Equations of Motion in the Linear Approximation†

by

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1. Introduction

There are some advantages in approaching the linear approximation to General Relativity through the field rather than the potential: geometry ends where gauge transformations begin.

This paper is a first step towards a gauge-invariant theory of the motion of singularities. The principal results were reported by one of us at the Haifa Seminar on Relativity and Gravitation in July 1969; and the work was completed† shortly afterwards. We have presented it elsewhere, but never previously in writing. We are happy indeed that this delinquency enables us to dedicate it now to Professor J. L. Synge, whose elegant geometrical approach to Relativity Theory has been the inspiration of several generations of scientists.

2. Linear Approximation

If g_{ab} is a metric of n-dimensional Riemannian space, the curvature tensor

$$R_{abcd} := \tfrac{1}{2}\{g_{pq,rs} + g^{km}[pq,k][rs,m]\}\delta^{ps}_{ab}\delta^{qr}_{cd}\ , \tag{2.1}$$

satisfies

$$R_{abcd} = R_{[cd][ab]} - R_{[abcd]}, \tag{2.2}$$

and

$$R_{ab[cd;e]} = 0. \tag{2.3}$$

Let us now consider some of the consequences of these identities, without taking account of the definition (2.1). The fourth rank tensor

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† So, at least, we supposed at the time; but Andrzej Trautman[1] pointed out a significant gap. We have tried to fill it in Section 9.

$$G_{abcd} := R_{abcd} + g_{ac}R_{bd} + g_{bd}G_{ac} - g_{ad}R_{bc} - g_{bc}G_{ad}, \tag{2.4}$$

where

$$R_{ab} := g^{pq}R_{apqb}, \quad G_{ab} := R_{ab} - \tfrac{1}{2}g^{pq}R_{pq}g_{ab}, \tag{2.5}$$

satisfies the algebraic identities (2.2), and has vanishing divergence,

$$G^{abcd}{}_{;d} = 0; \tag{2.6}$$

whence

$$R^{abcd}{}_{;d} = 0, \tag{2.7}$$

in the Ricci-flat case,

$$R_{ab} = 0. \tag{2.8}$$

All this holds in n dimensions. For n = 4, we introduce the duals,

$$\begin{aligned} {}^*R_{abcd} &:= \tfrac{1}{2}\eta_{abpq}R^{qp}{}_{cd}, \\ R^*{}_{abcd} &:= \tfrac{1}{2}R_{ab}{}^{pq}\eta_{qpcd}, \end{aligned} \tag{2.9}$$

where η_{abcd} is the Levi-Civita's totally antisymmetric tensor. The double duals are given by

$$\begin{aligned} {}^{**}R_{abcd} &= R^{**}{}_{abcd} = -R_{abcd}, \\ {}^*R^*{}_{abcd} &= -G_{abcd}. \end{aligned} \tag{2.10}$$

It follows directly that in four dimensions, equation (2.6) is not merely a consequence of Bianchi identities, but is actually equivalent to them. In Ricci-flat space (2.8), the left and right duals are equal, and the system is self-dual, in the sense that the basic equations (2.2) and (2.3) hold for $*R_{abcd}$; conversely, self-duality implies (2.8).

To obtain the linear approximation, we assume that the metric is the sum of a potential h_{ab} which is small, and a background which is flat. We assume further that covariant differentiation with respect to the background does not alter orders of smallness; and we discard anything smaller than the potential. Since this decomposition is not unique, the potential is subject to a gauge transformation

$$h_{ab} \to h_{ab} + \xi_{a;b} + \xi_{b;a} \, . \tag{2.11}$$

At this point we can avoid some tedious repetition by means of a small change in notation. From now onwards, the background metric will be denoted by g_{ab} and used for all operations of index shifting and covariant differentiation. In our new notation, the full metric is $g_{ab} + h_{ab}$. Instead of (2.1),

$$\{g_{pq,rs} + g^{km}[pq,k][rs,m]\}\delta^{ps}_{ab}\delta^{qr}_{cd} = 0, \tag{2.12}$$

and

$$R_{abcd} = \tfrac{1}{2}h_{pq;rs}\delta^{ps}_{ab}\delta^{qr}_{cd} \, . \tag{2.13}$$

From these two equations there now follow the basic identities (2.2) and (2.3), with all the consequences we have already drawn from them. Because of (2.12), we can introduce Cartesian co-ordinates x^r in which the components of the background metric are constants. Then, for any R_{abcd} subject to the basic equations (2.2) and (2.3), the general solution of (2.13) is given by

$$h_{ab} = 2x^p x^q \int_0^1 R(\lambda x)_{apqb}\lambda(1-\lambda)d\lambda + \xi_{a,b} + \xi_{b,a}, \tag{2.14}$$

provided that this integral exists. Locally, at least, we can discard the gravitational potential and its attendant gauge transformation.

We conclude by making two trivial formal changes: in the rigorous theory we shall regard R_{abcd} as an independent field, subject to the field equations (2.1); and in the linearized theory we shall drop the assumption that R_{abcd} is small†. Formally, then, our theories look almost identical: all the basic equations for the metric and the field are the same in both theories, except that (2.1) in the rigorous theory is replaced by (2.12) in the approximation. The difference is of some significance in our computations.

3. Geometrical preliminaries

Suppose that we have a parametrized time-like line in Minkowski space. We confine our attention to the subspace consisting of future null-cones springing from this source-line. It may be the whole of space, as for example, when the source is bounded by a three-dimensional cylinder. In any case, if the source is sufficiently smooth the subspace will be a four-dimensional manifold. From these ingredients alone, without utilizing any extraneous elements, we shall construct a number of fields throughout the region.

On the line itself, we have the parameter σ and various functions of it, such as the tangent vector and its squared magnitude $\underline{K}$. To define these fields throughout the region, we require that they shall be constant on each of the special half-cones. The procedure is consistent, because the half-cones do not intersect. Next we consider the null displacement from the source line to a point of the subspace. By contracting it with the tangent and its first derivative respectively, we form the scalars ρ and ρH: thus if the source line is, in Cartesian co-ordinates,

$$x^k = X^k(\sigma) , \tag{3.1}$$

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†We could have avoided this condition in the first place by making an expansion in terms of a small parameter. So we are no worse off than in a more laborious formulation of the approximation theory.

we have

$$g_{ab}[x^a - X^a(\sigma)][x^b - X^b(\sigma)] = 0, \tag{3.2}$$

$$\rho := \dot{X}_r(\sigma)[x^r - X^r(\sigma)] \geqslant 0, \tag{3.3}$$

$$H := \ddot{X}^r \sigma_{,r}, \tag{3.4}$$

$$K := \dot{X}^r \dot{X}_r , \tag{3.5}$$

where the dot denotes differentiation with respect to σ.

We extend this operation to an arbitrary scalar (or tensor) field ψ by writing

$$\dot{\psi} := g^{ab} \psi_{;a}[\dot{X}_b - H\rho\sigma_{,b}]. \tag{3.6}$$

It is convenient also to introduce the 2-space Laplacian

$$\Delta\psi := \rho^2 ({*M}^{ap} {*M}_p{}^b \psi_{;b})_{;a}, \tag{3.7}$$

where

$$M_{ab} := 2\rho_{,[a}\sigma_{,b]}, \tag{3.8}$$

and ${*M}_{ab}$ is its dual,

$${*M}_{ab} := \tfrac{1}{2}\eta_{abcd}M^{dc}. \tag{3.9}$$

From M_{ab} and the null bivector

$$N_{ab} := K^{-1}(2\ddot{X}_{[a}\sigma_{,b]} - HM_{ab}), \tag{3.10}$$

we can construct three linearly independent solutions of the algebraic conditions (2.2) and (2.8):

$$D_{abcd} := M_{ab}M_{cd} - {}^*M_{ab}{}^*M_{cd} - (g_{ad}g_{bc} - g_{ac}g_{bd})/3, \tag{3.11}$$

$$III_{abcd} := M_{ab}N_{cd} + N_{ab}M_{cd} - {}^*M_{ab}{}^*N_{cd} - {}^*N_{ab}{}^*M_{cd}, \tag{3.12}$$

$$N_{abcd} := N_{ab}N_{cd} - {}^*N_{ab}{}^*N_{cd}. \tag{3.13}$$

These fields, like the scalars σ and $\underline{H}$, are constant on any future null ray from the source. They are of types (2,2), (3,1) and (4) respectively, in the Penrose classification. All have $\sigma,_k$ as propagation vector.

We can construct similar fields by substituting higher derivatives for $\ddot{X}_a$. To describe the process in general terms, we introduce a vector-field $\ell_a(\sigma)$ without describing specifically how it is formed from the tangent field and its derivatives. We project ℓ_a and $\sigma,_a$ into the subspace orthogonal to $\dot{X}_a$, and take the cosine λ of the angle between these projections; assuming, of course, that the first of the projections does not vanish:

$$\lambda := \Lambda^{-\frac{1}{2}}(\sigma,_q - K^{-1}\dot{X}_q)\ell^q, \tag{3.14}$$

where

$$\Lambda := K^{-2}[(\ell_p\dot{X}^p)^2 - K\ell_p\ell^p] \geqslant 0. \tag{3.15}$$

Since $K^{-1}\Delta$ is the Laplacian on a unit sphere, and λ is a direction cosine,

$$(\Delta + 2K)\lambda = 0. \tag{3.16}$$

We now write

$$L_{ab} := (1-\lambda^2)^{-1}[2\ell_{[a}\sigma,_{b]} - \ell^p\sigma,_p M_{ab}], \tag{3.17}$$

$$L_{abcd} := L_{ab}L_{cd} - {}^*L_{ab}{}^*L_{cd}. \tag{3.18}$$

The first tensor is linearly dependent on N_{ab} and $^*N_{ab}$; the second on N_{abcd} and $^*N_{abcd}$. We see that $K\sigma,_a\sigma,_b$ is invariant under transformations of the parameter σ, and that

$$L_{ap}L^p{}_b = (1-\lambda^2)^{-1}\Lambda K\sigma,_a\sigma,_b: \tag{3.19}$$

thus we associate an invariant magnitude $(1-\lambda^2)^{-1}\Lambda(\sigma)$ with the null bivector L_{ab} and the source-line. In each null cone of constant σ, L_{ab} is singular on the two rays defined by $\lambda = \pm 1$.

In the special case $\ell_a = \ddot{X}_a$, λ is proportional to $H - \dot{K}/2K$, and (3.16) reduces to

$$(\Delta + 2K)H = \dot{K}. \tag{3.20}$$

4. Non-geodetic particle in the linearized field theory

In special relativity, a particle is represented by a time-like world-line, together with certain functions on the line which describe intrinsic properties, such as mass and spin. One would expect gravitational effects to be infinite at the world-line, to propagate with the speed of light, and to disappear at infinity. We shall be concerned with a very simple kind of particle, in which all the intrinsic functions reduce to a single constant, the mass. We shall demand also that at very close quarters its field becomes indistinguishable from that of the Schwarzschild solution. To make this idea more precise, we shall define a Schwarzschild field for an arbitrary time-like world-line. For a straight world-line, this is simply the linear approximation to the curvature tensor of Schwarzschild solution, with its source on the world-line. At any point of an arbitrary time-like world-line, we construct the straight line tangent to it, and the Schwarzschild solution for a unit mass located on the straight line. This we restrict to the future null-cone with its vertex at the point. Piecing together the fields obtained in this way, we get the *Schwarzschild field*,

$$3(\rho/\sqrt{K})^{-3}D_{abcd}. \tag{4.1}$$

The algebraic conditions (2.2) and (2.8) are satisfied identically. The differential equations (2.3) are satisfied if and only if the source-line is straight. We are now faced with a well defined problem: to find a solution of the basic algebraic and differential equations with the following three properties:

1) It is constructed from the source line, the future null-cones emanating from it, a scale factor, and nothing else;
2) It consists of terms of degree -3, -2 and-1, in ρ;
3) Its leading term is the Schwarzschild field (4.1).

The Schwarzschild *solution*, incidentally, may be written as

$$P_{abcd} = \rho K^{-\frac{1}{2}}[3(P_{ab}P_{cd} - {}^*P_{ab}{}^*P_{cd}) + \tfrac{1}{2}P^{k\ell}P_{k\ell}(g_{ad}g_{bc} - g_{ac}g_{bd})], \tag{4.2}$$

where P_{ab} is the unit Coulomb field. It is easy to see that the field

$$P_{abcd} = 3K\sqrt{K}(\rho^{-3}D_{abcd} + \rho^{-2}III_{abcd} + \rho^{-1}N_{abcd}), \tag{4.3}$$

which we obtain by substituting the Liénard-Wiechert solution†

$$P_{ab} := K(\rho^{-2}M_{ab} + \rho^{-1}N_{ab}) \tag{4.4}$$

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†This form of the solution is essentially the same as that given by Synge[2].

into (4.2), has all the features we are looking for, apart from the vanishing of its divergence. For that, one calculates

$$P_{abc}{}^{d}{}_{;d} = -3\rho^{-2}K^{2}[2\ell_{[a}{}^{\sigma}{}_{,b]} - \ell^{p}{}_{\sigma,p}M_{ab}]\sigma_{,c}, \tag{4.5}$$

where

$$\ell_a := K^{-\frac{1}{2}}[K^{-\frac{1}{2}}(K^{-\frac{1}{2}}\dot{X}_a)^{\cdot}]^{\cdot}, \tag{4.6}$$

which vanishes for a line of constant curvature. We have thus succeeded in constructing a very natural generalization of Schwarzschild solution, in which the acceleration of the source is constant but not necessarily zero.

Could we construct a solution about an *arbitrary* base-line by making a better choice of the field in $1/\rho$? A straightforward calculation shows that, for any $\ell_a(\sigma)$,

$$(\rho^{-1}\Psi L_{abc}{}^{d})_{;d} = \rho^{-1}L_{abc}{}^{d}\Psi_{;d}: \tag{4.7}$$

hence, if ℓ_a is given by (4.6), and

$$\Psi = K\lambda(3-\lambda^2)/3\sqrt{\Lambda}, \tag{4.8}$$

the divergence of $3\Psi K^{-\frac{1}{2}}\rho L_{abcd}$ just cancels out that of P_{abcd}.

There is one difficulty. By taking Ψ in equation (4.7) to be a function of σ only, we obtain a spherically-fronted null solution emanating from a time-like source; and it is well known that such a solution has at least one singular direction on each null cone. In this case, we have seen that there are two such singular directions, corresponding to the intersection of the null cone with the plane containing $\dot{X}_a$ and ℓ_a; that is, to the values ± 1 of λ. These singularities survive when Ψ is given by (4.8). We can get rid of either one, but not both at once, by adding $+2K/3\sqrt{\Lambda}$ to Ψ. This has the effect of adding a spherically-fronted wave originating on the source-line. By adding the most general wave of this kind, we obtain

$$R_{abcd} = 3K\sqrt{K}(\rho^{-3}D_{abcd} + \rho^{-2}III_{abcd} + \rho^{-1}\hat{N}_{abcd}), \qquad (4.9)$$

where
$$\hat{N}_{abcd} := N_{abcd} + \Psi L_{abcd} + W_{ab}W_{cd} + \bar{W}_{ab}\bar{W}_{cd}, \qquad (4.10)$$

$$W_{ab} := \rho W_{,[a}\sigma_{,b]}, \qquad (4.11)$$

and

$$W = W(\sigma, \ell n \tan \tfrac{1}{2}\theta + i\phi); \qquad (4.12)$$

θ,ϕ being polar coordinates on each of the unit spheres for which σ is constant and $\rho = \sqrt{K}$. On those null cones where $\ell_a \neq 0$, however, there is no way of choosing W so that $\hat{N}_{abcd}$ is bounded in all directions. The only acceptable solution,of the class constructed here, has a line of constant curvature as its source.

5. Comparison with the electromagnetic field

There is a very close formal resemblance between the electromagnetic and the linearized gravitational fields, outside sources. In Maxwell's theory, we start with a potential ϕ_a, defined up to a gauge transformation,

$$\phi_a \to \phi_a + \psi_{,a} . \qquad (5.1)$$

The field

$$F_{ab} := \phi_{a,b} - \phi_{b,a} \qquad (5.2)$$

then satisfies

$$F_{ab} = F_{[ab]}, \quad F_{[ab,c]} = 0, \tag{5.3}$$

identically. Conversely, given any solution of the basic field equations, we can recover the potential up to a gauge transformation. The remaining field equations for empty space,

$$F^{ab}{}_{;b} = 0, \tag{5.4}$$

are necessary and sufficient conditions for the system to be self-dual that is, for $*F_{ab}$ to satisfy the basic equations (5.3). This being the case, it is instructive to consider how we might have proceeded if the analogy went even further, and we knew the Coulomb solution but not the Liénard-Wiechert potential,

$$\phi_a = (K\rho^{-1} - H)\sigma_{,a} + (\ell n\rho)_{,a}. \tag{5.5}$$

Following the procedure used in Section 4, we should define a Coulomb field for an arbitrary time-like line. This turns out to be $\rho^{-2} KM_{ab}$ It satisfies (5.3) but not (5.4); so we should look for an additional term in ρ^{-1} which satisfies the first two equations and annihilates the charge current vector of the Coulomb field. An obvious solution is given by

$$F_{ab} = P_{ab} + (W_{ab} + \bar{W}_{ab})/\rho, \tag{5.6}$$

with (4.4), (4.11), (4.12), and some further restriction on W to ensure that the field can be constructed out of the source-line and its future half-cones only. The last term in (5.6) is made zero to avoid singular generators on the null cones. Thus the difference between the two cases reduces to this: *in the electromagnetic case we can avoid singularities for an arbitrarily moving source; in the gravitational case, only for a source moving with a constant acceleration.*

6. A special class of metrics

The field obtained in Section 4 belongs to a class of curvature tensors which has been extensively investigated in the rigorous theory. The metric in such cases is formed out of functions

$$m(\sigma), \quad f(\sigma,\xi), \quad a(\sigma,\xi), \quad p(\sigma,\xi,\eta), \tag{6.1}$$

subject to

$$\partial f/\partial\sigma + f^2\partial a/\partial\xi = 0. \tag{6.2}$$

It may be written as

$$ds^2 = (K - 2H\rho - 2m/\rho)d\sigma^2 + 2d\rho d\sigma$$
$$-\rho^2 p^{-2}[f^{-1}(d\xi + afd\sigma)^2 + fd\eta^2], \tag{6.3}$$

with

$$H := (\ell np - \tfrac{1}{2}\ell nf)^{\cdot}, \tag{6.4}$$

$$K := \Delta(\ell np - \tfrac{1}{2}\ell nf), \tag{6.5}$$

where, for any $\psi(\xi,\eta,\sigma)$,

$$\psi' := \partial\psi/\partial\xi, \tag{6.6}$$

$$\dot\psi := \partial\psi/\partial\sigma - af\psi', \tag{6.7}$$

$$\Delta\psi := p^2[(f\psi')' + f^{-1}\partial^2\psi/\partial\eta^2]. \tag{6.8}$$

From these definitions, (3.20) again follows identically. For empty space

$$\dot m - 3Hm = \tfrac{1}{4}\Delta K, \tag{6.9}$$

whence

$$\tfrac{1}{4}(\Delta + 2K)\Delta K = 2K\dot{m} - 3m\dot{K}; \tag{6.10}$$

for nullity or flatness,

$$m = 0, \tag{6.11}$$

$$K = K(\sigma). \tag{6.12}$$

The system of equations is invariant under the transformation

$$\rho \to \rho/\dot{\Sigma}(\sigma), \quad \sigma \to \Sigma(\sigma), \tag{6.13}$$

with $[\xi,\eta] \to [\xi,\eta]$: for positive $\underline{K}$, therefore, we may replace (6.12) by the stronger condition

$$K = 1. \tag{6.14}$$

Flat space-time is characterized by (6.10), (6.11) and

$$\begin{aligned} &[F + p^{-1}(f^2p'' - \partial^2 p/\partial\eta^2)]^{\cdot} = 0, \\ &[p^{-1}(f'\partial p/\partial\eta - 2f\partial p'/\partial\eta]^{\cdot} = 0, \end{aligned} \tag{6.15}$$

where

$$F := \tfrac{1}{4}f'^2 - \tfrac{1}{2}ff''. \tag{6.16}$$

The scalars ρ,σ, $\underline{H}$, $\underline{K}$, together with the operators Δ and $^{\cdot}$, here revert to the meanings they were given in Section 3.

For any surface of constant ρ and σ, the Gaussian curvature is K/ρ^2. A singularity in $\underline{K}$ not only shows that the corresponding subspaces are singular, but also gives rise to a directional singularity in the curvature tensor. This involves the coefficient of ρ^{-2}, unlike the singularities discussed in

Section 4, which are confined to the term in ζ^{-1}.

These are all well known results for the case

$$f = 1, \quad a = 0. \tag{6.17}$$

To establish them more generally, it is sufficient to observe that (6.17) is a co-ordinate condition, arising from the transformation

$$[\rho,\sigma,\xi,\eta] \to [\rho,\sigma,\Xi(\xi,\sigma,),\eta], \tag{6.18}$$

with

$$d\Xi = f^{-1}d\xi + a d\sigma, \tag{6.19}$$

which is integrable on account of (6.4).

If p is independent of η, however, it is sometimes more convenient to put $\Xi' = 1/p$ in the transformation (6.18), so that

$$p = 1, \quad H = \tfrac{1}{2}(af)', \quad K = -\tfrac{1}{2}f'' ; \tag{6.20}$$

and (6.9), (6.12), (6.15) reduce to

$$[f(f''' - 12\, am) + 8\xi dm/d\sigma]' = 0, \tag{6.21}$$

$$F' = 0, \tag{6.22}$$

$$\partial F/\partial\sigma = 0, \tag{6.23}$$

respectively. We still have at our disposal the transformations (6.13) combined with

$$[\xi,\eta] \to [\xi/\dot{\Sigma}^2,\eta]; \tag{6.24}$$

(6.18) with

$$\Xi(\xi,\sigma) = \xi + \Xi(\sigma); \tag{6.25}$$

and

$$[\rho,\sigma,\xi,\eta] \to [\rho,\sigma,\mu\xi,(\eta+\nu)/\mu], \tag{6.26}$$

with μ and ν constant. Hence we can reduce the most general null or flat field with positive K to

$$f = F(\sigma) - \xi^2, \quad a = A(\sigma) + \partial G/\partial\sigma,$$
$$G := \{\ell n[(\sqrt{F} + \xi)/(\sqrt{F} - \xi)]\}/2\sqrt{F}, \tag{6.27}$$

with the further reduction to

$$f = 1 - \xi^2, \quad a = a(\sigma), \tag{6.28}$$

in the flat case. Here $\rho = 0$ can be shown to represent a line, of curvature $a(\sigma)$, in a fixed space-time plane.

7. Gravitational potential.

We divide the metric into a background and a potential, requiring the background to satisfy (6.11), (6.14),(6.15). Hence

$$(\Delta + 2)H = 0, \tag{7.1}$$

from (3.20); and

$$\Delta(\delta K - 6H\delta m) = 4\delta\dot{m}, \tag{7.2}$$

from the variation of (6.9). There are now two possibilities: either

$$\delta\dot{m} = 0, \tag{7.3}$$

and

$$\delta K = 6H\delta m + \kappa(\sigma); \tag{7.4}$$

or δK is singular. We have seen, however, that a singularity in $\underline{K}$ leads to an unacceptable singularity in the curvature tensor; and this effect does not disappear in the linear approximation. We therefore opt for (7.3) and (7.4), adjoining

$$\kappa(\sigma) = 0, \tag{7.5}$$

by means of an infinitesimal transformation (6.13).

This is a crucial decision: without it, the linearized curvature tensor would include the most general field which is algebraically degenerate and subject to the conditions laid down in Section 4; when (7.3) and (7.4) are satisfied, however, the linearized curvature tensor coincides precisely with the solution (4.9) for a particle in arbitrary motion.

In the co-ordinates (6.17) the potential satisfies

$$h_{k\ell} = T\sigma_{,k}\sigma_{,\ell} + \upsilon\rho\sigma_{;k\ell} \tag{7.6}$$

for any background, where

$$T := \tfrac{1}{2}\delta K + (\dot{\upsilon} + \upsilon H)\rho - (\delta m + \upsilon m)/\rho \tag{7.7}$$

and

$$\upsilon := \tfrac{1}{2}\delta f/f - \delta p/p. \tag{7.8}$$

From (6.5), (6.8), (6.14)

$$\delta K + (\Delta + 2)\upsilon = 0. \tag{7.9}$$

Hence, using (7.4), (7.5), and substituting

$$\upsilon = 2H\, \ell n|h - H| + 2h + \omega, \quad h := \pm\sqrt{(\ddot{X}_a \ddot{X}^a)}, \tag{7.10}$$

we obtain

$$(\Delta + 2)\omega = 0, \quad g^{pq}\omega_{,p}\sigma_{,q} = 0. \tag{7.11}$$

Equations (7.4)-(7.7) and (7.10), together with the background conditions, constitute an algorithm for constructing the potential out of an arbitrary source-line, the future null cones springing from it, a parameter δm, a function ω subject to (7.11), and nothing else: all the auxilliary apparatus has disappeared. To complete the operation, we must select an intrinsic solution of (7.11): a simple example is

$$\omega = 0. \tag{7.12}$$

For any σ such that $\ddot{X}_a\delta m \neq 0$, there is at least one generator of the future half-cone on which υ is singular: that follows immediately from (7.1), (7.4) and (7.9). In the special case (7.12), there is only one singular generator: an intersection of the half cone with the plane of $\dot{X}_a$ and $\ddot{X}_a$.

For motion in which this plane is independent of σ, we shall construct another form of potential. We could do so by means of a gauge transformation (2.11); but it is easier to start from different coordinate conditions: (6.20) instead of (6.17), with (6.28) for the background. We then have

$$(\delta f - aff'\delta m)'' = 0, \tag{7.13}$$

from the (7.4), (7.5), and

$$\partial\delta f/\partial\sigma + f^2\delta a' = 0, \tag{7.14}$$

from (6.2); whence

$$\delta f = aff'\delta m + \phi(\sigma) + f'\psi(\sigma), \tag{7.15}$$

$$\delta a + \dot{a}\delta m \ell n f = \tfrac{1}{4}\phi\{\ell n[(1-\xi)/(1+\xi)] + f'/f\} + \psi/f + \alpha(\sigma). \tag{7.16}$$

A transformation (6.25) gives

$$\psi(\sigma) = 0. \tag{7.17}$$

The remaining functions of integration, $\phi(\sigma)$ and $\alpha(\sigma)$, come directly from the functions F and A of the exact solution (6.27).

For δa to be nonsingular, both $\dot{a}$ and $\dot{\phi}$ must be zero. We then have a particle of constant background acceleration, unaccompanied by a wave. Suppose that $\dot{\alpha}$ vanishes as well. We remove ϕ by a transformation (6.26), and change the notation slightly: in place of $m + \delta m$, $f + \delta f$, $a + \delta a$, we write m, f, a. The metric which we thereby reassemble out of the background and the potential is given by (6.3), (6.20) and

$$f = (1-\xi^2)(1-2am\xi), \tag{7.18}$$

with constant a and m. It satisfies (6.2) and (6.21) exactly.

8. Singularities in the potential.

Let S be a surface where $\rho = 1$ and σ is constant. The line-element $d\Omega$ induced on S by the background metric is that of a unit sphere. When a gravitational potential is added to the background, the induced line-element becomes $(1+\upsilon)d\Omega$, correct to the first order in υ. To the same accuracy, however, this is the line-element of the distorted sphere

$$r = 1+\upsilon, \tag{8.1}$$

in Euclidean 3-space. A singularity in υ corresponds to a spike in the distortion; and we see from (7.9) that such a spike develops whenever we introduce a spherical harmonic of degree one into δK.

In the case of axial symmetry, we can deal with such singularities more precisely. Consider the line-element

$$ds^2 = f^{-1}d\xi^2 + fd\eta^2, \qquad |\xi| \leq 1, \tag{8.2}$$

where $f(\xi)$ is C^2 and positive in the interval $(-1,1)$, vanishing at the end points, so that

$$\int_{-1}^{1} \xi K d\xi = -\tfrac{1}{2}[f'(1) + f'(-1)]. \tag{8.3}$$

Let

$$u(\xi) := \int \surd\{f^{-1}[1-(f'/2\mu)^2]\}d\xi, \qquad v(\xi) := \surd f, \tag{8.4}$$

where 2μ is the upper bound of $|f'|$ in $(-1,1)$. In a Euclidian plane, the curve

$$x = u(\xi), \qquad y = v(\xi), \tag{8.5}$$

has the slope

$$dy/dx = \pm[(2\mu/f')^2 - 1]^{-\frac{1}{2}} \; ; \tag{8.6}$$

and the surface formed by rotating it about the x-axis (through an angle $\phi = \mu\eta$) has the line-element (8.2). By adjusting the range of η, we can always arrange for the surface of rotation to be regular at either one of its intersections with the axis: because of (8.3) and (8.6), the condition for it then to be regular at the other end point is

$$\int_{-1}^{1} \xi K d\xi = 0. \tag{8.7}$$

Viewing (8.7) against the slightly unnatural background of the unit sphere (6.28), (8.2), we might describe it as the condition for there to be no surface harmonic of degree one in K.

In the exact solution (7.18), this condition is violated; and the subspaces of constant ρ and σ have conical singularities at $\xi = 1$, or $\xi = -1$, or both, depending on the range of η.

Thus, in both approximate and exact solutions, any acceleration gives rise to directional singularities.

9. Surface Integrals.

Reverting, for the moment, to the rigorous theory, and writing

$$\Delta^{rab}{}_{cds} := \tfrac{1}{2}g\delta^{abr}_{cds}, \tag{9.1}$$

$$t^{abcd} := \eta^{abpq}\eta^{cdrs}\Gamma^{k}_{pr}\Gamma^{\ell}_{qs}g_{k\ell}, \tag{9.2}$$

we have

$$g(G^{abcd} + t^{abcd}) = \Delta^{rabcds}{}_{,rs}, \tag{9.3}$$

from (2.1) and (2.10).

Suppose that E is a time-like hypersurface with the normal n_a, that S is a closed surface in E with the oriented surface element dS_a, and that

$$J^{ab} := -\int_S g(G^{abpq} + t^{abpq})\, n_q dS_p, \tag{9.4}$$

in co-ordinates for which the components of n_a are constants. Because of (9.3), and the antisymmetry of Δ_{rabcds} in its last three indices, each component of J_{ab} is the curl of a vector integrated over a closed surface; and therefore

$$J_{ab} = 0. \tag{9.5}$$

This is an exact result.

In the linear approximation, if E is a hyperplane, we may put

$$t^{abcd} = 0 \tag{9.6}$$

by taking Cartesian co-ordinates. Suppose, in particular, that we define E by fixing the scalars ρ and σ in (3.3). Let S be its intersection with the null cone given, for the same value of σ, by (3.2). Demanding further that

$$n_a n^a = 1 = -g, \tag{9.7}$$

we find that

$$J_{ab} = \int \tfrac{1}{2} K^{-1} \rho^2 G_{abcd} M^{dc} d\omega \tag{9.8}$$

over a unit sphere.

For the fields defined by (4.9)-(4.12), the integrand turns out to be

$$(2\rho^{-1} M_{ab} + 3N_{ab})\sqrt{K} \tag{9.9}$$

or

$$6K^{-3/2} \delta^p_{[a} \dot{X}_{b]} \ddot{X}^q (g_{pq} + r_p r_q) + 2r_{[a} s_{b]}, \tag{9.10}$$

where

$$\begin{aligned} r_a &:= (\dot{X}_a - K\sigma_{,a})/\sqrt{K}, \\ s_a &:= 2\rho^{-1} \dot{X}_a + 3K^{-\frac{1}{2}} [K^{-\frac{1}{2}} \dot{X}_a]^{\cdot}. \end{aligned} \tag{9.11}$$

On integrating, we obtain

$$J_{ab} = 8\pi K^{-3/2} (\ddot{X}_a \dot{X}_b - \dot{X}_a \ddot{X}_b), \tag{9.12}$$

which, by (9.5), entails vanishing acceleration.

Thus, following Trautman's suggestion, we can complete our derivation of the equations of motion without calculating the potential.

10. Remarks

In Section 4 we set up criteria for a simple particle, examined a class of possible solutions, and showed that it contained just one acceptable member: a particle in constant acceleration. By a slight extension of the argument developed in Section 7, we can show that this is the only acceptable solution which can be constructed out of the bivectors N_{ab}, M_{ab} and their duals. We made no use, however, of the null bivectors with

$$\dot{X}_a - \tfrac{1}{2}K\sigma_{,a}$$

as their principal null direction, which are needed to complete the basis. E. T. Newman[3], who has investigated the problem more generally, informs us that we lose nothing by this omission. Thus, the gauge invariant field theory set up in Section 2 gives rise to an equation of motion: the acceleration of a simple particle must be constant but not necessarily zero.

P. G. Bergmann[4] has pointed out that two of the three ingredients of this curious result are invariant under the 15-parameter conformal group: the linearized equations for empty space, (2.2), (2.3), (2.8), (2.12), from which we start, and the hyperbolic trajectory with which we end. The only thing missing is a conform-invariant definition of a simple particle; but this we have been unable to provide.

We have already remarked on the close formal resemblance between the gravitational and electromagnetic solutions for sources with constant acceleration. Here, as in the electromagnetic case, it is convenient to combine a retarded solution for one branch of the hyperbola with an advanced solution,

of the opposite charge, for the other. The resulting field is defined throughout space-time, except on the null hyperplane through one of the asymptotes of the hyperbola. At each point of the field, the four principal null directions reduce to two pairs: the two null rays connecting the point with the hyperbola.†
It is easy to see that

$$R^{abcd}R_{abcd} > 0, \quad {*R}^{abcd}R_{abcd} = 0. \tag{10.1}$$

In general, however, the principal null directions of an empty-space Riemann tensor determine it up to a change of scale and a duality rotation; while (2.3) reduces the two functions involved in these transformations to constants, unless the Riemann tensor is null. In the present case, the Riemann tensor is not null; and (10.1) excludes duality rotation. For any space-time hyperbola, therefore, our two conditions are sufficient to determine the field up to a constant factor. Both conditions, however, are manifestly conform-invariant; and both hold in the limiting case where the trajectory becomes a straight line.†† The solution is thus a conformal transformation of the linear approximation to Schwarzschild's metric.

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† See Appendix.

†† As Trautman remarks[1], Elie Cartan[5] must have known all about this in 1922, when he wrote: Nous pouvons convenir d'appeler *Univers optique d'Einstein* l'espace conforme généralisé normal défini en annulant le ds^2 de l'Univers d'Einstein. C'est conformément aux propriétés géométriques de cet Univers optique que se fait la propagation de la lumière. La courbure de rotation de cet Univers est définie en chaque point par dix quantités scalaires, ou encore par une forme quadratique ternaire à coefficients complexes, qu'un changement due système de référence transforme par une substitution orthogonale. Au point de vue géométrique, la propriété suivante mérite d'être signalée. Il existe en chaque point A quatre directions *optiques* (c'est-à'dire annulant le ds^2) privilégiées. Elles sont caractérisées par la propriété que si AA' est l'une d'elles, elle se conserve par le déplacement associé à un parallélogramme élémentaire admettant comme côtés AA' et une autre direction optique *quelconque* issue de A. Dans le cas du ds^2 d'une seule masse attirante (ds^2 de Schwarzschild), ces quatre directions optiques privilégiées se réduisent à deux (doubles): les deux rayons lumineux qui leur correspondent iraient au centre d'attraction ou en viendraient. [In his geometrical characterization of privileged directions, Cartan evidently had the degenerate case in mind].

The rigorous solution (7.18) was discovered by Levi-Civita[6] in 1918, rediscovered by Newman and Tamburino[7], mentioned by Robinson and Trautman[8], described (as the C metric) by Ehlers and Kundt[9], and again by Kinnersley[10] in his survey of type D solutions. It was first "tentatively identified as the gravitational analog of the runaway solutions encountered in electrodynamics" by Kinnersley[11], on the basis of its asymptotic Killing vectors. We arrived independently at a more definite identification by writing the metric in the form given here, which exhibits clearly its connection with the linear approximation. Physically the solution is unacceptable on account of the singularities described in Section 8. A generalization by Kinnersley and Walker[12] includes metrics free from this defect.

There is always some pleasure in looking at old results from a new point of view. It is our hope, however, that the present work provides something more: a technique that can be used to investigate the motion of quite complicated systems in a surveyable manner.

This paper has grown out of a long series of seminars and discussions. The authors are happy to take this opportunity of thanking all those who have participated in them. They are most grateful to Mrs. Helen Armstrong, without whom the paper could not have been produced.

Appendix: on principal null directions.

The Lienard-Wiechert bivector (4.4) may be written as

$$P_{ab} = 2V^{-3} v_{[a} u_{b]}, \tag{A.1}$$

where

$$u_a := \rho\sigma_{,a} \tag{A.2}$$

$$v_a := V^{-1}[(K - H\rho)\dot{X}_a + \rho\ddot{X}_a] - u_a, \tag{A.3}$$

and V is a disposable scalar. Hence, using (3.2) and (3.3), we have

$$x^a = x^a + u^a. \tag{A.4}$$

At any point of the field, therefore, u^a is a null displacement from the source. By taking

$$V = \tfrac{1}{2}\rho\,[(H - \dot{K}/2K - K/\rho)^2 - KA^2], \tag{A.5}$$

where A is the scalar of acceleration,

$$A := [K^{-1}(\dot{K}/2K)^2 - K^{-2}\ddot{X}_r\ddot{X}^r]^{\frac{1}{2}}, \tag{A.6}$$

we make v_a null too. Then u_a and v_a are both solutions of the equations

$$k_r k^r = 0, \quad k_{[a}R_{b]pqc}k^p k^q = 0, \tag{A.7}$$

for the field given by (4.2), (4.4) and $R_{abcd} = P_{abcd}$: consequently, the four principal null directions of this field reduce to two pairs, along u_a and v_a.

Suppose that $A \neq 0$. Writing

$$Y_a := A^{-2}K^{-\frac{1}{2}}(K^{-\frac{1}{2}}\dot{X}_a)^{\cdot}, \tag{A.8}$$

$$Z_a := Y_a + u_a + v_a, \tag{A.9}$$

we have

$$\dot{X}_a Y^a = 0, \quad Y_a Y^a = -A^{-2}, \tag{A.10}$$

and

$$X_{[a}Y_b Z_{c]} = 0, \quad Z_a Z^a = -A^{-2}. \tag{A.11}$$

We turn now to the special case in which the field satisfies the Bianchi equations (2.3). As we have seen, the source is then one branch of a space-time hyperbola. We can choose the parameter and Cartesian coordinates so that the full hyperbola is given by

$$X^a(\sigma) = \varepsilon(P^a \mathrm{ch}A\sigma + Q^a \mathrm{sh}A\sigma) \tag{A.12}$$

whereP_r and Q_r are constant vector fields subject to

$$P_r Q^r = 0, \quad P_r P^r = -Q_r Q^r = A^{-2} \tag{A.13}$$

while

$$\varepsilon = \pm 1. \tag{A.14}$$

Alternatively, without using a parameter, we may write the hyperbola as the intersection of a hyperboloid

$$X_a X^a = -A^{-2} \tag{A.15}$$

and a space-time plane

$$P_{[a}Q_b X_{c]} = 0. \qquad (A.16)$$

It is easy to verify that the constant $\underline{A}$ satisfies (A.6) and that

$$Y^a = X^a. \qquad (A.17)$$

In fact, (A.17) is sufficient to characterize a hyperbola: from (A.10) we then have that $\underline{A}$ is constant, and that X_a lies on the hyperboloid (A.15); while (A.8), with $\underline{A}$ constant, shows that X_a lies in a fixed space-time plane.

From (A.11) we now see that Z_a lies on the same hyperbola. From (A.4), (A.17) and the definition (A.9), however,

$$Z^a = X^a + v^a : \qquad (A.18)$$

thus v_a, like u_a, is a null displacement from the hyperbola.

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