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NASA Headquarters
400 Maryland Avenue, S.W.
Washington, D.C. 20546

Dear Sir:

Transmitted herewith are five copies of the Final Technical Report for NASA Contract NGR 05-007-138.

Yours sincerely,

William M. Kaula
Principal Investigator
Professor of Geophysics

WMK:jk

Encls.

xc w/encl. M.J. Swetnick

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Los Angeles, California

FINAL TECHNICAL REPORT

TIDAL THEORY AND ORBITAL PERTURBATIONS

NASA Contract No. NGR 05-007-138

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Submitted to Headquarters, National Aeronautics and Space Administration

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I INTRODUCTION

The formal period for Contract NGR 05-007-138 was June 1, 1967 to February 28, 1969. The inspiration for linking tides to satellite orbits was mainly the prospect of satellites with drag free capability good enough to enable the separation of long term body force effects from drag and other surface effects. A second motivation for the work was the development of procedures for the manipulation of spherical harmonics on a variety of problems ranging from the elastic distortions of the earth to the perturbations of satellite orbits. It seemed as though such techniques might be extendable to the ocean tide problem to accomplish a purely analytical solution. It was hoped that such a solution could be carried to a high enough degree to obtain the very low degree harmonics of the tide accurately enough to make a comparison with tides inferred from satellite orbit analysis.

The work thus fell naturally into two parts: ocean tides and satellite orbits.

In attacking the ocean tide problem, it was found that the initial ideas of expanding the Laplace tidal equation in spherical harmonics, converting products to sums, and then inverting the equations to obtain a solution for the harmonic coefficients of the tide heights, were naive. A satisfyingly converging solution was never attained. Hence the theory was recast in a form similar to that of Longuet-Higgins using stream function & velocity potential as the prime variables. This work was almost entirely carried out by Mr. M.A. Joncich, who is the author of Sec. II. This tidal computation took much more effort than anticipated, and the work stretched out two years beyond the original termination date.

The satellite orbit analyses were more straightforward, and obtained qualitatively interesting results pertaining to the effects of non-uniform tidal amplitude and dissipation on both the evolution of the moon's orbit and the perturbations of artificial satellite orbits. Quantitatively, the results were in a sense disappointing, in that for plausible values of the parameters, (1) the evolutionary effects on the moon were slight; and (2) the perturbations of artificial satellite orbits were too small to be observable for all but the very lowest degree harmonics. This work was partially supported by NASA Contract NAS 12-695, and was continued under it after February 1969.

Another development under this contract was the analysis of the appropriate treatment of the earth's gravity field and orbit perturbations for new observing systems which can observe the field virtually continuously, such as the radar altimeter and satellite-to-satellite tracking. The final portion of the report is a discussion of these considerations. This work has since April 1970 been continued under NASA Contract NGR 05-007-280.

II OCEAN TIDES

Summary. Section II considers the possibility of analytical, rather than numerical, solution of world ocean tide equations. In subsection A the basic applicability of the Laplace tidal equations, if corrected to incorporate dissipation and perhaps other factors, is outlined. The need to determine more precisely these correction terms calls for analytical, rather than numerical methods. The strength and weakness of analytical solutions is their continuity; computation can be efficient, but discrete behavior can only be approximated. On a sphere, when structure is basically large scaled and small scale structure is beyond current theory, as is the case with world ocean tides, a representation in spherical harmonics is appropriate. Such a representation is ideal for global geophysical studies using satellites. In subsection B an efficient method for studying the tidal problem is outlined. Dissipative and other corrective terms are treated as perturbations of a basic Laplacian system of equations. In subsection C the Laplace tidal equations are reduced to a set of simultaneous equations for the spherical harmonic coefficients of velocity potential. These coefficients converge more rapidly than those for tide height. The latter are related to velocity potential through a divergence relationship -- the continuity condition over a vertical column of water. In subsection D a typical dissipative term is reduced in a way amenable to the perturbation method of subsection B. In subsection E an earlier reduction to simultaneous equations

for tide height is outlined. Some test calculations using these equations are discussed. The slow convergence of these solutions led to the reworking of the problem.

A LAPLACE TIDAL EQUATIONS

All efforts at calculating gravitationally induced ocean tides are based on the tidal equations of Laplace:

$$\frac{\partial U_\theta}{\partial t} - 2\Omega \cos\theta\, U_\varphi + \frac{1}{R}\frac{\partial}{\partial\theta}\left(g\zeta - V(\zeta) - V_e\right) = 0 \qquad (1a)$$

$$\frac{\partial U_\varphi}{\partial t} + 2\Omega \cos\theta\, U_\theta + \frac{1}{R\sin\theta}\frac{\partial}{\partial\varphi}\left(g\zeta - V(\zeta) - V_e\right) = 0 \qquad (1b)$$

together with the continuity condition

$$\frac{\partial \zeta}{\partial t} + \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}\left(h U_\theta \sin\theta\right) + \frac{\partial}{\partial\varphi}\left(h U_\varphi\right)\right] = 0 \qquad (1c)$$

Where R, θ, φ are spherical polar coordinates; U_θ, U_φ are fluid velocities in the θ-, φ-directions respectively; h is the ocean depth; Ω is the angular rate of rotation of the sphere; ζ is the tide height above mean sea level; g is the mean gravitational acceleration at the surface; and $V(\zeta)$ and V_e are the potential energies due to tidal self-gravitation and the external tide generating body respectively.

Equations (1) can be considered a set of Eulerian fluid equations simplified from a Navier-Stokes equation in a rotating reference frame. The simplifications consist in assuming an incompressible fluid of constant density constrained by the earth's gravitation to a layer thin in comparison to the earth's radius and subjected to geostrophic (coriolis) and time-varying tidal forces of approximately equal magnitude. The viscosity and inertia of the fluid flow are ignored along with radial components of geostrophic force and fluid acceleration. The radial fluid force equation then easily integrates to obtain a Bernoulli-type expression for the pressure and centrifugal force in terms of the tidal force and time rate of change of tidal potential energy. Classical derivations can be found in Hough (1897), Lamb (1945, pp. 330-334) and Kaula (1968, pp. 190-196). See Doodson (1958) or Cartwright (1969) for a general review of progress in ocean tide research.

The assumptions made above are similar to those in a first order analysis of gravity waves. They are reasonable when the wavelengths are large in comparison with the depth, and the depth does not vary appreciably across a distance of one wavelength. A welcome feature of the equations is that they are linear and allow the separate derivation of the tide for each frequency of the sun and moon (normal modes).

Critique of the Equations

Continents give rise to appreciable dissipation on continental shelves. Reintroduction of the Navier-Stokes viscosity term with ν equal to the molecular viscosity of salt water will not work; it is too small. The mechanisms whereby small scale turbulence and internal waves interact to produce effects on the scale of the ocean depth and tidal wavelength are unclear. One approach is to introduce a boundary layer flow and a "kinematic" viscosity on the order of a thousand times the molecular value; another is to employ empirical dissipation forces proportional to velocity or velocity squared. All models today employ one or another of these dissipative terms.

In the continental regions the depth varies appreciably and may give rise to significant vertical accelerations. The procedure discussed in subsection B can be used to take into account the vertical forces and accelerations through correction terms.

The other assumption most likely to need revision is the one of uniform density with depth. The procedures below can also be modified to incorporate a multi-layer model. However, until more results are obtained from all models currently being developed, and until large amounts of open ocean tide measurements are obtained, such a refinement is best postponed.

Advantages of Analytical Methods

Most methods currently in use and projected, involve numerical integration of tidal equations. Two related difficulties are inevitable: the capacity of the largest computers is easily exhausted; and it is hard to keep the parameters physically meaningful without making concessions to numerical expediency and stabilization of the solution.

The difficulty in incorporating the variable ocean depth and presence of continents has discouraged the search for an analytic method not using numerical integration. If this were achieved, the difficulties mentioned above would be greatly lessened. In addition, analytical methods allow more flexibility in testing the parameters of a tidal model and in studying the inclusion of additional mathematical terms.

Appropriateness of Spherical Harmonics

The orthogonality of spherical harmonics with respect to integration over a sphere make them the natural representations for continuous functions on a sphere. Their appropriateness for the tidal problem must be considered in two ways: how a tide represented in spherical harmonics would interface with other geophysical studies; and how adequate such a representation of the tide would be from a computational point of view.

Appropriateness of Spherical Harmonics to Geophysical Studies

A knowledge of ocean tides will be very valuable to studies using low orbit geophysical satellites, especially those carrying radar altimeters with accuracies to better than one meter. A computed total real-time tide can be added to the altimeter reading to reference the satellite to the mean ocean geoid. Any form of tidal representation is equally advantageous for this purpose. However, a spherical harmonic tidal representation is the best form for calculating the tidal perturbation of satellite orbits. The gravitational potential of the tide at satellite heights is a simple function of the tide coefficients. Moreover, the gravitational effects of higher order terms quickly damp out at satellite heights, thus minimizing the effect of errors in these terms.

Appropriateness of Spherical Harmonics to Efficient Tidal Analysis

A parallel advantage is obtained by calculating in spherical harmonics. The self-gravitation of the tide is a simple function of the tide coefficients and can be directly incorporated in the calculations for the tide. Hough (1897, p. 240) showed that self-gravitation accounts for approximately 5% of the tide.

The choice of a representation for tides is large -- from completely general power series in sines and cosines of longitude and latitude to the particular functions which are solutions of the homogeneous Laplace tidal equation (zero tide force). Power series, while completely general, have slow convergence (Hough, 1897, pp. 202-203). Homogeneous solutions have been obtained as sums of spherical harmonics for constant (Longuet-Higgins, 1968) and simple, longitude-independent ocean profiles (Hough, 1897, 1899) covering the regime appropriate to the world oceans; namely, when the dimensionless quantity $\varepsilon = \frac{4R\Omega^2}{g} \cdot \frac{R}{H_o} \approx 35$ is not very large $\left(H_o = \text{mean ocean depth; for rest, see after eq. (1)}\right)$. But for realistic ocean profiles, their computation would be extremely difficult, and they would not be useful for general ocean model investigations. Spherical harmonics have similar convergence properties; their orthogonality on a sphere allow straightforward computation, and as a result parametric manipulation is easily managed.

Method of Computation in Spherical Harmonics

The computational method using spherical harmonics is the following: for a continuously variable ocean profile the Laplace tidal equations can be viewed as containing two conjugate functions; the depth h, and tide ζ, which have an equal footing -- any variation in one necessitates a change in the other. Representing the depth and tide by series of spherical harmonics, the equations can be combined and reduced to a single series of spherical harmonics with constant coefficients set equal to zero; these constant coefficients consist of sums and products of the coefficients of the depth and tide series in conjunction with the parameters of the equations and the tide force. The mathematical requirement that these constant coefficients be individually zero results in a set of simultaneous equations which the tide coefficients must satisfy for given depth profile (or vice versa). The equations may be analyzed directly, inverted, or solutions may be obtained by iteration. Variation of parameters and the inclusion of additional terms may be studied in these three ways, or by perturbation methods.

Complexity of the Tide

It is also important to note that the method of reduction to a single series of spherical harmonics determines the character of the resulting simultaneous equations. This, in turn, affects ease of analysis and computation. Also, complexities in the depth result in complexities in the tide. Mathematically speaking, higher order terms in the series for the depth necessitates higher order terms in the tide, and these additional depth terms cause "scattering" off the diagonal of the simultaneous equations. Physically speaking, rises of the ocean profile cause deflection and upwelling of the tide, resulting in a more complex tide.

The principal aspect of the ocean profile in determining the character of the world's tides is the continental division into three major oceans. This results in the tides of each frequency rotating about amphidromic (null) points, with a consequent increment to the mean vorticity of the oceans.

Requirements of a Solution

A criterion for success of a global analytical solution is that there be insignificant tide over the continents. If appropriate _infinite_ series for the depth and tide satisfy the Laplace equations, then this condition will be met: elimination of the velocity components from the equations shows that an exact zero depth, zero profile slope results in a zero tide at that point. However, the corresponding system is an infinite set of simultaneous equations. Truncation must be utilized. The greatest error would be expected in coastal regions where rapid changes in profile occur. Also, the error at shallower depths is further enhanced by the absence of a dissipative term in the Laplace tidal equations.

A properly chosen dissipative term can be expected to lessen both inaccuracies and, besides resulting in more realistic tides, it should lead to a better estimate of the rate of dissipation of energy in the oceans. The accuracy of each tidal model and areas for improvement will ultimately be determined by comparison with measurements from satellites and deep ocean sensors. A flexible model is best at this stage of relative ignorance both of the details in the ocean tides and of the full gamut of physical factors that determine these detailed features.

B PERTURBATION METHOD

To achieve the goal of flexibility in studying various tidal models at minimum computational expense a perturbation procedure can be worthwhile. This subsection outlines the procedure and weighs its computational efficiency. Details of application remain to be done.

The first, and therefore the simplest mathematical model that will result in adequate tides will be based on the Laplace tidal equations with various correction terms included. If the exact form and parametric values of these correction terms were known, attention could be concentrated on calculating the tide for each of the principal frequencies. Without this knowledge, attention should be concentrated on analyzing the possible correction terms and parametric values.

Since the correction terms are small in comparison with the Laplace terms, and can be additive, the use of perturbation techniques is possible. Perturbation techniques are usually used when direct calculation appears impossible; they can also be useful to decrease the burden of computation.

Assume that a particular reduction of the Laplace equations to a set of simultaneous equations has the form

$$\underline{M}\zeta = b \qquad (2)$$

where $\underline{M}$ is a matrix, and b and the tidal coefficients ζ are column vectors. Assume further that modified Laplacian equations reduce similarly to the system

$$(\underline{M} + p\,\underline{C})\zeta' = kb \qquad (3)$$

where $\underline{C}$ is a correction matrix, and p and k are scalars with p proportional to the parameters and variable at will, but always <<1. To solve system (3) in terms of the solution to system (2), namely

$$\zeta = \underline{M}^{-1} b \qquad (4)$$

where $\underline{M}^{-1}$ is the inverse matrix to $\underline{M}$, write ζ' as

$$\zeta' = k(\zeta + p\zeta_1 + p^2\zeta_2 + \cdots) \tag{5}$$

Then substitute this expression in (3). Since eq. (3) must be true for all values of p, the coefficients of the power series in p must separately vanish.

The zero values require that

$$\begin{aligned} \underline{M}\zeta &= b \\ \underline{M}\zeta_1 &= -\underline{c}\zeta \\ \underline{M}\zeta_2 &= -\underline{c}\zeta_1 \\ &\vdots \end{aligned} \tag{6}$$

Multiplying these equations from the left by $\underline{M}^{-1}$, we obtain a sequential procedure for calculating higher order corrections to the original solution, ζ:

$$\begin{aligned} \zeta_1 &= -\underline{M}^{-1}\underline{c}\zeta \\ \zeta_2 &= -\underline{M}^{-1}\underline{c}\zeta_1 \\ \zeta_3 &= -\underline{M}^{-1}\underline{c}\zeta_2 \\ &\vdots \end{aligned} \tag{7}$$

Estimate of Efficiency

This procedure can result in decreased computations when the necessary number of higher order terms is relatively small.

It is easy to estimate the relative amount of computation by straightforward inversion and by perturbation approximation. Assume that $(\underline{M} + p\underline{C})$ in system (3) is found to give adequate solutions when truncated to an $N \times N$ dimensional matrix. Solution by inversion for each value of p requires approximately N^3 arithmetic operations. The perturbation method requires N^3 operations to invert $\underline{M}$, N^3 operations to compute $\underline{M}^{-1}\underline{C}$ and N^2 operations to compute each correction term. If n_p is the largest number of correction terms necessary to study the pertinent values of p, then this study will require approximately $\left(2N + 1 + n_p\right)N^2$ arithmetic operations. If $n_p \ll N$, this involves little more than the effort to invert the system (3) for two specific values of p.

The situation for more than one independent parameter is only slightly more complicated. Cross terms must be computed at each level of correction. Consider the matrix system $(\underline{M} + \alpha\underline{A} + \beta\underline{B})\zeta = b$, where α and β are parameters of comparable magnitude. The independence of α and β requires that, at the first order of correction,

$$\underline{M}\,\zeta_1(\alpha) = -\underline{A}\,\zeta_0(\alpha)$$

and

$$\underline{M}\,\zeta_1(\beta) = -\underline{B}\,\zeta_0(\beta) \tag{8}$$

At the next level, there are cross terms such as

$$\underline{M}\,\zeta_2(\alpha) = -\underline{A}\,\zeta_1(\beta) \tag{9}$$

The number of terms at each level follows the sequence 2,4,6,8, ... , and the approximate number of arithmetic operations is $\left(2N+1+n_\alpha^2+n_\beta\right)N^2$, under the assumption $n_\alpha \leq n_\beta$. For three independent parameters the number of terms at each level follows the sequence 3,6,9,12, ... , and the number of operations is approximately $\left[2N+1+3n_\alpha\left(n_\alpha+1\right)/2\right]N^2$ if, for simplicity, $n_\alpha \sim n_\beta \sim n_\gamma$. Provided each parameter necessitates corrections numbering significantly less than the dimension of the matrix system, the perturbation method is thus efficient for any number of independent parameters.

C REDUCTION OF THE LAPLACE TIDAL EQUATIONS

Velocity Potential and Stream Function

To obtain a reduction of the Laplace equations to a set of simultaneous equations we follow Love (1913) and Lonquet-Higgins (1968) by introducing functions Φ and Ψ such that

$$u_\vartheta = \frac{\partial \Phi}{\partial \vartheta} - \frac{1}{\sin\vartheta}\frac{\partial \Psi}{\partial \varphi}$$

$$u_\varphi = \frac{1}{\sin\vartheta}\frac{\partial \Phi}{\partial \varphi} + \frac{\partial \Psi}{\partial \vartheta} \qquad (10)$$

Under the assumption that radial components are negligible, Φ is velocity potential and Ψ is stream function, as shown by the computations

$$\frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(u_\vartheta \sin\vartheta) + \frac{\partial u_\varphi}{\partial\varphi}\right] = \nabla^2\Phi$$

and

$$\frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(u_\varphi \sin\vartheta) - \frac{\partial u_\vartheta}{\partial\varphi}\right] = \nabla^2\Psi \qquad (11)$$

where ∇^2 is the angular part of the Laplacian operator,

$$\nabla^2 \equiv \frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin\vartheta}\frac{\partial^2}{\partial\varphi^2}\right] \qquad (12)$$

$\nabla^2\Psi$ represents vorticity about the axis (0, R). $\nabla^2\Phi$ represents horizontal divergence. It is not necessarily zero, fluid may flow outward if the tide height is decreasing or if the ocean depth is variable. The first case is evident when h = constant is substituted into the continuity equation (1c), giving

$$\frac{\partial\zeta}{\partial t} + \frac{h}{R}\nabla^2\Phi = 0 \qquad (13)$$

In general, the continuity equation becomes

$$\frac{\partial\zeta}{\partial t} + \frac{1}{R}\Big[h\nabla^2\Phi + \frac{\partial h}{\partial\vartheta}\left(\frac{\partial\Phi}{\partial\vartheta} - \frac{1}{\sin\vartheta}\frac{\partial\Psi}{\partial\varphi}\right) + \frac{1}{\sin\vartheta}\frac{\partial h}{\partial\varphi}\left(\frac{1}{\sin\vartheta}\frac{\partial\Phi}{\partial\varphi} + \frac{\partial\Psi}{\partial\vartheta}\right)\Big] = 0 \qquad (14)$$

In (14), the Φ-, Ψ-brackets represent u_θ and u_φ respectively.

To reduce the Laplace tidal equations ((1a) and (1b)) to expressions involving Φ and Ψ, perform the operations

$$\frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\varphi}(1b) + \frac{\partial}{\partial\vartheta}\{(1a)\sin\vartheta\}\right]$$

and

$$\frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\varphi}(1a) - \frac{\partial}{\partial\vartheta}\{(1b)\sin\vartheta\}\right]$$

Use (11) to identify factors and obtain the intermediate equations

$$\frac{\partial}{\partial t}\nabla^2\Phi - 2\Omega\cos\vartheta\,\nabla^2\Psi + 2\Omega\sin\vartheta\,u_\varphi = \frac{\nabla^2}{R}(V_e + V(\zeta) - g\zeta)$$

and

$$\frac{\partial}{\partial t}\nabla^2\Psi + 2\Omega\cos\vartheta\,\nabla^2\Phi - 2\Omega\sin\vartheta\,u_\varphi = 0$$

Then substitute for u_θ and u_φ from eqs. (10) to obtain

$$\left(\frac{\partial}{\partial t}\nabla^2 + 2\Omega\frac{\partial}{\partial\varphi}\right)\Phi - 2\Omega\left(\cos\vartheta\,\nabla^2 - \sin\vartheta\frac{\partial}{\partial\vartheta}\right)\Psi$$

$$= \frac{\nabla^2}{R}(V_e + V(\zeta) - g\zeta) \qquad (15a)$$

and

$$\left(\frac{\partial}{\partial t}\nabla^2 + 2\Omega\frac{\partial}{\partial\varphi}\right)\Psi + 2\Omega\left(\cos\vartheta\,\nabla^2 - \sin\vartheta\frac{\partial}{\partial\vartheta}\right)\Phi = 0 \qquad (15b)$$

Reduction of the Perturbing Potential

We seek solutions separately for each tide. Let the one under consideration have period $2\pi/\sigma$. We represent each quantity V_e, Φ, Ψ and ζ as the product of a complex spherical harmonic series times $e^{-i\sigma t}$, with the physical value represented by the real part of the quantity.

It is necessary next to explicate the expression $V_e + V(\zeta) - g\zeta$. See Kaula (1968, pp. 185-187, 191-192) for details of derivation. Using the continuity condition, this quantity can then be expressed as a function of Φ and Ψ.

The semi-diurnal potential energy of the sun or moon at the reference surface of the earth involves the associated Legendre polynomial $P_2^2(\cos\theta)$.* The diurnal tide involves $P_1^1(\cos\theta)$, and the semi-monthly tide involves $P_2^0(\cos\theta)$. These are the principal ones under consideration. We represent the one entering the computations by

$$v_{en^*}^{s^*} = g\,\overline{Z}_{n^*}^{s^*}\,P_{n^*}^{s^*}(\cos\vartheta)\,e^{i(s^*\varphi-\sigma t)} \qquad (16)$$

If we take into account the elastic response of the earth to the potential $v_{en^*}^{s^*}$, then correction terms are necessary. Since the response is small enough to be linear and essentially radial, we can account for displacement of internal earth mass with the term $k_{n^*}v_{en^*}^{s^*}$, where k_{n^*} is a <u>Love number</u> of order n^*. The largest and the one of interest is $k_2 \approx 0.3$. However, the ocean bottom is also displaced upwards an amount $h_{n^*}v_{en^*}^{s^*}/g$, which in effect drops the surface** equipotential

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*We employ the superscript convention in this part of the report to increase the legibility of the equations.

**The ellipticity of the ocean geoid can be incorporated as a perturbation of the basic spherical system. Any other deviations from sphericity, such as those due to steady currents, are insignificantly small.

an equivalent amount. The Love number $h_2 \approx 0.6$. These terms combine to give the effective perturbing potential energy term

$$V_e = (1 + k_{n*} - h_{n*}) g \bar{Z}_{n*}^{s*} P_{n*}^{s*}(\cos\vartheta) e^{i(s*\varphi - \sigma t)} \tag{17}$$

We represent the tide height as a series with complex coefficients $Z_n^{\underline{s}}$:

$$\zeta = \sum_{n=1} \sum_{\underline{s}=-n}^{n} Z_n^{\underline{s}} P_n^{s}(\cos\vartheta) e^{i(\underline{s}\varphi - \sigma t)} \tag{18}$$

We employ the convention, here and below, of underlining those expansion integers which take on negative, as well as positive values; for example, s is the absolute value of $\underline{s}$.

The integrated gravitational potential energy at the point (R, θ, φ), due to the tidal fluid layer with thickness ζ, reduces to

$$\mathcal{V}(\zeta) = 3g \frac{\rho}{\rho_e} \sum_{n,\underline{s}} \frac{Z_n^{\underline{s}}}{2n+1} P_n^{s}(\cos\vartheta) e^{i(\underline{s}\varphi - \sigma t)} \tag{19}$$

where $\frac{\rho}{\rho_e}$ is the ratio of water density to mean earth density and has the approximate value $(5.5)^{-1}$.

The elastic response of earth mass to the tide will give rise to a correction potential $\sum_{n,\underline{s}} k'_n v_n^{\underline{s}}(\zeta)$ (k'_n negative).

The tide also loads the sea bottom causing a downward displacement $-\sum_{n,\underline{s}} h'_n v_n^{\underline{s}}(\zeta)/g$ (h'_n negative) and a corresponding rise in equipotential. The parameters k'_n and h'_n are called load deformation coefficients. The largest are $k'_2 \approx -0.3$ and $h'_2 \approx -1.0$. The perturbing self-gravitation of the tide, $V(\zeta)$, then takes the form

$$V(\zeta) = g \sum_{n=1} \sum_{\underline{s}=-n}^{n} (1+k'_n-h'_n) \frac{3}{2n+1} \frac{\rho}{\rho_e} Z_n^{\underline{s}} P_n^{s}(\cos\vartheta) e^{i(\underline{s}\varphi-\sigma t)} \quad (20)$$

The complete perturbing potential $V_e + V(\zeta) - g\zeta$ can now be written

$$V_e + V(\zeta) - g\zeta = g \sum_{n=1} \sum_{\underline{s}=-n}^{n} [\delta_n^{n*} \delta_{\underline{s}}^{s*} \mathcal{K}^*_{n*} Z_{n*}^{s*} - \mathcal{K}_n Z_n^{\underline{s}}] P_n^{\underline{s}}(\cos\vartheta) e^{i(\underline{s}\varphi-\sigma t)} \quad (21)$$

where

$$\mathcal{K}^*_{n*} = 1 + k_{n*} - h_{n*} \quad (22)$$

$$\mathcal{K}_n = 1 - (1+k'_n-h'_n) \frac{\rho}{\rho_e} \frac{3}{2n+1} \quad (23)$$

and δ_n^{n*}, $\delta_{\underline{s}}^{s*}$ are Kronecker deltas:

$$\delta_a^b = \begin{cases} 1 & \text{when } a=b \\ 0 & \text{when } a \neq b \end{cases} \quad (24)$$

Reduction of the Continuity Equation

Next, we find an expression for $Z_n^{\underline{s}}$ in terms of the expansion coefficients of Φ, Ψ and the depth h, by reducing the continuity equation (14).

First, we define the series expansions of Φ, Ψ, and h:

$$\Phi = i \sum_{n=1} \sum_{s=-n}^{n} A_n^s P_n^s(\cos\vartheta) e^{i(s\varphi - \sigma t)} \tag{25}$$

$$\Psi = -\sum_{n=1} \sum_{s=-n}^{n} B_n^s P_n^s(\cos\vartheta) e^{i(s\varphi - \sigma t)} \tag{26}$$

$$h = H_0 + \sum_{j=1} \sum_{k=-j}^{j} H_j^k P_j^k(\cos\vartheta) e^{i k\varphi} \tag{27}$$

In eq. (25), i is a phase factor which allows the elimination of i from the final equations which A_n^s and B_n^s must satisfy.

Define

$$\mu = \cos\vartheta \tag{28}$$

In terms of μ, eq. (14) becomes

$$\frac{\partial \zeta}{\partial t} + \frac{1}{R}\Big[h \nabla^2 \Phi + \sqrt{1-\mu^2}\frac{\partial h}{\partial \mu}\Big(\sqrt{1-\mu^2}\frac{\partial \Phi}{\partial \mu} + \frac{1}{\sqrt{1-\mu^2}}\cdot\frac{\partial \Psi}{\partial \varphi}\Big) + \frac{1}{\sqrt{1-\mu^2}}\frac{\partial h}{\partial \varphi}\Big(\frac{1}{\sqrt{1-\mu^2}}\frac{\partial \Phi}{\partial \varphi} - \sqrt{1-\mu^2}\frac{\partial \Psi}{\partial \mu}\Big)\Big] = 0 \tag{29}$$

where the Laplacian operator is now independent of φ:

$$\nabla^2 = \frac{d}{d\mu}\Big[(1-\mu^2)\frac{d}{d\mu}\Big] - \frac{s^2}{1-\mu^2} \tag{30}$$

We need the following operators on the Associated Legendre polynomial:

$$\nabla^2 P_n^s(\mu) = -n(n+1) P_n^s(\mu) \tag{31}$$

(for s > 0):

$$\frac{s}{\sqrt{1-\mu^2}} P_n^s(\mu) = \frac{1}{2}\left[P_{n+1}^{s+1}(\mu) + (n-s+1)(n-s+2)\, P_{n+1}^{s-1}(\mu) \right] \tag{32}$$

and

$$-\frac{d}{d\vartheta} P_n^s(\mu) = \sqrt{1-\mu^2}\,\frac{d}{d\mu} P_n^s(\mu) = \frac{1}{2}\left[(1+\delta_0^s)\, P_n^{s+1}(\mu) - (n-s+1)(n+s)\, P_n^{s-1}(\mu) \right] \tag{33}$$

Eq. (31) follows immediately from the differential equation defining $P_n^s(\mu)$.

Eq. (32) follows from eliminating the second term of

$$P_{n+1}^{s+1} + \frac{(n-s+1)\mu}{\sqrt{1-\mu^2}} P_{n+1}^s - \frac{n+s+1}{\sqrt{1-\mu^2}} P_n^s = 0 \tag{34}$$

by using the equation (valid for $s > 0$)

$$P_n^{s+1} - \frac{2s\mu}{\sqrt{1-\mu^2}} P_n^s + (n-s+1)(n+s) P_n^{s-1} = 0 \tag{35}$$

with $n+1$ replacing n.

When $s = 0$, eq. (33) follows immediately from

$$\sqrt{1-\mu^2}\,\frac{d}{d\mu} P_n^s = -\frac{s\mu}{\sqrt{1-\mu^2}} P_n^s + P_n^{s+1} \tag{36}$$

When $s \neq 0$, use eq. (35) to eliminate the first term on the right. The signs in eqs. (32) - (36) are based on P_n^s defined as

$$P_n^s(\mu) = (1-\mu^2)\frac{d^s}{d\mu^s} P_n(\mu) \tag{37a}$$

where $P_n(\mu)$ is the Legendre polynomial. The associated Legendre polynomial is assumed above and in all subsequent formulas to have the property

$$P_n^s(\mu) = 0 \text{ when } s > n \text{ or } s < 0 \tag{37b}$$

The distribution of μ-factors in eq. (29) was chosen to maintain the physical meaning of the operators. The expressions

$$\frac{\partial}{\partial\vartheta}\Phi = -\sqrt{1-\mu^2}\,\frac{\partial}{\partial\mu}\Phi$$

and

$$\frac{1}{\sqrt{1-\mu^2}}\frac{\partial}{\partial\varphi}\Phi = i\ \mathrm{sign}(\underline{s})\cdot\frac{s}{\sqrt{1-\mu^2}}\Phi$$

represent the directional derivatives of Φ in the θ- and φ-directions. In this way, the parentheses in eq. (29) continue to represent the velocity components and the operators on h, the directional derivatives.

To reduce eq. (29), represent ζ using eq. (18), Φ using (25), Ψ using (26), and h using (27). Perform the partial differentiations on t and φ, and use eqs. (31) - (33) to reduce the remaining differentiations. Shift the n-index of terms involving $A_n^{\underline{s}}P_{n+1}^{s}$, $B_n^{\underline{s}}P_{n+1}^{s}$ and $H_j^{\underline{k}}P_{j+1}^{k}$. Then divide the equations by $i\sigma$ to obtain

$$\begin{aligned}
\sum_{n=1}\sum_{\underline{s}=-n}^{n}\Big\{&\Big(-Z_n^{\underline{s}}-\frac{H_0\,n(n+1)}{R\sigma}A_n^{\underline{s}}\Big)P_n^{s}\\
&+\frac{1}{R\sigma}\sum_{j=1}\sum_{\underline{k}=-j}^{j}\Big[-n(n+1)A_n^{\underline{s}}H_j^{\underline{k}}P_j^{k}P_n^{s}\\
&+H_j^{\underline{k}}\big(\alpha_j^{+k}P_j^{k+1}-\beta_j^{+k}P_j^{k-1}\big)\cdot\Big(A_n^{\underline{s}}\big[\alpha_n^{+s}P_n^{s+1}-\beta_n^{+s}P_n^{s-1}\big]\\
&\qquad-B_{n-1}^{\underline{s}}\big[\alpha_n^{-\underline{s}}P_n^{s+1}-\beta_n^{-\underline{s}}P_n^{s-1}\big]\Big)\\
&+H_{j-1}^{\underline{k}}\big(\alpha_j^{-\underline{k}}P_j^{k+1}-\beta_j^{-\underline{k}}P_j^{k-1}\big)\cdot\Big(B_n^{\underline{s}}\big[\alpha_n^{+s}P_n^{s+1}-\beta_n^{+s}P_n^{s-1}\big]\\
&\qquad-A_{n-1}^{\underline{s}}\big[\alpha_n^{-\underline{s}}P_n^{s+1}-\beta_n^{-\underline{s}}P_n^{s-1}\big]\Big)\Big]\\
&\cdot e^{i\underline{k}\varphi}\Big\}e^{i(\underline{s}\varphi-\sigma t)}=0
\end{aligned}\qquad (38)$$

where

$$\alpha^{-k} = \frac{1}{2}\,\mathrm{sign}(k)\cdot(1-\delta_0^{\;k})$$

$$\alpha^{+k} = \frac{1}{2}(1+\delta_0^{\;k})$$

$$\beta_j^{-k} = \alpha^{-k}(k-j)(j-k+1)$$

$$\beta_j^{+k} = \frac{1}{2}(1-\delta_0^{\;k})(k+j)(j-k+1) \qquad (39)$$

The summations on n, $\underline{s}$ span all meaningful coefficients of $A_n^{\underline{s}}$ and $B_n^{\underline{s}}$, namely $n \geq |\underline{s}|$ provided $n \geq 1$.

The spherical harmonic product $P_j^{k+\delta_1} P_n^{s+\delta_2} e^{i(\underline{k}+\underline{s})}$, under the restrictions $\delta_1 = \delta_2 = 0$ or $\delta_1 = \delta_2 = \pm 1$ or $\delta_1 = -\delta_2 = \pm 1$, represents $e^{i(\underline{k}+\underline{s})}$ times $(1-\mu^2)^{(j+n-|\underline{k}+\underline{s}|)}$ times an even or odd polynomial in μ whose highest power is $\mu^{(j+n-|\underline{k}+\underline{s}|)}$. Such a product is uniquely expressible as $e^{i(\underline{k}+\underline{s})}$ times a series of spherical harmonics $P_{j+n}^{|\underline{k}+\underline{s}|}$, $P_{j+n-2}^{|\underline{k}+\underline{s}|}, \ldots$. The symbols Q below are defined by, and derivable from, the equations

$$P_j^{k} P_n^{s} e^{\pm i(k\pm s)\varphi} = \sum_{r'=0} Q_{jkns\,(j+n-2r')|k\pm s|}\, P_{j+n-2r'}^{|k\pm s|}\, e^{\pm i(k\pm s)\varphi}$$

$$P_j^{k+1} P_n^{s+1} \quad \ldots \quad = \sum Q^{++} \quad \ldots$$

$$P_j^{k-1} P_n^{s-1} \quad \ldots \quad = \sum Q^{--} \quad \ldots$$

$$P_j^{k+1} P_n^{s-1} \quad \ldots \quad = \sum Q^{+-} \quad \ldots$$

$$P_j^{k-1} P_n^{s+1} \quad \ldots \quad = \sum Q^{-+}_{jkns\,(j+n-2r')|k\pm s|}\, P_{j+n-2r'}^{|k+s|}\, e^{\pm i(k\pm s)\varphi} \qquad (40)$$

Appendix A details the derivation of the Q-coefficients. Q-functions for s = 0 are also given in Sec. III, p. 74.

Eq. (38) can then be resumed in terms of a single series of spherical harmonics:

$$\sum_{n,\underline{s}} \Big\{ \Big[-Z_n^{\underline{s}} - \frac{H_0\, n(n+1)}{R\sigma} A_n^{\underline{s}}\Big] P_n^{s} e^{i\underline{s}\varphi} + \frac{1}{R\sigma}\sum_{j,\underline{k}}\sum_{r'=0} \Big\{ -n(n+1) A_n^{\underline{s}} H_j^{\underline{k}} Q_{jkns(j+n-2r')|\underline{k}+\underline{s}|}$$
$$+ A_n^{\underline{s}} H_j^{\underline{k}} [\alpha^{+s}\xi_1^{+} - \beta_n^{+s}\xi_2^{+}] - B_{n-1}^{\underline{s}} H_j^{\underline{k}} [\alpha^{-\underline{s}}\xi_1^{+} - \beta_n^{-\underline{s}}\xi_2^{+}]$$
$$+ B_n^{\underline{s}} H_{j-1}^{\underline{k}} [\alpha^{+s}\xi_1^{-} - \beta_n^{+s}\xi_2^{-}] - A_{n-1}^{\underline{s}} H_{j-1}^{\underline{k}} [\alpha^{-\underline{s}}\xi_1^{-} - \beta_n^{-\underline{s}}\xi_2^{-}]\Big\}$$
$$\cdot P_{j+n-2r'}^{|\underline{k}+\underline{s}|}\, e^{i(\underline{k}+\underline{s})\varphi} \Big\} e^{-i\sigma t} = 0 \qquad (41)$$

where

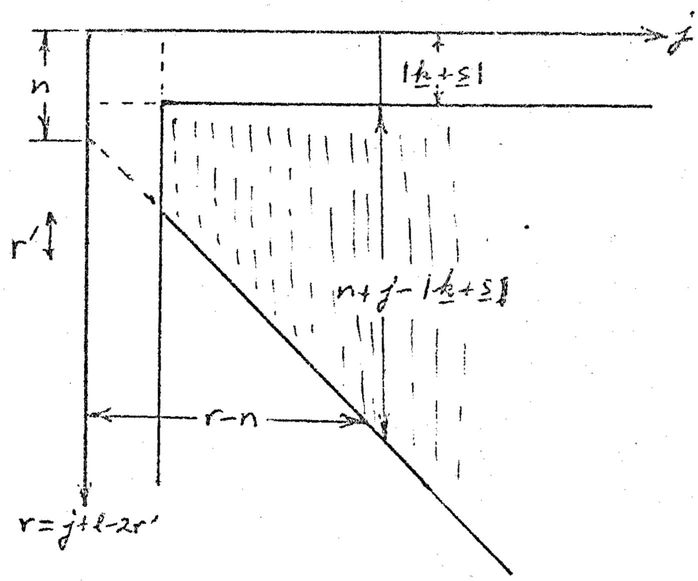
$$\xi_1^{+} = \alpha^{+k} Q^{++}_{jkns(j+n-2r')|\underline{k}+\underline{s}|} - \beta_j^{+k} Q^{-+}_{jkns(j+n-2r')|\underline{k}+\underline{s}|}$$
$$\xi_2^{+} = \alpha^{+k} Q^{+-} \ \ldots \qquad - \beta_j^{+k} Q^{--} \ \ldots$$
$$\xi_1^{-} = \alpha^{-\underline{k}} Q^{++} \ \ldots \qquad - \beta_j^{-\underline{k}} Q^{-+} \ \ldots$$
$$\xi_2^{-} = \alpha^{-\underline{k}} Q^{+-}_{jkns(j+n-2r')|\underline{k}+\underline{s}|} - \beta_j^{-\underline{k}} Q^{--}_{jkns(j+n-2r')|\underline{k}+\underline{s}|} \qquad (42)$$

In the latter part of eq. (41) we need to interchange the order of summation on r′ and $\underline{k}$ with respect to n, $\underline{s}$, and j. The diagram below is an n = const., k = const., $\underline{s}$ = const. plane of the summation scheme. The first step is to make r′ the outermost summation. The summation becomes

$$\sum_{r \geq |\underline{k}+\underline{s}|}\ \sum_{n,\underline{s}}\ \sum_{j=0} \cdots P_r^{|\underline{k}+\underline{s}|} e^{i(\underline{k}+\underline{s})\varphi - i\sigma t}$$

where $r = j+n-2r'$ and j in the innermost sum is now

$$j = \begin{cases} r-n+2j' & \text{when } r \geq n \\ (n-r)\,\text{modulo}(2)+2j' & \text{when } r < n \end{cases} \quad \text{provided } j \geq k$$



n = const., $\underline{k}$ = const., $\underline{s}$ = const., plane.

Next, resum on $\underline{m} = \underline{k}+\underline{s}$ in place of $\underline{k}$ and bring this sum out to the left, which requires the replacement of $\underline{k}$ with $\underline{m}-\underline{s}$, $|\underline{k}+\underline{s}|$ with m and k with $|\underline{m}-\underline{s}|$, giving

$$\sum_{r,\underline{m}} \sum_{n,\underline{s}} \sum_{j'=0} \cdots \; P_r^m e^{i(\underline{m}\varphi - \sigma t)}$$

Finally, interchanging the labels $r \leftrightarrow n$, $\underline{s} \leftrightarrow \underline{m}$ gives a scheme compatible for the whole expression

$$\sum_{n,\underline{s}} \sum_{r,\underline{m}} \sum_{j'=0} \cdots \; P_n^s e^{i(\underline{s}\varphi - \sigma t)}$$

This requires the following replacements in the latter summations of eq. (41):

$n \to r$

$j + n - 2r' \to n$

$\underline{s} \to \underline{m}$

$s \to m$

$\underline{k} \to \underline{s} - \underline{m}$

$k \to |\underline{s} - \underline{m}|$

$|\underline{k} + \underline{s}| \to s$

and

$$j = \left\{ \begin{matrix} n - r + 2j' & \text{when } n \geq r \\ (r-n)\,\text{modulo}(2) + 2j' & \text{when } n < r \end{matrix} \right\} \text{ provided } j \geq |\underline{s} - \underline{m}|$$

Eq. (41) then becomes

$$\sum_{n=1} \sum_{\underline{s}=-n}^{n} \Big\{ -Z_n^{\underline{s}} - \frac{H_0\, n(n+1)}{R\sigma} A_n^{\underline{s}}$$
$$+ \frac{1}{R\sigma} \sum_{r=1} \sum_{\underline{m}=-r}^{r} \sum_{j'=0} \Big[-r(r+1) A_r^{\underline{m}} H_j^{\underline{s}-\underline{m}} Q_{j\,|\underline{s}-\underline{m}|\,r\,m\,n\,s}$$
$$+ A_r^{\underline{m}} H_j^{\underline{s}-\underline{m}} (\alpha^{+m} \xi_1^{+} - \beta_r^{+m} \xi_2^{+}) - B_{r-1}^{\underline{m}} H_j^{\underline{s}-\underline{m}} (\alpha^{-\underline{m}} \xi_1^{+} - \beta_r^{-\underline{m}} \xi_2^{+})$$
$$+ B_r^{\underline{m}} H_{j-1}^{\underline{s}-\underline{m}} (\alpha^{+m} \xi_1^{-} - \beta_r^{+m} \xi_2^{-})$$
$$- A_{r-1}^{\underline{m}} H_{j-1}^{\underline{s}-\underline{m}} (\alpha^{-\underline{m}} \xi_1^{-} - \beta_r^{-\underline{m}} \xi_2^{-}) \Big] \Big\} P_n^{s}(\mu)\, e^{i(\underline{s}\varphi - \sigma t)} = 0 \qquad (43)$$

where the ξ's are now subscripted to read

$$\xi_1^+ = \alpha^{+|\underline{s}-\underline{m}|}\, Q^{++}_{j|\underline{s}-\underline{m}|rmns} - \beta_j^{+|\underline{s}-\underline{m}|}\, Q^{-+}_{j|\underline{s}-\underline{m}|rmns}$$

$$\xi_2^+ = \alpha^{+|\underline{s}-\underline{m}|}\, Q^{+-} \qquad -\beta_j^{+|\underline{s}-\underline{m}|}\, Q^{--}$$

$$\xi_1^- = \alpha^{-(\underline{s}-\underline{m})}\, Q^{++} \qquad -\beta_j^{-(\underline{s}-\underline{m})}\, Q^{-+}$$

$$\xi_2^- = \alpha^{-(\underline{s}-\underline{m})}\, Q^{+-}_{j|\underline{s}-\underline{m}|rmns} - \beta_j^{-(\underline{s}-\underline{m})}\, Q^{--}_{j|\underline{s}-\underline{m}|rmns} \qquad (44)$$

The bracket multiplying each $P_n^s(\mu)e^{i(\underline{s}\varphi-\sigma t)}$ in eq. (43) is a constant. For such a series of spherical harmonics to equal zero, it is necessary that each bracket be individually zero. As a result, we have an expression for each $Z_n^{\underline{s}}$ coefficient of ζ in terms of a series in the expansion coefficients for Φ, Ψ, and h.

Reduction of the Fluid Equations

In the fluid equations (15), we substitute for Φ from eq. (25), Ψ from (26), and the perturbing potential from eq. (21), then compute $\partial/\partial t$ and $\partial/\partial\varphi$, and divide the first equation by 2Ω and the second by $2\Omega i$ to obtain

$$\sum_{n,\underline{s}} \Big\{ (f\nabla^2-\underline{s})A_n^{\underline{s}} + (\mu\nabla^2+D)B_n^{\underline{s}} + \frac{g}{2\Omega R}\nabla^2(\varkappa_n Z_n^{\underline{s}} - \delta_n^{n^*}\delta_{\underline{s}}^{s^*}\varkappa_{n^*}^* \bar{Z}_{n^*}^{s^*})\Big\} P_n^s e^{i(\underline{s}\varphi-\sigma t)} = 0 \qquad (45a)$$

and

$$\sum_{n,\underline{s}} \Big\{ (f\nabla^2-\underline{s})B_n^{\underline{s}} + (\mu\nabla^2+D)A_n^{\underline{s}} \Big\} P_n^s e^{i(\underline{s}\varphi-\sigma t)} = 0 \qquad (45b)$$

where

$$f = \frac{\sigma}{2\Omega} \tag{46}$$

The operator D is defined as

$$D = -\sin\vartheta \frac{\partial}{\partial\vartheta} = (1-\mu^2)\frac{\partial}{\partial\mu} \tag{47}$$

and has the property

$$D P_n^s(\mu) = \frac{(n+1)(n+s)}{2n+1} P_{n-1}^s(\mu) - \frac{n(n-s+1)}{2n+1} P_{n+1}^s(\mu) \tag{48}$$

Again using eq. (31) and

$$\mu P_n^s(\mu) = \frac{n+s}{2n+1} P_{n-1}^s(\mu) + \frac{n-s+1}{2n+1} P_{n+1}^s(\mu) \tag{49}$$

we obtain

$$(\mu\nabla^2 + D) P_n^s(\mu) = -\frac{(n-1)(n+1)(n+s)}{2n+1} P_{n-1}^s - \frac{n(n+2)(n-s+1)}{2n+1} P_{n+1}^s \tag{50}$$

We substitute for Z_n^s by equating each bracket in eq. (43) to zero, and apply the operators ∇^2 and $(\mu\nabla^2 + D)$, using (31) and (50) to obtain

$$\sum_{n=1}\sum_{s=-n}^{n}\Big\{\Big[f n(n+1)+s-\frac{\chi_n}{\epsilon f}n^2(n+1)^2\Big]A_n^s+\frac{n(n+2)(n+s+1)}{2n+3}B_{n+1}^s$$

$$+\frac{(n-1)(n+1)(n-s)}{2n-1}B_{n-1}^s$$

$$+\frac{\chi_n}{\epsilon f}n(n+1)\sum_{r=1}\sum_{m=-r}^{r}\sum_{j=0}\Big\{A_r^m\frac{H_j^{s-m}}{H_0}\Big[-r(r+1)Q+\alpha^{+m}\xi_1^+-\beta_r^{+m}\xi_2^+\Big]$$

$$-A_{r-1}^m\frac{H_{j-1}^{s-m}}{H_0}\Big[\alpha^{-m}\xi_1^- -\beta_r^{-m}\xi_2^-\Big]+B_r^m\frac{H_{j-1}^{s-m}}{H_0}\Big[\alpha^{+m}\xi_1^- -\beta_r^{+m}\xi_2^-\Big]$$

$$-B_{r-1}^m\frac{H_j^{s-m}}{H_0}\Big[\alpha^{-m}\xi_1^+-\beta_r^{-m}\xi_2^+\Big]\Big\}$$

$$-n^*(n^*+1)\delta_n^{n^*}\delta_s^{s^*}\frac{g}{2\Omega R}Z_{n^*}^{s^*}\Big\}P_n^s e^{i(s\varphi-\sigma t)}=0 \qquad (51a)$$

and

$$\sum_{n=1}\sum_{s=-n}^{n}\Big\{\Big[f n(n+1)+s\Big]B_n^s+\frac{n(n+2)(n+s+1)}{2n+3}A_{n+1}^s$$

$$+\frac{(n-1)(n+1)(n-s)}{2n-1}A_{n-1}^s\Big\}P_n^s e^{i(s\varphi-\sigma t)}=0 \qquad (51b)$$

where

$$\epsilon=\frac{4R^2\Omega^2}{gH_0} \qquad (52)$$

$\epsilon \approx 35$ for the earth.

Each bracket in (51) must be separately zero. Setting each to zero, we have simultaneous equations which the coefficients of Φ and Ψ must satisfy. We divide each equation by $n(n+1)$. Using the symbols

$$L_n^s=f+\frac{s}{n(n+1)} \qquad K_n^s=L_n^s-\frac{\chi_n n(n+1)}{\epsilon f}$$

$$p_n^s=\frac{(n+1)(n+s)}{n(2n+1)}$$

$$q_n^s=\frac{n(n-s+1)}{(n+1)(2n+1)} \qquad (53)$$

eqs. (51) become

$$q_{n-1}^{s} B_{n-1}^{\underline{s}} + K_n^{\underline{s}} A_n^{\underline{s}} + P_{n+1}^{s} B_{n+1}^{\underline{s}}$$

$$+ \frac{\chi_n}{\epsilon f} \sum_{r,m} \sum_{j} \Big\{ A_r^{\underline{m}} \frac{H_j^{\underline{s-m}}}{H_0} [-r(r+1)Q + \alpha^{+m} \xi_1^{+} - \beta_r^{+m} \xi_2^{+}]$$

$$- A_{r-1}^{\underline{m}} \frac{H_{j-1}^{\underline{s-m}}}{H_0} [\alpha^{-m} \xi_1^{-} - \beta_r^{-m} \xi_2^{-}] + B_r^{\underline{m}} \frac{H_{j-1}^{\underline{s-m}}}{H_0} [\alpha^{+m} \xi_1^{-} - \beta_r^{+m} \xi_2^{-}]$$

$$- B_{r-1}^{\underline{m}} \frac{H_j^{\underline{s-m}}}{H_0} [\alpha^{-m} \xi_1^{+} - \beta_r^{-m} \xi_2^{+}] \Big\} = \delta_n^{n*} \delta_{\underline{s}}^{s*} \frac{g \bar{Z}_{n*}^{s*}}{2\Omega R} \qquad (54a)$$

and

$$q_{n-1}^{s} A_{n-1}^{\underline{s}} + L_n^{\underline{s}} B_n^{\underline{s}} + P_{n+1}^{s} A_{n+1}^{\underline{s}} = 0 \qquad (54b)$$

Consider the case where h = constant. Then eq. (54a) becomes

$$q_{n-1}^{s} B_{n-1}^{s} + K_n^{\underline{s}} A_n^{\underline{s}} + P_{n+1}^{s} B_{n+1}^{\underline{s}} = \delta_n^{n*} \delta_{\underline{s}}^{s*} \frac{g \bar{Z}_{n*}^{s*}}{2\Omega R} \qquad (54c)$$

It is not hard to see that, unless the choice of parameters makes the equations singular, only coefficients of order m* are non-zero. Moreover, with respect to degree, only one of two sets of coefficients are non-zero:

$$A_{|m*|}^{m*},\ B_{|m*|+1}^{m*},\ A_{|m*|+2}^{m*},\ B_{|m*|+3}^{m*},\ \cdots \qquad (55a)$$

when $\ell*$ is even, or

$$B_{|m*|}^{m*},\ A_{|m*|+1}^{m*},\ B_{|m*|+2}^{m*},\ A_{|m*|+3}^{m*},\ \cdots \qquad (55b)$$

when $\ell*$ is odd. If the depth is a function of θ only, both sets (55) will be present. And in the general case, terms of all degree and order are non-zero.

There are two methods of solution. Either solve the full set of eqs. (54), or use (54b) to eliminate the $B_n^{\underline{s}}$ from (54a) or (54c), and then solve the resulting half-system of equations. The latter method requires less computer resources. If we employ substitution, the set of simultaneous equations $A_n^{\underline{s}}$ must satisfy are:

$$
\begin{aligned}
&-\frac{q_{n-2}^{s}\,q_{n-1}^{s}}{L_{n-1}^{\underline{s}}}A_{n-2}^{\underline{s}} + \left[K_n^{\underline{s}} - \frac{p_{n+1}^{s}\,q_n^{s}}{L_{n+1}^{\underline{s}}} - \frac{q_{n-1}^{s}\,p_n^{s}}{L_{n-1}^{\underline{s}}}\right]A_n^{\underline{s}} - \frac{p_{n+1}^{s}\,p_{n+2}^{s}}{L_{n+1}^{\underline{s}}}A_{n+2}^{\underline{s}} \\
&+\frac{K_n}{\varepsilon f}\sum_{r=1}\sum_{\underline{m}=-r}^{r}\sum_{j'=0}\Bigg\{-\frac{p_{r+1}^{m}}{L_r^{\underline{m}}}\left(\alpha^{+m}\xi_1^{-}-\beta_r^{+m}\xi_2^{-}\right)\frac{H_{j-1}^{\underline{s}-\underline{m}}}{H_0}A_{r+1}^{\underline{m}} \\
&\quad+\left[-r(r+1)\,Q_{j\,|\underline{s}-\underline{m}|\,rmns}+\alpha^{+m}\xi_1^{+}-\beta_r^{+m}\xi_2^{+}+\frac{p_r^{m}}{L_{r-1}^{\underline{m}}}\left(\alpha^{-\underline{m}}\xi_1^{+}-\beta_r^{-\underline{m}}\xi_2^{+}\right)\right]\frac{H_j^{\underline{s}-\underline{m}}}{H_0}A_r^{\underline{m}} \\
&\quad-\left[\alpha^{-\underline{m}}\xi_1^{-}-\beta_r^{-\underline{m}}\xi_2^{-}+\frac{q_{r-1}^{m}}{L_r^{\underline{m}}}\left(\alpha^{+m}\xi_1^{-}-\beta_r^{+m}\xi_2^{-}\right)\right]\frac{H_{j-1}^{\underline{s}-\underline{m}}}{H_0}A_{r-1}^{\underline{m}} \\
&\quad+\frac{q_{r-2}^{m}}{L_{r-1}^{\underline{m}}}\left(\alpha^{-\underline{m}}\xi_1^{+}-\beta_r^{-\underline{m}}\xi_2^{+}\right)\frac{H_j^{\underline{s}-\underline{m}}}{H_0}A_{r-2}^{\underline{m}}\Bigg\} = \delta_n^{n^*}\delta_{\underline{s}}^{s^*}\frac{g\,\bar{Z}_{n^*}^{s^*}}{2\Omega R}
\end{aligned}
\tag{56}
$$

D A DISSIPATIVE MODEL

In this subsection we give one example of a dissipative model amenable to the perturbation method of subsection B.

Bottom Force

The analysis of energy dissipation in narrow seas and rivers, and the measurement of total drag force on flat bodies moving through moderate, steady fluid flow are found to be consistent with the assumption of an average boundary layer drag force per unit area of the form

$$\underline{\underline{F}} = -0.002 \rho u \underline{\underline{u}} \tag{57}$$

(Taylor, 1919). By action and reaction, such a force can be assumed to also act on the fluid and be incorporated in the equations if small scale dynamics of the fluid and the generation of vorticity by the boundary layer can be assumed to have little effect on the large scale fluid behavior. The bottom force (57) was incorporated in the equations used by Doodson and Proudman (1924) in their study of the North Sea, and by Grace (1931) in his study of the Gulf of Suez. So far no one has tackled the formidable problem of applying (57) to larger bodies of water.

Linear Bottom Force

A linear bottom force.

$$\underline{\underline{F}} = -\alpha \rho \underline{\underline{u}} \tag{58}$$

has been found to give adequate results in small bodies of water if α is carefully picked to meet circumstances of depth and tidal frequency (Grace, 1931; Doodson, 1956). Pekeris (1969) used a modification of this bottom force,

$$\underline{\underline{F}} = -\alpha \rho \frac{h_0}{h} \underline{\underline{u}} \tag{59}$$

to improve tide heights on continental shelves and tide phases in his numerical integration of the semi-diurnal tide M_2 over the world oceans. A linear law was feasible for numerical integration on such a vast scale, and serves our purpose of a simple illustration of perturbation analysis.

Reduction of Linear Bottom Force

If the bottom force (58) is assumed to give rise to an average drag force on the fluid column of depth h, then an additional acceleration $-\alpha\underline{\underline{u}}/h$ acts on the column and enters the fluid equations. Eqs. (1a,b) become

$$\frac{\partial u_\vartheta}{\partial t} - 2\Omega \cos\vartheta\, u_\varphi + \frac{\alpha}{h} u_\vartheta + \frac{1}{R}\frac{\partial}{\partial\vartheta}(g\zeta - V(\zeta) - V_e) = 0 \tag{60a}$$

and

$$\frac{\partial u_\varphi}{\partial t} + 2\Omega \cos\vartheta\, u_\vartheta + \frac{\alpha}{h} u_\varphi + \frac{1}{R\sin\vartheta}\frac{\partial}{\partial\varphi}(g\zeta - V(\zeta) - V_e) = 0 \tag{60b}$$

After performing the operations below eq. (14), we have, instead of eqs. (15),

$$\left(\frac{\partial}{\partial t}\nabla^2 + 2\Omega\frac{\partial}{\partial\varphi}\right)\Phi - 2\Omega\left(\cos\vartheta\,\nabla^2 - \sin\vartheta\frac{\partial}{\partial\vartheta}\right)\Psi$$

$$+ \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\left(\alpha\frac{U_\varphi}{h}\right) + \left(\frac{\partial}{\partial\vartheta} + \cot\vartheta\right)\left(\alpha\frac{U_\vartheta}{h}\right)$$

$$= \frac{\nabla^2}{R}\left(V_e + V(\zeta) - g\zeta\right) \qquad (61a)$$

and

$$\left(\frac{\partial}{\partial t}\nabla^2 + 2\Omega\frac{\partial}{\partial\varphi}\right)\Psi + 2\Omega\left(\cos\vartheta\,\nabla^2 - \sin\vartheta\frac{\partial}{\partial\vartheta}\right)\Phi$$

$$+ \frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\left(\alpha\frac{U_\vartheta}{h}\right) - \left(\frac{\partial}{\partial\vartheta} + \cot\vartheta\right)\left(\alpha\frac{U_\varphi}{h}\right) = 0 \qquad (61b)$$

Under the assumption h = constant, these equations reduce to

$$\left[\left(\frac{\partial}{\partial t} + \frac{\alpha}{h}\right)\nabla^2 + 2\Omega\frac{\partial}{\partial\varphi}\right]\Phi - 2\Omega\left(\cos\vartheta\,\nabla^2 - \sin\vartheta\frac{\partial}{\partial\varphi}\right)\Psi$$

$$= \frac{\nabla^2}{R}\left(V_e + V(\zeta) - g\zeta\right) \qquad (62a)$$

and

$$\left[\left(\frac{\partial}{\partial t} + \frac{\alpha}{h}\right)\nabla^2 + 2\Omega\frac{\partial}{\partial\varphi}\right]\Psi + 2\Omega\left(\cos\vartheta\,\nabla^2 - \sin\vartheta\frac{\partial}{\partial\varphi}\right)\Phi = 0 \qquad (62b)$$

The drag force introduces a phase shift relative to $\partial/\partial t$ that is inversely proportional to depth. In general, the drag force modifies the tide locally both in this way and in proportion to the gradient of 1/h.

We need operator formulas for the effect of $\partial/\partial\theta$, $\frac{1}{\sin\theta}\partial/\partial\varphi$, and cotθ on a quantity expressed in spherical harmonics. In this way we can obtain expressions for u_θ and u_φ in terms of Φ and Ψ (see eqs. (10)) and ultimately reduce the additional terms in eqs. (61) to a single series of spherical harmonics.

To reduce $\partial/\partial\theta$ we write

$$\sum_{n,\underline{s}} C_n^{\underline{s}} P_n^s(\mu) e^{i\underline{s}\varphi} = \frac{\partial}{\partial\vartheta} \sum_{m,\underline{s}} B_m^{\underline{s}} P_m^s(\mu) e^{i\underline{s}\varphi} \tag{63}$$

Using eq. (33), integration over the unit sphere gives

$$\begin{aligned} C_n^{\underline{s}} &= -\frac{(2n+1)(n-s)!}{2\,(n+s)!} \sum_m B_m^{\underline{s}} \int_{-1}^{1} P_n^s(\mu) \sqrt{1-\mu^2} \frac{d}{d\mu} P_m^s(\mu)\, d\mu \\ &= -\frac{(2n+1)(n-s)!}{2\,(n+s)!} \sum_m B_m^{\underline{s}} \int_{-1}^{1} P_n^s(\mu) \tfrac{1}{2}\left[(1+\delta_0^s) P_m^{s+1} \right. \\ &\qquad \left. - (m-s+1)(m+s) P_m^{s-1} \right] d\mu \end{aligned} \tag{64}$$

We define

$$q_{nm}^{s}(\vartheta) = \tfrac{1}{2}\left[(m-s+1)(m+s)\, q_{nm}^{s-1} - (1+\delta_0^s)\, q_{nm}^{s+1}\right] \tag{65}$$

together with

$$q_{nm}^{s+\alpha} = \frac{(2n+1)(n-s)!}{2\,(n+s)!} \int_{-1}^{1} P_n^s(\mu)\, P_m^{s+\alpha}(\mu)\, d\mu \tag{66}$$

to rewrite (64), then substitute from this into (63) for $C_n^{\underline{s}}$, and rename the index n→m on the right side of (63) to obtain

$$\frac{\partial}{\partial\vartheta} \sum_{n,\underline{s}} B_n^{\underline{s}} P_n^s(\mu) e^{i\underline{s}\varphi} = \sum_{n,\underline{s}} \left(\sum_m q_{nm}^{s}(\vartheta) B_m^{\underline{s}}\right) P_n^s(\mu)\, e^{i\underline{s}\varphi} \tag{67}$$

The same procedure using eq. (32) results in

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\sum_{n,\underline{s}} B_n^{\underline{s}} P_n^s(\mu)e^{i\underline{s}\varphi}$$
$$= \sum_{n,\underline{s}\neq 0}\sum_m i\, q_{nm}^s(\varphi)\, B_m^{\underline{s}} P_n^s(\mu)e^{i\underline{s}\varphi} \tag{68}$$

where

$$q_{nm}^s(\varphi) = \mathrm{sign}(\underline{s})\cdot\frac{1}{2}\left[(m-s+1)(m-s+2)q_{n\,m+1}^{s-1} + q_{n\,m+1}^{s+1}\right] \tag{69}$$

With the definition

$$q_{nm}^s(\cot\vartheta) = \frac{(2n+1)(n-s)!}{2\,(n+s)!}\int_{-1}^{1}\frac{\mu}{\sqrt{1-\mu^2}}P_n^s(\mu)P_m^s(\mu)\,d\mu \tag{70}$$

we have

$$\cot\vartheta\sum_{n,\underline{s}} B_n^{\underline{s}} P_n^s(\mu)e^{i\underline{s}\varphi} = \sum_{n,\underline{s}}\sum_m q_{nm}^s(\cot\vartheta)\, B_m^{\underline{s}} P_n^s(\mu)e^{i\underline{s}\varphi} \tag{71}$$

all terms $q_{nm}^s(\cot\theta)$, except for $s = 0$ can be reduced to terms of the form $q_{nm}^{s+\alpha}$. Using eq. (35) with m replacing n to reduce $(u/\sqrt{1-u^2})P_m^s$ in eq. (70), there results

$$q_{nm}^s(\cot\vartheta) = \begin{cases} \frac{1}{2s}\left[(m-s+1)(m+s)\,q_{nm}^{s-1} + q_{nm}^{s+1}\right] & , s>0 \\ \frac{2n+1}{2}\int_{-1}^{1}\frac{\mu}{\sqrt{1-\mu^2}}P_n(\mu)P_m(\mu)\,d\mu & , s=0 \end{cases} \tag{72}$$

Moreover, all the q-factors of this subsection can be expressed in terms of the general Q-factors in Appendix A. A comparison of eqs. (66) and (70) with eq. (A1) shows that

$$q_{nm}^{s+\alpha} = Q_{00m(s+\alpha)ns}^{00} \tag{73}$$

and

$$q_{nm}^{0}(\cot\vartheta) = Q_{00mon0}^{11} \tag{74}$$

Now we can write u_θ and u_φ as a series of spherical harmonics with the coefficients obtained from those for velocity potential and vorticity. In eq. (10), substitute for Φ and Ψ using eqs. (25) and (26). Then use eqs. (67) and (68) to reduce the differential operators. There results

$$u_\vartheta = i\sum_{n,\underline{s}}\sum_{m}\left\{q_{nm}^{s}(\vartheta)A_m^{\underline{s}} + q_{nm}^{s}(\varphi)B_m^{\underline{s}}\right\}P_n^{s}(\mu)e^{i(\underline{s}\varphi-\sigma t)} \tag{75a}$$

and

$$u_\varphi = -\sum_{n,\underline{s}}\sum_{m}\left\{q_{nm}^{s}(\varphi)A_m^{\underline{s}} + q_{nm}^{s}(\vartheta)B_m^{\underline{s}}\right\}P_n^{s}(\mu)e^{i(\underline{s}\varphi-\sigma t)} \tag{75b}$$

u_θ and u_φ, since they are differentially related to velocity potential and vorticity, have slower convergence, as does tide height. This is reflected in eqs. (75).

Now, assume that u_θ has a bounded character such that we can write

$$\frac{u_\vartheta}{h} = \sum_{n,\underline{s}} F_n^{\underline{s}} P_n^{s}(\mu)e^{i(\underline{s}\varphi-\sigma t)} \tag{76}$$

Then

$$\left(\frac{u_v}{h}\right) h = u_v \tag{77}$$

implies that

$$\sum_{n,\underline{s}} F_n^{\underline{s}} P_n^{s}(\mu) e^{i\underline{s}\varphi} \cdot \sum_{j=0,\underline{k}} H_j^{\underline{k}} P_j^{k}(\mu) e^{i\underline{k}\varphi}$$
$$= i \sum_{\ell,\underline{m}} \sum_{k} \left\{ q_{\ell k}^{m}(v) A_k^{\underline{m}} + q_{\ell k}^{m}(\varphi) B_k^{\underline{m}} \right\} \cdot P_\ell^{m}(\mu) e^{i\underline{m}\varphi} \tag{78}$$

Integration over the unit sphere gives

$$\sum_{n,\underline{s}} F_n^{\underline{s}} \sum_{j=0} H_j^{\underline{m}-\underline{s}} \frac{(2\ell+1)(\ell-m)!}{2(\ell+m)!} \int_{-1}^{1} P_j^{|\underline{m}-\underline{s}|}(\mu) P_n^{s}(\mu) P_\ell^{m}(\mu) d\mu$$
$$= i \sum_{k} \left\{ q_{\ell k}^{m}(v) A_k^{\underline{m}} + q_{\ell k}^{m}(\varphi) B_k^{\underline{m}} \right\} \tag{79}$$

The integration over φ was used to reduce the summation over $\underline{k}$. These u-integrations can be expressed in terms of the Q-factors of subsection C (see eqs. (40) or (A12)):

$$\sum_{n,\underline{s}} F_n^{\underline{s}} \sum_{j=0} H_j^{\underline{m}-\underline{s}} Q_{j\,|\underline{m}-\underline{s}|\,n s \ell m} = i \sum_{k} \left\{ q_{\ell k}^{m}(v) A_k^{\underline{m}} + q_{\ell k}^{m}(\varphi) B_k^{\underline{m}} \right\} \tag{80}$$

This equation can be written in matrix notation. Define the matrix element for row (n,$\underline{s}$) and column (ℓ,$\underline{m}$) as

$$\mathcal{H}_{\binom{s}{n}\binom{m}{\ell}} = \sum_{j=0} H_j^{\underline{m}-\underline{s}} Q_{j\,|\underline{m}-\underline{s}|\,n s \ell m} \tag{81}$$

Also consider $F_n^{\underline{s}}$ to be the $(n,\underline{s})$th element of u_θ/h and

$$U_{\vartheta\ell}^{\underline{m}} = i\sum_k \{q_{\ell k}^{m}(\vartheta)A_k^{\underline{m}} + q_{\ell k}^{m}(\varphi)B_k^{\underline{m}}\} \qquad (82)$$

as the $(\ell,\underline{m})$th element of u_θ. Then, eq. (80) takes on the matrix form

$$\sum_{n,\underline{s}} F_n^{\underline{s}} \mathcal{H}_{(n^{\underline{s}})(\ell^{\underline{m}})} = U_{\vartheta\ell}^{\underline{m}} \qquad (83)$$

Inverting the matrix $\underline{\mathcal{H}}$, we obtain the elements of u_θ/h:

$$F_n^{\underline{s}} = \sum_{\ell,\underline{m}} \mathcal{H}^{-1}_{(n^{\underline{s}})(\ell^{\underline{m}})} U_{\vartheta\ell}^{\underline{m}} \qquad (84)$$

Substituting from this equation into the equation for the expansion of u_θ/h, we obtain

$$\frac{u_\vartheta}{h} = i\sum_{n,\underline{s}} \sum_{\ell,\underline{m}} \mathcal{H}^{-1}_{(n^{\underline{s}})(\ell^{\underline{m}})} \sum_k \{q_{\ell k}^{m}(\vartheta)A_k^{\underline{m}} + q_{\ell k}^{m}(\varphi)B_k^{\underline{m}}\} \cdot P_n^s(\mu)e^{i(\underline{s}\varphi-\sigma t)} \qquad (85)$$

By the same procedure we obtain

$$\frac{u_\varphi}{h} = -\sum_{n,\underline{s}} \sum_{\ell,\underline{m}} \mathcal{H}^{-1}_{(n^{\underline{s}})(\ell^{\underline{m}})} \sum_k \{q_{\ell k}^{m}(\varphi)A_k^{\underline{m}} + q_{\ell k}^{m}(\vartheta)B_k^{\underline{m}}\} \cdot P_n^s(\mu)e^{i(\underline{s}\varphi-\sigma t)} \qquad (86)$$

We can now perform the differentiations on these quantities to obtain the dissipative terms in eqs. (60). We use (67), (68) and (72), as before, to reduce the differentiations. To keep the notation concise and to indicate a practical sequence for matrix multiplications we define

$$M^{(\varphi)}_{(j^s)(k^m)} = \sum_{\ell} \mathcal{H}^{-1}_{(j^s)(\ell^m)} g^{m}_{\ell k}(\varphi) \tag{87a}$$

and

$$M^{(\vartheta)}_{(j^s)(k^m)} = \sum_{\ell} \mathcal{H}^{-1}_{(j^s)(\ell^m)} g^{m}_{\ell k}(\vartheta) \tag{87b}$$

In terms of these products we define

$$M^{D}_{(n^s)(k^m)} = \sum_{j} \left\{ -g^{s}_{nj}(\varphi) M^{(\varphi)}_{(j^s)(k^m)} + \left(g^{s}_{nj}(\vartheta) + g^{s}_{nj}(\cot\vartheta) \right) M^{(\vartheta)}_{(j^s)(k^m)} \right\} \tag{88a}$$

and

$$M^{V}_{(n^s)(k^m)} = \sum_{j} \left\{ -g^{s}_{nj}(\varphi) M^{(\vartheta)}_{(j^s)(k^m)} + \left(g^{s}_{nj}(\vartheta) + g^{s}_{nj}(\cot\vartheta) \right) M^{(\varphi)}_{(j^s)(k^m)} \right\} \tag{88b}$$

Then, the dissipative term in eq. (61a) becomes

$$\frac{1}{\sin\vartheta} \frac{\partial}{\partial\varphi} \left(\alpha \frac{u_\varphi}{h} \right) + \left(\frac{\partial}{\partial\vartheta} + \cot\vartheta \right) \left(\alpha \frac{u_\vartheta}{h} \right)$$

$$= i\alpha \sum_{n,s} \sum_{k,m} \left(M^{D}_{(n^s)(k^m)} A^{m}_{k} + M^{V}_{(n^s)(k^m)} B^{m}_{k} \right) P^{s}_{n}(\mu) e^{i(s\varphi - \sigma t)} \tag{89a}$$

and the term in (61b) becomes

$$\frac{1}{\sin\vartheta} \frac{\partial}{\partial\varphi} \left(\alpha \frac{u_\vartheta}{h} \right) - \left(\frac{\partial}{\partial\vartheta} + \cot\vartheta \right) \left(\alpha \frac{u_\varphi}{h} \right)$$

$$= -\alpha \sum_{n,s} \sum_{k,m} \left(M^{V}_{(n^s)(k^m)} A^{m}_{k} + M^{D}_{(n^s)(k^m)} B^{m}_{k} \right) P^{s}_{n}(\mu) e^{i(s\varphi - \sigma t)} \tag{89b}$$

It is then clear that full reduction of eqs. (61) results in equations identical to (54) except for there being on the left side of (54a) the term

$$i\alpha \sum_{r,m} \left(M^{D}_{(n^{s})(r^{m})} A_r^{m} + M^{V}_{(n^{s})(r^{m})} B_r^{m} \right) \tag{90a}$$

and on the left side of (54b) the term

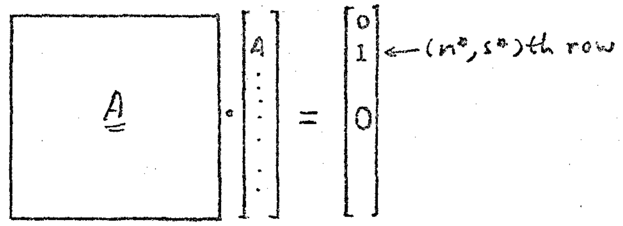
$$-\alpha \sum_{r,m} \left(M^{V}_{(n^{s})(r^{m})} A_r^{m} + M^{D}_{(n^{s})(r^{m})} B_r^{m} \right) \tag{90b}$$

Values of α less than 0.5 should give plausible correction to the semi-diurnal tides. If these values prove too small, other models should be pursued. With $\alpha \leq 0.5$, convergence by the perturbation method should be practicable.

Reduced System Equation

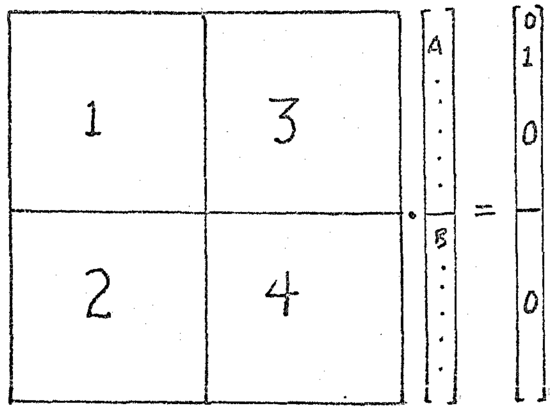
One observation is worthwhile with regard to applying the method. At the close of subsection C, it was stated that the vorticity equation (54b) can be used to eliminate the B_n^{s} from the velocity potential equation (54a), and thereby reduce the size of the equation-set. At first sight, the introduction of (90b) into the vorticity equation would seem to argue against this procedure since an inverse for the larger system is needed to calculate the higher order corrections to the coefficients of Φ and Ψ. However, the inverse for the larger system can be obtained from the inverse for the reduced system.

For simplicity we can replace the constant on the right side of eq. (54a) with unity. Then the final solution can be multiplied by $g\bar{Z}_{n*}^{s*}/2\Omega R$. The reduced and full systems take on the structures



(91)

and



(92)

respectively.

Multiplying each system from the left by its inverse, we see that the solution vector in each case is equal to the (n*,s*)th column of its respective inverse. Thus, block 1 of the inverse of the full system is equal to the inverse of the reduced system. We can obtain block 2 from block 1 by using eq. (54b).

Blocks 3 and 4 of the inverse of the full system are the solution vectors when unity is substituted (non-physically) in each location separately of the lower half of the vector on the right side of (92). This is equivalent to inserting the factor of unity times the Kronecker deltas on the right in eq. (54b) instead of (54a). Doing this, we can obtain the (n*s*)th column of block 3 by substituting from (54b) into (54a). The equation reduces to a form like eq. (56) on the left, but with various constant factors on the right. Multiplying the resulting matrix equation by the inverse of the reduced system, we obtain the desired column, and from this, using (54b), we obtain the corresponding column of block 4. Repeating the procedure we obtain each column of blocks 3 and 4 of the large inverse. This more complicated procedure requires less arithmetic operations than taking a direct inverse of the full system, and the computer memory requirements are quartered.

E REDUCTION IN TERMS OF TIDE HEIGHT

This section presents an earlier reduction of the Laplace tidal equations in terms of the tide height ζ. There is the drawback that solution in terms of a spherical harmonic series in ζ converges slower than in terms of a series in Φ.

Slower Convergence of Tide Height

The continuity equation, in the form (38), together with the operator equations (31) - (33) explicate the relationship between the coefficients of ζ and those of Φ, Ψ, and h. ζ is obtained from spatial differentiation of these quantities; this is reflected in multiplication of their coefficients by index polynomials. The simplest case, h = constant, relates Z_n^s directly to A_n^s, namely

$$Z_n^s = \frac{h}{i\sigma R} n(n+1) A_n^s \tag{93}$$

(see eq. (13)) and graphically shows the relatively slow convergence of Z_n^s. These equations show that ζ is much more sensitive to both physical factors and computational errors than are velocity potential and vorticity.

A practical effect is that a solution for ζ requires a larger system of equations to prevent truncation error. Theoretically, a larger system in ζ serves to give emphasis in the computations to the higher coefficients of ζ. However, if these higher coefficients are to have real physical meaning, terms which were left out of the Laplace equations should be reintroduced.

Other techniques, including iteration and perturbation analysis, but centered on solution in terms of velocity potential, would probably be more practical for obtaining this tide height detail than direct solution for ζ. Velocity potential and vorticity accurately reflect the assumptions made in deriving the Laplace tidal equations; they would continue to reflect any improvements in the physical model.

Solution in Terms of Tide Height

The reduction of eqs. (1) to an equation for ζ is straightforward. Perform the differentiation $\partial/\partial t$. Solve eqs. (1a,b) for u_θ and u_φ, and substitute in the continuity equation (1c). Make the change of variable

$$\mu = \cos\vartheta \tag{28}$$

to obtain

$$4\Omega^2 R^2 \zeta = \frac{\partial}{\partial\mu}\left\{\frac{h(1-\mu^2)}{f^2-\mu^2}\frac{\partial U}{\partial\mu} - \frac{\mu h}{if(f^2-\mu^2)}\frac{\partial U}{\partial\varphi}\right\} + \frac{\partial}{\partial\varphi}\left\{\frac{\mu h}{if(f^2-\mu^2)}\frac{\partial U}{\partial\mu} + \frac{h}{(1-\mu^2)(f^2-\mu^2)}\frac{\partial U}{\partial\varphi}\right\} \quad (94)$$

where

$$f = \frac{\sigma}{2\Omega} \quad (46)$$

and

$$U = V_e + V(\zeta) - g\zeta \quad (95)$$

For most tide frequencies the factor $f^2-\mu^2 = \frac{\sigma^2}{4\Omega^2} - \cos^2\vartheta$ is zero at a pair of colatitudes. The singularity, resulting from the presence of this factor in eq. (94), is regular. But when the second line of the equation is zero, namely when h and U are not functions of φ (e.g., for semi-monthly tides), the singularity vanishes. For all other cases, U cannot be expanded directly as a series of spherical harmonics since the singularity in eq. (94) cannot be validly removed by multiplying each side by $(f^2 - \mu^2)^2$. Instead we define

$$U = -(f^2-\mu^2)^2 \sum_{n=1} \sum_{s=-n}^{n} U_n^s P_n^s(\mu)\, e^{i(s\varphi-\sigma t)} \quad (96)$$

The polynomial factor generates additional scattering in the reduced equations. This complication is attributable to not obtaining an initial integration of the Laplace equations when solving in terms of ζ in general.

It is necessary to obtain an expression relating the coefficients of U and ζ. $(f^2 - \mu^2)^2 P_n^s(\mu)$ reduces to a sum of five associated Legendre polynomials by repeated use of eq. (49). After substituting in (96) and index shifting, there results

$$U = -\sum_{n=1} \sum_{\underline{s}=-n}^{n} \sum_{m=n-4}^{n+4} E_{nm}^{(s)} U_m^{\underline{s}} P_n^s(\mu) e^{i(\underline{s}\varphi - \sigma t)} \tag{97}$$

Expressions for the coefficients $E_{nm}^{(s)}$ can be found in Appendix B. These coefficients are non-zero only when $n+m$ is even, $-4 \le n-m \le 4$, $m \ge s$, and $m \ge 1$. Next, in the defining equation for the disturbing potential (eq. (95)), we substitute the expansion (97) for U and (21) for the right side of the equation. After collecting the constant factors multiplying each $P_n^s(\mu)$, it follows that each must be separately equal to zero, or

$$g\, \mathcal{K}_n Z_n^{\underline{s}} = \delta_n^{n^*} \delta_{\underline{s}}^{s^*} g\, \overline{Z}_{n^*}^{s^*} \mathcal{K}_{n^*}^{*} + \sum_{m=n-4}^{n+4} E_{nm}^{(s)} U_m^{\underline{s}} \tag{98}$$

Because of the sum on $U_m^{\underline{s}}$ in this connective formula, it is better to reduce eq. (94) in terms of $U_n^{\underline{s}}$, rather than $Z_n^{\underline{s}}$. U, in eq. (95) is analogous to $g(\overline{\zeta} - \zeta)$ where $\overline{\zeta}$ is the equilibrium tide, as seen in the literature on ocean tides.

The first algebraic step to achieve the reduction is to substitute into the Laplace equation (94) for U, using (96), and for h, using (27). Next, perform the differentiations $\partial/\partial\mu$

on factors of μ and $\partial/\partial\varphi$. After collecting terms and using eqs. (30) and (31) to simplify, there results

$$
\begin{aligned}
4R^2\Omega^2\zeta = \sum_{n=1}\sum_{\underline{s}=-n}^{n} U_n^{\underline{s}} \sum_{j=0}\sum_{\underline{k}=-j}^{j} H_j^{\underline{k}} \Big\{ P_j^k \Big\{ 4(1-3\mu^2) \\
+ n(n+1)(f^2-\mu^2) + \frac{s}{f}(f^2+\mu^2)\Big] P_n^s + 6\mu(1-\mu^2)\frac{dP_n^s}{d\mu} \Big\} \\
- \frac{dP_j^k}{d\mu} \Big\{ (f^2-\mu^2)(1-\mu^2)\frac{dP_n^s}{d\mu} \\
- \mu\Big[4(1-\mu^2) + \frac{s}{f}(f^2-\mu^2)\Big] P_n^s \Big\} \\
- ik P_j^k \Big\{ \Big[is\,\frac{f^2-\mu^2}{1-\mu^2} - \frac{4\mu^2}{if}\Big] P_n^s \\
+ \frac{\mu}{if}(f^2-\mu^2)\frac{dP_n^s}{d\mu} \Big\} \Big] e^{i(\underline{s}+\underline{k})\varphi} e^{-i\sigma t}
\end{aligned} \tag{99}
$$

If factors are grouped to represent operators for directional derivatives (see below eq. (37)) and this equation is compared with eq. (29), an additional factor $\sqrt{1-\mu^2}$ remains. $\sqrt{1-\mu^2}$ ultimately arises from the presence in eq. (94) of $1/(f^2 - \mu^2)$, a factor which does not appear in the continuity condition itself.

A resummation of the equation above in a single series of spherical harmonics requires that the order of each harmonic be $|\underline{s} + \underline{k}|$ by reason of the term $e^{i(\underline{s} + \underline{k})}$. The factor $\sqrt{1-\mu^2}$ cannot, therefore, be interpreted as a shift in the order of spherical harmonics. The correct interpretation is to consider $\sqrt{1-\mu^2}$ expanded in powers of μ^2 and, therefore, giving rise to an infinite series with respect to the degree of the spherical harmonics: $P_{n+j}^{|\underline{s}+\underline{k}|}$, $P_{n+j+2}^{|\underline{s}+\underline{k}|}$, $P_{n+j+4}^{|\underline{s}+\underline{k}|}$, ... Therefore,

$U_n^{\underline{s}}$ for at least all alternate n can be expected to enter into each of the infinite number of simultaneous equations. This is another way of saying that tide height converges slower than velocity potential and vorticity, and is more sensitive to mathematical and physical variants.

The differential operator on u that was used is

$$\frac{d}{d\mu} P_n^s = -\frac{s\mu}{1-\mu^2} P_n^s + \frac{1}{\sqrt{1-\mu^2}} P_n^{s+1} \tag{100}$$

After applying this operator and simplifying terms, there results

$$\begin{aligned} 4R^2\Omega^2\zeta = \sum_{n,\underline{s}} U_n^{\underline{s}} \sum_{j,\underline{k}} H_j^{\underline{k}} \Big[& P_j^k P_n^s \{4 + n(n+1)f^2 + \underline{s}f \\ & -\mu^2(12 + n(n+1) - \underline{s}/f + 6s - 4\underline{k}/f + 4k) \\ & -(f^2-\mu^2)[-\frac{\underline{k}\underline{s}}{1-\mu^2} + \frac{\mu^2}{1-\mu^2}(\frac{k\underline{s} - \underline{k}s}{f} + ks)]\} \\ & + P_j^k P_n^{s+1}\{6\mu\sqrt{1-\mu^2} + \frac{\mu(f^2-\mu^2)}{\sqrt{1-\mu^2}}(k - \underline{k}/f)\} \\ & + P_j^{k+1} P_n^s \{4\mu\sqrt{1-\mu^2} + \frac{\mu(f^2-\mu^2)}{\sqrt{1-\mu^2}}(s + \underline{s}/f)\} \\ & - P_j^{k+1} P_n^{s+1}(f^2-\mu^2)\Big] e^{i(\underline{k}+\underline{s})\varphi} e^{-i\sigma t} \end{aligned} \tag{101}$$

To express the right side of the equation as a single series of spherical harmonics we resum on $\underline{k}$, using index $\underline{m}$:

$$\sum_{\underline{k}} \{\cdots, \underline{k}, \cdots\} e^{i(\underline{k}+\underline{s})\varphi} \Rightarrow \sum_{\underline{m}=\underline{k}+\underline{s}} \{\cdots, \underline{m}-\underline{s}, \cdots\} e^{i\underline{m}\varphi}$$

The same result is obtained from the φ-part of the following summation-integration operation:

$$\sum_{\ell,\underline{m}} \frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!} P_\ell^m(\mu) e^{i\underline{m}\varphi} \int_{-1}^{1}\int_0^{2\pi} \{\text{equation } 101\} P_\ell^m(\mu) e^{-i\underline{m}\varphi} d\varphi\, d\mu$$

The left side of the equation is unchanged, except for the expansion parameters now being $\ell,\underline{m}$ instead of $n,\underline{s}$. We therefore define the general Q-factors:

$$Q^{ab}_{jkns\ell m} = \frac{(2\ell+1)(\ell-m)!}{2(\ell+m)!} \int_{-1}^{1} P_j^k(\mu) P_n^s(\mu) P_\ell^m(\mu) \frac{\mu^a}{(1-\mu^2)^{b/2}} d\mu \qquad (102)$$

When we perform the operation above to eq. (101), divide each side of the equation by 2π, and use the Q-symbols to represent the μ-integrations, we obtain

$$4R^2\Omega^2 \sum_{\ell=1}\sum_{\underline{m}=-\ell}^{\ell} Z_\ell^{\underline{m}} P_\ell^m(\mu) e^{i\underline{m}\varphi}$$

$$= \sum_{\ell=1}\sum_{\underline{m}=-\ell}^{\ell} \sum_{n=1}\sum_{\underline{s}=-n}^{n} u_n^{\underline{s}} \sum_{j\geq|\underline{m}-\underline{s}|} H_j^{\underline{m}-\underline{s}}$$

$$\cdot\Big\{\big[4+f^2(n(n+1)+(\underline{m}-\underline{s})s\beta\gamma f^2+s\gamma)\big] Q^{00}_{j|\underline{m}-\underline{s}|ns\ell m}$$

$$-\big[12+n(n+1)+6s+|\underline{m}-\underline{s}|(4+s\beta\gamma f^2-4\beta)$$

$$-s\gamma+\delta^{\gamma}_{-\beta}\, 2|\underline{m}-\underline{s}|s(1+\gamma)\big] Q^{20}_{j|\underline{m}-\underline{s}|ns\ell m}$$

$$-\delta^{\gamma}_{-\beta}\, 2|\underline{m}-\underline{s}|s(f^2-1)(1+\gamma) Q^{22}_{j|\underline{m}-\underline{s}|ns\ell m}$$

$$+\big[6+|\underline{m}-\underline{s}|(1-\beta)\big] Q^{1-1}_{j|\underline{m}-\underline{s}|n(s+1)\ell m}$$

$$+|\underline{m}-\underline{s}|(1-\beta)(f^2-1) Q^{11}_{j|\underline{m}-\underline{s}|n(s+1)\ell m}$$

$$+\big[4+s(1+\gamma)\big] Q^{1,-1}_{j(|\underline{m}-\underline{s}|+1)ns\ell m}$$

$$+s(1+\gamma)(f^2-1) Q^{11}_{j(|\underline{m}-\underline{s}|+1)ns\ell m} \qquad (103)$$

$$-f^2 Q^{00}_{j(|\underline{m}-\underline{s}|+1)n(s+1)\ell m}$$

$$+Q^{20}_{j(|\underline{m}-\underline{s}|+1)n(s+1)\ell m}\Big\} P_\ell^m(\mu) e^{i\underline{m}\varphi}$$

where

$$\beta = \text{sign}(\underline{m}-\underline{s}) \times 1/f \tag{104}$$

and

$$\gamma = \text{sign}(\underline{s}) \times 1/f \tag{105}$$

have been defined to simplify the notation.

The constant factors multiplying each spherical harmonic must be individually zero. After index renaming, substitute from eq. (98) for Z_ℓ^{m} into each zero quantity to obtain the matrix equation

$$\underline{\underline{C}}_\ell{}^{\underline{m}}{}_n{}^{\underline{s}} \, U_n{}^{\underline{s}} = \delta_\ell{}^{n*} \delta_{s*}{}^{\underline{m}} \, Z_{n*}{}^{s*} X_{n*}^{*} \tag{106}$$

with the elements $\underline{\underline{C}}$ defined as

$$\underline{\underline{C}}_\ell{}^{\underline{m}}{}_n{}^{\underline{s}} = \frac{1}{4R^2\Omega^2} \sum_{j \geq |\underline{m}-\underline{s}|} H_j{}^{\underline{m}-\underline{s}}$$

$$\cdot \Big\{ \big[4 + f^2(n(n+1) + (\underline{m}-\underline{s}) s\beta\gamma f^2 + s\gamma)\big] Q_{j\,|\underline{m}-\underline{s}|\,n\,s\,\ell\,m}^{0\,0}$$

$$- \big[12 + n(n+1) + 6s + |\underline{m}-\underline{s}|(4 + s\beta\gamma f^2 - 4\beta)$$

$$- s\gamma + \delta_{-\beta}^{\gamma}\, 2|\underline{m}-\underline{s}| s(1+\gamma)\big] Q_{j\,|\underline{m}-\underline{s}|\,n\,s\,\ell\,m}^{2\,0}$$

$$- \delta_{-\beta}^{\gamma}\, 2|\underline{m}-\underline{s}| s(f^2-1)(1+\gamma) Q_{j\,|\underline{m}-\underline{s}|\,n\,s\,\ell\,m}^{2\,2}$$

$$+ \big[6 + |\underline{m}-\underline{s}|(1-\beta)\big] Q_{j\,|\underline{m}-\underline{s}|\,n\,(s+1)\,\ell\,m}^{1\,-1}$$

$$+ |\underline{m}-\underline{s}|(1-\beta)(f^2-1) Q_{j\,|\underline{m}-\underline{s}|\,n\,(s+1)\,\ell\,m}^{1\,1}$$

$$+ \big[4 + s(1+\gamma)\big] Q_{j\,(|\underline{m}-\underline{s}|+1)\,n\,s\,\ell\,m}^{1\,-1}$$

$$+ s(1+\gamma)(f^2-1) Q_{j\,(|\underline{m}-\underline{s}|+1)\,n\,s\,\ell\,m}^{1\,1}$$

$$- f^2 Q_{j\,(|\underline{m}-\underline{s}|+1)\,n\,(s+1)\,\ell\,m}^{0\,0}$$

$$+ Q_{j\,(|\underline{m}-\underline{s}|+1)\,n\,(s+1)\,\ell\,m}^{2\,0} \Big\} - \delta_n^{\begin{pmatrix}\ell+4,\\ \ell+2,\\ \ell,\\ \ell-2,\\ \ell-4\end{pmatrix}} \frac{E_{\ell n}{}^{(m)}}{X_\ell} \tag{107}$$

$E_{\ell n}(m)$ is zero when $n < m$ or $n < 1$ or $(\ell + n)$ is odd. There is at most one $E_{\ell m}(m)$ factor in each matrix element, by reason of the Kronecker delta. After solving eq. (106) for the coefficients $U_n^{\underline{s}}$, the coefficients of tide height are obtained by applying eq. (98).

Numerical Tests

The solution for a general depth model requires the inclusion of all orders for each degree of the associated Legendre polynomials. If, for example, to obtain an accurate solution all terms through degree 14 are required, the system will consist of $14 \times (14 + 2) = 224$ equations and there is the need to manipulate over 50,000 complex numbers.

When a test model consisting of the first eight degrees of an expansion for the earth ocean depth (Kaula, 1966) was tested for convergence, terms well beyond degree 14 were found necessary. Instead of attempting to directly obtain a solution for the full ocean depth expansion through degree 35, with no prior knowledge of the computational size of the task, it was decided to study on a more manageable scale the questions of convergence and how terms in the depth expansion interact to produce terms in the tide height.

When the constant b in a depth model of the form $a + b\sin^2\theta$ takes on the value $\frac{4R^2\Omega^2}{gn(n+1)\kappa_n}$ with n even, the semi-monthly tide series is zero above term Z_n. In effect, a standing wave tide is formed. It was decided to take advantage of the lessened computational requirement and test for the effects of adding one or more spherical harmonic terms to such a depth model.

Following Hough (1897), we define the depth model

$$h_4 = \left(\frac{1}{5} + \frac{\sin^2\vartheta}{4.5 K_n}\right)\frac{4R^2\Omega^2}{g} \tag{108}$$

and using values approximating his, together with the relationship

$$P_2 = \frac{1}{2}(-1 + 3\cos^2\vartheta) \tag{109}$$

we obtain

$$h_4 = 20843\,P_0(\cos\vartheta) - 1404\,P_2(\cos\vartheta) \text{ meters} \tag{110}$$

Fig. 1 summarizes results for the depth models

$$h_4 + H_1^{\circ} P_1^{\circ}(\cos\vartheta) = h_4 + H_1^{\circ}\cos\vartheta \tag{111}$$

with H_1° taking values up to 900 meters. When truncating the equations above the eighth degree, only terms up to the fifth degree are accurate. This is attributable to the expansion coefficients $E_{8(10)}^{(s)}$ and $E_{8(12)}^{(s)}$ being left out of the system of equations. Otherwise this truncation causes little error, even for rather large values of H_1°. The odd degree harmonics entering the tide expansion increase in linear proportion with H_1°. The contribution to the tide from this term could be quite accurately obtained by treating $H_1^{\circ}\cos\theta$ as a perturbation of the system h_4 and computing one perturbation correction.

Fig. 2 summarizes the results for the models

$$h_4 + H_1^1 P_1^1(\cos\vartheta)\cos\varphi = h_4 + H_1^1 \sin\vartheta \cos\varphi \qquad (112)$$

Truncating terms above the eighth degree is accurate only when H_1^1 is less than 100 meters, and then only for tide terms of degree four and lower. The term $\sin\theta \cos\varphi$ in the depth introduces a small quadratic change in the even degree terms; the odd degree terms continue to vary linearly.

The models (Fig. 3)

$$\begin{aligned} h_5 &+ H_1^1 P_1^1(\cos\vartheta)\cos\varphi \\ &= \left(\frac{1}{5} + \frac{\sin^2\vartheta}{5.6\, X_5}\right)\frac{4R^2\Omega^2}{g} + H_1^1 P_1^1 \cos\varphi \\ &= 19724\, P_0 - 923\, P_2 + H_1^1 P_1^1 \cos\varphi \quad \text{meters} \end{aligned} \qquad (113)$$

indicate how quickly convergence can deteriorate. Depth model h_5 fails to truncate the tide series. But a comparison of solutions for the simple h_5 model with those for models h_4 and h_6 (see Hough, 1897, p.242) shows that the tide terms Z_2 and Z_4 of h_5 are close to those for h_4, the term Z_6 is somewhat similar to Z_6 for model h_6, and higher terms for model h converge rapidly. Therefore, this depth model generates a tide that is approximately a standing wave. Fig. 3 shows that convergence continues good for small (less than 400 meters) intrusions of H_1^1, but requires a larger system of equations when H_1^1 is large.

We are forced to conclude that a general depth model would require a very large system of equations.

Table 1 gives another indication of how depth terms interact. The last two columns cover two models combining terms H_1^0 and H_1^1. These tests led us to explore a system wherein the depth variation is treated as a perturbation of a constant depth model. However, doubts with regard to convergence compelled us to look elsewhere.

Table 1. Tide Heights for Various Models (in units of equilibrium tide)

Model: Order, Degree	$h_4 + 500P_1^0$	$h_4 + 100P_1^1$	$h_4 + 500P_1^0 + 50P_1^1 \cos\varphi$	$h_4 + 500P_1^0 + 100P_1^1 \cos\varphi$
1,0	0.00434		0.00434	0.00434
1,1		0.00126	0.00060	0.00126
2,0	0.74243	0.74259	0.74244	0.74246
2,1			0.00007	-0.00003
2,2		---	---	---
3,0	0.00458		0.00457	0.00459
3,1		0.00141	0.00113	0.00142
3,2			---	---
3,3		---	---	---
4,0	-0.09794	-0.09796	-0.09796	-0.09806
4,1			-0.00058	0.00015
4,2		---	---	---
4,3			---	---
4,4		---	---	---
5,0	0.00152		0.00150	0.00163
5,1		0.00053	-0.00140	0.00053
5,2			---	---
5,3		---	---	---
5,4			---	---
5,5		---	---	---
6,0	0.00444	-0.00019	0.00426	-0.00023

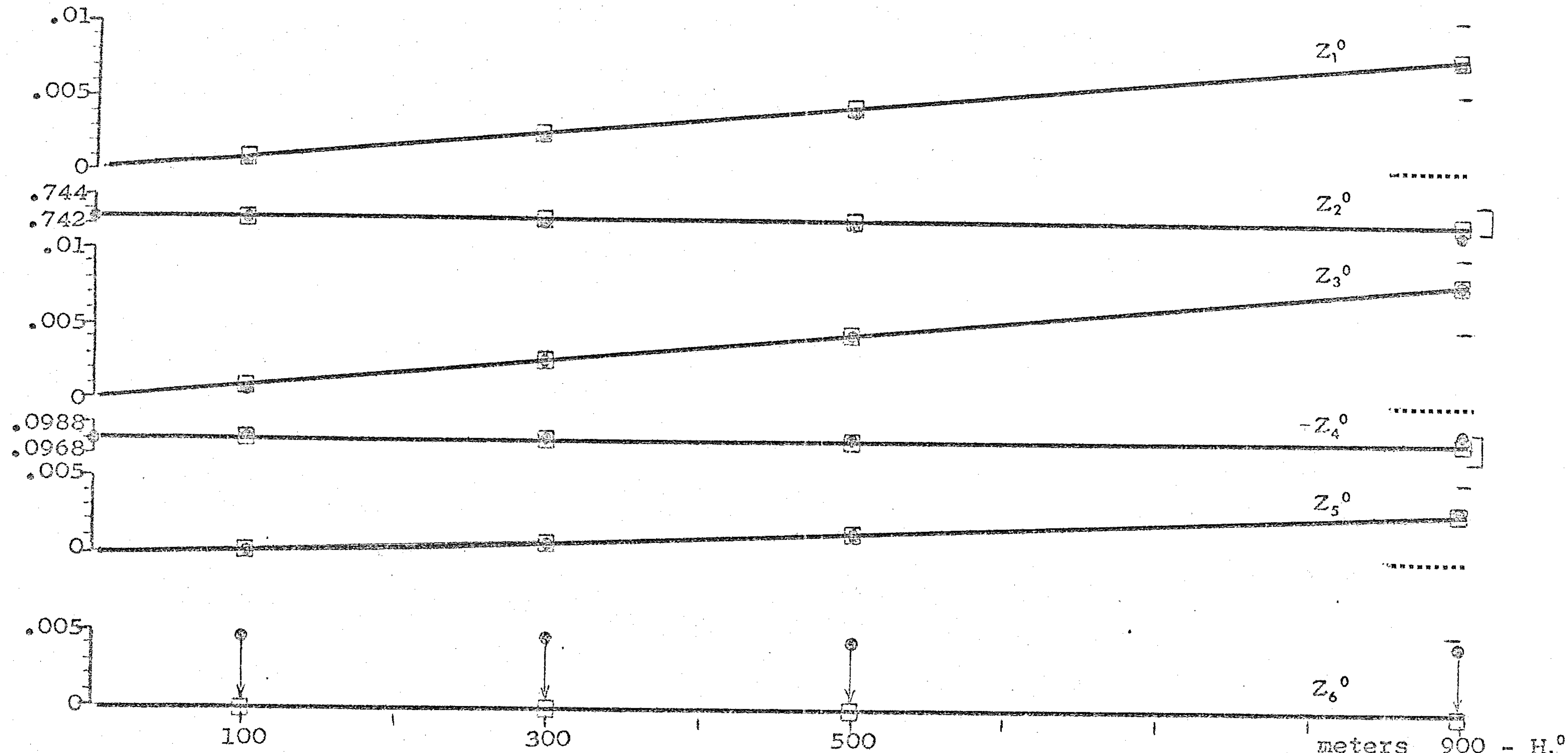


Fig. 1 Tide heights for depth models $h_4 + H_1^0 P_1^0 = 20843 P_0^0 + 1404 P_2^0(\cos\theta) + H_1^0 P_1^0(\cos\theta)$. Dots are for solutions with degree $\ell \le 8$; squares are for $\ell \le 10$. Tide heights are expressed as multiples of the equilibrium tide.

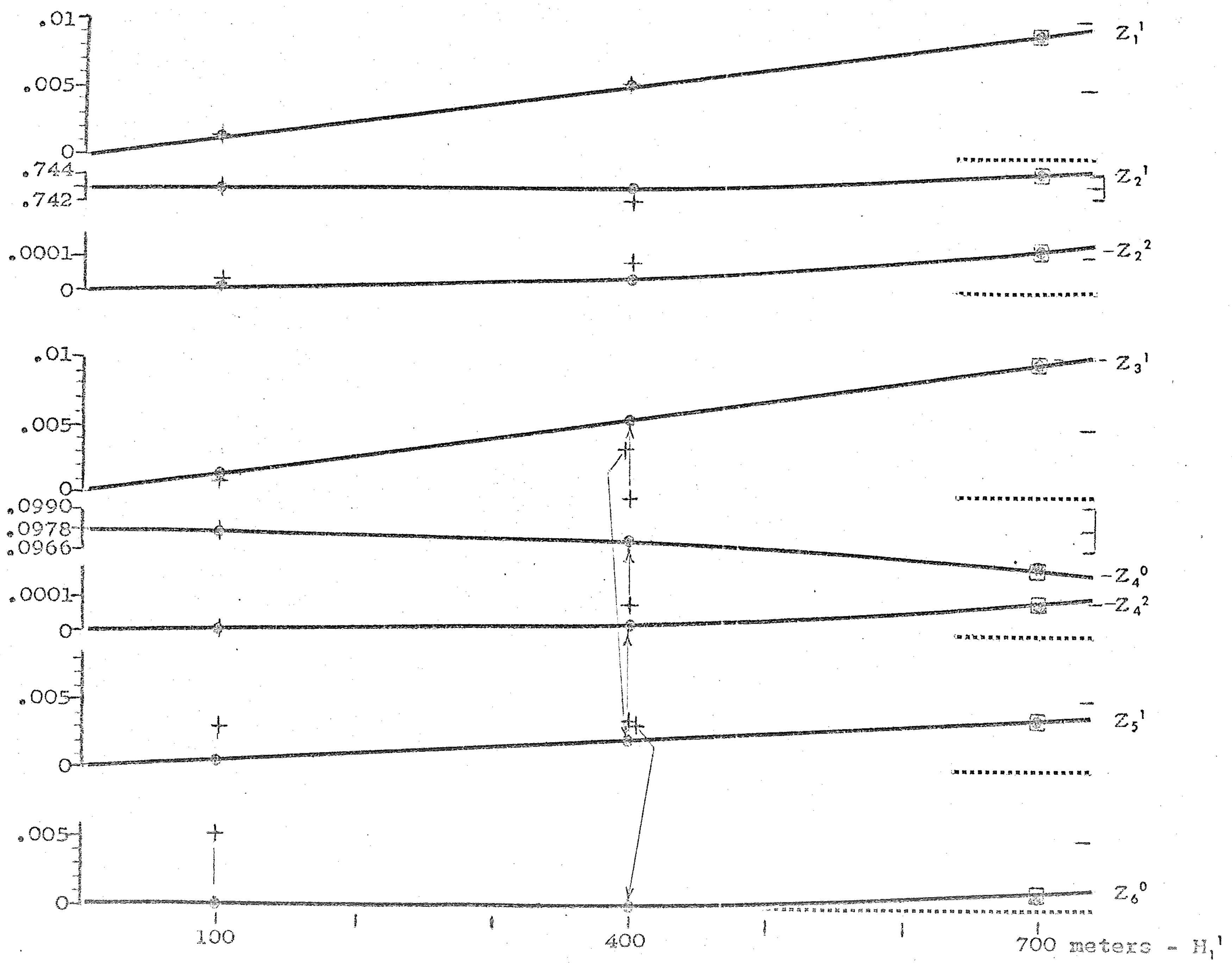


Fig. 2 Tide heights for depth models $h_4 + H_1^1 P_1^1 = 2083P_0^0 - 1404P_2^0 + H_1^1 P_1^1 \cos\varphi$. Squares at $H_1^1 = 700$ meters are for solutions for degree $\ell \leq 12$; dots are for $\ell \leq 11$; pluses are for $\ell \leq 8$.

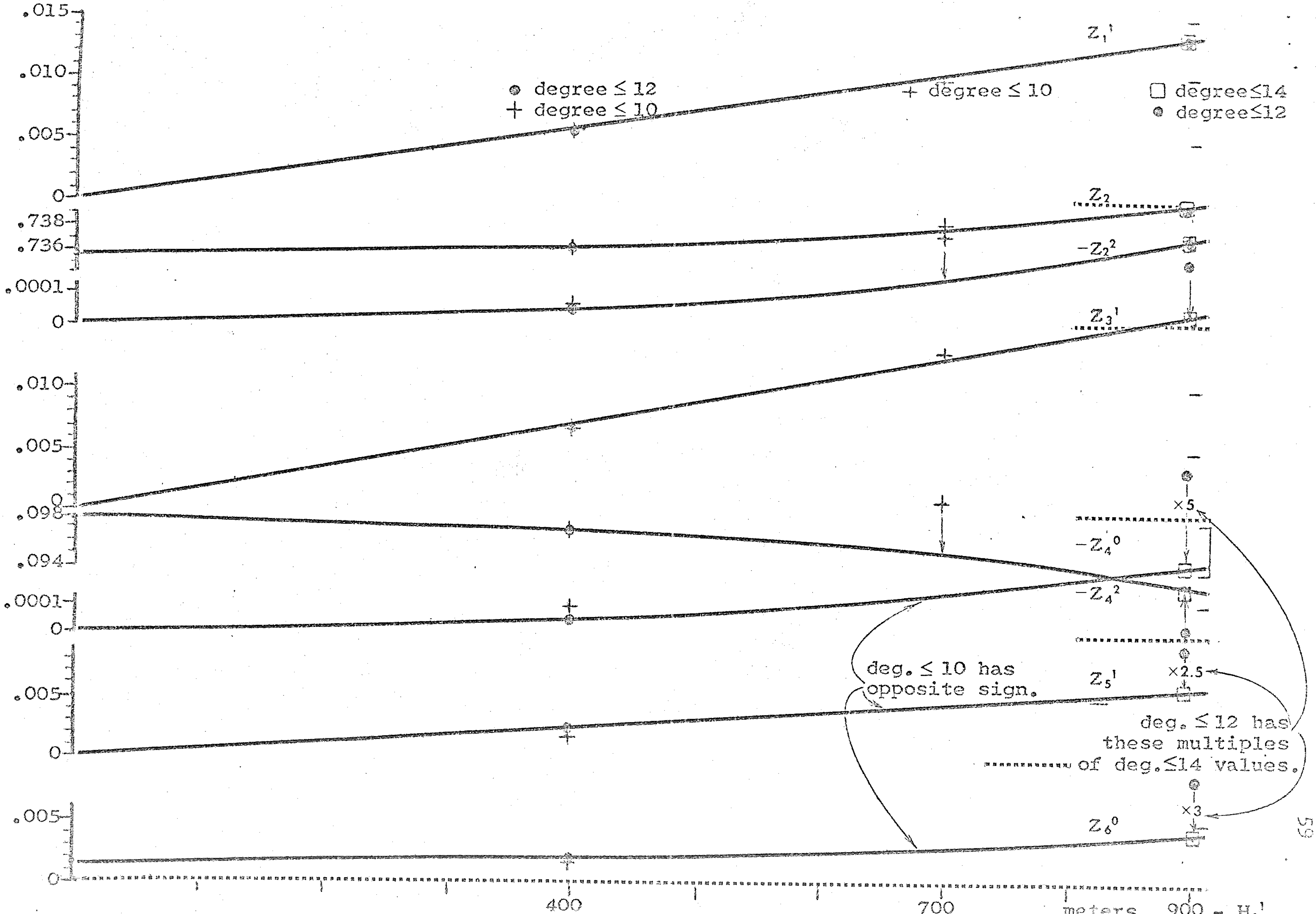


Fig. 3 Tide heights for models $h_s + H_1^1 P_1^1 = 19724 P_0^0 - 923 P_2^0 + H_1^1 P_1^1 \cos\varphi$. The degree of the solutions represented by +, ● and □ are at the top of columns for H_1^1 = 400, 700, and 900 meters.

APPENDIX A

This appendix consists of two parts; (1) a computation of the general Q-factors (eq. (102)) used in the reduction for tide height, and (2) the expression in terms of these factors of the Q-factors (eq. (40)) used in the reduction for velocity potential.

Computation of Factors Q^{ab}

These factors, or matrix elements, are defined by the integration

$$Q^{ab}_{jknslm} = \frac{(2l+1)(l-m)!}{2(l+m)!}\int_{-1}^{1} P_j^k(\mu)\, P_n^s(\mu)\, P_l^m(\mu)\, \mu^a (1-\mu^2)^{-b/2} d\mu \qquad \text{(A1)}$$

and are evaluated as

$$Q^{ab}_{jknslm} = \frac{\left[1+(-1)^{n+l+j}\right]}{2}\;\frac{(l-m)!\;(2l+1)}{2(l+m)!}\;\Gamma\left(\frac{\varkappa}{2}\right)2^{\varkappa/2}$$

$$\cdot \sum_{c=0}^{\bar{c}} T_{lm(\bar{c}-c)} \sum_{d=0}^{\bar{d}} T_{ns(\bar{d}-d)}\,'$$

$$\cdot \sum_{e=0}^{\bar{e}} T_{jk(\bar{e}-e)}\; F(\nu, A, c+d+e) \qquad \text{(A2)}$$

where

$$\varkappa = m+s+k+2-b$$

$$A = a+1+(\ell-m) \text{ modulo } (2) + (n-s) \text{ modulo } (2) + (j-k) \text{ modulo } (2)$$

$$\bar{c} = \left[\frac{\ell-m}{2}\right] = \text{"greatest interger in" } (\ell-m)/2$$

$$\bar{d} = \left[\frac{n-s}{2}\right]$$

$$e = \left[\frac{j-k}{2}\right]$$

$$T_{abc} = \frac{(-1)^c \; (2a-2c)!}{2^a c! \; (a-c)! \; (a-b-2c)!}$$

$$F(\varkappa, A, c+d+e) = 1 \Big/ \prod_{i=c+d+e}^{c+d+e+\frac{\varkappa}{2}-1} (A+2i) \qquad \text{(A3)}$$

Moreover, in the computer program the following recurrence relation for F is used:

$$F(\varkappa, A, c+d+e+1) = \frac{A+2c+2d+2e}{A+2c+2d+2e+\varkappa} F(\varkappa, A, c+d+e) \qquad \text{(A4)}$$

The derivation (A2) was possible because in the problem there occurs the following restrictions on the ranges of the parameters:

$$a \pm (m+s+k), \; a-b, \text{ and } m+s+k-b \text{ are always even} \qquad \text{(A5)}$$

The sequence of computations resulting in Equation (A2) is: a) substitute for each associated Legendre polynomial from the definition

$$P_a^{\,b}(\mu) = (1-\mu^2)^{b/2} \sum_{c=0}^{\left[\frac{a-b}{2}\right]} T_{abc} \, \mu^{a-b-2c} \qquad \text{(A6)}$$

b) split the integration range into positive- and negative-u ranges and combine the negative range with the positive using the substitution $u \to -u$; c) make the change of variable $x = u^2$; d) use the integral (Bateman, 1954)

$$\int_0^y x^{a-1} (y-x)^{b-1}\, dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} y^{a+b-1}$$

$$\text{Re } a > 0, \quad \text{Re } b > 0 \qquad \text{(A7)}$$

e) noting that, as used in the problem, the argument of each gamma function is half-integral, substitute into the gammas from the relationship

$$\Gamma(n+\tfrac{1}{2}) = \frac{\Gamma(2n+1)\,\Gamma(\tfrac{1}{2})}{2^{2n}\Gamma(n+1)} \qquad \text{(A8)}$$

At this stage the formula simplifies to

$$Q^{ab}_{jkns\ell m} = \frac{(\ell-m)!\,(2\ell+1)}{2(\ell+m)!} \cdot \frac{\left[1+(-1)^{n+\ell+j}\right]}{2} \Gamma\left(\frac{m+s+k+2-b}{2}\right) 2^{(m+s+k+2-b)}$$

$$\cdot \sum_{c=0}^{\bar{c}} T_{\ell mc} \sum_{d=0}^{\bar{d}} T_{nsd} \sum_{e=0}^{\bar{e}} T_{jke}$$

$$\cdot \frac{\Gamma(n+\ell+j+a+1-m-s-k-2c-2d-2e)}{\Gamma\left(\frac{n+\ell+j+a+2-m-s-k-2c-2d-2e}{2}\right)}$$

$$\cdot \frac{\Gamma\left(\frac{n+\ell+j+4+a-b-2c-2d-2e}{2}\right)}{\Gamma(n+\ell+j+3+a-b-2c-2d-2e)} \qquad \text{(A9)}$$

Note that under the conditions outlined above, (A5), the arguments of the gamma functions are integers;

f) make the variable changes

$$c' = \bar{c}-c, \quad d' = \bar{d}-d, \quad e' = \bar{e}-e \tag{A10}$$

and then drop the primes. At this stage the symbols $\varkappa$ and A in (A2) can be used;

g) the gamma functions are factorials. Expand them out and cancel terms that occur in numerator and denominator. The four gammas simplify to

$$1 \Big/ \left\{ 2^{\varkappa/2} \cdot \prod_{i=c+d+e}^{c+d+e+\frac{\varkappa}{2}-1} (A+2i) \right\} \tag{A11}$$

h) after substituting from (A10) and simplifying, the final result, (A2), is obtained.

Factors of Type Q^{+-}

The Q-factors defined by eqs. (40) can be expressed as particular forms of the Q-factors evaluated above. We multiply both sides of eqs. (40) by $P_\ell^m(\mu)$ and integrate over the unit sphere. The integration over φ requires that, other than for trivial results, $m = k \pm s$. We are left with

$$\int_{-1}^{1} P_j^{k+\alpha}(\mu)\, P_n^{s+\beta}(\mu)\, P_\ell^{|k\pm s|}(\mu)\, d\mu$$

$$= \sum_{r'=0} \int_{-1}^{1} Q^{s(\alpha)s(\beta)}_{jknsl|k\pm s|}\, P_{j+n-2r'}^{|k\pm s|}(\mu)\, P_\ell^{|k\pm s|}(\mu)\, d\mu$$

$$= \quad Q^{s(\alpha)s(\beta)}_{jknsl|k\pm s|}\, \frac{2(\ell+|k\pm s|)!}{(2\ell+1)(\ell-|k\pm s|)!} \tag{A12}$$

where, for conciseness we have defined

$$s(\alpha) = \text{sign}(\alpha) \qquad , \alpha = \pm 1 \tag{A13}$$

except for the special case of using no superscript when $\alpha = \beta = 0$.

A comparison of this computation with eqs. (102) or (A1) shows that

$$Q^{s(\alpha)s(\beta)}_{jknsl|k\pm s|} = Q^{00}_{j(k+\alpha)n(s+\beta)\ell|k\pm s|} \tag{A14}$$

APPENDIX B

Expressions for the Coefficients $E_{nm}(s)$

The coefficients arise from the repeated application of

$$\mu P_n^s(\mu) = \frac{n+s}{2n+1} P_{n-1}^s(\mu) + \frac{n-s+1}{2n+1} P_{n+1}^s(\mu) \qquad (49)$$

to reduce the factor $(f^2 - \mu^2)^2$ in the expression

$$U = -\sum_{n=1} \sum_{s=-n}^{n} U_n^s (f^2-\mu^2)^2 P_n^s(\mu) e^{i(s\varphi-\sigma t)} \qquad (59)$$

then index-shifting, to obtain

$$U = -\sum_{n=1} \sum_{s=-n}^{n} \sum_{m=n-4}^{n+4} E_{nm}(s) U_m^s P_m^s(\mu) e^{i(s\varphi-\sigma t)} \qquad (97)$$

The coefficients $E_{nm}(s)$ are non-zero only when $m = n-4$, $n-2$, n, $n+2$, $n+4$ and $m \geq$ (s and 1).

They can be expressed symbolically as

$$E_{nm}(s) = F(N,n,s) - 2f^2 G(N,n,s) + \delta_n^m f^4 \qquad (B1)$$

where

$$N = \tfrac{1}{2}(6+m-n) = 1,2,3,4,5 \qquad (B2)$$

and

$$G(1,n,s) = 0$$

$$G(2,n,s) = g(n-1,\ 1-s) \cdot g(n-2,\ 1-s)$$

$$G(3,n,s) = B+C$$

$$G(4,n,s) = g(n+2,\ s) \cdot g(n+1,\ s)$$

$$G(5,n,s) = 0$$

$$F(1,n,s) = g(n-3,\ 1-s) \cdot g(n-4,\ 1-s) \cdot G(2,n,s)$$

$$F(2,n,s) = [G(3,n,s) + D + g(n-2,\ s) \cdot g(n-3,\ 1-s)] \cdot G(2,n,s)$$

$$F(3,n,s) = (G(3,n,s))^2 + A \cdot B + C \cdot D$$

$$F(4,n,s) = [g(n+3,\ s) \cdot g(n+2,\ 1-s) + A + G(3,n,s)] \cdot G(4,n,s)$$

$$F(5,n,s) = g(n+4,\ s) \cdot g(n+3,\ s) \cdot G(4,n,s) \qquad \text{(B3)}$$

The symbols used above are

$$g(n,m,) = (n+m)/(2n+1) \qquad \text{(B4)}$$

and

$$A = g(n+2,\ s) \cdot g(n+1,\ 1-s)$$

$$B = g(n+1.\ s) \cdot g(n,\ 1-s)$$

$$C = g(n,\ s) \cdot g(n-1,\ 1-s)$$

$$D = g(n-1,\ s) \cdot g(n-2,\ 1-s) \qquad \text{(B5)}$$

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III SATELLITE ORBITS

A TIDAL PERTURBATIONS

Summary. It was very early concluded that only zonal variations of the earth's tidal properties might have a significant effect on orbits. Hence effort was concentrated thereon, with the principal results summarized below.

Tidal disturbing functions were developed in which the amplitude factor k and lag angle ϵ are expressed as sums of zonal spherical harmonics.

In regard to the current evolution of the moon's orbit, the existence of a second degree harmonic in the lag angle could make a significant contribution to energy transfer to the moon; it is unlikely, however, that it has an important effect on the overall time-scale of the orbit evolution.

If the moon formed in an equatorial orbit about the earth, tidal friction could do nothing to incline the orbit. Once the orbit was inclined, tidal friction could increase the inclination further if there was a commensurability between the earth's rotation and the moon's revolution.

Latitudinal variations in tidal properties would have appreciable effects on close satellite orbits. An appreciable second degree harmonic in the phase lag is needed, however, to reconcile the available data with the rate of the earth's rotational deceleration. Further progress requires satellites free from surface force effects.

Introduction

As has been established by various studies in recent years, integration of the moon's orbit back in time under the influence of tidal friction with the dissipation factor 1/Q inferred from observations of lunar & solar motion bring the moon close to the earth only 1.0 to 2.0 x 10^9 years ago (Gerstenkorn, 1955; MacDonald, 1964; Goldreich, 1966). Furthermore, at the closest point the moon's orbit has an appreciable inclination with respect to the earth's equator, which would rule out a fission origin. Goldreich (1966) and Munk (1968) suggested, however, than an asymmetric tidal response of the earth may have had a significant effect on the orbital evolution of the moon.

Additional data which have recently been developed use the effects of tidal potentials on artificial satellite orbits. Kozai (1968) analyzed perturbations of the inclination of three satellites of 33° to 50° inclination. He obtained Love numbers k_2 varying from 0.23 to 0.33 with uncertainties less than ± .03, and phase lags from 0° to 9° with uncertainties of ± 5 to 7°. Newton (1968) analyzed perturbations of the node and inclination of four polar satellites. He obtained Love numbers k_2 varying from 0.28 to 0.44 with uncertainties less than ± .03, and phase lags from 0.7° to 2.5°, with uncertainties less than ± 0.8°. Newton emphasized that there appeared to be significant rate-dependence of both amplitude and phase angle. However, we still might hope to explain some of the differences among artificial satellite analyses, as well as the discrepancies from the 2° - 2.5°

lags inferred from the deceleration of the earth's rotation by an asymmetric tidal response: i.e., one which is a function of location within the earth.

Development of Disturbing Function

Let the disturbing function of the sun or moon be represented in terms of the Kepler elements of the disturbing body and the spherical coordinates of the point of calculation of the potential (Kaula, 1964):

$$U = \sum_{\ell,m,p,q} B^*_{\ell m}\, C^*_{\ell mpq}\, r^{\ell}\, P_{\ell m}(\sin\varphi) \cdot \begin{Bmatrix}\cos\\ \sin\end{Bmatrix}_{\ell-m\ \text{odd}}^{\ell-m\ \text{even}} \left\{v^*_{\ell mpq} - m(\lambda+\theta)\right\}, \qquad (1)$$

where r, φ, λ are radius, latitude, and longitude respectively; θ is the Greenwich sidereal time; $P_{\ell m}(\sin\varphi)$ is the Legendre Associated function; and

$$B^*_{\ell m} = Gm^* \frac{(\ell-m)!}{(\ell+m)!}\left(2-\delta_{om}\right) \qquad (2)$$

$$C^*_{\ell mpq} = \frac{1}{a^{*\,\ell+1}}\, F_{\ell mp}(I^*)\; G_{\ell pq}(e^*) \qquad (3)$$

$$v^*_{\ell mpq} = (\ell-2p)\omega^* + (\ell-2p+q)M^* + m\Omega^* \qquad (4)$$

where G is the gravitational constant; m^*, a^*, e^*, I^*, M^*, ω^*, Ω^* are the mass and Kepler elements of the disturbing body; and $F_{\ell mp}(I^*)$ and $G_{\ell pq}(e^*)$ are polynomials as derived by Kaula (1966).

Now, regardless of the nature of the tidal response of the earth -- oceanic, bodily, or otherwise -- so long as there is no significant non-linearity (i.e., tidal terms interacting with themselves or other terms), then the tidal potential <u>at the earth's surface</u> r = R can be written as:

$$T(R, \varphi, \lambda) = \sum_{\ell mpq} k_\ell(\varphi, \lambda)\, R^\ell\, B^*_{\ell m}\, C^*_{\ell mpq}$$

$$\cdot\ P_{\ell m}(\sin\varphi) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix}_{\ell - m \text{ odd}}^{\ell - m \text{ even}}$$

$$\cdot \left\{ v^*_{\ell mpq} - \varepsilon_{\ell mpq}(\varphi, \lambda) - m(\lambda + \theta) \right\} \quad (5)$$

We should expect the Love number $k_\ell(\varphi, \lambda)$ and phase lag $\varepsilon_{\ell mpq}(\varphi, \lambda)$ to be rather smoothly varying functions of position, and hence representable by spherical harmonics; this would certainly be true of that portion of their effect which would affect satellite orbits. The full development of Eq. (5) would involve products of tesseral harmonics, and hence would be most conveniently done in complex representation, along the lines of Section II, p.60. But a fiarly elementary consideration shows that any longitude-dependent part of the product $k_\ell \varepsilon_{\ell mpq}$ will have its effect on any orbit "averaged out" by the earth's rotation.

Hence for the purpose of orbital analysis let us assume k_ℓ and $\epsilon_{\ell mpq}$ to be functions of latitude only, which allows us to retain a real representation:

$$k_\ell = \sum_h \varkappa_{\ell h} P_{ho}(\sin\varphi) \tag{6}$$

and

$$\epsilon_{\ell mpq} = \sum_n \epsilon_{n(\ell mpq)} P_{no}(\sin\varphi) \tag{7}$$

In derivations where the frequency dependence of ϵ_n is not being emphasized, we shall drop the argument (ℓmpq).

If it is assumed that all the ϵ_n are small enough that $\sin\left(\epsilon_{\ell mpq}\right) = \epsilon$, $\cos\left(\epsilon_{\ell mpq}\right) = 1$, then the tidal potential at the surface can be written:

$$T(R) = \sum_{\ell mpqhn} K_{\ell mpq} \varkappa_{\ell h} P_{ho}(\sin\varphi) P_{\ell m}(\sin\varphi) \cdot \left[\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}} \left\{ v^*_{\ell mpq} - m(\lambda+\theta) \right\} + \epsilon_n P_{no}(\sin\varphi) \begin{Bmatrix}\sin\\ -\cos\end{Bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}} \left\{ v^*_{\ell mpq} - m(\lambda+\theta) \right\} \right] \tag{8}$$

where

$$K_{\ell mpq} = R^{\ell} B^{*}_{\ell m} C^{*}_{\ell mpq} \tag{9}$$

The product $P_{\ell m}(\sin\varphi)\, P_{jo}(\sin\varphi)$ can be converted to a sum:

$$P_{\ell m}P_{jo} = \sum_{k=m}^{\ell+j} Q_{\ell jkm}\, P_{km} \tag{10}$$

Values of $Q_{\ell kjm}$ are given in Table I. Applying Eq. (10) to all products in Eq. (8) allows us to write the tidal potential T(r) at any radius r > R as:

$$T(r) = \sum_{\ell mpqhnks} K_{\ell mpq}\, \kappa_{\ell h}\, Q_{\ell hkm} \cdot \left[\left(\frac{R}{r}\right)^{k+1} P_{km}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}^{\ell-m \text{ even}}_{\ell-m \text{ odd}}\left\{v^{*}_{2mpq} - m(\lambda+\theta)\right\} + \varepsilon_n\, Q_{knsm}\left(\frac{R}{r}\right)^{s+1} P_{sm}\begin{Bmatrix}\sin\\ -\cos\end{Bmatrix}^{\ell-m \text{ even}}_{\ell-m \text{ odd}}\left\{v^{*}_{2mpq} - m(\lambda+\theta)\right\}\right] \tag{11}$$

Using the usual conversion from spherical coordinates to Kepler elements of the perturbed satellite orbit (Kaula, 1966, p. 37),

$$\left(\frac{R}{r}\right)^{\ell+1} P_{\ell m}(\sin\varphi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\lambda = \left(\frac{R}{a}\right)^{\ell+1} \sum_{jg} F_{\ell mj}(I)\, G_{\ell jg}(e) \begin{Bmatrix}\cos & \sin\\ \sin & -\cos\\ \text{even} \;(\ell-m) & \text{odd}\end{Bmatrix} \cdot \left\{v_{\ell mpq} - m\theta\right\} \tag{12}$$

TABLE I FACTORS FOR PRODUCT TO SUM CONVERSION OF LEGENDRE ASSOCIATED POLYNOMIALS*

$$P_{\ell m} P_{no} = \sum_k Q_{\ell nkm} P_{km}$$

ℓ	m	n	$Q_{\ell n0m}$	$Q_{\ell n1m}$	$Q_{\ell n2m}$	$Q_{\ell n3m}$	$Q_{\ell n4m}$	$Q_{\ell n5m}$	$Q_{\ell n6m}$
2	0	0			1				
2	0	1		2/5		3/5			
2	0	2	1/5		2/7		18/35		
2	0	3		9/35		4/15		10/21	
2	0	4			2/7		20/77		5/11
2	1	0			1				
2	1	1		3/5		2/5			
2	1	2			10/21		9/35		
2	1	3		-9/35		1/15		4/21	
2	1	4			-4/21		3/77		5/33
2	2	0			1				
2	2	1				1/5			
2	2	2			-2/7		3/35		
2	2	3				-2/15		1/21	
2	2	4			1/21		-6/77		1/33
3	0	0				1			
3	0	1			3/7		4/7		
3	0	2		9/35		4/15		10/21	
3	0	3	1/7		4/21		18/77		100/231
3	1	0				1			
3	1	1			4/7		3/7		
3	1	2		18/35		1/5		2/7	
3	1	3			2/21		9/77		50/231

*Neumann or Ferrers type:

$$P_{\ell m}(\cos\theta) = \sin^m\theta \sum_t \frac{(-1)^t \, (2\ell-2t)!}{2^{\ell} t!(\ell-t)! \, (\ell-m-2t)!} \cos^{\ell-m-2t}\theta$$

we get

$$T = \sum_{\ell mpqhnksjg} K_{\ell mpq}\, \varkappa_{\ell h}\, Q_{\ell hkm}$$

$$\cdot \left[\left(\frac{R}{a}\right)^{k+1} F_{kmj}(I)\, G_{kjg}(e) \begin{Bmatrix} \cos \\ (-1)^m \sin \end{Bmatrix}_{(k-\ell)\text{ odd}}^{(k-\ell)\text{even}}\right.$$

$$\cdot \left\{v_{kmjg} - v^*_{\ell mpq}\right\} + \varepsilon_n\, Q_{knsm}\left(\frac{R}{a}\right)^{s+1} F_{smj}(I)$$

$$\left.\cdot\, G_{sjg}(e) \begin{Bmatrix} -\sin \\ (-1)^m \cos \end{Bmatrix}_{(s-\ell)\text{ odd}}^{(s-\ell)\text{ even}} \left\{v_{smjg} - v^*_{\ell mpq}\right\}\right] \quad (13)$$

To simplify, let us substitute $(k_\ell \varepsilon)_h$ for $\varkappa_{\ell h} \sum_n \varepsilon_n Q_{knsm}$: then we can write

$$T = \sum_{\ell mpqhkjg} K_{\ell mpq}\left(\frac{R}{a}\right)^{k+1} F_{kmj}(I)\, G_{kjg}(e)\, Q_{\ell hkm}$$

$$\cdot \left[\varkappa_{\ell h}\begin{Bmatrix} \cos \\ (-1)^m \sin \end{Bmatrix}_{(k-\ell)\text{odd}}^{(k-\ell)\text{ even}} + (k_\ell \varepsilon)_h \begin{Bmatrix} -\sin \\ (-1)^m \cos \end{Bmatrix}_{(k-\ell)\text{ odd}}^{(k-\ell)\text{ even}}\right]$$

$$\cdot \left\{v_{kmjg} - v^*_{\ell mpq}\right\} \quad (14)$$

For analysis of the moon's orbital evolution, we require

(1) zero rate:

$$\dot{v}_{kmjg} - \dot{v}^*_{\ell mpq} = 0, \quad (15)$$

or, from Eq. (4),

$$j = \tfrac{1}{2}(k-\ell) + p \tag{16}$$

$$g = q; \tag{17}$$

and (2) small "damping" factor $\left(a_*^{\ell+1}\, a^{k+1}\right)^{-1}$, or

$$\ell k = 20,\ 22,\ 24,\ 33,\ \text{or } 42 \tag{18}$$

The combination 31 for ℓk is excluded because a first degree harmonic would represent a shift of the origin, the earth's center of mass. For lunar orbit evolution then let us write

$$T = T_0 + T_2 + T_4$$

where

$$\left.\begin{aligned} T_0 &= \sum_{qh} K_{201q}\left(\frac{R}{a}\right) F_{000}\, G_{00q}\, Q_{2h00}\, \psi_{201hq} \\ T_2 &= \sum_{qhmp} K_{2mpq}\left(\frac{R}{a}\right)^3 F_{2mp}\, G_{2pq}\, Q_{2h2m}\, \psi_{22mphq} \\ T_4 &= \sum_{\ell=2}^{4} \sum_{qhmp} K_{\ell mpq}\left(\frac{R}{a}\right)^{7-\ell} F_{(6-\ell)mj}\, G_{2jq} \\ &\quad \cdot Q_{(6-\ell)h\ell m}\, \psi_{(6-\ell)\ell mphq} \end{aligned}\right\} \tag{19}$$

in which

$$\psi_{\ell kmphq} = \left(\varkappa_{\ell h} \cos - (k_\ell \epsilon)_h \sin\right) \left(v_{kmjq} - v^*_{\ell mpq}\right)$$

and j is related to k, ℓ, and p by Eq. (16).

For analysis of tidal perturbations of artificial satellites, we require (1) no short period terms, or

$$g = 2j - k \tag{20}$$

and (2) small "damping" factor $\left(a*^{\ell+1}\right)^{-1}$, or

$$\ell = 2 \tag{21}$$

Eq. (14) thus becomes

$$T = \sum_{mpqhkj} K_{2mpq}\left(\frac{R}{a}\right)^{k+1} F_{kmj} G_{kj(2j-k)} Q_{2hkm}$$

$$\cdot \left[\varkappa_{2h}\left\{\begin{matrix}\cos \\ (-1)^m \sin\end{matrix}\right\}_{k\ \mathrm{odd}}^{k\ \mathrm{even}} + (k_\ell \epsilon)_h\left\{\begin{matrix}-\sin \\ (-1)^m \cos\end{matrix}\right\}_{k\ \mathrm{odd}}^{k\ \mathrm{even}}\right]$$

$$\cdot \left[v_{kmj(2j-k)} - v^*_{2mpq}\right] \tag{22}$$

Values of the amplitude factors $K_{\ell mpq}$ and associated rates $\dot{v}^*_{\ell mpq}$ are given in Table II.

TABLE II AMPLITUDE FACTORS OF TIDAL TERMS

"Planetary" units: length 6.37×10^8 cm, mass 5.97×10^{27} g, time 806.8 sec.

ℓ	m	p	q	MOON $K_{\ell mpq}$	SUN $K_{\ell mpq}$
2	0	0	-1	$.632 \cdot 10^{-10}$	$.929 \cdot 10^{-11}$
			0	$-.229 \cdot 10^{-8}$	$-.111 \cdot 10^{-8}$
			1	$-.440 \cdot 10^{-9}$	$-.651 \cdot 10^{-10}$
		1	-1	$-.122 \cdot 10^{-8}$	$-.180 \cdot 10^{-9}$
			1	$-.123 \cdot 10^{-8}$	$-.180 \cdot 10^{-9}$
		2	-1	$-.440 \cdot 10^{-9}$	$-.651 \cdot 10^{-10}$
			0	$-.229 \cdot 10^{-8}$	$-.111 \cdot 10^{-8}$
			1	$.632 \cdot 10^{-10}$	$.929 \cdot 10^{-11}$
2	1	0	-1	$-.122 \cdot 10^{-8}$	$-.180 \cdot 10^{-9}$
			0	$.441 \cdot 10^{-7}$	$.213 \cdot 10^{-7}$
			1	$.852 \cdot 10^{-8}$	$.125 \cdot 10^{-8}$
		1	-1	$-.350 \cdot 10^{-8}$	$-.515 \cdot 10^{-9}$
			0	$-.427 \cdot 10^{-7}$	$-.204 \cdot 10^{-7}$
			1	$-.350 \cdot 10^{-8}$	$-.515 \cdot 10^{-9}$
		2	-1	$-.365 \cdot 10^{-9}$	$-.540 \cdot 10^{-10}$
			0	$-.189 \cdot 10^{-8}$	$-.921 \cdot 10^{-9}$
			1	$.525 \cdot 10^{-10}$	$.774 \cdot 10^{-11}$
2	2	0	-1	$-.584 \cdot 10^{-8}$	$-.859 \cdot 10^{-9}$
			0	$.212 \cdot 10^{-6}$	$.103 \cdot 10^{-6}$
			1	$.409 \cdot 10^{-7}$	$.606 \cdot 10^{-8}$
		1	-1	$.153 \cdot 10^{-8}$	$.223 \cdot 10^{-9}$
			0	$.186 \cdot 10^{-7}$	$.888 \cdot 10^{-8}$
			1	$.153 \cdot 10^{-8}$	$.223 \cdot 10^{-9}$
		2	-1	$.634 \cdot 10^{-10}$	$.112 \cdot 10^{-10}$
			0	$.393 \cdot 10^{-9}$	$.190 \cdot 10^{-11}$
			1	$-.109 \cdot 10^{-10}$	$-.160 \cdot 10^{-11}$

Effects on Lunar Orbit Evolution

Of the terms in Eq. (19), those with $\ell mpq = 2010$ are meaningless because $\dot{v}_{0000}$ is zero. The lowest value of h which gives a non-zero Q_{2h00} is 2. From Table II, K_{2011} is appreciably larger than K_{201-1}. G_{001} is the coefficient of cos M in the expansion of a/r, or e. Writing out the parts of K_{2011} and evaluating Q_{2200} from Table I we get

$$T_o = \frac{Gm^*}{Ra^*}\left[\frac{3}{4}\sin^2 I^* - \frac{1}{2}\right]\left[\frac{3}{2}e^* + \frac{27}{16}e^{*3} + \ldots\right]$$

$$\cdot \left(\frac{R}{a^*}\right)^3 \left(\frac{R}{a}\right)e\frac{1}{5}(\kappa_{22}\cos - \kappa_{20}\ \epsilon_2 \sin)\ (M-M^*) \quad (23)$$

From the Lagrangian planetary equations-of-motion, we get (dropping asterisks after differentiating T_o):

$$\dot{a}_o = \frac{2}{na}\frac{\partial T_o}{\partial M}$$

$$= -\frac{Gm^*}{na^2}\frac{2e}{5R}\left(\frac{R}{a}\right)^4\left[\frac{3}{4}\sin^2 I - \frac{1}{2}\right]\left[\frac{3}{2}e + \frac{27}{16}e^3 + \ldots\right](k_2\epsilon_2)$$

$$\approx \frac{3Gm^* e^2}{10na^2R}\left(\frac{R}{a}\right)^4 \kappa_{20}\ \epsilon_2 \quad (24)$$

and

$$\dot{e}_0 = \frac{1-e^2}{na^2 e} \frac{\partial T_0}{\partial M}$$

$$= - \frac{Gm^*}{na^3} \frac{(1-e^2)}{5R} \left(\frac{R}{a}\right)^4 \left[\frac{3}{4}\sin^2 I - \frac{1}{2}\right] \left[\frac{3}{2}e + \frac{27}{16}e^3 + . . .\right]$$

$$\cdot \; (k_2 \epsilon_2)$$

$$\approx \frac{3Gm^* e}{20 na^3 R} \left(\frac{R}{a}\right)^4 \kappa_{20} \; \epsilon_2 \qquad (25)$$

These terms express the effect of the interaction of the eccentricity of the moon's orbit with a variation of the tidal dissipation in the earth which is symmetric about the equator. Since the earth's rotation is not involved, there is no angular momentum transfer: only energy transfer. Eq. (24) states that if dissipation is concentrated near the poles in the earth, the moon will move away faster; while if it is concentrated near the equator it will move away more slowly. Eq. (25) is merely consistent with conservation of angular momentum, or $a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}$.

Putting in numbers, we have for the present lunar orbit, from Eq. (24), in planetary units

$$\dot{a}_0 = 1.08 \times 10^{-13} \; \kappa_{20} \; \epsilon_2 (2011) \qquad (26)$$

From Eq. (41) of Kaula (1964),

$$\dot{a}_2 \approx \frac{3Gm^*}{na^2R}\left(\frac{R}{a}\right)^6 \kappa_{20}\ \epsilon_0(2200)$$

$$= 0.97 \times 10^{-13}\kappa_{20}\ \epsilon_0(2200) \qquad (27)$$

The frequency (in an earth reference frame) for ℓmpq = 2200 is about 55 times that for ℓmpq = 2011. Hence, since a significant part of the tidal dissipation is in the oceans, probably

$$\epsilon_0(2011) << \epsilon_0(2200) \qquad (28)$$

Furthermore, we would expect $\epsilon_0 + \epsilon_2 P_2$ to be greater than zero everywhere, so probably

$$-\epsilon_0(2011) < \epsilon_2(2011) < 2\epsilon_0(2011) \qquad (29)$$

Thus it seems unlikely that the effect on the lunar orbit evolution of non-uniformity of dissipation can explain the discrepancy between the lag angle inferred from lunar & solar motions (most recently by Curott, 1966) and that from artificial satellite orbits by Newton (1968). Since $\dot{n}_☽/\dot{\omega}$ is positive, for non-uniform dissipation to explain some of the discrepancy its contribution to $\dot{n}_☽$ should be negative and hence $\dot{a}_0$ should be positive, corresponding to a predominance around the poles (see Kaula, 1968, p. 197).

The most significant result of Goldreich (1966) was that if dissipation has been uniform over the earth's surface, the inclination to the equator could never have been less than 10°

within 10 earth radii. Goldreich suggested that strong local dissipation in a few places in the earth's oceans or crust may lead to unanticipated deviations from his results. No matter how "local" was this dissipation, and no matter how close the moon, the main effect on the orbit must be expressible in spherical harmonics -- just as is the case for satellite orbit perturbations by gravity anomalies fixed in the earth. A faster rate of rotation such as prevailed early in the earth's history would make this all the more true.

For the inclination to be changed by the disturbing function of Eq. (19), the combination $F_{\ell mp} F_{kmj}\left[(k-2j)\cot I - m \csc I\right]$ must be non-zero (see, e.g., Kaula, 1966, p. 40). For I = 0, this never occurs: $F_{\ell mp} F_{kmj}$ contains a non-zero factor only for $m = k-2j$. If $m \neq k-2j$, $F_{\ell mp} F_{kmj}$ is at least order I^2. All of which goes to say that purely latitudinal variations in tidal properties can do nothing to wrench the moon out of an equatorial orbit. What is necessary is an interaction such that

$$\left[(k-2j) - (\ell-2p)\right] (\dot{\omega}+\dot{M}) + (s-m) (\dot{\Omega}-\dot{\theta}) \approx 0 \qquad (30)$$

with $s \neq m$: i.e., a resonance. To maintain such a resonance long enough to have significant effect, $\dot{n}/\ddot{\theta}$ must have about equaled $n/\dot{\theta}$, where n is the mean motion, related to $\mu = G(M+m^*)$ by Kepler's third law,

$$n = \mu^{\frac{1}{2}} a^{-3/2} \qquad (31)$$

From conservation of angular momentum (Eq. (59), Kaula, 1964):

$$C\ddot{\theta} = \frac{1}{3}m^* \mu^{2/3} n^{-4/3} \dot{n} \tag{32}$$

where C is the earth's moment of inertia, or

$$\dot{n}/\ddot{\theta} = 3Cm^{*-1} \mu^{-2/3} n^{4/3}$$

$$= 3Cm^{*-1} a^{-2}$$

$$\approx 81/a^2 \tag{33}$$

for a in earth radii. Integrating the moon back close to the earth gives a rotation period of about five hours (Fig. 9, Goldreich, 1966), or 0.28 for $\dot{\theta}$ in the "planetary" units of radians/807 sec. Setting n as equal to 0.28 x $81/a^2$ and equating it to n by Kepler's law (Eq. (31)), we get an absurdly large semi-major axis a. Hence it seems unlikely that any longitudinal variation in tidal properties ever had a significant effect on the lunar orbit. This holds true even for nonlinearities in the tides, since in Eq. (30) the tidal harmonic ks now has a completely independent specification from ℓm.

The next, more desperate, possibility is that an irregularity in tidal properties would affect $\dot{n}$ so much in a direction counter to the central tidal term as to hold the moon at a resonance point where its inclination could be changed. A situation might have existed which is mathematically similar to that of Mimas and Tethys, with a tidal bulge taking the place of the inner satellite Mimas. However, the treatment of Allan (1968) indicates that the inclinations must have been non-zero <u>before</u> these satellites became coupled, and that the subsequent increase in the inclinations depends on a factor of order I^2.

We thus seem to be forced back to the conclusion that at least part of the moon was captured. The interesting question then is how small a portion of the moon needs to be captured, taking into account that resonance with longitudinal variations in tidal properties may help to further feed the inclination. Since the mixed capture-fission hypothesis would involve several more ramifications that either theory alone (see the discussion & conclusions of Goldreich, 1966), it seems appropriate to defer further consideration thereof to another paper.

In regard to the time-scale problem of the lunar orbit evolution, the conclusion of Eqs. (26) - (27) that latitudinal variation in tidal lag can account for only a minor part of the current evolution indicates that it was of even less significance in the past, and hence a less important effect than changes in the extent of shallow seas, as suggested by the most recent paleontological work (Panella et al., 1968).

Effects on Close Satellite Orbits

We wish to examine the bahavior of artificial satellites of small a/R under the influence of the disturbing function given by Eq. (22) with a view to explaining results already obtained and suggesting specification of future orbits to determine tidal properties of the earth.

In his analysis, Kozai (1968) determined k and ϵ from the perturbation of the inclination ΔI of argument Ω. Hence he used only the term mpqhkj = 110021 of Eq. (22), or

$$T_2 = K_{2100}\left(\frac{R}{a}\right)^3 \left[-\frac{3}{2}\sin I \cos I\right] (1-e^2)^{-3/2}$$

$$\cdot \kappa_{20}(\cos - \epsilon_0 \sin)(\Omega - \Omega*) \qquad (34)$$

Newton (1968) determined k and ϵ from the perturbations of inclination ΔI and node $\Delta\Omega$ with arguments containing Ω and 2Ω. He obtained the lunar and solar-orbit dependent factors by numerical integration, which would be equivalent to using all 6 terms m = 1,2 and p = 0,1,2, with qhkj = 0021 in Eq. (22):

$$T_2 = \sum_{mp} K_{2mp0}\left(\frac{R}{a}\right)^3 F_{2m1}(I)\ (1-e^2)^{-3/2}$$

$$\cdot \kappa_{20}(\cos - \epsilon_0 \sin)\ m(\Omega - \Omega*) \qquad (35)$$

Since the perigee argument ω is absent from the disturbing functions (34) and (35), odd zonal variations h = 1,3,5 . . . . in k_2 and ϵ could not have given rise to perturbations of the same period as equations (34) and (35); only even variations h = 2,4 . . . Hence in the expressions for perturbation of

ΔI (dependent on $\partial R/2\Omega$) containing the Love number k we can make the substitution

$$\left(\frac{R}{a}\right)^3 F_{2m1}\, G_{210}\begin{Bmatrix} k_I \\ k_I\, \epsilon \end{Bmatrix} = \sum_{k,h} \left(\frac{R}{a}\right)^{k+1} F_{kmk/2}\, G_{k(k/2)0} \cdot Q_{2hkm}\begin{Bmatrix} \kappa_{2h} \\ (k_2\, \epsilon)_h \end{Bmatrix}, \qquad (36)$$

$m = 1,2;\quad h,k$ even

and in the expressions for the perturabtion $\Delta\Omega$ (dependent on $\partial R/\partial I$), the substitution

$$\left(\frac{R}{a}\right)^3 \frac{\partial F_{2m1}}{\partial I} G_{210}\begin{Bmatrix} k_\Omega \\ k_\Omega \epsilon \end{Bmatrix} = \sum_{k,h} \left(\frac{R}{a}\right)^{k+1} \frac{\partial F_{kmk/2}}{\partial I}\, G_{k(k/2)0} \cdot Q_{2hkm}\begin{Bmatrix} k_\Omega \\ (k_2\, \epsilon)_h \end{Bmatrix} \qquad (37)$$

$m = 1,2;\quad h,k$ even

Let us define

$$J_{hm} = \sum_k \left(\frac{R}{a}\right)^{k+1} F_{kmk/2}\, G_{k(k/2)0}\, Q_{2hkm} \qquad (38)$$

and

$$J'_{hm} = \partial J_{hm}/\partial I \qquad (39)$$

Then (36) and (37) become

$$J_{om}\begin{Bmatrix} k_l \\ k_l\epsilon \end{Bmatrix} = \sum_h J_{hm}\begin{Bmatrix} \varkappa_{2h} \\ (k_2\epsilon)_h \end{Bmatrix} \tag{40}$$

$$J'_{om}\begin{Bmatrix} k_\Omega \\ k_\Omega\epsilon \end{Bmatrix} = \sum_h J'_{hm}\begin{Bmatrix} \varkappa_{2h} \\ (k_2\epsilon)_h \end{Bmatrix} \tag{41}$$

Kozai (1968) used satellites of essentially two specifications: $a/R = 1.30$, $I = 33°$, $e = 0.16$; and $a/r = 1.22 \pm .04$, $I = 48.6° \pm 1.5°$, $e = .01$. Newton (1968) used satellites of one specification $a/R = 1.157 \pm .016$, $I = 90.2° \pm 0.3°$, $e < .008$. Values of J_{hm} corresponding to the m values of terms used in these analyses are given in Table III.

In Table IV are summarized the results obtained by Kozai and Newton. Newton's values for k and $k\delta$ are based mainly on solar perturbations and are combinations of values ranging from 0.31 to 0.38 which are weighted inversely proportionate to their radiation pressure perturbations. In combining Kozai's values for his $47.2°$ and $50.1°$ inclination satellites we have utilized a similar weighting: 1/.21 for the $47.2°$ (ECHO 1 ROCKET CASE) and 1/.07 for the $50.1°$ (ANNA 1B). In defining the lag angle we have accepted the Darwinian assumption, used by Newton, that it is proportionate to frequency: i.e., $\epsilon = \delta$ for m = 1 and $\epsilon = 2\delta$ for m = 2.

We also have included in Table IV the lag inferred from the earth's deceleration, using the rate derived by Curott (1966). It appears in the column $k_l\delta$, since it depends on $\partial R/\partial M$ and the ℓmpq = 2200 term is dominant (Kaula, 1968, pp. 198-203).

TABLE III INFLUENCE COEFFICIENTS OF TIDAL PARAMETERS ON SATELLITE ORBITS

a	e	I	m	J_{0m}	J'_{0m}	J_{2m}	J'_{2m}	J_{4m}	J'_{4m}
1.30	.16	33°	1	- .310	- .276	- .086	- .284	.042	.303
1.20	.01	49°	1	- .429	.121	- .204	- .278	.126	- .123
1.16	.00	90°	1	.000	.986	.000	- .008	.000	- .026
			2	.986	.000	.155	.000	- .304	.000
60.27	.055	23.5°	2	1.04 $\times 10^{-6}$	---	- .30 $\times 10^{-6}$	---	.05 $\times 10^{6}$	---

TABLE IV SUMMARY OF ORBIT ANALYSES FOR TIDAL PARAMETERS

a	e	I	k_1	$k_1\delta$	k_Ω	$k_\Omega\delta$
1.30	.16	33°	.312	.0000	--	--
1.20	.01	49°	.250	.0348	--	--
1.16	.00	90°	.342	.0054	.351	.0074
60.27	.055	23.5°	--	.0107	--	--

Applying Eqs. (40) - (41) to the close satellite data in Table IV we get as observation equations:

$$\begin{Bmatrix} -.310, & -.086, & +.042 \\ -.429, & -.204, & +.126 \\ .986, & .155, & -.304 \\ .986, & -.008, & -.026 \end{Bmatrix} \begin{Bmatrix} \varkappa_{20}, (\varkappa_2\delta)_0 \\ \varkappa_{22}, (\varkappa_2\delta)_2 \\ \varkappa_{24}, (\varkappa_2\delta)_4 \end{Bmatrix} = \begin{Bmatrix} -.0967, & .0000 \\ -.1072, & -.0149 \\ +.337, & .0053 \\ +.346, & .0073 \end{Bmatrix} \tag{42}$$

Which yield

$$k_2 = .372 - .240\, P_{20} - .013\, P_{40} \tag{43}$$

$$(k\delta)_2 = .006 - .093\, P_{20} - .079\, P_{40} \tag{44}$$

The large $\varkappa_{22}$ of $-.240$ makes significant terms $\varkappa_{\ell h} e_n$, when both h and n are non-zero, appearing in Eq. (13). If we let

$$\sum_{\ell, n} Q_{\ell nko}\, \varkappa_{2\ell}\, \delta_{2n} = (k\delta)_{2k} \tag{45}$$

we get

$$\delta_2 = .003 - .028\, P_{20} - .263\, P_{40}, \tag{46}$$

which is even more implausible.

An alternative procedure would be to use Newton's data alone for κ_{20} and κ_{22}, and then use Newton's data with the moon for $(k\delta)_{20}$, $(k\delta)_{22}$, $(k\delta)_{24}$. We get

$$k_2 = .351 - .055\, P_{20} \tag{47}$$

$$(k\delta)_2 = .0072 - .0121\, P_{20} - .0004\, P_{40} \tag{48}$$

and

$$\delta_2 = .0195 - .0331\, P_{20} - .0028\, P_{40} \tag{49}$$

Eq. (47) is much more reasonable than Eq. (43), and Eq. (49) is somewhat more reasonable than Eq. (46). Physical necessity requires only that δ_{20} is positive; the existence of ocean tides makes it possible that one of the zonal coefficients can be larger than δ_{20}. Eq. (49) says that beyond the latitude where P_{20} is .0195/.0331, 58°, phase leads should predominate over phase lags in the tide meter readings. A statistical analysis would be worthwhile. A major effect by ocean tides would invalidate the Darwinian assumption and make it impossible to extrapolate any inferences to rates other than semi-diurnal & diurnal, such as the monthly rate which Eq. (26) suggests might be important in lunar orbit evolution.

With the data on hand we get only a tentative indication of significant latitudinal variation in the earth's tidal properties. But it is certainly a good enough indication to warrant further effort in two directions: (1) the placing in orbit of artificial satellites which are instrumented to compensate for any surface forces, in order to determine long-periodic (fortnightly or more) variations in the orbit; and (2) the transformation of tidal models of the earth to a form compatible with the potential of Eqs. (13) or (14).

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B APPROPRIATE TREATMENT FOR NEW OBSERVING SYSTEMS

Summary. Because of the characteristics of close satellite orbit dynamics and orbit determination from ground tracking, spherical harmonics will continue to be the most suitable representation of the main part of the gravitational field indefinitely. However, qualitatively different satellite measurements such as precise satellite-to-satellite tracking; radar altimetry; and gravity gradiometry make appropriate spatial representations for the residual field. The significantly different spectral response of these methods will minimize the distortions which might arise from such hybrid representations.

Introduction

Several recent developments raise the question of whether spherical harmonic coefficients will continue to be the most convenient way to represent irregularities in the earth's gravity field for close satellite purposes: the increasingly large number of coefficients required to represent the field compatible with current accuracy of tracking and variety of orbit specifications (280 in the latest analysis by Gaposhkin [1969]); the great success of Muller & Sjogren [1968] in obtaining a spatial representation of the gravitational field for the front face of the moon; the coming of new techniques which will obtain greatly increased detail, such as the radar altimeter (Lundquist, [1967]); and the increased detail of terrestrial gravimetry, making desirable a more effective combination with satellite data than heretofore (Kaula, [1966a]; Rapp, [1969]). It therefore is appropriate (1) to review the reasons why spherical harmonics have been used in satellite orbit analyses up to now, and (2) to examine what modifications might be appropriate when more precise techniques are employed to measure variations in the gravity field through artificial satellites.

The first part of this paper will be largely a restatement of principles previously published several times (e.g., Kaula, [1966b]). For the purposes of the present paper, we wish to emphasize the idea of satellite orbit perturbations as being a particular transformation of the spectrum of variations in the gravitational field. The other means of measuring gravity constitute different transformations of this same spectrum, and an optimum technique of combination must necessarily consider these differences.

The long-wave length part of the spectrum is succinctly expressed by the rule

$$\delta_{\ell}(V^*) = \sigma\{\overline{C}_{\ell m}, \overline{S}_{\ell m}\} \approx \pm\ 10^{-5}/\ell^2 \qquad (1)$$

where ℓ is the spherical harmonic degree and $\overline{C}_{\ell m}$, $\overline{S}_{\ell m}$ are fully-normalized non-dimensionalized potential coefficients. The asterisk on V^* denotes that it is non-dimensionalized. The rule (1) was first suggested by Kaula [1963] and has been most strongly confirmed by Anderle & Smith [1968]. It is reliable up to at least degree ℓ = 15. For higher degrees, estimates can be made from the statistical analyses of gravimetry (Kaula, [1959]). It works out quite well that

$$\sigma^2\left\{\Delta g_{si}\right\} - \sigma^2\left\{\Delta g_{sj}\right\} = \sum_{\ell=\ell_j+1}^{\ell_i} \sigma_\ell^{\,2}\left\{\Delta g\right\} \tag{2}$$

where Δg_{si} is the mean free air gravity anomaly for a square of side length s_i, ℓ_i is π/s_i, and $\sigma_\ell^{\,2}\left\{\Delta g\right\}$ is the degree variance of the gravity acceleration

$$\sigma_\ell^{\,2}\left\{\Delta g\right\} = (2\ell+1)\ \delta_\ell^{\,2}\left\{\Delta g\right\}$$

$$= g^2(2\ell+1)\ (\ell-1)^2\ \delta_\ell^{\,2}\left\{V^*\right\} \tag{3}$$

Estimates of $\delta_\ell(V^*)$ calculated on the assumption that separate power laws k/ℓ^n applied between the ℓ's of 15, 36, 90, and 180 are given in Figure 1. The calculations are summarized in Table 1. An integral approximation was employed

$$\sum_{\ell=\ell_i}^{\ell_j} \frac{k}{\ell^n} \approx \frac{k}{1-n}\left[\left(\ell_j+\tfrac{1}{2}\right)^{1-n} - \left(\ell_i-\tfrac{1}{2}\right)^{1-n}\right] \tag{4}$$

Satellite Orbit Dynamics

For analysis of satellite orbit perturbations using the Lagrangian equations, the customary spherical harmonic representation of the gravitational potential V is transformed to (Kaula, [1966b])

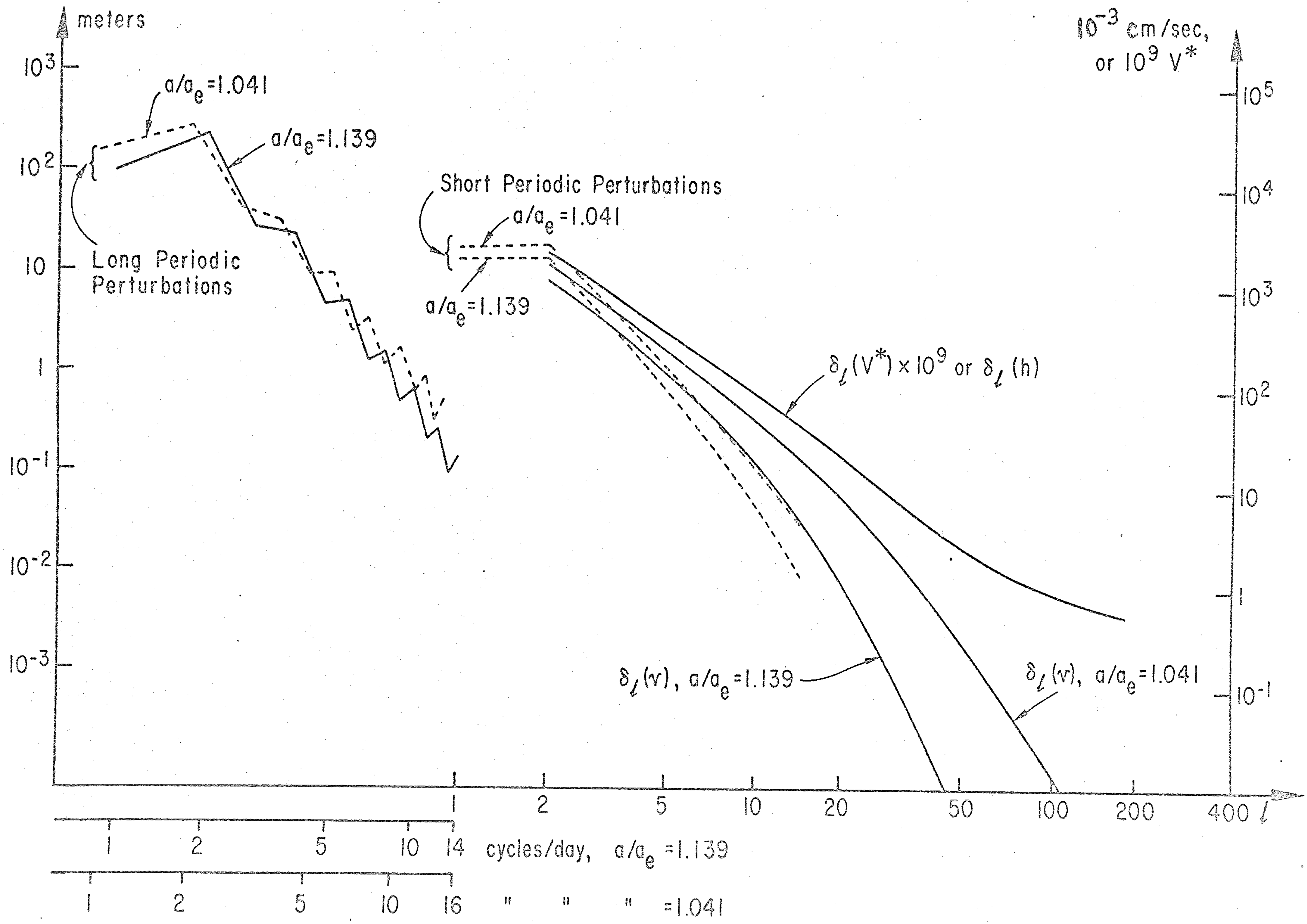

Fig. 1 Spectra related to the earth's gravitational field.

TABLE 1. CALCULATION OF GEOPOTENTIAL SPECTRA

Assuming $\sigma_\ell^2\{\Delta g\} = k/\ell^n$

s	$\sigma^2\{\Delta g_s\}$ mgal²	ℓ	$\sum_{\ell_1}^{\ell_2} \sigma_\ell^2\{\Delta g\}$ mgal²	$\sigma_\ell^2\{\Delta g\}$ mgal²	k	n	$\delta_\ell(V^*)$ 10^{-9}	$\delta_\ell(h)$ m	$\delta_\ell\{v\}$ $a/a_e=1.139$ 10^{-3} cm/sec	$\delta_\ell\{v\}$ $a/a_e=1.041$ 10^{-3} cm/sec
12°	214	15		12.0			44.4	0.283	4.390	18.290
			136		362	1.26				
5°	350	36		3.9			6.6	0.042	0.042	1.154
			163		53.4	0.73				
2°	513	90		2.0			1.2	0.008	0.000	0.023
			263		.0245	-0.98				
1°	776	180		4.0			0.6	0.004	0.000	0.000

$$V = \frac{GM^*}{a_e} \sum_{\ell mpq} \left(\frac{a_e}{a}\right)^{\ell+1} F_{\ell mp}(I)\; G_{\ell pq}(e)$$

$$\cdot \left[\begin{Bmatrix} C_{\ell m} \\ -S_{\ell m} \end{Bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}} \cos\psi_{\ell mpq} + \begin{Bmatrix} S_{\ell m} \\ C_{\ell m} \end{Bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}} \sin\psi_{\ell mpq} \right] \quad (5)$$

where

$$\psi_{\ell mpq} = (\ell-2p)\omega + (\ell-2p+q)M + m(\Omega-\theta) \qquad (6)$$

a, I, e, ω, M, Ω are Kepler elements; G is the gravitational constant; M^* and a_e are the earth's mass and equatorial radius; and θ is Greenwich sidereal time. The polynomials $F_{\ell mp}(I)$ and $G_{\ell pq}(e)$ are defined on pp. 34-37 of Kaula [1966b]. For analysis using the equations-of-motion in rectangular coordinates, a form may be used such as that of Kaula et al., [1966]

$$V = \frac{GM^*}{a_e} \sum_{\ell mth} \left(\frac{a_e}{r}\right)^{\ell+1} J_{\ell m} \frac{(-1)^t\,(2\ell-2t)!}{2^\ell\, t!\,(\ell-t)!\,(\ell-m-2t)!}$$

$$\cdot \binom{m}{2h} (-1)^h \frac{U_{\ell m1}^{m-2h}\; U_{\ell m2}^{2h}\; U_{\ell m3}^{\ell-m-2t}}{r^{\ell-2t}} \qquad (7)$$

where the $U_{\ell mi}$'s are rectangular coordinates referred to the equator and the meridian of the maximum of the ℓm harmonic; $J_{\ell m}$ is the amplitude $\left[C_{\ell m}^2 + S_{\ell m}^2\right]^{\frac{1}{2}}$ of the ℓm harmonic; and r is the radial coordinate.

The principal reasons why spherical harmonics are useful for orbit analysis are:

1. the gravitational field converges rather rapidly as expressed by equation (1);

2. the field damps with altitude as a function of wavelength, as expressed by the factor $\left(a_e/a\right)^{\ell+1}$ in equation (5) or $\left(a_e/r\right)^{\ell+1}$ in equation (7);

3. the effect of accelerations upon integration to obtain position is strongly a function of the wavelength, $\left(2\pi a_e/\ell\right)$: gravitational anomalies which persist over a great distance (such as those associated with ocean basins) are more important than greater anomalies of smaller extent (such as those associated with trenches); this wave-number ℓ related effect is more significant than the damping with altitude mentioned above;

4. the effect of that part of the gravitational field not "averaged out" in a single revolution is strongly dependent on the order m, since the time in which this effect is averaged out by the earth's rotation will be about 24/m hours (except for near resonant terms);

5. any near resonances in the gravitational field are sharply isolated as a set of a few harmonics: for a satellite orbit of small eccentricity and mean motion about m revolutions/day, the sets: ℓm, $(\ell+2)m$, $(\ell+4)m$, $(\ell+6)m$, ... , where ℓ is the odd wave number equal to m or m+1; (2m)2m, (2m+2)2m, (2m+4)2m, ... ; k3m, (k+2)3m, (k+4)3m, ..., where k is the odd wave number equal to 3m or 3m+1; etc. In addition, if the eccentricity of the orbit is appreciable, there will be significant near resonances

for $(\ell+1)m$, $(\ell+3)m$, $(2m+1)2m$, $(2m+3)2m$, $(k+1)3m$, etc.;

6. the nature of gravitational perturbations of a satellite orbit is largely expressible as a function of the spherical harmonic indices ℓm; specifically,

a. even degree (ℓ even) harmonics always cause predominantly along-track perturbations, but for small eccentricity ($e < .05$) orbits some odd-degree harmonics cause predominantly radial perturbations;

b. for a given degree ℓ, low inclination orbits are perturbed more by low order (m/ℓ near 0) terms and are insensitive to higher order (m/ℓ near 1), while high inclination orbits are sensitive to both; the overall magnitude is roughly proportional to $(1+\sin I)$.

Based on actual partial derivatives times the δ_ℓ of equation (1), a rough rule-of-thumb has been derived for the m cycles/day perturbations of a close satellite orbit by a non-resonant tesseral harmonic $C_{\ell m}$, $S_{\ell m}$

$$\Delta_{\ell m}(\underline{r}) \approx \left(\frac{a_e}{a}\right)^{\ell+1} \frac{\ell+m(\sin I-1)}{\ell^3 m}\Big[1-2(.25-e) \cdot \{\ell (\text{mod } 2)\}\Big] 3000 \text{ meters} \qquad (8)$$

There will also be short-periodic perturbations of k cycles per revolution which for a satellite making N revolutions per day will be m/kN times as great. For a small eccentricity orbit, $k-\ell$ is even, and $k \le \ell$. In Figure 1 we have plotted the spectra of $\Delta_{\ell\ell}$ for orbits of inclination 60°, small eccentricity, and semi-major axes 1.139 a_e (14 revolutions/day) and 1.041 a_e (16 revolutions/day). Note that since a wave number ℓ in the potential corresponds to a frequency ℓ cycles/revolution in terms of the orbit, it falls more than an order-of-magnitude distant from its principal perturbations.

In addition there will be resonant effects which are, roughly

$$\Delta_{res.}(\underline{r}) \approx \pm \frac{1600N}{(N-m)^2 m^2} \text{meters} \qquad (9)$$

with a period of $1/|N-m|$ days.

The essential succintness of expression of the gravitational perturbations by spherical harmonics results in greater efficiency in orbit calculation. A test was carried out by Kaula et al. [1966] in which the "real" gravity field was taken to be a set of 184 15° square mean anomalies. A reference orbit of 1.3 a_e semi-major axis, 0.2 eccentricity, and 46° inclination was integrated using this field extrapolated by the formulae of Hirvonen and Moritz [1963]. Two approximations to this field were then calculated: 1) the set of 42 harmonic coefficients through ℓm=66; and 2) the set of 48 30° square mean anomalies. The orbit integrations using these two approximations differed less than 2 percent in the computer time required. However, the error of the orbit integration using the 30° square mean anomalies grew to 600 meters in less than 5 hours orbit time, while the error using the spherical harmonics never was greater than 90 meters. Plainly, the area means failed to represent well some significant long wave effects on the orbit which were precisely sorted out by the harmonics.

For the purpose of determining the variations of the gravitational field as expressed by the harmonic coefficients, the magnitude of the effects is not the sole significant factor. Both station distribution and variety of orbital specifications have appreciable influence.

Solving the spherical triangle formed by the equator, orbit, and tracking station meridian, we get that the principal argument of the gravitational perturbations, $\Omega - \theta$, is a function of quantities essentially invariant, λ, I, and φ

$$\Omega - \theta = \lambda - \cot^{-1}\left(\pm\sqrt{\tan^2 I \csc^2\varphi - \sec^2 I}\right) \qquad (10)$$

Hence a distribution of stations which does not give a variety of values for a particular $m(\Omega - \theta)$ will yield a weak determination of the corresponding harmonics ℓm.

Variety of orbital specifications is the only way of overcoming the ambiguity arising from the fact that the same angular argument $m(\Omega - \theta)$ pertains to several harmonics ℓm. The lack of adequate variety has resulted in some harmonics $\ell 1$, $\ell 2$ being less well determined than higher orders of the same degree such as $\ell\ell$, $\ell(\ell - 1)$ because of the much greater number of terms sharing the lower order periodicity.

New Observational Techniques

The techniques of measurement which appear most likely to be developed in the near future are radar altimetry and satellite-to-satellite Doppler tracking, most likely between a synchronous and close satellite. The advance represented by both these techniques is a much more detailed coverage of the orbit, rather than a greater accuracy.

A satellite altimeter would detect variations in the mean sea level almost directly proportionate to the variations in potential, of course. Hence the mean sea level -- or, more precisely, the geoid -- has a spectrum directly proportionate to that of the potential at the earth's surface, already given in Figure 1

$$\delta_{\ell}(h) = a_e \, \delta_{\ell}(V^*) \qquad (11)$$

There will be a portion not "averaged out" in a revolution, but it would <u>not</u> be magnified in effect, like the orbit perturbations.

Satellite-to-satellite Doppler tracking would detect mainly variations in one component of the close satellite velocity. Velocity is the integral with respect to time of acceleration, potential is the integral with respect to space of acceleration. Hence the velocity spectrum will be directly proportionate to the potential <u>at close satellite altitude</u>. The proportionality factor will be the satellite velocity v, or $[GM^*/a]^{\frac{1}{2}}$, times the damping factor $\left(a_e/a\right)^{\ell}$

$$\delta_{\ell}(v) = \left[GM^*/a_e\right]^{\frac{1}{2}} \left(a_e/a\right)^{\ell+1} \delta_{\ell}(V^*) \qquad (12)$$

This spectrum is also shown in Figure 1 for a/a_e of 1.139 and 1.041.

The total error of recent determinations of the geopotential by satellite orbit analysis is on the order of $\pm$ 15 meters, in terms of geoid height. The equivalent error in close satellite velocity is v/a_e times as much, or about $\pm$ 2 cm/sec. An altimeter with an error small compared to $\pm$ 15 meters, or a satellite-to-satellite range-rate system with an error small compared to $\pm$ 2 cm/sec, would yield an appreciable increase in our information about the gravitational field. There is no doubt that accuracies an order-of-magnitude or more better than these values are possible; the main limitation on both techniques is the expense of the weight, power, and reliability required for the satellite-borne systems.

The improvement in determination of the gravity spectrum obtained by one of the new techniques will depend, of course, on the error spectrum of the technique. For both systems data points at 10-second intervals -- commensurate with reasonable power and data storage requirements -- would be essentially uncorrelated. 10 seconds in orbit is equivalent to about 64 km at the earth's surface. Uncorrelated altitudes with error $\sigma\{h\}$ and spacing s over the entire earth's surface yield an error $\delta_{\ell}(\epsilon^*)$ a normalized potential coefficient $\bar{C}_{\ell m}$ or $\bar{S}_{\ell}$, of

$$\delta_{\ell}(\epsilon^*) = \frac{\sigma\{h\}}{a_e}\left[\frac{s^2}{4\pi a_e^2}\right]^{\frac{1}{2}} \qquad (13)$$

For s=64 km and $\sigma\{h\} = \pm 1$ meter, $\delta_\ell(\epsilon^*)$ is $\pm 0.44 \times 10^{-9}$, equivalent to over a 200 degree harmonic by Figure 1. In actuality, however, small systematic errors will place the limit on the resolution of detail: in particular, atmospheric refraction for the altimeter.

Manner of Representation

We trust the spectrum shown in Figure 1 makes it evident that the variations of the gravitational field look radically different in their effect on a satellite orbit than on a device to measure potential directly. Hence if the altimetry or the satellite-to-satellite range-rate is used to determine the orbital constants-of-integration and the lower degree spherical harmonics over a span of, say, one day, the residuals in altitude or range-rate should give the shorter wavelength variations with negligible distortion.

The drop-off of the range-rate spectrum with respect to the surface potential, particularly for the a/a_e of 1.139, indicates that the orbits should be as low as possible, and hence instrumented with the drag-free capability. In addition, if the oceanographer's aspiration of distinguishing mean sea level from the geoid is to be attained, then a more sensitive means of distinguishing gravitation from geometry at the satellite might be required: either a satellite-borne gradiometer, or extrapolation of surface gravimetry to satellite altitude.

Given that a separate mode of representation for the altitude or Doppler residuals is feasible because of the significantly different spectral response, there remains the question of whether a more elaborate representation such as the polynomials proposed by Lundquist & Giacaglia [1968] is appropriate. The elaboration of spherical harmonics is justified for orbit perturbations because the physics of satellite orbits makes them highly selective in their response to the variations of the gravitational field, as summarized by equations (8) and (9). But the altimeter and Doppler range-rate essentially observe potential directly: the former at sea level, and hence with no selectivity at all; the latter at altitude, and hence with only the selectivity of the $\left(a_e/a\right)^{\ell+1}$ damping factor. Hence the physics which would justify a special representation would have to be that which generates the density irregularities causing the geopotential variations. This physics is of a greatly different and more complicated sort, and to suggest any more elaborate mode of representation in essence entails guessing beforehand the geophysical explanations which are the end purpose of the entire analysis.

In conclusion, we can say that a change from the present representation of the earth's gravity field customary for satellite orbit analysis of spherical harmonics will be justified when qualitatively different systems of observation come into application. We can further say that the likely new systems come close to measuring potential directly, and hence are so different in their spectral response that they can be represented in the form of residuals with respect to a reference model. Furthermore, their lack of selectivity makes a more elaborate representation of dubious value.

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