RESEARCH REPORT

STUDY OF NON-LINEAR OPTIMIZATION TECHNIQUES

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1. The study performed under this grant began on May 1, 1969. Most of the research was carried out during the summers of 1969 and 1970.

A study of the present status of non-linear optimization techniques in dynamic programming, especially algebraic, combinatorial, and enumerative methods in integer programming was carried out. This field is very important and broad and in it only limited progress has been made. A bibliography of books examined and studied in this effort are listed next.

Bibliography

1. Mathematics of the Decision Sciences, Part I; Lectures in Applied Mathematics Volume II; George B. Dantzig and Arthur F. Veinott, Jr., Editors; 1968 AMS, Providence, Rhode Island; Especially Section III and IV.

   III. Convex Polyhedra and Integer Programs.

   IV. Combinatories.

2. Progress in Oper. Res. Vol III; Publications in Operations Research No. 16; Edited by Julius S. Aronofsky; John Wiley and Sons, Inc.; Especially Chap. 7, Methods for Integer Programming Algebraic, Combinatorial and Enumerative.

3. Mathematics in Science and Engineering, Vol. 37; Dynamic Programming; Edited by A. Kaufmann and R. Cruon; 1967 Academic Press, New York, London.

4. Mathematics of Automatic Control; Edited by George M. Kranc; Editor is Toshie Takahashi; Holt, Rinehart, and Winston, Inc.

5. Dynamic Programming; Richard Bellman, Editor; 1957, Princeton

University Press, Princeton, New Jersey.

6. Management Science, Vol 5, 1958-59; C. West Churchman, Editor-in-Chief; Printed by the Waverly Press, Inc.; Especially two articles:

   I. Sequencing n Jobs on Two Machines with Arbitrary Time Lags; Edited by L. G. Mitten, p. 293-298.

   II. Discussion: Sequencing n Jobs on Two Machines with Arbitrary Time Lags; Edited by S. M. Johnson, p. 299-303.

2. More effort and research was carried on in the area of solving ordinary, non-linear differential equations, especially by Runge-Kutta type methods as reported earlier by the author. In this area, progress was made in optimizing such formulas that use nine, ten, or eleven evaluations. Specifically, there was developed for the first time a formula of this type of the eighth order with only eleven evaluations. This formula (S 8-11) also has a built-in error control (regulator). These formulas are more efficient, according to tests that are reported, than any known to exist and will be described next.

3. <u>NOTATION</u>. Consider the analytic function $f(x,y)$ and the differential equation $y' = f(x,y)$ at the initial point (x_o,y_o) . Let f_i be defined by the equation

$$f_i = f(x_o + \alpha_i h \ , \ y_o + \alpha_i h \sum_{j=0}^{i-1} \beta_{ij} f_j),$$

where $f_o = f(x_o,y_o)$. Consider the finite series

$$Y = y_o + h \sum_{i=0}^{n} \gamma_i f_i \ .$$

It is immediate that a necessary and sufficient condition that the Taylor series for Y and the Taylor series for the solution $y(x_o + h)$ of the given differential equation agree through terms in h^m is that

$$(f^{(k-1)})_o = k \sum_{i=0}^{n} \gamma_i (f_i^{(k-1)})_o$$

for $k = 1,2,\ldots,m$, where the symbol $f^{(k)}$ is used for the $k^{\underline{th}}$ derivative of the function $f(x_o + h , y(x_o + h))$.

In order to specify a given formula, one may detach the coefficients as follows:

Formula

$$\begin{array}{ll}
\alpha_1 & \alpha_1\beta_{10} \\
\alpha_2 & \alpha_2\beta_{20} + \alpha_2\beta_{21} \\
\vdots & \\
\alpha_n & \alpha_n\beta_{n0} + \alpha_n\beta_{n1} + \ldots + \alpha_n\beta_{n,n-1}
\end{array}$$

$$\gamma_0 + \gamma_1 + \ldots + \gamma_n$$

To make this clear, consider a third order formula with three evaluations of the function. The detached coefficients are

$$\begin{array}{ll}
\alpha_1 & \alpha_1\beta_{10} \\
\alpha_2 & \alpha_2\beta_{20} + \alpha_2\beta_{21}
\end{array}$$

$$\gamma_0 + \gamma_1 + \gamma_2$$

This array would imply the formula

$$Y = y_o + h(\gamma_o f_o + \gamma_1 f_1 + \gamma_2 f_2) ,$$

where

$$f_o = f(x_o, y_o)$$

$$f_1 = f(x_o + \alpha_1 h \ , \ y_o + h\alpha_1 \beta_{10} f_0)$$

$$f_2 = f(x_o + \alpha_2 h \ , \ y_o + h(\alpha_2 \beta_{20} f_0 + \alpha_2 \beta_{21} f_1)) .$$

If the expression $\alpha_2\beta_{20} + \alpha_2\beta_{21} = \frac{m}{n}\beta_{20} + \frac{p}{n}\beta_{21}$, this will be written $\frac{1}{n}(m\beta_{20} + p\beta_{21})$. In other words, common factors may be factored out.

These formulas, of course, are used also for systems of differential equations $y_i' = f_i(x; y_i)$ at the initial point $(x_0, y_{10}, \ldots, y_{p0})$.

4. Listed next are the new formulas with nine, ten, and eleven evaluations, respectively. With each formula there is included an approximation to the error. This approximation is of high order and its absolute value is called the regulator and denoted by R. At each step of the integration, R is easily calculated since it uses only evaluations calculated already for that step. The stepsize for the next step is determined to be 2h, h, or $\frac{1}{2}h$, where h is the current stepsize, according to whether $R < m$, $m \leq R \leq M$, or $M < R$, respectively, where the interval $[m, M]$ is appropriately chosen to give the tolerance desired. By experience, an interval $[10^{-n-4}, 10^{-n}]$ for an appropriate positive integer n works well, where n is chosen to give the desired tolerance. Once n is chosen, it usually remains constant for a given series of steps in an integration problem. The larger n is chosen, the more accurate will be the integration (at the cost of more computational effort, of course). The comparative data

given below will illustrate this relationship between accuracy and time adequately. In the designation of each formula, the author's initial will be used, followed by two numbers separated by a dash, the first number gives the order of the formula and the second the number of evaluations required for one step.

Formula S 7-9

$\frac{4}{27} \quad \frac{4}{27}$

$\frac{2}{9} \quad \frac{1}{18}(1 + 3)$

$\frac{1}{3} \quad \frac{1}{12}(1 + 0 + 3)$

$\frac{1}{7} \quad \frac{1}{1372}(109 + 0 + 135 - 48)$

$\frac{1}{4} \quad \frac{1}{2048}(206 + 0 + 594 - 141 - 147)$

$1 \quad \frac{1}{20}(-97 + 0 + 189 + 462 + 490 - 1024)$

$\frac{7}{9} \quad \frac{1}{708588}(-356391 + 0 - 137781 + 2857680 + 2524480 - 4358144 + 21280)$

$1 \quad \frac{1}{149492}(359879 + 0 + 68229 - 1944726 - 2013753 + 3526656 - 23940 + 177147)$

$\frac{1}{4596480}(65664 + 0 + 0 + 4072194 + 2235331 - 3670016 + 0 + 1594323 + 298984)$

$$R = \left| \frac{h}{4596480}(f_8 - f_6) \right|$$

Formula S 7-10

$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{4}$	$\frac{1}{16}(1 + 3)$
$\frac{3}{8}$	$\frac{1}{32}(3 + 0 + 9)$
$\frac{1}{2}$	$\frac{1}{18}(2 + 0 + 3 + 4)$
$\frac{1}{8}$	$\frac{1}{576}(73 + 0 - 120 + 200 - 81)$
$\frac{5}{8}$	$\frac{1}{1728}(-933 + 0 + 168 - 1944 + 1469 + 2320)$
1	$\frac{1}{1107}(18870 + 0 - 5604 + 57024 - 39839 - 37660 + 8316)$
$\frac{7}{8}$	$\frac{1}{59040}(106442 + 0 - 52395 + 340648 - 222390 - 188280 + 66528 + 1107)$
1	$\frac{1}{7155}(-40083 + 0 + 22830 - 131472 + 94610 + 78760 - 21672 - 738 + 4920)$

$$\frac{1}{13230}(477 + 0 + 0 + 4032 - 1036 + 2624 + 4032 + 0 + 2624 + 477)$$

$$R = \left| \frac{h(f_9 - f_7)}{13230} \right|$$

Formula S 8-11

$\frac{2}{9}$	$\frac{2}{9}$
$\frac{1}{3}$	$\frac{1}{12}(1 + 3)$
$\frac{1}{2}$	$\frac{1}{8}(1 + 0 + 3)$
$\frac{2}{3}$	$\frac{1}{27}(4 + 0 + 6 + 8)$
$\frac{1}{6}$	$\frac{1}{5400}(548 + 0 + 687 - 416 + 81)$
$\frac{1}{6}$	$\frac{1}{5400}(818 + 0 + 1767 - 956 + 171 - 900)$
$\frac{1}{3}$	$\frac{1}{108}(-103 + 0 - 420 + 208 - 33 + 768 - 384)$
1	$\frac{1}{20}(63 + 0 + 228 - 232 + 73 - 3632 + 3400 + 120)$
$\frac{5}{6}$	$\frac{1}{1080}(20 + 0 - 285 + 70 + 345 - 5586 + 5916 + 405 + 15)$
1	$\frac{1}{820}(35 + 0 + 444 + 1616 - 1107 + 21816 - 21384 - 1260 - 60 + 720)$

$$\frac{1}{4200}(205 + 0 + 0 + 1360 + 135 + 972 + 108 + 135 + 0 + 1080 + 205)$$

$$R = \left| \frac{h}{4200}(f_{10} - f_8) \right|$$

5. Listed in this section are some formulas previously published by the author, usually without regulators. The regulator for each formula is included as was done for the formulas in the previous section. The prefix 0 (old) will be added to distinguish these from the previous ones.

<u>Formula OS 7-9</u>

$$\frac{4}{27} \qquad \frac{4}{27}$$

$$\frac{2}{9} \qquad \frac{1}{18}(1 + 3)$$

$$\frac{1}{3} \qquad \frac{1}{12}(1 + 0 + 3)$$

$$\frac{1}{7} \qquad \frac{1}{1372}(109 + 0 + 135 - 48)$$

$$\frac{1}{4} \qquad \frac{1}{2048}(206 + 0 + 594 - 141 - 147)$$

$$\frac{7}{9} \qquad \frac{1}{78732}(-124719 + 0 - 15309 + 604800 + 686000 - 1089536)$$

$$\frac{7}{9} \qquad \frac{1}{78732}(-15279 + 0 - 15309 + 173880 + 138572 - 229376 + 8748)$$

$$1 \qquad \frac{1}{298984}(227278 + 0 + 136458 - 980742 - 1153509 + 1892352 - 177147 + 354294)$$

$$\frac{1}{4596480}(65664 + 0 + 0 + 4072194 + 2235331 - 3670016 + 0 + 1594323 + 298984)$$

$$R = \left| \frac{h}{4596480}(f_7 - f_6) \right|$$

Formula OS 7-10

$$\frac{4}{27} \qquad \frac{4}{27}$$

$$\frac{2}{9} \qquad \frac{1}{18}(1 + 3)$$

$$\frac{1}{3} \qquad \frac{1}{12}(1 + 0 + 3)$$

$$\frac{1}{2} \qquad \frac{1}{8}(1 + 0 + 0 + 3)$$

$$\frac{2}{3} \qquad \frac{1}{54}(13 + 0 - 27 + 42 + 8)$$

$$\frac{1}{6} \qquad \frac{1}{4320}(389 + 0 - 54 + 966 - 824 + 243)$$

$$1 \qquad \frac{1}{20}(-231 + 0 + 81 - 1164 + 656 - 122 + 800)$$

$$\frac{5}{6} \qquad \frac{1}{288}(-127 + 0 + 18 - 678 + 456 - 9 + 576 + 4)$$

$$1 \qquad \frac{1}{820}(1481 + 0 - 81 + 7104 - 3376 + 72 - 5040 - 60 + 720)$$

$$\frac{1}{840}(41 + 0 + 0 + 27 + 272 + 27 + 216 + 0 + 216 + 41)$$

$$R = \left| \frac{h}{840}(f_9 - f_7) \right|$$

Formula OS 8-12

$\frac{1}{9}$ $\frac{1}{9}$

$\frac{1}{6}$ $\frac{1}{24}(1 + 3)$

$\frac{1}{4}$ $\frac{1}{16}(1 + 0 + 3)$

$\frac{1}{10}$ $\frac{1}{500}(29 + 0 + 33 - 12)$

$\frac{1}{6}$ $\frac{1}{972}(33 + 0 + 0 + 4 + 125)$

$\frac{1}{2}$ $\frac{1}{36}(-21 + 0 + 0 + 76 + 125 - 162)$

$\frac{2}{3}$ $\frac{1}{243}(-30 + 0 + 0 - 32 + 125 + 0 + 99)$

$\frac{1}{3}$ $\frac{1}{324}(1175 + 0 + 0 - 3456 - 6250 + 8424 + 242 - 27)$

$\frac{5}{6}$ $\frac{1}{324}(293 + 0 + 0 - 852 - 1375 + 1836 - 118 + 162 + 324)$

$\frac{5}{6}$ $\frac{1}{1620}(1303 + 0 + 0 - 4260 - 6875 + 9990 + 1030 + 0 + 0 + 162)$

1 $\frac{1}{4428}(-8595 + 0 + 0 + 30720 + 48750 - 66096 + 378 - 729 - 1944 - 1296 + 3240)$

$\frac{1}{840}(41 + 0 + 0 + 0 + 0 + 216 + 272 + 27 + 27 + 36 + 180 + 41)$

$$R = \left| \frac{h}{840}(f_{10} - f_9) \right|$$

6. Error control procedure. In addition to the procedure for error control already outlined, it is usually useful to limit the stepsize h. In the examples programmed for this paper, h has been required to be between .0025 and .80 . Other bounds could be used if desired. The entire procedure will be recapitulated at this point.

a. Choose bounds for the regulator R. These will be positive numbers L and U, $L < U$, rather small (for example $L = 10^{-n-4}$ and $U = 10^{-n}$ may be used as suggested above, where n is a suitable integer). The choice of U will greatly influence the size of the error and will vary from one formula to another. It is suggested that a trial run on a known differential equation be made to adjust these differences.

b. Choose convenient bounds for the step size h (for example, $.0025 < h < .80$).

c. Choose an initial h between these bounds for the first step (for example, $h_0 = .01$ or $h_0 = \frac{1}{128}$).

d. Calculate the first step, using the initial values and the desired formula, in the usual way.

e. After each step has been calculated, before proceeding to the next step, calculate R. If $R < L$ and $h < .40$, double the current step size and use the result for the new step size. If $R > U$ and $h > .005$ halve the current step size and use the result for the new step size. In all other cases, use the current step size for the new step size.

f. Proceed to calculate the next step and continue as far as desired.

7. Comparative data. The data here will pertain to the solution of the system of two differential equations:

$$y' = -2xy \log z \ , \ z' = 2xz \log y$$

Initial values: $x_0 = 0, \ y_0 = e, \ z_0 = 1$

Exact solution: $y = e^{\cos (x^2)} \ , \ z = e^{\sin (x^2)}$.

Integration interval: $x = 0$ to $x = 5$.

This system is not the best one to bring out the advantages of integration runs using the regulator (regulated runs) due to the rapid oscillation of the solution, but it has already been used [1] to make comparisons that might easily lead to erroneous conclusions.

On page 66 [1], Fehlberg writes

"58. For the numerical comparison of our formula RK7(8) with SHANKS's formula we applied these formulas again to our problem (53) in exactly the same way as in Part I and Part II. We again used RICHARDSON's principle as stepsize control procedure for SHANKS's formula since no other satisfactory stepsize control procedure seems to be known for SHANKS's formulas. Table XI shows the results of our comparison.

Table XI: Comparison of seventh-order Methods for Example (53)

Method	Number of Substitutions per Step	Results for x = 5 and Tolerance 10^{-16}				
		Number of Steps	Total Number of Evaluations	Running Time on IBM-7094 (min)	Accumulated Errors in y and z	
					Δy	Δz
SHANKS	17	1423	24 191	2.49	$-0.1332 \cdot 10^{-13}$	$-0.7377 \cdot 10^{-13}$
RK7(8)	13	818	10 634	1.12	$-0.2509 \cdot 10^{-13}$	$-0.5135 \cdot 10^{-13}$

As a matter of fact, two years before publication of Fehlberg's article in 1968, in 1966 (see [2] where a report to NASA for 1966 is included) and in 1967, the author developed stepsize control procedures for all of his published formulas and some of these were programmed on the IBM 7094 at the Computation Division of the Marshall Space Flight Center (MSFC) and at the Vanderbilt Computer Center and made available to personnel at MSFC.

The following example will illustrate the situation. Consider the table

RK 6(7)	10	22450	$.46\ 10^{-13}$	$.11\ 10^{-12}$
OS 7-10	10	13640	$.30\ 10^{-13}$	$.20\ 10^{-14}$,

where the first column names the method, the second column states the number of evaluations per step, the third column states the total number of evaluations for the run and the last two columns give the errors for y and z respectively (this order will be followed in all such tables). The first row is given in [1] and the second row was programmed at MSFC, Computation Division. This constitutes a fair comparison since both formulas require 10 evaluations per step. Moreover, this very formula OS 7-10 with its regulator given appeared in the 1966 resumé mentioned above. It gives a better result than Fehlberg's formula and uses less than 61% of the evaluations used in that formula.

Another example that will further illuminate Fehlberg's comparisons is given next.

RK 7(8)	13	10634	$.25\ 10^{-13}$	$.51\ 10^{-13}$		
OS 8-12	12	8268	$.11\ 10^{-14}$	$.14\ 10^{-14}$	10^{-14}	$\frac{1}{128}$

The first line in the table is found on page 66 of [1] as already given above. The second line is a run of Shanks' old 8-12 formula with $U = 10^{-14}$ and $h_0 = \frac{1}{128}$ as the last two entries (this order will be followed hereafter). The formula OS 8-12 gives a much better approximation and uses less than 78% of the evaluations used in Fehlberg's formula.

From these examples, one can easily see that Fehlberg's comparisons were extremely distorted by his use of Richardson's principle only on the formulas of others and not on his own formulas. It is also obvious from the examples that his formulas run a poor second. This is made dramatically evident when it is seen that the very best run in his whole report (made with a higher order formula) comes in second to the OS 8-12 run just given. Note the comparison given next.

RK 8(9)	17	8670	$.18\ 10^{-14}$	$.36\ 10^{-13}$		
OS 8-12	12	8268	$.11\ 10^{-14}$	$.14\ 10^{-14}$	10^{-14}	$\frac{1}{128}$

Listed below are other runs using the same system of equations. Regulated runs programmed for this paper will indicate U and h_0 as mentioned above. If one should decide to use one formula, it might well be S 8-11 as the data shows. If one uses more than one formula, S 7-9 and S 7-10 might be used along with S 8-11. In general, formulas of this type with regulators are much better than formulas without regulators.

The formulas below come from [1] or [2] or the present paper.

RK 8(9)	17	16286	$.68\ 10^{-15}$	$.31\ 10^{-14}$	10^{-19}	$\frac{1}{128}$
OS 9-16	16	14224	$.31\ 10^{-15}$	$.75\ 10^{-15}$	10^{-19}	$\frac{1}{128}$
OS 8-12	12	11412	$.53\ 10^{-15}$	$.71\ 10^{-15}$	10^{-15}	$\frac{1}{128}$

RK 8(9)	17	10982	$.19\ 10^{-14}$	$.29\ 10^{-14}$	10^{-17}	$\frac{1}{128}$
OS 9-16	16	10816	$.77\ 10^{-15}$	$.58\ 10^{-15}$	10^{-18}	$\frac{1}{128}$
OS 8-12	12	8268	$.11\ 10^{-14}$	$.14\ 10^{-14}$	10^{-14}	$\frac{1}{128}$
RK 8(9)	17	4403	$.12\ 10^{-10}$	$.97\ 10^{-12}$	10^{-13}	$\frac{1}{128}$
OS 9-16	16	4000	$.11\ 10^{-12}$	$.21\ 10^{-12}$	10^{-15}	$\frac{1}{128}$
OS 8-12	12	4476	$.22\ 10^{-11}$	$.95\ 10^{-13}$	10^{-12}	$\frac{1}{128}$
RK 8(9)	17	2210	$.11\ 10^{-7}$	$.29\ 10^{-9}$	10^{-10}	$\frac{1}{128}$
OS 9-16	16	2080	$.13\ 10^{-8}$	$.22\ 10^{-9}$	10^{-13}	$\frac{1}{128}$
OS 10-21	21	9870	$.11\ 10^{-15}$	$.24\ 10^{-15}$	-10^{-18}	$\frac{1}{128}$
		7644	$.31\ 10^{-14}$	$.29\ 10^{-14}$	-10^{-17}	$\frac{1}{128}$
		6006	$.14\ 10^{-14}$	$.42\ 10^{-13}$	-10^{-16}	$\frac{1}{128}$
		4536	$.10\ 10^{-12}$	$.60\ 10^{-12}$	-10^{-15}	$\frac{1}{128}$
		3759	$.38\ 10^{-12}$	$.48\ 10^{-11}$	-10^{-14}	$\frac{1}{128}$
		3003	$.49\ 10^{-12}$	$.69\ 10^{-10}$	-10^{-13}	$\frac{1}{128}$
		2079	$.17\ 10^{-8}$	$.18\ 10^{-8}$	-10^{-12}	$\frac{1}{128}$
		1785	$.29\ 10^{-7}$	$.43\ 10^{-7}$	-10^{-11}	$\frac{1}{128}$

OS 9-16	16	14224	$.31\ 10^{-15}$	$.75\ 10^{-15}$	-10^{-19}	$\frac{1}{128}$
		10816	$.77\ 10^{-15}$	$.58\ 10^{-15}$	-10^{-18}	$\frac{1}{128}$
		7584	$.86\ 10^{-14}$	$.20\ 10^{-15}$	-10^{-17}	$\frac{1}{128}$
		5328	$.81\ 10^{-13}$	$.25\ 10^{-13}$	-10^{-16}	$\frac{1}{128}$
		4000	$.11\ 10^{-12}$	$.21\ 10^{-12}$	-10^{-15}	$\frac{1}{128}$
		2880	$.19\ 10^{-10}$	$.14\ 10^{-11}$	-10^{-14}	$\frac{1}{128}$
		2080	$.13\ 10^{-8}$	$.22\ 10^{-9}$	-10^{-13}	$\frac{1}{128}$
		1536	$.11\ 10^{-7}$	$.29\ 10^{-8}$	-10^{-12}	$\frac{1}{128}$
		1088	$.96\ 10^{-6}$	$.17\ 10^{-6}$	-10^{-11}	$\frac{1}{128}$
		864	$.18\ 10^{-4}$	$.11\ 10^{-4}$	-10^{-10}	$\frac{1}{128}$
OS 8-12	12	11412	$.53\ 10^{-15}$	$.71\ 10^{-15}$	-10^{-15}	$\frac{1}{128}$
		8268	$.11\ 10^{-14}$	$.14\ 10^{-14}$	-10^{-14}	$\frac{1}{128}$
		5844	$.93\ 10^{-13}$	$.44\ 10^{-13}$	-10^{-13}	$\frac{1}{128}$
		4476	$.22\ 10^{-11}$	$.95\ 10^{-13}$	-10^{-12}	$\frac{1}{128}$
		3000	$.48\ 10^{-9}$	$.87\ 10^{-10}$	-10^{-11}	$\frac{1}{128}$
		2256	$.12\ 10^{-8}$	$.14\ 10^{-9}$	-10^{-10}	$\frac{1}{128}$
OS 7-10	10	14510	$.70\ 10^{-14}$	$.4\ 10^{-15}$	-10^{-10}	$\frac{1}{128}$
		8760	$.22\ 10^{-12}$	$.35\ 10^{-13}$	-10^{-9}	$\frac{1}{128}$
		5570	$.10\ 10^{-10}$	$.15\ 10^{-11}$	-10^{-8}	$\frac{1}{128}$

08 7-9	9	18720	$.10\ 10^{-14}$	$.15\ 10^{-14}$	-10^{-14}	$\frac{1}{128}$
		17127	$.49\ 10^{-16}$	$.18\ 10^{-14}$	-10^{-13}	$\frac{1}{128}$
		13329	$.18\ 10^{-14}$	$.18\ 10^{-14}$	-10^{-12}	$\frac{1}{128}$
		9297	$.40\ 10^{-13}$	$.27\ 10^{-14}$	-10^{-11}	$\frac{1}{128}$
		6210	$.74\ 10^{-12}$	$.51\ 10^{-13}$	-10^{-10}	$\frac{1}{128}$
		4320	$.39\ 10^{-11}$	$.27\ 10^{-11}$	-10^{-9}	$\frac{1}{128}$
		3042	$.90\ 10^{-10}$	$.17\ 10^{-10}$	-10^{-8}	$\frac{1}{128}$
		1962	$.17\ 10^{-8}$	$.41\ 10^{-9}$	-10^{-7}	$\frac{1}{128}$
8 7-9	9	16452	$.14\ 10^{-14}$	$.15\ 10^{-14}$	10^{-17}	$\frac{1}{64}$
		12834	$.60\ 10^{-14}$	$.89\ 10^{-15}$	10^{-16}	$\frac{1}{64}$
		7839	$.15\ 10^{-12}$	$.43\ 10^{-13}$	10^{-15}	$\frac{1}{64}$
		5553	$.59\ 10^{-11}$	$.92\ 10^{-12}$	10^{-14}	$\frac{1}{64}$
		3204	$.11\ 10^{-9}$	$.18\ 10^{-10}$	10^{-13}	$\frac{1}{64}$
		1962	$.24\ 10^{-8}$	$.83\ 10^{-9}$	10^{-12}	$\frac{1}{64}$

S 7-10	10	17350	$.26\ 10^{-14}$	$.18\ 10^{-14}$	10^{-13}	$\frac{1}{64}$
		11990	$.47\ 10^{-14}$	$.20\ 10^{-14}$	10^{-12}	$\frac{1}{64}$
		7220	$.64\ 10^{-12}$	$.47\ 10^{-13}$	10^{-11}	$\frac{1}{64}$
		4520	$.74\ 10^{-12}$	$.99\ 10^{-12}$	10^{-10}	$\frac{1}{64}$
		3120	$.36\ 10^{-9}$	$.47\ 10^{-11}$	10^{-9}	$\frac{1}{64}$
		1800	$.76\ 10^{-9}$	$.42\ 10^{-9}$	10^{-8}	$\frac{1}{64}$
S 8-11	11	12738	$.59\ 10^{-15}$	$.79\ 10^{-15}$	10^{-12}	$\frac{1}{64}$
		7656	$.29\ 10^{-13}$	$.43\ 10^{-14}$	10^{-11}	$\frac{1}{64}$
		4741	$.14\ 10^{-12}$	$.32\ 10^{-12}$	10^{-10}	$\frac{1}{64}$
		3355	$.43\ 10^{-10}$	$.45\ 10^{-11}$	10^{-9}	$\frac{1}{64}$
		1947	$.16\ 10^{-8}$	$.43\ 10^{-9}$	10^{-8}	$\frac{1}{64}$

APPENDIX

Other formulas programmed at the Computation Division at Marshall Space Flight Center are given below. (Sincere thanks is given to Mr. Audie E. Anderson for his help and cooperation in programming these and other formulas). In general, the efficiency of these formulas are a little below those discussed in the main body of the paper.

S 7-10 A

$\frac{4}{63}$ $\frac{4}{63}$

$\frac{2}{21}$ $\frac{1}{42}(1 + 3)$

$\frac{1}{7}$ $\frac{1}{28}(1 + 0 + 3)$

$\frac{2}{7}$ $\frac{1}{7}(1 + 0 - 3 + 4)$

$\frac{3}{7}$ $\frac{1}{28}(-9 + 0 + 54 - 48 + 15)$

$\frac{4}{7}$ $\frac{1}{455}(176 + 0 - 975 + 1162 - 317 + 214)$

$\frac{5}{7}$ $\frac{1}{5460}(12227 + 0 - 63765 + 64264 - 12654 + 2008 + 1820)$

$\frac{6}{7}$ $\frac{1}{33215}(-45402 + 0 + 256230 - 246969 + 41829 + 30972 - 25935 + 17745)$

1 $\frac{1}{195260}(563047 + 0 - 2942940 + 2814644 - 3199 - 915152 + 944580 - 398580 + 132860)$

$\frac{1}{17280}(751 + 0 + 0 + 3577 + 1323 + 2989 + 2989 + 1323 + 3577 + 751)$

No Regulator

S 7-10 B

$\frac{1}{9}$ $\frac{1}{9}$

$\frac{1}{6}$ $\frac{1}{24}(1 + 3)$

$\frac{1}{4}$ $\frac{1}{16}(1 + 0 + 3)$

$\frac{1}{2}$ $\frac{1}{4}(1 + 0 - 3 + 4)$

$\frac{3}{4}$ $\frac{1}{16}(3 + 0 + 0 + 0 + 9)$

$\frac{2}{3}$ $\frac{1}{729}(242 + 0 - 540 + 528 + 288 - 32)$

$\frac{1}{3}$ $\frac{1}{2916}(343 + 0 - 216 + 864 + 96 + 128 - 243)$

$\frac{1}{3}$ $\frac{1}{2916}(373 + 0 - 216 + 288 - 624 - 64 + 243 + 972)$

1 $\frac{1}{604}(83 + 0 - 1176 + 3360 + 1968 + 1472 - 2187 + 2916 - 5832)$

$\frac{1}{2520}(151 + 0 + 0 + 2048 + 2496 + 2048 - 2187 + 0 - 2187 + 151)$

$$R = \left| \frac{h\,(f_8 - f_7)}{2520} \right|$$

S 8-11 A

$\frac{2}{9}$ $\frac{2}{9}$

$\frac{1}{3}$ $\frac{1}{12}(1 + 3)$

$\frac{1}{2}$ $\frac{1}{8}(1 + 0 + 3)$

$\frac{8}{9}$ $\frac{1}{729}(232 + 0 - 480 + 896)$

$\frac{1}{9}$ $\frac{1}{268272}(15649 + 0 + 44448 - 35392 + 5103)$

$\frac{1}{9}$ $\frac{1}{46656}(-97 + 0 + 2208 - 3488 + 729 + 5832)$

$\frac{2}{3}$ $\frac{1}{264600}(-59143 + 0 + 79968 - 3104 + 20655 - 475416 + 613440)$

1 $\frac{1}{196000}(422625 + 0 - 164640 + 402400 - 79785 + 3790152 - 4292352 + 117600)$

$\frac{1}{3}$ $\frac{1}{1058400}(61005 + 0 - 305760 + 545440 + 93555 - 52272 + 324432 - 264600 - 49000)$

1 $\frac{1}{1654240}(-152635 + 0 + 3128160 - 14486560 - 1228365 + 27162216 - 29232576 + 7408800 + 588000 + 8467200)$

$$\frac{1}{1646400}(51695 + 0 + 0 + 158720 + 295245 + 150903 + 144342 + 396900 + 0 + 396900 + 51695)$$

$$R = \left| \frac{h}{1646400}(f_{10} - f_8) \right|$$

S 8-12 A

$\frac{5}{27}$	$\frac{5}{27}$
$\frac{5}{18}$	$\frac{1}{72}(5 + 15)$
$\frac{5}{12}$	$\frac{1}{48}(5 + 0 + 15)$
$\frac{1}{6}$	$\frac{1}{300}(29 + 0 + 33 - 12)$
$\frac{1}{2}$	$\frac{1}{20}(1 + 0 + 0 + 4 + 5)$
$\frac{2}{3}$	$\frac{1}{135}(2 + 0 + 0 - 32 + 50 + 70)$
$\frac{1}{6}$	$\frac{1}{108}(13 + 0 + 0 + 60 - 7 - 58 + 10)$
$\frac{1}{3}$	$\frac{1}{1620}(-1197 + 0 + 0 - 13568 - 740 + 12780 - 2135 + 5400)$
$\frac{5}{6}$	$\frac{1}{1620}(369 + 0 + 0 + 1156 - 785 - 2070 + 1060 + 0 + 1620)$
$\frac{5}{6}$	$\frac{1}{1620}(207 + 0 + 0 + 1156 - 785 - 450 + 250 + 810 + 0 + 162)$
1	$\frac{1}{164}(-41 + 0 + 0 - 384 + 212 + 484 - 107 + 0 - 72 - 48 + 120)$

$$\frac{1}{840}(41 + 0 + 0 + 0 + 0 + 272 + 27 + 216 + 27 + 36 + 180 + 41)$$

$$R = \left| \frac{h\,(f_{10} - f_9)}{840} \right|$$

S 8-12

$\frac{5}{18}$ $\frac{5}{18}$

$\frac{5}{12}$ $\frac{1}{48}(5 + 15)$

$\frac{5}{8}$ $\frac{1}{32}(5 + 0 + 15)$

$\frac{1}{4}$ $\frac{1}{200}(29 + 0 + 33 - 12)$

$\frac{3}{4}$ $\frac{1}{40}(3 + 0 + 0 + 12 + 1$ [illegible]

$\frac{1}{2}$ $\frac{1}{180}(14 + 0 + 0 + 16 +$ [illegible] $5)$

$\frac{1}{4}$ $\frac{1}{288}(41 + 0 + 0 - 224$ [illegible] $3 + 73 + 1$ [illegible]

$\frac{2}{3}$ $\frac{1}{10935}(478 + 0 + 0$ [illegible] $2 + 7240 -$ [illegible] $0 -$ [illegible] $1440)$

$\frac{1}{3}$ $\frac{1}{43740}(6953 + 0 + 0$ [illegible] $4288 + 272$ [illegible] 16000 [illegible] $600 - 5$ [illegible] $- 3645)$

$\frac{1}{3}$ $\frac{1}{43740}(8753 + 0$ [illegible] $44288 + 272$ [illegible] $80 - 600 -$ [illegible] $25515 + 58320)$

1 $\frac{1}{27180}(-4057 +$ [illegible] $0 + 32512 + 1$ [illegible] $0080 + 81960$ [illegible] $0 - 98415 - 65610 - 65610$ [illegible]

$\frac{1}{10080}(604 + 0 + 0 + 0 + 0 + 8192 + 9$ [illegible] $192 - 8748 -$ [illegible] $2187 + 604)$

$$R = \left| \frac{h\,(f_{10} - \ldots}{10080} \ldots \right|$$

S 8-12 C

$\frac{4}{63}$	$\frac{4}{63}$
$\frac{2}{21}$	$\frac{1}{42}(1 + 3)$
$\frac{1}{7}$	$\frac{1}{28}(1 + 0 + 3)$
$\frac{2}{7}$	$\frac{1}{7}(1 + 0 - 3 + 4)$
$\frac{3}{7}$	$\frac{1}{28}(5 + 0 - 9 + 8 + 8)$
$\frac{4}{7}$	$\frac{1}{35}(-15 + 0 + 24 + 21 - 33 + 23)$
$\frac{4}{7}$	$\frac{1}{63}(-20 + 0 + 36 + 14 - 2 - 6 + 14)$
$\frac{5}{7}$	$\frac{1}{756}(523 + 0 + 81 - 892 - 12 + 1148 - 812 + 504)$
1	$\frac{1}{32}(194 + 0 - 210 - 502 + 1193 - 913 + 396 - 153 + 27)$
$\frac{6}{7}$	$\frac{1}{3577}(335 + 0 - 1449 + 1036 + 4165 - 4123 + 3731 - 1260 + 567 + 64)$
1	$\frac{1}{3004}(-1059 + 0 + 2415 + 3416 - 11368 + 12740 - 8708 + 4536 - 756 - 256 + 2044)$

$$\frac{1}{17280}(751 + 0 + 0 + 3577 + 1323 + 2989 + 1225 + 1764 + 1323 + 0 + 3577 + 751)$$

$$R = \left| \frac{h\,(f_{11} - f_9)}{17280} \right|$$

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