cylinder, $GJ(x)$ is proportional to the thickness $T(x)$, then equation (2.2.4) becomes

$$\frac{d}{dx}\left(\frac{T(x)}{T_o}\frac{d\theta}{dx}\right) + \left(\frac{m}{m_o}\right)\left(\frac{\pi^2}{2}\right)\theta(x) = 0 \qquad (2.2.5)$$

Now, define a nondimensional thickness parameter

$$t(x) = \frac{T(x)}{T_o} \qquad (2.2.6)$$

and note that we may express m_o as

$$m_o = \rho T_o + \gamma$$

where γ is a nonstructural moment of inertia contribution and ρ is a constant which depends only on geometry so that

$$m(x) = \rho T_o\left(\frac{T(x)}{T_o}\right) + \gamma \qquad (2.2.7)$$

Finally, we can express the ratio of $m(x)$ to m_o as

$$\frac{m(x)}{m_o} = \delta_1 t(x) + \delta_2 \qquad (2.2.8)$$

where

$$\delta_1 = \rho\frac{T_o}{m_o} \quad \text{and} \quad \delta_2 = \frac{\gamma}{m_o} = 1 - \delta_1$$

The constraint equation now may be expressed as

$$(t\theta')' + (\pi/2)^2(\delta_1 t + \delta_2)\theta(x) = 0 \qquad 0 \le x \le 1 \tag{2.2.9}$$

The addition of the nonstructural mass term is seen as a necessary condition for a meaningful answer. An examination of equation (2.2.9) will show that, if the δ_2 term were not there, a possible solution to equation (2.2.9) could be one for which $t(x) = 0$. It can also be seen that, if GJ and m are linear functions of T(x), the frequency ω_n is independent of the thickness. Thus, one may expect that the zero thickness solution might be mathematically possible if $\delta_2 = 0$.

The boundary conditions for the above problem are

$$\begin{aligned} \theta(0) &= 0 \\ t\theta'(1) &= 0 \end{aligned} \tag{2.2.10}$$

The first condition requires zero rotation at the fixed end and the second condition states that no torque is applied at $x = 1$.

The above second-order eigenvalue constraint equation can be expressed as two simultaneous first-order equations by introducing a new variable

$$s = t\theta' \tag{2.2.11}$$

Thus equation (2.2.9) can be written as

$$s' = -(\alpha t + \beta)\theta \tag{2.2.12}$$

$$\theta' = s/t \tag{2.2.13}$$

where $\alpha = (\pi/2)^2\delta_1$ and $\beta = (\pi/2)^2\delta_2$ and $\theta(0) = s(1) = 0$.

Thus, we have taken the free vibration equation for a nonuniform thin-walled cylinder having an eigenvalue (the lowest frequency of free vibration) equal to that of a reference structure eigenvalue and reduced it to two first-order ordinary differential equations which are dimensionless and contain the dimensionless thickness parameter $t(x)$. This is the method which will be followed in all the problems which involve the use of continuum structural models.

2.3 Optimal Control Theory as Applied to Systems With First-Order, Ordinary Differential Equation Constraints With No Constraints on the Thickness Parameter

Some definitions of nomenclature are necessary before beginning the discussion of optimal control theory. These definitions are consistent with Bryson and Ho (Ref. 9). A scalar J, called the "performance index" or, "merit function" is used to define the quantity which is to be minimized. This quantity J is also variously known in the literature as the "return function" or the "payoff function." The applicability of these terms to controls problems is obvious.

In the field of structural optimization we look for an optimal distribution of a thickness or weight parameter such as $t(x)$, which will minimize a performance index J which is itself a function of the weight or thickness parameter. For the one-dimensional structure discussed in Section 2.2, the performance index can be written as

$$J' = B\int_0^1 \delta_1 \frac{T(x)}{T_o} dx + \delta_2 C \tag{2.3.1}$$

where B and C are constants related to the geometry of the structure. Since $\delta_2 C$ is a fixed constant, equation (2.3.1) can be just as simply stated as

$$J = \int_0^1 t(x)dx = \frac{J' - \delta_2 C}{B\delta_1} \tag{2.3.2}$$

The problem now involves minimizing J subject to a given set of differential equation constraints and boundary conditions such as those described in Section 2.2.

A brief review of optimal control theory as detailed in Bryson and Ho (Ref. 9) will be given now. Some symbols will be changed from their notation in order to conform to structural terminology. Also, while the discussion will not be rigorous, it will nonetheless be correct. Let us assume that the structural system is described by a set of N first-order, ordinary differential equations. (Each element of $\tilde{v}(x)$ is called a "state variable" because it partially defines the physical state of the system.)

$$\tilde{v}'(x) = \tilde{f}(v, t, x) \qquad x_o \le x \le x_f \tag{2.3.3}$$

The $(\tilde{\ })$ represents a vector or vector function. The boundary conditions are in general split, just as we saw in the simple torsion problem, with some specified at $x = x_o$ while others are specified at $x = x_f$.

Let us consider the so-called Bolza problem where the performance index is of the form

$$J = \phi(\tilde{v}(x_f)) + \int_{x_o}^{x_f} L(t, \tilde{v}, x)dx \tag{2.3.4}$$

For the previously discussed torsion problem, $\phi = 0$ and $L = t(x)$. Our problem is to find a function $t(x)$ which minimizes J subject to the specified constraint equations $\tilde{v}(x)$. We can accomplish this by "adjoining" the constraint equations (2.3.3) to the performance index J with multiplier functions $\lambda(x)$ (also called adjoint variables or Lagrange Multipliers) in the following manner.

$$J = \phi + \int_{x_o}^{x_f} [L + \tilde{\lambda}^T \left\{ \tilde{f} - \tilde{v}' \right\}]dx \tag{2.3.5}$$

For convenience, as in equation (2.3.3), the adjoint variables are written in vector form so that

$$\tilde{\lambda}^T = \lfloor \lambda_i \rfloor$$

Now, define a scalar function H (called the Hamiltonian)

$$H(t, \tilde{v}, x) = L + \tilde{\lambda}^T \tilde{f} \tag{2.3.6}$$

If we integrate the last term on the right-hand side of equation (2.3.5) by parts, we will have

$$J = \phi - \tilde{\lambda}^T(x_f)\tilde{v}(x_f) + \tilde{\lambda}^T(x_o)\tilde{v}(x_o) + \int_{x_o}^{x_f} (H + \tilde{\lambda}'^T \tilde{f})dx \tag{2.3.7}$$

Finally we approximate — to first order — the variation of J due to admissible weak variations or first order changes in the "control variable" $t(x)$ for fixed values of x_o and x_f.

$$\delta J = \left(\frac{\partial \tilde{\phi}}{\partial v} - \tilde{\lambda}^T\right)\delta\tilde{v}\,\Bigg|_{x = x_f} + \left(\tilde{\lambda}^T \delta\tilde{v}\right)\Bigg|_{x = x_o}$$

$$+\int_{x_o}^{x_f} \left\{\left(\frac{\partial \tilde{H}}{\partial v} + \tilde{\lambda}'^{\,T}\right)\delta\tilde{v} + \frac{\partial H}{\partial t}\delta t\right\} dx \tag{2.3.8}$$

where $\delta\tilde{v}$ are any "admissible" variations of the state variables $\tilde{v}$. The thickness parameter $t(x)$ is called the control variable because its behavior "controls" the state variable equations. From the constraint equations and boundary conditions, t determines $\tilde{v}$ in a complicated manner, but we do not want to go through the tedious process of determining the variations $\delta\tilde{v}$ caused by δt. Therefore, we arbitrarily choose the functions $\tilde{\lambda}(x)$ in such a way as to force the coefficients of the $\delta\tilde{v}$'s in equation (2.3.8) to vanish. We see then that

$$\tilde{\lambda}'^{\,T} = -\frac{\partial \tilde{H}}{\partial v} = -\frac{\partial \tilde{L}}{\partial v} - \tilde{\lambda}^T \frac{\overline{\partial f}}{\partial v} \tag{2.3.9}$$

If some "geometric" or state variable boundary conditions are specified in the form

$$v_j(x_o) - v_j^o = 0 \tag{2.3.10a}$$

$$v_j(x_f) - v_j^f = 0 \tag{2.3.10b}$$

then

$$\delta v_j(x_i) = 0 \tag{2.3.10c}$$

and

$$\delta v_j(x_f) = 0 \tag{2.3.10d}$$

Now, with the above conditions specified, the variation δJ in equation (2.3.8) becomes

$$\delta J = \left(\frac{\partial \tilde{\phi}}{\partial v} - \tilde{\lambda}^T\right)\delta\tilde{v}\,\Bigg|_{x = x_f} + \left(\tilde{\lambda}^T \delta\tilde{v}\right)\Bigg|_{x = x_o}$$

$$+\int_{x_o}^{x_f} \left(\frac{\partial H}{\partial t}\right)(\delta t)dx \tag{2.3.11}$$

If $\delta v_k \neq 0$ at $x = x_f$ then $\lambda_k = \frac{\partial \phi}{\partial v_k}$ at $x = x_f$, if δJ is to be zero. If $\delta v_i(x_o) \neq 0$, as would be the case if no geometric or state variable boundary condition is specified at $x = x_o$, then the multiplier of λ_i must be zero at $x = x_o$ if δJ is to be zero.

$$\lambda_i(x_o) = 0 \text{ if } \delta v_i(x_o) \neq 0 \tag{2.3.12}$$

In control theory, the function $\frac{\partial H}{\partial t}$ is referred to as the "impulse response function" since it represents the variation in J caused by a unit impulse in δt at position x when $v(x_o), \lambda(x_o)$ and the constraint equations are held fixed. Finally, for an extremum of the performance index we must have δJ equal to zero for any admissible $\delta t(x) \neq 0$. As a result, we see from equation (2.3.11) that we must have

$$\frac{\partial H}{\partial t} = 0 \qquad x_o \leq x \leq x_f \tag{2.3.13}$$

if $\delta t(x) \neq 0$. Equation (2.3.13) is known as the "control equation" and is always an algebraic rather than a differential equation. It should be noted that equation (2.3.13) is only necessary if $\delta t(x) \neq 0$, that is, if there is no constraint placed on the control variable $t(x)$.

To summarize this development, we have found that, in the absence of constraints on the control variable t, we must solve the following set of differential equations to find a thickness parameter distribution $t(x)$ which produces an extremum of J, the performance index.

$$\tilde{v}'(x) = \tilde{f}(v, t, x) \tag{2.3.14}$$

$$\tilde{\lambda}' = -(\overline{f_v})^T \tilde{\lambda} - \left(\frac{\partial \tilde{L}}{\partial v}\right)^T \tag{2.3.15}$$

$$\frac{\partial H}{\partial t} = 0 = \left(\frac{\partial \tilde{f}}{\partial t}\right)^T \tilde{\lambda} + \frac{\partial L}{\partial t} \tag{2.3.16}$$

Equations (2.3.15) are known in the calculus of variations as the Euler-Lagrange equations.

In most structural optimization work the function $\phi(x_f, v(x_f))$ is zero and the resulting optimization problem is referred to as the "Lagrange problem."

The boundary conditions for this problem are easy to remember, since if

$$\left.\begin{array}{l} v_i(x_o) \text{ unspecified then } \lambda_i(x_o) = 0 \\ v_i(x_f) \text{ unspecified then } \lambda_i(x_f) = 0 \end{array}\right\} \qquad (2.3.17)$$

It may be seen from the above that we have N variables describing the system behavior together with N Lagrange multipliers and one control variable t. The Euler-Lagrange equations plus the constraint equations and the control equation yield 2N first-order, nonlinear differential equations which are coupled together through the control equation. The 2N necessary boundary conditions related to these equations are given in equation (2.3.17). These boundary conditions are split equally, with N being given at x_o, and the N others at x_f. This problem is referred to as a two-point boundary problem and is sufficient to solve for the 2N variables and $t(x)$.

Because of the nonlinear nature of the problem and the split boundary conditions, these problems are difficult to solve, even with numerical techniques. An interesting characteristic of the Hamiltonian function, H, is that if H is not an explicit function of x, then it will be a constant over the entire interval $x_o \le x \le x_f$ provided that t is unconstrained. This is easily seen from the relations

$$\frac{dH}{dx} = H_x + \tilde{H}_v \frac{d\tilde{v}}{dt} + H_t \frac{dt}{dx} + \frac{d\tilde{\lambda}^T}{dx}\tilde{f} \qquad (2.3.18a)$$

$$\frac{dH}{dx} = H_x + H_t t' + (\tilde{H}_v + \tilde{\lambda}'^T)\tilde{f}$$

$$\frac{dH}{dx} = H_x + H_t \frac{dt}{dx} \qquad (2.3.18c)$$

Thus, if $H \neq H(x)$ and if $H_t = 0$ (meaning that we are on a "path" which has optimal thickness t) then

$$\frac{dH}{dx} = 0 \qquad (2.3.19)$$

$$H = \text{constant} \qquad x_o \le x \le x_f \qquad (2.3.20)$$

This constancy of H when $t(x)$ is unconstrained and $H \neq H(x)$ provides a good check to see that one has indeed found an extremum of J.

2.4 Terminal Constraints and Optimization Methods With Inequality Constraints on the Thickness Parameter

The discussion in Section 2.3 dealt with the problem of finding the conditions necessary to ensure an extremum of a performance index in which the state variables may or may not have prescribed end conditions and for which no constraint was placed on the thickness parameter. A slightly more complicated problem arises if a function of the state variables is prescribed at $x = x_f$. Such a condition might arise if the optimization problem involved a cantilever beam with a discrete tip mass at $x = x_f$. A complicated boundary condition occurs at this point because the shear, which is one of the state variables, must be proportional to the acceleration of the tip mass at $x = x_f$. If we are dealing with harmonic motion at a frequency ω then the relation would be of the form

$$r(x_f) = -\omega^2 M_t w(x_f) \tag{2.4.1}$$

where $r(x)$ and $w(x)$ are nondimensional modal shear and deflection amplitudes respectively and M_t is the concentrated mass at the tip.

The treatment of this type of end or terminal condition is discussed at length by Bryson (Ref. 9) and is treated by adding additional or side conditions to the optimization problem. These M side conditions may be expressed as an M-dimensional vector

$$\tilde{\psi}(v(x_f), x_f) = 0 \tag{2.4.2}$$

This equation may be adjoined to the performance index by another M-dimensional vector $\tilde{\eta}$ to form a problem similar to that discussed in Section 2.3. Since $\phi = 0$, the performance index for the Lagrange problem becomes

$$J = \tilde{\eta}^T \tilde{\psi} - \tilde{\lambda}^T(x_f)\tilde{v}(x_f) + \tilde{\lambda}^T(x_o)\tilde{v}(x_o) + \int_{x_o}^{x_f} \left\{ H + \tilde{\lambda}'^T \tilde{v} \right\} dx \tag{2.4.3}$$

If the product $\tilde{\eta}^T\tilde{\psi}$ is treated in the same manner as the function ϕ in the development of the Bolza problem, the necessary conditions for an extremum of J are found to be

$$\tilde{v}' = \tilde{f}(v, t, x) \tag{2.4.4}$$

$$\tilde{\lambda}' = -\tilde{f}_v^T\tilde{\lambda} - \tilde{L}_v^T \tag{2.4.5}$$

$$\frac{\partial H}{\partial t} = 0 \tag{2.4.6}$$

with

$$v_i(x_o) = 0 \quad \text{or} \quad \lambda_i(x_o) = 0 \qquad (i = 1, 2, ----N) \tag{2.4.7}$$

and

$$\tilde{\lambda}^T(x_f) = \tilde{\eta}^T \overline{\frac{\partial\psi}{\partial v}} \Bigg|_{x = x_f} \tag{2.4.8}$$

with

$$\tilde{\psi}(v(x_f), x_f) = 0 \tag{2.4.9}$$

The only further development necessary before solving an example problem is the discussion of the method of handling constraints on the thickness parameter $t(x)$. The most common constraint is the inequality or minimum gauge constraint, that is, the requirement that the thickness parameter be greater than (or less than) a given value. These constraints may be expressed mathematically as

$$\left.\begin{array}{l} C_{min}(t) = t_{min} - t(x) \le 0 \\ \text{or} \\ C_{max}(t) = t(x) - t_{max} \le 0 \end{array}\right\} \quad x_o < x < x_f$$

where t_{min} is a minimum thickness value while t_{max} is a maximum thickness value.

If a new Hamiltonian functional is defined as

$$H^* = L + \tilde{\lambda}^T\tilde{f} + \mu C \tag{2.4.10}$$

it can be shown (Ref. 9) that the necessary conditions for an extremum of J, using the above Hamiltonian, will be identical to those previously derived. Thus, the Euler-Lagrange equations are the same, but the control equation will be slightly different since

$$\frac{\partial H^*}{\partial t} = L_t + \tilde{\lambda}^T \tilde{f}_t + \mu C_t = 0 \tag{2.4.11}$$

It can further be shown that

$$\mu \text{ is } \begin{cases} \geq 0 & \text{if } C = 0 \\ = 0 & \text{if } C < 0 \end{cases} \tag{2.4.12}$$

The functional H plays an important part in optimal control theory. It has been shown that, when there is no constraint on $t(x)$, the variation of J may be written as

$$\delta J = \int_{x_o}^{x_f} H_t \delta t dx = \int_{x_o}^{x_f} \delta H dx \tag{2.4.13}$$

Since our entire problem has been cast in the form of finding a <u>minimum</u> of J, it is readily seen that δJ must be either positive or zero for any admissible variation of $t(x)$. This then implies that $\delta H(x)$ itself must be likewise either greater than or equal to zero for all admissible $\delta t(x)$. This is all expressed in Pontryagin's Minimum Principle which states that, for an extremum of the performance index J, the Hamiltonian also must be minimized over a set of all possible $t(x)$. That is, H is decreased until either a minimum is found or a constraint boundary for $t(x)$ is encountered.

The need for the positive sign on μ is readily seen. Since $C_t \leq 0$, equation (2.4.11) becomes

$$L_t + \tilde{\lambda}^T \tilde{f} = \text{sign}(\mu) * |\mu C_t| \tag{2.4.14}$$

The left hand side of equation (2.4.14) is the partial derivative (with respect to t) of the standard "unconstrained" Hamiltonian, H. If $t(x)$ is unconstrained then $\mu = 0$ from our previous work but, if we should encounter the constraint boundary then further improvement can only be made if $\delta t < 0$ that is, if we decrease t

below t_{min}. But, from our discussion above, this decrease in δt should also cause a decrease in δH, or $\delta H < 0$ and thus $H_t > 0$. Thus, the sign of μ should be positive. Similar reasoning can be applied to the $t \le t_{max}$ case.

If there are inequality constraints, optimization problems will have two or more solution regions. Although the necessary differential equations will be identical in all regions, the control equation will be different in regions where the control variable is restrained. At the junction points between the constrained and unconstrained solutions, the thickness parameter $t(x)$ will, by the nature of the inequality constraint, always be continuous although its first derivative with respect to x will not. Because of this continuity, the state variables and the adjoint variables are continuous as are all their first derivatives with respect to x. In addition, the Hamiltonian and its derivative with respect to $t(x)$ (the control equation) are continuous at the junction points. A full treatment of the analytic solution of some simple optimization problems having inequality constraints is given by Armand and Vitte (Ref. 8).

2.5 An Example Problem-Free Torsional Vibration With a Single Frequency Held Fixed

As a simple example of the optimization of a simple structure, let us consider the thin-walled cylinder discussed in Section 2.2. The equations governing this problem have discussed and solved in several previous works including Turner (Ref. 4), Ashley and McIntosh (Ref. 7) and Armand and Vitte (Ref. 8). We wish to find the least possible weight of a cylinder with similar geometry which has an identical first frequency. When the overall geometry is held fixed, the only allowable structural variation will be the wall thickness. From equation (2.2.9), the optimal thickness must satisfy the eigenvalue equation

$$(t\theta')' + (\pi/2)^2(\delta_1 t + \delta_2)\theta = 0 \qquad 0 \le x \le 1 \tag{2.5.1}$$

with $\theta(0) = t\theta'(1) = 0$, if the optimum structure is to have its lowest frequency identical to that of the uniform structure.

Equation (2.5.1) may be expressed, as discussed before, as two first-order differential equations having two boundary conditions.

$$s' = -(\alpha t + \beta)\theta \tag{2.5.2}$$

$$\theta' = s/t \tag{2.5.3}$$

$$\theta(0) = s(1) = 0 \tag{2.5.4}$$

The performance index may be written as

$$J = \int_0^1 t dx \tag{2.5.5}$$

while the Hamiltonian is written as

$$H = t + \lambda_s(-(\alpha t + \beta)\theta) + \lambda_\theta(\frac{s}{t}) \tag{2.5.6}$$

$$\alpha = \delta_1(\pi/2)^2 \qquad \beta = \delta_2(\pi/2)^2$$

If $t(x)$ is unconstrained, then, using equations (2.3.15) and (2.3.16) the Euler-Lagrange equations may be formed together with the control equation

$$\lambda_s' = -\frac{\partial H}{\partial s} = -\lambda_\theta/t \tag{2.5.7}$$

$$\lambda_\theta' = -\frac{\partial H}{\partial \theta} = \lambda_s(\alpha t + \beta) \tag{2.5.8}$$

$$\frac{\partial H}{\partial t} = 0 = 1 - \frac{\lambda_\theta s}{t} - \alpha\lambda_s\theta \tag{2.5.9}$$

and, since $s(0) \neq 0$ then $\lambda_s(0) = 0$ and since $\theta(1) \neq 0$ then $\lambda_\theta(1) = 0$.

Equation (2.5.9) can be solved for the thickness, yielding

$$t^2 = \frac{\lambda_\theta s}{1 - \alpha\lambda_s\theta} \tag{2.5.10}$$

Equations (2.5.7) through (2.5.9) and equations (2.5.2) and (2.5.3) plus the associated boundary conditions are sufficient to determine the optimal thickness distribution such that J is an extremum, hopefully a minimum. A closer examination of these equations reveals a similarity between the state variable equations and the multiplier equations. In fact, it can be demonstrated that

$$\begin{Bmatrix} \lambda_\theta(x) \\ \lambda_s(x) \end{Bmatrix} = A \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \theta(x) \\ s(x) \end{Bmatrix} \tag{2.5.11}$$

The constant A is a modal or undetermined constant which will be discussed later. Similar relations are found to occur in optimization problems involving eigenvalues of conservative systems. The state variables are said to be "self-adjoint" since they satisfy their own adjoint or multiplier equations and the respective boundary conditions. This self-adjoint property considerably simplifies the problem since it reduces the 2Nth order system to Nth order. Our fourth order system becomes a second order system with the introduction of equation (2.5.11).

$$\theta' = s/t \tag{2.5.12}$$

$$s' = -[\alpha t + \beta]\theta \tag{2.5.13}$$

$$t^2 = \frac{As^2}{1 + A\theta^2} \tag{2.5.14}$$

with $\theta(0) = s(1) = 0$.

The solution to these equations is found to be (Ref. 8) given by

$$\theta = \left(\frac{1}{\alpha A}\right)^{1/2} \sinh(\sqrt{\alpha}\,x) \tag{2.5.15}$$

$$s = \left(\frac{1-\delta_1}{\sqrt{A}}\right)\frac{\cosh(\sqrt{\alpha}\,x)}{2\delta_1}\left(\left[\frac{\cosh\sqrt{\alpha}}{\cosh(\sqrt{\alpha}\,x)}\right]^2 - 1\right) \tag{2.5.16}$$

The resulting thickness distribution is given by

$$t(x) = \frac{1-\delta_1}{2\delta_1}\left(\left[\frac{\cosh\sqrt{\alpha}}{\cosh\sqrt{\alpha}\,x}\right]^2 - 1\right) \tag{2.5.17}$$

In addition, the author has found that

$$H = H_o = \left(\frac{1-\delta_1}{\delta_1}\right)(\sinh\sqrt{\alpha})^2 \tag{2.5.18}$$

Note that, although the expressions for $\theta(x)$ and $s(x)$ contain the constant "A," the thickness distribution and the Hamiltonian are independent of this constant.

This constant, which occurs in the relation between the multipliers and the state variables, plays the role of a "modal amplitude factor" in the equations. Thus, one would not only expect, but require, that the thickness not be a function of the eigenvector amplitude but only of the eigenvector modal shape and the value of the eigenvalue itself. Thus, we can and will set $A = 1$ in all of the future investigations since it does not affect the solution for $t(x)$. Note, however, that it does affect the magnitude of the solution for the state variables and the multiplier functions. Note also, from equation (2.5.14) that A must be greater than zero if $t^2(0)$ is to be positive.

It is also interesting to note that, in the absence of constraints on the thickness, the value of the thickness goes to zero at the end, $x = 1$. This will be seen to be a common characteristic of problems where no torque or moment is prescribed at a point. It may also be seen that at $x = 1$, equation (2.5.12) is indeterminate since both $s(1)$ and $t(1)$ are zero.

An examination of the expression for H_o in equation (2.5.18) shows that

$$\lim_{\delta_1 \to 1} H_o = 0 \tag{2.5.19}$$

and, using L'Hospital's rule

$$\lim_{\delta_1 \to 0} H_o = 0 \tag{2.5.20}$$

Between these upper and lower bounds the Hamiltonian is positive. This behavior at the upper and lower limits occurs because the problem is not well posed at these extremes. At the lower limit, $\delta_1 = 0$, no structure exists to be optimized so the minimum value of H is simply zero. At the upper limit, $\delta_1 = 1$, something more subtle occurs. As $\delta_1 \to 1$, the magnitude of the thickness as given in equation (2.5.17) grows smaller and finally vanishes as $\delta_1 \to 1$. This seems to say that, in the absence of any nonstructural moment of inertia, the optimum cylinder is one with zero thickness. Physical reasoning leads one to assume that the problem is poorly posed.

The reason for this problem's being poorly posed is that the frequency, ω_o, is not an explicit function of the thickness parameter $t(x)$ since both $GJ(x)$

and $m(x)$, on whose ratio ω_o depends, are both linear functions of $t(x)$. Therefore, the frequency is unaffected by changes in $t(x)$. Also, the state variable or constraint equations will be linear and homogeneous in t, if there is no nonstructural mass or moment of inertia. Thus, for $\delta_1 = 1$, $t(x) = 0$ is an allowable solution. With the addition of a nonstructural mass, the constraint equations are no longer homogeneous in $t(x)$ and the problem is no longer poorly posed.

The expression for $t^2(x)$ in equation (2.5.14) has a resemblance to the Rayleigh Quotient encountered in mechanics. The numerator is the square of the strain energy per unit volume while the denominator is equal to the kinetic energy per unit volume due to elastic deformation at a fixed frequency plus a constant, "one."

This concludes the brief discussion of the formulation of eigenvalue constraint equations and the theory of optimization of functions which have constraints in the form of first-order, ordinary differential equations. This also will be the last time that an analytic solution will be obtained since the discussion in the next section will focus on numerical estimation techniques, solution methods, and accuracy of these methods.

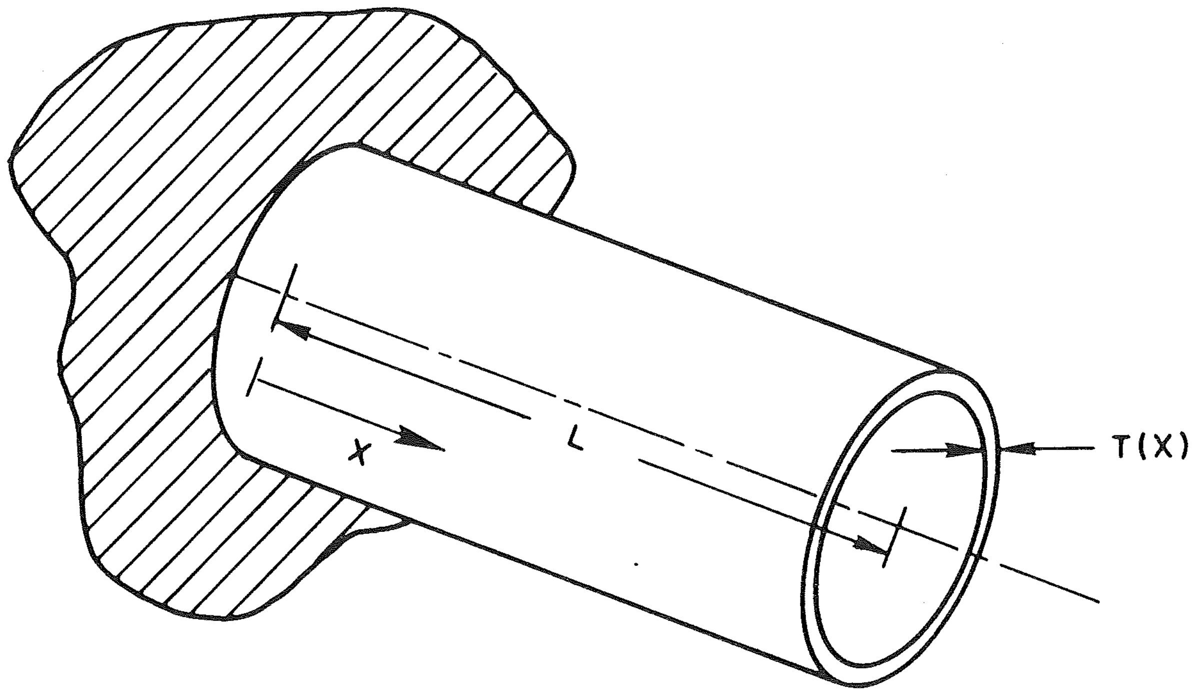

Fig. 2.1 Thin-walled Cylinder Model —— Free Torsional Vibration.

3. NUMERICAL TECHNIQUES FOR PROBLEMS WITH DIFFERENTIAL EQUATION CONSTRAINTS

3.1 Introduction

Chapter 2 discussed the development of a set of nonlinear differential equations and boundary conditions which are necessary to determine an optimal thickness parameter distribution. The formulation of the necessary conditions was seen to be quite simple, however, the actual solution of these equations poses a real obstacle. Unfortunately, only the very simplest of problems, such as the torsion example discussed in Section 2.5, have analytic or exact solutions. In most cases, one is forced to use numerical or approximate methods to solve the problem.

Chapter 3 will describe and discuss numerical methods which will yield "exact" solutions to a wide variety of one-dimensional structural optimization problems with differential equation constraints. Optimal control techniques again provide the basis for these numerical schemes. The sections which follow will discuss the theory behind these methods and will conclude with an example involving the torsional frequency problem discussed in Section 2.5.

3.2 The Transition Matrix Method

One of the first successful numerical solution techniques encountered in this investigation was the "transition matrix" method. The basic discussion of the transition matrix technique as applied to optimal control problems is discussed in Bryson (Ref. 9). In Chapter 2, optimization problems with N differential equation constraints were shown to lead to a problem involving $2N$ nonlinear differential equations with N boundary conditions specified at $x = x_o$ and the remaining N boundary conditions specified at $x = x_f$. Thus, at either x_o or x_f there are N "specified" and N "unspecified" or "undetermined" boundary conditions.

The terms "unspecified" or "undetermined" can be a source of some confusion since these initial or final conditions are not, in fact, undetermined or unspecified. They are determined by the N boundary conditions at $x = x_o$ <u>and</u>

the N boundary conditions at $x = x_f$ and the 2N differential equations. However, since all numerical integration schemes for solving a system of simultaneous first order differential equations require a number of starting or initial values which is equal to the number of simultaneous equations, these N constants are initially unknown. In any solution scheme these N constants will be determined by the requirement that they have values which, when used together with the N "specified" initial conditions, will yield the final N specified conditions once the 2N equations have been integrated from x_o to x_f.

The terms "initial" and "final" in the above discussion may be interchanged since the same numerical technique will work for forward as well as backward integration. Most solution techniques for nonlinear, two-point boundary value problems have one characteristic in common; a search for these "undetermined" constants and the subsequent determination, by numerical integration, of the control variable (thickness parameter) distribution.

In the discussion which follows, the term "optimality equations" will be used to refer to the control equation plus all the first-order differential equations necessary to solve for $t(x)$, the thickness parameter. These equations may be composed of both state and adjoint variables or, as was shown in the torsional vibration problem, only the coupled state variable equations. The coupling of the state variable equations is seen to occur in the control equation. The optimality equations have been shown to be of the form

$$\frac{d\tilde{y}_i}{dx} = \tilde{y}_i' = \tilde{f}_i(y_j, t) \qquad (i = 1, 2, \ldots 2N) \tag{3.2.1}$$

where

$$t^2 = g(y_i) \tag{3.2.2}$$

with N boundary conditions specified at x_o and N conditions required at x_f. The variables $\tilde{y}_i(x)$ will be referred to as system variables. The term "control constants" will be used to denote the N "undefined" or "unspecified" constants at $x = x_o$. This term is used because these N constants "control" the value of $\tilde{y}_i(x_f)$, the values of the system variables at $x = x_f$. If these control constants are adjusted correctly, the numerical solution to the optimality equations should yield the N specified boundary conditions at $x = x_f$.

Let us define the values of the control constants at $x = x_o$ as an N-dimensional vector $\tilde{\mu}$. Also, let $\tilde{\beta}_e$ by an N-dimensional vector composed of the N prescribed values of $\tilde{y}(x_f)$. This development will concern itself only with the case where each element of the vector $\tilde{\beta}_e$ is a function of a single prescribed $y_i(x_f)$ only, although the transition matrix method is not restricted to this case only.

If an initial or "guessed" value of $\tilde{\mu}$ is chosen then, this value of $\tilde{\mu}$ together with the N prescribed boundary conditions at $x = x_o$ can be used to start the integration of the optimality differential equations (3.2.1) from $x = x_o$. At $x = x_f$ the result of the integration will be a vector $\tilde{y}_i(x_f)$. There will be N values of this vector which are to be made equal to their respective elements in $\tilde{\beta}_e$, the specified values of the system variables.

In general, these corresponding values of $\tilde{y}_i(x_f)$ which are numerically calculated using an assumed or initial $\tilde{\mu}$ (together with the N prescribed initial conditions) will not be identical to $\tilde{\beta}_e$. What is needed then, is a method to perturb $\tilde{\mu}$ in such a way that the <u>calculated values</u> of the specified variables, defined as $\tilde{\beta}_c$, can be made to approach $\tilde{\beta}_e$. It should be noted that, at $x = x_o$, we always start with N specified boundary conditions and use them, together with the optimality equations to generate values of $\tilde{y}_i(x_f)$. Thus, no matter what values are initially chosen for $\tilde{\mu}$, <u>every</u> necessary optimality condition is satisfied by the numerical solution <u>except</u> the N final specified boundary conditions. The solution technique will involve solving a series of problems in which $\tilde{\beta}_c$ takes on different values and finally approaches $\tilde{\beta}_e$. Thus, one may think of the final numerical solution as being found through a transition from an initial problem, in which $\tilde{\beta}_c \neq \tilde{\beta}_e$, to a final problem with $\tilde{\beta}_c \cong \tilde{\beta}_e$. Each of these transition solutions will satisfy the optimality equations (3.2.1) and have identical values of the N prescribed boundary conditions at $x = x_o$. By perturbing $\tilde{\mu}$ in a proper manner, a transition from an initial problem with $\tilde{\mu}_{int}$ and $\tilde{\beta}_c \neq \tilde{\beta}_e$ to a problem with $\tilde{\mu} = \tilde{\mu}_{final}$ and $\tilde{\beta}_c \cong \tilde{\beta}_e$ can be obtained.

In practice, because of numerical "round-off" error, $\tilde{\beta}_c$ will never exactly equal $\tilde{\beta}_e$. Thus, the problem becomes one of convergence, i.e., obtaining a series of intermediate or transitional solutions until

$$\left|\tilde{\beta}_c - \tilde{\beta}_e\right| \leq \tilde{\epsilon} \tag{3.2.3}$$

where $\tilde{\epsilon}$ is an "error" or "tolerance" vector and has all elements $\epsilon_i > 0$.

The question remains, "how do we perturb $\tilde{\mu}$ to bring $\tilde{\beta}_c$ to $\tilde{\beta}_e$?" Let us suppose that a linear relation could be found between a perturbation of $\tilde{\mu}$ and a resulting change of the <u>difference</u> between $\tilde{\beta}_c$ and $\tilde{\beta}_e$. We are primarily interested in the difference $\tilde{\beta}_c - \tilde{\beta}_e$ since this gives us one measure of how close we are to the otpimal solution. Adding to the list of definitions, let us call this difference

$$\tilde{\beta}_c - \tilde{\beta}_e = \tilde{e} \tag{3.2.4}$$

If $\delta\tilde{\mu}$ is a perturbation of $\tilde{\mu}$ then <u>assume</u> that this linear relation can be written as

$$\delta\tilde{e} = \overline{T}\,\delta\tilde{\mu} \tag{3.2.5}$$

where $\delta(\ )$ presents the numerical perturbation of $(\ )$. $\overline{T}$ is called the transition matrix and is $N \times N$. Each of the elements of the $\overline{T}$ matrix represents the change in an element of δe for a unit change in an element of $\tilde{\mu}$ $(\delta\mu_i = 1)$ holding all other changes equal to zero, e.g.

$$T_{ij} = \frac{\delta e_i}{\delta\mu_j} \cong \frac{\partial e_i}{\partial\mu_j} \tag{3.2.6}$$

If we wish to decrease $\tilde{\beta}_c - \tilde{\beta}_e$ by an amount κ i.e., $\delta(\tilde{\beta}_c - \tilde{\beta}_e) = -\kappa\tilde{e} = \delta e$ then, by inverting $\overline{T}$ and premultiplying (3.2.5) by $\overline{T}^{-1}$ we have

$$\delta\mu = -\kappa\,\overline{T}^{-1}\tilde{e} \tag{3.2.7}$$

Therefore, if a relation such as (3.2.5) does exist, then each element of the error vector $\tilde{e}$ can be reduced by a uniform amount $\kappa\tilde{e}$ by perturbing $\tilde{\mu}$

by an amount $\delta\tilde{\mu}$ as given in (3.2.7). If (3.2.5) were, in fact, valid then $\overline{T}$ could be found and, by letting $\kappa = 1$, we could solve for $\delta\tilde{\mu}$. However, since the system of differential equations is nonlinear, (3.2.5) may be regarded only as a first-order approximation and thus $0 < \kappa < 1$. In practice, it is the adjustment of this scalar, κ, which separates the experienced researcher from the amateur.

The matrix $\overline{T}$ is called the "transition matrix" for reasons which should now be clear. It is this matrix which permits one to calculate the necessary $\delta\tilde{\mu}$ to achieve a transition from an initial or trial solution to the "exact" or optimal solution. As mentioned, each element T_{ij} represents the change in the difference $\beta_{c_i} - \beta_{e_i}$ for a unit change in $\mu_j (\delta\mu_j = 1)$. The transition matrix is thus always an $N \times N$ matrix. The calculation of the elements of the transition matrix is usually straightforward and will be discussed in the last part of Section 3.2. By using the above technique we can obtain a transition from a trial solution to a final numerical solution which satisfies the optimality equations and has the specified initial and final boundary conditions.

From the above discussion we can now outline a solution technique for solving a 2Nth order system of nonlinear differential equations with split boundary conditions. The technique is as follows:

(1) Using the N specified boundary conditions and an initial, assumed set of control constants $\tilde{\mu}$, integrate the necessary differential equations of optimality from $x = x_o$ to x_f.

(2) Record $\tilde{\beta}_c$ at $x = x_f$, calculate $\tilde{e} = \tilde{\beta}_c - \tilde{\beta}_e$. If $|\tilde{e}| \leq \tilde{\epsilon}$, stop the procedure here —— the solution obtained is the numerical approximation to the optimal solution. If $|\tilde{e}| > \tilde{\epsilon}$ then continue.

(3) Calculate the transition matrix $\overline{T}$.

(4) Calculate $\delta\tilde{\mu}$ using

$$\delta\tilde{\mu} = -\kappa \overline{T}^{-1} \tilde{e}$$

where κ is pre-selected.

(5) Form a new set of control constants using the equation

$$\tilde{\mu}_{new} = \tilde{\mu}_{old} + \delta\tilde{\mu} = \tilde{\mu}_{old} - \kappa \overline{T}^{-1}\tilde{e}$$

(6) If $|\tilde{e}|$ has been found to be greater than $\tilde{\epsilon}$ in step (2) above, begin calculations at step (1) using the $\tilde{\mu}_{new}$. Continue this process until convergence has been obtained, i.e. $|\tilde{e}| \leq \tilde{\epsilon}$.

The calculation of the transition matrix can be accomplished in several ways. The method to be used in this study involves solving a set of perturbation differential equations simultaneously with the optimality equations. Let $\delta(\ )$ represent a perturbation of a variable. The operator δ has the same properties as the differential operator $d(\ )$. The perturbation of the optimality equations (3.2.1) and (3.2.2) yields:

$$\frac{d}{dx}(\delta\tilde{y}_i) = \overline{\frac{\partial f_i}{\partial y_j}}\delta\tilde{y}_j + \frac{\partial \tilde{f}_i}{\partial t}\delta t \tag{3.2.8}$$

$$\delta t = \left(\frac{1}{2t}\right)\overline{\frac{\partial g}{\partial y_j}}^T \delta\tilde{y}_j \tag{3.2.9}$$

Although the optimality equations are nonlinear, the perturbation equations are linear in the perturbation variables. "N" of the variables $\delta\tilde{y}$ correspond to system variables whose initial values at $x = x_o$ are elements in the perturbation vector $\delta\tilde{\mu}$. For instance, if $\delta y_k(x_o)$ is set equal to "one" while all other $\delta y_i(x_o) = 0$, $(i \neq k)$, and the combined set of differential equations (3.2.1) and (3.2.8) are integrated (using equations (3.2.2) and (3.2.9)) from $x = x_o$ to x_f with an assumed $\tilde{\mu}$, then N of the final values of $\delta\tilde{y}(x_f)$ can be used to construct $\delta\tilde{e}$ due to a unit perturbation in $y_k(x_o)$

$$\delta\tilde{e} = \delta(\tilde{\beta}_c - \tilde{\beta}_e) = \delta\tilde{\beta}_c \tag{3.2.10}$$

Thus

$$T_{ik} = \frac{\delta e_i}{\delta y_k} = \frac{(\delta\beta_c)_i}{\delta y_k} = (\delta\beta_c)_i \tag{3.2.11}$$

but $(\delta\beta_c)_i$ is the value of one of the perturbation variables in $\delta\tilde{y}(x_f)$. Thus the value of an entire column of the transition matrix due to a unit change in the control constant $y_k(x_o)$ can be found in the following manner:

(1) Set the value of $\delta y_k(x_o)$ — $y_k(x_o)$ is an element of the control constant vector $\tilde{\mu}$ — equal to "one" and all other $\delta y_i(x_o) = 0$; $i = 1$ to $2N$; $i \neq k$).

(2) Using these initial perturbation values, together with the specified initial conditions and $\tilde{\mu}$, integrate the optimality equations <u>and</u> the perturbation equations simultaneously from $x = x_o$ to x_f.

(3) Record the values of $\delta\tilde{y}(x_f)$. These values represent the change in the variables $\tilde{y}(x_f)$ for a unit change in the variable $y_k(x_o)$. Thus, N of these elements represent a column of the transition matrix since

$$T_{ik} = \frac{\partial e_i}{\partial y_k} \cong \frac{\delta e_i}{\delta y_k(x_o)} = \delta e_i = \delta\beta_{c_i}$$

and β_{c_i} represents a calculated value of a particular variable $y_i(x_f)$ which has its value prescribed. Thus, N of the values of $\delta\tilde{y}(x_f)$ will compose a column of the transition matrix $\bar{T}$.

If the above scheme is carried out N times, each time with a different value of $\delta y_i(x_o)$ set equal to "one" and all others zero, then the entire transition matrix may be calculated.

In all cases studied in this thesis $\tilde{\beta}_e = \left\{0\right\}$ and each element of $\tilde{\beta}_c$ is a function of one system variable only. An extension of this procedure to calculate a transition matrix for a problem in which $\tilde{\beta}_c$ involves <u>functions</u> of the variables is easy. If there are N functions $\psi_i(y_j(x_f), t(x_f))$ which are to have prescribed values then let

$$\tilde{\beta}_c = \tilde{\psi}_c \tag{3.2.12a}$$

and

$$\tilde{\beta}_e = \tilde{\psi}_e \tag{3.2.12b}$$

where the subscripts c and e refer to the calculated and exact values respectively. Since

$$\tilde{e} = \tilde{\beta}_c - \tilde{\beta}_e = \tilde{\psi}_c - \tilde{\psi}_e \tag{3.2.13}$$

then

$$\delta\tilde{e} = \frac{\overline{\partial\psi_c}^T}{\partial y_j(x_f)}\delta\tilde{y}_j(x_f) \tag{3.2.14}$$

In this case, each element $\delta\tilde{e}_i$ is a function of more than one $\delta y_j(x_f)$, however, the transition matrix procedure is the same since

$$T_{ik} = \delta e_i \tag{3.2.15}$$

for a particular choice of $\delta y_k(x_o) = 1$.

The transition matrix is therefore not restricted to systems where δe_i is a function of a single $\delta y_j(x_f)$.

3.3 Example of the Transition Matrix Solution

As an example of the general procedure described in Section 3.2, consider the simple torsional vibration problem discussed in Chapter 2. The necessary optimality conditions were found to be

$$\theta' = s/t \qquad 0 \le x \le 1 \tag{3.3.1}$$

$$s' = -\theta(\alpha t + \beta) \tag{3.3.2}$$

where

$$t^2 = \frac{s^2}{1 + \alpha\theta^2} \tag{3.3.3}$$

with $\theta(0) = s(1) = 0$.

The perturbation equations for the above system are

$$(\delta\theta)' = \frac{\delta s}{t} - \frac{s}{t^2}\delta t \tag{3.3.4}$$

$$(\delta s)' = -\delta\theta(\alpha t + \beta) - \theta\alpha(\delta t) \tag{3.3.5}$$

$$\delta t = \frac{1}{t}\left(\frac{s\delta s}{1+\alpha\theta^2} - \frac{\alpha\theta s^2\delta\theta}{(1+\alpha\theta^2)^2}\right) \tag{3.3.6}$$

Equation (3.3.6) can also be written as

$$\delta t = t\left(\frac{\delta s}{s} - \alpha\frac{\theta\delta\theta}{s^2}\right) \tag{3.3.7}$$

For the numerical solution of these equations, the previously defined vectors $\tilde{\mu}$, $\tilde{\beta}_e$, and $\tilde{\beta}_c$ are equal to the following

$$\tilde{\mu} = s(0) = S_o \tag{3.3.8}$$

$$\tilde{\beta}_e = s(1) = 0 \tag{3.3.9}$$

$$\tilde{\beta}_c = S_1 \tag{3.3.10}$$

$$\tilde{e} = \tilde{\beta}_c - \tilde{\beta}_e = \tilde{\beta}_c = S_1 \quad \text{so} \quad \delta e = \delta s(1) = dS_1 \tag{3.3.11}$$

where S_1 and dS_1 are numerical values of $s(1)$ and $\delta s(1)$ respectively. The task is to force S_1 to zero by perturbing the value of S_o. The relation between changes in $\delta s(0) = S_o$ and $dS_1 = \delta s(0)$ is written as:

$$\delta s(1) = T_{11}\delta s(0) \tag{3.3.12}$$

Using the boundary condition $\theta(0) = 0$ and an assumed value for $s(0)$:

$s(0) = S_o$; with $\delta\theta(0) = 0$

and $\delta s(0) = 1$ the system equations (3.3.1) to (3.3.6) may be numerically integrated from $x = 0$ to $x = 1$. In general, the calculated value of $s(1)$, S_1, will be unequal to zero. If the calculated value of $\delta s(1)$ is called dS_1 then, since

$$\delta(\tilde{\beta}_c - \tilde{\beta}_e) = \delta\tilde{\beta}_c = \delta\tilde{e} = dS_1 \tag{3.3.13}$$

and $\delta s(0) = 1$, then

$$T_{11} = \frac{\partial s(1)}{\partial s(0)} = \frac{\delta e_1}{\delta s(0)} = dS_1 \tag{3.3.14}$$

A change in S_o can then be calculated such that the error $e_1 = S_1$ is reduced by an amount κ. Let

$$\delta e_1 = -\kappa e_1 = -\kappa S_1 \tag{3.3.15}$$

then from equation (3.3.12)

$$S_o = -\frac{\kappa S_1}{dS_1} \tag{3.3.16}$$

Now, let

$$S_{o_{new}} = S_{o_{old}} + \delta S_o = S_{o_{old}} - \frac{\kappa S_1}{dS_1} \tag{3.3.17}$$

The system equations can be numerically integrated once more from $x = 0$ to 1 using

$$\left.\begin{aligned} s(0) &= S_{o_{new}} \\ \theta(0) &= 0 \\ \delta s(0) &= 1 \\ \delta\theta(0) &= 0 \end{aligned}\right\} \tag{3.3.18}$$

At $x = 1$, new values will be obtained for both $s(1)$ and $\delta s(1)$. If $S_{1_{new}}$ denotes the new value of $s(1)$ then from equation (3.3.15) the following relation should hold

$$\left| \frac{S_{1_{new}} - S_{1_{old}}}{S_{1_{old}}} \right| \cong \kappa \tag{3.3.19}$$

if κ is being chosen correctly. Since the transition matrix definition is a linear approximation, if equation (3.3.19) does not hold then we have exceeded the limits of the linear approximation. If this is the case, κ has been chosen too large and should be decreased on the next integration iteration. On the other hand, if κ is very closely equal to that given in equation (3.3.19) then a larger value of κ should be chosen to drive $s(1)$ to zero more quickly. In practice, it is found that a relation such as equation (3.3.19) is satisfied for

$$0 < \kappa \le \kappa_{max} < 1 \tag{3.3.20}$$

where κ_{max} varies from one iteration to the next. As the error $\tilde{e}$ decreases, κ_{max} is found to increase. As the solution begins to converge, that is, when $s(1)\rightarrow 0$, it is found that $\kappa_{max}\rightarrow 1$. In problems with more variables, the problem of adjusting κ becomes more complicated and is largely one of experience with particular problems being solved. After a number of iterations, S_1 will be driven to zero and the numerical solution will satisfy all the necessary optimality differential equations and the boundary conditions.

The results of an application of this method are shown in Figure 3.1. The nondimensional thickness distribution is plotted vs. x (no minimum thickness constraint is imposed on t(x)). Using the numerical solution with a starting value $s(0) = S_1 = 1.0$ and κ from between .8 and .95, convergence of the numerically calculated thickness distribution to the analytic solution was achieved in five iterations.

If a comparison between the numerical and analytic solutions for t(x) were given on a figure, the results would be indistinguishable. Table 3.1 gives a numerical comparison between these two solutions. In the table, D1 is the amount by which the nth iterative value of S_o would change if κ equalled unity. If the method is converging satisfactorily then this value of D1 should decrease nearly proportional to κ.

As a historical note, in early studies by Ashley and McIntosh (Ref. 7), this method was so successful on the torsional vibration problem that some over-optimism was expressed with regard to the transition matrix procedure. Since, in the torsional vibration problem

$$t^2(0) = \frac{s^2(0)}{1 + \alpha\theta^2(0)} = s^2(0) \tag{3.3.21}$$

one might deduce from physical intuition that, even in an optimum thickness distribution, the thickness at $x = 0$ will not be far from unity.

Thus, a good starting value for the control constant s(0) would be unity. Except for cases involving large values of δ_1 this observation holds true. Early researchers used this starting value for s(0) and the adjoint $\lambda_\theta(0)$ and were highly successful. In more complicated problems, a bit more insight is needed and the transition matrix procedure becomes more difficult due to the difficulty in estimating the control constants.

3.4 Transition Matrix Solution With Minimum Thickness Constraints

The previous discussion of the transition matrix made no mention of thickness constraints. A simple logical statement or test must be added to a computation scheme if thickness constraints are to be imposed. For instance, in Section 3.3, in the absence of thickness constraints the torsional vibration problem was solved using the equations

$$\theta' = s/t \qquad 0 \le x \le 1 \tag{3.4.1}$$

$$s' = -\theta(\alpha t + \beta) \tag{3.4.2}$$

$$t^2 = \frac{s^2}{1 + \alpha\theta^2} \tag{3.4.3}$$

$$(\delta\theta)' = \frac{\delta s}{t} - \frac{s}{t^2}\delta t \tag{3.4.4}$$

$$(\delta s)' = -(\delta\theta)(\alpha t + \beta) - (\theta\alpha)\delta t \tag{3.4.5}$$

$$\delta t = t\left(\frac{\delta s}{s} - \alpha\left(\frac{\theta}{s}\right)\left(\frac{\delta\theta}{s}\right)\right) \tag{3.4.6}$$

with $\theta(0) = s(1) = 0$; $\delta s(0) = 1$; $\delta\theta(0) = 0$.

Suppose, as an example, that the constraint

$$t \ge t_{min} \qquad 0 \le x \le 1 \tag{3.4.7}$$

is specified. The numerical solution method for this type problem is nearly the same as described in Section 3.3. If, at a value $x = x_c$ in the numerical solution process, a value of t, as calculated in equation (3.4.3) is found such that

$$t \le t_{min}$$

then $t(x_c)$ is set equal to t_{min} and δt must be set equal to zero. The value of δt must be zero because no variation of thickness is permitted. The integration procedure continues using these values of t and δt until such time when $t(x)$, as calculated from equation (3.4.3), again exceeds t_{min}. If and when this occurs, this value of $t(x)$ is again used in the optimality equations, together with $\delta t(x)$ as calculated in equation (3.4.6). The logic may be expressed as

(1) Let $C = \dfrac{s^2}{1 + \alpha\theta^2}$

(2) If $\left\{ \begin{array}{c} C > t_{min} \\ \text{or} \\ C \le t_{min} \end{array} \right\}$ Then $\left\{ \begin{array}{c} t = C \\ \delta t \ne 0 \\ \text{or} \\ t = t_{min} \\ \delta t = 0 \end{array} \right\}$

This completes Chapter 3 and the bulk of the discussion of the numerical techniques used to solve the optimization problems which have ordinary differential equation constraints. In the chapters which follow, additional techniques will be discussed as is necessary. The following chapters present solutions to problems whose analytic solutions are as yet unknown. These numerical solutions have been obtained using the transition matrix method or a modification of this basic method.

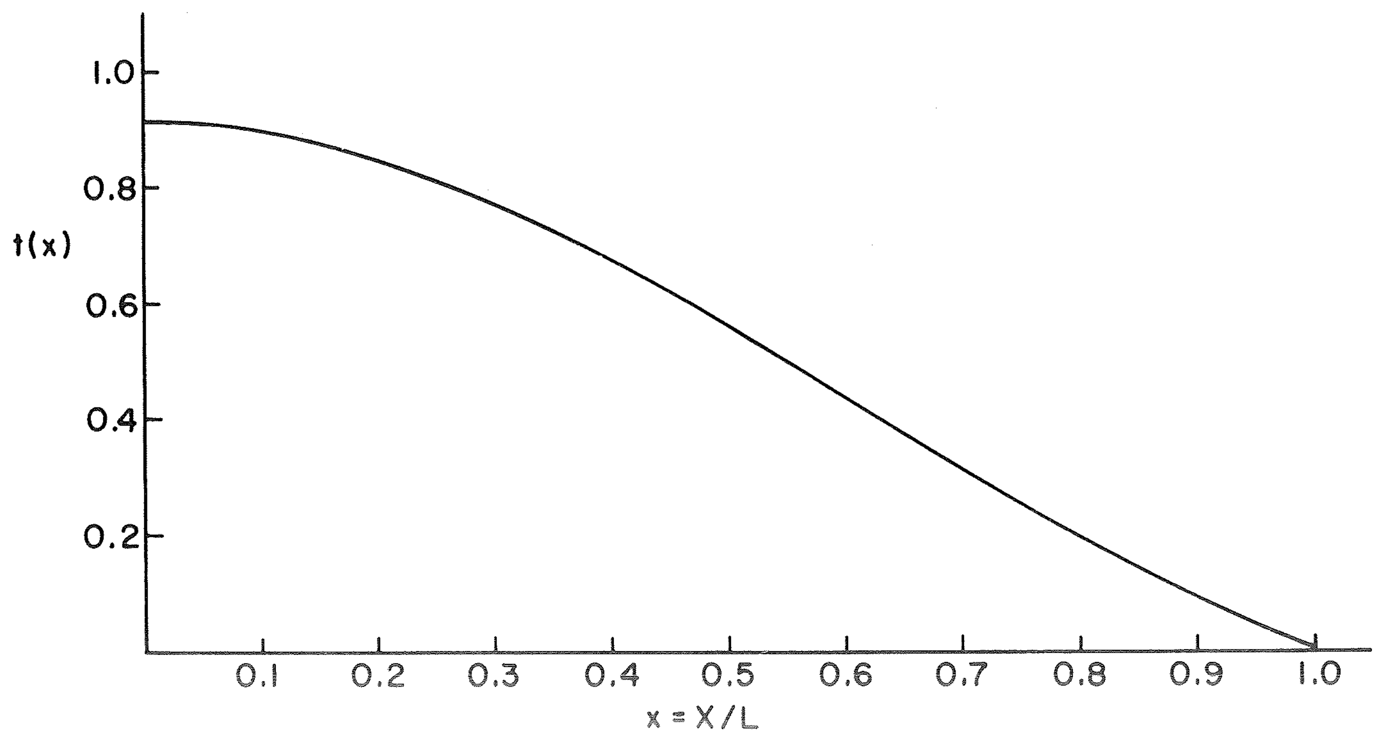

Fig. 3.1 Nondimensional Thickness Distribution — Minimum Weight Cylinder With First Torsional Frequency Held Constant — $\delta_1 = .50$.

TABLE 3.1

Comparison of a Numerically Obtained Solution to an Exact Solution for the Thickness Parameter —— Torsional Vibration Problem

$\delta_1 = 0.50;\ t_{min} = 0.0;\ \theta(0) = 0$

Iteration Number	s(0)	t(x) x = 0.0	x = 0.20	x = 0.40	x = 0.60	x = 0.80	x = 1.00	D1
1	1.0000	1.0000	0.9283	0.7389	0.4908	0.2424	0.0296	0.0836
2	0.93291	0.9329	0.8645	0.6835	0.4465	0.2092	0.0059	0.01677
3	0.91781	0.9178	0.8501	0.6710	0.4365	0.2017	0.0006	0.00167
4	0.91631	0.9163	0.8487	0.6698	0.4355	0.2010	0.0001	0.000164
5	0.91616	0.9161	0.8485	0.6697	0.4354	0.2009	0.0	0.0000157
Exact Solution	0.91613	0.9161	0.84848	0.66965	0.43542	0.20087	0.0	

4. OPTIMIZATION OF SIMPLE BEAMS WITH ONE FLEXURAL VIBRATION FREQUENCY HELD FIXED

4.1 Introduction

This section will discuss the solution of problems involving the search for a least-weight design of a beam whose first or fundamental frequency of flexural vibration is held fixed. As in the torsional vibration problem, the constraint equation will be expressed in nondimensional form. The results which are obtained will then apply to any beam with similar end conditions and whose length is held fixed. A nondimensional thickness parameter will be defined in each case discussed and will be used to relate the structural and inertia properties of the optimal beam to a uniform thickness reference beam.

Two structural configurations will be discussed. The first is a beam composed of two thin face-sheets with a nonstiffening core sandwiched between them. The problem will be to find an optimum face-sheet thickness distribution, without altering the core, which results in an optimum or least-weight design and still has a fundamental frequency identical to a beam of similar geometry and uniform face-sheet thickness. The second problem will deal with a beam of fixed width and length having a solid rectangular cross-section. The problem will involve finding a distribution of thickness such that a least-weight design is found which has the same fundamental frequency as a uniform thickness beam with similar geometry.

Two sets of boundary conditions will be considered for the sandwich beam problem. One set of boundary conditions will be those due to clamping one end of the beam while allowing the other end to be free. The second set of boundary conditions will be due to fixing the beam at each end and allowing rotations at these ends. This set of end conditions is referred to as "pinned-pinned." The solid cross-section beam will be studied only for pinned-pinned boundary conditions.

4.2 Least Weight Optimization of a Cantilever Beam Composed of Two Thin Face-Sheets With its Fundamental Frequency Held Fixed

The equilibrium equation for the free flexural vibration of a simple beam with inextensional bending is given by:

$$\frac{d^2}{dX^2}\left(EI(X)\frac{d^2W}{dX^2}\right) - \omega^2 m(X)W(X) = 0 \qquad (4.2.1)$$

$0 \le X \le L$

where the lateral motion is assumed to be of the form

$$W^*(X,\tau) = W(X)e^{i\omega\tau}$$

For a sandwich beam, that is, a beam with thin face-sheets top and bottom, the bending stiffness $EI(X)$ is given as

$$EI(X) = 2Ebh^2T(X) \qquad (4.2.2)$$

where: b = width of face-sheet

h = distance of face sheet above a hypothetical neutral axis or midplane

E = Young's modulus

$T(X)$ = face sheet thickness

Now, we nondimensionalize equation (4.2.1) by letting

$w(x) = W(x)/L$

$$x = X/L \qquad (\ )' = \frac{d}{dx} = \frac{1}{L}\frac{d}{d(X/L)}$$

to get the following equation:

$$(EI(x)w(x)'')'' - (\omega^2L^4)m(x)w(x) = 0 \qquad (4.2.3)$$

Furthermore, if $EI_o = 2Ebh^2T_o$ is a reference stiffness due to a sandwich beam with uniform face-sheet thickness then, dividing by EI_o, equation (4.2.3) can be written as

$$\left(\frac{EI(x)}{EI_o}w''\right)'' - \omega^2\left(\frac{m_oL^4}{EI_o}\right)\left(\frac{m(x)}{m_o}\right)w(x) = 0 \qquad (4.2.4)$$

where m_o is the mass per unit length of the uniform thickness beam. Since

$m(x)$ is composed both of the contributions due to face-sheet inertia and non-structural core inertia, the ratio $m(x)/m_o$ may be written as

$$\frac{m(x)}{m_o} = \delta_1 t(x) + \delta_2 \tag{4.2.5}$$

where $\delta_2 = 1 - \delta_1$ and $t(x) = T(x)/T_o$. The quantity δ_2 is the ratio between the nonstructural or core mass and the total mass of the original structure and is a constant. Expression (4.2.5) is similar to that encountered in the torsional vibration problem in Chapter 2. From equation (4.2.2) it is seen that the ratio $EI(x)/EI_o$ can be written as:

$$\frac{EI(x)}{EI_o} = \frac{T(x)}{T_o} = t(x) \tag{4.2.6}$$

Using the above expression, equation (4.2.4) becomes:

$$(tw'')'' - \omega_o^2\left(\frac{m_o L^4}{EI_o}\right)(\delta_1 t + \delta_2)w(x) = 0 \tag{4.2.7}$$

Equation (4.2.7) is the constraint equation which any variable face-sheet thickness distribution must satisfy if it is to have a frequency ω_o equal to that of the uniform structure. One further simplification can be made by noting that ω_o^2 is, for a cantilever beam, given by

$$\omega_o^2 = (.597\pi)^4\left(\frac{EI_o}{m_o L^4}\right) \tag{4.2.8}$$

By letting: $\alpha = (.597\pi)^4 \delta_1$

$\beta = (.597\pi)^4 \delta_2$

the constraint equation (4.2.7) may be written as:

$$(tw'')'' - (\alpha t + \beta)w(x) = 0 \tag{4.2.9}$$

$$0 \le x \le 1$$

Using a change of variables, this fourth order equilibrium equation can be written as a set of four simultaneous, first-order, ordinary differential equations

$$w' = p \tag{4.2.10a}$$

$$p' = q/t \tag{4.2.10b}$$

$$q' = r \tag{4.2.10c}$$

$$r' = (\alpha t + \beta)w \tag{4.2.10d}$$

These equations are now in the form suitable for optimization theory.

The constraint equations in equations (4.2.10a,b,c,d) could have been formulated without reference to the boundary conditions. Until we specify that $\alpha = (.597\pi)^4 \delta_1$ and give the boundary conditions for a cantilever, these constraint equations are perfectly general. These general equations will also appear in the section which discusses the beam on simple supports.

For the cantilever beam, the state variable boundary conditions are

$$w(0) = p(0) = q(1) = r(1) = 0 \tag{4.2.10e}$$

The state variables w, p, q, r represent the nondimensional modal deflection, slope, bending moment, and shear respectively. The performance index or merit function whose minimum is sought is given by

$$J = \int_0^1 t\,dx \tag{4.2.11}$$

The Hamiltonian for this problem is given by

$$H = t + \lambda_w p + \lambda_p q/t + \lambda_q r + \lambda_r(\alpha t + \beta)w \tag{4.2.12}$$

where $\lambda_w, \lambda_p, \lambda_q$ and λ_r are the adjoint variables or Lagrange multipliers for their respective state variables w, p, q, r.

The necessary set of differential equations for an extremum of J is composed of the constraint or state variable equations (4.2.10a,b,c,d) and the differential equations for the adjoint variables. These adjoint equations are given by

$$-\frac{\partial H}{\partial w} = \lambda_w' = -\lambda_r(\alpha t + \beta) \tag{4.2.13a}$$

$$-\frac{\partial H}{\partial p} = \lambda_p' = -\lambda_w \tag{4.2.13b}$$

$$-\frac{\partial H}{\partial q} = \lambda'_q = -\lambda_p/t \tag{4.2.13c}$$

$$-\frac{\partial H}{\partial r} = \lambda'_r = -\lambda_q \tag{4.2.13d}$$

The control equation gives an algebraic relation between the control variable $t(x)$ and the state variables and adjoint variables.

$$\frac{\partial H}{\partial t} = 0 = 1 - \frac{\lambda_p q}{t^2} + \alpha\lambda_r w \tag{4.2.13e}$$

or

$$t^2(x) = \frac{\lambda_p q}{1 + \alpha\lambda_r w} \tag{4.2.13f}$$

Since $t(x)$ occurs both in the state variable and the adjoint variable differential equations, they are both coupled and nonlinear.

The boundary conditions for the adjoint variables may be simply expressed as

$$\begin{aligned} &\text{at } x = 0 \text{ either } \lambda_a(0) \text{ or } a(0) = 0 \\ &\text{at } x = 1 \text{ either } \lambda_a(1) \text{ or } a(1) = 0 \end{aligned} \tag{4.2.14a}$$

where "a" is a particular state variable such as w. Therefore, from the boundary conditions in equation (4.2.10e) it can be seen, using equations (4.2.14a), that the adjoint variable boundary conditions are

$$\lambda_q(0) = \lambda_r(0) = \lambda_w(1) = \lambda_p(1) = 0 \tag{4.2.14b}$$

Equations (4.2.10a,b,c,d) and (4.2.13a,b,c,d) together with the control equation and the boundary conditions (4.2.14a,b) define an eighth order, nonlinear, two-point, boundary value problem which must be solved to find the thickness parameter distribution $t(x)$ for the least weight beam. Fortunately, an examination of the state variable equations and the adjoint equations and the boundary conditions shows that a solution exists for which the adjoints or multipliers are linear functions of the state variables. That is:

$$\begin{Bmatrix} w(x) \\ p(x) \\ q(x) \\ r(x) \end{Bmatrix} = A \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_w(x) \\ \lambda_p(x) \\ \lambda_q(x) \\ \lambda_r(x) \end{Bmatrix} \tag{4.2.15}$$

The constant A is arbitrary, but must be greater than zero for the present problem. The adjoint variables are seen to be proportional to the modal amplitude of the state variables. Using equation (4.2.15a), the control equation may be written

$$t^2(x) = Aq^2/(1 + \alpha A w^2) \tag{4.2.16a}$$

If $A > 0.0$ then we can see that $t^2(x) > 0$. Now, since we are working with eigenvalue constraints, we would suspect that the thickness would depend on the various state variable mode <u>shapes</u>, but not the <u>amplitude</u>. Thus, for a set of A's the state variables such as $q(x)$ will have the same shape but their amplitudes will vary with A. Thus, at $x = x_1$ the value of $q(x_1)$ will vary with A but, the <u>product</u> $Aq(x_1)$ will be invariant for the optimum solution. For ease of numerical computation, the value of A is set equal to unity. Then, equation (4.2.16a) becomes:

$$t^2(x) = q^2(x)/(1 + \alpha w^2) \tag{4.2.16b}$$

Using equation (4.2.16b) together with the state variable equations, the eighth-order problem can be reduced to a fourth-order problem in the state variables. Using equation (4.2.15) permits us to uncouple the state variables from their adjoints, however, note that the state variables are still coupled through the control equation.

The problem of finding a thickness distribution for a minimum weight beam which has its first or fundamental frequency equal to that of a uniform thickness reference beam now becomes one of solving the first-order equations

$$w' = p \tag{4.2.17a}$$

$$p' = q/t \tag{4.2.17b}$$

$$q' = r \qquad 0 \le x \le 1 \tag{4.2.17c}$$

$$r' = (\alpha t + \beta)w \tag{4.2.17d}$$

with $w(0) = p(0) = q(1) = r(1) = 0$ and with $t(x)$ given in equation (4.2.16b).

A computer program was written to numerically integrate the above equations from $x = 0$ to $x = 1$ using as boundary conditions $w(0) = p(0) = 0$ and assumed initial values of the control constants $q_o = q(0)$ and $r_o = r(0)$. For the optimum solution, these constants q_o and r_o must have values which, together with the boundary conditions $w(0) = p(0) = 0$ will yield $q(1) = r(1) = 0$ when the optimality differential equations (4.2.17a,b,c,d) are integrated from $x = 0$ to $x = 1$.

An analytic function solution to equations (4.2.10a,b,c,d) when $t(x)$ is given by equation (4.2.16b) has not yet been found. A wide variety of possible solutions have been attempted, but with no success. However, a transition matrix solution has been devised to solve this problem. Using equations (4.2.17a,b,c,d) with $t(x)$ given in equation (4.2.16b) and the boundary conditions $w(0) = p(0) = 0$, a numerical method was programmed which perturbs an initial set of assumed values of $q(0)$ and $r(0)$ in such a way as to eventually force $q(1)$ and $r(1)$ close to zero. The result of these cyclic iterations is a numerical solution to the set of optimality equations (4.2.10a,b,c,d; 4.2.16b) for which the boundary conditions are $w(0) = p(0) = q(1) = r(1) = 0$. These are the necessary conditions for J to be a minimum and thus our numerical solution satisfies the optimality conditions. To force $q(1)$ and $r(1)$ to zero, perturbations in $q(1)$ and $r(1)$ must be related to perturbations in q_o and r_o. These relations may be written, to a first order approximation, as

$$\delta \begin{Bmatrix} q(1) \\ r(1) \end{Bmatrix} = \begin{bmatrix} \frac{\partial q(1)}{\partial q(0)} & \frac{\partial q(1)}{\partial r(0)} \\ \frac{\partial r(1)}{\partial q(0)} & \frac{\partial r(1)}{\partial r(0)} \end{bmatrix} \begin{Bmatrix} \delta q_o \\ \delta r_o \end{Bmatrix} \tag{4.2.18a}$$

or

$$\delta \begin{Bmatrix} q(1) \\ r(1) \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \delta q_o \\ \delta r_o \end{Bmatrix} \tag{4.2.18b}$$

The 2 x 2 matrix above is seen to be nothing more than a transition matrix such as described in Chapter 2. For a given choice of q_o and r_o and with $w(0) = p(0) = 0$, the integration of the state variable equations from $x = 0$ to 1 using the expression for $t(x)$ in equation (4.2.16b) will yield values of $q(1)$ and $r(1)$ which are either near zero, to within some error tolerance, or which are outside this tolerance, i.e.

$$\begin{Bmatrix} q_o \\ r_o \end{Bmatrix} \Rightarrow \begin{Bmatrix} q(1) \\ r(1) \end{Bmatrix} \leq \begin{Bmatrix} \epsilon_q \\ \epsilon_r \end{Bmatrix} \qquad (4.2.19a)$$

or

$$\begin{Bmatrix} q_o \\ r_o \end{Bmatrix} \Rightarrow \begin{Bmatrix} q(1) \\ r(1) \end{Bmatrix} > \begin{Bmatrix} \epsilon_q \\ \epsilon_r \end{Bmatrix} \qquad (4.2.19b)$$

If the latter case, equation (4.2.19b), is true then perturbations in q_o and r_o must be found such that the next integration cycle brings the values of $q(1)$ and $r(1)$ closer to zero. If we let

$$\delta \begin{Bmatrix} q(1) \\ r(1) \end{Bmatrix}_M = - \kappa \begin{Bmatrix} q(1) \\ r(1) \end{Bmatrix}_M \qquad (4.2.20)$$

where $q(1)_M$ and $r(1)_M$ are the values of $q(1)$ and $r(1)$ found at the end of the Mth integration cycle, then from equation (4.2.18b) we find the perturbations δq_o and δr_o to be

$$\begin{Bmatrix} \delta q_o \\ \delta r_o \end{Bmatrix}_{M+1} = - \kappa_M [T_{ij}]^{-1}_M \begin{Bmatrix} q(1) \\ r(1) \end{Bmatrix}_M \qquad (4.2.21)$$

where the subscript "M" refers to the integration cycle number. Thus, on the (M + 1)st cycle, the values of q_o and r_o will be

$$\begin{Bmatrix} q_o \\ r_o \end{Bmatrix}_{M+1} = \begin{Bmatrix} q_o \\ r_o \end{Bmatrix}_M + \begin{Bmatrix} \delta q_o \\ \delta r_o \end{Bmatrix}_M \qquad (4.2.22)$$

2⊕

This type of iterative integration has been found to be extremely successful when the initial values of q_o and r_o are chosen properly. This initial choice of q_o and r_o will be discussed below. However, before the discussion of the estimation of q_o and r_o, it would be well to discuss the generation of the transition matrix for this problem. As defined in Chapter 3, the perturbation equations for this problem are:

$$(\delta w)' = \delta p \tag{4.2.23a}$$

$$(\delta p)' = \frac{\delta q}{t} - \frac{q}{t^2}\delta t \tag{4.2.23b}$$

$$(\delta q)' = \delta r \tag{4.2.23c}$$

$$(\delta r)' = (\alpha t + \beta)(\delta w) + \alpha w(\delta t) \tag{4.2.23d}$$

where

$$\delta t = \frac{1}{t}\left[\frac{q\delta q}{1+\alpha w^2} - \frac{\alpha q^2 w\delta w}{(1+\alpha w^2)^2}\right] \tag{4.2.24}$$

or

$$\delta t = t\left[\frac{\delta q}{q} - \alpha t^2 \frac{w\delta w}{q^2}\right] \tag{4.2.25}$$

The perturbation equations involve values of the state variables. The above perturbation equations are integrated, together with the state variable equations and the chosen boundary conditions q_o and r_o, with:

$$w(0) = p(0) = \delta w(0) = \delta p(0) = \delta r(0) = 0; \quad \delta q(0) = 1 \tag{4.2.26}$$

then, at $x = 1$ the values of $\delta r(1)$ and $\delta q(1)$ are seen to be

$$T_{11} = \frac{\partial q(1)}{\partial q(0)} = \delta q(1) \tag{4.2.27a}$$

$$T_{21} = \frac{\partial r(1)}{\partial q(0)} = \delta r(1) \tag{4.2.27b}$$

Similarly, with $q(0) = q_o$; $r(0) = r_o$ and $w(0) = p(0) = \delta w(0) = \delta p(0) = \delta q(0) = 0$; $\delta r(0) = 1$ the integration of the system equations and the perturbation equations yields

$$T_{12} = \frac{\partial q(1)}{\partial r(0)} = \delta q(1) \tag{4.2.28a}$$

$$T_{22} = \frac{\partial r(1)}{\partial r(0)} = \delta r(1) \tag{4.2.28b}$$

It will be noted that, since the perturbation equations are linear in the perturbation variables and only depend on the system variables, the generation of all columns of $\overline{T}$ may be done simultaneously if computer storage space permits.

Placing a minimum thickness constraint on this problem is quite useful since, at $x = 1$, in the absence of such a constraint

$$t^2(1) = \frac{q^2(1)}{1 + \alpha w^2(1)} = 0 \tag{4.2.29}$$

If, during the iterative integration process, a value of $q(x)$ should be encountered which is equal to zero, then the numerical solution process may diverge. This occurs because, from equation (4.2.10b)

$$p' = q/t = 0/0 \tag{4.2.10b}$$

$p'(x)$ will then become numerically indeterminate. Thus, in most cases, at least a small minimum-thickness constraint is imposed to facilitate a solution.

The estimation of initial values of q_o and r_o requires a little analysis. The previous investigations have shown that good first approximations to the state variables, i.e. $w(x)$, $p(x)$, $q(x)$, $r(x)$, can be found by using the eigenfunction solutions from the uniform thickness solution.

For a cantilever beam of uniform stiffness and mass per unit length, the mode shape $w(x)$ is found to be (Ref. 14):

$$w(x) = C[(\frac{\sin \Omega_n - \sinh \Omega_n}{\cosh \Omega_n + \cos \Omega_n})(\sinh \Omega_n x - \sin \Omega_n x) + (\cosh \Omega_n x - \cos \Omega_n x)] \tag{4.2.30}$$

Letting $t(x) = 1$ and using the definitions for w, p, q, r in equations (4.2.10a,b,c,d), we find that the ratio $r(0)/q(0)$ obtained using equation (4.2.30) as an approximation for $w(x)$ is

$$\frac{r(0)}{q(0)} = \Omega_n(\frac{\sin \Omega_n - \sinh \Omega_n}{\cosh \Omega_n + \cos \Omega_n}) \tag{4.2.31}$$

where, for the fundamental frequency

$$\Omega_n = .597\pi$$

Thus, a good trial value for the ratio in the numerical solution might be:

$$\frac{r(0)}{q(0)} = \frac{r_o}{q_o} = -1.37734 \qquad (4.2.32)$$

Now that an estimate of the ratio of control constants has been obtained, only q_o or r_o need be estimated. Since $w(0) = 0$, an estimate for $q_o = q(0)$ can be easily obtained from equation (4.2.16b)

$$t^2(x) = q^2(x)/(1 + \alpha w^2(x)) \qquad (4.2.16b)$$

For any iteration, $t(0) = q(0) = q_o$. Thus, the estimate involves the sizing of $t(0)$. Since, initially, the reference structure has $t(0) = 1$, a likely initial value of q_o is $q_o = 1$. From the above ratio r_o/q_o we find $r_o = -1.3774$.

These estimates were used in a computer program which uses iterative integration to solve the optimization problem. Using these numbers, an initial distribution of $t(x)$ is obtained as shown in Figure 4.1. Using values of $\kappa = .5, .9, .9, .9$, the final converged solution of thickness distribution was obtained and is also shown in Figure 4.1. For this case, δ_1, the initial structural mass ratio, is $\delta_1 = 0.50$. In addition, no minimum thickness constraint is imposed. A quantity, called the "mass ratio," is defined as

$$MR = \delta_1 \int_0^1 t dx + \delta_2 \qquad (4.2.33)$$

and is used to indicate weight savings from the optimization process. The mass ratio is the ratio of the total weight of the optimum beam to the total weight of the reference beam. In the case shown in Figure 4.1, the mass ratio is $MR = 0.6632$. Thus, the weight of the cantilever beam after optimization is 66.32% of the reference beam so that the result of optimization is a 33.68% savings.

Although no analytic or functional solution for this optimization problem has yet been obtained, an interesting check solution has been found. If $\alpha t << \beta$ and $\alpha w << 1$, then the nonlinear state variable equations may be estimated by:

$$w' = p \tag{4.2.34a}$$

$$p' \cong 1.0 \tag{4.2.34b}$$

$$q' = r \tag{4.2.34c}$$

$$r' \cong \beta w \tag{4.2.34d}$$

$$w(0) = p(0) = q(1) = r(1) = 0 \tag{4.2.34e}$$

where

$$t^2(x) = [q^2/(1 + \alpha w^2)] \cong q^2(x) \tag{4.2.34f}$$

The exact solution to the equations above is given by

$$w(x) = x^2/2 \tag{4.2.35a}$$

$$p(x) = x \tag{4.2.35b}$$

$$q(x) = t(x) = \frac{\beta}{24}(x^4 - 4x + 3) \tag{4.2.35c}$$

$$r(x) = \frac{\beta}{6}(x^3 - 1) \tag{4.2.35d}$$

where $\beta = (0.597\pi)^4(1 - \delta_1)$.

Note that this approximate solution for $t(x)$ is linear in $\delta_2 = (1 - \delta_1)$. This approximation is remarkably accurate, as can be seen in Figure 4.2. In this figure, the topmost curve shows the results of expression (4.2.35c) for $\delta_1 = 0$ or $\delta_2 = 1 - \delta_1 = 1.0$. An approximation for any other value of δ_1 can be found by multiplying each point on this curve marked "approximation" by $(1 - \delta_1)$. The "exact" solutions for $\delta_1 = .1, .5, .9$ are also given in Figure 4.2 and show the behavior of the thickness distribution as a function of δ_1. As $\delta_1 \to 1.0$, or $(1 - \delta_1) \to 0.0$, the nonstructural mass disappears. At the same time, as in the case of the torsional vibration problem, this problem becomes poorly posed, i.e. $t(x) = 0$ is a solution, and the load carrying structure disappears, that is $\mathcal{MR} = 0$.

At the opposite extreme, $\delta_1 \to 0.0$, as δ_1 decreases there is less total structure available to be optimized. Because of this, as $\delta_1 \to 0.0$, $\mathcal{MR} \to 1.0$. Figure 4.3 shows the behavior of the mass ratio between the two extremes, $\delta_1 = 0$ and $\delta_1 = 1.0$.

If a constraint is placed on the thickness in the form $t(x) \geq t_{min}$, the results of this minimum thickness constraint at any value of δ_1 for which the problem is well posed will be similar to those shown in Figure 4.4. This figure shows three different minimum thickness constraints together with the case for which $t_{min} = 0$. These cases all have the common parameter $\delta_1 = 0.50$. As is seen in Figure 4.4, as $t_{min} \to 1.0$, the optimum distribution of $t(x)$ approaches the uniform thickness reference case. In addition, as $t_{min} \to 1.0$, MR $\to 1.0$.

Table 4.1 shows some values of the control constants q_o and r_o which were obtained in the previously discussed work. The values of the ratio r_o/q_o are also shown to give an indication of the accuracy of the estimation procedure suggested in this section. In all cases shown in the table, this estimated ratio was within 4% of the "exact" value.

This completes the discussion of the optimum cantilever beam. Further reference will be made to this configuration after the discussion of the pinned-pinned beam in the next section.

4.3 Least Weight Optimization of a Sandwich Beam on Simple Supports With the Fundamental Frequency of Flexural Vibration Held Constant

The nondimensional equilibrium equations for a sandwich beam on simple supports has the same general form as that given for the cantilever in Section 4.2. These first-order equations are:

$$w' = p \tag{4.3.1a}$$

$$p' = q/t \tag{4.3.1b}$$

$$q' = r \tag{4.3.1c}$$

$$r' = (\alpha t + \beta)w \qquad 0 \leq x \leq 1 \tag{4.3.1d}$$

For this case, α and β are found to be

$$\alpha = (\pi^4)\delta_1$$

$$\beta = (\pi^4)(1 - \delta_1)$$

The boundary conditions reflect the fact that, at the supports, there is no deflection or bending moment. This may be expressed mathematically as:

$$w(0) = q(0) = w(1) = q(1) = 0 \tag{4.3.2}$$

Since the quantity to be minimized is again

$$J = \int_0^1 t\,dx \tag{4.3.3}$$

the Hamiltonian may again be written as

$$H = t + \lambda_w p + \lambda_p q/t + \lambda_q r + \lambda_r(\alpha t + \beta)w \tag{4.3.4}$$

Except for the state variable boundary conditions and different numerical values of α and β, the governing equations necessary to achieve a minimum of J are identical to those encountered in the previous section. Therefore, the adjoint variable differential equations are:

$$-\frac{\partial H}{\partial w} = \lambda'_w = -\lambda_r(\alpha t + \beta) \tag{4.3.5a}$$

$$-\frac{\partial H}{\partial p} = \lambda'_p = -\lambda_w \tag{4.3.5b}$$

$$0 \le x \le 1$$

$$-\frac{\partial H}{\partial q} = \lambda'_q = -\lambda_p/t \tag{4.3.5c}$$

$$-\frac{\partial H}{\partial r} = \lambda'_r = -\lambda_q \tag{4.3.5d}$$

while the control equation can again be written as

$$t^2 = \lambda_p q/(1 + \alpha\lambda_r w) \tag{4.3.6}$$

Given the state variable boundary conditions in equation (4.3.2), the adjoint boundary conditions must be:

$$\lambda_p(0) = \lambda_r(0) = \lambda_p(1) = \lambda_r(1) = 0 \tag{4.3.7}$$

It can be shown that, given the above eighth-order system with its boundary conditions, a solution exists for which the following algebraic relation between the state variables and the multipliers will hold.

$$\begin{Bmatrix} w(x) \\ p(x) \\ q(x) \\ r(x) \end{Bmatrix} = A \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_w(x) \\ \lambda_p(x) \\ \lambda_q(x) \\ \lambda_r(x) \end{Bmatrix} \tag{4.3.8}$$

Here again, the modal constant A appears in the state variable-adjoint relation. If $t^2(x)$ is to be positive, then A must be greater than zero. For ease of computation, let A equal one. Because of equation (4.3.8), the control equation becomes a function only of the state variables (or equivalently, only of the multipliers):

$$t^2(x) = q^2(x)/(1 + \alpha w^2(x)) \tag{4.3.9}$$

An examination of the boundary conditions for the problem shows that, at $x = 0$ and $x = 1$, the thickness parameter vanishes. This vanishing of the thickness at the end points gives rise to singularities in the differential equations. For this reason, a minimum thickness constraint was used in all cases to be discussed.

It is worthwhile to note that the differential equations and the control equation for the pinned-pinned beam are the same as those for the clamped-free beam. The boundary conditions are, of course, different and it will be seen that these differing boundary conditions give rise to markedly different thickness distributions and mass ratios.

The problem now becomes one of finding a numerical solution to the differential equations (4.3.1a,b,c,d), with $t(x)$ given by equation (4.3.9), and the boundary conditions:

$$w(0) = q(0) = w(1) = q(1) = 0$$

The control constants for this problem are $p(0) = p_o$, the nondimensional slope and $r(0) = r_o$, the nondimensional shear. These control constants must be perturbed in such a way that, after several iterative integration cycles, the values of $w(1)$ and $q(1)$ are near zero.

A first-order approximation of the relation between perturbations in $p(0)$ and $r(0)$ and the resulting perturbations in $w(1)$ and $q(1)$ may be expressed as:

$$\delta \begin{Bmatrix} w(1) \\ q(1) \end{Bmatrix} = \begin{bmatrix} \frac{\partial w(1)}{\partial p(0)} & \frac{\partial w(1)}{\partial r(0)} \\ \frac{\partial q(1)}{\partial p(0)} & \frac{\partial q(1)}{\partial r(0)} \end{bmatrix} \begin{Bmatrix} \delta p_o \\ \delta r_o \end{Bmatrix} \tag{4.3.10a}$$

$$\delta \begin{Bmatrix} w(1) \\ q(1) \end{Bmatrix} = \begin{bmatrix} T_{ij} \end{bmatrix} \begin{Bmatrix} \delta p_o \\ \delta r_o \end{Bmatrix} \qquad (4.3.10b)$$

The matrix $\overline{T}$ is again seen to be the transition matrix for the problem. The method of solution in this problem is similar to that used for the problem in Section 4.2.

The control constants p_o and r_o must first be estimated in a manner similar to that shown in Section 4.2. Using these approximations and the boundary conditions $w(0) = q(0) = 0$, the state variable equations may be integrated from $x = 0$ to $x = 1$. At $x = 1$, the values of $w(0)$ and $q(0)$ are recorded and compared to the error vector $\tilde{\epsilon}$. If

$$\begin{Bmatrix} w(1) \\ q(1) \end{Bmatrix}_M > \tilde{\epsilon}$$

then we choose

$$\begin{Bmatrix} \delta w(1) \\ \delta q(1) \end{Bmatrix}_M = -\kappa \begin{Bmatrix} w(1) \\ q(1) \end{Bmatrix}_M$$

where the subscript M refers to the iteration cycle number. During each of these integration cycles, the transition matrix is determined by simultaneously integrating the system perturbation equations. For the pinned-pinned beam, the perturbation differential equations are identical to those used for the cantilever beam, equations (4.2.22a,b,c,d; 4.2.23b). There are two sets of perturbation boundary conditions necessary to determine the transition matrix. These are:

$$\delta w(0) = \delta q(0) = \delta r(0) = 0; \quad \delta p(0) = 1 \qquad (4.3.11a)$$

and

$$\delta w(0) = \delta p(0) = \delta q(0) = 0; \ \delta r(0) = 1 \qquad (4.3.11b)$$

The first set of perturbation boundary conditions, equation (4.3.11a), will yield

$$T_{11} = \frac{\partial w(1)}{\partial p(0)} = \delta w(1); \qquad T_{21} = \frac{\partial q(1)}{\partial p(0)} = \delta q(1) \qquad (4.3.12a)$$

while the second set will give

$$T_{12} = \frac{\partial w(1)}{\partial r(0)} = \delta w(1); \qquad T_{22} = \frac{\partial q(1)}{\partial r(0)} = \delta q(1) \tag{4.3.12b}$$

When $\overline{T}$ has been determined, the control constant perturbations can be calculated from:

$$\begin{Bmatrix} \delta p_o \\ \delta r_o \end{Bmatrix}_M = - \kappa_M [T_{ij}]_M^{-1} \begin{Bmatrix} w(1) \\ q(1) \end{Bmatrix}_M \tag{4.3.13}$$

On the $(M+1)$st integration, the system boundary conditions will be:

$$\begin{Bmatrix} p(0) \\ r(0) \end{Bmatrix}_{M+1} = \begin{Bmatrix} p(0) \\ r(0) \end{Bmatrix}_M + \begin{Bmatrix} \delta p_o \\ \delta r_o \end{Bmatrix}_M \tag{4.3.14}$$

If at any time the value of $t(x)$, as given by equation (4.3.9) falls below t_{min} then the value of $t(x)$, as used in the state variable equations, is then set equal to t_{min}. In addition, the value of $\delta t(x)$ is set equal to zero when the constraint boundary, $t = t_{min}$, is encountered. The solution to this problem is basic to the solution of the panel flutter optimization problem discussed in Chapter 6. For this reason it was researched carefully and a wide variety of solutions was obtained.

First of all, let us discuss the estimation of p_o and r_o. The estimation procedure is similar to that discussed in Section 4.2 and involves using the mode shape of the reference structure as an approximation for the deflection $w(x)$ encountered in the optimization equations. The solution for $w(x)$ with $t(x) = 1$ and equations (4.3.1a,b,c,d) is found to be:

$$w(x) = C \sin \pi x \tag{4.3.15}$$

Since $t(x) = 1$, the equations are linear and the constant C is, as yet, undetermined. Once $w(x)$ has been approximated, the expressions for $p(x)$, $q(x)$ and $r(x)$ also follow from equations (4.3.1a,b,c,d) if $t(x) = 1$. With these expressions, the ratio r_o/p_o can be shown to be

$$\frac{r(0)}{p(0)} = \frac{r_o}{p_o} = -\pi^2 = -9.8696 \tag{4.3.16}$$

If the approximate solutions obtained above are inserted into the control equation we get

$$t^2(x) = \frac{\pi^4 C^2 \sin^2 \pi x}{1 + \pi^4 C^2 \delta_1 \sin^2 \pi x} \tag{4.3.17}$$

It can be shown that this solution for $t(x)$ is symmetric about $x = 1/2$ and reaches its maximum value there. If $t(1/2) = t_{max}$, then the solution for C becomes

$$C = \left(\frac{t_{max}}{\pi^2}\right) \left(\frac{1}{1 - \delta_1 t_{max}^2}\right)^{1/2} \tag{4.3.18}$$

For a given value of δ_1 and for an assumption that $t_{max} \cong 1.0$ a good set of approximations for $p(0)$ and $r(0)$ are:

$$p_o = p(0) = C\pi; \; r_o = r(0) = - C\pi^3 \tag{4.3.19}$$

where C is determined from equation (4.3.18).

It will be noted that the approximations for the state variables have an arbitrary modal constant multiplier C. However, once these approximations are used in the expression for $t(x)$ and the value of t_{max} is set, the arbitrary constant C is determined. This can be seen to be parallel to the discussion of the constant A encountered in the previous discussion. With $t(x)$ not equal to a function of the state variables, the state variable equations are linear in the state variables. However, once $t(x)$ becomes a definite function of the state variables the equations become nonlinear. Thus, for the nonlinear state variable equations, the solutions are unique and have no arbitrary constant multiplier.

When $\beta >> \alpha t$ and $1 >> \alpha w^2$, an approximate solution can also be obtained in a manner similar to that shown in Section 4.2. If the above assumptions hold, the nonlinear state variable equations become:

$$w' = p \tag{4.3.20a}$$

$$p' = q/t \cong 1.0 \tag{4.3.20b}$$

$$q' = r \tag{4.3.20c}$$

$$r' = (\alpha t + \beta)w \cong \beta w \tag{4.3.20d}$$

and

$$r^2(x) = q^2/(1 + \alpha w^2) \cong q^2(x) \tag{4.3.20e}$$

with boundary conditions

$$w(0) = q(0) = w(1) = q(1) = 0 \tag{4.3.20f}$$

The solutions to the above equations are:

$$w(x) = x(x - 1)/2 \tag{4.3.21a}$$

$$p(x) = x - 1/2 \tag{4.3.21b}$$

$$q(x) = t(x) = \pi^4(1 - \delta_1)(x^4/2 - x^3 + x/2)/12 \tag{4.3.21c}$$

$$r(x) = \pi^4(1 - \delta_1)(x^3/6 - x^2/4 + 1/24) \tag{4.3.21d}$$

The solutions in equations (4.3.21a,b,c,d) are symmetric about $x = 1/2$ and the maximum value of $t(x)$ also occurs at $x = 1/2$ and is

$$t_{max} = t(1/2) = 1.2683(1 - \delta_1) \tag{4.3.22}$$

Although the above solution for thickness was obtained assuming that $t_{min} = 0.0$, that is, no thickness constraint, it compares rather closely with the exact solution obtained numerically for which $\delta_1 = .1$ and $t_{min} = .01$. These two solutions are plotted in Figure 4.5 for comparison.

The effect on the thickness distribution of varying δ_1 can be seen in Figure 4.6. This figure shows three different distributions for δ_1 equal to 0.9, 0.5 and 0.1 and a minimum-thickness constraint of $t_{min} = 0.10$. The mass ratio for these configurations is also shown on the figure.

The mass ratio **MR** is shown vs. δ_1 in Figure 4.7 for several values of δ_1 and a thickness constraint $t_{min} = .10$. The weight savings for the pinned-pinned beam are seen to be in the area of six to ten percent. This result differs greatly from the large savings seen to be possible for the cantilever beam.

The effect of varying the minimum thickness of a beam with a fixed value of δ_1 is shown in Figure 4.8. As $t_{min} \to 1$ it is seen that the thickness distribution begins to approach $t(x) = 1$ as it should.

Table 4.2 lists, for reference, the values of the control constants obtained for the cases shown in the figures, together with the ratio r_o/p_o. These control constants exhibit a continuous behavior when plotted vs. δ_1 or t_{min}. This behavior is extremely useful because once one or more solutions are known we may extrapolate to find others.

This completes the discussion of the pinned-pinned sandwich beam for the moment. More will be said about this configuration in Chapter 5 as regards the uniqueness of the thickness distributions found and the solution formulation when one or more eigenvalues other than the fundamental frequency are held fixed. The final portion of this section will treat the problem of optimizing a pinned-pinned beam whose cross-section is a rectangular solid. Thus, the bending stiffness will be found to be proportional to t^3.

4.4 Least Weight Optimization of a Pinned-Pinned Beam of Solid Rectangular Cross Section With its Fundamental Flexural Frequency Held Constant

The previous sections have dealt with beams for which the structural stiffness was a linear function of the nondimensional thickness and the thickness enters linearly in the equilibrium equation. If a beam has a solid rectangular cross-section, then the bending stiffness is given by

$$EI(x) = \frac{EbT^3(x)}{4} \tag{4.4.1}$$

where b is the cross-sectional width. Since the ratio of a variable bending stiffness to a reference bending stiffness can be written as

$$\frac{EI(x)}{EI_o} = \frac{bT^3(x)}{bT_o^3} = t^3(x) \tag{4.4.2}$$

it can be shown, with reference to equation (4.2.7) that the state variable eigenvalue constraint equations can be written as:

$$w' = p \tag{4.4.3a}$$

$$p' = q/t^3 \qquad 0 \le x \le 1 \tag{4.4.3b}$$

$$q' = r \tag{4.4.3c}$$

$$r' = (\alpha t + \beta)w \tag{4.4.3d}$$

In the above equations, α and β have the same values as in Section 4.3. For a pinned-pinned beam the boundary conditions and values for α and β are:

$$w(0) = w(1) = q(0) = q(1) = 0 \tag{4.4.3e}$$

$$\alpha = \delta_1 \pi^4$$

$$\beta = (1 - \delta_1)\pi^4$$

An interesting facet of this problem is that a solution for $\delta_1 = 1.0$ can be obtained, that is, an optimum beam can be found for which there is no nonstructural mass. This occurs because the frequency of a solid section beam depends upon the thickness. For a uniform beam, the square of the frequency is given by:

$$\omega_o^2 = \pi^4\left(\frac{EI_o}{m_o L^4}\right) = \pi^4\left(\frac{ET_o^2}{\rho L^4}\right) \tag{4.4.4}$$

where ρ is the density/unit volume. Although the expression for the frequency of a nonuniform beam will differ from equation (4.4.4), it is reasonable that the dependence of the frequency on beam thickness is similar to equation (4.4.4). Thus, the frequency will not be independent of thickness as it was for the sandwich beam for $\beta = 0$, and $t(x) = 0$ will not be an allowable solution.

The merit function to be minimized is again written as

$$J = \int_0^1 t dx$$

and, therefore, the Hamiltonian may be written as

$$H = t + \lambda_w p + \lambda_p \frac{q}{t^3} + \lambda_q r + \lambda_r(\alpha t + \beta)w$$

The adjoint equations become

$$-\frac{\partial H}{\partial w} = \lambda'_w = -\lambda_r(\alpha t + \beta) \tag{4.4.5a}$$

$$-\frac{\partial H}{\partial p} = \lambda'_p = -\lambda_w \tag{4.4.5b}$$

$$-\frac{\partial H}{\partial q} = \lambda'_q = -\lambda_p/t^3 \tag{4.4.5c}$$

$$-\frac{\partial H}{\partial r} = \lambda'_r = -\lambda_q \tag{4.4.5d}$$

with boundary conditions

$$\lambda_p(0) = \lambda_p(1) = \lambda_r(0) = \lambda_r(1) = 0 \tag{4.4.5e}$$

The control equation for this problem is

$$\frac{\partial H}{\partial t} = 1 - \frac{3\lambda_p q}{t^4} + \alpha\lambda_r w \tag{4.4.6a}$$

or

$$t^4 = \frac{3\lambda_p q}{1 + \alpha\lambda_r w} \tag{4.4.6b}$$

A comparison of these equations with the state variable and adjoint equations in Section 4.3 reveals a marked similarity. As in Section 4.3, it can be shown that a solution exists for which the relation between the state variables and the adjoint variables is

$$\begin{Bmatrix} w \\ p \\ q \\ r \end{Bmatrix} = A \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_w \\ \lambda_p \\ \lambda_q \\ \lambda_r \end{Bmatrix} \tag{4.4.7}$$

If the state variables are chosen as the dependent variables and $A = 1$, the control equation can be written as:

$$t^4(x) = \frac{3q^2(x)}{1 + \alpha w^2(x)} \tag{4.4.8}$$

Then, for this problem, it is necessary only to find that solution to the set of state variable equations (4.4.3a, b, c, d) together with equation (4.4.8) which satisfies the boundary conditions given in equation (4.4.3e).

The numerical solution process for this problem is nearly identical to that shown for the sandwich beam in Section 4.3. The state variable and perturbation equations differ slightly from those in Section 4.3 but the transition matrix relation is of the same form. The perturbation equations are:

$$(\delta w)' = \delta p \tag{4.4.9a}$$

$$(\delta p)' = \frac{\delta q}{t^3} - \frac{3q}{t^4}(\delta t) \tag{4.4.9b}$$

$$(\delta q)' = \delta r \tag{4.4.9c}$$

$$(\delta r)' = (\alpha t + \beta)(\delta w) + \alpha w(\delta t) \tag{4.4.9d}$$

with

$$\delta t = \frac{t}{2q}\left[\delta q - \alpha t^4\left(\frac{w}{q}\right)(\delta w)\right] \tag{4.4.9e}$$

The above perturbation equations and the nonlinear state variable equations were used with a transition matrix procedure exactly like that in Section 4.3. This method was used to generate optimal thickness distributions for several problems. Because of the control equation, this problem is highly nonlinear, a fact which makes the accuracy of the numerical techniques critical. The slope of the thickness distribution is very high near $x = 0$ and $x = 1$ causing additional numerical problems. To remedy these problems, a large minimum thickness constraint was used in the examples shown in Figure 4.9.

Figure 4.9 shows two cases solved using the transition matrix method. These cases have $\delta_1 = 1$ and $\delta_1 = 0.5$; $t_{min} = 0.50$. There is little difference between the distributions having the same t_{min} but different values of δ_1.

4.5 Summary of Results

Much has already been said regarding the quantitative results of the analyses in Chapter 4. There are, however, several topics which remain to be explored. The subject of the uniqueness of these solutions is very interesting. It would be well to ask; "are there any solutions other than those already found which also satisfy the constraints placed on the problem?" The related question is "what does the application of additional frequency constraints do to the problem?" Suppose, for instance, that the first two frequencies are held fixed or that only the second frequency is held fixed —— what results may we expect? These are interesting and important problems that will be discussed in the next section, Chapter 5.

As regards qualitative comments on the previous studies, probably the most interesting observation to be made is one regarding weight savings for the same structural configuration with different boundary conditions. It appears that quite sizable savings can be realized when clamped-free conditions are enforced and only the vibration frequency is held constant. This is of particular engineering interest since these are the type structures commonly encountered in aircraft wing design. Thus, if there are no constraints such as strength, a great savings can be realized for cantilever structures. The results for pinned-pinned structures are less impressive but are still significant and are seen to be of the order of 10%.

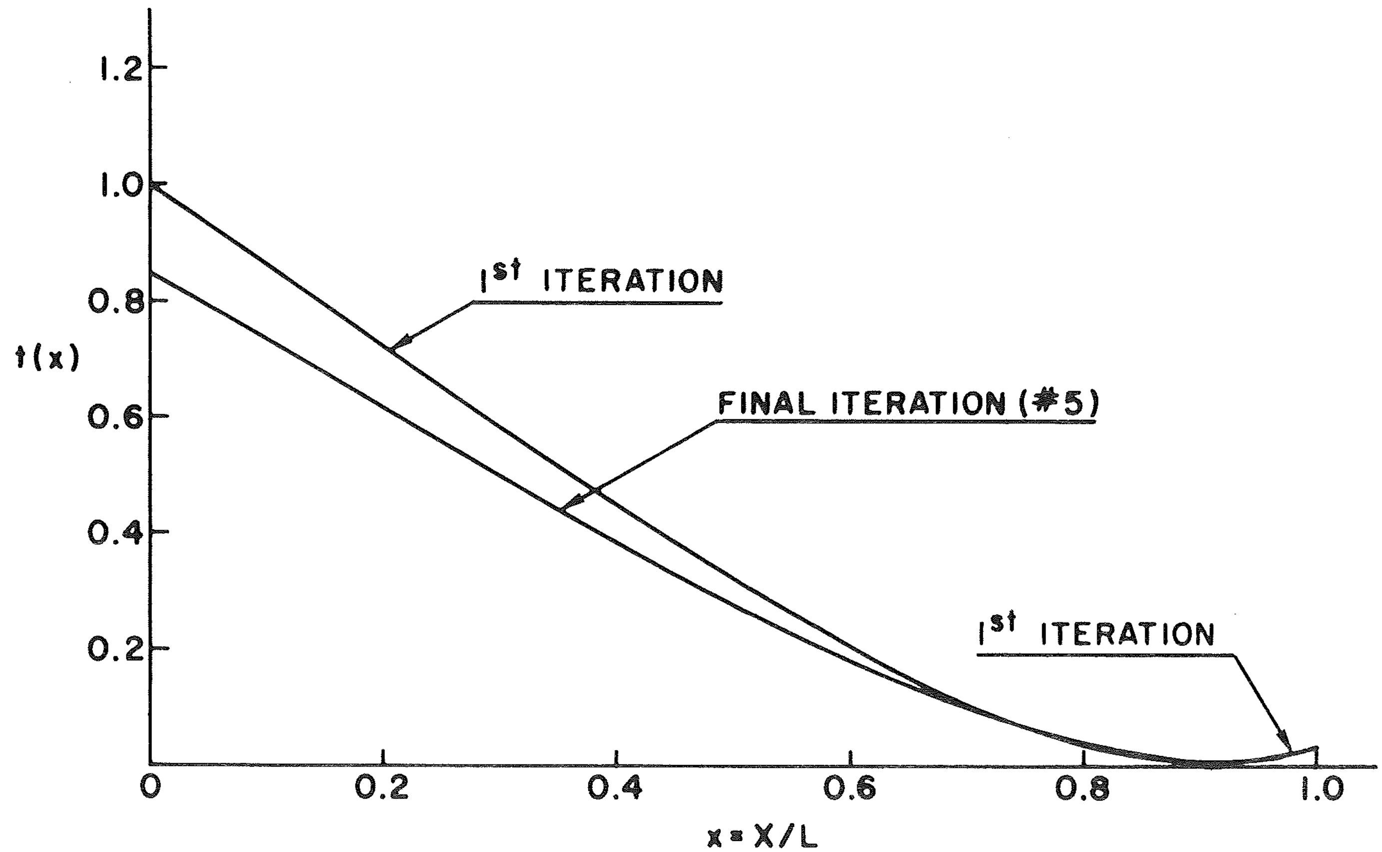

Fig. 4.1 Nondimensional Thickness Distribution Convergence — Minimum Weight Cantilever Beam — Sandwich Construction — $\delta_1 = .50$.

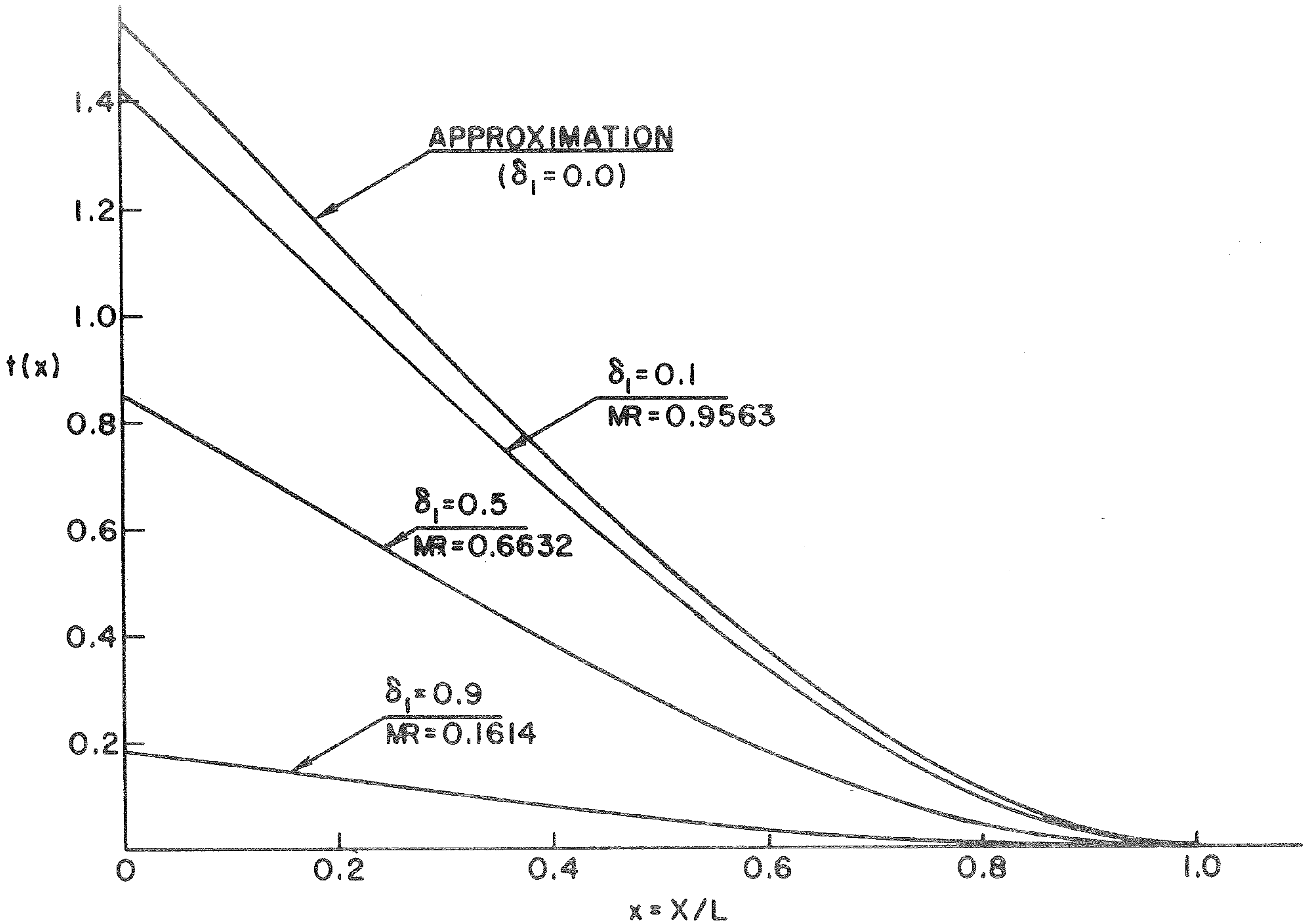

Fig. 4.2 Nondimensional Thickness Distributions —— Minimum Weight Cantilever Beam —— Sandwich Construction —— Varying Values of δ_1 —— First Frequency Held Constant.

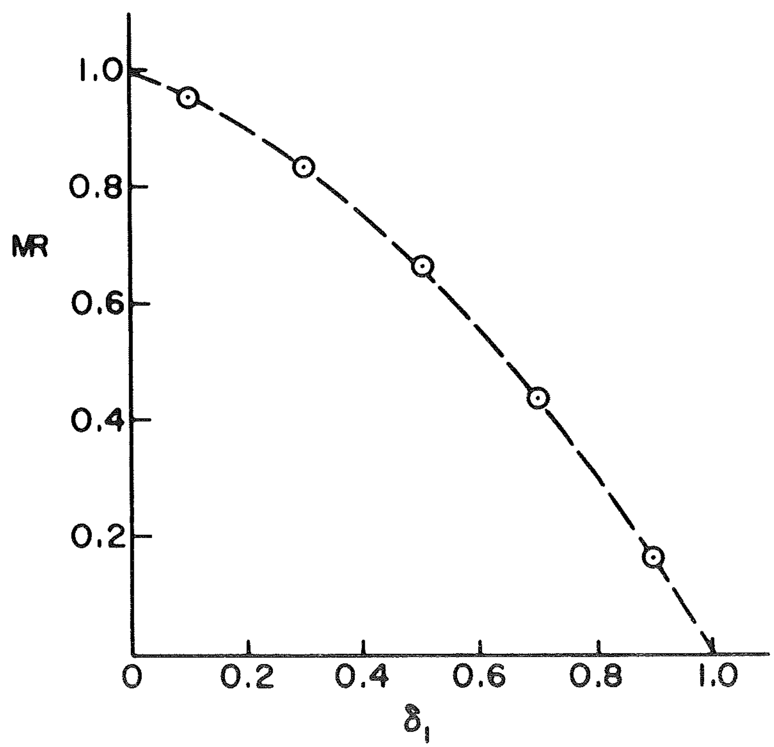

Fig. 4.3 Mass Ratio vs. δ_1 — Minimum Weight Cantilever Beam — Sandwich Construction — $t_{min} = 0.0$ — First Frequency Held Constant.

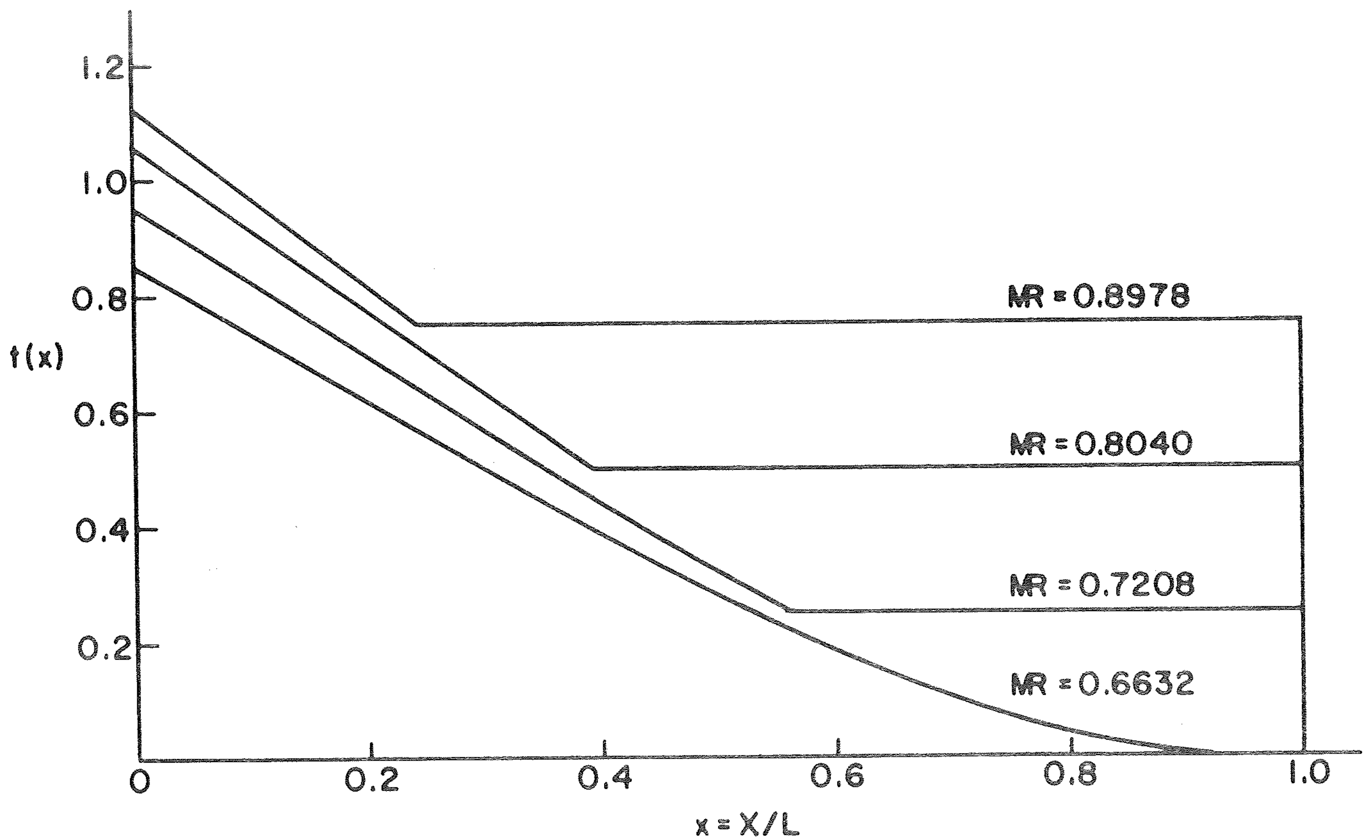

Fig. 4.4 Nondimensional Thickness Distribution — Minimum Weight Cantilever Beam — Sandwich Construction — $\delta_1 = 0.50$ — Several Minimum Thickness Constraints — First Frequency Held Constant.

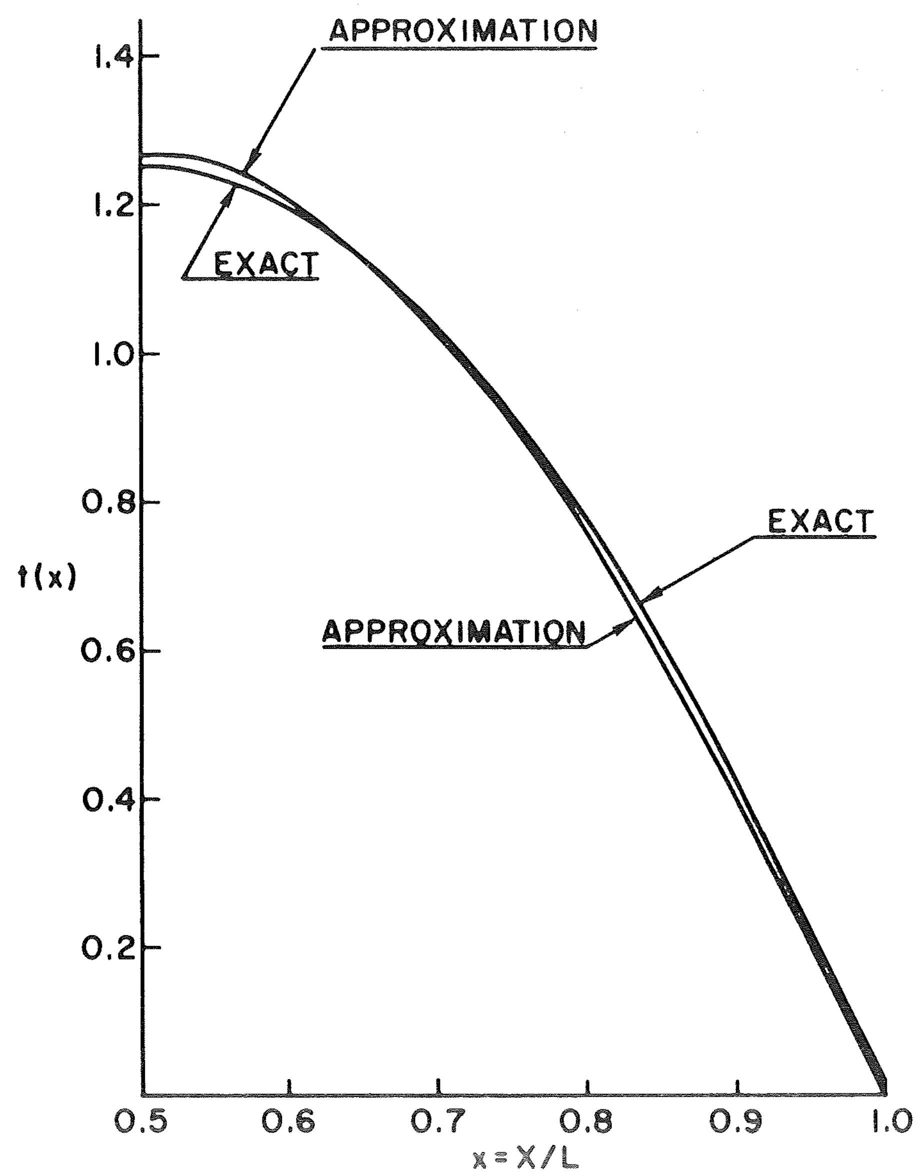

Fig. 4.5 Nondimensional Thickness Distributions (Symmetrical) Showing an Approximation Using $\delta_1 \simeq 0.0$ and $t_{min} = 0.0$ Compared to an Exact Solution for Which $\delta_1 = 0.10$ and $t_{min} = 0.01$ —— Minimum Weight Sandwich Beam on Simple Supports —— Fundamental Frequency Held Constant.

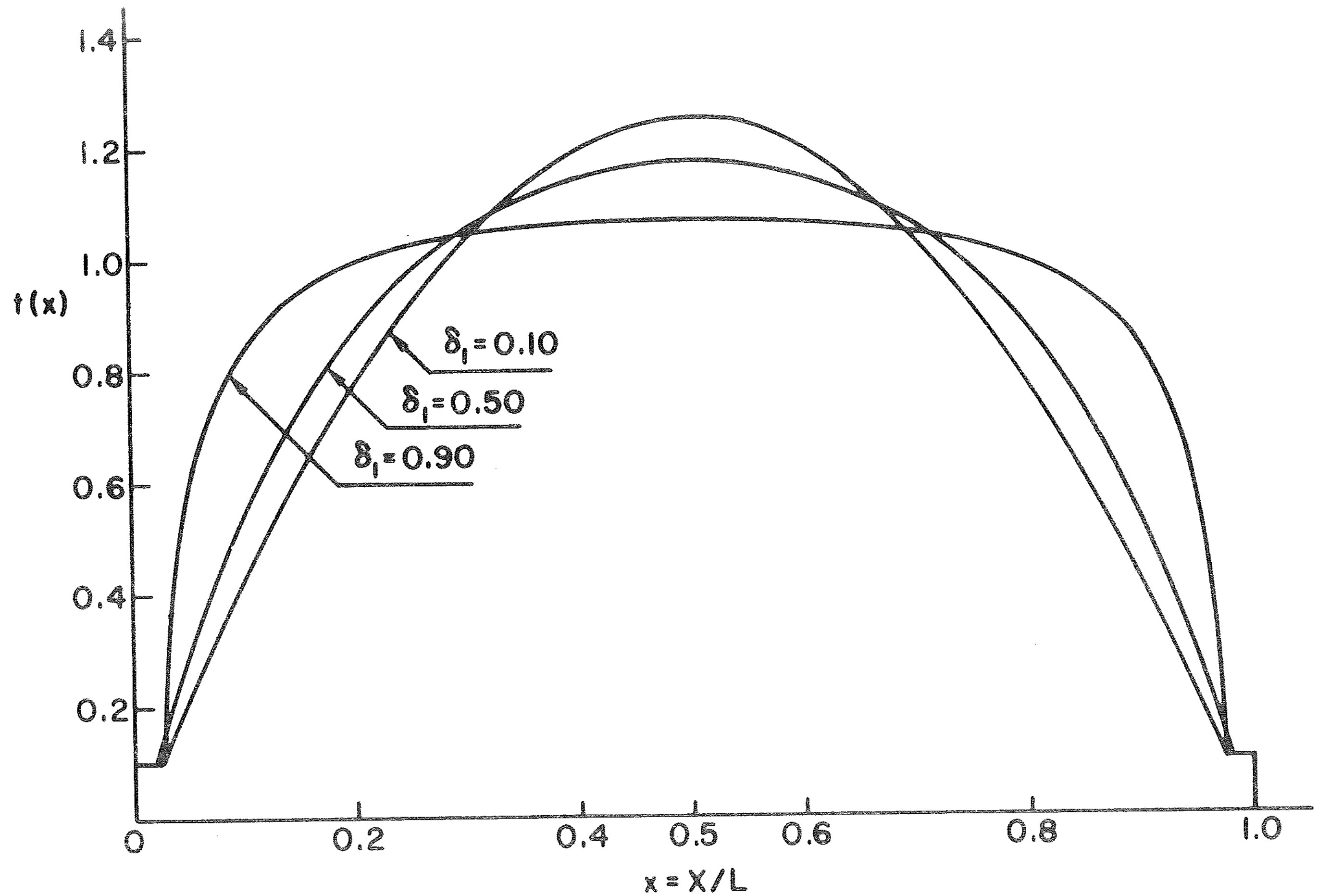

Fig. 4.6 Nondimensional Thickness Distributions — Minimum Weight Sandwich Beam on Simple Supports — Fundamental Frequency Held Constant — $t_{min} = .10$ — Various Values of δ_1.

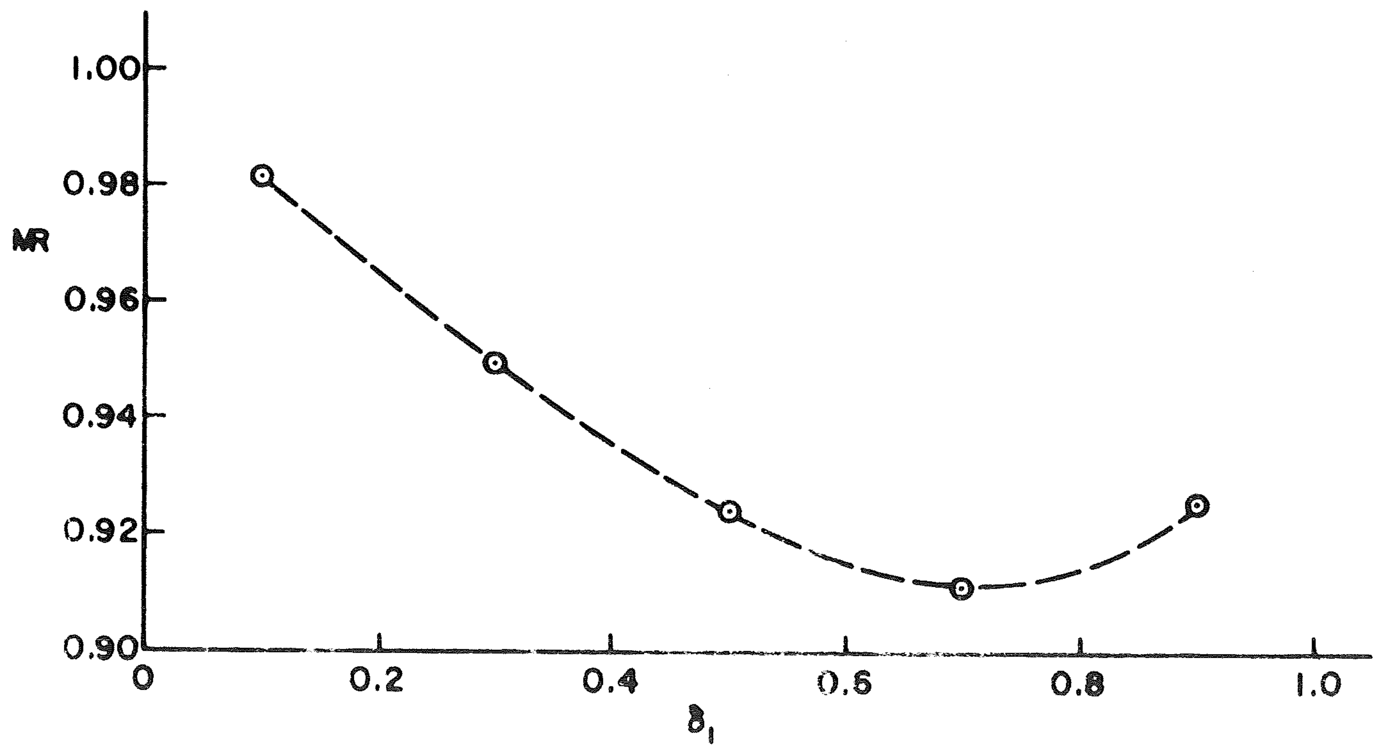

Fig. 4.7 Mass Ratio vs. δ_1 — Minimum Weight Sandwich Beam on Simple Supports — Fundamental Frequency Held Constant — $t_{min} = .10$.

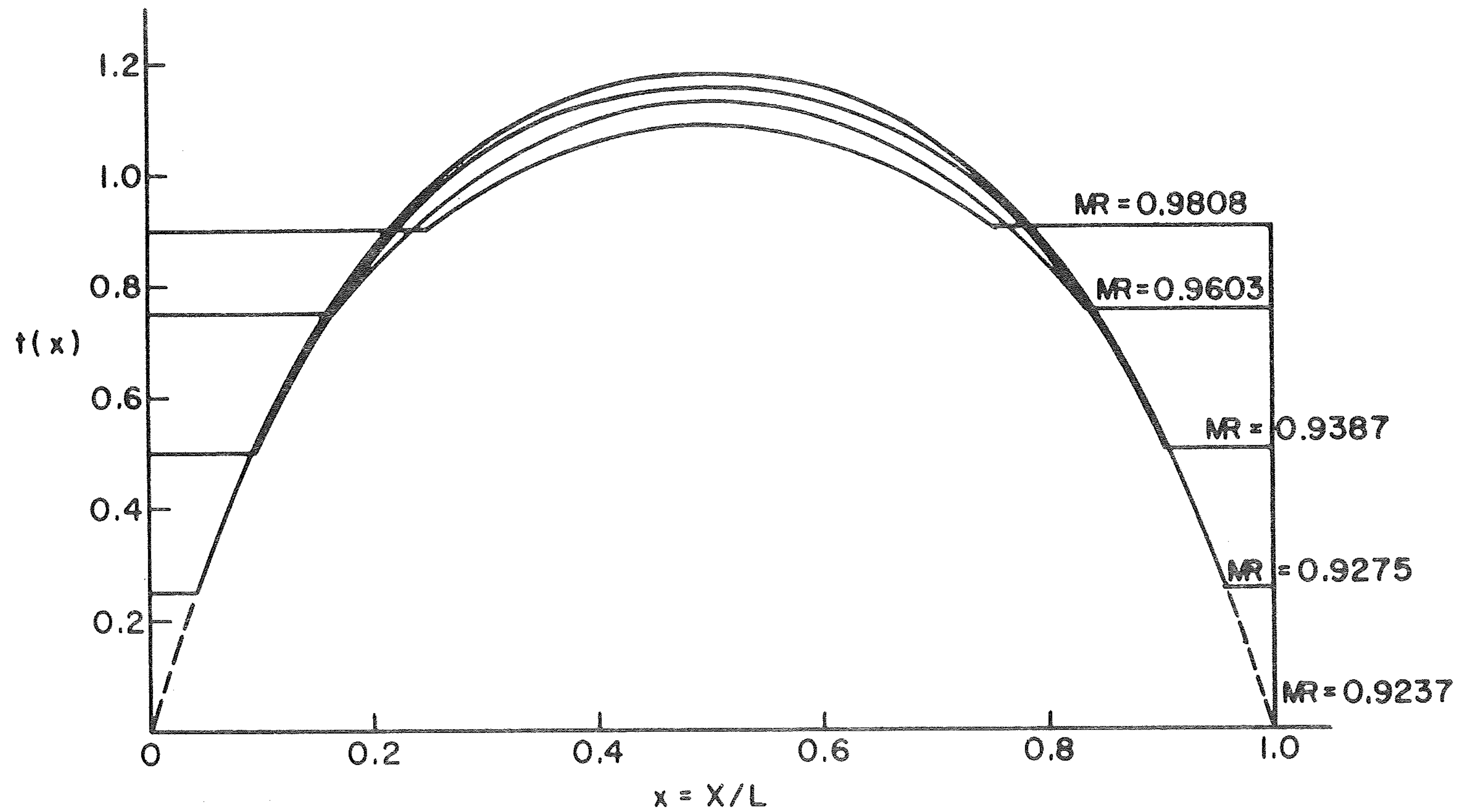

Fig. 4.8 Nondimensional Thickness Distribution — Minimum Weight Sandwich Beam on Simple Supports — $\delta_1 = 0.50$ — Several Minimum Thickness Constraints — Fundamental Frequency Held Constant.

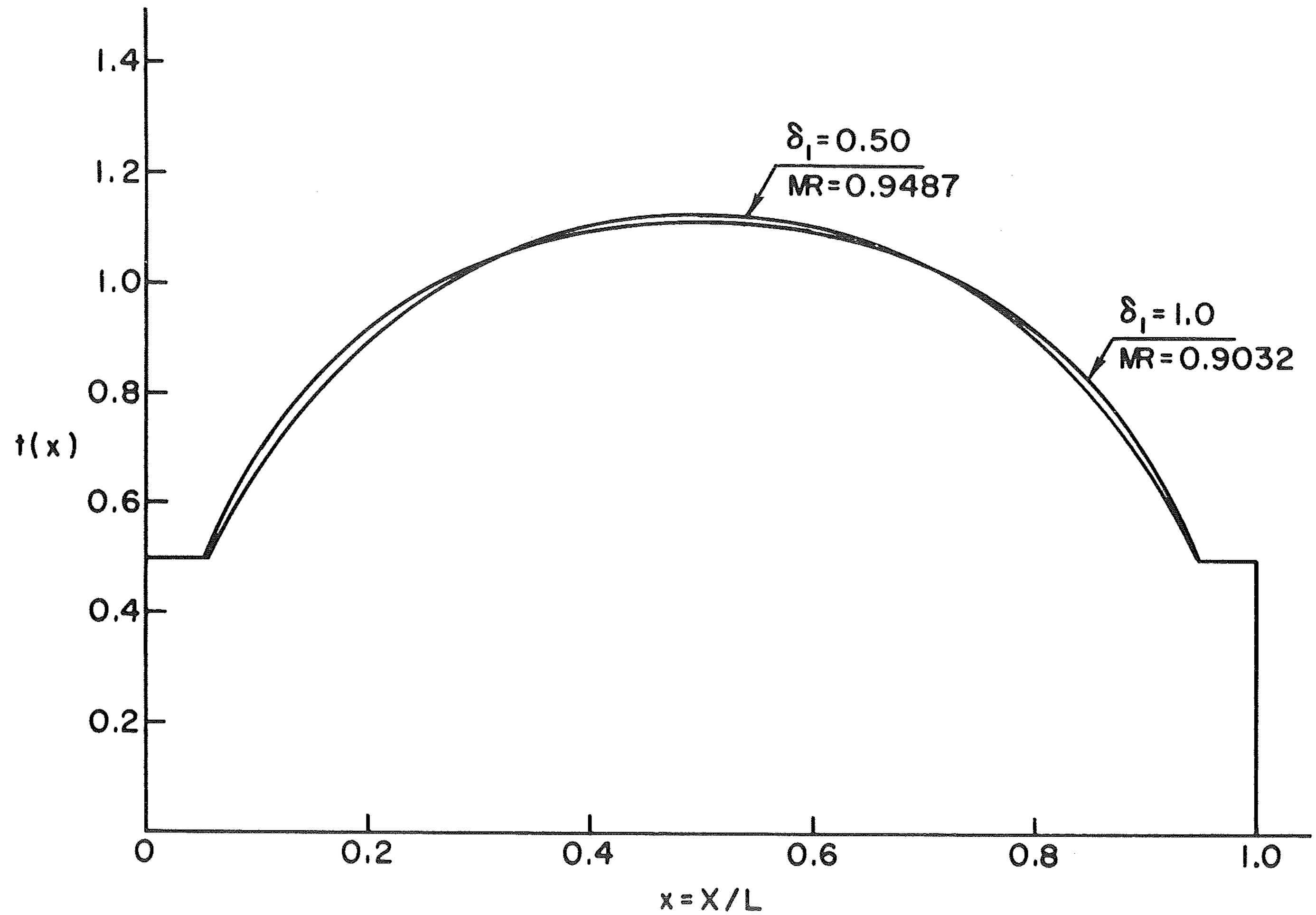

Fig. 4.9 Nondimensional Thickness Distributions —— Minimum Weight Solid Beam With Rectangular Cross-Section —— Simple Supports —— Fundamental Frequency Held Constant —— $t_{min} = 0.50$.

TABLE 4.1

Control Constants —— Clamped-Free Beam Comparison

$t_{min} = 0.0$

δ_1	r_o	q_o	r_o/q_o	MR
0.1	-1.90491	1.42031	-1.34119	0.95630
0.3	-1.55647	1.14746	-1.35645	0.83429
0.5	-1.16474	0.849191	-1.37159	0.66319
0.7	-0.72974	0.526270	-1.38663	0.43963
0.9	-0.254255	0.181853	-1.39813	0.16141

Estimated value of $r_o/q_o = -1.37734$

TABLE 4.2

Control Constants —— Pinned-Pinned Beam Comparison

$t_{min} = 0.10$

δ_1	r_o	p_o	r_o/p_o	MR
0.1	4.27976	-0.510427	-8.38467	0.98187
0.3	4.87837	-0.569647	-8.56385	0.94946
0.5	5.82908	-0.662153	-8.80322	0.92419
0.7	7.65440	-0.837900	-9.13522	0.91138
0.9	13.6490	-1.42012	-9.61116	0.92600

Estimated value of $\frac{r_o}{p_o}$ is: $-\pi^2 = -9.86960$

5. MULTIPLE FREQUENCY CONSTRAINTS AND CONSTRAINTS ON FREQUENCIES OTHER THAN THE FUNDAMENTAL

5.1 Introduction

This section will examine problems where constraints are placed on natural frequencies other than the fundamental frequency. In addition, the question of optimizing weight while holding two or more natural frequencies fixed simultaneously will be considered. The results of these studies will provide valuable insight into the question of the uniqueness of the solutions found in the preceding sections. In addition, it will be shown that, in certain cases, once one solution has been found holding a single frequency constant, solutions involving other frequency constraints may be constructed.

5.2 Multiple Frequency Constraints — Torsional Vibration

A simple example of multiple frequency constraints can be illustrated with a torsional vibration example. As shown in Chapter 3, the nondimensional eigenvalue equilibrium equation for free torsional vibration can be written as:

$$\theta' = s/t \tag{5.2.1a}$$

$$s' = -(\alpha_m t + \beta_m)\theta \tag{5.2.1b}$$

with

$$\theta(0) = s(1) = 0 \tag{5.2.1c}$$

For the mth eigenvalue, α_m and β_m are equal to

$$\alpha_m = \delta_1\left(\frac{2m-1}{2}\right)^2 \pi^2 \tag{5.2.2a}$$

$$\beta_m = (1-\delta_1)\left(\frac{2m-1}{2}\right)^2 \pi^2 \qquad m = 1,2,3,\ldots\infty \tag{5.2.2b}$$

Therefore, any thickness distribution which satisfies the eigenvalue equations (5.2.1a,b) with one or more values of α_m and β_m prescribed by equations (5.2.2a,b) is an admissible candidate in the search for a least weight configuration. Let us fix the lowest two eigenvalues (given by $m = 1$ and $m = 2$) and apply optimization theory to find a least weight thickness distribution with the first two natural frequencies of torsional vibration equal to those of a given reference structure. The nondimensional thickness distribution $t(x)$ is constrained to satisfy two eigenvalue equations given by

$$\theta_1' = s_1/t \tag{5.2.3a}$$

$$s_1' = -(\alpha_1 t + \beta_1)\theta_1 \tag{5.2.3b}$$

$$\theta_2' = s_2/t \tag{5.2.3c}$$

$$s_2' = -(\alpha_2 t + \beta_2)\theta_2 \tag{5.2.3d}$$

with

$$\theta_1(0) = \theta_2(0) = s_1(1) = s_2(1) = 0 \tag{5.2.3e}$$

and

$$\alpha_1 = (\frac{\pi}{2})^2 \delta_1; \quad \alpha_2 = (\frac{3\pi}{2})^2 \delta_1 \tag{5.2.3f}$$

$$\beta_1 = (\frac{\pi}{2})^2 (1 - \delta_1); \; \beta_2 = (\frac{\pi}{2})^2 (1 - \delta_1) \tag{5.2.3g}$$

Note that the differential equations above have the control variable $t(x)$ in common, as well as the structural mass ratio, δ_1. We wish to minimize the merit function, J, given by

$$J = \int_o^1 t dx \tag{5.2.4}$$

Therefore, the Hamiltonian for this problem may be expressed as:

$$\begin{aligned} H = t + \lambda_{\theta_1}(\frac{s_1}{t}) + \lambda_{s_1}[-(\alpha_1 t + \beta_1)\theta_1] \\ + \lambda_{\theta_2}(\frac{s_2}{t}) + \lambda_{s_2}[-(\alpha_2 t + \beta_2)\theta_2] \end{aligned} \tag{5.2.5}$$

The multiplier equations are found to be:

$$-\frac{\partial H}{\partial \theta_1} = \lambda_{\theta_1}' = \lambda_{s_1}(\alpha_1 t + \beta) \tag{5.2.6a}$$

$$-\frac{\partial H}{\partial s_1} = \lambda'_{s_1} = -\lambda_{\theta_1}/t \tag{5.2.6b}$$

$$-\frac{\partial H}{\partial \theta_2} = \lambda'_{\theta_2} = \lambda_{s_2}(\alpha_2 t + \beta_2) \tag{5.2.6c}$$

$$-\frac{\partial H}{\partial s_2} = \lambda'_{s_2} = -\lambda_{\theta_2}/t \tag{5.2.6d}$$

with

$\lambda_{s_1}(0) = \lambda_{s_2}(0) = \lambda_{\theta_1}(1) = \lambda_{\theta_2}(1) = 0.$ The control equation is found to be:

$$\frac{\partial H}{\partial t} = 0 = 1 - \frac{\lambda_{\theta_1} s_1}{t^2} - \alpha_1 \lambda_{s_1} \theta_1 - \frac{\lambda_{\theta_2} s_2}{t^2} - \alpha_2 \lambda_{s_2} \theta_2 \tag{5.2.6e}$$

or

$$t^2 = \left(\lambda_{\theta_1} s_1 + \lambda_{\theta_2} s_2\right) \Big/ \left(1 - \alpha_1 \lambda_{s_1} \theta_1 - \alpha_2 \lambda_{s_2} \theta_2\right) \tag{5.2.6f}$$

These equations are consistent with the previous torsional vibration problem which had only one frequency held fixed. In fact, this problem may be generalized to "z" frequency constraints with the result that we have "z" sets of state variable equations like (5.2.3a,b) with "z" adjoint or multiplier equations similar to equations (5.2.6a,b) and a control equation of the form

$$t^2 = \left[\sum_{i=1}^{z} (\lambda_{\theta_i} s_i)\right] \Big/ \left[1 - \sum_{i=1}^{z} (\alpha_i \lambda_{s_i} \theta_i)\right] \tag{5.2.7}$$

The boundary conditions for this general problem would be

$$\theta_i(0) = s_i(1) = \lambda_{s_i}(0) = \lambda_{\theta_i}(1) = 0 \qquad i = 1, 2, \ldots . z \tag{5.2.8}$$

If $z = 2$, it can be shown, as in the case of the single frequency constraint, that an algebraic relation exists between the state variables and the multipliers.

$$\begin{Bmatrix} \theta_1(x) \\ s_1(x) \\ \theta_2(x) \\ s_2(x) \end{Bmatrix} = A \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_{\theta_1}(x) \\ \lambda_{s_1}(x) \\ \lambda_{\theta_2}(x) \\ \lambda_{s_2}(x) \end{Bmatrix} \qquad (5.2.8)$$

$$A < 0$$

Thus, the solution for the optimum thickness distribution involves solving the state variable equations (5.2.3a,b,c,d) with the thickness given as a function of these state variables. If we choose $A = -1$, then

$$t^2 = \left((s_1)^2 + (s_2)^2\right) / \left(1 + \alpha_1(\theta_1)^2 + \alpha_2(\theta_2)^2\right) \qquad (5.2.9)$$

To obtain a numerical solution we must solve the state variable differential equations with assumed initial conditions for $s_1(0)$, $s_2(0)$ and with the specified initial conditions $\theta_1(0) = \theta_2(0) = 0$. The control constants $s_1(0)$ and $s_2(0)$ then must be perturbed, using a transition matrix, in a way such that a solution to the state variable differential equations will be obtained for which $t(x)$ is given by equation (5.2.9) and for which the boundary conditions at $x = 1$ are given by:

$$\theta_1(0) = \theta_2(0) = 0; \quad s_1(1) \cong s_2(1) \cong 0$$

The transition matrix relation which specifies the relation between initial and final perturbations is given by

$$\begin{Bmatrix} \delta s_1(1) \\ \delta s_2(1) \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial s_1(1)}{\partial s_1(0)} & \dfrac{\partial s_1(1)}{\partial s_2(0)} \\[2ex] \dfrac{\partial s_2(1)}{\partial s_1(0)} & \dfrac{\partial s_2(1)}{\partial s_2(0)} \end{bmatrix} \begin{Bmatrix} \delta s_1(0) \\ \delta s_2(0) \end{Bmatrix} \qquad (5.2.10a)$$

or

$$\begin{Bmatrix} \delta s_1(1) \\ \delta s_2(1) \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \delta s_1(0) \\ \delta s_2(0) \end{Bmatrix} \qquad (5.2.10b)$$

The results of a numerical computation for the above case with $\delta_1 = 0.5$ is compared with a problem for which only the fundamental is held fixed. These results are shown in Figure 5.1. The state variable behavior for the "two fixed frequencies" problem is shown in Figure 5.2. These mode shapes are seen to be similar to those which are found for the uniform reference case. A close look at Figures 5.1 and 5.2 together with equation (5.2.9) for $t(x)$ reveals that the thickness first decreases with x and then, because of $s_2(x)$, increases again before finally falling to t_{min}. Thus, a "hump" in the thickness distribution is formed because of the influence of the second frequency constraint.

Once the numerical method has been programmed for the computer it is easy to numerically vary each of the parameters in the problem. An extremely interesting result is found by varying α_1 and β_1. These parameters are functions of two other parameters, the nondimensional frequency and the structural mass ratio, δ_1. If α_2 and β_2 are held fixed and α_1 and β_1 are reduced by one fourth, then the problem posed is one in which weight is minimized while holding the fundamental frequency equal to one half the fundamental frequency $(\omega^2_{1_{new}} = (\frac{1}{2}\omega_{1_{old}})^2 = \frac{1}{4}(\omega_{1_{old}})^2)$ of the reference structure and holding the second frequency equal to the second frequency of the reference structure. As we decrease the values of α_1 and β_1, the influence on $t(x)$ of the first set of state variables becomes less and less. In fact, with $\alpha_1 = \beta_1 = 0$, the numerical solution gives $\theta_1(x)$ and $s_1(x)$ equal to zero. With $\alpha_1 = \beta_1 = \theta_1(x) = s_1(x) = 0$, we no longer have a multiple constraint but instead have only a constraint involving the second frequency. The mode shapes for $\theta_2(x)$ and $s_2(x)$ in this problem are similar to those shown in Figure 5.2. The thickness distribution for this case, shown in Figure 5.3, has a minimum thickness constraint for an obvious reason. If only the second frequency is held fixed, the nondimensional torque, s, is zero at two places in the region $0 \le x \le 1$. For this reason, in the absence of thickness inequality constraints the thickness will also be zero because of equation (5.2.9). Figure 5.3 shows two thickness distributions, one with only the fundamental frequency held fixed and the other with only the second frequency held fixed. Both have the same minimum thickness constraint. The mass ratios, MR, are shown for each problem and are seen to be identical to each other. This fact, when first encountered, was believed to be coincidence. However, after a close study, it was proven to be true for this and certain other classes of problems. A proof for this will be given in Section 5.4. Notice too that there is a similarity between the thickness distributions shown in Figure 5.3, and that the maximum value of $t(x)$ in each case is identical. This will also prove to be true for certain classes of problems. Also, the ratio between the control constants in each case is equal to unity.

$$\frac{s_1(0)}{s_2(0)} = 1 \tag{5.2.11}$$

It will be shown later that, if the optimal thickness distribution has been found for a problem in which only the fundamental frequency is held fixed, one may construct the solution to a problem in which any single eigenvalue is held fixed. The saving in weight for any single frequency constraint, having the same t_{min} and δ_1, is invariant. This is truly a surprising result and will be studied in Section 5.4.

In the absence of a thickness inequality constraint, holding the second frequency fixed while optimizing weight will cause the thickness to go to zero between $x = 0$ and $x = 1$ as well as at $x = 1$. At the point where $t = 0$, the rotation $\theta(x)$ is continuous but, its first derivative $\theta'(x)$ is discontinuous and takes a value, in the limit as $t(x) \to 0$, of plus or minus one. Thus the optimum configuration has a discontinuity in the mode shape where the reference structure has none. In addition, the first elastic frequency of the optimal structure is equal to the second elastic frequency of the reference structure.

A counter-example for a solution uniqueness proof can readily be seen from the above discussion. Once we have obtained a "two humped" solution of the type shown in Figure 5.3 we can reduce the parameter α_2 until it becomes numerically equal to α_1. The resulting thickness distribution will have a vibrational frequency numerically equal to the fundamental frequency of the reference structure. However, the mode shape $\theta_1(x)$ for the reference structure <u>never crosses the x axis</u> while the thickness distribution in the counter-example has an elastic mode shape $\theta_1(x)$ which <u>does cross</u> the x axis. The mass ratio in the counter-example is substantially less than the classic optimal distribution as shown in Figure 5.3.

If the problem statement is of the form: "Find a minimum weight thickness distribution which has its lowest frequency identical to the lowest frequency of a uniform reference structure," then the counter-example is not acceptable. The counter-example has a rigid body frequency and thus its second frequency is identical to the lowest frequency of the reference structure. Thus the crucial words are underlined above. If the words "a frequency" are substituted for those first underlined then the problem solution is nonunique.

To summarize, it seems apparent from numerical results that the optimization problem is unique only if one insists that the ordering of the constrained frequencies in the optimum configuration is identical to that of the reference structure. If we wish to hold the nth frequency fixed then the optimum structure will have its nth frequency identical to the nth frequency of the reference structure. The phenomenon in which the (n + 1)st frequency for the optimal structure is equal to the nth frequency in the reference structure has been termed by Ashley as "frequency slippage" and by the author as "solution slippage." When frequency slippage or solution slippage occurs, the solution obtained is said to be "superoptimal." Guarding against superoptimal solutions is extremely important in practical engineering work, particularly in such work as flutter analysis, since one obtains many flutter speeds or instability points. Although many instability points occur, only the lowest speed is of interest because it will be the first encountered by the aircraft. If one is minimizing weight while holding this lowest flutter speed or eigenvalue constant, he must guard against superoptimal solutions. These solutions not only have a flutter speed equal to the lowest flutter speed of the reference structure, but also have an even lower flutter speed which will be encountered in flight first. Thus, it is usually important that not only the eigenvalue itself be fixed during optimization but also the eigenvalue ordering must be preserved. Thus, the superoptimal flutter solutions are not only light weight but also less stable and do not satisfy the design requirement. An excellent example of this phenomenon, for wing torsional divergence, is given by Armand and Vitte (Ref. 8).

5.3 Multiple Frequency Constraints —— Flexural Vibration and Combined Bending-Torsional Vibration

The nondimensional equilibrium equations for free flexural (bending) vibration of a sandwich beam have been shown to be

$$w' = p \qquad (5.3.1a)$$

$$p' = q/t \qquad 0 \le x \le 1 \qquad (5.3.1b)$$

$$q' = r \qquad (5.3.1c)$$

$$r' = (\alpha_m t + \beta_m) w \qquad (5.3.1d)$$

The boundary conditions are determined by the type of end restraints, as are the values of α_m and β_m. If "z" of the frequencies are held fixed, while the weight is minimized, the Hamiltonian becomes

$$H = t + \sum_{i=1}^{z} \left[\lambda_{w_i} p_i + \lambda_{p_i} q_i/t + \lambda_{q_i} r_i + \lambda_{r_i} (\alpha_i t + \beta_i) w_i \right] \qquad (5.3.2)$$

The multiplier equations become

$$-\frac{\partial H}{\partial w_i} = \lambda'_{w_i} = -\lambda_{r_i} (\partial_i t + \beta_i) \qquad (5.3.3a)$$

$$-\frac{\partial H}{\partial p_i} = \lambda'_{p_i} = -\lambda_{w_i} \qquad (5.3.3b)$$

$$-\frac{\partial H}{\partial q_i} = \lambda'_{q_i} = -\lambda_{p_i}/t \qquad (5.3.3c)$$

$$-\frac{\partial H}{\partial r_i} = \lambda'_{r_i} = -\lambda_{q_i} \qquad (5.3.3d)$$

$i = 1, 2, 3, \ldots. z$

The multiplier variables will have "z" boundary conditions determined by the "z" state variable boundary conditions. The control equation can be written

$$\frac{\partial H}{\partial t} = 0 = 1 + \sum_{i=1}^{z} \left[(\alpha_i \lambda_{r_i} w_i) - \frac{\lambda_{p_i} q_i}{t^2} \right] \qquad (5.3.4a)$$

or

$$t^2(x) = \left[\sum_{i=1}^{z} (\lambda_{p_i} q_i)\right] \Big/ \left[1 + \sum_{i=1}^{z} (\alpha_i \lambda_{r_i} w_i)\right] \tag{5.3.4b}$$

An algebraic relation between the state variables and the adjoint or multiplier variables can be shown to be similar to that given in Section 4.3.

No multiple frequency examples for flexural vibration have been attempted. However, typical solutions for which the second elastic frequency was held fixed are shown in Figures 5.4 and 5.5 for the cantilever beam and the pinned-pinned sandwich beam. For comparison, the solutions for configurations having the same values of δ_1 and t_{min} are shown which have only their fundamental frequencies held fixed. The two solutions for the cantilever are seen to be dissimilar, whereas, the solutions for the pinned-pinned beam are markedly similar in form. They both have the same t_{max} and the same mass ratio and, in fact, they appear to be periodic solutions with one solution having twice the period in x as the other.

The pinned-pinned beam solutions will be shown to be related to each other just as the solutions for torsional vibration were, and the arguments against uniqueness, unless the problem is correctly stated, will also apply. The solution similarity also will be discussed in Section 5.4.

Let us return to the beam on simple supports, for which a single frequency is held fixed. We have previously shown that there is a relation between the state variables and the multipliers given by:

$$\begin{Bmatrix} w(x) \\ p(x) \\ q(x) \\ r(x) \end{Bmatrix} = A \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_w(x) \\ \lambda_p(x) \\ \lambda_q(x) \\ \lambda_r(x) \end{Bmatrix} \tag{5.3.5}$$

A similar relation using the independent variable $\zeta = (1 - x)$ may be found to be

$$\begin{Bmatrix} w(x) \\ p(x) \\ q(x) \\ r(x) \end{Bmatrix} = B \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_w(\zeta) \\ \lambda_p(\zeta) \\ \lambda_q(\zeta) \\ \lambda_r(\zeta) \end{Bmatrix} \tag{5.3.6}$$

if, and only if, one also assumes that

$$t(x) = t(1 - x) = t(\zeta) \tag{5.3.7}$$

The relation in equation (5.3.7) states that the thickness distribution is symmetric about the coordinate $x = 1/2$.

Unlike the modal constant A, which was required to be positive if a solution was to exist, the constant B may take on both positive and negative values. If $B > 0$, it can be shown that the state variables $w(x)$ and $q(x)$ are symmetric about $x = 1/2$ while $p(x)$ and $r(x)$ are antisymmetric about this point. If $B < 0$, it can also be shown that $w(x)$ and $p(x)$ must be antisymmetric while $p(x)$ and $r(x)$ are symmetric in order for a solution for $t(x)$ to exist.

As a final example of multiple frequency constraints, let us formulate the problem, without actually solving it, for a cantilever beam whose bending stiffness <u>and</u> torsional stiffness are linear functions of the nondimensional thickness parameter $t(x)$. Tne nondimensional equilibrium equations for uncoupled bending and torsion may be expressed as:

$$w' = p \tag{5.3.8a}$$

$$p' = q/t \tag{5.3.8b}$$

$$q' = r \qquad 0 \le x \le 1 \tag{5.3.8c}$$

$$r' = (\alpha_b t + \beta_b)w \tag{5.3.8d}$$

$$\theta' = s/t \tag{5.3.8e}$$

$$s' = -(\alpha_t t + \beta_t)\theta \tag{5.3.8f}$$

where

$$\alpha_b = \delta_1(.597\pi)^4 \qquad \alpha_t = \delta_1(\pi/2)^2$$

$$\beta_b = (1 - \delta_1)(.597\pi)^4 \qquad \beta_t = (1 - \delta_1)(\pi/2)^2$$

the boundary conditions for the problem are

$$w(0) = p(0) = \theta(0) = q(1) = r(1) = s(1) = 0 \tag{5.3.9}$$

The performance index, or merit function is,

$$J = \int_0^1 t dx \tag{5.3.10}$$

so that the Hamiltonian becomes:

$$H = t + \lambda_w p + \lambda_p q/t + \lambda_q r + \lambda_r(\alpha_t t + \beta_t)w + \lambda_\theta s/t + \lambda_s(-(\alpha_t t + \beta_t)\theta) \tag{5.3.11}$$

The multiplier equations become

$$-\frac{\partial H}{\partial w} = \lambda'_w = -\lambda_r(\alpha_b t + \beta_b)w \tag{5.3.12a}$$

$$-\frac{\partial H}{\partial p} = \lambda'_p = -\lambda_w \tag{5.3.12b}$$

$$-\frac{\partial H}{\partial q} = \lambda'_q = -\lambda_p/t \tag{5.3.12c}$$

$$-\frac{\partial H}{\partial r} = \lambda'_r = -\lambda_q \tag{5.3.12d}$$

$$-\frac{\partial H}{\partial s} = \lambda'_s = -\lambda_\theta/t \tag{5.3.12e}$$

$$-\frac{\partial H}{\partial \theta} = \lambda'_\theta = (\alpha_t t + \beta_t)\lambda_s \tag{5.3.12f}$$

while the boundary conditions are

$$\lambda_q(0) = \lambda_r(0) = \lambda_s(0) = \lambda_w(1) = \lambda_p(1) = \lambda_\theta(1) = 0 \tag{5.3.13}$$

The equations are coupled together by the control equation:

$$\frac{\partial H}{\partial t} = 0 = 1 - \frac{\lambda_p q}{t^2} + \alpha_b \lambda_r w - \alpha_t \lambda_s \theta - \frac{\lambda_\theta s}{t^2} \tag{5.3.14a}$$

or

$$t^2(x) = (\lambda_p q + \lambda_\theta s)/(1 + \alpha_b \lambda_r w - \alpha_t \lambda_s \theta) \tag{5.3.14b}$$

Therefore, the solution to this problem involves a 12th order nonlinear system of differential equations with six boundary conditions specified at $x = 0$ and six at $x = 1$. As in all the other cases studied involving conservative vibration problems, there is an algebraic relation between the state variables and the Lagrange multipliers.

$$\begin{Bmatrix} w(x) \\ p(x) \\ q(x) \\ r(x) \\ s(x) \\ \theta(x) \end{Bmatrix} = A \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} \lambda_w(x) \\ \lambda_p(x) \\ \lambda_q(x) \\ \lambda_r(x) \\ \lambda_s(x) \\ \lambda_\theta(x) \end{Bmatrix} \tag{5.3.15}$$

where $A > 0$. Thus, the control equation becomes, with $A = 1$,

$$t^2(x) = (q^2(x) + s^2(x))/(1 + \alpha_b w^2(x) + \alpha_t \theta^2(x)) \tag{5.3.16}$$

and one need only consider the set of coupled state variable equations and boundary conditions to find a solution to the problem.

The numerical solution method for the problem is quite similar to those shown previously. A linear relation between the changes in the control constants, in this case $q(0)$, $r(0)$, $s(0)$, and changes in the specified boundary conditions at $x = 1$, $q(1)$, $r(1)$, $s(1)$ is postulated to be:

$$\delta \begin{Bmatrix} q(1) \\ r(1) \\ s(1) \end{Bmatrix} = \begin{bmatrix} \frac{\partial q(1)}{\partial q(0)} & \frac{\partial q(1)}{\partial r(0)} & \frac{\partial q(1)}{\partial s(1)} \\ \frac{\partial r(1)}{\partial q(0)} & \frac{\partial r(1)}{\partial r(0)} & \frac{\partial r(1)}{\partial s(1)} \\ \frac{\partial s(1)}{\partial q(0)} & \frac{\partial s(1)}{\partial r(0)} & \frac{\partial s(1)}{\partial s(0)} \end{bmatrix} \begin{Bmatrix} \delta q_o \\ \delta r_o \\ \delta s_o \end{Bmatrix} \tag{5.3.17}$$

Although this problem has not been programmed for the computer, because of budget considerations, it is felt that the solution is not difficult. This problem and related problems of coupled bending and torsion are of interest in solving wing bending-torsion flutter optimization problems.

5.4 Similarity Transformations in the Construction of Higher Eigenvalue Constraint Solutions From a Known Optimal Solution

Several of the solutions found in Sections 5.2 and 5.3 appear to be similar. This similarity will be shown to occur only when the structure has reference eigenvalues which are integer multiples of one another. For a structural configuration of this type, the following statement will be shown to be true:

"For a given thickness constraint and value of δ_1, the total possible weight savings for a structure, when any single frequency is held fixed during the optimization, is a constant, independent of the fixed frequency. Furthermore, the optimal thickness distribution for any fixed frequency can be shown to be a periodic extension of the distribution found holding the fundamental frequency fixed. Finally, the control constants and state variables for any fixed frequency differ from those obtained while holding the fundamental frequency constant by a constant multiplicative factor which is a function of the ratio between the higher eigenvalue and the fundamental eigenvalue."

Now let us prove this lengthy statement and demonstrate the ideas to which it refers. Suppose that we know the solution to the optimization problem for the beam on simple supports with its fundamental frequency held fixed. Such a solution has been shown to satisfy the equations:

$$w_1' = p_1 \tag{5.4.1a}$$

$$p_1' = q_1/t_1 \tag{5.4.1b}$$

$$q_1' = r_1 \tag{5.4.1c}$$

$$r_1' = (\alpha_1 t_1 + \beta_1) w_1 \tag{5.4.1d}$$

$$t_1^2(x) = \frac{q_1^2(x)}{1 + \alpha_1 w_1^2(x)} \tag{5.4.1e}$$

$$w_1(0) = w_1(1) = q_1(0) = q_1(1) \tag{5.4.1f}$$

$$t_1(x) \geq t_{min} \tag{5.4.1g}$$

$$0 \leq x \leq 1 \qquad \alpha_1 = \delta_1 \pi^4 \qquad \beta_1 = (1 - \delta_1)\pi^4$$

We wish to find a solution to a problem which has its <u>nth frequency</u> of free vibration held fixed. The equations which specify this problem are:

$$w_n'(x) = p_n(x) \tag{5.4.2a}$$

$$p_n'(x) = \frac{q_n(x)}{t_n(x)} \tag{5.4.2b}$$

$$q_n'(x) = r_n'(x) \tag{5.4.2c}$$

$$r_n'(x) = (\alpha_n t_n(x) + \beta_n) w_n(x) \tag{5.4.2d}$$

$$t_n^2(x) = \frac{q_n^2(x)}{1 + \alpha_n w_n^2(x)} \tag{5.4.2e}$$

$$t_n(x) \le t_{min} \tag{5.4.2f}$$

$$w_n(0) = w_n(1) = q_n(0) = q_n(1) = 0 \tag{5.4.2g}$$

$$0 \le x \le 1 \qquad \alpha_n = \delta_1 (n\pi)^4 \qquad \beta_n = (1 - \delta_1)(n\pi)^4$$

Let us define a new dependent variable $\rho = \frac{x}{n}$ and let the range of x be extended

$$0 \le x \le n \tag{5.4.3a}$$

so that

$$0 \le \rho \le 1 \tag{5.4.3b}$$

and

$$\frac{d}{dx} = \frac{1}{n}\frac{d}{d\rho} = \frac{1}{n}(\dot{\ }) \tag{5.4.4}$$

Now, assume that the solution given by equations (5.4.1a, b, c, d, e, f, g), called the fundamental solution for $0 \le x \le 1$, can be extended periodically for $0 \le x \le n$ so that $w_1(x)$ and $q_1(x)$ will be equal to zero for $x = 0, 1, 2, 3, \ldots n$. Since $t_1(x)$ is a function of the <u>squares</u> of the state variables, it too will be periodic and always positive.

Now, <u>assume</u> that

$$w_n(\zeta) = Cw_1(x) \qquad 0 \le x \le n \tag{5.4.5}$$

where C is, as yet, an undetermined constant. From equations (5.4.1a) and (5.4.2a) we get the following

$$Cw_1'(x) = Cp_1(x) = \frac{1}{n}\dot{w}_n(\rho) = \frac{1}{n}p_n(\rho) \tag{5.4.6a}$$

Thus, we see that, because of the definition in equation (5.4.5)

$$Cp_1(x) = \frac{1}{n}p_n(\rho) \tag{5.4.6b}$$

$$0 \le x \le n; \qquad 0 \le \rho \le 1$$

Similarly, using the state variable equations for $p_1(x)$ and $p_n(x)$,

$$Cp_1'(x) = \frac{Cq_1(x)}{t_1(x)} = \frac{1}{n^2}\dot{p}_n(\rho) = (\frac{1}{n^2})\frac{q_n(\rho)}{t_n(\rho)} \tag{5.4.7}$$

Assume, for the moment, that

$$t_1(x) = t_n(\rho) \tag{5.4.8}$$

then, from equation (5.5.7) we find that

$$n^2Cq_1(x) = q_n(\rho) \tag{5.4.9}$$

Using the state variable equations for $q_1(x)$ and $q_n(\rho)$ we find

$$n^2Cq_1'(x) = n^2Cr_1(x) = \frac{1}{n}\dot{q}_n(\rho) = \frac{r_n}{n}(\rho) \tag{5.4.10}$$

Thus, from equation (5.4.10) we get

$$r_n(\rho) = n^3Cr_1(x) \tag{5.4.11}$$

Finally, using relation (5.4.11) we get

$$n^3Cr_1'(x) = n^3C(\alpha_1 t + \beta_1)w_1 \tag{5.4.12}$$

$$= \frac{1}{n}\dot{r}_n(\rho) = \frac{1}{n}(\alpha_n t_n + \beta_n)w_n(\rho)$$

Thus, we are left with the relation

$$n^4C(\alpha_1 t_1(x) + \beta_1)w_1(x) = (\alpha_n t_n(\rho) + \beta_n)w_n(\rho) \tag{5.4.13}$$

For a pinned-pinned beam, $\alpha_n = n^4\alpha_1$

(5.4.14)

while $\beta_n = n^4\beta_1$

Substituting these relations into equation (5.4.13) and remembering that $w_n(\rho) = Cw_1(x)$ we get

$$n^4 C(\alpha_1 t_1(x) + \beta_1) w_1(x) = n^4 C(\alpha_1 t_n(\rho) + \beta_1) w_1(x) \tag{5.4.15}$$

We have postulated that $t_1(x) = t_n(\rho)$ and, if this is true, equation (5.4.15) is an identity. Let us now calculate $t_n^2(\rho)$ in terms of the fundamental solution. From the above identities

$$t_n^2(\rho) = \frac{q_n^2(\rho)}{1 + \alpha_n w_n^2(\rho)} = \frac{n^4 C^2 q_1^2(x)}{1 + n^4 \alpha_1 C^2 w_1^2(x)} \tag{5.4.16}$$

Thus, if:

$$C^2 = \frac{1}{n^4}, \qquad C = \frac{1}{n^2} \tag{5.4.17}$$

we will have the relation

$$t_n^2(\rho) = t_1^2(x)$$

$$0 \le \rho \le 1 \qquad 0 \le x \le n$$

Now, exactly what has this long derivation shown? If we have the solution to the fundamental problem and if this solution is periodic from $x = 0$ to n, where n is an integer, then the solution

$$w_n(x) = (\frac{1}{n^2}) w_1(nx) \tag{5.4.18a}$$

$$p_n(x) = (\frac{1}{n}) p_1(nx) \tag{5.4.18b}$$

$$q_n(x) = q_1(nx)$$

$$r_n(x) = (n) r_1(nx) \tag{5.4.18d}$$

$$t_n^2(x) = \frac{q_1^2(nx)}{1 + \alpha_1 w_1^2(nx)} = t_1^2(nx) \tag{5.4.18e}$$

$$t_n(x) \geq t_{min}$$

is a solution to the weight optimization problem where the nth natural frequency is held fixed. Since the solution is periodic, the thickness will take on values of t_{min} at $x = 0$, $x = \frac{1}{n}, \ldots, 1$, and all the boundary conditions will be satisfied. Also, because the modal deflection shapes of the optimal distribution will have the same number of crossing points on the x-axis, we can assure ourselves that there will not be "frequency slippage." This then is an optimal solution and not a superoptimal solution.

The above results were, in fact, tested numerically and found to be correct. For instance, the control constants for the problem with only the third elastic frequency held fixed were found to be:

$$p_{30} = p_3(0) = \frac{p_{10}}{3} = \frac{p_1(0)}{3}$$

$$r_{30} = r_3(0) = 3r_{10} = 3r_1(0)$$

The key to the above demonstration was the fact that $w_1(x)$ and $q_1(x)$ were periodic such that they took on values of zero at $x = 0, 1, 2, 3, \ldots. n$. This will only happen if the ratio between the eigenvalues are integer multiples of one another. For this reason, the above statements do not hold for the cantilever beam problem.

The invariance of the mass ratio **MR** can be seen from the following:

$$J_n = \int_0^1 t_n(x)dx \tag{5.4.19a}$$

Changing variables gives

$$J_n = \frac{1}{n}\int_0^n t_1(nx)d(nx) \tag{5.4.19b}$$

Because of periodicity, of period unity, of $t_1(nx)$, we find

$$J_n = \frac{1}{n}\left(n\int_0^1 t_1(x)dx\right) \tag{5.4.19c}$$

Thus,

$$J_n = J_1 \tag{5.4.19d}$$

If we have the solution to the fundamental torsional vibration weight optimization problem, then, for the nth free vibration frequency held fixed, the state variables are given by

$$\theta_n(x) = \frac{1}{m}\theta_1(mx) \tag{5.4.20a}$$

$$s_n(x) = s_1(mx) \tag{5.4.20b}$$

$$t_n(x) = t_1(mx) \qquad 0 \le x \le 1 \tag{5.4.20c}$$

$$(m = 2n - 1) \tag{5.4.20d}$$

where the state variables θ_1 and s_1 are the fundamental solutions.

5.5 Summary

This section has discussed a group of solutions involving multiple eigenvalue constraints and constraints on single eigenvalues other than the lowest. These solutions and their behavior are of more than passing academic interest. They show, for instance, that, using numerical techniques which are available, the solution may not be unique. Thus, if an estimate of control constants is incorrectly chosen, one might obtain a superoptimal solution. In practical engineering work this could be dangerous where the eigenvalues involve instability parameters.

An observation which will be of great importance for the panel flutter optimization problem in Chapter 6 is that the shape of the optimal thickness distribution for a particular fixed frequency is similar to that obtained by substituting the reference structure mode shapes into the control equation. For instance, because the modal deflection and bending moment go to zero at $x = 1/2$ for the second elastic frequency for the reference structure, one expects similar behavior for the optimal structure. Thus, by examining the control equation, one can predict the qualitative behavior of the thickness distribution. In all cases studied in this thesis, the reference mode shapes always gave an accurate qualitative indication of the thickness distribution behavior. Thus, if one has a pinned-pinned beam to be optimized, an examination of the mode shapes for $w(x)$ and $q(x)$ for the first frequency should indicate that the thickness distribution he will finally obtain will have only one maximum and that it will occur at $x = 1/2$.

The next section will examine a problem which involves a structure with nonconservative aerodynamic loading. This solution evaded researchers for several years because little was known about the basic behavior of solutions such as those shown in this section. Finally, the experience with problems such as these provided the groundwork necessary to solve this difficult problem.

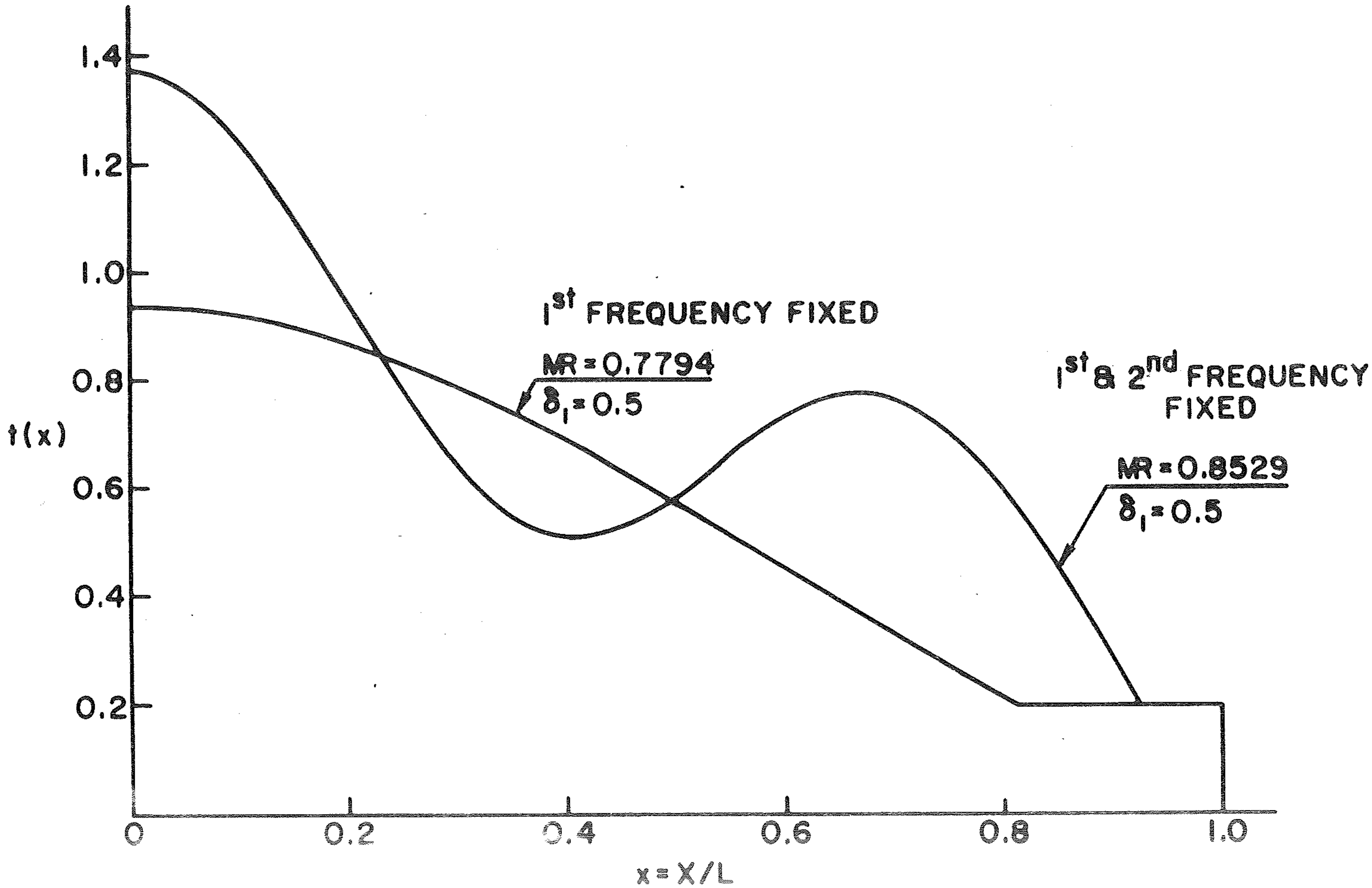

Fig. 5.1 Nondimensional Thickness Distributions — Minimum Weight Thin-walled Cylinder — $\delta_1 = 0.50$; $t_{min} = 0.20$ — First and Second Torsional Frequencies Held Constant.

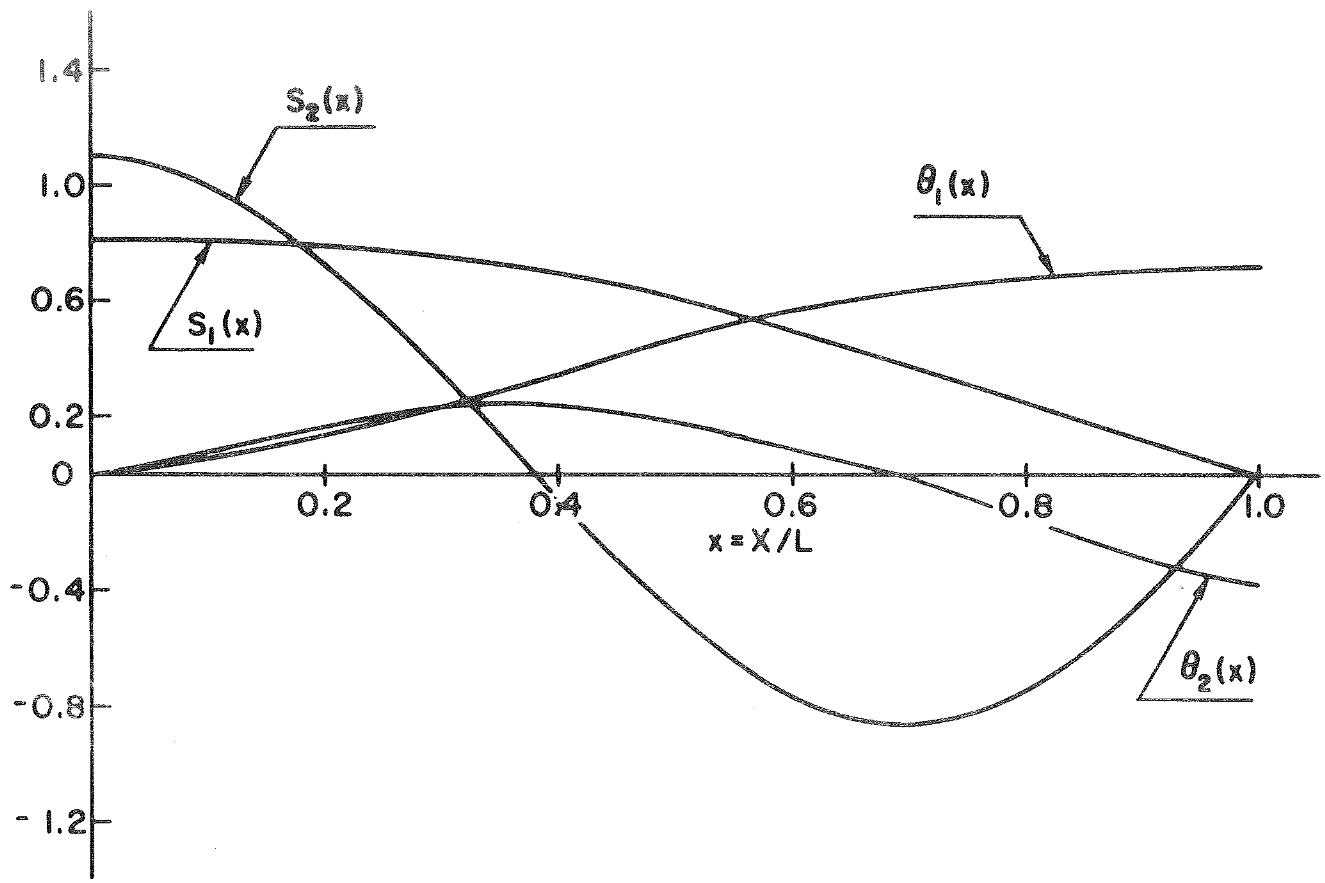

Fig. 5.2 State Variables —— Minimum Weight Thin-walled Cylinder —— $\delta_1 = 0.50$; $t_{min} = 0.20$ —— First and Second Torsional Frequencies Held Constant.

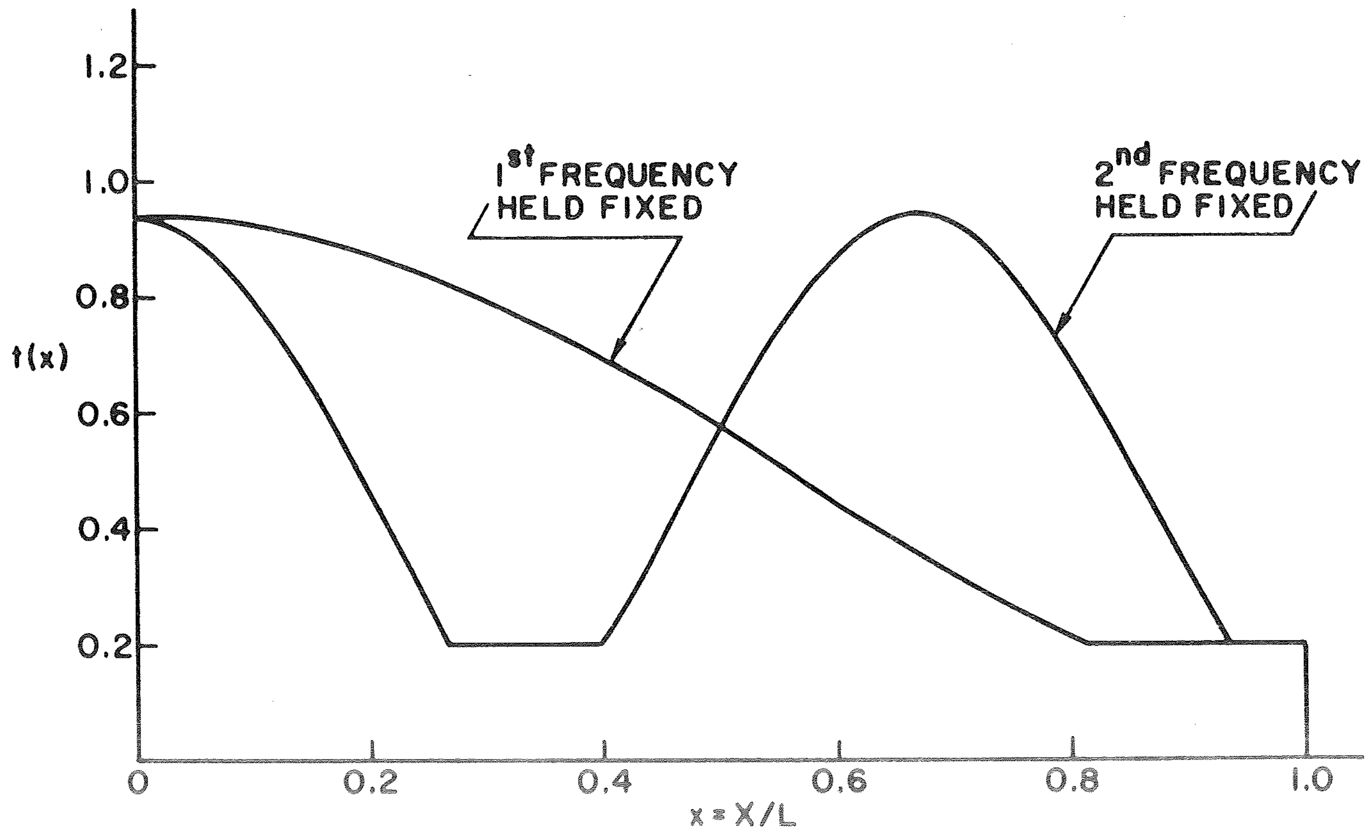

Fig. 5.3 Nondimensional Thickness Distributions — Minimum Weight Thin-walled Cylinder — First or Second Frequencies Held Constant — $\delta_1 = 0.50$; $t_{min} = 0.20$.

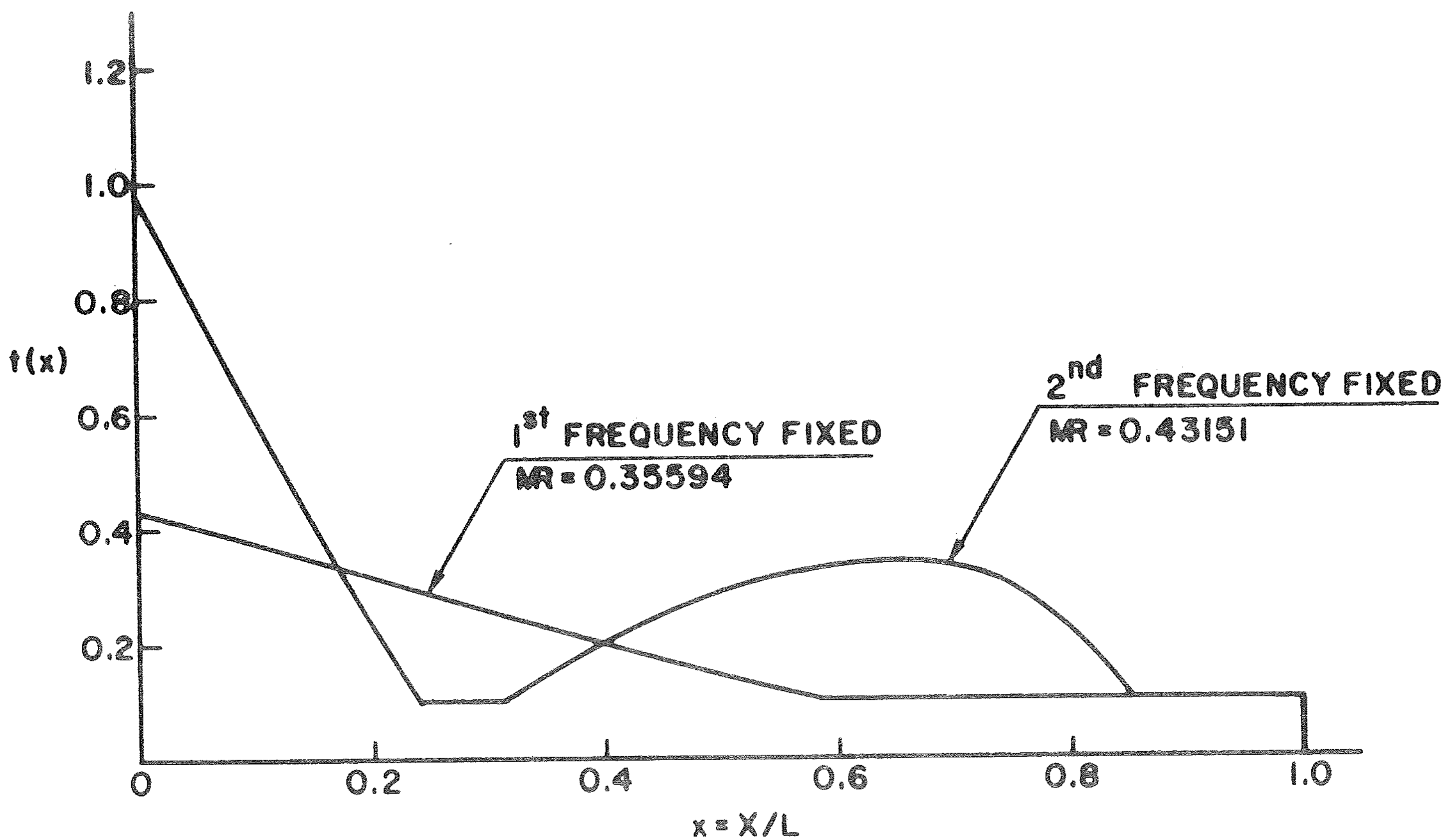

Fig. 5.4 Nondimensional Thickness Distributions — Minimum Weight Cantilever Beam — Sandwich Construction — $\delta_1 = 0.80$; $t_{min} = 0.10$ — First or Second Flexural Frequency Held Constant.

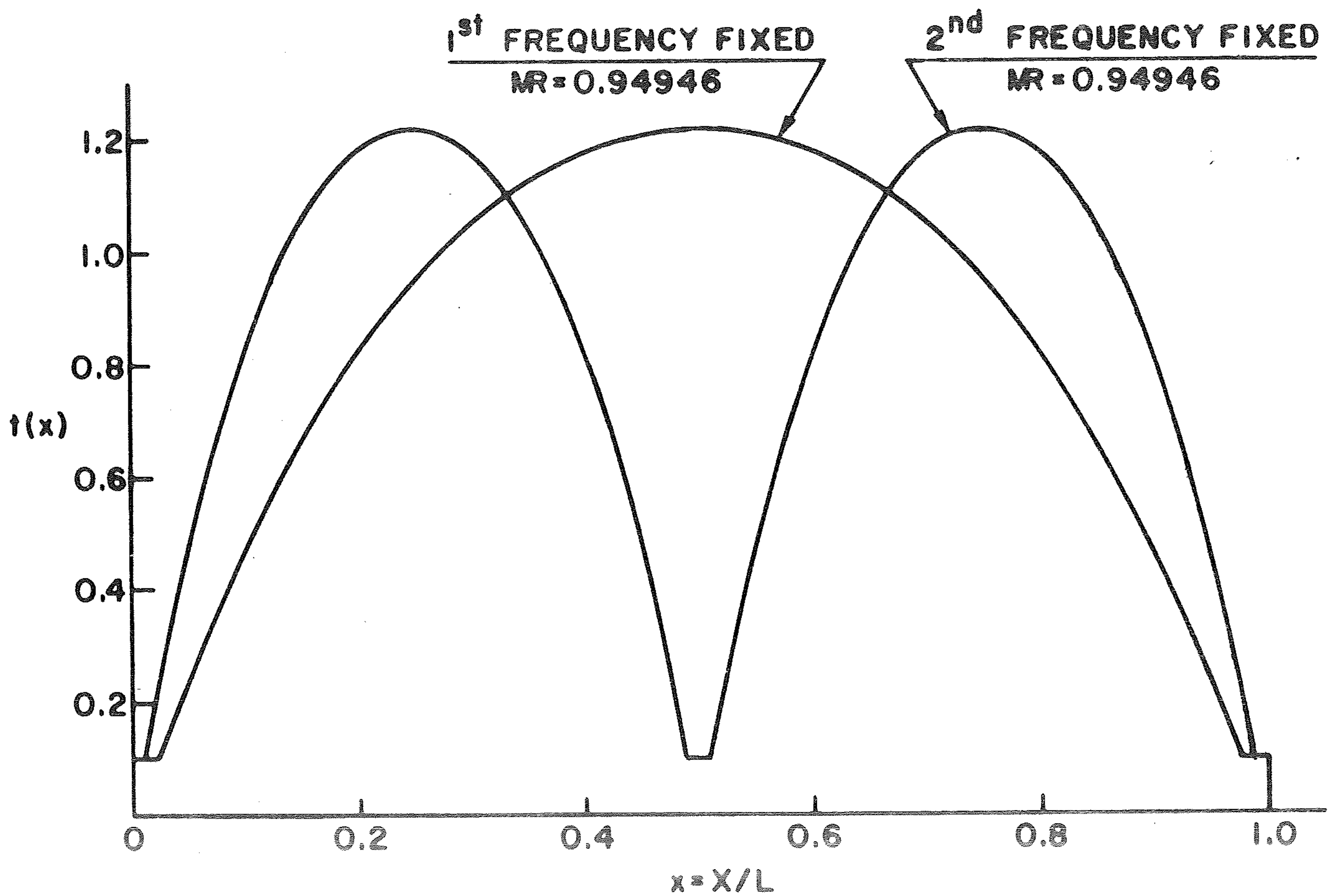

Fig. 5.5 Nondimensional Thickness Distributions — Minimum Weight Sandwich Beam on Simple Supports — First or Second Flexural Frequency Held Constant — $\delta_1 = 0.30$; $t_{min} = 0.10$.

6. OPTIMIZATION OF AN INITIALLY FLAT PANEL IN HIGH MACH NUMBER SUPERSONIC FLOW WITH ITS AERODYNAMIC FLUTTER PARAMETER HELD CONSTANT

6.1 Introduction

This chapter will discuss panel flutter optimization, the search for a least-weight thickness distribution for an initially flat plate-beam with the constraint that its critical flutter parameter in supersonic flow is held constant during the optimization. Only the simplest case of panel flutter will be discussed, that is, one for which the following assumptions are made.

(a) The panel is initially flat.

(b) There are no inplane stresses.

(c) The panel rests on simple supports.

(d) The aerodynamic forces act on only one side of the panel.

(e) The Mach number is sufficiently high so that quasi-steady linearized supersonic flow may be used $(M > 1.6)$.

(f) The panel is of infinite dimension in a direction perpendicular to the free stream direction (spanwise), i.e. the problem is one-dimensional.

(g) The panel is of sandwich construction with thin face sheets top and bottom and a nonstructural core in between.

Figure 6.1 shows the structural configuration and nomenclature for this problem. The general problem of panel flutter is discussed fully in Ref. 15.

The following section will discuss numerical techniques used to determine the instability parameter for this type of problem. In addition, other sections will illustrate methods for obtaining initial estimates of the thickness distribution before finally using a modified transition matrix procedure to solve the necessary optimization equations. The end result of this study will be a thickness distribution for a sandwich panel which has a least total weight.

6.2 The Determination of the Critical Aerodynamic Parameter For Panel Flutter

Before beginning the discussion of the panel flutter optimization problem, it would be well to review the method of solution for simple panel flutter problems.

For the panel configuration shown in Figure 6.1, the governing nondimensional differential equation of equilibrium with quasi-steady linearized supersonic aerodynamics and simple harmonic motion can be written as

$$\frac{d^2}{dx^2}\left(\frac{D(x)}{D_o}w''\right) + R_{xx}\frac{d^2w}{dx^2} + \lambda_o\frac{dw}{dx} + \lambda_o\left(\frac{M^2-2}{M^2-1}\right)\left(\frac{a}{U}\right)(-i\omega)w(x)$$
$$+\frac{m(x)}{D_o}(a^4\omega^2)w(x) = 0 \tag{6.2.1}$$

Equation (6.2.1) is written using the variables

$x = X/a$

$w(x) = \dfrac{W(x)}{a}$

$W(X,\tau) = W(X)e^{i\omega\tau}$

$R_{xx} = -\dfrac{N_{xx}a^2}{D_o}$

D_o = reference plate stiffness (constant thickness panel)

$\lambda_o = 2q_oa^3/D_o\sqrt{M^2-1}$ (aerodynamic parameter)

q_o = dynamic pressure

If the assumptions listed in Section 6.1 are to be applied, then we must have

$R_{xx} = 0$

$\dfrac{D(x)}{D_o} = \dfrac{T(x)}{T_o} = t(x)$

For a panel on simple supports, the boundary conditions will be

$w(0) = w(1) = 0$

$tw''(0) = tw''(1) = 0$

Equation (6.2.1) is a complex equation because of the aerodynamic damping term (bracketed in equation (6.2.1)). Studies have shown (Ref. 15) that this term may be neglected without great loss of accuracy in a great number of cases. Therefore, this study will neglect this damping term in order to simplify the calculations. With aerodynamic damping neglected, equation (6.2.1) can be written as

$$(tw'')'' + \lambda_o w' - (z_o \pi)^4 (\delta_1 t + \delta_2) w(x) = 0 \tag{6.2.2}$$

where

$$\frac{m(x)}{m_o} = \delta_1 t + \delta_2 \tag{6.2.3a}$$

$$(z_o \pi)^4 = \left(\frac{m_o a^4}{D_o}\right)\omega^2 \tag{6.2.3b}$$

Relation (6.2.3b) occurs because

$$\frac{D_o}{m_o a^4} = \frac{\omega_o^2}{\pi^4}$$

where ω_o is the fundamental frequency of free vibration of the panel. Thus, the variable z_o is seen to be

$$z_o = \sqrt{\frac{\omega}{\omega_o}}$$

If $\lambda_o = 0$ and $z_o = 1$, equation (6.2.2) reduces to the nondimensional equation for free flexural vibration in the fundamental mode that we encountered in Chapter 4. If $\lambda_o = 0$ and $z_o = 2$, the constraint equation corresponds to the equation for free vibration with the second frequency fixed.

With $\lambda_o \neq 0$, equation (6.2.2) belongs to a class of equations termed "non-self-adjoint." These equations arise in nonconservative elastic systems, that is, systems where the work done during any single cycle of oscillation is a function of the path taken during that cycle. Free vibration systems, in the absence of damping, are termed conservative because the energy of the system is conserved over any cycle of oscillation. A characteristic of nonconservative systems such as the panel, as discussed by Bolotin (Ref. 17), is that, as the parameter causing the system to be nonconservative is increased, the vibration frequencies — which are real — are changed in such a way that they approach each other in pairs. At a certain critical point, if there is no damping, one set of real frequencies will "merge" or become equal to each other. If the parameter is raised further, these merged frequencies no longer remain real, but become complex conjugates of one another. Because the motion is assumed to be of the form

$$w(x,\tau) = w(x)e^{i\omega\tau}$$

and if there is no system damping present, if $\omega = \alpha \pm i\beta$, then the behavior of $w(x,\tau)$ will be divergent and thus unstable. If the nonconservative parameter is increased further, additional pairs of frequencies will merge. In physical problems, the nonconservative parameter is often equal to an airspeed or a follower load, therefore, we are usually only interested in the lowest value of the parameter which causes the system to become unstable.

For the panel flutter problem, it is expected that, for a given value of λ_o,

$$0 \leq \lambda_o \leq \lambda_o^*$$

$$(\lambda_o^* = \text{critical value of } \lambda_o \text{ for instability})$$

the frequencies in equation (6.2.2) are real and distinct — in this case z_o^2 is real. For values of $\lambda_o > \lambda_o^*$, no solution to equation (6.2.2) with the prescribed boundary conditions is possible — unless the equation and the variables are assumed complex, i.e.

$$w(x) = w_r(x) + iw_i(x)$$

One method of solution, given a thickness distribution, is to fix λ_o and solve for z_o. Since z_o is a free vibration frequency parameter it is multivalued. If several values of λ_o are chosen, a graph can be drawn showing the behavior of λ_o vs. z_o, and eventually a point will be found where $z_o^{(1)} = z_o^{(2)}$, where $z_o^{(n)}$ refers to the nth value of the frequency parameter for a fixed λ_o. If, on the other hand, we choose z_o and choose it to be a real number, then we may solve for the corresponding λ_o and construct a graph of λ_o vs. z_o. This latter technique has the advantage that, for a fixed value of z_o, the parameter λ_o will be single valued. Also, since we are concerned only with the merging of real frequencies, the merging point can be seen to be the point at which λ_o reaches its maximum value for real values of z_o. It can further be shown that $\frac{d\lambda_o}{dz_o} = 0$ at this merging point. A graph such as that described above is shown in Figure 6.2.

There are many ways to solve the eigenvalue problem posed in equation (6.2.2), but one numerical method which will give an exact solution (exact in the numerical sense) was used by the author in this study. This method uses the unit solutions to equation (6.2.2) to generate a determinant which must be forced to zero if boundary conditions are to be met. The solution technique begins by defining auxiliary variables as:

$$p = w' \tag{6.2.4a}$$

$$q = tw'' \tag{6.2.4b}$$

$$r = (tw'')' \tag{6.2.4c}$$

Equation (6.2.2) then can be written as four simultaneous, first-order, linear differential equations.

$$w' = p \tag{6.2.5a}$$

$$p' = q/t \tag{6.2.5b}$$

$$q' = r \tag{6.2.5c}$$

$$r' = (\alpha t + \beta)w - \lambda_o p \tag{6.2.5d}$$

$$0 \leq x \leq 1$$

where

$$\alpha = (z_o \pi)^4 \delta_1$$

$$\beta = (z_o \pi)^4 (1 - \delta_1)$$

and

$$w(0) = w(1) = q(0) = q(1) = 0$$

Assume, for the moment, that $t(x)$ is known analytically; equations (6.2.5a, b, c, d) are then linear functions of the boundary conditions $p(0) = p_o$ and $r(0) = r_o$ since $w(0) = q(0) = 0$. The solutions for $w(x)$ and $q(x)$ may be symbolically written, by linear superposition, as

$$w(x) = p_o w_p(x) + r_o w_r(x) \tag{6.2.6a}$$

$$q(x) = p_o q_p(x) + r_o q_r(x) \tag{6.2.6b}$$

The functions $w_p(x)$, $w_r(x)$, $q_p(x)$, $q_r(x)$ are called unit solutions for the following reason. The function $w_p(x)$ is the solution for $w(x)$ in equation (6.2.5a) with initial conditions

$$p(0) = 1;\ w(0) = q(0) = r(0) = 0$$

while $q_p(x)$ is equal to $q(x)$ in equation (6.2.5c) for the same boundary conditions. The functions $w_r(x)$ and $q_r(x)$ are similarly formed using the boundary conditions

$$r(0) = 1;\ w(0) = p(0) = q(0) = 0$$

The solutions (6.2.6a, b) must satisfy the boundary conditions $w(1) = q(1) = 0$ or

$$\begin{Bmatrix} w(1) \\ q(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} w_p(1) & w_r(1) \\ q_p(1) & q_r(1) \end{bmatrix} \begin{Bmatrix} p_o \\ r_o \end{Bmatrix} \tag{6.2.7}$$

For a nontrivial solution, $p_o \neq 0$, $r_o \neq 0$, the determinant of the matrix in equation (6.2.7) must be zero. Thus

$$\begin{vmatrix} w_p(1) & w_r(1) \\ q_p(1) & q_r(1) \end{vmatrix} = f(t(x), \delta_1, \lambda_o, z_o) \equiv 0 \tag{6.2.8}$$

A graph of λ_o vs. z_o can easily be constructed in the following manner:

(a) For a given function $t(x)$ and fixed parameters z_o and δ_1, guess a value of λ_o.

(b) Obtain — numerically — the unit solution values necessary to construct the determinant in equation (6.2.8) — in general $f(t(x), \delta_1, \lambda_o, z_o) = f(\lambda_o) \neq 0$.

(c) Perturb λ_o in such a way as to reduce the value of $f(\lambda_o)$.

(d) Begin again at step (b) and iterate on the value λ_o until $f(\lambda_o)$ is close enough to zero that λ_o may be considered exact. This gives a (λ_o, z_o) point for the configuration.

Note that since the determinant is a function $f(t(x), \delta_1, \lambda_o, z_o)$, we could choose any one of the parameters to force the unit solution determinant to zero.

Figure 6.2 shows two typical plots of λ_o vs. z_o. The solid curve is that for a uniform thickness panel $(t(x) = 1)$ while the broken line is a plot showing the λ_o vs. z_o behavior of an optimum panel to be shown later. Note that the frequency merging occurs at a point where $\frac{d\lambda_o}{dz_o} = 0$. The classic paper by Hedgepeth (Ref. 16) shows, for a uniform thickness panel, that $\lambda_o^* = 343.2$. This present study has found $\lambda_o^* = 343.2$ and $z_o^* = 1.82$.

6.3 Constraining a Non-Uniform Panel to Have its Flutter Parameter Equal to that of a Uniform Thickness Reference Panel

The formulation of the constraints for the panel flutter optimization problem is different than that encountered in the fixed frequency problems discussed in previous sections. One of the necessary conditions for the flutter problem is that the thickness distribution satisfy equations (6.2.5a,b,c,d) with the associated boundary conditions and with $\lambda_o = 343.2$. The constraint equation set is a two parameter eigenvalue problem but z_o is not explicitly required to assume any fixed value. The satisfaction of equations (6.2.5a,b,c,d) is, however, only a <u>necessary</u> condition and <u>is not sufficient</u> to specify the flutter problem constraints.

If we set $\lambda_o = 343.2$ and let z_o take on a set of values near z_o^* then, using equations (6.2.5a,b,c,d) as the constraint conditions, we can solve a series of optimization problems each having $\lambda_o = 343.2$ but with different values of z_o. The distributions found in each of these optimization problems will all have one characteristic in common. If we construct the (λ_o, z_o) curves for each of them, they will have flutter parameters — places where $\frac{d\lambda_o}{dz_o} = 0$ — which are equal to or greater than $\lambda_o^* = 343.2$. This must occur because of the general shape of the (λ_o, z_o) curve. By picking $\lambda_o = 343.2$ we thus ensure the fact that a configuration has a point on its (λ_o, z_o) curve where λ_o is at least 343.2. Each of these optimal solutions will have a mass ratio associated with it. Since $\lambda_o = 343.2$ is a common characteristic of each solution, the mass ratio, for a given t_{min} and δ_1, is a function only of z_o.

If the problem is to have a solution, there will be a well defined minimum value of MR for a certain $z_o = z_{opt}$. This configuration for (λ_o, z_{opt}) will have a (λ_o, z_{opt}) curve tangent to the line $\lambda_o = 343.2$. Thus, the free parameter z_o

in the constraint equations is adjusted to find an optimum solution which has $\lambda_o = 343.2$ and $\frac{d\lambda_o}{dz_o} = 0$. This then is the true optimum solution. If should be noted that this procedure only applies for a system with no damping, that is, one which has merging frequencies.

To summarize, the optimum thickness distribution satisfies the constraint equations (6.2.5a, b, c, d) with $\lambda_o = 343.2$ <u>and</u> with z_o determined such that the resulting (λ_o, z_o) graph is <u>tangent</u> to the line $\lambda_o = 343.2$. This parameter z_o must be determined through a series of suboptimal problems which have $\lambda_o^* \geq 343.2$, until one value z_{opt} is found for which the mass ratio is a minimum. Any further increase or decrease in z_o will result in configurations with larger mass ratios and with $\lambda_o^* > 343.2$.

6.4 Governing Equations for Panel Flutter Optimization

Since our panel model has infinite span, the term "minimum weight" has no meaning. If, however, we choose to minimize the weight of a strip of panel of unit spanwise width, the merit function may be written as:

$$J = \int_o^1 t dx \tag{6.4.1}$$

With equations (6.2.5a, b, c, d) as the constraint equations, the Hamiltonian becomes

$$H = t + \lambda_w p + \lambda_p q/t + \lambda_q r + \lambda_r [(\alpha t + \beta)w - \lambda_o p] \tag{6.4.2}$$

The multiplier equations are:

$$-\frac{\partial H}{\partial w} = \lambda_w' = -\lambda_r(\alpha t + \beta) \tag{6.4.3a}$$

$$-\frac{\partial H}{\partial p} = \lambda_p' = -\lambda_w + \left\{\lambda_o \lambda_r\right\} \tag{6.4.3b}$$

$$-\frac{\partial H}{\partial q} = \lambda_q' = -\lambda_p/t \tag{6.4.3c}$$

$$-\frac{\partial H}{\partial r} = \lambda_r' = -\lambda_q \tag{6.4.3d}$$

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The control equation is

$$\frac{\partial H}{\partial t} = 0 = 1 - \lambda_p q/t^2 + \alpha\lambda_r w \tag{6.4.3e}$$

or

$$t^2 = \lambda_p q/(1 + \alpha\lambda_r w) \tag{6.4.3f}$$

with the boundary conditions

$$\begin{aligned} &\lambda_p(0) = \lambda_p(1) = \lambda_r(0) = \lambda_r(1) = 0 \\ &w(0) = w(1) = q(0) = q(1) = 0 \end{aligned} \tag{6.4.3g}$$

Except for the term in brackets in equation (6.4.3b), these adjoint equations are identical to those seen previously in fixed frequency optimization problems. These equations were first presented by Ashley and McIntosh (Ref. 7) and are, by now, well-known. While the equations are well-known, their solution is not well-known and has been the subject of a great deal of discussion since 1968. Following a suggestion by Turner, Armand (Ref. 8) has shown that, with the above necessary conditions for an extremum, there will be at least one solution to the problem in which $t(x)$ is symmetric, that is

$$t(x) = t(1 - x) \tag{6.4.4}$$

It follows that, if the solution is unique, then this type of solution is <u>the</u> solution. At this time, no rigorous mathematical proof exists which shows that there is or is not a unique solution. Also, since optimal control theory only guarantees an <u>extremum</u>, this solution may lead to a <u>maximum</u> or a <u>minimum</u>. If $t(x) = t(1 - x)$, a relation between the state variables and the multipliers exists and is found to be

$$\begin{Bmatrix} w(x) \\ p(x) \\ q(x) \\ r(x) \end{Bmatrix} = B \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -\lambda_o \end{bmatrix} \begin{Bmatrix} \lambda_w(1 - x) \\ \lambda_p(1 - x) \\ \lambda_q(1 - x) \\ \lambda_r(1 - x) \end{Bmatrix} \tag{6.4.5}$$

where B is a modal constant, which apparently may be positive or negative. It may be noted that, if $\lambda_o = 0$, our constraint equations reduce to the beam free vibration equations. Equation (6.4.5) is identical to equation (5.4.6) if $\lambda_o = 0$. It will be remembered that, for $\lambda_o = 0$, and $B > 0$, the state variables $w(x)$

and $q(x)$ never cross the x-axis and the solution corresponds to the fixed fundamental frequency problem. If $B < 0$ and $\lambda_o = 0$, both $w(x)$ and $q(x)$ change signs when they cross the axis and the solution corresponds to the fixed second frequency problem. This observation will be of great importance when solving the problem later on.

Using equation (6.4.5) we can write the control equation as

$$t^2(x) = t^2(1 - x) = Bq(x)q(1 - x)/(1 + \alpha Bw(x)w(1 - x)) \qquad (6.4.6)$$

Note that the aerodynamic parameter does not enter explicitly into this expression for $t(x)$. With $\lambda_o \neq 0$, no reduction in the number of dependent variables can be made since we can make no further assumption about the behavior of $w(x)$ and $w(1 - x)$ (for $\lambda_o = 0$, $w(x) = w(1 - x)$ or $w(x) = -w(1 - x)$). However, since x is the independent variable in the problem, we see that, as x goes from 1 to 1/2, $\rho = (1 - x)$ goes from 0 to 1/2 therefore we may reduce the integration interval on any numerical scheme by defining a new variable $0 \le \rho \le 1/2$ where $\rho = 1 - x$.

The state variable equations are

$$w'(x) = p(x) \qquad (6.4.7a)$$

$$p'(x) = q(x)/t \qquad (6.4.7b)$$

$$q'(x) = r(x) \qquad (6.4.7c)$$

$$r'(x) = (\alpha t + \beta)w(x) - \lambda_o p(x) \qquad (6.4.7d)$$

$$0 \le x \le 1$$

with

$$w(0) = w(1) = q(0) = q(1)$$

Let

$$\bar{w} = w(1 - x)$$

$$\bar{p} = p(1 - x)$$

$$\bar{q} = q(1 - x)$$

$$\bar{r} = r(1 - x)$$

and

$$(\dot{\ }) = \frac{d(\ )}{d\rho} = \frac{d(\ )}{d(1 - x)}$$

$$(\ )' = \frac{d(\ )}{dx} = \frac{d(\ )}{d\rho}\frac{d\rho}{dx} = -\frac{d(\ )}{d\rho} \tag{6.4.8a}$$

Therefore, from equations (6.4.7a, b, c, d) we get

$$\mathring{\overline{w}} = -\overline{p}$$

$$\mathring{\overline{p}} = -\overline{q}/t \tag{6.4.8b}$$

$$\mathring{\overline{q}} = -\overline{r} \tag{6.4.8c}$$

$$\mathring{\overline{r}} = -(\alpha t + \beta)\overline{w} + \lambda_o \overline{p} \tag{6.4.8d}$$

$$0 \le \rho \le 1$$

$$\overline{w}(0) = \overline{w}(1) = \overline{q}(0) = \overline{q}(1) = 0$$

with

$$t = \overline{t} = \frac{Bq\overline{q}}{1 + \alpha w\overline{w}} \tag{6.4.8e}$$

Note that $\overline{w}(1) = w(0)$, etc. and that, at $x = 1/2$, $\overline{w}(1/2) = w(1/2)$; $\overline{p}(1/2) = p(1/2)$;

$$\overline{q}(1/2) = q(1/2);\ \overline{r}(1/2) = r(1/2). \tag{6.4.9}$$

As the problem now is written, we have eight dependent variables with four independent boundary conditions.

$$w(0) = \overline{w}(0) = \overline{q}(0) = q(0) = 0$$

or

$$w(1) = \overline{w}(1) = \overline{q}(1) = q(1) = 0$$

and four undetermined control constants

$$p(0);\ \overline{p}(0);\ r(0);\ \overline{r}(0)$$

or

$$p(1);\ \overline{p}(1);\ r(1);\ \overline{r}(1)$$

From continuity of the state variables, we must have the relations in equation (6.4.9) hold at $x = 1/2$. The transition matrix, to be discussed later, will then involve integrating equations (6.4.7a, b, c, d) and (6.4.8a, b, c, d) with equation (6.4.8e) and the boundary conditions over $0 \le x \le 1/2$ and $0 \le \rho \le 1/2$ and forcing continuity at $x = 1/2$. This transition matrix will be a 4 x 4 matrix. Although the observation that $t = \overline{t}$ has not reduced the number of variables, it will enable us to reduce the range of integration.

6.5 An Estimation Technique —— Sine Series Approximation for $t(x)$

Before beginning the discussion of the transition matrix solution, it is necessary to discuss techniques for the estimation of our unknown parameters. In previous sections this estimation discussion centered on the estimation of the control constants. In this section, for reasons which will later become apparent, the discussion will dwell on the estimation or approximation of the optimal thickness distribution, $t(x)$.

The panel flutter problem was at first thought to be a mere extension of the previously solved beam problems. However, the eigenvalues in the problem, $\lambda_o^* = 343.2$, $(z_o^*\pi)^4 \cong 1100$ are far larger than anything that had been previously encountered. Because of the size of λ_o^* and z_o^*, the constraint equations are extremely sensitive to initial conditions. In addition, the paper by Turner (Ref. 12) presented a finite element solution which led the author to believe that the eventual solution for $t(x)$ would be similar in form to that found holding the fundamental frequency constant. In fact, several flutter analyses were done on these configurations and it was found that beams which were optimized while holding the fundamental frequency constant had flutter parameters of from 325 to 335, depending on the values of δ_1 and t_{min}. In general, for values of δ_1 close to 1, the flutter parameter λ_o^* was only three percent lower than that for the uniform thickness reference case. Initial assaults on the problem using the state variable equations <u>and</u> the multiplier equations and with no assumptions as to the form of $t(x)$ ended in failure for what was then a curious reason. No set of estimated control constants could be found which would keep the thickness distribution from equaling zero in the range $0 \leq x \leq 1$. For any set of estimated control constants, the resulting thickness distribution given then by equation (6.4.3f) would first rise to a maximum value near $x = .25$, then fall sharply in the vicinity of $x = .45$. Even with a minimum thickness constraint, the integration method would diverge because numbers would soon go out of range. This strange behavior led to the adoption of the assumption that $t(x) = t(1 - x)$ and a reformulation of the problem as it is presented in Section 6.4.

After a great many trials and failures, the thought came to mind that perhaps Turner's results, although obviously correct for the level of complexity that he chose, were inconclusive. Perhaps the optimum panel did have some strange, unanticipated shape. If so, the task was to find this shape. Let us examine the expression for the thickness as given in equation (6.4.8e)

$$t = \bar{t} = \frac{Bq\bar{q}}{1 + \alpha Bw\bar{\bar{w}}} \tag{6.4.8e}$$

In the absence of minimum thickness constraints, the optimum thickness distribution must be zero at $x = 0$ and at $x = 1$. This occurs because $q(0) = q(1) = 0$. In the language of variational calculus, an <u>admissible</u> trial function for $t(x)$ must be such that $t(0) = t(1) = 0$. In addition, we have chosen $t(x) = t(1 - x)$. An obvious choice for a valid approximation for $t(x)$ is seen to be

$$t(x) = t(1 - x) = \sum_{\substack{m=1 \\ (m\ \text{odd})}}^{N} t_m \sin(m\pi x) \tag{6.5.1}$$

From our definition of the merit function, this expression for $t(x)$ yields

$$J = \int_0^1 t dx = \sum_{\substack{m=1 \\ (m\ \text{odd})}}^{N} \frac{2}{m\pi} t_m \tag{6.5.2}$$

In addition, $t(x)$ is also required to satisfy the constraint equations

$$w' = p \tag{6.5.3a}$$

$$p' = q/t \tag{6.5.3b}$$

$$q' = r \tag{6.5.3c}$$

$$r' = (\alpha t + \beta)w - \lambda_o p \tag{6.5.3d}$$

with

$$w(0) = w(1) = q(0) = q(1) \tag{6.5.3e}$$

Section 6.2 described a numerical method for determining parameters such that the above eigenvalue equations are satisfied. This method involves forcing to zero a determinant, given symbolically by

$$f(t_1, t_3, \text{----} t_N, \delta_1, \lambda_o, z_o) \qquad (6.5.4)$$

Since the reference panel flutters at $\lambda_o = 343.2$ and $z_o = 1.82$ let us set the values of (λ_o, z_o) in the equation (6.5.3d) to these values. For a given value of δ_1, the eigenvalue determinant is now a function only of $(t_1, t_3, \ldots . t_N)$. If we set $N - 1$ of these thickness parameter values equal to specific numbers, then the remaining value t_i can be determined by choosing it in such a way that

$$f(t_i) \equiv 0$$

With this logic, a preliminary study was done using as an approximation

$$t(x) = .01 + t_1 \sin \pi x + t_3 \sin 3\pi x \qquad (6.5.5)$$

The constant .01 in equation (6.5.5) is added to ensure that equation (6.5.3b) is not numerically indeterminate. Thus equation (6.5.4) defines a function which is "almost admissible." For any combination of values of t_1 and t_3, the mass ratio for a strip of panel of unit spanwise width is given by:

$$MR = \delta_1(.01 + \frac{2t_1}{\pi} + \frac{2t_3}{3\pi}) + (1 - \delta_1) \qquad (6.5.6)$$

For the analysis, the values

$$\lambda_o = 343.2$$

$$z_o = 1.82$$

$$\delta_1 = .7$$

were used. A value t_1 was selected; then the value of t_3 was determined by numerically forcing the determinant

$$f(t_3) \equiv 0$$

After t_3 had been determined, the mass ratio was calculated with equation (6.5.6). This mass ratio is plotted as a function of t_1 in Figure 6.3 and as a function of t_3 in Figure 6.4. Four typical thickness distributions found using this method are shown in Figure 6.5. Two significant conclusions can be drawn from these figures. The first conclusion can be drawn from the behavior of in Figures 6.3 and 6.4 where it is seen that mass ratio (and thus J) has a very smooth behavior and a well defined minimum. This well-behaved nature of led to the belief that the exact solution would be similarly well behaved and that an absolute minimum would be found without encountering local minima in the process. A second and less favorable conclusion can be drawn from Figure 6.5. Although the mass ratio changes only slightly in the four cases shown, the shapes of the distributions change radically. From this, it was concluded that the problem might be slow to converge when an exact solution was attempted. The approximation to the minimum weight distribution with two terms of a sine series shows very pronounced peaks in the vicinity of $x = .2$ and $x = .8$. This behavior was compared to the results of initial attempts at the transition matrix procedure which had failed near $x = .5$. Perhaps the exact optimal thickness distribution has two maxima, instead of one maximum at $x = 1/2$?

Figure 6.6 shows a plot of the mode shapes (the amplitude is unspecified here) for $q(x)$ and $w(x)$ and also the products $q(x)q(1 - x)$ and $w(x)w(1 - x)$ obtained from the analysis of the minimum weight "sine panel" found in the above study. Although the signs of $w(x)$ and $q(x)$ may be plus or minus times those shown in Figure 6.6, the signs of the products $w(x)w(1 - x)$ and $q(x)q(1 - x)$ are invariant because they are products. Since $q(x)q(1 - x)$ is the numerator in the expression for $t(x)$ and because this product is negative while $t(x)$ must be positive, one may tentatively conclude that, because of the behavior of the mode shapes in the thickness approximation, the arbitrary constant B in equation (6.4.8e) should be negative for a meaningful solution to exist.

One may see that, with $B < 0$, the state variable $q(x)$ will have to change signs somewhere over the range $0 \le x \le 1$. If $B < 0$ and if the exact optimum value of $q(x)$ is to be similar to the approximation shown in Figure 6.6, this sign change can only occur at $x = 1/2$ because of the symmetry of $t(x)$. If the product $q(x)q(1 - x)$ is positive with $B < 0$ then there must be a constraint $t(x) \ge t_{min}$, otherwise a negative thickness would result. Therefore, all the evidence accumulated from the "sine panel" study pointed to a two-peaked panel as the final solution. With this evidence in mind, the study again returned to the transition matrix procedure which had earlier been abandoned.

6.6 Transition Matrix Solutions for the Optimum Panel With its Flutter Speed Fixed

Section 6.4 showed that a necessary condition for a thickness distribution for minimum weight is that the following differential equations are satisfied:

$$w'(x) = p(x) \tag{6.6.1a}$$

$$p'(x) = q(x)/t \tag{6.6.1b}$$

$$q'(x) = r(x) \tag{6.6.1c}$$

$$r'(x) = (\alpha t + \beta)w(x) - \lambda_o p(x) \tag{6.6.1d}$$

$$0 \leq x \leq 1/2$$

$$\dot{\overline{w}}(\rho) = -\overline{p}(\rho) \tag{6.6.1e}$$

$$\dot{\overline{p}}(\rho) = -\overline{q}(\rho)/t \tag{6.6.1f}$$

$$\dot{\overline{q}}(\rho) = -\overline{r}(\rho) \tag{6.6.1g}$$

$$\dot{\overline{r}}(\rho) = -(\alpha t + \beta)\overline{w}(\rho) + \lambda_o \overline{p}(\rho) \tag{6.6.1h}$$

$$0 \leq \rho \leq 1/2$$

$$t(x) = t(\rho) = \frac{Bq(x)\overline{q}(\rho)}{1 + \alpha B w(x)\overline{w}(\rho)} \tag{6.6.1i}$$

with boundary conditions

$$\overline{w}(0) = w(0) = \overline{q}(0) = q(0) = 0 \tag{6.6.1j}$$

and the continuity condition

$$\begin{Bmatrix} \overline{w} \\ \overline{p} \\ \overline{q} \\ \overline{r} \end{Bmatrix} = \begin{Bmatrix} w \\ p \\ q \\ r \end{Bmatrix} \quad \text{at } x = \rho = 1/2 \tag{6.6.1k}$$

We suspect, from Section 6.5, that $B = -1$, and therefore will use this value in all numerical calculations. The problem above is a form of two-point boundary value problem much like that encountered previously, with the ranges of x and ρ both being 0 to 1/2. Equation set (6.6.1) can be integrated simultaneously, using estimates of the control constants $p(0)$, $\overline{p}(0)$, $\overline{r}(0)$, $r(0)$, to generate trial values of $t(x)$, which has been forced to be symmetric. In general, for any numerical computation the values of the respective state variables at $x = 1/2$ will not be equal, that is, equation (6.6.1k) will not be satisfied. Our solution technique will involve perturbing p_o, r_o, $\bar{p}_o$, $\bar{r}_o$ in such a way as to force equation (6.6.1k) to be approximately satisfied.

We begin by defining the quantities

$$\begin{Bmatrix} \Delta w \\ \Delta p \\ \Delta q \\ \Delta r \end{Bmatrix} = \begin{Bmatrix} w(1/2) \\ p(1/2) \\ q(1/2) \\ r(1/2) \end{Bmatrix} - \begin{Bmatrix} \bar{w}(1/2) \\ \bar{p}(1/2) \\ \bar{q}(1/2) \\ \bar{r}(1/2) \end{Bmatrix} \tag{6.6.2}$$

For any numerical integration cycle, the components of the column matrix in equation (6.6.2) will have numerical values. The object of the iteration is to reduce these $\Delta(\ \ )$ quantities to zero. We postulate a first order relation

$$\delta \begin{Bmatrix} \Delta w \\ \Delta p \\ \Delta q \\ \Delta r \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \delta p_o \\ \delta r_o \\ \delta \bar{p}_o \\ \delta \bar{r}_o \end{Bmatrix} \tag{6.6.3}$$

where each element T_{ij} gives the change in one of these same quantities for a unit perturbation in one of the control constants, with all other control constant perturbations set zero. The elements of the matrix T_{ij} are obtainable from the perturbation equations of the system. These perturbation equations are

$$(\delta w)' = \delta p \tag{6.6.4a}$$

$$(\delta p)' = \delta q/t - (q/t^2)\delta t \tag{6.6.4b}$$

$$(\delta q)' = \delta r \tag{6.6.4c}$$

$$(\delta r)' = (\alpha t + \beta)(\delta w) - \lambda_o(\delta p) + \alpha w \delta t \tag{6.6.4d}$$

$$(\delta \bar{w})' = -\delta \bar{p} \tag{6.6.4e}$$

$$(\delta \bar{p})' = -\frac{\delta \bar{q}}{t} + \frac{\bar{q}}{t^2}(\delta t) \tag{6.6.4f}$$

$$(\delta \bar{q})' = -\delta \bar{r} \tag{6.6.4g}$$

$$(\delta \bar{r})' = -(\alpha t + \beta)(\delta \bar{w}) + \lambda_o(\delta \bar{p}) - \alpha \bar{w}(\delta t) \tag{6.6.4h}$$

$$\delta t = \delta \bar{t} = \frac{B}{2t}\left\{\frac{q\delta\bar{q} + \bar{q}\delta q}{1 + \alpha Bw\bar{w}} - \frac{\alpha q\bar{q}(w\delta\bar{w} + \bar{w}\delta w)}{(1 + \alpha Bw\bar{w})^2}\right\} \tag{6.6.4i}$$

Element T_{11} will be the change in Δw for a unit perturbation in $p_o(\delta p_o = 1)$ with all other initial perturbations set equal to zero. The change in Δw is simply

$$\delta(\Delta w) = \delta w(1/2) - \delta\bar{w}(1/2)\Bigg|_{\substack{\delta p_o = 1 \\ \delta w_o = \delta q_o = \delta r_o = \delta\bar{w}_o \\ = \delta\bar{p}_o = \delta\bar{r}_o = \delta\bar{q}_o = 0}}$$

The other transition matrix elements have a similar obvious definition. The perturbation equations may be integrated, with four different sets of initial perturbation boundary conditions, simultaneously with equation set (6.6.1) for a given set of estimated control constants $p_o, r_o, \bar{p}_o, \bar{r}_o$.

Several interesting observations should be noted. By assuming the thickness symmetric about $x = 1/2$, the integration interval has been reduced by one-half. But, on the other hand, the number of dependent variables has not been reduced, having remained at eight. The transition matrix for this problem is 4 by 4 and is thus four times as large as any previously encountered. Thus, the simple inclusion of the airload parameter λ_o into the beam problem has greatly changed the complexity of the problem.

Our previous discussion and analysis has led us to believe that the constant B is negative. We must not however, disallow the fact that B may be positive. For this reason, two solutions were attempted, one with $B = 1$ and the other with $B = -1$. The first solution attempts with $\lambda_o = 343.2$ and $z_o = 1.82$ were failures, no matter what value of B was chosen, because the errors, Δw, etc., were extremely large for the estimated values of the control constants. Once again, the transition matrix method failed to be an effective approach.

Previous experience with beam vibration problems and the torsional vibration problem has shown that the control constants will be continuous functions of the problem parameters such as δ_1 and t_{min}. It was safe to assume that the control constants for the flutter problem should be continuous functions of the problem parameters λ_o and z_o. This, in fact, is the case and was finally the approach used to achieve the final answer. If $B = -1$ and $\lambda_o = 0$, the problem reduces to one in which we minimize the weight of a strip of panel of unit width while holding the second frequency of free vibration constant. The solution to this problem, as well as the control constants, was readily available from previous work. If λ_o is made slightly positive, say $\lambda_o = 5$, the control constants for $\lambda_o = 0$ are good initial estimates and can be used to solve this problem. As the parameter λ_o is increased, we can graphically estimate the new values of the control constants and obtain extremely accurate estimates. Thus, we can obtain a series of solutions to equation set (6.6.1), each having λ_o different, but with $z_o = 2$.

If $z_o = 2$, an increase in λ_o results in a solution which has a higher $\mathcal{MR}$. At $\lambda_o = 200$, $\mathcal{MR}$ was close to 1. At this point z_o was reduced to 1.99 and equation set (6.6.1) was solved using $(\lambda_o, z_o) = (200, 1.99)$. This solution produced an $\mathcal{MR}$ which was less than that for (200, 2.0). This technique was continued until a solution for (343.2, 1.96) was obtained. This solution had an $\mathcal{MR} = 1.231$, meaning that it weighed 23.1% more than a similar uniform thickness panel. It also had a higher value of λ_o^* than $\lambda_o = 343.2$. Since the requirement that $\lambda_o^* = 343.2$ had been surpassed, z_o was varied to satisfy the requirement that flutter occur at $\lambda_o = 343.2$, that is, 343.2 will be the maximum value taken on by the (λ_o, z_o) curve. A decrease in z_o, and subsequent solution of the optimization problem, resulted in a lower $\mathcal{MR}$. The control constants for decreasing z_o were again easily estimated because they exhibited a continuous behavior for changes in z_o. A graph of $\mathcal{MR}$ with $\lambda_o = 343.2$ and for various values of z_o is presented in Figure 6.7. From this figure, it is seen that a minimum value of $\mathcal{MR}$ is reached at (343.2, 1.87) for this configuration.

The thickness distribution corresponding to this minimum value is shown in Figure 6.8. For comparison, the nondimensional deflection $w(x)$ and the nondimensional bending moment $q(x)$ are shown in Figures 6.9 and 6.10 respectively.

This thickness distribution behavior is highly unorthodox, to say the least, and was met, initially, with suspicion by the author and his advisor. There are, however, several reasons for believing these results to be valid:

(1) The flutter mode shape is similar to that encountered in the analysis of the reference panel.

(2) The sine series approximation for the optimal thickness distribution exhibits similar behavior.

(3) The mass ratio MR is of the same order of magnitude as that previously encountered in pinned-pinned beam fixed frequency problems and in the sine series approximation.

(4) Finally, this thickness distribution shape has been shown to give a λ_o, z_o curve similar to that for the reference panel.

Figure 6.1 shows a λ_o vs. z_o curve for a panel similar to that shown in Figure 6.11. This panel was optimized using $z_o = 1.87$ with $\lambda_o = 343.2$ and $\delta_1 = 1.0$, $t_{min} = 0.50$. Next, the λ_o vs. z_o curve shown in Figure 6.1 was generated and it was found that λ_o^* for the panel was 343.22, slightly greater than the required value. At first, it was thought that this distribution was close to the optimum. However, subsequent analysis showed that for $\delta_1 = 1.0$ and $t_{min} = 0.50$, an additional weight savings of 1% could be found by reducing z_o from 1.87 to 1.824. This distribution is shown in Figure 6.11 and is the actual optimal distribution. This study showed that for a given δ_1 and t_{min} and with $\lambda_o = 343.2$ the mass ratio was not very sensitive to changes in z_o. A careful analysis is required to find the actual z_o^* which yields the optimal thickness distribution.

Several distributions with varying structural parameters δ_1 are shown in Figure 6.12. An interesting characteristic of this problem is that an optimum beam, with its second frequency held fixed, weighs more than a similarly supported optimum beam whose flutter parameter is fixed. A beam with

$\delta_1 = .7$

$t_{min} = .10$

has mass ratios

$\mathcal{MR}$ = 0.91138 (fixed frequency)

$\mathcal{MR}$ = 0.88493 (fixed λ_o^*)

This difference of over 2.5% is seen to be too large to be attributable to any difference in solution accuracy. Also, although the uniform thickness panel flutters with $\lambda_o^* = 343.2$, and $z_o^* = 1.82$ the optimum panels flutter with $\lambda_o^* = 343.2$ but with z_o^* from 1.82 to 1.87 for the cases analyzed. Although a case for which $t_{min} = 0$ was not computed, it is expected that flutter for this limiting case would involve the coalescence of a rigid body frequency ($z_o = 0$) and the first elastic frequency, and that $t(1/2) = 0$, with the bending moment $q(x)$ crossing the x-axis at $x = 1/2$.

The analysis above was carried out with the modal constant $B = -1$. The optimization equations were also used with $B = +1$, but with no meaningful results. For $\lambda_o = 0$ and $z_o = 1$ with $B = +1$, the problem solution corresponds to the minimum weight beam with its fundamental frequency held constant. If z_o is fixed at $z_o = 1$ and λ_o is slowly increased, a series of solutions are obtained with the mass ratio, $\mathcal{MR}$, decreasing with increasing λ_o. The nondimensional deflection $w(x)$ and nondimensional bending moment $q(x)$ keep their respective signs over $0 \le x \le 1$.

This technique was successful for low values of λ_o. However, at a value of λ_o near 50, convergence problems were encountered. These convergence problems could only be remedied by raising the value of z_o, with the result that $\mathcal{MR}$ also increased. The trouble was traced to the fact that at certain values of λ_o and z_o the solution apparently wanted to change form and let $w(x)$ and $q(x)$ have a cross-over point similar to the cases encountered with $B = -1$. However, because $B = +1$, this is not possible. Thus, the only solutions obtainable with $B = +1$ were very high $\mathcal{MR}$ solutions. Because of this, the solution with $B = +1$ was abandoned.

No theoretical reason can be offered for $B = -1$ being a solution while $B = +1$ is not. It may be remembered that the theory we are using guarantees us an <u>extremum</u> of our merit function. Of course, it is hoped that this extremum is a <u>minimum</u>, but it may be that, with $B = -1$, we get an extremum which is a minimum, while $B = +1$ will also give us an extremum which corresponds to a maximum.

6.7 Summary

The panel flutter problem is by far the most interesting problem solved during this study because the behavior of the system of governing equations makes the solution extremely difficult. For this reason, a great deal of computer time was expended studying the behavior of the system equations and of the optimization problem itself. However, the greatest difficulty and obstacle to overcome was the closed-mindedness of the author himself. When one "knows intuitively" what the answer should look like, he renders the problem doubly difficult. In this case, the belief that the optimum must look like previously derived solutions with fixed frequency constraints automatically disallowed the eventual and final solution.

If there is any value in structural optimization at all, it lies in discovering unanticipated solutions. If we "know" the optimum solution, why go to the trouble of the generation of optimal solutions? It is the author's opinion that not even an experienced aeroelastician could have anticipated this optimal panel solution. If for no other reason, optimization has proven valuable in showing the trend or form of the panel shape. From the above solutions we can draw the conclusion that, to raise the flutter speed of a uniform panel with a small change in weight we should stiffen the panel near $x = .30$ and $x = .70$. These positions correspond to places where the optimum panel is thickest. We may also deduce that adding a stiffener at midchord $(x = 1/2)$ is not as efficient as adding it elsewhere.

The numerical difficulties encountered in this problem suggest that this size problem is as large as can be conveniently handled by a transition matrix method. Unless a sophisticated numerical scaling technique is used, one will encounter difficulties integrating the equations and inverting the transition matrix. Let it be noted, however, that the transition matrix procedure has never failed in any of the problems treated thus far. It would therefore be a great mistake to automatically dismiss it.

The use of estimation techniques such as the sine series approximation for

$t(x)$ appears to be an inexpensive, effective, easy-to-use technique for most eigenvalue constraint problems. It may be noted that although the shape of the "exact" optimal thickness distribution differs greatly from the two term sine series approximation, the mass ratios differ by only 2% to 3%. The results in this chapter should be qualified by a final remark. No damping was considered in these studies. It may well be that the addition of the aerodynamic damping term to the constraint equations will make a significant difference in the results. On the other hand, this zero damping solution is a limiting case and the author doubts that the addition of a slight bit of damping would substantially change the solution. This is certainly an area for future research.

This completes the treatment of systems which have differential equation constraints. The following section will discuss parameter optimization, that is, the optimization of a system described by a finite set of structural parameters.

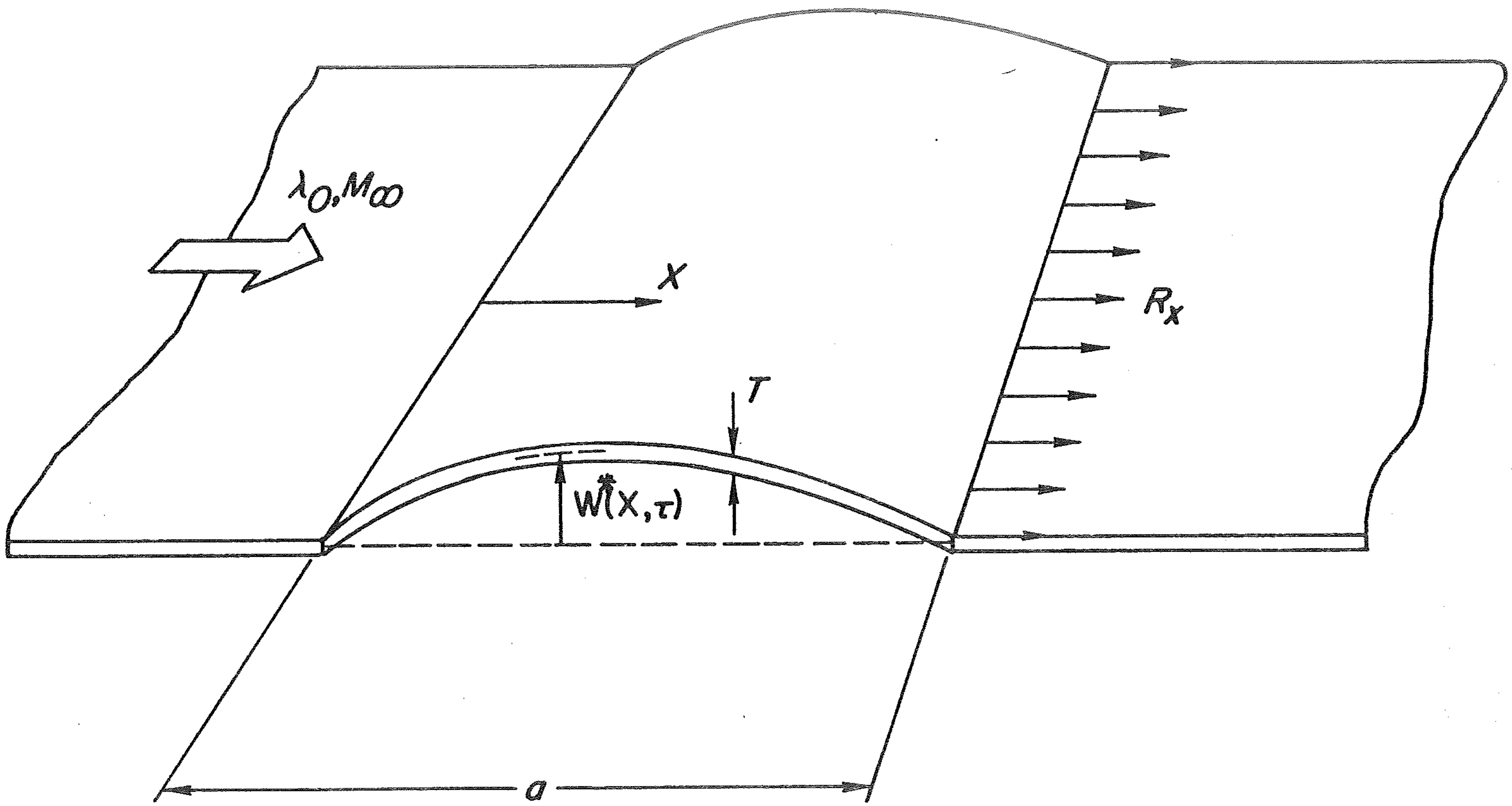


Fig. 6.1 One-dimensional Panel Flutter Model.

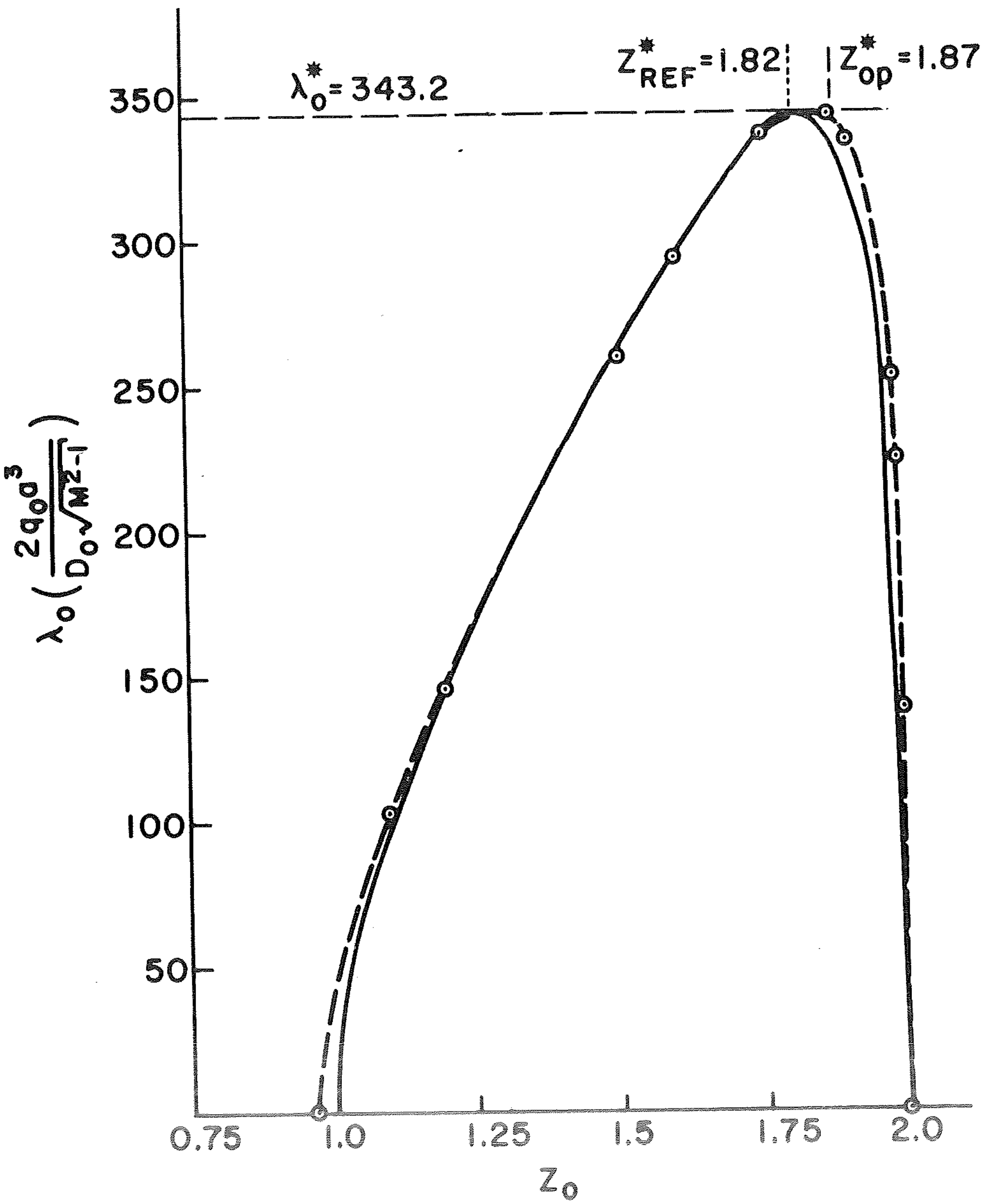


Fig. 6.2 Typical λ_o vs. z_o Diagram —— Uniform Thickness Panel Shown by Solid Line.

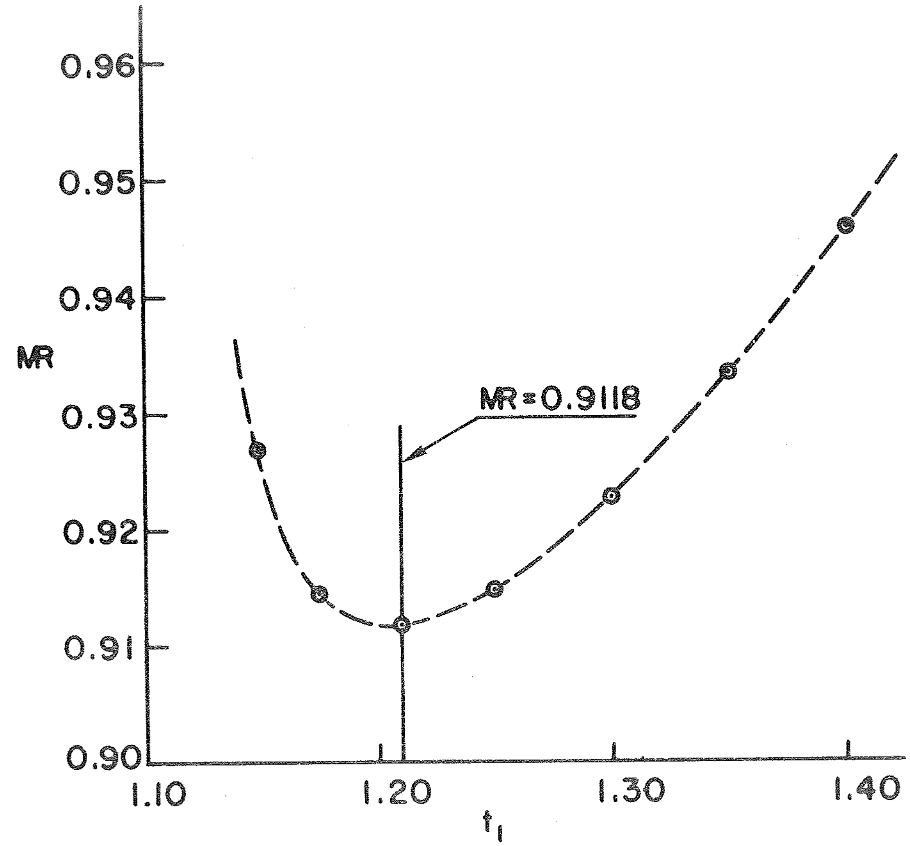


Fig. 6.3 Mass Ratio vs. t_1 for Two Term Sine Series Panel — $\delta_1 = 0.7$; $z_o = 1.82$; $\lambda_o = 343.20$.

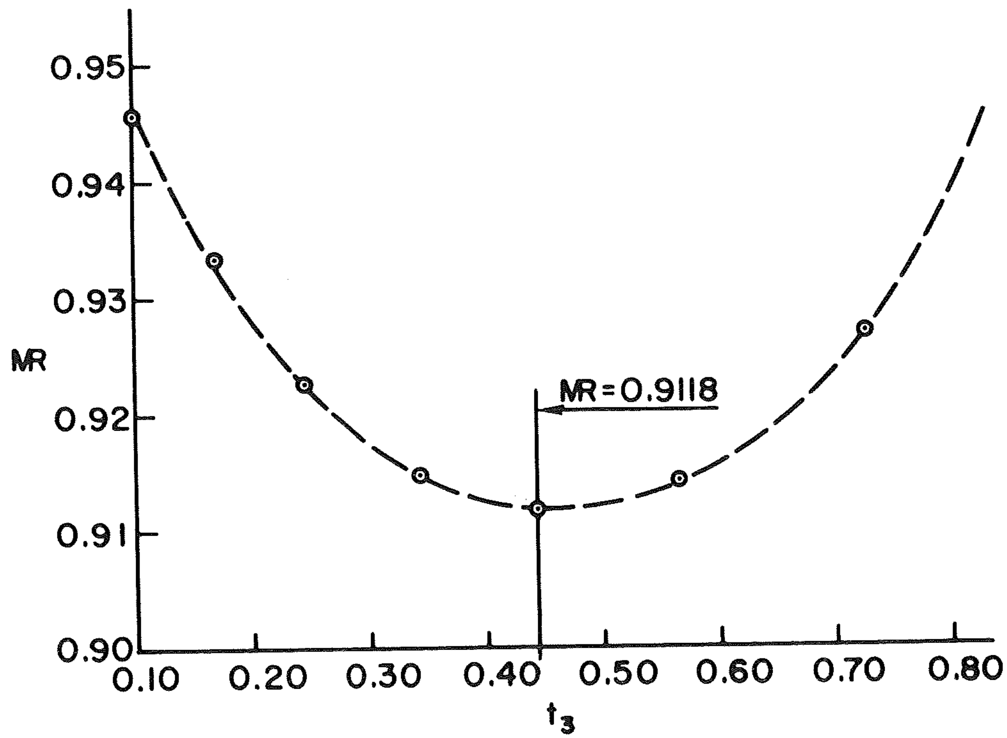


Fig. 6.4 Mass Ratio vs. t_3 for Two Term Sine Series Panel —— $\delta_1 = 0.7$; $z_o = 1.82$; $\lambda_o = 343.20$.

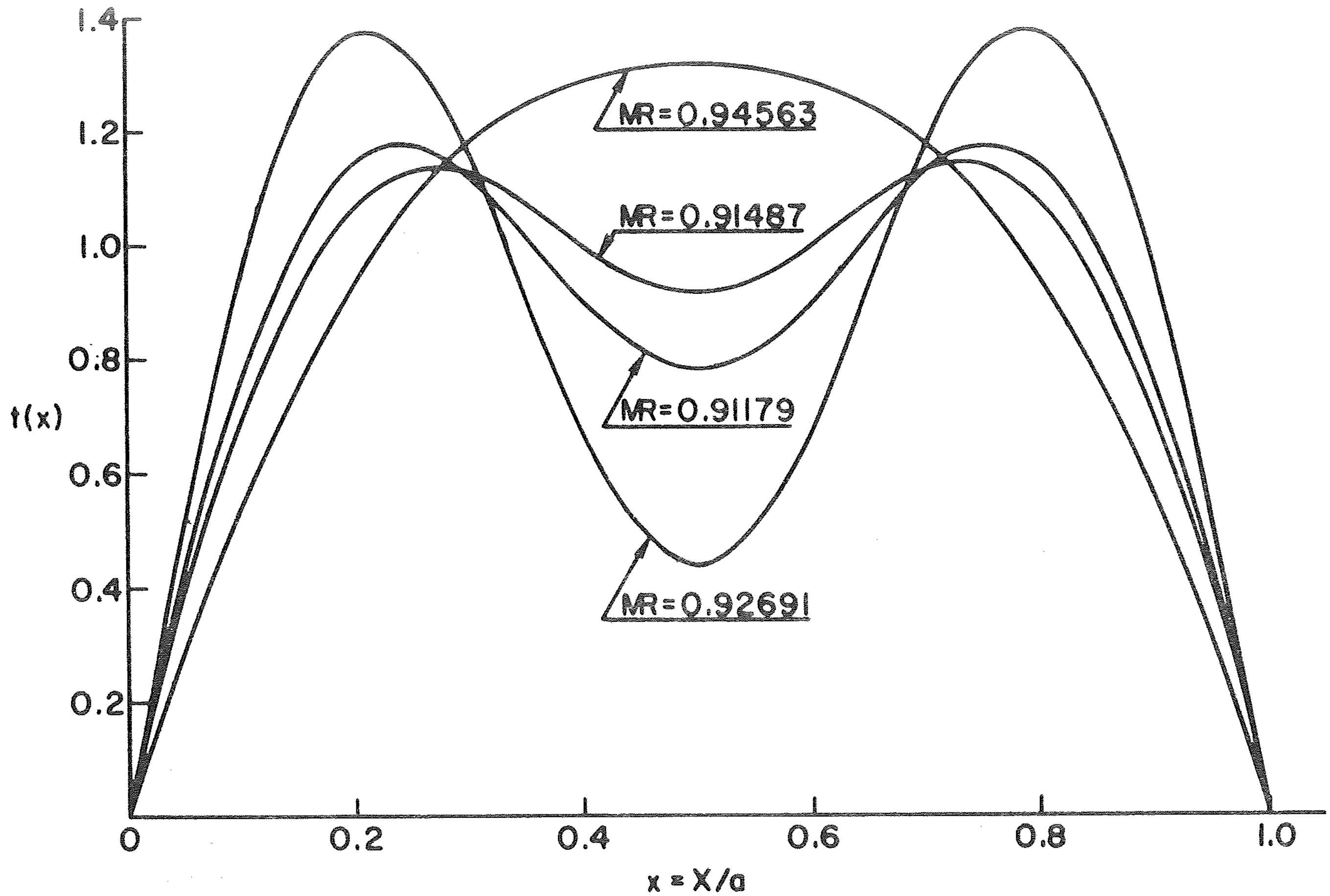


Fig. 6.5 Nondimensional Thickness Distributions for Two Term Sine Series Panel — $\delta_1 = 0.7$; $z_o = 1.82$; $\lambda_o = 343.20$.

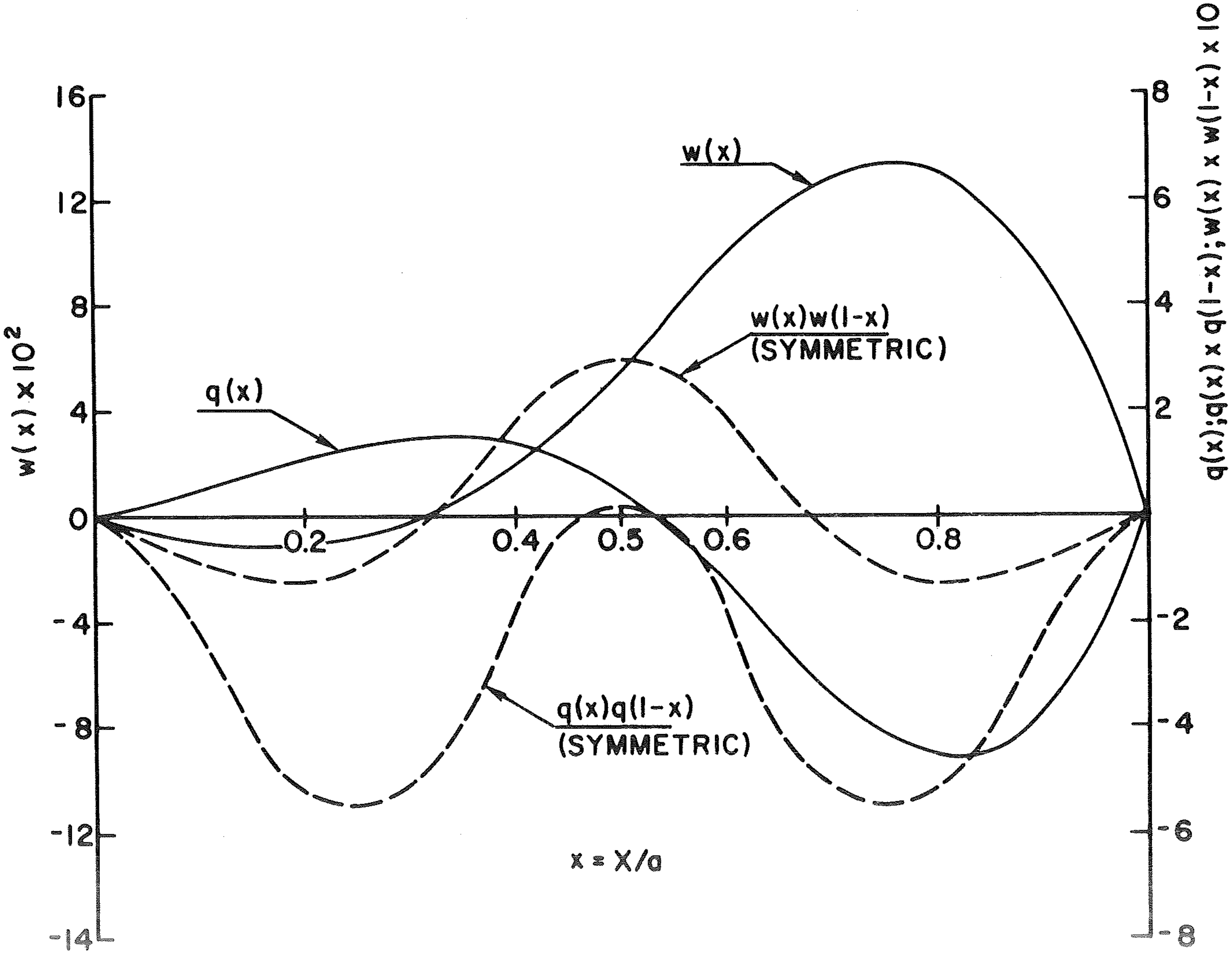


Fig. 6.6 Deflection and Bending Moment Mode Shapes for a Least Weight Two Term Sine Series Panel —— $\delta_1 = 0.7$; $z_o = 1.82$; $\lambda_o = 343.20$.

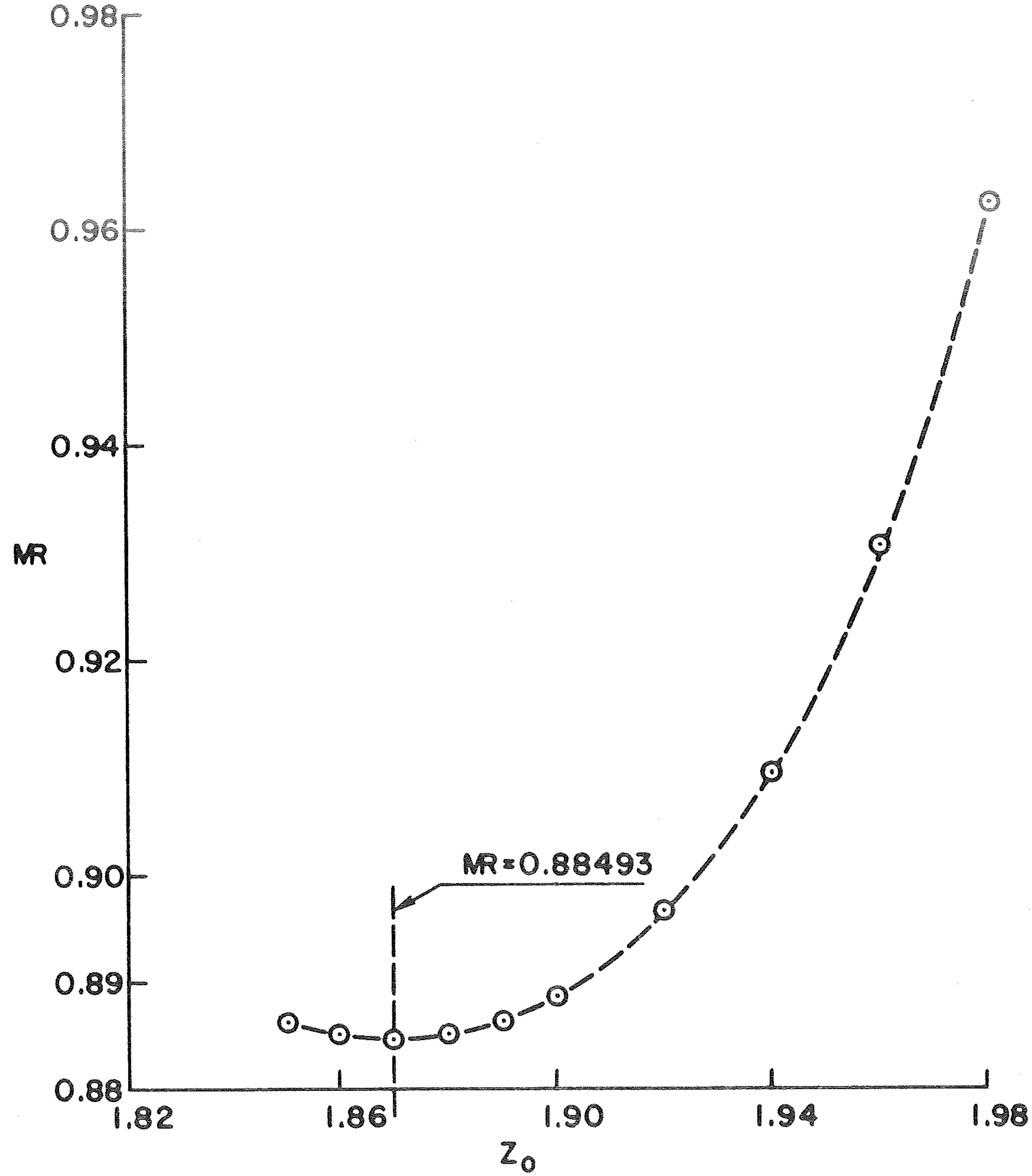


Fig. 6.7 Mass Ratio vs. z_o for a Series of Optimized Panels Having $\lambda_o = 343.20$ — $\delta_1 = 0.7$; $t_{min} = 0.10$.

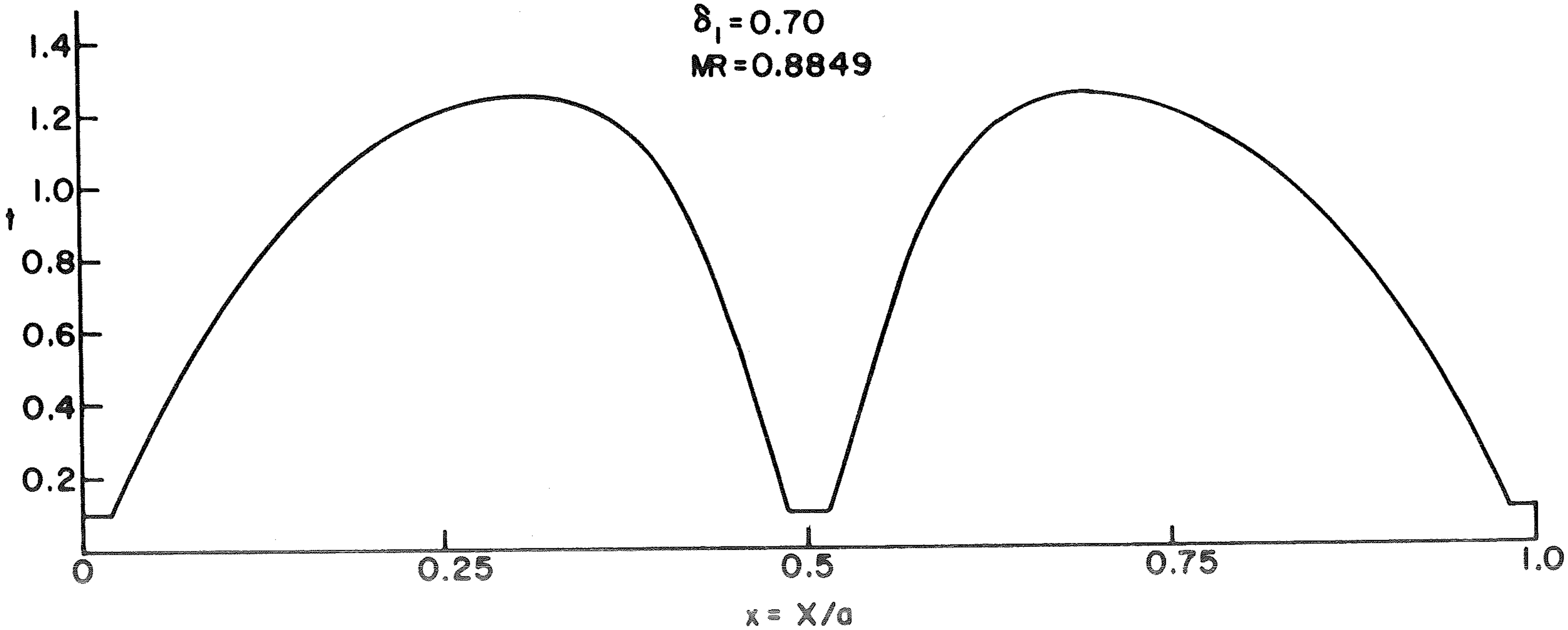


Fig. 6.8 Optimal Thickness Distribution for a Panel With $\lambda_o^* = 343.20$ —— $\delta_1 = 0.70$; $z_o^* = 1.87$.

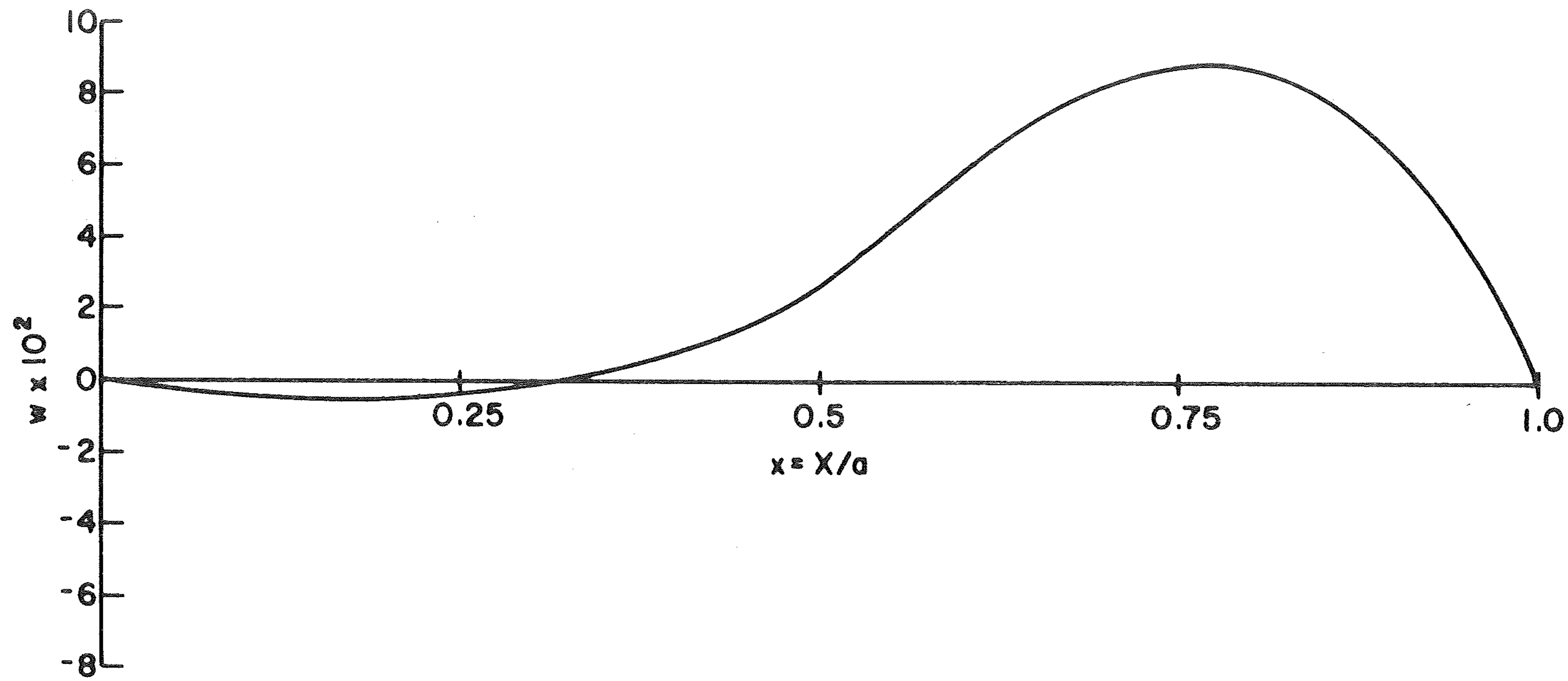


Fig. 6.9 Nondimensional Deflection for the Least Weight Panel With $\lambda_o^* = 343.20$ — $\delta_1 = 0.70$; $z_o^* = 1.87$; $t_{min} = 0.10$.

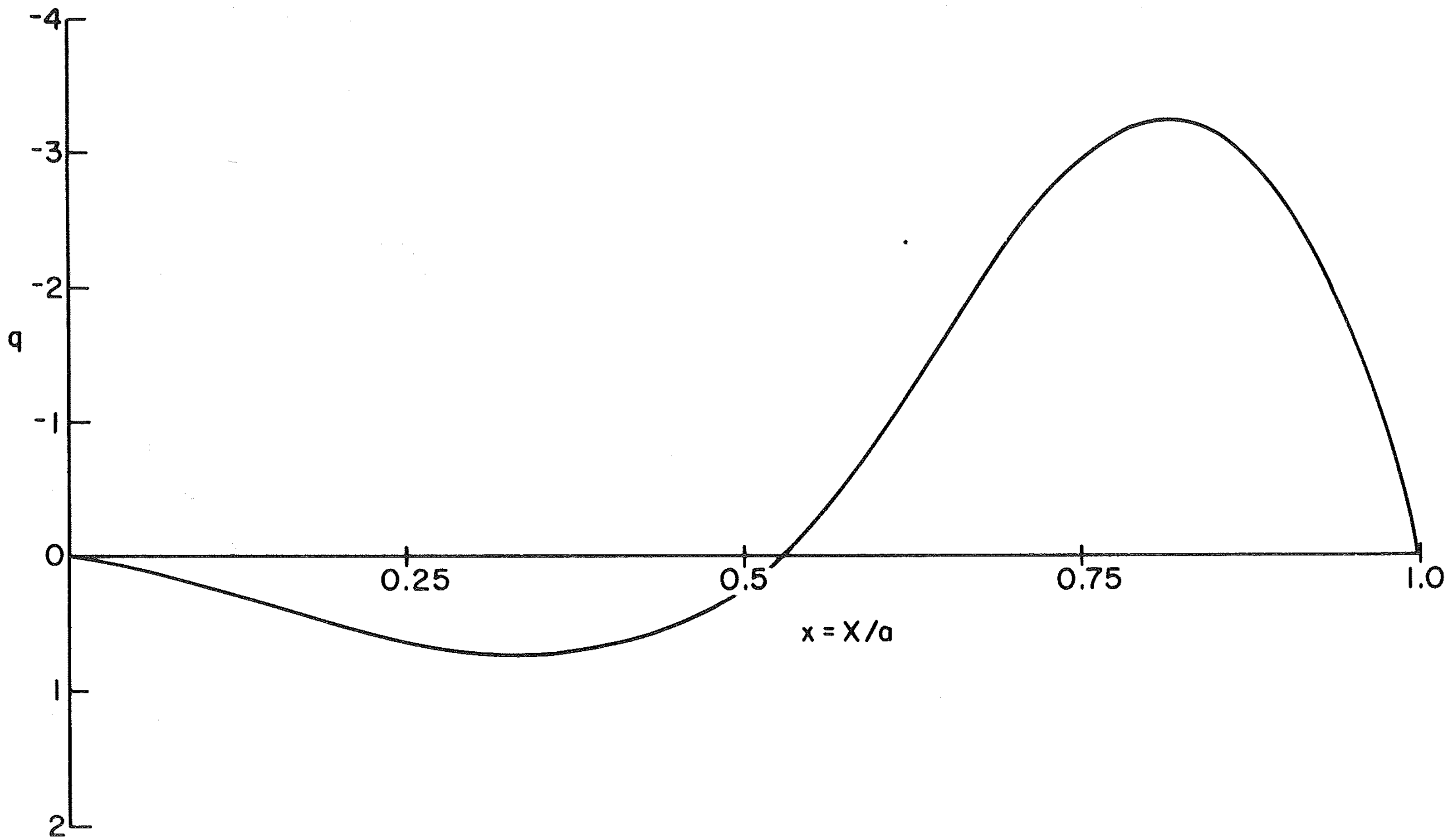


Fig. 6.10 Nondimensional Bending Moment for the Least Weight Panel With $\lambda_o^* = 343.20$ — $\delta_1 = 0.70$; $z_o^* = 1.87$; $t_{min} = 0.10$.

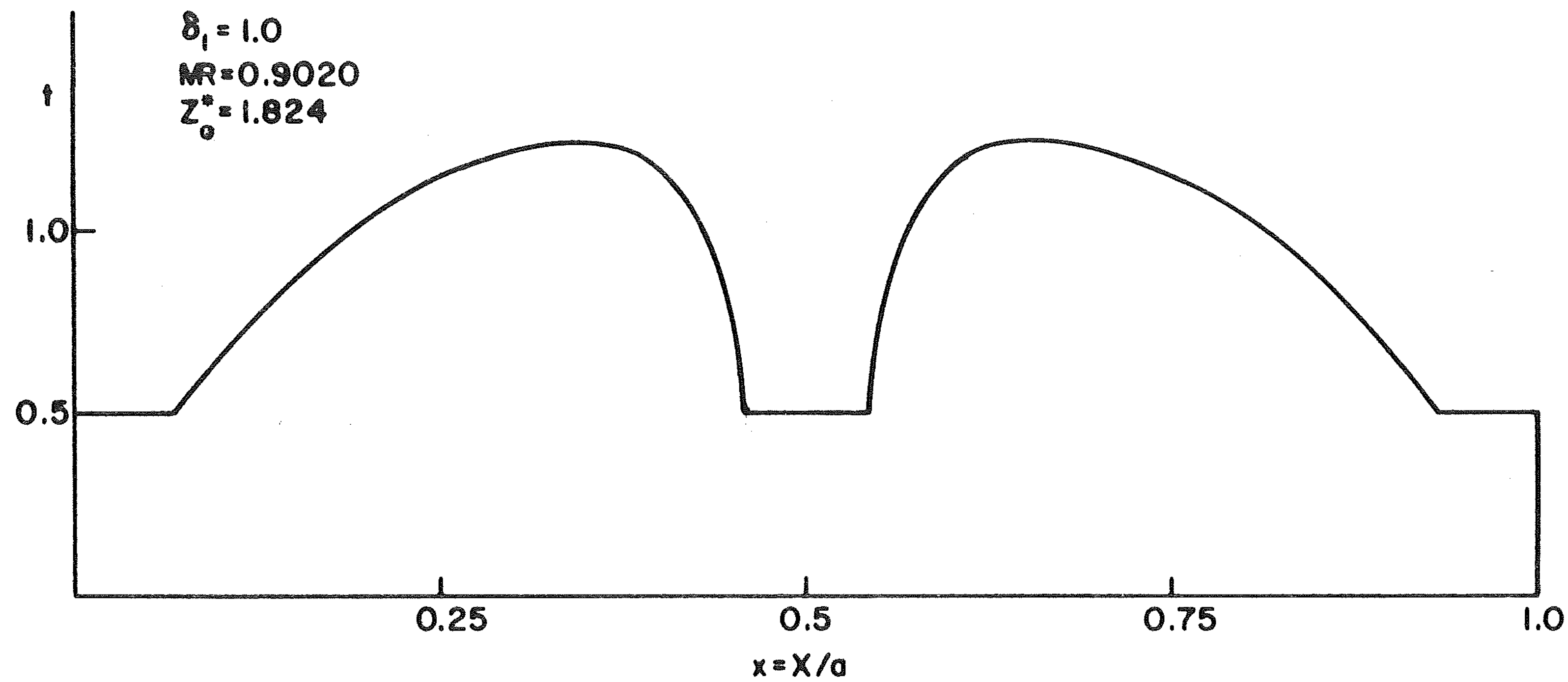


Fig. 6.11 Optimal Thickness Distribution for a Panel With $\lambda_o^* = 343.20$ —— $\delta_1 = 1.00$; $z_o^* = 1.824$; $t_{min} = 0.50$.

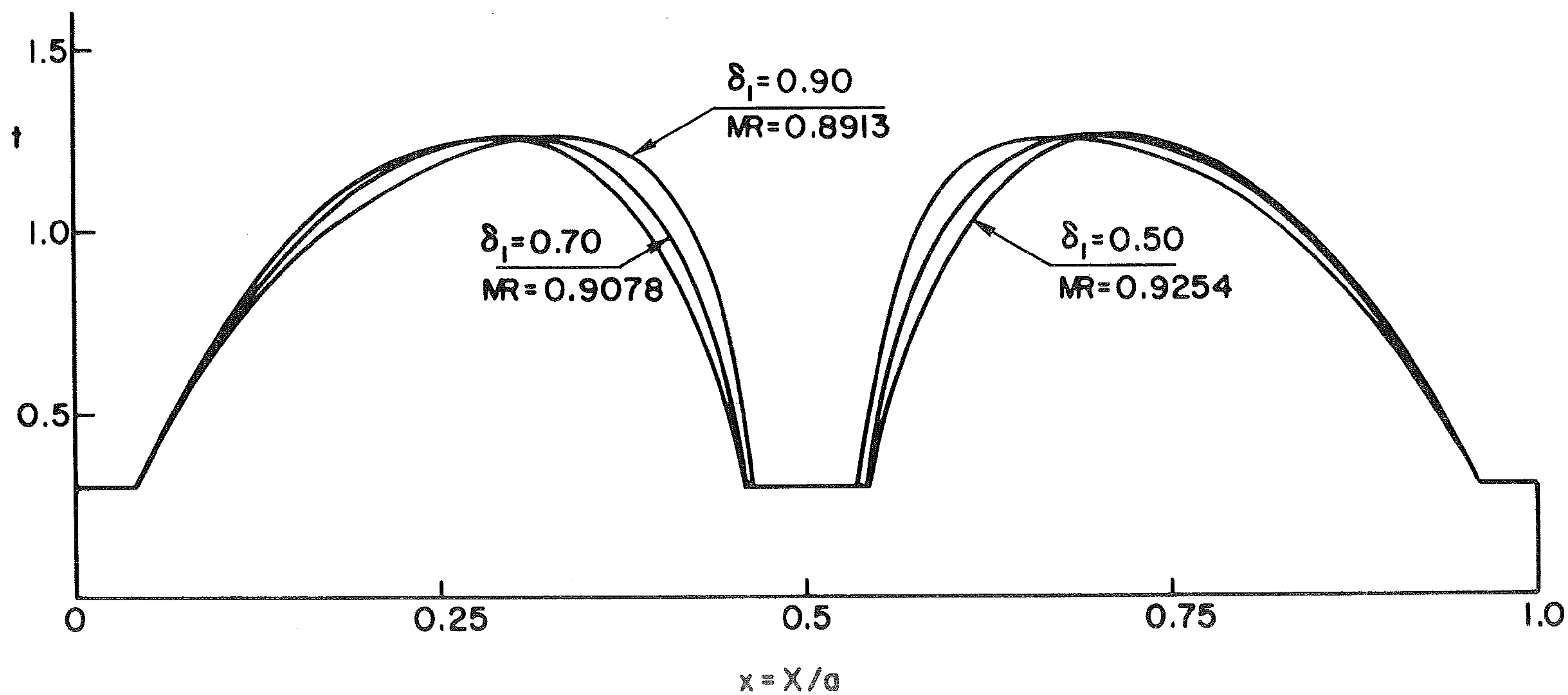


Fig. 6.12 Optimal Panel Thickness Distributions for Values of δ_1 Equal to 0.50, 0.70, 0.90 —— $\lambda_o^* = 343.20$; $t_{min} = 0.30$.

7. LEAST-WEIGHT DESIGNS USING FINITE ELEMENT STRUCTURAL MODELS AND FIRST-ORDER GRADIENT OPTIMIZATION METHODS

7.1 Introduction

This final chapter of analysis will concern itself with realistic design methods such as might be used in actual engineering design problems. The structural models for the studies will be obtained using finite element analysis techniques while the optimization technique will be patterned after one described by Rubin (Ref. 13). For comparison with previous results found using continuous one-dimensional structural models and optimal control theory, two problems will be studied in this chapter. The first problem discussed will be the least weight optimization of a beam on simple supports with a frequency constraint. The other study will involve solving the panel flutter optimization problem while constraining the critical aerodynamic flutter parameter λ^*_c. Both of these problems will use finite element techniques to describe the elastic and inertia properties of the one-dimensional structures involved.

Unlike the constraints imposed on the problems in previous chapters, the constraint requirements in this chapter will be imposed such that a parameter, either the frequency or the aerodynamic flutter parameter, is held close to a reference parameter to within a specified tolerance. The optimization technique suggested by Rubin (Ref. 13) was chosen because of its simplicity and the ease of programming it for the digital computer.

7.2 Finite Element Structural Modeling

The field of structural modeling using discrete parameters or finite elements is extremely broad and complex. The end result of any finite element method is, however, always the same. For problems involving dynamic response, a mass matrix and a stiffness matrix are calculated to describe the inertia and elastic properties of the structure. Any structure is considered to be an assemblage of smaller structural elements which have their own elemental mass matrices and stiffness matrices. These elemental matrices are combined to form the total system matrices by enforcing a requirement of geometrical continuity of displacement and slope at the boundary between each element. The analysis then proceeds using only the displacements and rotations of a finite number of points on the structure as variables. These variables are termed "generalized displacements" and the discrete points are termed "nodes."

Within each element a continuous displacement pattern or function is chosen which is a function only of the generalized displacements. This function is restricted by the requirement that the satisfaction of geometric compatibility at the nodes must ensure geometric compatibility along any element boundary. Using the displacement pattern or function, elemental mass and stiffness matrices are generated using energy methods similar to Rayleigh-Ritz techniques. These techniques are well described in books such as Przemieniecki (Ref. 18) and Zienkiewicz (Ref. 19). It will be assumed that the reader has at least a rudimentary knowledge of matrix techniques for structural analysis. This study will be concerned only with simple beam elements. These elements are well known in the literature and, for this study, the author derived elemental mass and stiffness matrices for a beam element whose bending stiffness and mass varied linearly along the length of the element. These matrices are listed in the Appendix. These beam elements are called "tapered elements" and are useful for analyzing a beam whose mass and stiffness properties change rapidly along its length.

7.3 A First-Order Gradient Technique

The description of the method which follows was described in detail by Rubin (Ref. 13) and belongs to a class of optimization schemes referred to as "first-order gradient techniques." While it does have some drawbacks which will be discussed later, this method is an excellent technique for optimizing the type of structures to be studied.

To begin, let us define the merit function (also called the "objective function") as the total nondimensional weight of the beam

$$W = \sum_{i=1}^{N} (t_i w_i) + W_o \qquad (7.3.1)$$

The N variables t_i are design parameters which define a beam dimension and which are free to be varied to achieve an extremum of W. In the simple beam cases which follow, t_i is the nondimensional thickness at a node point and w_i has values such that W is a nondimensional weight of the structure. W_o is the total weight of any nonstructural mass. The change in the total weight is expressed as a function of changes in the design variables Δt_i.

$$\Delta W = \sum_{i=1}^{N} (w_i \Delta t_i) \tag{7.3.2}$$

Let us multiply equation (7.3.2) by -1 to get

$$-\Delta W = -\sum_{i=1}^{N} (w_i \Delta t_i) \tag{7.3.3}$$

If the right-hand side of equation (7.3.3) can be made to be positive, then ΔW will be negative, that is, the change in total weight, for a given set of changes in the design variables, will be negative.

In the examples which will follow, an eigenvalue parameter, Ω, will be fixed. Since this eigenvalue is a function of the design variables, the total change in this eigenvalue parameter may be written as

$$d\Omega = \sum_{i=1}^{N} \frac{\partial \Omega}{\partial t_i} dt_i \tag{7.3.4}$$

If small design variable changes are used and only first-order terms are retained, then

$$d\Omega \cong \sum_{i=1}^{N} \frac{\partial \Omega}{\partial t_i} \Delta t_i \tag{7.3.5}$$

The constraint that Ω is equal to a constant may then be written as

$$d\Omega \equiv 0 \equiv \sum_{i=1}^{N} \frac{\partial \Omega}{\partial t_i} \Delta t_i = \sum_{i=1}^{N} g_i \Delta t_i \tag{7.3.6}$$

where

$$g_i = \frac{\partial \Omega}{\partial t_i}$$

Thus, where there were originally N independent design variables, the constraint equation (7.3.6) has reduced this number to $N - 1$. Let us consider the jth element in the above summation and solve for Δt_j.

$$\Delta t_j = - \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{g_i}{g_j} \Delta t_i \qquad (7.3.7)$$

Now, substitute equation (7.3.7) into equation (7.3.3) to get

$$- \Delta W = - \sum_{\substack{i=1 \\ i \neq j}}^{N} \left\{ \frac{w_i g_j - w_j g_i}{g_j} \right\} \Delta t_i \qquad (7.3.8)$$

or

$$- \Delta W = \sum_{\substack{i=1 \\ i \neq j}}^{N} G_i \Delta t_i \qquad (7.3.9a)$$

where

$$\underset{i \neq j}{G_i} = - \left\{ \frac{w_i g_j - w_j g_i}{g_j} \right\} \qquad (7.3.9b)$$

Equation (7.3.9a) is an expression for the negative change in weight as a function of changes in $(N - 1)$ design variables. Let us <u>define</u> these changes as

$$\Delta t_i = \epsilon \left[\frac{G_i}{|G_i|_{max}} t_i \right] \quad \begin{array}{l} i \neq j \\ i = 1, N \end{array} \qquad (7.3.10)$$

where ϵ is a <u>positive</u> constant and $|G_i|_{max}$ is the maximum absolute value of the set of gradients G_i, $i \neq j$. If we substitute equation (7.3.10) into equation (7.3.9a) we get the following equation.

$$- \Delta W = \epsilon \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{(G_i)^2}{|G_i|_{max}} t_i = \epsilon S \qquad (7.3.11)$$

where S is the value of the above summation. Since $t_i \geq 0$, the value of this summation can be seen to always be

$$S \geq 0$$

and thus $\Delta W \leq 0$. Furthermore, the value of ϵ can be seen to be

$$\epsilon = -\frac{\Delta W}{S} \tag{7.3.12}$$

If a 2% decrease in weight is desired, then we must have

$$\Delta W = -0.02W$$

and

$$\epsilon = (0.02)(\frac{W}{S}) \tag{7.3.13}$$

The above method will provide a set of Δt_i which causes a decrease in W as long as there is a value $G_i (i \neq j)$ which is non-zero. In turn, G_i will be unequal to zero as long as $g_i w_j \neq g_j w_i$.

The calculation of the $\frac{\partial \Omega}{\partial t_i}$ terms is essential to this method and is easily accomplished for the fixed frequency case. These terms are gradients of Ω with respect to the design variables, t_i. Although several other authors have detailed similar or more general techniques for calculating $\frac{\partial \Omega}{\partial t_i}$ when Ω is a frequency, a recent paper by Zarghamee (Ref. 20) gives a clear, concise, specialized example of an analytic expression which may be used to give these derivatives. This frequency gradient generation method will be discussed in Section 7.4.

The continuous problems treated in previous chapters were constrained to have one or more frequencies <u>exactly</u> equal to those of a given reference structure. Because of the slight inaccuracies and the assumptions of linearity of the optimization techniques, the frequency may "drift" or vary from that of the reference structure during the design process. Because of this, the frequency constraint is expressed as a <u>frequency band</u> constraint, that is, the frequency is constrained to be held within a certain tolerance or range on either side of a specified frequency. In the cases studied in this investigation the <u>square</u> of the frequency of the optimized structure was held to within $\pm .5\%$ of the square of the frequency of the original or reference structure.

If the frequency drifts outside this tolerance band, the optimization design cycle must be stopped. A design cycle is then used to bring the frequency back to within the specified tolerances. Rubin also provides a scheme for performing this task using the least possible weight, that is, changing the frequency a given amount using the least possible weight.

If the frequency is to be modified, changes in the design variables equal to

$$\Delta t_i = \kappa t_i \left(\frac{g_i}{|g_i|_{max}}\right) \tag{7.3.14}$$

are used, where κ is a positive or negative constant. The total change in frequency is seen to be

$$\Delta\omega^2 = \sum_{i=1}^{N} g_i \Delta t_i = \kappa \sum_{i=1}^{N} \frac{(g_i)^2 t_i}{|g_i|_{max}} \tag{7.3.15}$$

with

$$\kappa = \frac{\Delta\omega^2}{\sum_{i=1}^{N} \left[\frac{(g_i)^2}{|g_i|_{max}} t_i\right]} \tag{7.3.16}$$

7.4 Weight Minimization of a Finite-Element Beam on Simple Supports With its Fundamental Frequency Constrained

A beam composed of thin face-sheets (a sandwich beam) with a nonstructural core may be modeled as a finite element beam with Z equal segments as shown in Figure 7.1. The beam rests on simple supports so that the translational displacements at the beam ends are zero. The total number of unrestrained generalized displacements is

$$n = 2Z$$

If $[M]$ is the mass matrix for the structure while $[K]$ is the stiffness matrix, the matrix equation of free vibration may be written as

$$[-\omega^2[M] + (K]]\ q = 0 \tag{7.4.1}$$

where $[M]$ and $(K]$ are $n \times n$ symmetric matrices. The vector $\{q\}$ is $n \times 1$ and represents the vector of generalized displacements as shown in Figure 7.1.

Equation (7.4.1) represents an eigenvalue problem and has n real eigenvalues and eigenvectors

$$\omega_p^2; \quad \{q^{(p)}\}$$

From matrix theory, the eigenvectors $\{q^{(p)}\}$ are orthogonal with respect to $[M]$. That is

$$\lfloor q^{(r)} \rfloor\ [M]\{q^{(p)}\} = \begin{cases} \overline{M}^p; & r = p \\ 0; & r \neq p \end{cases} \tag{7.4.2a}$$

$$\lfloor q^{(r)} \rfloor\ [K]\{q^{(p)}\} = \begin{cases} \omega_p^2 \overline{M}^p & r = p \\ 0 & r \neq p \end{cases} \tag{7.4.2b}$$

$\overline{M}^p$ is termed the generalized mass for the pth eigenvector. If equation (7.4.1) is nondimensionalized, the eigenvectors and eigenvalues will be nondimensional also. For a sandwich structure, the system mass and stiffness matrices can be written

$$\underset{n \times n}{[M]} = \underset{n \times n}{[M_o]} + \sum_{i=1}^{N} t_i \underset{n \times n}{[m_i]} \tag{7.4.3}$$

$$\underset{n \times n}{[K]} = \underset{n \times n}{[K_o]} + \sum_{i=1}^{N} t_i \underset{n \times n}{[k_i]} \tag{7.4.4}$$

where t_i is a nondimensional face-sheet thickness. The matrices $[m_i]$ or $[k_i]$ are matrices which give the mass or stiffness contribution to the system due to the design variable t_i. Note that there will be many zero or null elements in these matrices. Also, note that

$$\frac{\partial}{\partial t_i}[M] = [m_i] \tag{7.4.5}$$

$$\frac{\partial}{\partial t_i}[K] = [k_i] \tag{7.4.6}$$

Using the technique described by Zarghamee (Ref. 20), let us differentiate the matrix expression (7.4.1) (for a particular value of ω_p^2) with respect to a design variable t_i.

$$[-\omega_p^2[M] + (K]] \left\{\frac{\partial q^{(p)}}{\partial t_i}\right\} + [-\frac{\partial \omega_p^2}{\partial t_i}[M] - \omega_p^2[m_i] + [k_i]]\left\{q^{(p)}\right\} = 0 \tag{7.4.7}$$

Because the transpose of the eigenvalue problem posed in equation (7.4.1) has the same eigenvalues and eigenvectors ([M] and [K] are real and symmetric), premultiplication of equation (7.4.7) by $\left\{q^{(p)}\right\}$ will eliminate the first matrix term on the left-hand side of equation (7.4.7). It will reduce the second term to a scalar.

$$-\frac{\partial \omega_p^2}{\partial t_i}\overline{M}^{(p)} - \omega_p^2\overline{m}_i^{(p)} + \overline{k}_i^{(p)} = 0 \tag{7.4.8}$$

The symbol $\overline{(\quad)}^{(p)}$ refers to the result of the matrix operation

$$\lfloor q^{(p)} \rfloor (\quad)\left\{q^{(p)}\right\} = \overline{(\quad)}^{(p)}$$

Equation (7.4.8) may be rearranged to give

$$\frac{\partial \omega_p^2}{\partial t_i} = \frac{\omega_p^2\overline{m}_i^{(p)} - \overline{k}_i^{(p)}}{\overline{M}^{(p)}} \tag{7.4.9}$$

Thus, the change in the <u>square</u> of the pth frequency with respect to t_i is given as a function of the mode shape $\left\{q^{(p)}\right\}$, the frequency ω_p and the reference matrices which describe the system. Expression (7.4.9) is "exact" in the sense that $\frac{\partial \omega^2}{\partial t_i}$ is a calculated matrix function of the system properties, eigenvalues and eigenvectors.

For a fixed frequency beam vibration problem, the frequency constraint is expressed as

$$d(\omega^2) = \sum_{i=1}^{N} \frac{\partial(\omega^2)}{\partial t_i} \Delta t_i = \sum_{i=1}^{N} g_i(\Delta t_i) \equiv 0 \qquad (7.4.10)$$

For a tapered-beam-element model, the total nondimensional weight of a beam with Z elements as shown in Figure 7.1 is

$$W = \left(\delta_1 \left[\frac{t_1 + t_{Z+1}}{2} + \sum_{i=2}^{Z} t_i \right] \right) \Big/ Z + (1 - \delta_1) \qquad (7.4.11)$$

The design variables t_i are discrete nondimensional thickness parameters at each node. The parameter δ_1 is, as in the continuum case, the initial ratio of structural mass to total mass in the reference structure. Using the definition in equation (7.3.1)

$$w_i = \frac{1}{Z} \quad \text{for} \quad i = 2, 3, \ldots. Z$$

$$w_1 = w_{Z+1} = \frac{1}{2Z} \qquad (7.4.12)$$

In addition to the constraint expressed in equation (7.4.10), a minimum thickness constraint can be added such that

$$t_i \geq t_{min} \quad (i = 1, 2, \ldots. Z + 1) \qquad (7.4.13)$$

Using the method outlined in Section 7.3 and calculating the frequency gradients g_i as shown previously, a computer program was written to search for an optimal design variable vector $\{t_i\}$. The initial numerical cases concerned themselves with testing the operation of the computer routine. A uniform thickness case was chosen as a starting point, that is, $\{t_i\} = \{1\}$. Using the above techniques with the fundamental frequency constrained to be $0.995\pi^4 \leq \omega_1^2 \leq 1.005\pi^4$, the method always drove design variables t_1 and t_{Z+1} to the value t_{min}. At this point, no further changes in t_1 and t_{Z+1} were allowed, that is, $\Delta t_1 = \Delta t_{Z+1} = 0$. In addition, because of the symmetry of $\{q^{(1)}\}$ about $x = 1/2$, the frequency gradients g_i were symmetric about this point. Since $t_{i\ initial} = 1$, the design cycles always yielded a symmetric structure. After these characteristics were determined, the program was altered so that it automatically set t_1 and t_{Z+1} equal to t_{min} before beginning calculations. Only symmetric changes in t_i were then allowed during each optimization design cycle. To save computation time, initial estimated values of t_i which might be chosen in engineering work were chosen so that the initial design had a weight less than the uniform reference structure. If the fundamental frequency of the initial design chosen was outside the frequency tolerance band, a design cycle was taken to alter t_i to bring the frequency to within the allowable band. By doing this, the initial design, while not being a least weight design, satisfied the requirements of the problem and was at least a "lesser weight" design as compared to the reference structure. This initial design estimation and modification technique usually saved ten or more design optimization cycles and resulted in an approximation which had a fundamental frequency within the tolerance band and a total weight or mass ratio which differed from the exact minimum by only a few percent.

Two results of the above optimization work are shown in Figure 7.2. One model was composed of four elements while the other model used six elements. As mentioned before, during the weight minimization search, a series of "lesser weight" structures are generated which satisfy the requirement that their lowest frequency is close to that of a uniform thickness reference structure. As the finite element design approaches the least weight design, the method outlined above is very sensitive to the variable ϵ, defined in equation (7.3.12). If ϵ is large, we are in effect asking for large changes in the weight. As the value of the converged mass ratio, that is, the value of the mass ratio obtained using the differential equation model, is approached, the design variable distribution varies greatly. It appears that, near the optimum design, the merit or objective function is fairly insensitive to the constraint boundary while the design variables are very sensitive. For this reason, several designs of nearly the same weight are found at the end of the optimization process. This is far different from the differential equation approach where one and only one design satisfied the constraints.

Because of this, the author selected the design which most closely approximated the exact design shape. If ϵ was kept large, that is, if we ask for large changes in weight at a point where we are near the minimum mass ratio, the design variables began to increase near $x = 0.25$ and $x = 0.75$ and decrease near $x = 0.50$. Thus, it appeared that the design began to take the form of a superoptimal solution. When this began to occur, the computer calculations were terminated.

From Figure 7.2 it can be seen that, while the finite element thickness distribution — for four elements — is in error by large amounts in some places, the mass ratio differs only slightly from the exact solution. The reader is cautioned, however, in drawing conclusions on the accuracy of this mass ratio since the nondimensional frequencies in the finite element case are slightly lower than that from the continuous model, that is, they differ from $\omega_1^2 = \pi^4$. Also, if one were to build a beam in the exact shape of the finite element model, its frequency would differ from that shown in Figure 7.2 because of the inaccuracies of the finite element model itself. Still, the general conclusion that the discrete parameter model for this problem is consistent with the exact solution can be drawn. Although the design variables do differ substantially from the analytic solution, the minimum mass ratio is fairly accurate. It is interesting to note that the taper of the first and last elements in Figure 7.2 decreases with the number of elements in the model. This occurs because of the minimum thickness constraint, $t_{min} = 0.5$. This constraint was purposely chosen to demonstrate this characteristic.

Figure 7.3 shows a ten-element finite-element model of the exact optimal thickness distribution. Unlike the cases previously shown, this model is not the result of an optimization process, but it shows the accuracy one may expect from a multi-element structural model of the exact solution.

7.5 Panel Flutter Optimization Using a Finite Element Structural Model

The problem of finding a least weight thickness distribution for a sandwich panel on simple supports which has an aerodynamic flutter parameter λ_o^* held fixed has been discussed in Chapter 6. In Chapter 6, the constraint equations were expressed as a set of first-order differential equations. Therefore, optimal control techniques were used to formulate and solve the problem. The result of this formulation was an optimal thickness distribution for a least-weight panel. This section will discuss the identical problem posed in Chapter 6, but will use finite element methods to describe a one-dimensional plate-beam on simple supports and also to describe the aerodynamic forces using quasi-steady linearized supersonic aerodynamics.

The problem of finding a least-weight panel with a prescribed flutter parameter using finite element techniques was first discussed by Turner (Ref. 12). His structural model consisted of a series of equal length finite elements, each having a uniform thickness. In addition, his aerodynamic model included an aerodynamic damping term. The model used in this thesis consists of a series of equal length nondimensional tapered finite elements such as shown in Figure 7.1. The aerodynamic forces are taken from a paper by Olson (Ref. 21). The non-dimensional matrix equilibrium equation for this system is written

$$[-\omega^2[M] + [K] + \lambda_o[A]]\{q\} = 0 \qquad (7.5.1)$$

$[A]$ is the aerodynamic generalized force matrix given by Olson and shown in the Appendix and λ_o is, as in Chapter 6, equal to

$$\lambda_o = \frac{2q_o a^3}{D_o\sqrt{M_\infty^2 - 1}}$$

All aerodynamic damping is neglected in equation (7.5.1) and motion is assumed to be of the form

$$\left\{Q(x,\tau)\right\} = \left\{q(x)\right\}e^{i\omega\tau} \qquad (7.5.2)$$

As discussed in Chapter 6, for a range of values $0 \le \lambda_o \le \lambda^*$, the frequencies ω, as determined from equation (7.5.1), will be real. At $\lambda_o = \lambda^*_1$ frequencies ω_1 and ω_2 merge, and, for $\lambda_o > \lambda^*$, ω_1 and ω_2 will be complex conjugates of each other. Thus, from, equation (7.5.2) at $\lambda_o = \lambda^*$, the system is neutrally stable and for $\lambda_o > \lambda^*$ the panel becomes unstable. Using a four equal finite-element model with uniform thickness and neglecting damping, Olson found $\lambda^* = 342.343$. This value differed from the exact value of λ^* by 0.3%.

A computer program using the optimization logic described in Section 7.2 was written for this problem. For this case, the constraint equation is written

$$d\lambda^* = \sum_{i=1}^{N} \frac{\partial \lambda^*}{\partial t_i} \Delta t_i = 0 \qquad N = Z + 1 \tag{7.5.3}$$

For easy reference, the design variables t_i may be expressed in vector form

$$\left\{ t_i(x) \right\} = \left\{ \begin{matrix} t_1(0) \\ t_2(1/Z) \\ \vdots \\ t_{Z+1}(1) \end{matrix} \right\} \tag{7.5.4}$$

The variable Z is the number of equal length elements used in the finite element model. These design variables represent nondimensional face-sheet thicknesses at node points along the panel x as shown in Figure 7.1.

The calculation of the λ^* gradients, $\frac{\partial \lambda^*}{\partial t_i}$, was done numerically. First, λ^* was determined for the initial design panel. Next, the design variables were perturbed one by one. New flutter parameters λ_i^* were calculated for each perturbation Δt_i to give the gradient of λ^*, g_i.

$$g_i = \frac{\partial \lambda^*}{\partial t_i} \cong \frac{\Delta \lambda^*}{\Delta t_i} = \frac{\lambda_i^* - \lambda_{REF}^*}{\Delta t_i} \tag{7.5.5}$$

$i = 1, 2, \ldots . Z + 1$

The design studied involved a case with $\delta_1 = 1$ and $t_{min} = 0.50$. Four tapered elements were used to model the plate-beam. An initial set of design parameters was chosen using previous experience as a guide. This design had a symmetrical shape with $t_3 = t(1/2)$ as the greatest element and $t_1 = t(0)$, $t_5 = t(1)$ equal to 0.50. This initial design was found to have $\lambda^* \cong 335$, a value which was lower than the reference value of $\lambda^* = 342.36$. By using the gradient of λ^*, as calculated in equation (7.5.5), this initial design was modified so that the value λ^* became 343.68. This design was chosen as a starting point for the optimization search. Table 7.1 shows the results of four design cycles. The thickness distributions for these results are shown in Figures 7.4 and 7.5.

Table 7.1 shows several interesting things. First of all, it shows that an initial design can be found with a mass ratio close to optimal by simply using the λ^* gradients to increase or decrease the λ^* of an initial design. More importantly however, the λ^* gradient elements are symmetric about $x = 1/2$ as hypothesized by Turner (Ref. 12) and Armand (Ref. 8). The results also show that, in the absence of aerodynamic damping, the optimal one-dimensional panel shape for minimum weight has a pronounced "dip" at $x = 1/2$.

The comparison of these results with those found using differential techniques is favorable and is shown in Figure 7.5. The exact value of $\mathcal{MR}$ is 0.9020 while the finite element analysis gives 0.9151. The exact value of the square of the flutter frequency is 1070.0 while the finite element analysis gives $\omega_1^2 = \omega_2^2 = 1059.3$.

7.6 Summary

The previous results using finite element analysis have only scratched the surface of the subject. They have, however, shown that excellent results may be obtained using finite element techniques in the problems covered. They have also shown that the designer or researcher must be able to interact with the optimization design process. In this way, his experience becomes an added constraint on the design process. This analysis has also shown that the design parameters may be relatively sensitive to the design constraint but, the mass ratio in insensitive to changes in the design parameters near the least weight design. Therefore, the designer may choose his final design on the basis of factors other than least weight.

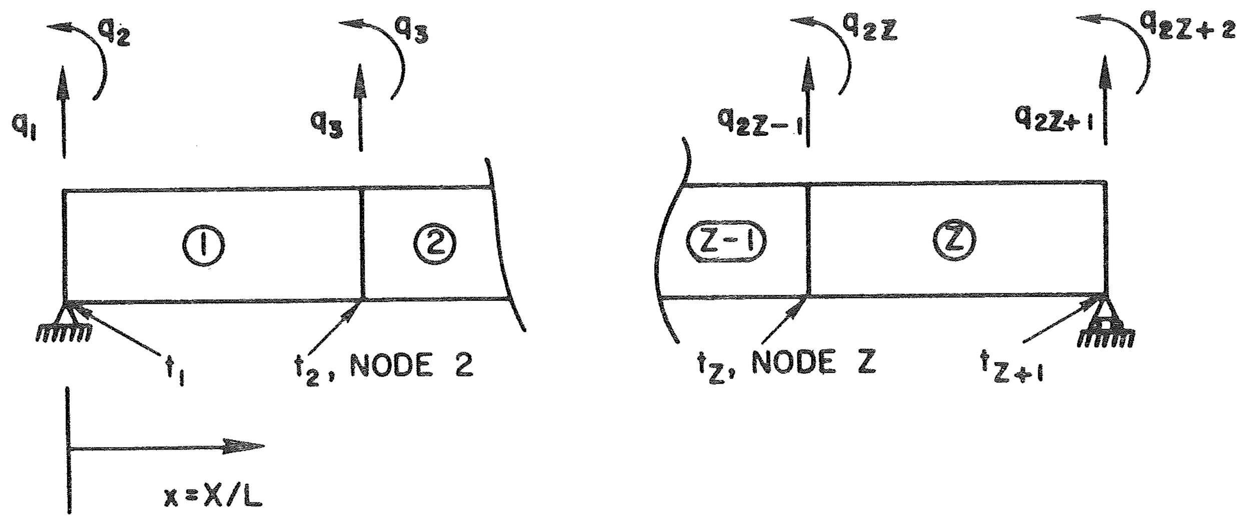


Fig. 7.1 Finite Element Model for a Beam With Z Elements, Showing Generalized Displacements and Design Variable Parameter Locations.

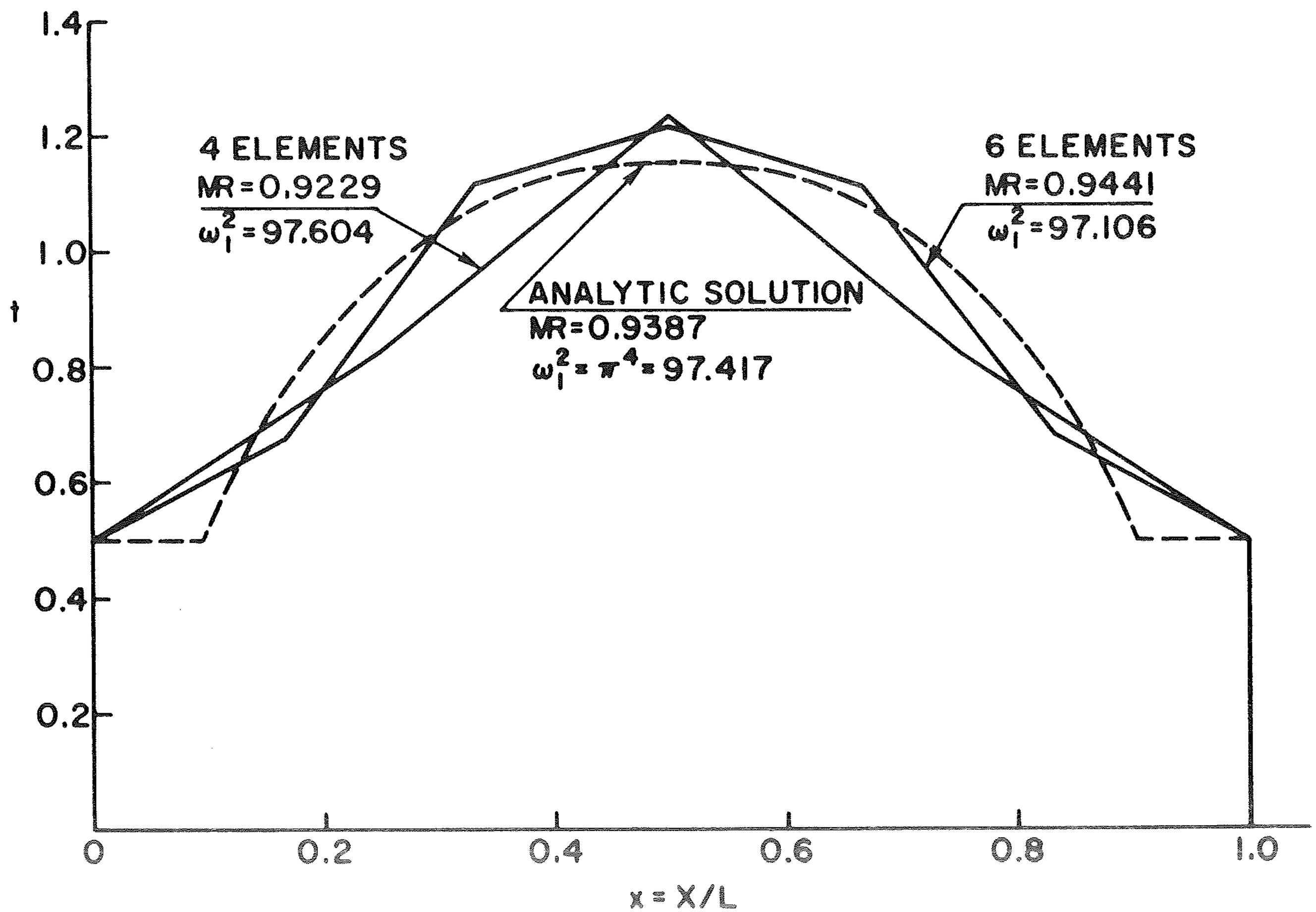


Fig. 7.2 A Comparison Between the Exact Thickness Distribution for a Minimum Weight Beam With its Fundamental Frequency Fixed and Four and Six Element Discrete Parameter Optimization Results — $\delta_1 = 0.50$; $t_{min} = 0.50$.

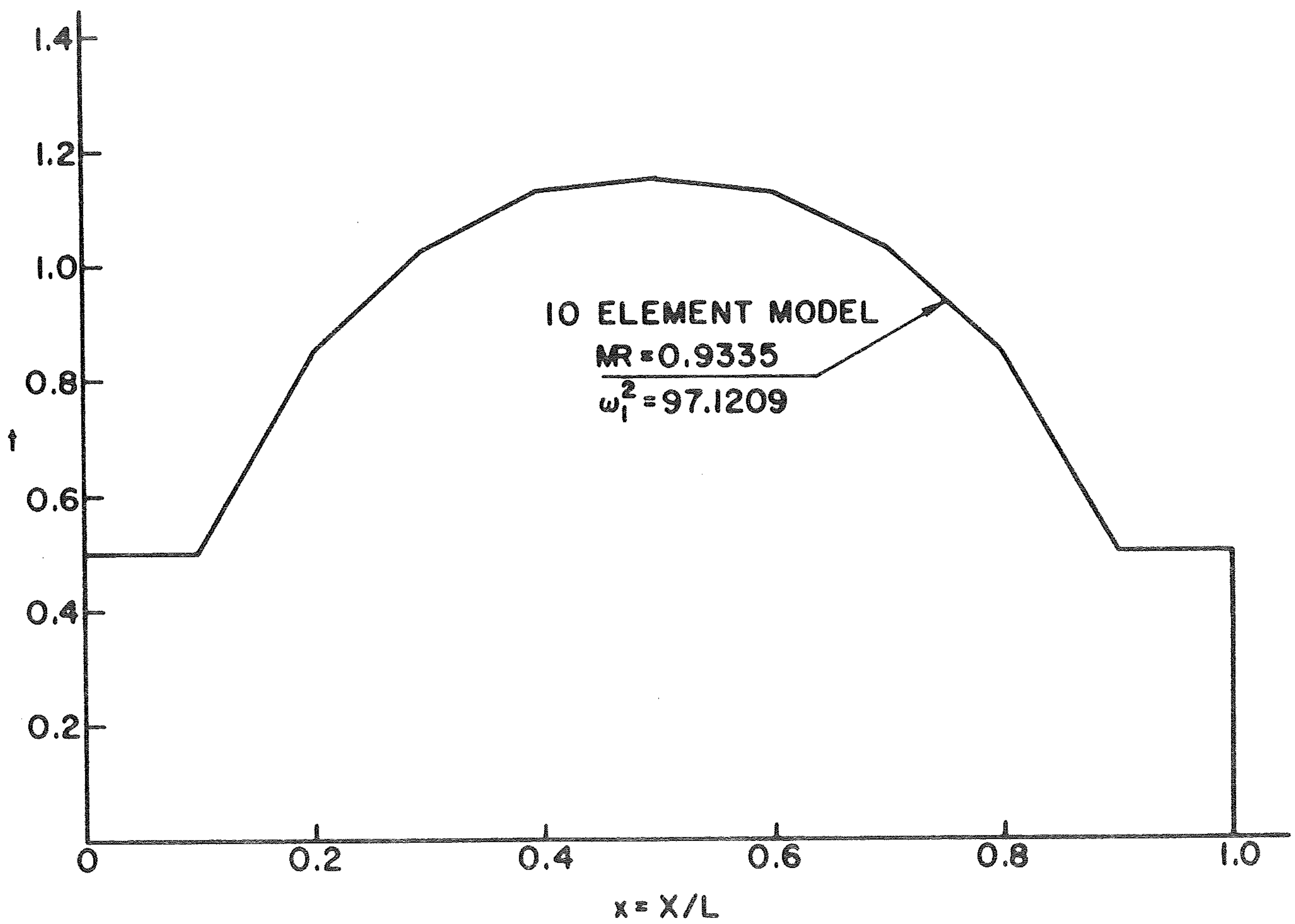


Fig. 7.3 A Ten Element Model of the Exact Thickness Distribution for a Minimum Weight Beam on Simple Supports With its Fundamental Frequency Fixed.

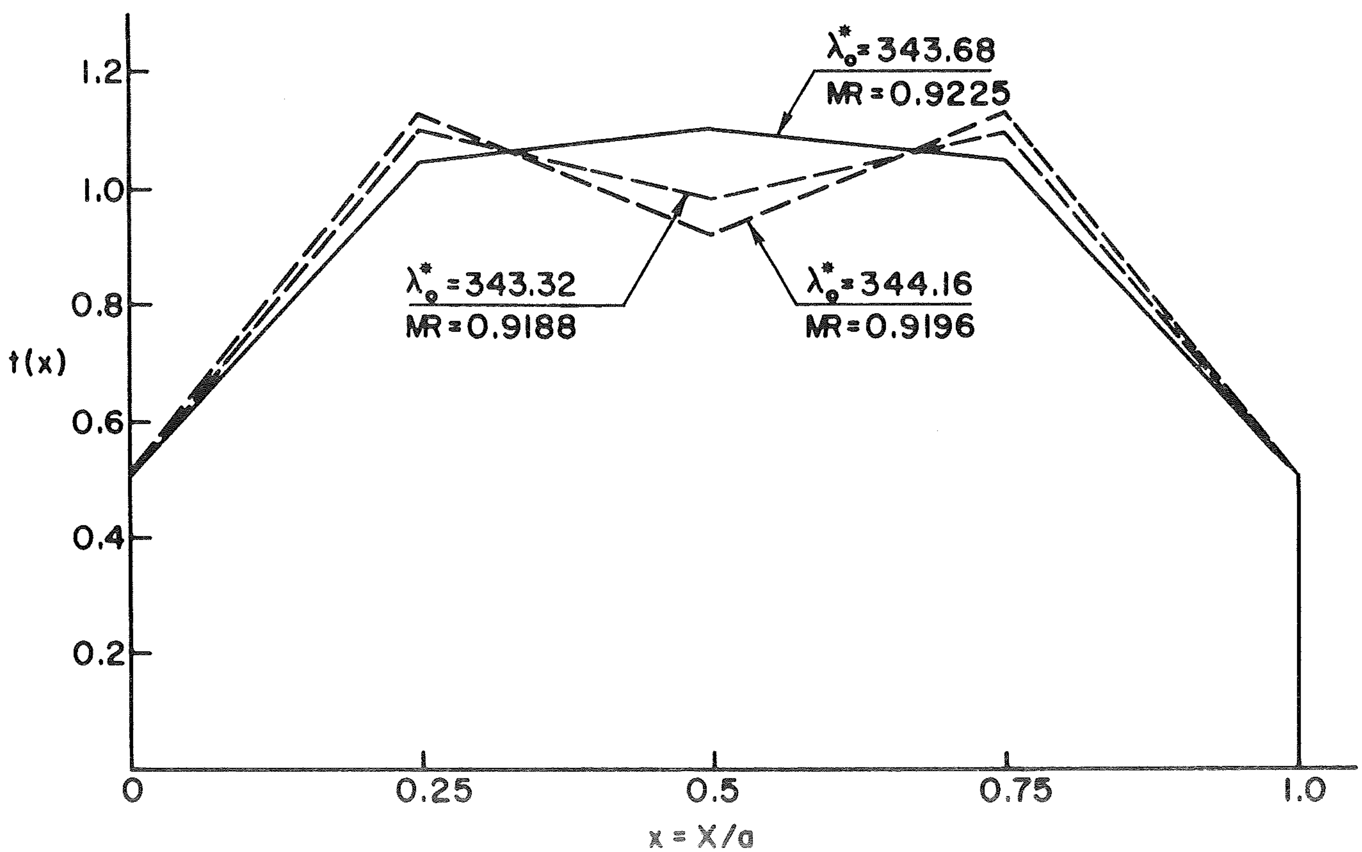


Fig. 7.4 Design Cycle Results Using a Four Element Model to Optimize Weight While Holding the Panel Flutter Parameter λ_o^* Fixed.

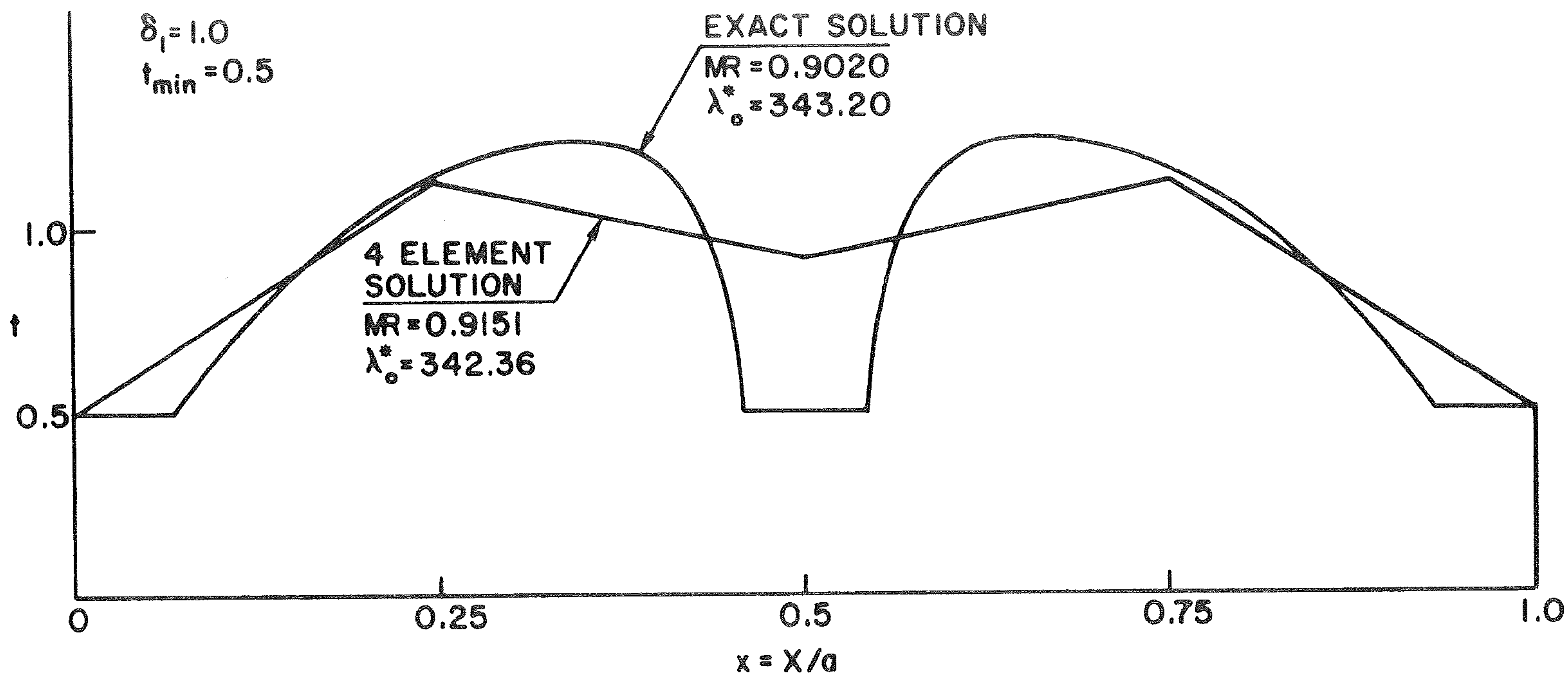


Fig. 7.5 A Comparison Between the Exact Thickness Distribution and a Four Element Discrete Parameter Optimization Model for Panel Flutter —— $\delta_1 = 1.0$; $t_{min} = 0.50$.

TABLE 7.1

Design Cycle	Initial	1	2	FINAL[1]
MR	0.9225	0.9188	.9196	0.9151
t_1	0.500	HELD FIXED		
t_2	1.045	1.0972	1.1302	1.1242
t_3	1.100	0.9798	0.9181	0.9121
t_4	1.045	1.09725	1.1302	1.1242
t_5	0.500	HELD FIXED		
g_1	-	NOT CALCULATED		
g_2	104.369	102.920	101.30	101.83
g_3	90.712	94.736	96.05	95.00
g_4	104.369	102.920	101.30	101.83
g_5	-	NOT CALCULATED		
λ^*	343.68	343.32	344.16	342.36
$\omega_1^2 = \omega_2^2$	1045.7	1054.0	1059.1	1059.3

[1] The final step was taken to lower λ^* from 344.16 to within 342.36 $\pm$ 0.5. Getting $\lambda^* = 343.36$ exactly was good fortune.

8. CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

If one has read the proceding chapters, he has by now formed his own conclusions. However, there are several general conclusions which should be noted and discussed. First of all, the optimal control techniques used in structural optimization are limited to systems which are easily described by a small number of first-order differential equations. This is not a method which one would apply to the design of a 747 or SST. On the other hand, the use of this technique does provide useful information about the behavior of simple systems such as one-dimensional panels. This information may be extrapolated to estimate the behavior of larger systems. As seen in Chapter 7, the knowledge of how the continuous system behaved proved very valuable in obtaining initial design parameters which satisfied the constraints.

The field of parameter optimization was only briefly discussed. This area obviously will be the most important and promising area of study for the next few years. It is important to recognize that many techniques such as Rubin's exist in the literature. They are well documented and are waiting for an enterprising researcher to adapt them to the many problems in the field of vibration and aircraft flutter. The adaptation of some of these methods to flutter problems is already underway at Stanford. The main difficulty with these parameter optimization problems is in calculating the constraint gradients such as $\frac{\partial \lambda^*}{\partial t_i}$. While this can be done by brute force methods for small problems, larger problems require more sophistication because of computer cost limitations.

There are several areas for future research that are of great interest to the author. They are listed below.

(1) For the panel flutter problem, add the aerodynamic damping term to the constraint equation. This is easily done for the finite element model.

(2) Add inplane stresses $(R_{xx} \neq 0)$ to the panel flutter equation. This may be easily done using the differential equation model.

(3) Extend the finite element technique to two-dimensional panel flutter by using triangular plate bending elements.

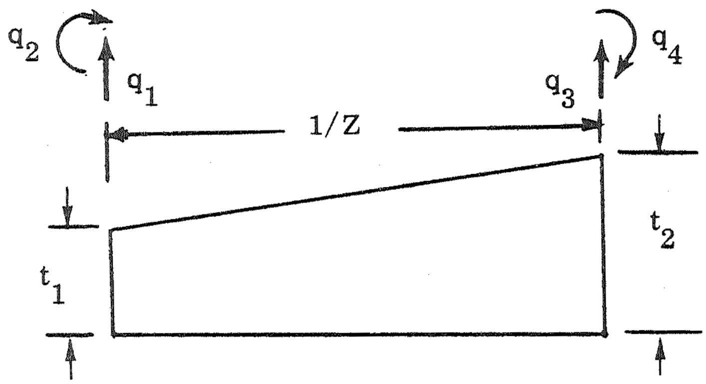
(4) Using additional theory covered in Bryson and Ho (Ref. 9), formulate and solve some simple beam problems having both static (maximum stress or deflection) and dynamic constraints.

(5) Use a series approximation for $t(x)$ to solve a bending-torsion-flutter weight-minimization problem such as covered in Ref. 22.

Some of these and other similar problems are under study by the author and others in the field. Their solution should prove not only interesting but helpful in the field of structural design.

APPENDIX

Elemental Mass, Stiffness, and Aerodynamic Matrices
Used in Finite Element Studies



$z = 1/Z$

Nondimensional Element

Mass and stiffness per unit length are proportional to the thickness which varies linearly with x.

Mass Matrix:

$$[m_{ij}] = \frac{zt_1}{420}\begin{bmatrix} 156 & 22z & 54 & -13z \\ & 4z^2 & 13z & -3z^2 \\ & & 156 & -22z \\ \text{(Symmetric)} & & & 4z^2 \end{bmatrix}$$

$$+\frac{(t_2 - t_1)z}{840}\begin{bmatrix} 72 & 14z & 54 & -12z \\ & 3z^2 & 14z & -3z^2 \\ & & 240 & -30z \\ \text{(Symmetric)} & & & 5z^2 \end{bmatrix}$$

Stiffness Matrix:

$$[k_{ij}] = \frac{2t_1}{z^3}\begin{bmatrix} 6 & 3z & -6 & 3z \\ & 2z^2 & -3z & z^2 \\ & & 6 & -3z \\ \text{(Symmetric)} & & & 2z^2 \end{bmatrix} + \frac{t_2 - t_1}{z^3}\begin{bmatrix} 6 & 2z & -6 & 4z \\ & z^2 & -2z & z^2 \\ & & 6 & -4z \\ \text{(Symmetric)} & & & 3z^2 \end{bmatrix}$$

Aerodynamic Matrix:

$$[a_{ij}] = \begin{bmatrix} \frac{1}{2} & \frac{z}{10} & \frac{1}{2} & -\frac{z}{10} \\ & 0 & \frac{z}{10} & -\frac{z^2}{60} \\ & & \frac{1}{2} & \frac{z}{10} \\ \text{(anti-symmetric)} & & & 0 \end{bmatrix}$$

Note that these elemental matrices are not the same as defined in Chapter 7. These present matrices must be used to generate the matrices discussed in Chapter 7.

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DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Stanford University Department of Aeronautics and Astronautics Stanford, California 94305	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

AN APPLICATION OF CONTROL THEORY METHODS TO THE OPTIMIZATION OF STRUCTURES HAVING DYNAMIC OR AEROELASTIC CONSTRAINTS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Interim

5. AUTHOR(S) (First name, middle initial, last name)

Terrence A Weisshaar

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
October 1970	173	22

8a. CONTRACT OR GRANT NO. F44620-68-C-0036	9a. ORIGINATOR'S REPORT NUMBER(S) SUDAAR NO. 412
b. PROJECT NO. 9781-02	
c. 61102F	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d. 681307	AFOSR 70-2862TR

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13. ABSTRACT A great deal of interest and attention has recently been focused on the optimal design of structures. By optimal design it is meant that a structure performs the same function as other similar structure while minimizing some performance index, usually the weight of the structure. This study investigates some simple structures whose weights are minimized subject to several types of constraints involving fixed eigenvalues. These eigenvalues may be related to free vibration, in which case a least weight structure is determined while holding one or more natural frequencies constant. Similarly, the eigenvalues may be related to aeroelastic instabilities where a least weight structure is found while holding the flutter speed constant. With one exception, the models are idealized one-dimensional structures with fixed geometry and spatial dimensions. These models are adequately described by a set of N simultaneous first-order ordinary differential equations which come from the general Nth order equilibrium equation. Methods adapted from optimal control theory are used to develop differential equations and boundary conditions which are necessary to ensure optimality. This optimization problem then becomes a two-point boudnary value problem with 2N simultaneous nonlinear differential equations. The solution method used to solve these equations is an adaption of a numerical technique used in optimal control theory which is referred to as the "transition matrix" procedure. This method involves perturbing the optimally equations and boundary conditions to find successive neighboring extremal solutions until the optimum design is reached. Solutions presented include optimum weight configurations for beams and thin-walled cylinders whose bending or torsional vibration frequencies are held fixed and which may or may not have minimum thickness constraints. The problem of finding an optimum panel whose aerodynamic

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimal design of structures						
Eigenvalues						

13. Abstract (continued)

flutter parameter in high supersonic flow is specified is also studied. Finally a simple study is presented which provides insight into the accuracy and usefulness of discrete or finite element methods when they are used to generate the structural model for an optimization s... search involving discrete parameters.

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