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ON DISPERSION AND CHARACTERISTIC MOTIONS OF TEMPERATURE-RATE DEPENDENT MATERIALS

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16. Abstract

A general three-dimensional theory of a thermomechanical material which can be a metallic or polymeric medium, or a structured composite, is developed using the modern techniques of axiomatic continuum mechanics and the laws of thermodynamics. One-dimensional linear spatial gradient temperature-rate dependent theories are presented for both thermoviscoelastic and thermoelastic materials. The characteristic motions are considered and it is shown that, due to the presence of temperature-rate effects, thermal propagation speeds have finite values. A comprehensive study of the dispersion relations is presented and illustrated graphically for typical values of the material constants. Analytical expressions are obtained for both high and low frequency responses. It is demonstrated that the characteristic speeds coincide with the high frequency asymptotic phase velocities in both cases. Physical and numerical limitations on the material constants are obtained for stable wave propagations. A class of self-similar solutions is obtained for the temperature-rate dependent thermoelastic medium using the theory of continuous group of transformations.

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This report was also a dissertation submitted by the first author in partial fulfillment of the requirement for the degree of Doctor of Philosophy in the Department of Engineering Mechanics, North Carolina State University, 1970.

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1. INTRODUCTION

Unsteady motion of rate dependent materials under high speed of loading is of fundamental theoretical and practical interest. Such a medium is inherently dissipative so that the waves propagating in the material are both attenuated and dispersed.

To study the phenomena of wave propagation, a thorough knowledge of the behavior of the material under investigation is essential. One of the drawbacks of the existing theories in thermoviscoelasticity and thermoelasticity is the prediction of infinite thermal speed of propagation. Various investigators[1] have attempted to modify the classical heat conduction law to alleviate this paradoxial result. Bogy and Naghdi (1969) considered a generalized axiomatic theory of heat conduction in rigid solids by allowing the constitutive relations to depend on the temperature-rate. In this treatise, a general nonlinear thermomechanical theory of a temperature-rate dependent thermoviscoelastic material is formulated using the modern techniques of axiomatic continuum mechanics and laws of thermodynamics. The formulation for a temperature-rate dependent thermoelastic medium is easily deduced from the general theory by neglecting certain strain-rate effects.

One-dimensional linear spatial gradient constitutive relations are presented to illustrate the basic concepts. Dispersion relations and asymptotic behaviors of the linearized longitudinal waves will be discussed, and the results illustrated graphically. It will also be shown that the temperature-rate dependent theories presented here predict finite speeds of propagation due to heat conduction.

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[1] See, *e.g.*, Chester (1963), Gurtin and Pipkin (1969), Horie (1970), Kaliski (1965), and Ulbrich (1961).

2. THEROMODYNAMIC FORMULATION

The fundamental equations of mechanics and thermodynamics[2], in Lagrangian form, are the continuity equation

$$\rho_0 = \rho J \quad , \tag{2.1}$$

the Kirchoff-Piola equation of motion

$$\rho_0 \, \dot{q}_i = \Sigma_{Ai,A} + \rho_0 \, F_i \quad , \tag{2.2}$$

equation of balance of energy

$$\rho_0 \, \dot{e} = q_{i,A} \, \Sigma_{Ai} - B_{A,A} + \rho_0 \, C + \rho_0 \, F_i \, q_i \quad , \tag{2.3}$$

and moment of momentum equation

$$y_{i,A} \, \Sigma_{Aj} = y_{j,A} \, \Sigma_{Ai} \quad , \tag{2.4}$$

where $\rho(Y_A,t)$ denotes the material density, $\rho_0 \overset{\Delta}{=} \rho(Y_A,t_0)$, $y_i(Y_A,t)$ is the deformation field, $J(Y_A,t) \overset{\Delta}{=} |y_{i,A}| > 0$, $q_i \overset{\Delta}{=} \dot{y}_i(Y_A,t)$ is the particle velocity, $\Sigma_{Ai}(Y_A,t)$ is the Kirchoff-Piola stress tensor, $F_i(Y_A,t)$ is the body force per unit mass, $e(Y_A,t)$ is the specific internal energy per unit mass, $B_A(Y_M,t)$ is the heat flux vector per unit original area due to conduction, C is the internal heat generation per unit mass per unit time, comma and superposed dot denote partial differential with respect to the reference coordinate system Y_A and time t, respectively, and t_0 is the original time of reference.

---

[2] See, *e.g.*, Green and Naghdi (1968).

We now postulate the local entropy inequality, deduced from the Clausius-Duhem inequality, in the following form:

$$\rho_0 \, \dot{s} \geq \frac{\rho_0 C}{T} - \left(\frac{B_A}{T}\right)_{,A} \tag{2.5}$$

where s is the specific entropy per unit mass. Using the conservation of energy (2.3), the inequality (2.5) may be re-written as

$$\rho_0 \, (\dot{e} - T \, \dot{s}) = \rho_0 \, (\dot{a} + \dot{T} \, s) \leq q_{i,A} \, \Sigma_{Ai} - \frac{1}{T} B_A \, T_{,A} \quad , \tag{2.6}$$

where

$$a = e - T \, s \tag{2.7}$$

is the specific Helmholtz free energy per unit mass.

To complete the thermodynamic formulation for a given material, specific knowledge of the constitutive relations characterizing the behavior of the medium is required. In the following section we will introduce such phenomenological relations for a temperature-rate dependent thermoviscoelastic material.

3. THE THREE-DIMENSIONAL TEMPERATURE-RATE DEPENDENT THERMOVISCOELASTIC CONSTITUTIVE RELATIONS

The thermoviscoelastic material considered in this treatise may be characterized by the response functions: Helmholtz free energy a, entropy s, internal energy e, heat flux vector B_A, and the Kirchoff-Piola stress tensor Σ_{Ai}. The response functions, in turn, are assumed to depend on the generalized thermodynamic variables: temperature T, temperature-rate $\dot{T}$, temperature gradient $T_{,A}$, deformation gradient $y_{i,A}$, and velocity gradient $q_{i,A}$. Therefore, we may write

$$\begin{aligned} a &= a\,(T, \dot{T}, T_{,A}, y_{i,A}, q_{i,A}) \quad , \\ s &= s\,(T, \dot{T}, T_{,A}, y_{i,A}, q_{i,A}) \quad , \\ e &= e\,(T, \dot{T}, T_{,A}, y_{i,A}, q_{i,A}) \quad , \\ B_A &= B_A(T, \dot{T}, T_{,B}, y_{i,B}, q_{i,B}) \quad , \\ \Sigma_{Ai} &= \Sigma_{Ai}(T, \dot{T}, T_{,B}, y_{j,B}, q_{j,B}) \quad , \end{aligned} \tag{3.1}$$

where we have made use of the principle of equipresence[3] which states that an independent variable present in one constitutive relation should appear in all unless it is excluded by the principles of continuum mechanics and laws of thermodynamics.

3.1. Consequences of the Second Law of Thermodynamics

For a thermoviscoelastic material whose constitutive relations are characterized by (3.1), the entropy inequality (2.6) becomes

---

[3] See, *e.g.*, Truesdell and Toupin (1960).

$$\rho_0 \left(\frac{\partial a}{\partial T} + s\right) \dot{T} + \left(\rho_0 \frac{\partial a}{\partial y_{i,A}} - \Sigma_{Ai}\right) q_{i,A} + \rho_0 \frac{\partial a}{\partial \dot{T}} \ddot{T}$$

$$+ \rho_0 \frac{\partial a}{\partial T_{,A}} \dot{T}_{,A} + \rho_0 \frac{\partial a}{\partial q_{i,A}} \dot{q}_{i,A} + \frac{1}{T} B_A T_{,A} \leq 0 \quad . \tag{3.2}$$

Following the procedure of Coleman and Noll (1963), we require this inequality to hold for all thermodynamically admissible processes and independent variations of $\ddot{T}$, $\dot{T}_{,A}$, and $\dot{q}_{i,A}$ which appear linearly with coefficients that are independent of these variables. Therefore,

$$\begin{aligned}
\frac{\partial a}{\partial \dot{T}} &= \frac{\partial}{\partial \dot{T}} \quad (e - T\,s) = 0 \quad , \\
\frac{\partial a}{\partial T_{,A}} &= \frac{\partial}{\partial T_{,A}} \quad (e - T\,s) = 0 \quad , \\
\frac{\partial a}{\partial q_{i,A}} &= \frac{\partial}{\partial q_{i,A}} (e - T\,s) = 0 \quad .
\end{aligned} \tag{3.3}$$

Hence, the Helmholtz free energy a is independent of $\dot{T}$, $T_{,A}$, and $q_{i,A}$ and the entropy inequality (3.2) reduces to

$$\rho_0 \left(\frac{\partial a}{\partial T} + s\right) \dot{T} + \left(\rho_0 \frac{\partial a}{\partial y_{i,A}} - \Sigma_{Ai}\right) q_{i,A} + \frac{1}{T} B_A T_{,A} \leq 0 \quad . \tag{3.4}$$

The constitutive relations (3.1) become, in view of (3.3),

$$\begin{aligned}
a &= a\,(T, y_{i,A}) \quad , \\
s &= s\,(T, \dot{T}, T_{,A}, y_{i,A}, q_{i,A}) \quad , \\
e &= e\,(T, \dot{T}, T_{,A}, y_{i,A}, q_{i,A}) \quad , \\
B_A &= B_A\,(T, \dot{T}, T_{,B}, y_{i,B}, q_{i,B}) \quad , \\
\Sigma_{Ai} &= \Sigma_{Ai}(T, \dot{T}, T_{,B}, y_{j,B}, q_{j,B}) \quad .
\end{aligned} \tag{3.5}$$

3.2. Invariance Requirements Under Superposed Rigid Body Motion

In this treatise, it will be assumed that the constitutive relations are form invariant with respect to a rigid body motion superposed on the spatial frame of reference.[4] If Q_{ij} denotes a time-dependent proper orthogonal transformation, then

$$y_i^* = Q_{ij}(t)\ y_j + p_i(t) \quad , \tag{3.6}$$

where y_i and y_i^* denote the spatial coordinates in the two reference frames, respectively, and p_i denotes the translation of the O-frame with respect to the O*-frame (Figure 3.1).

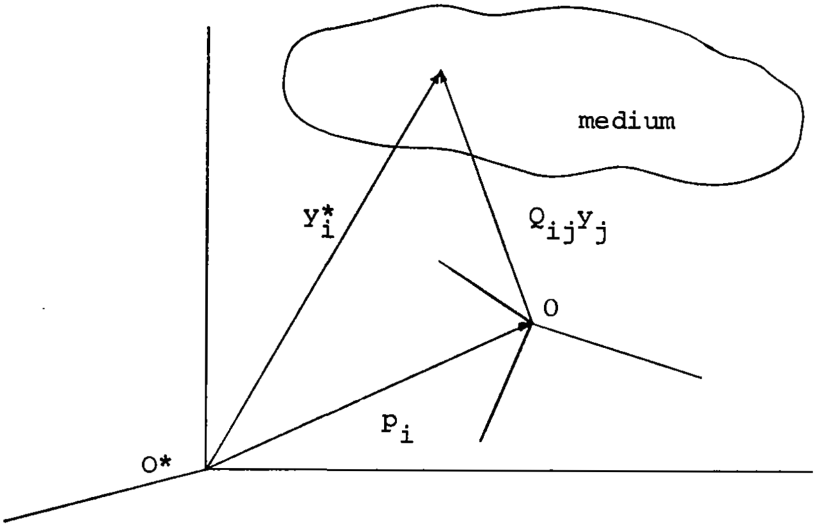

Figure 3.1. Change of the coordinate systems

By definition, the proper orthogonal tensor Q_{ij} satifies the following relations

---

[4] See, *e.g.*, Green and Rivlin (1957). It should be noted that this form of the invariance principle under superposed rigid body motion is slightly different from the so-called principle of frame indifference proposed by Noll (1955) who included inversion in the admissible orthogonal transformations.

$$Q_{ik}\,Q_{jk} = Q_{ki}\,Q_{kj} = \delta_{ij} \quad ,$$

$$\det Q_{ij} = 1 \quad . \tag{3.7}$$

Consequently,

$$\dot{Q}_{ik}\,Q_{jk} = -\,Q_{ik}\,\dot{Q}_{jk} \quad ,$$

$$\dot{Q}_{ki}\,Q_{kj} = -\,Q_{ki}\,\dot{Q}_{kj} \quad . \tag{3.8}$$

A quantity is said to be frame indifferent or objective if it is independent of the rigid body motion of the reference frame. For a scalar S, a vector V_i, and a second-order tensor T_{ij} in the O-frame, we must have in the O*-frame

$$S^* = S \quad , \quad V_i^* = Q_{ij}\,V_j \quad , \quad T_{ij}{}^* = Q_{im}\,Q_{jn}\,T_{mn} \quad . \tag{3.9}$$

Consider

$$y^*_{i,A} = Q_{ij}\,y_{j,A} \tag{3.10}$$

by (3.6), and form the following

$$\begin{aligned} y^*_{i,A}\,y^*_{i,B} &= Q_{im}\,y_{m,A}\,Q_{in}\,y_{n,B} \\ &= y_{m,A}\,y_{m,B} \end{aligned} \tag{3.11}$$

upon using (3.7). Thus, the so-called Cauchy-Green strain tensor G_{AB} defined by

$$G_{AB} \triangleq y_{i,A}\,y_{i,B} \tag{3.12}$$

is objective under (3.6). It is well known that for $J > 0$ any objective function which depends on $y_{i,A}$ can at most be a function of the six elements of G_{AB}. In a similar fashion we may demonstrate that $\dot{G}_{AB}$ is the objective quantity replacing $q_{i,A}$. Consider

$$q^*_{i,A} = \dot{Q}_{ij}\, y_{j,A} + Q_{ij}\, q_{j,A} \quad ,$$

and

$$q^*_{i,B} = \dot{Q}_{ij}\, y_{j,B} + Q_{ij}\, q_{j,B} \quad .$$

Multiplying the first equation by $y^*_{i,B}$ and the second one by $y^*_{i,A}$ and adding yields, upon using (3.10),

$$\begin{aligned} q^*_{i,A}\, y^*_{i,B} + y^*_{i,A}\, q^*_{i,B} &= Q_{ij}\, y_{j,B}\, (\dot{Q}_{im}\, y_{m,A} + Q_{im}\, q_{m,A}) + \\ &\quad + Q_{ij}\, y_{j,A}\, (\dot{Q}_{im}\, y_{m,B} + Q_{im}\, q_{m,B}) \\ &= Q_{ij}\, \dot{Q}_{im}\, y_{m,A}\, y_{j,B} + y_{i,B}\, q_{i,A} + \\ &\quad + Q_{ij}\, \dot{Q}_{im}\, y_{j,A}\, y_{m,B} + y_{i,A}\, q_{i,B} \quad , \end{aligned} \tag{3.13}$$

by employing (3.7). And finally, in view of (3.8), the right hand side of (3.13) is further simplified to give

$$q^*_{i,A}\, y^*_{i,B} + y^*_{i,A}\, q^*_{i,B} = q_{i,A}\, y_{i,B} + y_{i,A}\, q_{i,B} \quad , \tag{3.14}$$

which simply states

$$\dot{G}^*_{AB} = \dot{G}_{AB} \quad . \tag{3.15}$$

One may verify Equation (3.15) by direct differentiation of (3.12). In view of the restrictions imposed by the invariance principle of superposed rigid body motion, the constitutive relations (3.5) reduce to

$$
\begin{aligned}
a &= a\,(T,\ G_{AB}) \quad , \\
s &= s\,(T,\ \dot{T},\ T_{,A},\ G_{AB},\ \dot{G}_{AB}) \quad , \\
e &= e\,(T,\ \dot{T},\ T_{,A},\ G_{AB},\ \dot{G}_{AB}) \quad , \\
B_A &= B_A(T,\ \dot{T},\ T_{,A},\ G_{AB},\ \dot{G}_{AB}) \quad , \\
P_{AB} &= P_{AB}(T,\ \dot{T},\ T_{,A},\ G_{AB},\ \dot{G}_{AB}) \quad ,
\end{aligned}
\tag{3.16}
$$

where for convenience we have introduced the Piola stress tensor P_{AB} defined by the following relation

$$
\Sigma_{Ai} \overset{\Delta}{=} y_{i,B}\, P_{AB} \quad . \tag{3.17}
$$

3.3. Material Symmetry Restrictions

Solid-like materials may possess certain symmetry properties such that their constitutive response functions are form-invariant (in some reference frame) with respect to a time-independent group S which is a subgroup of the full orthogonal group of transformations Q. This imposes certain restrictions on the response functions. For example, the response functions $a(T,\ G_{AB})$, $s(T,\ \dot{T},\ T_{,A},\ G_{AB},\ \dot{G}_{AB})$ and $e(T,\ \dot{T},\ T_{,A},\ G_{AB},\ \dot{G}_{AB})$ should be scalar invariants under the symmetry group S. According to Wineman and Pipkin (1964), each of these scalar invariant functions can always be expressed explicitly as a single-valued scalar function of an irreducible integrity basis of its arguments under S. For an isotropic material, S is the full orthogonal group Q and the irreducible integrity bases[5] for each of the sets (T, G_{AB}) and $(T, \dot{T}, T_{,A}, G_{AB}, \dot{G}_{AB})$ under Q are:

$$T \quad , \quad G_{AB} \quad , \quad G_{AB}\,G_{BA} \quad , \quad G_{AB}\,G_{BC}\,G_{CA} \quad , \tag{3.18}$$

and

$$\begin{array}{llll}
T & , \; \dot{T} & , \; G_{AA} & , \\
G_{AB}\,G_{BA} & , \; G_{AB}\,G_{BC}\,G_{CA} & , \; T_{,A}\,T_{,A} & , \\
G_{AB}\,T_{,A}\,T_{,B} & , \; G_{AB}\,G_{BC}\,T_{,C}\,T_{,A} & , \; \dot{G}_{AA} & , \\
\dot{G}_{AB}\,\dot{G}_{BA} & , \; \dot{G}_{AB}\,\dot{G}_{BC}\,\dot{G}_{CA} & , \; \dot{G}_{AB}\,T_{,A}\,T_{,B} & , \\
\dot{G}_{AB}\,\dot{G}_{BC}\,T_{,C}\,T_{,A} & , \; G_{AB}\,\dot{G}_{BA} & , \; G_{AB}\,\dot{G}_{BC}\,\dot{G}_{CA} & , \\
\dot{G}_{AB}\,G_{BC}\,G_{CA} & , \; G_{AB}\,G_{BC}\,\dot{G}_{CD}\,\dot{G}_{DA} & , \; G_{AB}\,\dot{G}_{BC}\,T_{,C}\,T_{,A} & , \\
G_{AB}\,T_{,B}\,T_{,C}\,\dot{G}_{CD}\,\dot{G}_{DA} & , \; \dot{G}_{AB}\,T_{,B}\,T_{,C}\,G_{CD}\,G_{DA} & , \; G_{AB}\,G_{BC}\,\dot{G}_{CD}\,\dot{G}_{DE}\,T_{,E}\,T_{,A} & ,
\end{array} \tag{3.19}$$

respectively.

The canonical form of the heat flux vector $B_A(T, \dot{T}, T_{,M}, G_{MN}, \dot{G}_{MN})$ and Piola stress tensor $P_{AB}(T, \dot{T}, T_{,M}, G_{MN}\ \dot{G}_{MN})$ may also be obtained using the procedure suggested by Wineman and Pipkin (1964). They have shown that a tensor-valued response function of arbitrary rank, depending on an arbitrary number of tensor variables of arbitrary ranks, can be expressed as a linear combination of the basic form-invariants under S.

---

[5] The minimum isotropic integrity basis for an arbitrary number of three-dimensional second-order symmetric and skew-symmetric tensors, and axial and absolute vectors, under the full orthogonal group, are given by Smith (1965).

We shall adapt, however, the procedure of Rivlin (1959) in imposing material isotropy on the forms of $B_A(T, \dot{T}, T_{,M}, G_{MN}, \dot{G}_{MN})$ and $P_{AB}(T, \dot{T}, T_{,M}, G_{MN}, \dot{G}_{MN})$.

Consider an arbitrary, second-order, symmetric tensor Ψ_{AB} and define a scalar quantity Ψ by

$$\Psi \triangleq \Psi_{AB} P_{AB} \quad . \tag{3.20}$$

According to Wineman and Pipkin (1964), the scalar function Ψ can be expressed by

$$\Psi = \sum_{\beta=1}^{N} F_\beta G_\beta \quad , \tag{3.21}$$

where G_β are the elements (linear in Ψ_{AB}) of the irreducible integrity basis of T, $\dot{T}$, $T_{,A}$, G_{AB}, $\dot{G}_{AB}$, Ψ_{AB} under Q, and F_β are single-valued functions of the irreducible integrity basis of T, $\dot{T}$, $T_{,A}$, G_{AB}, $\dot{G}_{AB}$ under Q. Therefore,

$$P_{AB} = \frac{1}{2} \sum_{\beta=1}^{N} F_\beta \left(\frac{\partial G_\beta}{\partial \Psi_{AB}} + \frac{\partial G_\beta}{\partial \Psi_{BA}}\right) \quad . \tag{3.22}$$

One may also obtain the constitutive relation for the heat flux vector $B_A(T, \dot{T}, T_{,M}, G_{MN}\ \dot{G}_{MN})$ by forming the scalar product of B_A with an arbitrary vector and then follow the above procedure in a similar manner.

In the subsequent sections we make use of the polynomial canonical form of the constitutive relations when we consider the one-dimensional linear spatial gradient theory.

4. THE ONE-DIMENSIONAL, LINEAR SPATIAL GRADIENT, TEMPERATURE-RATE DEPENDENT, THERMOVISCOELASTICITY

Since we are primarily interested in small amplitude, longitudinal wave propagations, we consider, in this section, a one-dimensional linear spatial gradient theory of the thermoviscoelastic meterial formulated in the previous section.

4.1. Linear Gradient Assumption

To the first order approximation in the spatial gradient quantities, the one-dimensional polynomial canonical representation of the constitutive relations (3.16), using (3.17), reduce to

$$\rho_0\, e(T,\, \dot{T},\, T_X,\, G_{11},\, \dot{G}_{11}) = e_0(T,\, \dot{T}) + e_1(T,\, \dot{T})\, \varepsilon + e_2(T,\, \dot{T})\, \dot{\varepsilon} \quad , \qquad (4.1)$$

$$B(T,\, \dot{T},\, T_X,\, G_{11},\, \dot{G}_{11}) = -\, \kappa(T,\, \dot{T})\, T_X \quad , \qquad (4.2)$$

$$\Sigma(T,\, \dot{T},\, T_X,\, G_{11},\, \dot{G}_{11}) = -\, \pi(T,\, \dot{T}) + \mu(T,\, \dot{T})\, \varepsilon + \eta(T,\, \dot{T})\, \dot{\varepsilon} \quad , \qquad (4.3)$$

where

$$\varepsilon \triangleq \frac{\partial y_1}{\partial Y_1} - 1 = \sqrt{G_{11}} - 1 \triangleq \frac{\partial x}{\partial X} - 1 \quad ,$$

and (4.4)

$$\dot{\varepsilon} \triangleq \frac{\partial q_1}{\partial Y_1} = \frac{1}{2}\, \frac{\dot{G}_{11}}{\sqrt{G_{11}}} \triangleq \frac{\partial q}{\partial X}$$

are the Lagrangian strain and strain rate, respectively, the coordinate systems x and X correspond to y_1 and Y_1, respectively, the subscript X denotes partial differentiation with respect to the spatial coordinate, B denotes the heat flux vector in the X-direction, and Σ is the longitudinal stress.

One may show that π and μ are independent of $\dot{T}$. Let us substitute (4.2) through (4.3) in the one-dimensional form of the entropy inequality (3.4):

$$\left(- \pi + \mu \varepsilon - \rho_0 \frac{\partial a}{\partial \varepsilon}\right) \dot{\varepsilon} - \rho_0 \left(s + \frac{\partial a}{\partial T}\right) \dot{T} + \eta \dot{\varepsilon}^2 + \frac{1}{T} \kappa T_X^2 \geq 0 \quad . \qquad (4.5)$$

This inequality must hold for all thermodynamically admissible processes and independent variations of $\dot{\varepsilon}$, $\dot{T}$, and T_X. Therefore, we conclude that

$$- \pi + \mu \varepsilon - \rho_0 \frac{\partial a}{\partial \varepsilon} = 0 \quad , \qquad (4.6)$$

and

$$\eta \geq 0 \quad , \quad \kappa \geq 0 \quad . \qquad (4.7)$$

But a is independent of $\dot{T}$ and thus π and μ must also be independent of $\dot{T}$. The entropy inequality (4.5), in view of (4.6), reduces to

$$\eta \dot{\varepsilon}^2 + \frac{1}{T} \kappa T_X^2 - \rho_0 \left(s + \frac{\partial a}{\partial T}\right) \dot{T} \geq 0 \quad . \qquad (4.8)$$

Equation (4.6), upon integration with respect to ε and to the first order, yields the following

$$\rho_0 a = \psi(T) - \pi(T) \varepsilon \quad , \qquad (4.9)$$

where ψ is the constant of integration. Using Equations (4.1) and (4.9) in (2.7), we get

$$\rho_0\, s = \frac{1}{T}\rho_0 (e - a)$$

$$= \frac{1}{T}\{e_0(T, \dot{T}) - \psi(T) + [e_1(T, \dot{T}) + \pi(T)]\,\varepsilon + e_2(T, \dot{T})\,\dot{\varepsilon}\}\ . \quad (4.10)$$

The set (4.1) through (4.3) and (4.8) through (4.10) represents the thermodynamic formulation of the one-dimensional temperature-rate dependent thermoviscoelastic problem under consideration.

4.2. Basic Equations

The equations of motion and balance of energy (2.2) through (2.3) in the one dimensional form and in the absence of internal heat generation and body forces are,

$$\begin{aligned} \rho_0\, \dot{q} &= \Sigma_X \quad , \\ \rho_0\, \dot{e} &= q_X \Sigma - B_X \quad . \end{aligned} \qquad (4.11)$$

Substituting (4.1) through (4.3) into (4.11) gives, to the first order of approximation,

$$\begin{aligned} \rho_0\, u_{tt} - \mu\, u_{XX} - \eta\, u_{XXt} + \pi_T\, T_X &= 0 \quad , \\ (e_1 + \pi)\, u_{Xt} + e_2\, u_{Xtt} + e_{0T}\, T_t + e_{0\dot{T}}\, T_{tt} - \kappa\, T_{XX} &= 0 \quad , \end{aligned} \qquad (4.12)$$

where u is the longitudinal displacement and subscripts denote partial differentiation.

4.3. Characteristic Motions

To obtain the characteristic wave speeds, consider a discontinuity, represented by the curve S, propagating with the speed c in the Xt-plane

(Figure 4.1), and let $\Phi(X, t)$ denote any field variable having the values Φ_+ and Φ_- across the curve S with $\Phi_+ = \Phi_-$ whenever Φ is continuous.

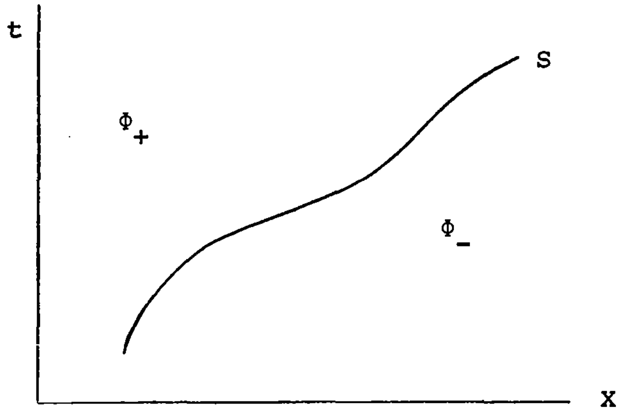

Figure 4.1. Discontinuous front

We further assume that although Φ may be discontinuous across S, the time rate of change Φ_+ (or Φ_-) along S may be evaluated according to Hadamard's lemma[6] as follows

$$\frac{\delta\Phi_\pm}{\delta t} = \frac{\partial\Phi_\pm}{\partial t} + c\,\frac{\partial\Phi_\pm}{\partial X} \quad . \tag{4.13}$$

Therefore,

$$\left[\frac{\delta\Phi}{\delta t}\right] = \left[\frac{\partial\Phi}{\partial t}\right] + c\,\left[\frac{\partial\Phi}{\partial X}\right] \quad , \tag{4.14}$$

where [] denotes the jump across S

$$[\Phi] \triangleq \Phi_+ - \Phi_- \quad .$$

---

[6] See, *e.g.*, Truesdell and Toupin (1960).

Let the temperature T and its first order derivatives, and the displacement u along with its first and second order derivatives, be continuous across S, *i.e.*,

$$
\begin{aligned}
&[T] = [u] = 0 \quad , \\
&[T_t] = [T_X] = [u_t] = [u_X] = 0 \quad , \\
\text{and} \\
&[u_{tt}] = [u_{tX}] = [u_{XX}] = 0 \quad .
\end{aligned}
\tag{4.15}
$$

Applying (4.14) to (4.15) gives

$$[T_{tt}] + c \; [T_{Xt}] = 0 \quad , \tag{4.16}$$

$$[T_{Xt}] + c \; [T_{XX}] = 0 \quad , \tag{4.17}$$

$$[u_{ttt}] + c \, [u_{ttX}] = 0 \quad , \tag{4.18}$$

$$[u_{tXt}] + c \, [u_{tXX}] = 0 \quad , \tag{4.19}$$

$$[u_{XXt}] + c \, [u_{XXX}] = 0 \quad , \tag{4.20}$$

Using (4.15), Equations (4.12) across S gives:

$$- \eta \, [u_{XXt}] = 0 \quad , \tag{4.21}$$

$$e_{0\dot{T}} \, [T_{tt}] - \kappa \, [T_{XX}] + e_2 \, [u_{Xtt}] = 0 \quad . \tag{4.22}$$

Therefore all the third-order derivatives of the displacement, in light of Equations (4.18) through (4.21), are continuous across S. Equations (4.16) through (4.17) and (4.22) may be re-written as:

$$
\begin{aligned}
&[T_{tt}] - c^2\ [T_{XX}] = 0 \quad , \\
&e_{0\dot{T}}\ [T_{tt}] - \kappa\ \ [T_{XX}] = 0 \quad .
\end{aligned}
\tag{4.23}
$$

For nontrivial solutions, the determinant of the coefficients must vanish. Therefore,

$$e_{0\dot{T}}\ c^2 - \kappa = 0 \quad ,$$

or

$$c = (\kappa / e_{0\dot{T}})^{1/2} \triangleq c_2 \quad . \tag{4.24}$$

Expression (4.24) represents the characteristic speed of wave propagation in the temperature-rate dependent thermoviscoelastic material under consideration. Furthermore, this characteristic speed becomes infinite as $e_{0\dot{T}}$ approaches zero, a well-known classical heat conduction result. Later on, we will also show that the characteristic speed c_2 corresponds to the asymptotic phase velocity at high frequency.

4.4. Dispersion Relations of Linearized Longitudinal Wave Propagation in an Initially Unstrained Thermoviscoelastic Material

In this section, we will employ the small perturbation technique to linearize the basic equations (4.12). Consider small perturbations T' and u' about some uniform equilibrium state $\bar{T}$ and $\bar{u}$ of the material such that

$$
\begin{aligned}
T &= \bar{T} + T' \quad , \\
u &= \bar{u} + u' \quad .
\end{aligned}
\tag{4.25}
$$

Using (4.25) in the equations of motion and balance of energy (4.12), it can be shown to the first order of T' and u':

$$\bar{\pi}_T \, T'_X + \rho_0 \, u'_{tt} - \bar{\mu} \, u'_{XX} - \bar{\eta} \, u'_{XXt} = 0 \quad ,$$

$$\bar{e}_{0T} \, T'_t + \bar{e}_{0\dot{T}} \, T'_{tt} - \bar{\kappa} \, T'_{XX} + (\bar{e}_1 + \bar{\pi}) \, u'_{Xt} + \bar{e}_2 \, u'_{Xtt} = 0 \quad , \tag{4.26}$$

where "bar" denotes quantities evaluated at the equilibrium state.

The set of linear partial differential equations (4.26), with appropriate initial and boundary conditions, may be solved using standard transform techniques or Fourier analysis. Instead of solving (4.26) for specific values of initial and boundary conditions, we will derive the dispersion relations and conditions under which stable waves may exist and propagate in the positive X-direction.

Consider longitudinal propagation of planar disturbances of the form

$$T' = T_0 \exp \, (i\omega t - kX) \quad ,$$

$$u' = u_0 \exp \, (i\omega t - kX) \quad , \tag{4.27}$$

where ω is the frequency, k is the complex wave number, and T_0 and u_0 are complex amplitudes. Substituting (4.27) into (4.26) yields

$$- \bar{\pi} \, k \, T_0 + \{\rho_0 \, (\omega i)^2 - \bar{\mu} \, k^2 - \bar{\eta} \, k^2 \, \omega i\} \, u_0 = 0 \quad ,$$

$$\{(\bar{e}_{0T} + \bar{e}_{0\dot{T}} \, \omega i) \, \omega i - \bar{\kappa} \, k^2\} \, T_0 - (\bar{e}_1 + \bar{\pi} + \bar{e}_2 \, \omega i) \, k \, \omega i \, u_0 = 0 \quad . \tag{4.28}$$

For non-trivial solutions, the determinant of the coefficients must vanish:

$$\begin{vmatrix} \bar{\pi}_T\, k & (\bar{\mu} + \bar{\eta}\, \omega i)\, k^2 - \rho_0 (\omega i)^2 \\ (\bar{e}_{0T} + \bar{e}_{0\dot{T}}\, \omega i)\, \omega i - \bar{\kappa}\, k^2 & -(\bar{e}_1 + \bar{\pi} + \bar{e}_2\, \omega i)\, k\, \omega i \end{vmatrix} = 0 \quad ,$$

which yields

$$\bar{\kappa}\, (\bar{\mu} + \bar{\eta}\, \omega i)\, k^4 - \{\rho_0\, \bar{\kappa}\, \omega i + (\bar{\mu} + \bar{\eta}\, \omega i)(\bar{e}_{0T} + \bar{e}_{0\dot{T}}\, \omega i) + + \bar{\pi}_T\, (\bar{e}_1 + \bar{\pi} + \bar{e}_2\, \omega i)\}\, k^2\, \omega i + \rho_0\, (\bar{e}_{0T} + \bar{e}_{0\dot{T}}\, \omega i)(\omega i)^3 = 0 \quad . \tag{4.29}$$

Considering a real frequency ω of propagation in the positive X-direction, the complex wave number k may be expressed as follows:

$$k = \alpha + \frac{\omega}{c}\, i \quad , \tag{4.30}$$

where the attenuation factor α and the phase velocity c are functions of ω. These are the dispersion relations. For a stable wave propagation both α and c must, however, be positive.

Equation (4.29) is satisfied if

$$(1 + Fi)\, H^4 - [\phi (Fi)^2 + (1 + \zeta + \phi + \chi)\, Fi + \gamma + \zeta]\, Fi\, H^2 + + (\zeta + \phi\, Fi)(Fi)^3 = 0 \quad , \tag{4.31}$$

where

$$F \triangleq \frac{\omega}{\bar{\mu}/\bar{\eta}} \quad , \quad H \triangleq \frac{(\bar{\mu}/\rho_0)^{1/2}}{\bar{\mu}/\bar{\eta}}\, k \tag{4.32}$$

are the dimensionless frequency and complex wave number; and,

$$\gamma \triangleq \frac{\bar{\eta}\ \bar{\pi}_T(\bar{e}_1 + \bar{\pi})}{\rho_0\ \bar{\mu}\ \bar{\kappa}} \quad , \tag{4.33}$$

$$\zeta \triangleq \frac{\bar{\eta}\ \bar{e}_{0T}}{\rho_0\ \bar{\kappa}} \quad , \tag{4.34}$$

$$\phi \triangleq \frac{\bar{\mu}\ \bar{e}_{0\dot{T}}}{\rho_0\ \bar{\kappa}} \quad , \tag{4.35}$$

$$\chi \triangleq \frac{\bar{\pi}_T\ \bar{e}_2}{\rho_0\ \bar{\kappa}} \tag{4.36}$$

are dimensionless material constants. Equation (4.31) yields

$$\begin{aligned} H_{\pm} = \{ & \frac{Fi}{2(1 + Fi)} [\phi(Fi)^2 + (1 + \zeta + \phi + \chi)\ Fi + \gamma + \zeta \pm \\ & \pm \sqrt{[\phi(Fi)^2 + (-1 + \zeta + \phi + \chi)\ Fi + \gamma + \zeta]^2 + 4\gamma\ Fi + 4\chi(Fi)^2}\,]\}^{1/2} \quad . \end{aligned} \tag*{$(4.37)_{\pm}$}$$

The dimensionless form of (4.30) is

$$H = A + \frac{F}{V}\, i \quad , \tag{4.38}$$

where A and V are the non-dimensional attenuation factor and phase velocity, respectively. Upon separation of the real and imaginary parts of the Equations $(4.37)_{\pm}$ one may obtain

$$A_{\pm} \triangleq \frac{(\bar{\mu}/\rho_0)^{1/2}}{\bar{\mu}/\bar{\eta}}\ \alpha_{\pm} = \mathrm{Re}\ (H_{\pm}) \quad , \tag*{$(4.39)_{\pm}$}$$

$$V_\pm \overset{\Delta}{=} \frac{c_\pm}{(\bar{\mu}/\rho_0)^{1/2}} = F/\mathrm{Im}\,(H_\pm) \quad , \qquad (4.40)_\pm$$

where only propagation in the positive X-direction has been considered, (±) corresponds to the sign preceding the radical in $(4.37)_\pm$, and $H_\pm$ is given by $(4.37)_\pm$.

4.4.1. Asymptotic Expansions

Attenuation factor A and phase velocity V must be positive at all real frequencies for a stable propagation in the positive X-direction. To fulfill this requirement, certain restrictions can be imposed on the dimensionless material constants γ, ζ, ϕ, and χ by studying the asymptotic behaviors of A and V.

(1) In the case of low frequency waves, when $F \ll 1$, Equations $(4.37)_\pm$ can be approximated as follows:

$$H_\pm \simeq \{\frac{Fi}{2}\,[\gamma + \zeta + (1 + \zeta + \phi + \chi)\,Fi \pm (\gamma + \zeta) \pm$$
$$\pm \left(-1 + \zeta + \phi + \chi + \frac{2\gamma}{\gamma + \zeta}\right) Fi + \ldots]\}^{1/2} \quad .$$

Hence, in view of $(4.39)_\pm$ and $(4.40)_\pm$,

$$A_+ \simeq \left(\frac{\gamma + \zeta}{2}\,F\right)^{1/2} + O(F^{3/2}) \quad , \qquad (4.41)$$

$$V_+ \simeq \left(\frac{2F}{\gamma + \zeta}\right)^{1/2} + O(F^{3/2}) \quad , \qquad (4.42)$$

provided

$$\gamma + \zeta > 0 \quad , \tag{4.43}$$

and

$$A_- \simeq 0 + O(F^2) \quad , \tag{4.44}$$

$$V_- \simeq \left(1 + \frac{\gamma}{\zeta}\right)^{1/2} + O(F^2) \quad , \tag{4.45}$$

provided

$$1 + \frac{\gamma}{\zeta} \quad 0 \quad . \tag{4.46}$$

Combining (4.43) and (4.46), it follows that

$$\zeta > 0 \quad . \tag{4.47}$$

In view of the definitions (4.32) through (4.36) and (4.39)-(4.40), the dimensional form of the expressions (4.41)-(4.42) and (4.44)-(4.45) are

$$\alpha_+ \simeq \left[\frac{\bar{\mu}\, \bar{e}_{0T} + \bar{\pi}_T(\bar{e}_1 + \bar{\pi})}{2\, \bar{\mu}\, \bar{\kappa}}\, \omega\right]^{1/2} + O(\omega^{3/2}) \quad , \tag{4.48}$$

$$c_+ \simeq \left[\frac{2\, \bar{\mu}\, \bar{\kappa}}{\bar{\mu}\, \bar{e}_{0T} + \bar{\pi}_T(\bar{e}_1 + \bar{\pi})}\, \omega\right]^{1/2} + O(\omega^{3/2}) \quad , \tag{4.49}$$

$$\alpha_- \simeq 0 + O(\omega^2) \quad , \tag{4.50}$$

and

$$c_- \simeq \left[\frac{\bar{\mu}}{\rho_0} + \frac{\bar{\pi}_T(\bar{e}_1 + \bar{\pi})}{\rho_0\, \bar{e}_{0T}}\right]^{1/2} + O(\omega^2) \quad . \qquad (4.51)$$

We note that expressions (4.48) through (4.51) are independent of the material constants $\bar{\eta}$, $\bar{e}_{0\dot{T}}$, and $\bar{e}_2$. This is in accordance with the physical intuition as one may expect that the effects of the rate of strain and temperature are negligible at low frequency oscillation.

We also observe that c_- given by Equation (4.51) is independent of the frequency ω (asymptotic value) and may be expressed as follows:

$$c_-^2 = \frac{\bar{\mu}}{\rho_0} + \frac{\bar{\pi}_T(\bar{e}_1 + \bar{\pi})}{\rho_0\, \bar{e}_{0T}}$$

$$\triangleq c_1^2 + c_3^2 \quad ,$$

where c_1 is the elastic wave speed $(\bar{\mu}/\rho_0)^{1/2}$ and c_3 denotes the wave speed due to the dependence on temperature of the constitutive relations. This analysis suggests that the low frequency asymptotic wave speed in a thermoviscoelastic material is always greater than that of the elastic one.

(2) In the case of high frequency waves, when F >> 1, then Equations $(4.37)_\pm$ can be approximated as follows:

$$H_\pm \simeq \{\tfrac{1}{2}\,[\phi(Fi)^2 + (1 + \zeta + \phi + \chi)\ Fi \pm \phi(Fi)^2 \pm$$

$$\pm (-1 + \zeta + \phi + \chi)\ Fi + \ldots][1 - \frac{1}{Fi} + \ldots]\}^{1/2} \quad .$$

Hence, in view of $(4.39)_{\pm}$ - $(4.40)_{\pm}$,

$$A_{+} \simeq \frac{\zeta + \chi}{2\sqrt{\phi}} + O(F^{-1}) \quad , \tag{4.52}$$

$$V_{+} \simeq \frac{1}{\sqrt{\phi}} + O(F^{-1}) \quad , \tag{4.53}$$

provided

$$\zeta + \chi \geq 0 \quad \text{and} \quad \phi > 0 \quad , \tag{4.54}$$

and

$$A_{-} \simeq (F/2)^{1/2} + O(F^{-1/2}) \quad , \tag{4.55}$$

$$V_{-} \simeq (2F)^{1/2} + O(F^{-1/2}) \quad . \tag{4.56}$$

The dimensional form of (4.52) - (4.53) and (4.55)-(4.56) may be obtained by employing definitions (4.32) - (4.36) and (4.39) - (4.40),

$$\alpha_{+} \simeq \frac{\bar{\pi}_T \bar{e}_2 + \bar{\eta} \bar{e}_{0T}}{2\ \eta \sqrt{\bar{\kappa}\ \bar{e}_{0\dot{T}}}} + O(\omega^{-1}) \quad , \tag{4.57}$$

$$c_{+} \simeq \left(\bar{\kappa}/\bar{e}_{0\dot{T}}\right)^{1/2} + O(\omega^{-1}) \quad , \tag{4.58}$$

$$\alpha_{-} \simeq \left(\frac{\rho_0}{2\ \bar{\eta}}\, \omega\right)^{1/2} + O(\omega^{-1/2}) \quad , \tag{4.59}$$

$$c_{-} \simeq \left(\frac{2\ \bar{\eta}}{\rho_0}\, \omega\right)^{1/2} + O(\omega^{-1/2}) \quad . \tag{4.60}$$

It is clear that expressions (4.57) through (4.60) depend on $\bar{\eta}$, $\bar{e}_2$, and $\bar{e}_{0\dot{T}}$, which are the coefficients of the strain-rate and temperature-rate. They are, however, independent of the elastic modulus $\bar{\mu}$. Expression (4.58), in light of (4.7), yields that

$$\bar{e}_{0\dot{T}} > 0 \quad . \tag{4.61}$$

We note that c_+ given by (4.58) is due to the temperature-rate effects and is identical to the characteristic speed (4.24) obtained earlier. Furthermore, the classical result of infinite wave speed due to heat conduction may be deduced by simply setting $\bar{e}_{0\dot{T}}$ equal to zero.

One may also observe that although c_- given by (4.51) is an asymptotic value, it is not a characteristic speed of the wave propagation.

In carrying out these asymptotic expansions we treated H as a real function of the variable (Fi) such that formally the expression for H(Fi) contains only real coefficients.

4.4.2. Possible Physical Limitations

There might be certain physical restrictions on the material constants. This may be best accomplished by drawing an analogy between the thermoviscoelastic medium considered here and those of the classical thermodynamics.

If one chooses Σ to correspond to the thermodynamic pressure p, then

$$\pi_T = -\left(\frac{\partial \Sigma}{\partial T}\right)_\varepsilon \sim \left(\frac{\partial p}{\partial T}\right)_\rho \quad ,$$

and comparison with a thermally perfect material; *e.g.*, for an ideal gas $\left(\frac{\partial p}{\partial T}\right)_\rho = \rho R$ -- indicates that $\pi_T > 0$. Also from thermodynamics

$$\left(\frac{\partial e}{\partial \nu}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_\nu - p \qquad \text{for any substance} \quad ,$$

$$= 0 \qquad \text{for an ideal gas} \quad ,$$

$$= \text{positive constant} \qquad \text{for van der Waals gas} \quad .$$

Therefore, using (4.1) and one dimensional form of the continuity equation (2.1), one obtains, applying the above argument,

$$\left(\frac{\partial e}{\partial \nu}\right)_T = \rho_0\left(\frac{\partial e}{\partial \varepsilon}\right)_T = e_1 \geq 0 \quad . \tag{4.62}$$

Now since $\pi \geq 0$, $\pi_T > 0$, $\kappa \geq 0$, and $e_1 \geq 0$, the definition (4.33) yields

$$\gamma \geq 0 \quad . \tag{4.63}$$

The specific heat at constant volume is defined by

$$c_\nu \triangleq \left(\frac{\partial e}{\partial T}\right)_\nu \quad ,$$

and is a positive quantity for most materials ($c_\nu > R$ for gases). Therefore, the constitutive equation (4.1) yields, to the zeroth order of the gradients,

$$\rho_0\left(\frac{\partial e}{\partial T}\right) = e_{0T} > 0 \quad . \tag{4.64}$$

Employing (4.64) in the definition of ζ given by (4.34), we get

$$\zeta > 0 \quad . \tag{4.65}$$

Incorporating the results (4.43), (4.46), (4.47), (4.54), (4.63), and (4.65) into one set, we obtain that

$$\begin{aligned} \gamma &\geq 0 \quad , \\ \zeta &> 0 \quad , \\ \phi &> 0 \quad , \end{aligned} \tag{4.66}$$

and

$$\zeta + \chi \geq 0$$

for a stable wave propagation.

These restrictions were used in numerical computations aimed at obtaining graphical illustrations of the dispersion relations.

4.4.3. Numerical Results and Graphs

Logarithmic values of the non-dimensional attentuation factor A and the non-dimensional phase velocity V were cross-plotted against the dimensionless frequency F (Figures 4.2 through 4.17). The following behaviors were observed:

(1) For the case of $F < 10^{-3}$:

(a) Both attentuation factors A_+ and A_-, and the phase velocity V_+ are directly proportional to the square root of the frequency F. The phase velocity V_-, however, is independent of the frequency F.

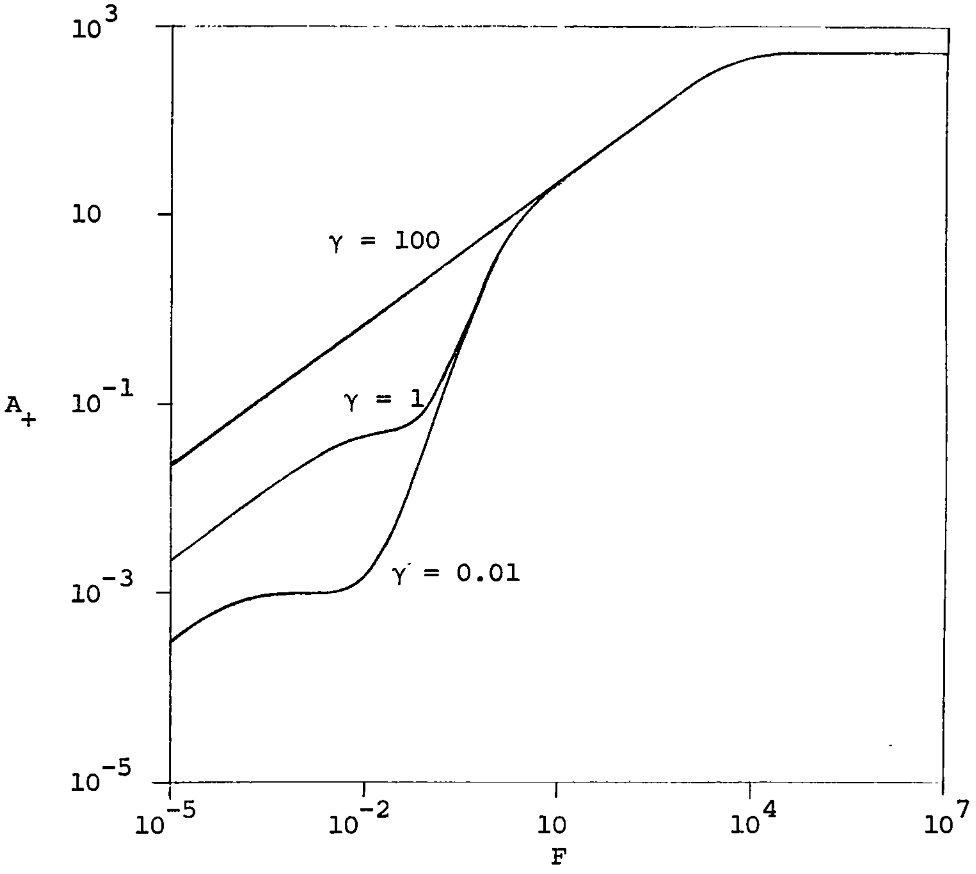

Figure 4.2. Effect of material constant γ on attenuation factor A_+ of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\zeta = \phi = 0.01$, $\chi = 100$

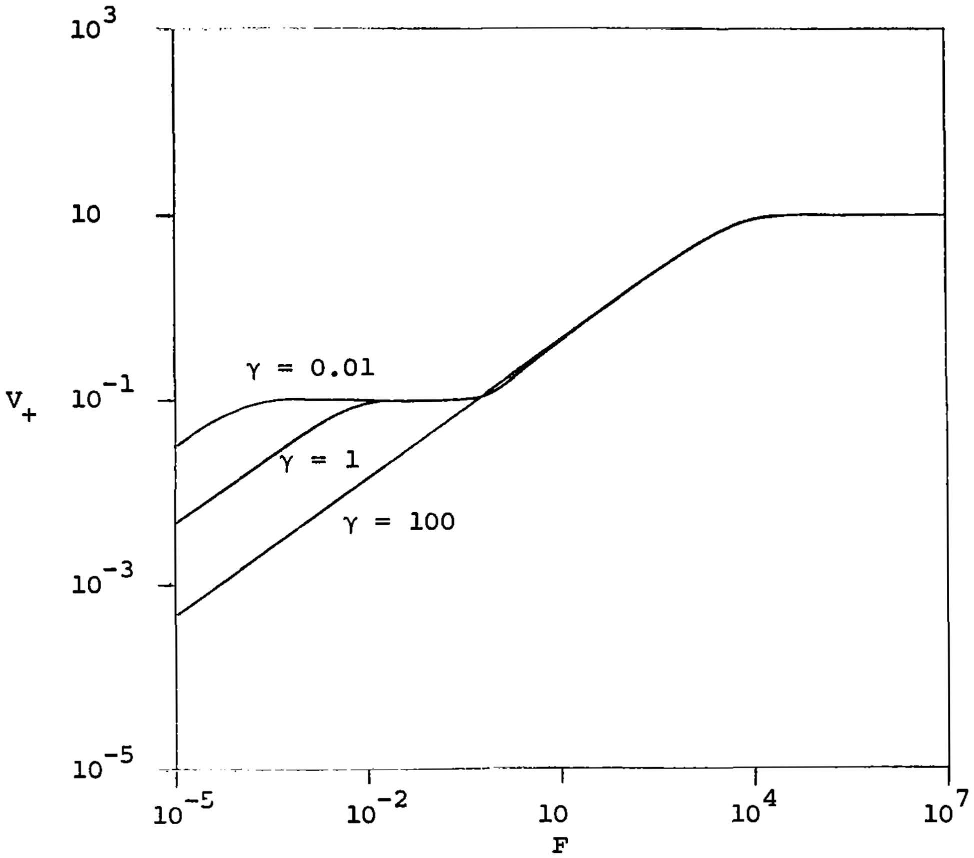

Figure 4.3. Effect of material constant γ on dispersion of phase velocity V_+ of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\zeta = \phi = 0.01$, $\chi = 100$

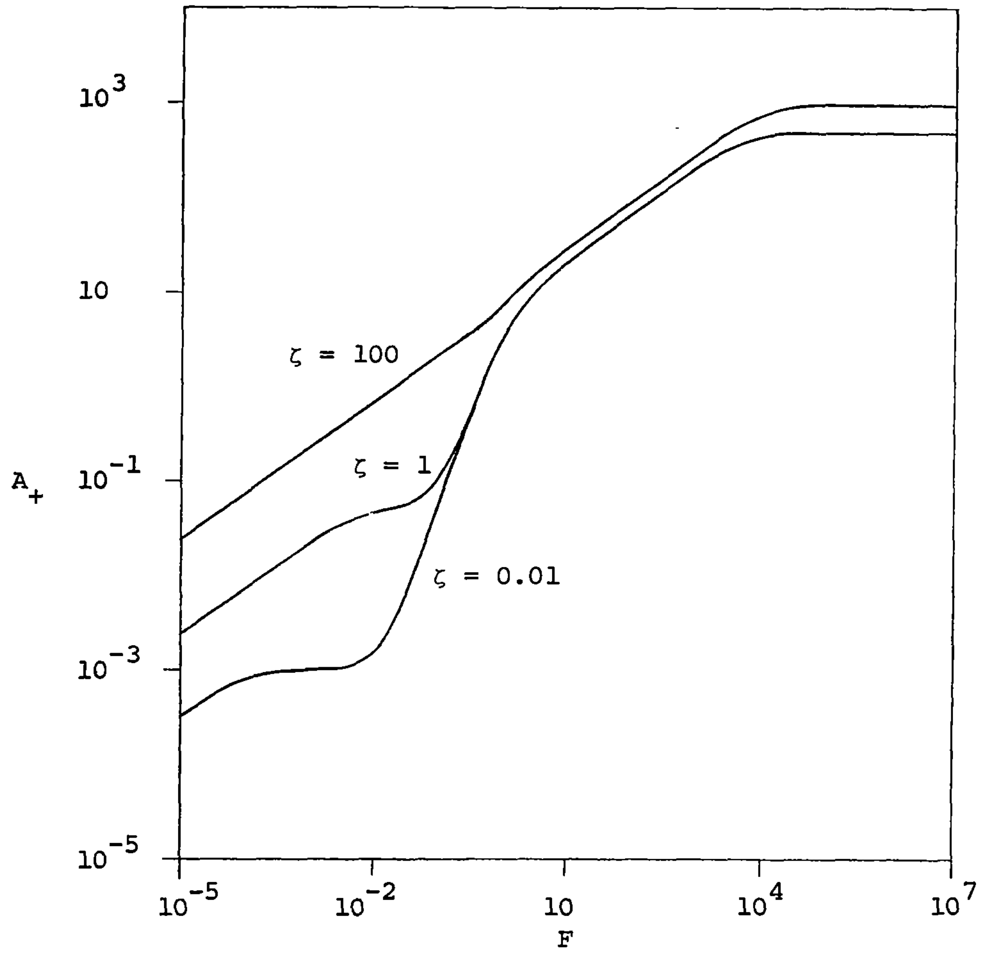

Figure 4.4. Effect of material constant ζ on attenuation factor A_+ of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \phi = 0.01$, $\chi = 100$

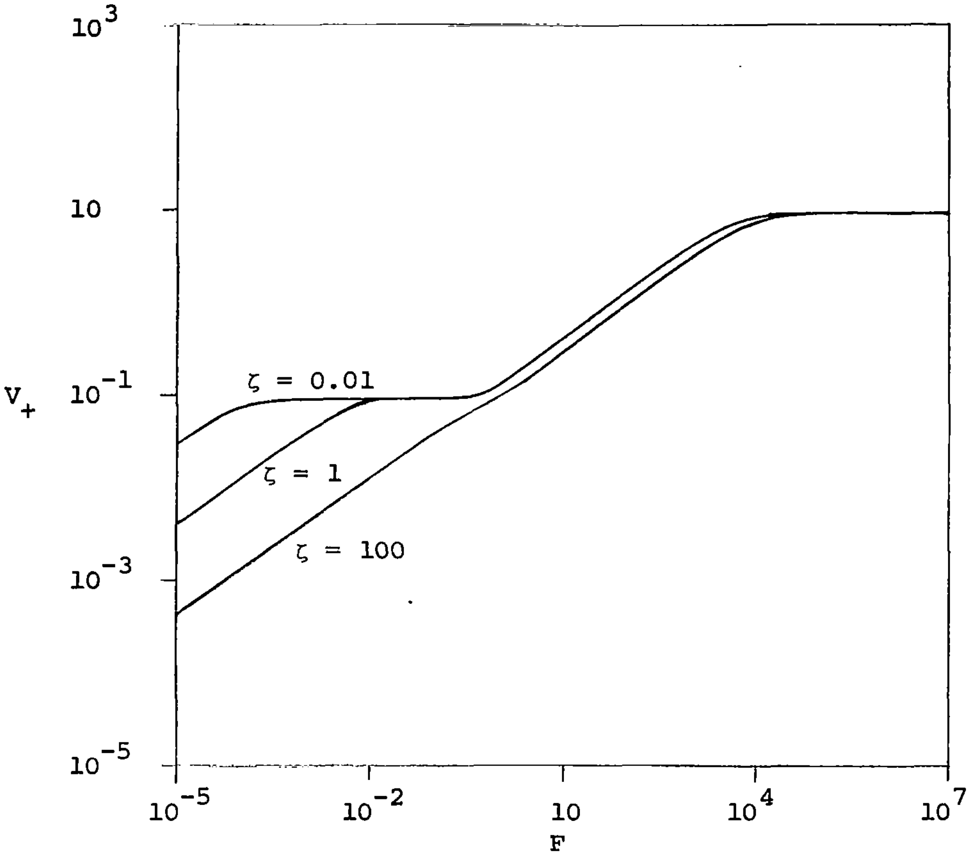

Figure 4.5. Effect of material constant ζ on dispersion of phase velocity V_+ of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \phi = 0.01$, $\chi = 100$

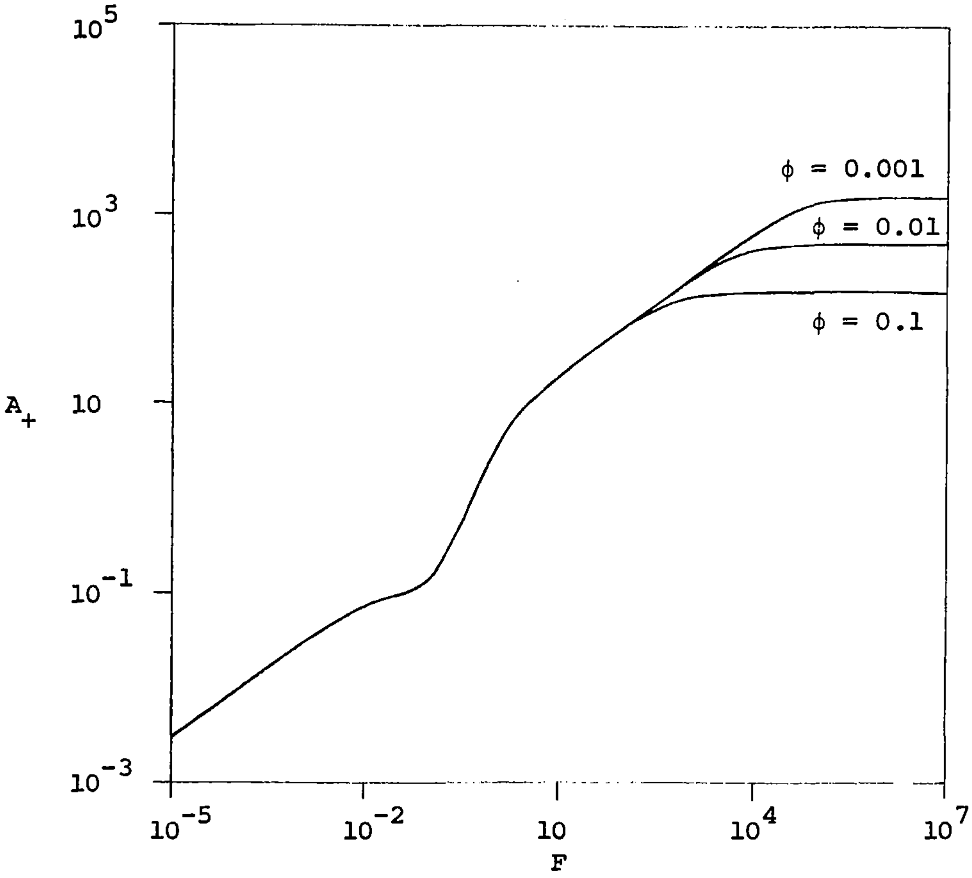

Figure 4.6. Effect of material constant ϕ on attenuation factor A_+ of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \zeta = 1$, $\chi = 100$

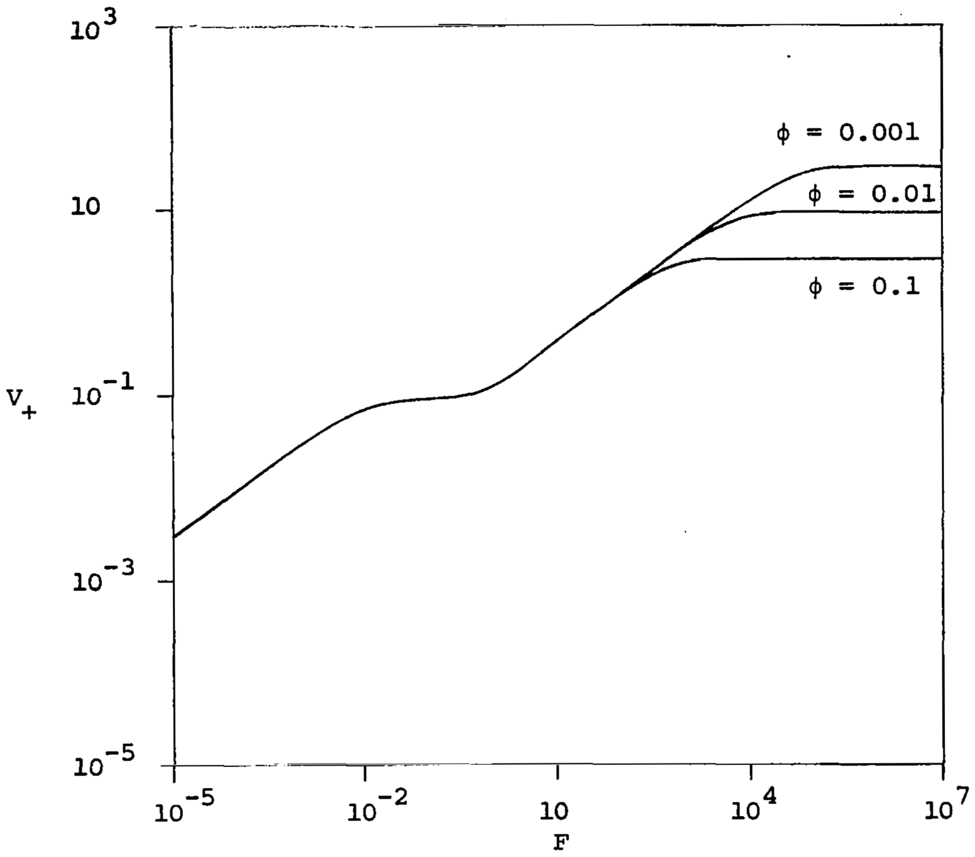

Figure 4.7. Effect of material constant ϕ on dispersion of phase velocity V_+ of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \zeta = 1$, $\chi = 100$

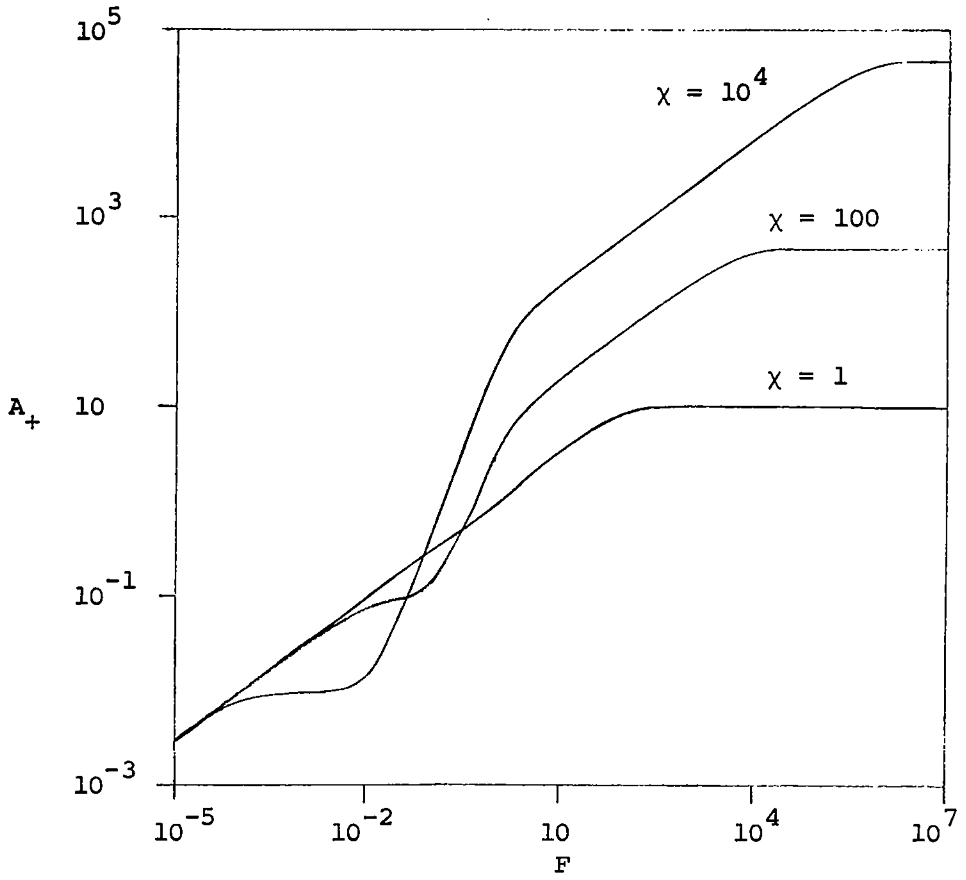

Figure 4.8. Effect of material constant χ on attenuation factor A_+ of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \zeta = 1$, $\phi = 0.01$

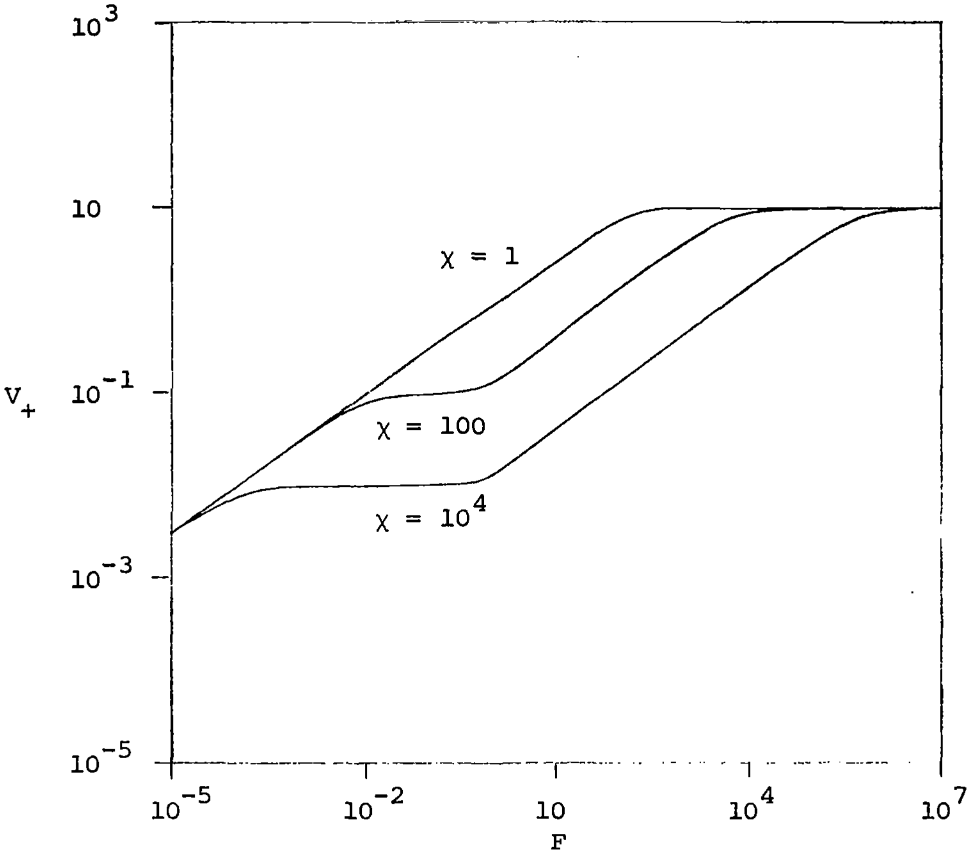

Figure 4.9. Effect of material constant χ on dispersion of phase velocity V_+ of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \zeta = 1$, $\phi = 0.01$

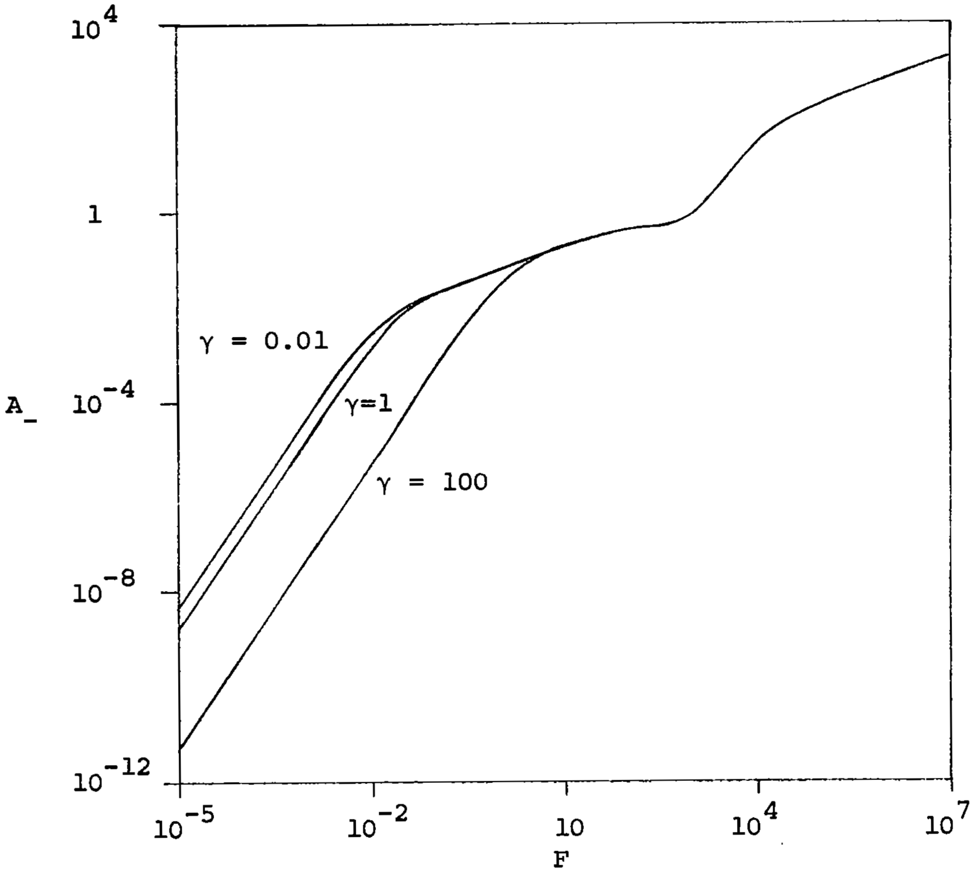

Figure 4.10. Effect of material constant γ on attenuation factor A_- of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\zeta = 1$, $\phi = 0.01$, $\chi = 100$

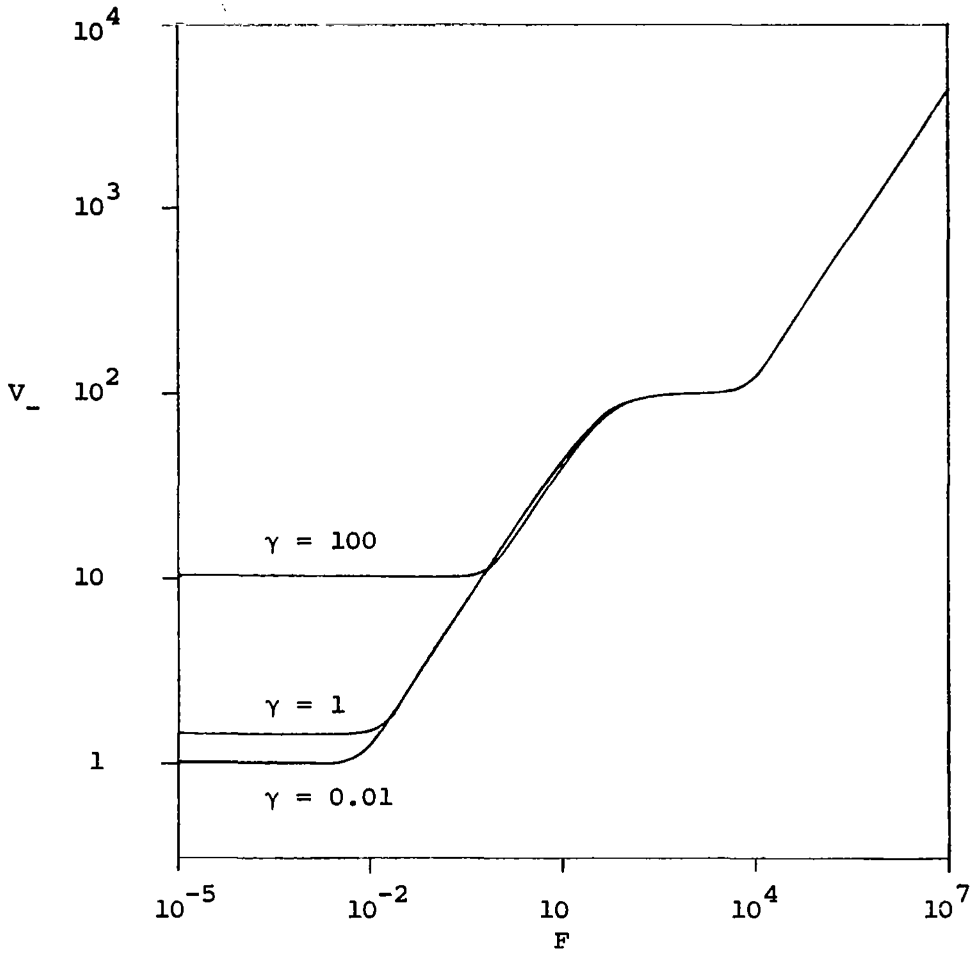

Figure 4.11. Effect of material constant γ on dispersion of phase velocity V_- of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\zeta = 1$, $\phi = 0.01$, $\chi = 100$

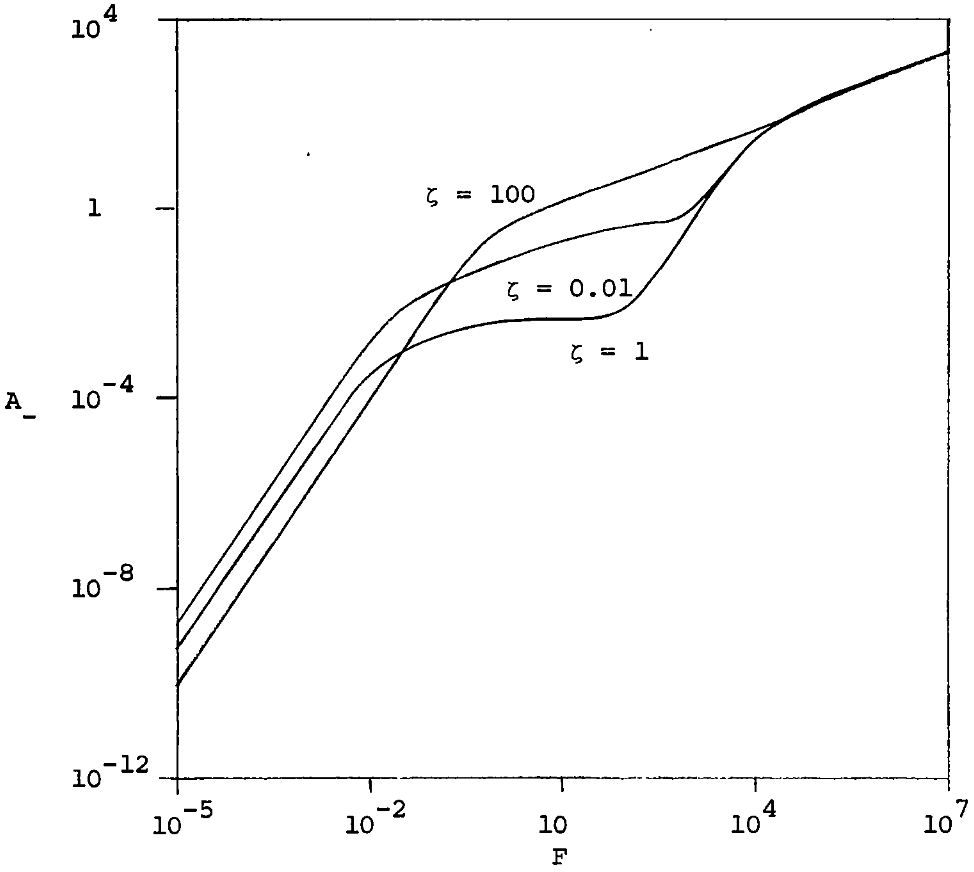

Figure 4.12. Effect of material constant ζ on attenuation factor A_- of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = 1$, $\phi = 0.01$, $\chi = 100$

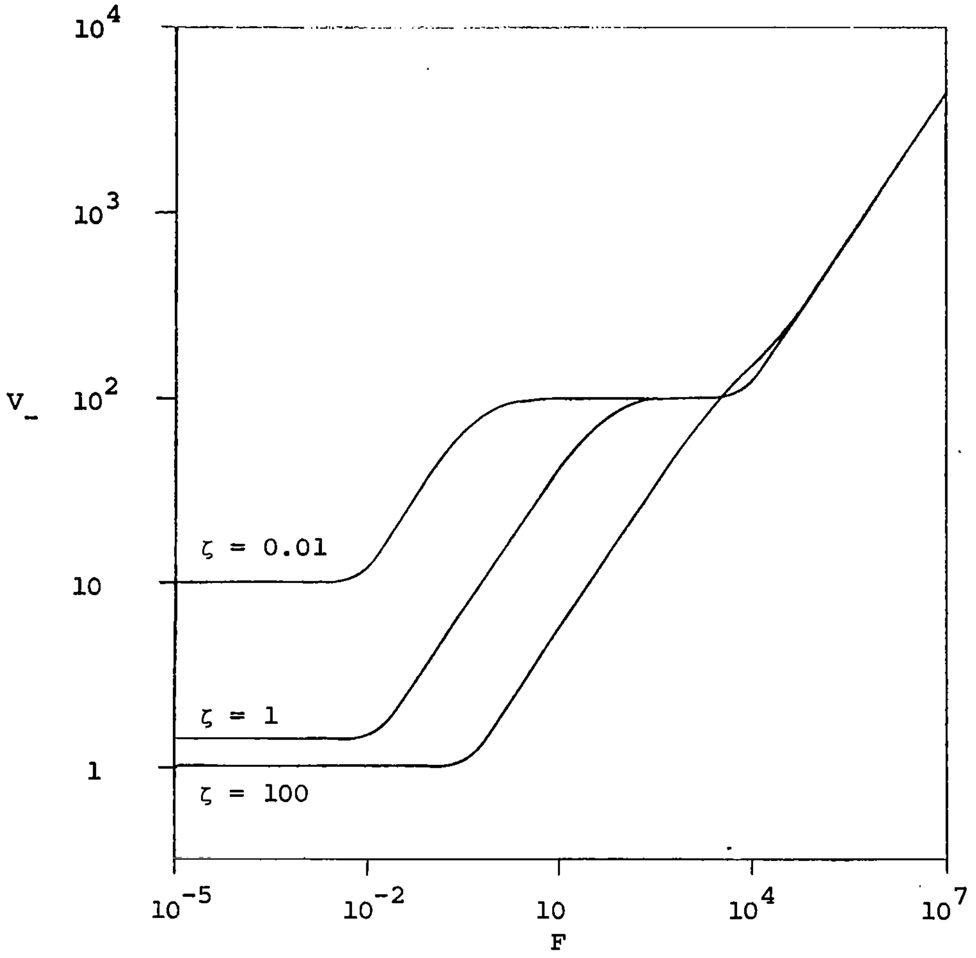

Figure 4.13. Effect of material constant ζ on dispersion of phase valocity V_- of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = 1$, $\phi = 0.01$, $\chi = 100$

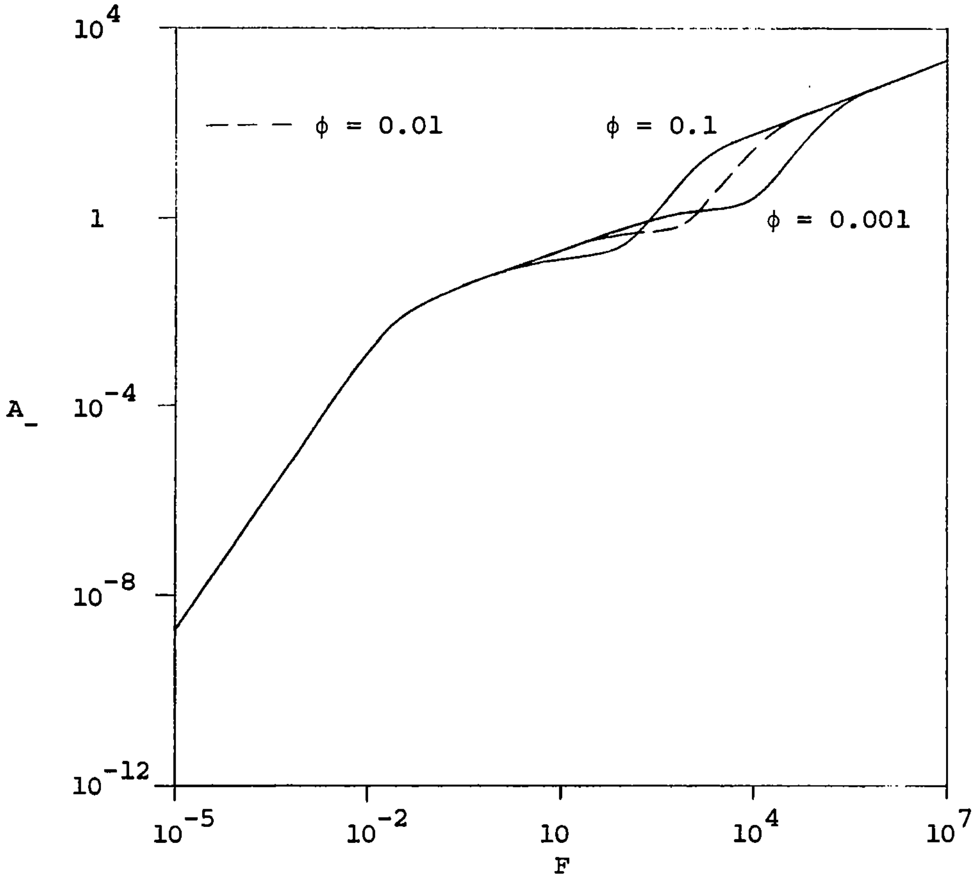

Figure 4.14. Effect of material constant ϕ on attenuation factor A_- of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \zeta = 1$, $\chi = 100$

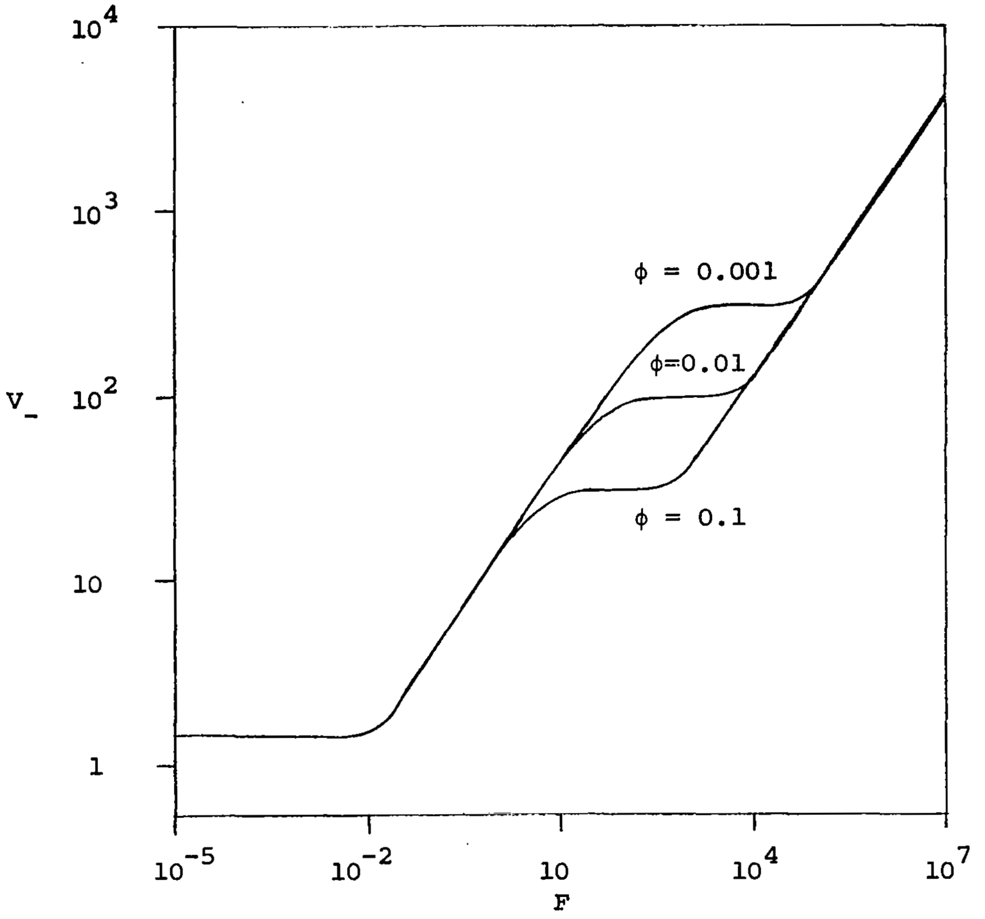

Figure 4.15. Effect of material constant ϕ on dispersion of phase velocity V_- of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \zeta = 1$, $\chi = 100$

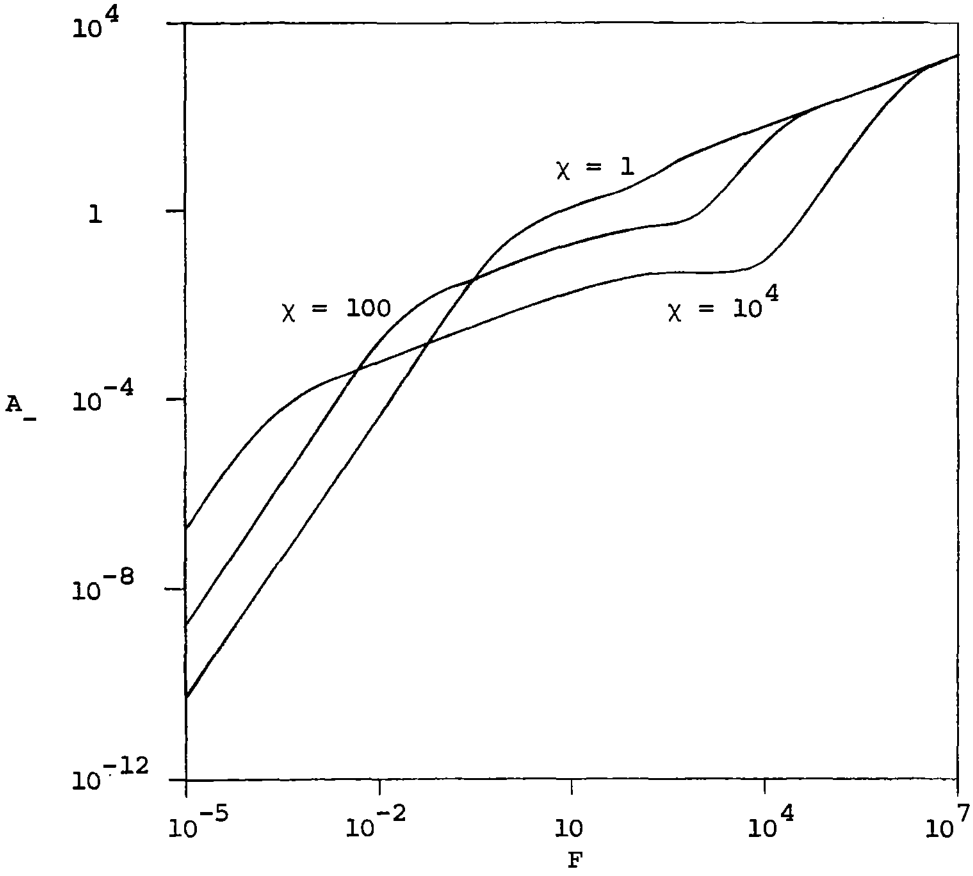

Figure 4.16. Effect of material constant χ on attenuation factor A_- of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \zeta = 1$, $\phi = 0.01$

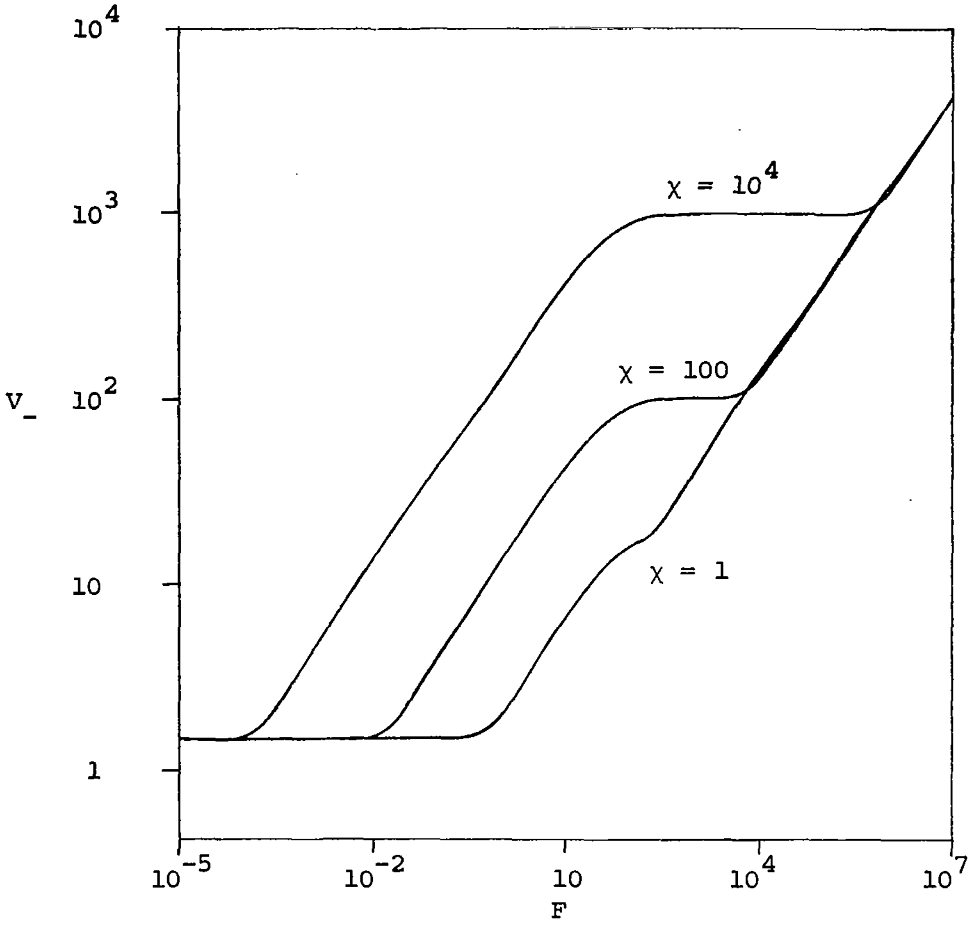

Figure 4.17. Effect of material constant χ on dispersion of phase velocity V_- of longitudinal waves in a temperature-rate dependent thermoviscoelastic material for $\gamma = \zeta = 1$, $\phi = 0.01$

(b) The attenuation factor A_+ and the phase velocity V_- increase while A_- and V_+ decrease with the material constant γ (Figures 4.2, 4.3, 4.10, and 4.11).

(c) The attenuation factors A_+ and A_- increase while the phase velocities V_+ and V_- decrease with the material constant ζ (Figures 4.4, 4.5, 4.12, and 4.13).

(d) The attenuation factors A_+, A_-, and the phase velocities V_+ and V_- are independent of the material constant ϕ (Figures 4.6, 4.7, 4.14, and 4.15).

(e) The attenuation factor A_- increases with the material constant χ while A_+, V_+, and V_- remain independent of it (Figures 4.8, 4.9, 4.16, and 4.17).

(2) For the case of $\sim 10^{-3} < F < 10^{3}$:

(a) In this range of frequency, the effects of the material constants γ, ζ, ϕ, and χ and the frequency F are mixed and a universal trend cannot be concluded.

(b) A more specific knowledge of the values of the parameters and range of the frequency is required in order to understand and establish the response of the material.

(3) For the case of $F \gtrsim 10^{3}$:

(a) The attenuation factor A_+ and the phase velocity V_+ are independent of the frequency F (Figures 4.2 through 4.9). A_- and V_- are directly proportional to the square root of F and are independent of all four material constants γ, ζ, ϕ, and χ as was shown earlier in the study of high frequency waves (Figures 4.10 through 4.17).

(b) The phase velocity V_+ is independent of all but one parameter. It decreases as the material constant ϕ increases (Figure 4.7).

(c) The attenuation factor A_+ increases with ζ as well as with χ, however, it decreases with ϕ and is independent of γ (Figures 4.4, 4.6, and 4.8).

5. TEMPERATURE-RATE DEPENDENT THERMOELASTIC CONSTITUTIVE RELATIONS

The formulation of Section 3 is quite general. One may wish to simplify these constitutive relations by reducing the number of generalized thermodynamic variables in (3.1). Such constitutive relations must again satisfy the principles of continuum mechanics and the laws of thermodynamics. One should not expect these simplified constitutive relations to coincide with those obtained from the thermoviscoelastic case by simply reducing the number of generalized thermodynamic variables in (3.16).

Several of these cases were explored during the course of this research. Among the ones studied the temperature-rate dependent thermoelasticity offers some interesting results. For such materials we may write, following the notation of Section 2,

$$\begin{aligned} a &= a\,(T,\ \dot{T},\ T_{,A},\ y_{i,A}) \quad , \\ s &= s\,(T,\ \dot{T},\ T_{,A},\ y_{i,A}) \quad , \\ e &= e\,(T,\ \dot{T},\ T_{,A},\ y_{i,A}) \quad , \\ B_A &= B_A(T,\ \dot{T},\ T_{,A},\ y_{i,A}) \quad , \\ \Sigma_{Ai} &= \Sigma_{Ai}(T,\ \dot{T},\ T_{,A},\ y_{i,A}) \quad . \end{aligned} \tag{5.1}$$

Upon application of the second law of thermodynamics and invariance principle of superposed rigid body motion, we obtain, for an isotropic temperature-rate dependent thermoelastic material,

$$\begin{aligned} a &= a\,(T,\ G_{AB}) \quad , \\ s &= s\,(T,\ \dot{T},\ T_{,A},\ G_{AB}) \quad , \end{aligned}$$

$$e = e(T, \dot{T}, T_{,A}, G_{AB}) \quad ,$$
$$B_A = B_A(T, \dot{T}, T_{,A}, G_{AB}) \quad ,$$
$$\Sigma_{Ai} = P_{AB}(T, \dot{T}, T_{,A}, G_{AB})\, y_{i,B} \quad . \tag{5.2}$$

Following the technique employed in Section 3.3, each of the response functions $a(T, G_{AB})$, $s(T, \dot{T}, T_{,A}, G_{AB})$, and $e(T, \dot{T}, T_{,A}, G_{AB})$ can always be expressed as a single-valued scalar function of irreducible integrity basis of its arguments under Q. For the isotropic, temperature-rate dependent, thermoelastic material under investigation, however, the irreducible integrity basis for each of the sets (T, G_{AB}) and $(T, \dot{T}, T_{,A}, G_{AB})$ are given by (3.18) and

$$T \;,\; \dot{T} \;,\; G_{AA} \;,\; G_{AB}G_{BA} \;,\; G_{AB}G_{BC}G_{CA}$$
$$T_{,A}T_{,A} \;,\; G_{AB}T_{,A}T_{,B} \;,\; G_{AB}G_{BC}T_{,C}T_{,A} \;, \tag{5.3}$$

respectively.

Similarly, we may form the scalar function Ψ given by (3.20), and note that in the present case the expression (3.21) becomes

$$\Psi = \sum_{\beta=1}^{M} H_\beta L_\beta \quad ,$$

where L_β are the elements (linear in Ψ_{AB}) of the irreducible integrity basis of T, $\dot{T}$, $T_{,A}$, G_{AB}, Ψ_{AB} under Q, and H_β are single-valued functions of the irreducible integrity basis of T, $\dot{T}$, $T_{,A}$, G_{AB} under Q. Thus

$$P_{AB} = \frac{1}{2} \sum_{\beta=1}^{M} H_\beta \left(\frac{\partial L_\beta}{\partial \Psi_{AB}} + \frac{\partial L_\beta}{\partial \Psi_{BA}}\right) \quad . \tag{5.4}$$

One may also obtain the constitutive expression for the heat flux vector $B_A(T, \dot{T}, T_{,M}, G_{MN})$ in a similar fashion by forming the scalar product of B_A with an arbitrary vector.

In the following sections we will study the one-dimensional linear spatial gradient theory of such formulation in a manner similar to that of Section 4. In addition, we will obtain a class of self-similar solutions in such a medium.

6. THE ONE-DIMENSIONAL LINEAR SPATIAL GRADIENT TEMPERATURE-RATE DEPENDENT THERMOELASTICITY

Since the one-dimensional theory presented in Section 4 has been developed in detail, we shall eliminate some of the analogous discussions to avoid unnecessary repetitions.

6.1. Linear Gradient Assumption

To the first-order approximation in the spatial gradient quantities we obtain

$$\rho_0 \; e = e_0(T, \dot{T}) + e_1(T, \dot{T}) \; \varepsilon \quad , \tag{6.1}$$

$$B = - \kappa(T, \dot{T}) \; T_X \quad , \tag{6.2}$$

$$\Sigma = - \pi(T) + \mu(T) \; \varepsilon \quad , \tag{6.3}$$

where we have already used the results of Section 4 in arriving at the last equation. Thus, the entropy inequality becomes

$$\kappa \; T_X^2 - \rho_0 \; T \left(s + \frac{\partial a}{\partial T}\right) \dot{T} \geq 0 \quad , \tag{6.4}$$

where again

$$\kappa \geq 0 \quad .$$

Similarly, the free energy may be expressed as

$$\rho_0 \; a = \psi(T) - \pi(T) \; \varepsilon \quad . \tag{6.5}$$

Using Equations (6.1) and (6.5) in (2.7) gives

$$\rho_0 \, s = \frac{1}{T} \{e_0(T, \dot{T}) - \psi(T) + [e_1(T, \dot{T}) + \pi(T)] \, \varepsilon\} \quad . \tag{6.6}$$

The set of expressions (6.1) through (6.6) and Equations (4.11) complete the thermodynamic formulation of the one-dimensional temperature-rate dependent thermoelastic problem on hand.

6.2. Basic Equations

Substituting the constitutive relations (6.1) through (6.3) in the equations of motion and balance of energy (4.11), we obtain to the first order of approximation,

$$\begin{aligned} &\rho_0 \, u_{tt} - \mu \, u_{XX} + \pi_T \, T_X = 0 \quad , \\ &(e_1 + \pi) \, u_{Xt} + e_{0T} \, T_t + e_{0\dot{T}} \, T_{tt} - \kappa \, T_{XX} = 0 \quad . \end{aligned} \tag{6.7}$$

Comparing the sets (6.7) and (4.12), we note that the former is free of the third order derivatives of the displacement.

6.3. Characteristic Motions

We shall follow the Hadamard's method and Section 4.3 with the exception that the second order derivatives of the displacement are no longer assumed to be continuous across S. Therefore, we have

$$[T] = [u] = [T_t] = [T_X] = [u_t] = [u_X] = 0 \quad , \tag{6.8}$$

yielding

$$[T_{tt}] + c \, [T_{Xt}] = 0 \quad , \tag{6.9}$$

$$[T_{Xt}] + c \, [T_{XX}] = 0 \quad , \tag{6.10}$$

$$[u_{tt}] + c\,[u_{Xt}] = 0 \quad , \tag{6.11}$$

$$[u_{Xt}] + c\,[u_{XX}] = 0 \quad . \tag{6.12}$$

The set of Equations (6.7) across S give

$$\rho_0\,[u_{tt}] - \mu\,[u_{XX}] = 0 \quad , \tag{6.13}$$

$$(e_1 + \pi)\,[u_{Xt}] + e_{0\dot{T}}\,[T_{tt}] - \kappa\,[T_{XX}] = 0 \quad . \tag{6.14}$$

Therefore, we have obtained six linear algebraic homogeneous equations in terms of six unknowns $[T_{tt}]$, $[T_{Xt}]$, $[T_{XX}]$, $[u_{tt}]$, $[u_{Xt}]$, and $[u_{XX}]$. For nontrivial solutions, the determinant of the coefficients in (6.9) through (6.14) must vanish.

$$(e_{0\dot{T}}\,c^2 - \kappa)(\rho_0\,c^2 - \mu) = 0 \quad ,$$

which is satisfied if

$$c = (\mu/\rho_0)^{1/2} \triangleq c_1 \quad , \tag{6.15}$$

or

$$c = (\kappa/e_{0\dot{T}})^{1/2} \triangleq c_2 \quad . \tag{6.16}$$

In the thermoelastic medium under consideration, c_1 and c_2 are the characteristic speeds of wave propagations. We note that c_2 is the same as that given by Equation (4.24) for the thermoviscoelastic case. Furthermore, c_1 is the elastic wave speed which was overwhelmed by the presence of the viscous terms in the thermoviscoelastic case. We shall show that these speeds c_1 and c_2 correspond to the high frequency asymptotic phase velocities. We also note that c_2 tends to infinity, the classical heat conduction wave speed, as $e_{0\dot{T}}$ approaches zero.

6.4. Dispersion Relations of Linearized Longitudinal Wave Propagation in an Initially Unstrained Thermoelastic Material

Following the procedure of Section 4.4, we obtain

$$\begin{aligned} &- \bar{\pi}_T\, k\, T_0 + \{\rho_0 (\omega i)^2 - \bar{\mu}\, k^2\}\, u_0 = 0 \quad , \\ &\{(\bar{e}_{0T} + \bar{e}_{0\dot{T}}\, \omega i)\, \omega i - \bar{\kappa}\, k^2\}\, T_0 - (\bar{e}_1 + \bar{\pi})\, k\, \omega i\, u_0 = 0 \quad , \end{aligned} \tag{6.17}$$

corresponding to (4.28).

For nontrivial solutions, the determinant of the coefficients must vanish. Again we consider a real frequency ω of propagation in the positive X-direction, and express the complex wave number by Equation (4.29). Thus, we have

$$\bar{\kappa}\, \bar{\mu}\, k^4 - \{\rho_0\, \bar{\kappa}\, \omega i + \bar{\mu}(\bar{e}_{0T} + \bar{e}_{0\dot{T}}\, \omega i) + \bar{\pi}_T(\bar{\pi} + \bar{e}_1)\}\, k^2\, \omega i + \rho_0 (\bar{e}_{0T} + \bar{e}_{0\dot{T}}\, \omega i)(\omega i)^3 = 0 \quad . \tag{6.18}$$

Equation (6.18) is satisfied if

$$H^4 - [\gamma + \zeta + (\phi + 1)\, Fi]\, H^2\, Fi + (\zeta + \phi\, Fi)(Fi)^3 = 0 \quad , \tag{6.19}$$

where

$$F \triangleq \frac{\omega}{\Omega} \quad , \quad H \triangleq \frac{(\bar{\mu}/\rho_0)^{1/2}}{\Omega}\, k \tag{6.20}$$

are the dimensionless frequency and complex wave-number with Ω as a characteristic frequency of the oscillation; and,

$$\gamma \triangleq \frac{\bar{\pi}_T \, (\bar{\pi} + \bar{e}_1)}{\rho_0 \, \Omega \, \bar{\kappa}} \quad , \tag{6.21}$$

$$\zeta \triangleq \frac{\bar{\mu} \, \bar{e}_{0T}}{\rho_0 \, \Omega \, \bar{\kappa}} \quad , \tag{6.22}$$

and

$$\phi \triangleq \frac{\bar{\mu} \, \bar{e}_{0\dot{T}}}{\rho_0 \, \bar{\kappa}} \tag{6.23}$$

are dimensionless material constants defined in a fashion similar to that of the thermoviscoelastic case. Equation (6.19) yields

$$H_\pm = \left\{\frac{Fi}{2} \left[\gamma + \zeta + (\phi + 1) \; Fi \pm \sqrt{[\gamma + \zeta + (\phi - 1) \; Fi]^2 + 4\gamma \; Fi}\right]\right\}^{1/2} \quad . \tag{6.24$_\pm$}$$

Separating the real and imaginary parts of $(6.24)_\pm$, we have

$$A_\pm \triangleq \frac{(\bar{\mu}/\rho_0)^{1/2}}{\Omega} \alpha_\pm = \mathrm{Re} \; (H_\pm) \quad , \tag{6.25$_\pm$}$$

$$V_\pm \triangleq \frac{c_\pm}{(\bar{\mu}/\rho_0)^{1/2}} = F/\mathrm{Im} \; (H_\pm) \quad , \tag{6.26$_\pm$}$$

where only propagation in the positive X-direction has been considered, ($\pm$) corresponds to the sign preceding the radical in $(6.24)_\pm$, and $H_\pm$ is given by $(6.24)_\pm$.

6.4.1. Asymptotic Expansions

Once again we shall follow Section 4.4.1 to derive certain restrictions on the material constants by means of asymptotic expansions of A and V.

(1) In the case of low frequency waves when F << 1, the asymptotic expansions are identical to the results obtained for the thermoviscoelastic case. This is not surprising since the viscous effects are negligible at low frequency anyway. We omit the details and state dimensional results of the attenuation factors and phase velocities:

$$\alpha_+ \simeq \left[\frac{\bar{\mu}\,\bar{e}_{0T} + \bar{\pi}_T(\bar{e}_1 + \bar{\pi})}{2\,\bar{\mu}\,\bar{\kappa}}\,\omega\right]^{1/2} + O(\omega^{3/2}) \quad , \qquad (6.27)$$

$$c_+ \simeq \left[\frac{2\,\bar{\mu}\,\bar{\kappa}}{\bar{\mu}\,\bar{e}_{0T} + \bar{\pi}_T(\bar{\pi} + \bar{e}_1)}\,\omega\right]^{1/2} + O(\omega^{3/2}) \quad , \qquad (6.28)$$

$$\alpha_- \simeq 0 + O(\omega^2) \quad , \qquad (6.29)$$

$$c_- \simeq \left[\frac{\bar{\mu}}{\rho_0} + \frac{\bar{\pi}_T(\bar{\pi} + \bar{e}_1)}{\rho_0\,\bar{e}_{0T}}\right]^{1/2} + O(\omega^2) \quad , \qquad (6.30)$$

which are identical to those given by (4.48) through (4.51) despite the differences in γ and ζ of the two cases. The discussion on decompositon of c_- will follow just the same way.

(2) In the case of high frequency waves when F >> 1, then Equations $(6.24)_\pm$ can be approximated as follows:

$$H_{\pm} \simeq \{\frac{Fi}{2} [\gamma + \zeta + (\phi+1)\ Fi \pm \frac{(\gamma+\zeta)(\phi-1) + 2\gamma}{|\phi - 1|} \pm |\phi - 1|\ Fi + \ldots]\}^{1/2} \quad . \tag{6.31}$$

The absolute value sign is placed on $(\phi - 1)$ wherever it represents square root of a real number $(\phi - 1)^2$ since we do not know whether $\phi > 1$ or $\phi < 1$. If we now separate the real and imaginary parts of (6.31) for $\phi < 1$ and $\phi > 1$, we observe that only $\phi < 1$ will yield positive values of A and V. Therefore,

$$A_+ \simeq \frac{\gamma}{2(1 - \phi)} + O(F^{-1}) \quad , \tag{6.32}$$

$$V_+ \simeq 1 + O(F^{-1}) \quad , \tag{6.33}$$

provided

$$\phi < 1 \quad , \quad \gamma \geq 0 \quad , \tag{6.34}$$

and

$$A_- \simeq \frac{1}{2\sqrt{\phi}} \left(\zeta - \frac{\gamma\phi}{1 - \phi}\right) + O(F^{-1}) \quad , \tag{6.35}$$

$$V_- \simeq \frac{1}{\sqrt{\phi}} + O(F^{-1}) \quad , \tag{6.36}$$

provided

$$\phi > 0 \quad \text{and} \quad \zeta - \frac{\gamma\phi}{1 - \phi} > 0 \quad . \tag{6.37}$$

The latter condition may be re-written

$$\frac{1}{\phi} > 1 + \frac{\gamma}{\zeta} \quad . \tag{6.38}$$

The dimensional form of the expressions (6.32) through (6.33) and (6.35) through (6.36) may be obtained by employing definitions (6.20) through (6.23) and (6.25) through (6.26).

$$\alpha_+ \simeq \frac{\bar{\pi}_T(\bar{\pi} + \bar{e}_1)}{2\,\Omega(\rho_0\,\bar{\kappa} - \bar{\mu}\,\bar{e}_{0\dot{T}})} + O(\omega^{-1}) \quad , \tag{6.39}$$

$$c_+ \simeq \sqrt{\frac{\bar{\mu}}{\rho_0}} + O(\omega^{-1}) = c_1 \quad , \tag{6.40}$$

$$\alpha_- \simeq \frac{\bar{\mu}}{2\rho_0\,\Omega}\left[\frac{\bar{e}_{0T}}{\bar{e}_{0\dot{T}}} - \frac{\bar{\pi}_T(\bar{\pi} + \bar{e}_1)}{\rho_0\,\bar{\kappa} - \bar{\mu}\,\bar{e}_{0\dot{T}}}\right] + O(\omega^{-1}) \quad , \tag{6.41}$$

$$c_- \simeq \sqrt{\frac{\bar{\kappa}}{\bar{e}_{0\dot{T}}}} + O(\omega^{-1}) = c_2 \quad . \tag{6.42}$$

Expressions (6.39) and (6.41) depend on the coefficient of the temperature-rate $\bar{e}_{0\dot{T}}$ as well as on the characteristic frequency Ω. The speed c_+ given by (6.40) is the elastic wave speed and c_- given by (6.42) is the dissipative heat wave speed. Furthermore, these wave speeds are identical to the characteristic speed c_1 and c_2 given by Equations (6.15) and (6.16). A close look at the condition (6.38) reveals that

$$\frac{\bar{\kappa}}{\bar{e}_{0\dot{T}}} > \frac{\bar{\mu}}{\rho_0} + \frac{\bar{\pi}_T(\bar{\pi} + \bar{e}_1)}{\rho_0 \, \bar{e}_{0T}} \quad ,$$

or

$$c_2^2 > c_1^2 + c_3^2 \quad . \tag{6.43}$$

Inequality (6.43) states that the heat wave speed is always greater than the elastic wave speed c_1 and the non-dissipative (low frequency) heat wave speed c_2. This conclusion is best illustrated on the dispersion curves (Section 6.4.3, pages 66, 68, and 70). Again, the classical infinite wave speed due to heat conduction may be deduced by setting $\bar{e}_{0\dot{T}}$ equal to zero.

6.4.2. Possible Physical Limitations

We may employ the same technique used in Section 4.4.2 and thus conclude that for most materials

$$\gamma \geq 0 \quad \text{and} \quad \zeta > 0 \quad . \tag{6.44}$$

Inequalities (6.44) are in perfect agreement with the results obtained earlier in the asymptotic expansions. We thus arrive at the following set of conditions for a stable wave propagation in the temperature rate dependent thermoelastic material under investigation:

$$\begin{gathered} \gamma \geq 0 \quad , \\ \zeta > 0 \quad , \\ 1 > \phi > 0 \quad , \\ \text{and} \qquad \frac{1}{\phi} > 1 + \frac{\gamma}{\zeta} \quad . \end{gathered} \tag{6.45}$$

These restrictions were appropriately introduced in the numerical computations performed to obtain the graphical illustrations of the dispersion relations.

6.4.3. Numerical Results and Graphs

Logarithmic values of the non-dimensional attenuation factor A and the non-dimensional phase velocity V were cross-plotted against the dimensionless frequency F (Figures 6.1 through 6.12). The following behaviors are observed:

(1) For the case of $F \overset{\sim}{<} 1$:

  (a) Both attenuation factors A_+ and A_-, and the phase velocity V_+ increase with the frequency F (Figures 6.1-6.6, 6.7, 6.9, and 6.11). The phase velocity V_-, however, is independent of the frequency (Figures 6.8, 6.10, and 6.12).

  (b) The attenuation factor A_+ and the phase velocity V_- increase while A_- and V_+ decrease with the material constant γ (Figures 6.1, 6.2, 6.7, and 6.8).

  (c) The attenuation factors A_+ and A_- increase while the phase velocities V_+ and V_- decrease with the material constant ζ (Figures 6.3, 6.4, 6.9, and 6.10).

  (d) The attenuation factor A_+ and the phase velocities V_+ and V_- are independent of the material constant ϕ (Figures 6.5, 6.6, and 6.12). The attenuation factor A_- decreases with ϕ (Figure 6.11).

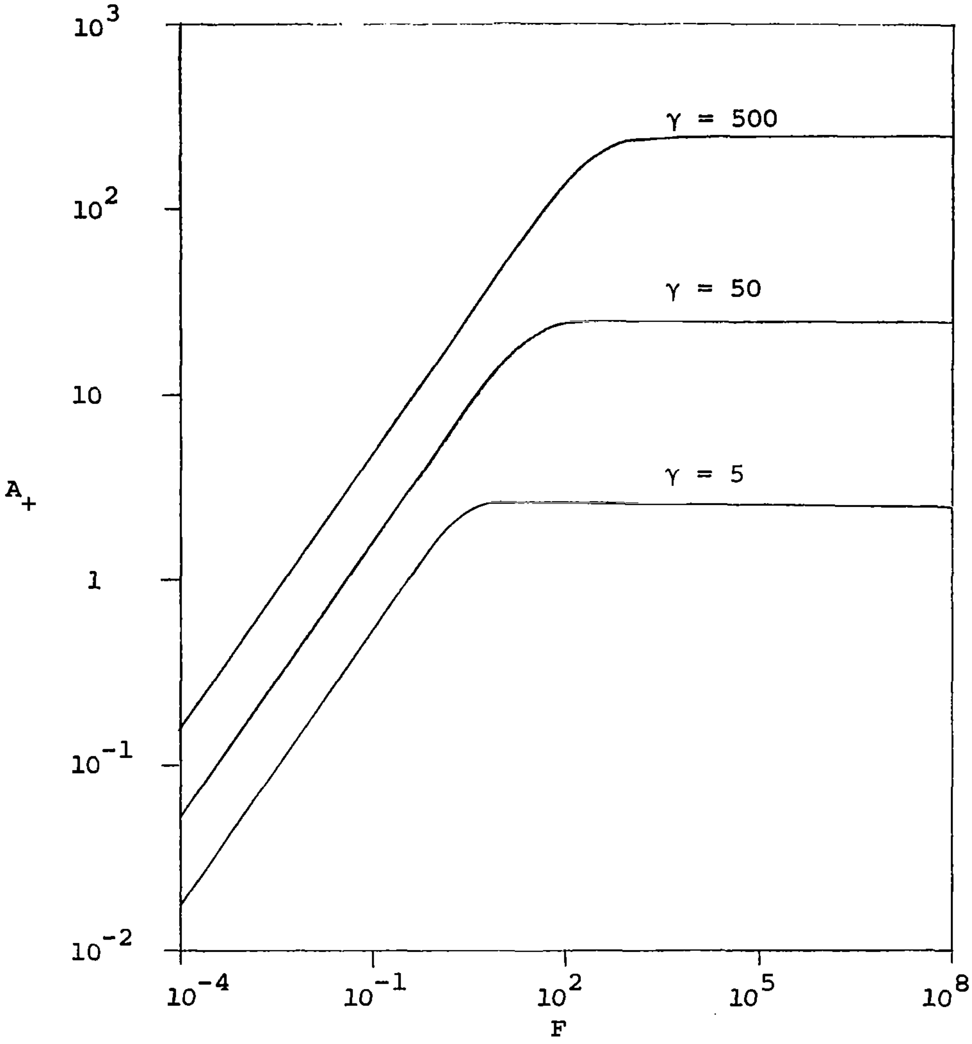

Figure 6.1. Effect of material constant γ on attenuation factor A_+ of longitudinal waves in a temperature-rate dependent thermoelastic material for $\zeta = 1$, $\phi = 0.001$

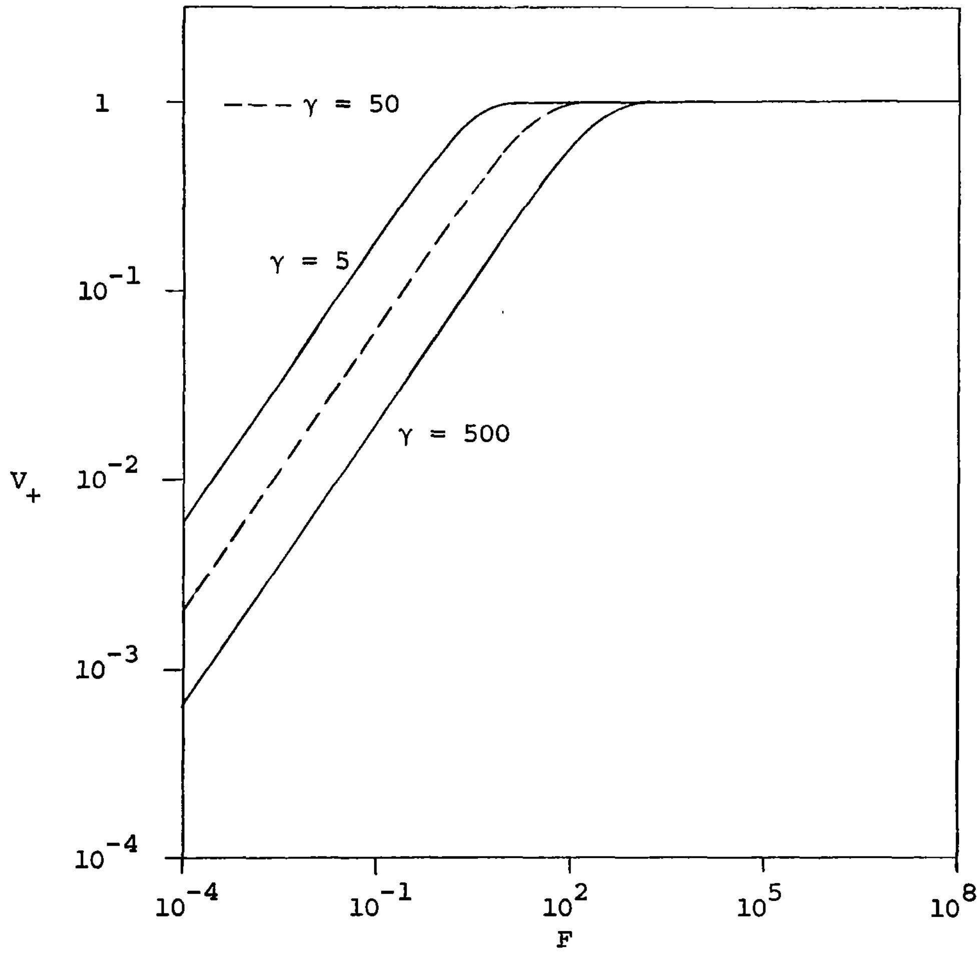

Figure 6.2. Effect of material constant γ on dispersion of phase velocity V_+ of longitudinal waves in a temperature-rate dependent thermoelastic material for $\zeta = 1$, $\phi = 0.001$

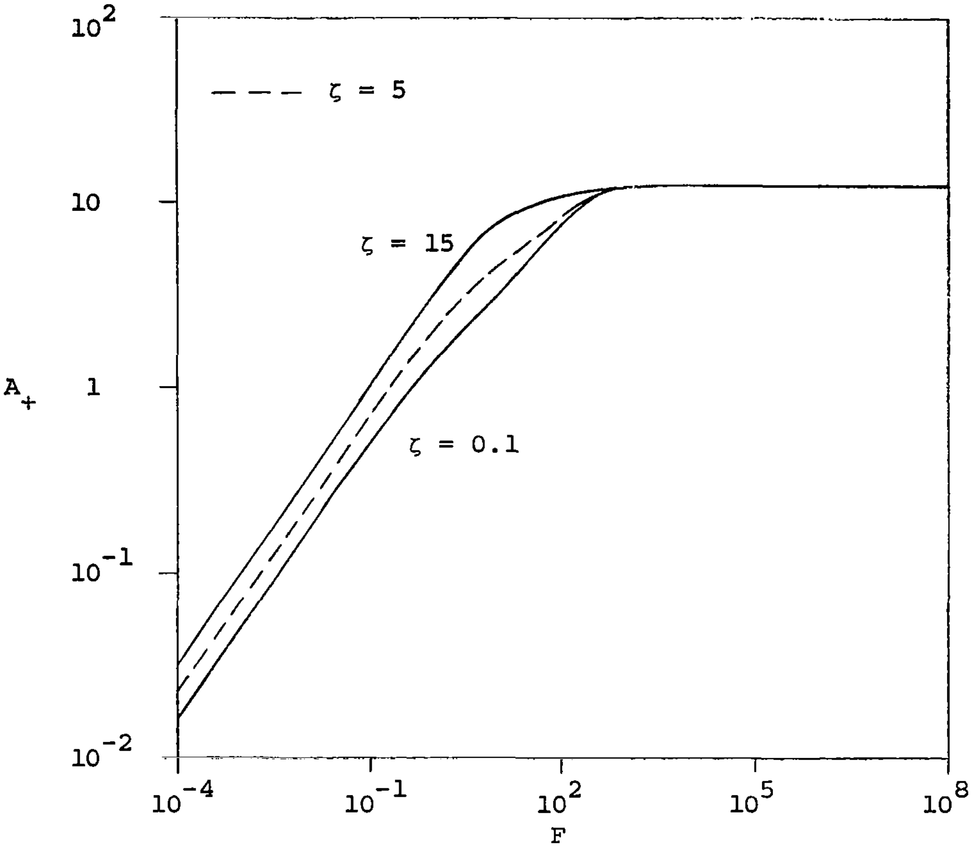

Figure 6.3. Effect of material constant ζ on attenuation factor A_+ of longitudinal waves in a temperature-rate dependent thermoelastic material for $\gamma = 5$, $\phi = 0.8$

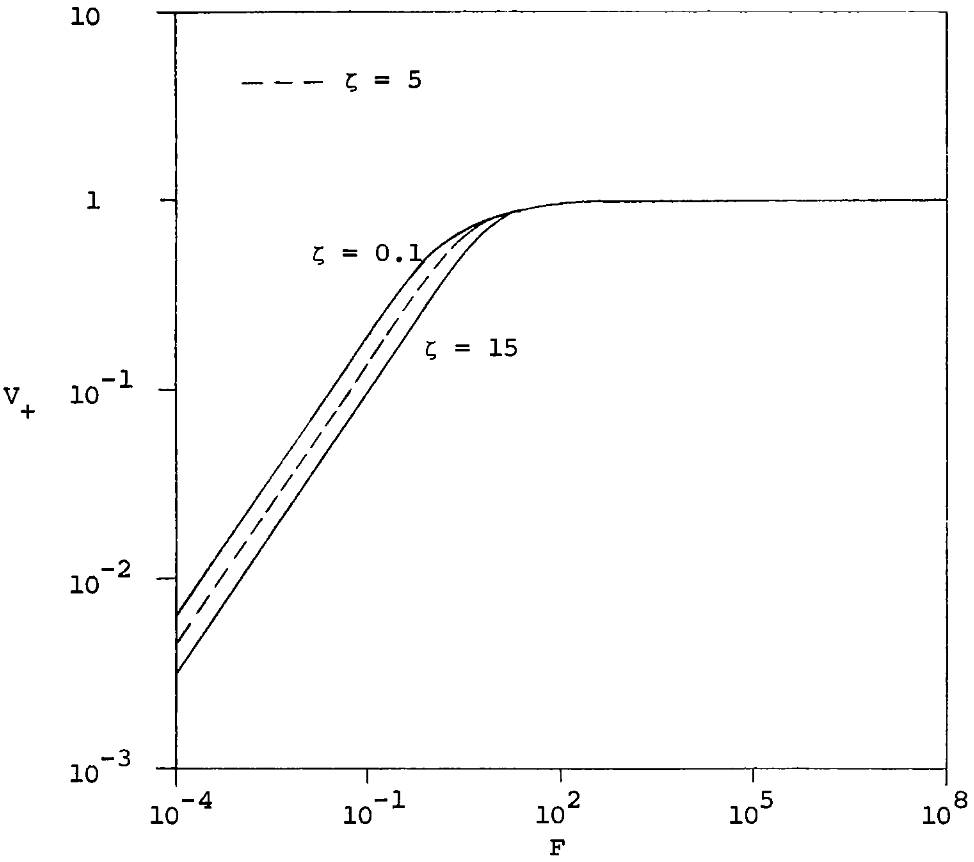

Figure 6.4. Effect of material constant ζ on dispersion of phase velocity V_+ of longitudinal waves in a temperature-rate dependent thermoelastic material for $\gamma = 5$, $\phi = 0.8$

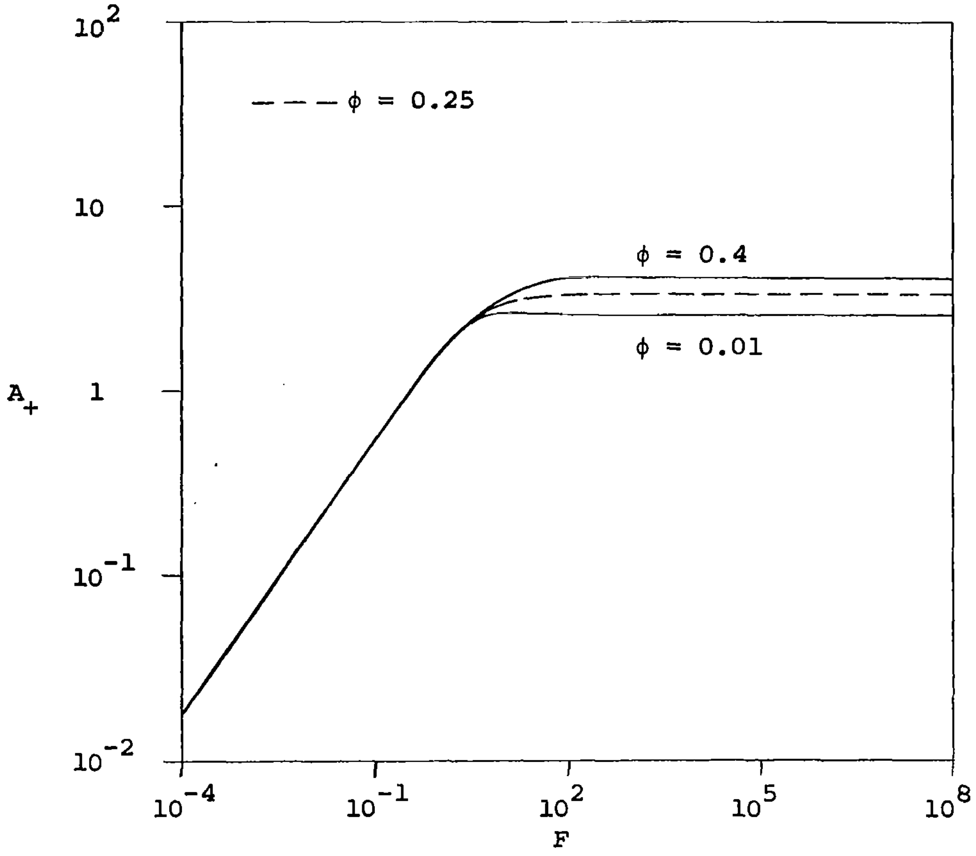

Figure 6.5. Effect of material constant ϕ on attenuation factor A_+ of longitudinal waves in a temperature-rate dependent thermoelastic material for $\gamma = 5$, $\zeta = 1$

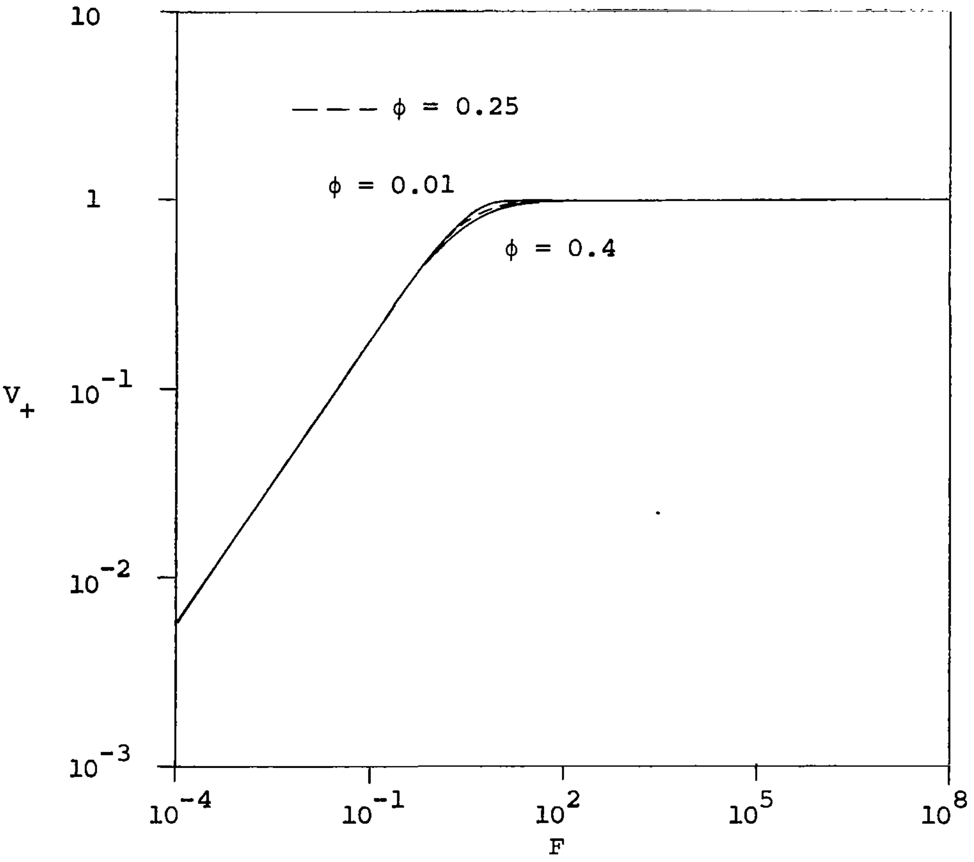

Figure 6.6. Effect of material constant ϕ on dispersion of phase velocity V_+ of longitudinal waves in a temperature-rate dependent thermoelastic material for $\gamma = 5$, $\zeta = 1$

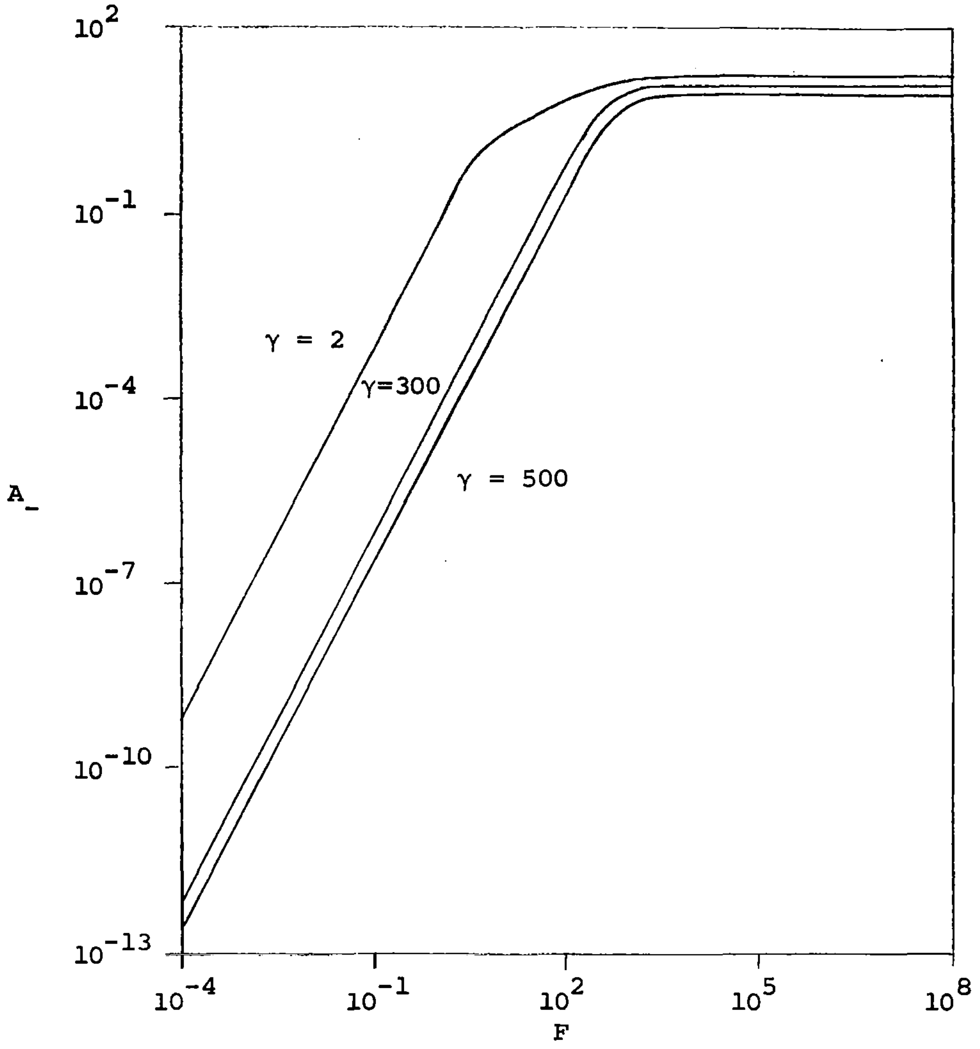

Figure 6.7. Effect of material constant γ on attenuation factor A_- of longitudinal waves in a temperature-rate dependent thermoelastic material for $\zeta = 1$, $\phi = 0.001$

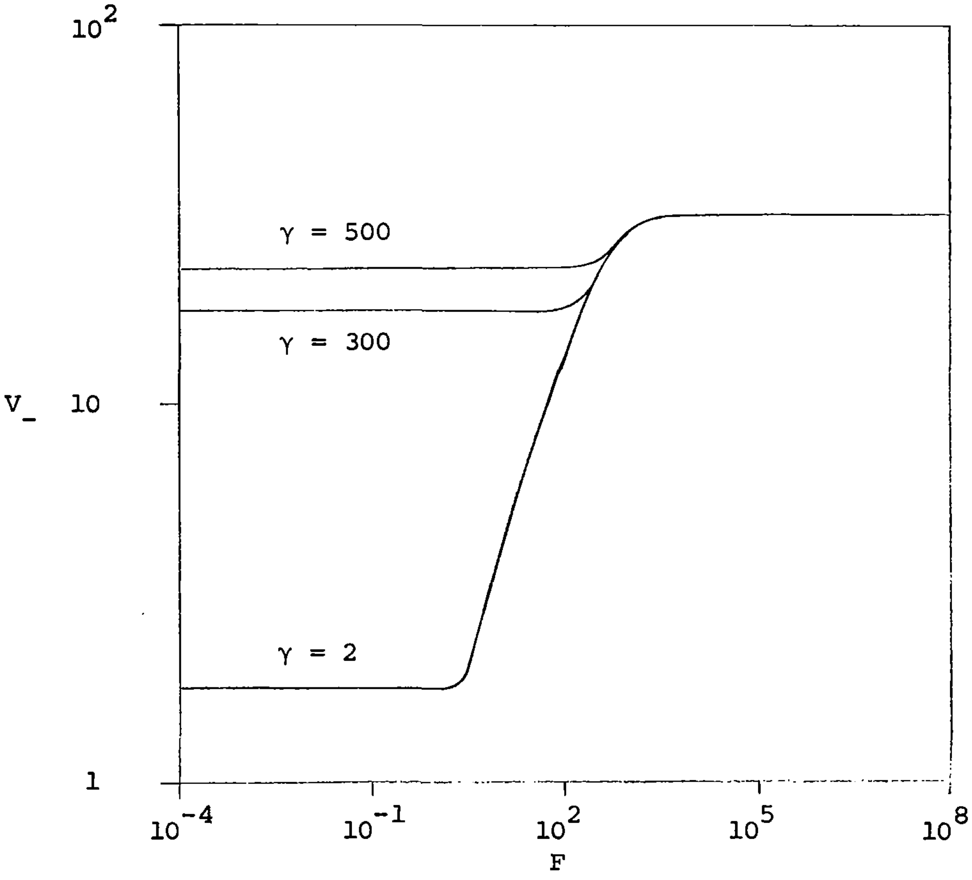

Figure 6.8. Effect of material constant γ on dispersion of phase velocity V_- of longitudinal waves in a temperature-rate dependent thermoelastic material for $\zeta = 1$, $\phi = 0.001$

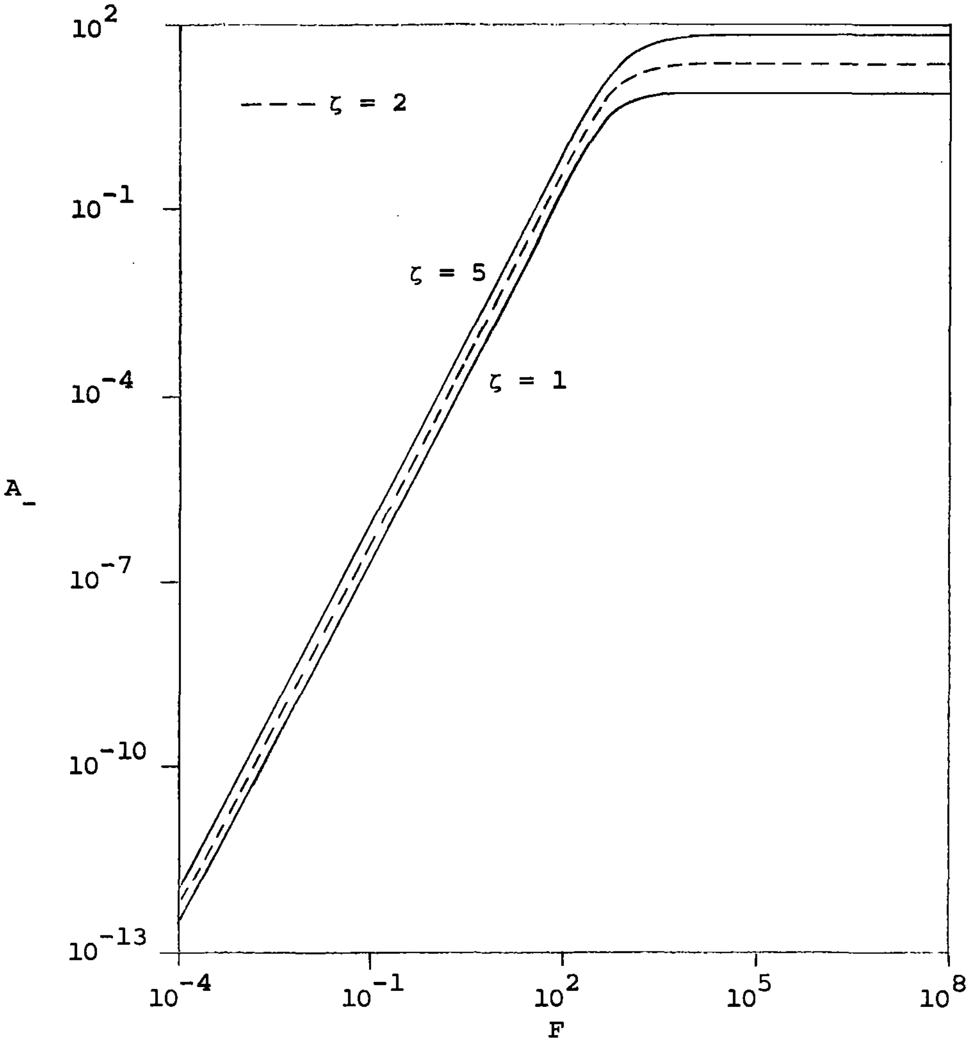

Figure 6.9. Effect of material constant ζ on attenuation factor A_- of longitudinal waves in a temperature-rate dependent thermoelastic material for $\gamma = 500$, $\phi = 0.001$

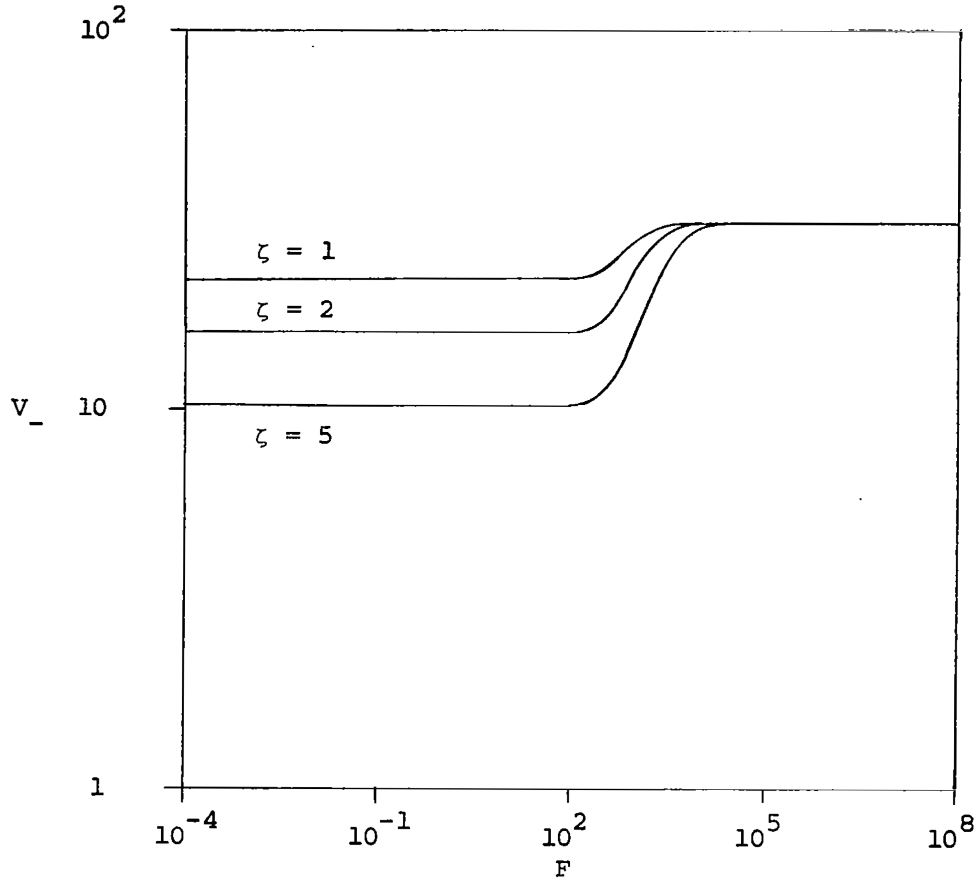

Figure 6.10. Effect of material constant ζ on dispersion of phase velocity V_- of longitudinal waves in a temperature-rate dependent thermoelastic material for $\gamma = 500$, $\phi = 0.001$

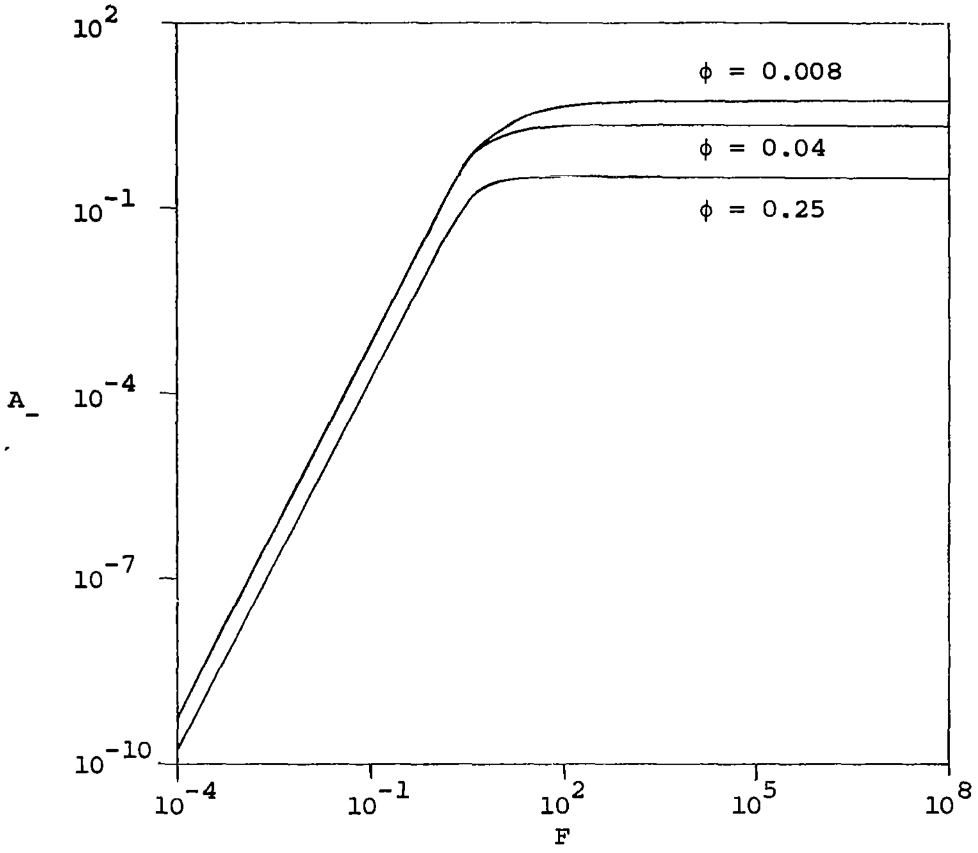

Figure 6.11. Effect of material constant ϕ on attenuation factor A_- of longitudinal waves in a temperature-rate dependent thermoelastic material for $\gamma = 2$, $\zeta = 1$

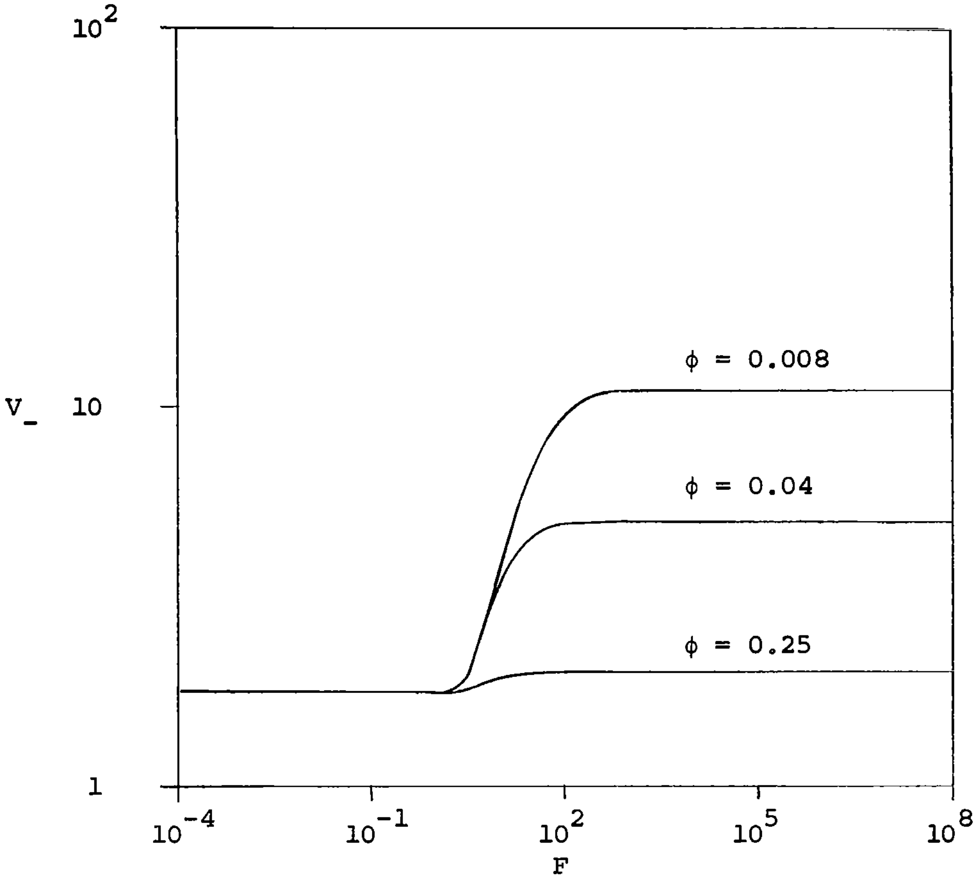

Figure 6.12. Effect of material constant ϕ on dispersion of phase velocity V_- of longitudinal waves in a temperature-rate dependent thermoelastic material for $\gamma = 2$, $\zeta = 1$

(2) For the case of $1 \lesssim F \lesssim 10^2$:

  (a) This is where the major difference between the thermoviscoelastic case and the thermoelastic case exists since there is no interaction between the material constants γ, ζ, and ϕ as the frequency F varies (Figures 6.1 through 6.12).

  (b) Except for a small decrease in V_+ with ϕ (Figure 6.6), the behavior of the other variables remain unchanged.

(3) For the case of $F \gtrsim 10^2$:

  (a) Unlike the thermoviscoelastic case, all quantities A_+, V_+, A_-, and V_- become independent of the frequency F and approach their asymptotic values (Figures 6.1 through 6.12). We recall that in the thermoviscoelastic case only A_+ and V_+ had asymptotic values while A_- and V_- increased indefinitely with the frequency F.

  (b) The attenuation factor A_+ increases with the material constant γ while A_- decreases. Both V_+ and V_- are independent of γ (Figures 6.1, 6.2, 6.7, and 6.8).

  (c) The attenuation factor A_- increases with the material constant ζ while A_+, V_+, and V_- are all independent of ζ (Figures 6.3, 6.4, 6.9, and 6.10)

  (d) The attenuation factor A_+ increases with the material constant ϕ while both A_- and V_- decrease. The phase velocity V_+ is independent of ϕ (Figures 6.5, 6.6, 6.11, and 6.12).

6.5. A Class of Self-Similar Solutions

In the previous sections we obtained conditions under which the partial differential equations (6.7) could have meaningful solution and stable wave propagations would exist. Characteristic speeds and typical dispersion relations were discussed in detail and analytical expressions describing the asymptotic behaviors of the attenuation factor and phase velocity associated with both high and low frequency oscillations were given.

In this section, we will employ the theory of continuous group of transformations to seek a class of self-similar solutions of the set (6.7).

Self-similar solutions are obtained by using appropriate transformations that reduce a system of partial differential equations to a system of ordinary differential equations. In general, the solutions are not unique and success of the method lies greatly on the choice of the transformation. Hansen (1964) discusses several methods for obtaining appropriate transformations.

We will follow the theory developed by Morgan (1952) to seek solutions to the set (6.7). To begin with, we must look for possible transformation groups so that the differential forms

$$\begin{aligned} \lambda_1 &= \rho_0 \, u_{tt} - \mu \, u_{XX} + \pi_T \, T_X \quad , \\ \lambda_2 &= (e_1 + \pi) \, u_{Xt} + e_{0T} \, T_t + e_{0\dot{T}} \, T_{tt} - \kappa \, T_{XX} \quad , \end{aligned} \tag{6.46}$$

are conformally invariant under such transformations. The problem may be simply formulated by using a one-parameter group of transformations defined by

$$(X^*,\ t^*,\ T^*,\ u^*) = (AX,\ A^{-m}\, t,\ A^{n}\, T,\ A^{r}\, u) \quad , \qquad (6.47)$$

where A is the parameter, (m, n, r) are constants; spatial coordinate X and time t are the independent variables, and temperature T and displacement u are the dependent variables. We also assume that the coefficients $\pi(T)$, $\mu(T)$, $e_0(T, \dot{T})$, $e_1(T, \dot{T})$, and $\kappa(T, \dot{T})$ can be expressed explicitly in terms of the products of the powers of their respective arguments. Consistent with our power law transformation, we consider

$$\begin{aligned} \pi(T) &= E_1\, T^{b_1} \quad , \\ \mu(T) &= E_2\, T^{b_2} \quad , \\ e_0(T, \dot{T}) &= E_3\, T^{b_3}\, \dot{T}^{d_3} \quad , \qquad (6.48) \\ e_1(T, \dot{T}) &= E_4\, T^{b_4}\, \dot{T}^{d_4} \quad , \\ \kappa(T, \dot{T}) &= E_5\, T^{b_5}\, \dot{T}^{d_5} \quad , \end{aligned}$$

where E_1, ... , d_5 are constants. Substituting (6.48) into (6.46) gives

$$\begin{aligned} \lambda_1 &= \rho_0\, u_{tt} - E_2\, T^{b_2}\, u_{XX} + E_1\, b_1\, T^{b_1-1}\, T_X \quad , \\ &\qquad\qquad\qquad\qquad\qquad\qquad (6.49) \\ \lambda_2 &= (E_1\, T^{b_1} + E_4\, T^{b_4}\, \dot{T}^{d_4})\, u_{Xt} + E_3\, b_3\, \dot{T}^{b_3-1}\, \dot{T}^{d_3}\, T_t + \\ &\quad + E_3\, d_3\, T^{b_3}\, \dot{T}^{d_3-1}\, T_{tt} - E_5\, T^{b_5}\, \dot{T}^{d_5}\, T_{XX} \quad . \end{aligned}$$

Since we require λ_1 and λ_2 to be conformally invariant under the group of transformations defined by (6.47) such that

$$\lambda_i\left(X^*,\ t^*;\ T^*,\ u^*;\ \frac{\partial T^*}{\partial X^*},\ \frac{\partial T^*}{\partial t^*},\ \frac{\partial^2 T^*}{\partial X^{*2}},\ \frac{\partial^2 T^*}{\partial t^{*2}},\ \frac{\partial^2 u^*}{\partial X^{*2}},\ \frac{\partial^2 u^*}{\partial t^* \partial X^*},\ \frac{\partial^2 u^*}{\partial t^{*2}}\right) =$$

$$= J_i\left(X,\ t;\ T,\ u;\ \frac{\partial T}{\partial X},\ \ldots\ ,\ \frac{\partial^2 u}{\partial t^2};\ A\right)\ \lambda_i\ \left(X,\ t;\ \frac{\partial T}{\partial X},\ \ldots\ ,\ \frac{\partial^2 u}{\partial t^2}\right)\quad , \qquad (6.50)$$

we must have

$$\begin{aligned} n\ b_1 &= 2m + r + 1 \quad , \\ n\ b_2 &= 2m + 2 \quad , \\ n\ b_3 + (m + n)\ d_3 &= 2m + 2r \quad , \\ n\ b_4 + (m + n)\ d_4 &= 2m + r + 1 \quad , \\ n\ b_5 + (m + n)\ d_5 &= 3m - n + 2r + 2 \quad . \end{aligned} \qquad (6.51)$$

Therefore, b_1, b_2, ... , d_5 are not entirely independent of one another.

According to Morgan (1952), the solution to (6.49) may be expressed in terms of functions $f(\xi)$ and $g(\xi)$ of an absolute invariant ξ of the subgroup of the transformations of the independent variables. Therefore, ξ must satisfy the condition

$$\xi\ (X^*,\ t^*) = \xi\ (X,\ t) \quad .$$

There are many ways to choose the form of ξ; several of which may yield satisfactory results. Since we have employed a power law transformation, we assume that ξ is a product of the powers of X and t also. Without loss of generality, we choose

$$\xi = t\,X^{m_1} \quad , \tag{6.52}$$

where m_1 may be determined by requiring ξ to remain invariant. Upon using (6.47), we obtain

$$t^*\,X^{*\,m_1} = A^{-m+m_1}\,t\,X^{m_1} \quad ,$$

which requires

$$m_1 = m \quad , \tag{6.53}$$

to insure absolute invariance. The functions f and g are, as proven by Morgan (1952), absolute invariants under the complete set of transformations (6.47). Again, the choice is unlimited as there may be many forms of f and g that would yield satisfactory answers. Following the power law employed so far, we assume

$$\begin{aligned} f(\xi) &= T\,X^{m_2} \quad , \\ g(\xi) &= u\,X^{m_3} \quad . \end{aligned} \tag{6.54}$$

To determine m_2 and m_3, we substitute the transformations (6.47) into (6.54) and set the powers of A equal to zero to insure absolute invariance of f and g. Thus, we obtain

$$\begin{aligned} m_2 &= -\,n \quad , \\ m_3 &= -\,r \quad . \end{aligned} \tag{6.55}$$

Substituting (6.53) into (6.52) and (6.55) into (6.54) gives

$$\xi = t\, X^m \quad ,$$
$$T = f(\xi)\, X^n \quad , \tag{6.56}$$
$$u = g(\xi)\, X^r \quad .$$

The system of partial differential equations (6.49), upon using (6.56) and the restrictions imposed by (6.51), simplify to a system of ordinary differential equations. It is not the purpose of this treatise to extract results from this class of self-similar solutions which satisfy specific initial and boundary conditions. The details of such an analyis, in general, are very involved.[7]

As a simple example, let us assume that both π and μ are constant. This requires that

$$b_1 = b_2 = 0 \quad . \tag{6.57}$$

Using (6.57) in (6.51) gives

$$m = -1 \quad \text{and} \quad r = 1 \quad . \tag{6.58}$$

Employing (6.56) through (6.58) in Equation $(6.49)_1$ gives

$$(\rho_0 - E_2\, \xi^2)\, g'' = 0 \quad , \tag{6.59}$$

where (') denotes differentiation with respect to ξ. The ordinary differential equation (6.59) is satisfied if

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[7] See, *e.g.*, Burniston and Chang (1970).

$$g(\xi) = D_1 \xi + D_2 \quad , \tag{6.60}$$

where D_1 and D_2 are constants of integration.

Using (6.60) in $(6.56)_3$ and noting that

$$\xi = t/X \quad , \tag{6.61}$$

we obtain:

$$u = D_1 t + D_2 X \quad . \tag{6.62}$$

This expression denotes that the displacement field is linear and thus resulting in a constant strain D_2 and a constant particle velocity D_1.

To simplify the Equation $(6.49)_2$, we further assume that

$$b_3 = b_4 = b_5 = 0 \quad . \tag{6.63}$$

Now the set (6.51) is satisfied if

$$n = 1 \quad , \tag{6.64}$$

and d_3, d_4, d_5 are arbitrary but do not vanish simultaneously. Thus, using (6.56) through (6.58) and (6.62) through (6.64) in Equation $(6.49)_2$, we have the following ordinary differential equation in f:

$$E_3 (f'^{\,d_3})' - E_5 \xi^2 f'^{\,d_5} f'' = 0 \quad . \tag{6.65}$$

Equation (6.65) is satisfied if $f' = 0$ or $f'' = 0$, both of which yield a linear temperature distribution; or if $f' \neq 0$, $f'' \neq 0$, then

$$E_3 d_3 f'^{d_3-d_5-1} - E_5 \xi^2 = 0 \quad , \tag{6.66}$$

provided that $d_3 \neq 0$. The ordinary first-order differential equation (6.66) may be re-written as

$$f' = (E_5/E_3 d_3)^{\ell} \xi^{2\ell} \quad , \tag{6.67}$$

where ℓ satisfies the condition

$$\ell(d_3 - d_5 - 1) = 1 \quad . \tag{6.68}$$

Integrating (6.67) for $\ell \neq 0$, $\ell \neq -\frac{1}{2}$, we obtain

$$f(\xi) = \frac{1}{2\ell + 1} (E_5/E_3 d_3)^{\ell} \xi^{2\ell+1} + D_3 \quad , \tag{6.69}$$

where D_3 is a constant of integration. The temperature distribution may be obtained by using (6.69) in $(6.56)_2$:

$$T = D_3 X + \frac{1}{2\ell + 1} (E_5/E_3 d_3)^{\ell} (t/X)^{2\ell+1} \quad . \tag{6.70}$$

Explicit expressions may also be obtained for the stress Σ, heat flux B, and internal energy e, by substituting the displacement field (6.62), the temperature distribution (6.70), and their derivatives, in the constitutive relations (6.1) through (6.3):

$$\Sigma = - E_1 + E_2 D_2 \quad ,$$

$$B = - E_5 \{D_3 - [(E_5/E_3 d_3)(t/X)^2]^{\ell} t^{-1}\}\{(E_5/E_3 d_3)(t/X)^2\}^{\ell d_5} \quad ,$$

$$\rho_0 e = E_3 \{(E_5/E_3 d_3)(t/X)^2\}^{\ell d_3} + E_4 D_2 \{(E_5/E_3 d_3)(t/X)^2\}^{\ell d_4} \quad .$$

(6.71)

The set (6.71) defines a constant stress temperature-rate dependent heat conducting thermoelastic medium.

7. CONCLUSIONS

A thermodynamic theory has been presented describing a class of temperature-rate dependent materials. Principles of modern axiomatic continuum mechanics were employed to impose certain restrictions on the constitutive relations.

One-dimensional linear-gradient theories for the thermoviscoelastic and thermoelastic cases satisfying the general theory were presented and compared. It was shown that there is only one characteristic speed associated with the thermoviscoelastic case and is due to the temperature rate effect. The thermoelastic medium, on the other hand, possesses an additional characteristic speed of the elastic wave. This difference is believed to be due to the overwhelming viscous effects which override the elasticity of the material at high frequency of oscillation in a thermoviscoelastic medium.

Dispersion relations were presented in dimensionless forms and analytical expressions for the asymptotic behaviors of the attenuation factors and phase velocities were derived for each case. It was demonstrated that the high frequency asymptotic phase velocities coincide with the characteristic speeds obtained earlier. Physical limitations were placed on the dimensionless material constants by using criteria for stable wave propagations at all frequency levels and by drawing analogy with results of the classical thermodynamics.

A class of self-similar solutions was obtained for the thermoelastic problem using the method of continuous group of transformations. Explicit expressions were obtained for the constitutive relations in the case of a constant-stress, heat-conducting medium.

8. LIST OF SYMBOLS

A	Similarity parameter appearing in (6.47).
A(F)	Non-dimensional attenuation factor.
B	Heat flux.
B_A	Heat flux vector.
C	Internal heat generation per unit mass per unit time.
D_j	Integration constants ($j = 1, 2, 3$).
E_j	Constant coefficients ($j = 1, 2, \ldots, 5$).
F	Non-dimensional frequency.
F_i	Body force per unit mass.
F_β	Functions of irreducible integrity basis.
G_{AB}	Cauchy-Green strain tensor.
G_β	Elements (linear in Ψ_{AB}) of irreducible integrity basis.
H(F)	Non-dimensional complex wave-number.
H_β	Functions of irreducible integrity basis.
J	Jacobian of the deformation gradient.
J_i	Functions of transformation variables and parameter.
L_β	Elements (linear Ψ_{AB}) of irreducible integrity basis.
P_{AB}	Piola stress tensor.
Q	Full orthogonal group.
Q_{ij}	Time dependent proper orthogonal transformations.
S	Symmetry group of material.
S	A scalar-valued function.

S_{AB}	Elements of the symmetry group $\mathcal{S}$.
T	Absolute temperature.
T_0	A complex amplitude coefficient.
T_{ij}	A second-order tensor-valued function.
$V(F)$	Non-dimensional phase velocity.
V_i	A vector-valued function.
X	Spatial coordinate.
Y_A	Reference coordinate system.
a	Specific Helmholtz free energy per unit mass.
b_i	Constant powers $(i = 1, 2, \ldots, 5)$.
c	Wave velocity.
d_i	Constant powers $(i = 3, 4, 5)$.
e	Specific internal energy per unit mass.
$e_i(T, \dot{T})$	Coefficients appearing in the expression for the internal energy.
$f(\xi)$	An absolute invariant function defined by (6.56).
$g(\xi)$	An absolute invariant function defined by (6.56).
i	Square root of (-1).
k	Complex wave-number.
ℓ	Constant power defined by Equation (6.68).
m	Constant power.
m_i	Constant powers $(i = 1, 2, 3)$.
n	Constant power.
p_i	Position vector.
q_i	Particle velocity.

r	Constant power.
s	Specific entropy per unit mass.
t	Time.
t_0	Original time of reference.
u	Displacement field.
u_0	Complex amplitude coefficient.
x	Spatial coordinate.
y_i	Spatial coordinate system.
κ	Coefficient of heat conductivity.
Σ	Kirchoff-Piola longitudinal stress.
Σ_{Ai}	Kirchoff-Piola stress tensor.
Φ	Any field variable.
Ψ	A scalar invariant given by (3.20)
Ψ_{AB}	An arbitrary second-order symmetric tensor.
Ω	A characteristic frequency.
α	Attenuation factor.
γ	Non-dimensional material constant.
δ_{ij}	Kronecker delta.
ε	Lagrangian strain.
ζ	Non-dimensional material constant.
$\eta(T, \dot{T})$	Coefficient appearing in the expression for the stress.
λ_i	Differential forms defined by Equations (6.46), $i = 1, 2$.
$\mu(T)$	Coefficient appearing in the expression for the stress.
ξ	Absolute invariant of the subgroup of transformations defined by (6.52).

$\pi(T)$	Coefficient appearing in the expression for the stress.
ρ	Density.
ρ_0	Initial density.
ϕ	Non-dimensional material constant.
χ	Non-dimensional material constant.
$\psi(T)$	Coefficient appearing in the expression for the specific Helmholtz free energy.
ω	Frequency.

9. LIST OF REFERENCES

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