NASA CR-72902

THE PREDICTION OF THE NONLINEAR BEHAVIOR OF UNSTABLE LIQUID ROCKETS

by

E. A. Powell and B. T. Zinn

GEORGIA INSTITUTE OF TECHNOLOGY

prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

NASA Lewis Research Center
Grant NGL 11-002-083
Richard J. Priem, Project Manager

Requests for copies of this report should be referred to

National Aeronautics and Space Administration
Scientific and Technical Information Facility
P. O. Box 33
College Park, Md. 20740

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-72902		

4. Title and Subtitle	5. Report Date
THE PREDICTION OF THE NONLINEAR BEHAVIOR OF UNSTABLE LIQUID ROCKETS	July 1971
	6. Performing Organization Code
7. Author(s): Eugene A. Powell and Ben T. Zinn	8. Performing Organization Report No.
9. Performing Organization Name and Address: Georgia Institute of Technology, Atlanta, Georgia 30332	10. Work Unit No.
	11. Contract or Grant No. NGL 11-002-083
	13. Type of Report and Period Covered: Contractor Report
12. Sponsoring Agency Name and Address: National Aeronautics and Space Administration, Washington, D. C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Technical Officer, Richard J. Priem, Chemical Rockets Division, NASA Lewis Research Center, Cleveland, Ohio

16. Abstract

An analytical technique is developed to solve nonlinear combustion instability problems associated with liquid-propellant rocket motors. The analysis produces the limit-cycle behavior of unstable motor operation and the threshold amplitude required to trigger a linearly stable motor into unstable operation by considering second order terms in the conservation equations. Calculated results indicate that limit-cycle amplitude increases with (1) increasing sensitivity of the combustion process to a pressure oscillation, (2) increasing chamber Mach number, and (3) decreasing chamber length-to-diameter ratio. Calculated pressure waveforms exhibit sharp peaks and shallow minima. The frequency is always within a few percent of the pure acoustic mode frequency.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Combustion instability Liquid rockets	Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	77	$3.00

* For sale by the National Technical Information Service, Springfield, Virginia 22151

FOREWORD

The research described herein, which was conducted at Georgia Institute of Technology, was supported by NASA Grant No. NGL-11-002-083. The work was done under the management of the NASA Project Manager, Dr. Richard J. Priem, Chemical Rockets Division, NASA-Lewis Research Center. Assistance in the preparation of this report was given by Mr. Allan J. Smith, Jr., Research Engineer, Georgia Institute of Technology.

ABSTRACT

An analytical technique is developed to solve nonlinear combustion instability problems associated with liquid-propellant rocket motors. The analysis produces the limit-cycle behavior of unstable motor operation and the threshold amplitude required to trigger a linearly stable motor into unstable operation by considering second order terms in the conservation equations. Calculated results indicate that limit-cycle amplitude increases with (1) increasing sensitivity of the combustion process to a pressure oscillation, (2) increasing chamber Mach number, and (3) decreasing chamber length-to-diameter ratio. Calculated pressure waveforms exhibit sharp peaks and shallow minima. The frequency is always within a few percent of the pure acoustic mode frequency.

TABLE OF CONTENTS

TABLE OF CONTENTS (CONTINUED)

LIST OF ILLUSTRATIONS

LIST OF TABLES

SUMMARY

An approximate analytical technique has been developed for the solution of nonlinear combustion instability problems that are frequently associated with liquid-propellant rocket motors. The application of this solution technique, that is based on the Method of Weighted Residuals, is demonstrated by considering the unstable behavior of a cylindrical liquid-rocket combustor with uniform injection of propellants at one end and a multi-orifice nozzle at the other end. Crocco's pressure sensitive time-lag model is used to describe the unsteady combustion process. It is shown that the behavior of unstable liquid-rocket combustors with low mean flow Mach numbers and moderate amplitude oscillations can be analyzed with the aid of a single nonlinear wave equation whose solution is considered in this report.

Calculated results indicate that the analysis can produce the transient behavior and nonlinear wave shape that have been observed during unstable motor operation, the limit-cycle amplitude and frequency typical of unstable motor operation, and the threshold amplitude required to trigger a linearly stable motor into unstable operation. These results establish the relationship that exists between the resulting instability (i.e., waveform, final amplitude, and final frequency), the combustion parameters (i.e., interaction index, n, and time-lag, $\bar{\tau}$), and the chamber Mach number and length-to-diameter ratio. Results indicate that the limit-cycle amplitude increases with increasing sensitivity of the combustion process to pressure oscillations. For most cases an increase in the chamber Mach number or a decrease in the chamber length-to-diameter ratio results in a larger limit-cycle amplitude. Calculated pressure waveforms exhibit sharp peaks and shallow minima, and the frequency of oscillation is always within a few percent of the frequency of one of the chamber's acoustic modes.

INTRODUCTION

Observation of the behavior of unstable rocket motors indicates that combustion instability can be divided into two categories; that is, linear and nonlinear instabilities. Linear instabilities are spontaneous in nature, and they are usually an outgrowth of the combustion noise present in the system. On the other hand, nonlinearly unstable motors require the introduction of a finite amplitude disturbance to produce combustion instability. In either case the instability, after a transient period, reaches a limiting maximum amplitude (i.e., limit-cycle amplitude) at which it oscillates with a frequency that is usually close to the frequency of one of the chamber's acoustic modes. Pressure measurements taken during test firings of unstable motors indicate that the limit-cycle waveforms are non-sinusoidal; that is, they exhibit sharp peaks and flattened minima[1]. These results indicate that nonlinearities need to be considered in the theoretical treatment of combustion instability.

In addition to being able to predict the nonlinear stability characteristics of liquid-propellant rocket motors, a practical theoretical treatment should endeavor to fulfill the following objectives:

1. It should be able to analyze multi-dimensional instabilities.
2. Its application should be straightforward.
3. It should require little computation time.

The theoretical treatment described in this report has the potential of meeting these objectives.

The application of the theory presented herein will be demonstrated by considering the behavior of an unstable liquid-propellant rocket combustor with uniform injection of propellants at one end and a multi-orifice nozzle at the other end. Crocco's pressure sensitive time lag model[2] is used to describe the unsteady combustion process. Only pure transverse modes are considered in this report.

In the sections to follow, the development of the wave equation for the analysis of nonlinear combustion instability in liquid-rockets will be briefly described, the solution of this problem will be outlined, and typical results will be presented and discussed. Two appendices, which represent a User's

Manual for the computer programs that are used to solve this problem, are also included with this report.

SYMBOLS

$A_{mn}(t)$, $B_{mn}(t)$	time-dependent coefficients in series given by Eq. (6)
$B(\tilde{\Phi})$	boundary residual
c^*	velocity of sound, ft/sec
C_1, C_2, C_3, C_4	coefficients of nonlinear terms in Eqs. (9) and (10)
$E(\tilde{\Phi})$	residual of Eq. (5)
f	dimensionless frequency, $f^*\left(\frac{R_c^*}{c_0^*}\right)$
J_m	Bessel function of the first kind, order m
K, K_T	coefficients of linear terms in Eqs. (9) and (10)
L/D	chamber length-to-diameter ratio
m	tangential mode number
n	pressure interaction index
$\underset{\sim}{n}$	unit outward normal vector
p	dimensionless pressure, $\gamma p^*/\rho_0^* c_0^{*2}$
Q_m'	unsteady nozzle response function (see Eq. (2))
r	dimensionless radial coordinate, r^*/R_c^*
R_c^*	motor radius, ft
S_{mn}	dimensionless transverse mode frequency
t	dimensionless time, $\frac{t^*}{(R_c^*/c_0^*)}$

$\bar{u}$	dimensionless steady state velocity, $\bar{u}^*/c_0^*$
$\underset{\to}{V}$	dimensionless velocity vector, $\underset{\to}{V}^*/c_0^*$
W_m'	unsteady combustion mass source
z	dimensionless axial coordinate, z^*/R_c^*
γ	ratio of specific heats
ϵ	ordering parameter
θ	azimuthal coordinate
ρ	dimensionless density, ρ^*/ρ_0^*
τ	dimensionless pressure sensitive time lag, $\frac{\tau^*}{(R_c^*/c_0^*)}$
Φ	velocity potential
$\varphi_{mn}(r,\theta)$, $\Psi_{mn}(r,\theta)$	weighting functions in Eqs. (7)
ω	dimensionless angular frequency, $2\pi f$

Subscripts:

e	evaluated at the nozzle entrance
n	radial mode number
r, t, z, θ	partial differentiation with respect to r, t, z, or θ
0	stagnation quantity

Superscripts:

′	perturbation quantity, differentiation with respect to argument
‒	steady state quantity
*	dimensional quantity
~	approximate solution

DEVELOPMENT AND SOLUTION OF THE EQUATIONS

Development of the Wave Equation

To keep the problem as simple as possible, yet still physically meaningful, the following assumptions are made. The gas phase in the combustor is assumed to consist of a single constituent which is thermally and calorically perfect. Transport phenomena, such as diffusion, viscosity, and heat conduction are neglected. The momentum and specific stagnation enthalpy of the unburned propellant is assumed constant throughout the chamber. It is also assumed that the Mach number of the combustor's mean flow is small and that the waves have moderate amplitudes.

As a result of the last two assumptions, the governing conservation equations may be combined and the unsteady flow in the combustor can be described by a single nonlinear partial differential equation. The derivation of this equation appears in Refs. 3 and 4, where it was assumed that each perturbation quantity and the mean flow Mach number were of $O(\epsilon)$, where ϵ is a small ordering parameter that is a measure of the wave amplitude. After neglecting all terms of $O(\epsilon^3)$ or higher and combining equations, one obtains the following nonlinear partial differential equation that describes the behavior of the velocity potential, Φ:

$$\nabla^2\Phi - \Phi_{tt} = 2\underset{\rightarrow}{\overline{V}}\cdot\nabla\Phi_t + \gamma(\nabla\cdot\underset{\rightarrow}{\overline{V}})\Phi_t + 2\nabla\Phi\cdot\nabla\Phi_t + (\gamma-1)\Phi_t\nabla^2\Phi + W_m' \qquad (1)$$

Equation (1) is the desired wave equation, and it is similar to the inhomogeneous wave equation used by Maslen and Moore[5]. This equation accounts for the following effects: (1) the effect of a steady state flow on the wave motion (viz., the first two terms on the right-hand side), (2) the coupling between the gas dynamical oscillations and the unsteady combustion process (viz., the last term on the right-hand side), and (3) the second order nonlinearities of the gas dynamical processes (viz., the third and fourth terms on the right-hand side).

In addition to satisfying Eq. (1), the desired solutions must satisfy rigid wall boundary conditions at the injector end of the chamber and at the chamber walls, while a boundary condition describing conservation of mass must be satisfied at the nozzle entrance. The nozzle boundary condition correct to $O(\epsilon^2)$ is given by:

$$B(\Phi) = Q_m'(\Phi) + [(1-\Phi_t)\nabla\Phi - \bar{u}\Phi_t]\cdot \underset{\rightarrow}{n} = 0 \tag{2}$$

where Q_m' is a quantity whose form depends upon the unsteady flow inside the nozzle; the remaining terms in Eq. (2) represent the perturbation of the mass flux leaving the chamber. The multi-orifice nozzle assumption permits use of the quasi-steady result that the Mach number at the nozzle entrance is constant[6]. Thus the quantity Q_m' in Eq. (2), becomes[3]

$$Q_m' = \frac{\gamma+1}{2}\,\bar{u}_e\Phi_t \tag{3}$$

The unsteady combustion process is represented by mass sources distributed throughout the volume of the chamber, and the response of the mass sources to pressure oscillations is assumed to be described by Crocco's pressure sensitive time-lag hypothesis[2]. The mass source perturbation, W_m', is then given by[3,7]:

$$W_m' = -\gamma n \frac{d\bar{u}}{dz}\left[\Phi_t(r,\theta,z,t) - \Phi_t(r,\theta,z,t-\bar{\tau})\right] \tag{4}$$

where n and $\bar{\tau}$ are the two parameters that Crocco used to describe the unsteady combustion process. Here n is a pressure "interaction index" that describes the sensitivity of the combustion process to pressure oscillations. The parameter $\bar{\tau}$, commonly referred to as the sensitive time-lag, is the part of the total combustion time-lag during which the combustion process is sensitive to pressure oscillations.

Substituting Eq. (4) into Eq. (1) and expressing the resulting equation in a cylindrical coordinate system (see Fig. 1) yields the following wave equation:

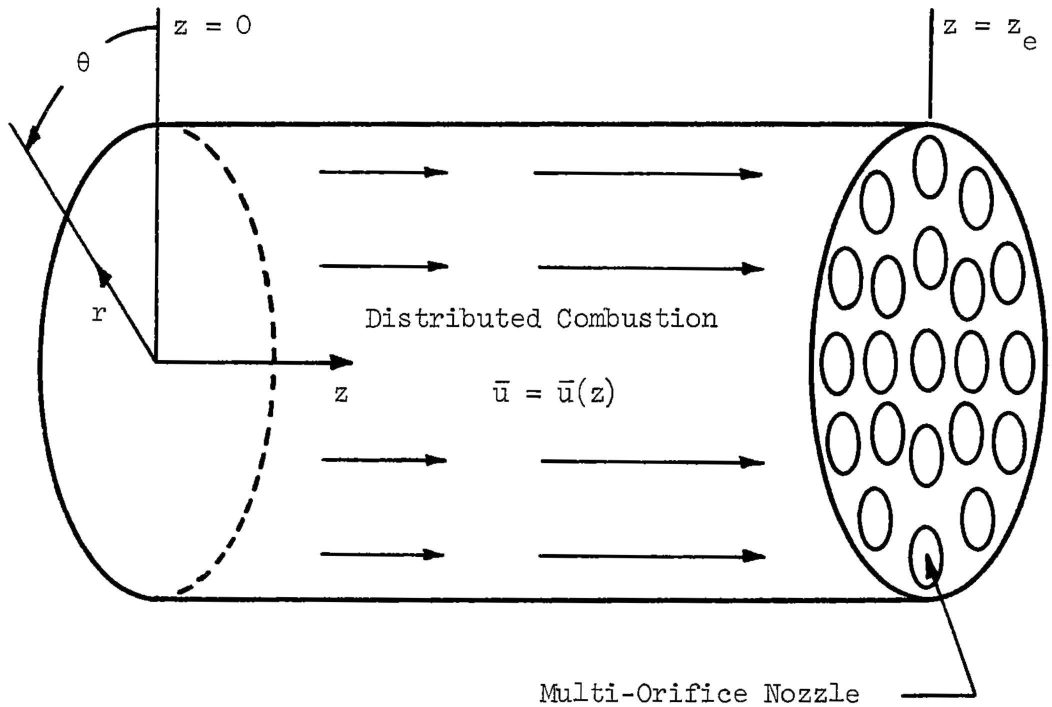


Figure 1. Combustor configuration and coordinate system.

$$E(\Phi) = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} + \Phi_{zz} - \Phi_{tt}$$

$$- 2\Phi_r\Phi_{rt} - \frac{2}{r^2}\Phi_\theta\Phi_{\theta t} - 2\Phi_z\Phi_{zt}$$

$$- (\gamma-1)\Phi_t\left(\Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} + \Phi_{zz}\right)$$

$$- 2\bar{u}\Phi_{zt} - \gamma\Phi_t\frac{d\bar{u}}{dz}$$

$$+ \gamma n\frac{d\bar{u}}{dz}\left[\Phi_t(r,\theta,z,t) - \Phi_t(r,\theta,z,t-\bar{\tau})\right] = 0 \qquad (5)$$

<u>Method of Solution</u>

Equation (5) is a nonlinear wave equation for which there is no known closed-form mathematical solution. Consequently it is necessary to resort to the use of either numerical solution techniques or approximate analytical techniques. Since the numerical solution techniques generally require excessive computer time, the latter approach is used. The experience of previous investigators in the fields of structural stability and aeroelasticity indicates that an approximate solution technique known as the Method of Weighted Residuals[8,9] may be effective in the solution of this nonlinear wave equation.

In order to employ the Method of Weighted Residuals in the solution of Equation (5), it is first necessary to express the velocity potential, Φ, as an approximating series expansion, $\tilde{\Phi}$. The question naturally arises as to what form of series expansion should be used. Inasmuch as the experimentally observed pressure oscillations during combustion instability usually resemble the natural acoustic modes of the chamber, the velocity potential, Φ, is expressed as a series expansion of the natural acoustic modes of the chamber with unknown time-dependent coefficients. Restricting attention to pure transverse modes the approximate velocity potential, $\tilde{\Phi}$, is expressed in the following form:

$$\tilde{\Phi} = \sum_{m=0}^{M}\sum_{n=1}^{N}\left[A_{mn}(t)\sin m\theta + B_{mn}(t)\cos m\theta\right]J_m(S_{mn}r) \qquad (6)$$

Each term in the expansion satisfies the solid wall boundary conditions at the injector end (i.e., $z = 0$) and at the chamber wall (i.e., $r = 1$); however, the boundary condition imposed at the nozzle end (i.e., at $z = z_e$ where $z_e = 2(L/D)$) is not exactly satisfied by the individual terms. Including both the $\sin m\theta$ and $\cos m\theta$ terms in the expansion of $\tilde{\Phi}$ allows for the possibility of either spinning or standing wave solutions.

In order to obtain a solution, the unknown time-dependent mode-amplitudes (i.e., $A_{mn}(t)$ and $B_{mn}(t)$) are determined by the following mathematical procedure. The assumed series expansion, $\tilde{\Phi}$, (i.e., Eq. (6)) is substituted into the wave equation (i.e., Eq. (5)) to form the equation residual, $E(\tilde{\Phi})$. Similarly, substituting the series expansion into the nozzle boundary condition (i.e., Eqs. (2) and (3)) yields the boundary residual, $B(\tilde{\Phi})$. In the event that these residuals are both identically zero, the solution is an exact solution. The residuals $E(\tilde{\Phi})$ and $B(\tilde{\Phi})$ are the errors incurred by using the approximate solution, $\tilde{\Phi}$.

According to the Method of Weighted Residuals, the residuals, $E(\tilde{\Phi})$ and $B(\tilde{\Phi})$, must satisfy the following orthogonality conditions[3,7]:

$$\int_0^{z_e}\int_0^{2\pi}\int_0^1 E(\tilde{\Phi})\,\varphi_{jk}(r,\theta)\,r\,dr\,d\theta\,dz - \int_0^{2\pi}\int_0^1 B(\tilde{\Phi})\,\varphi_{jk}(r,\theta)\,r\,dr\,d\theta = 0$$

$$\int_0^{z_e}\int_0^{2\pi}\int_0^1 E(\tilde{\Phi})\,\Psi_{jk}(r,\theta)\,r\,dr\,d\theta\,dz - \int_0^{2\pi}\int_0^1 B(\tilde{\Phi})\,\Psi_{jk}(r,\theta)\,r\,dr\,d\theta = 0 \qquad (7)$$

where the weighting functions $\varphi_{jk}(r,\theta)$ and $\Psi_{jk}(r,\theta)$ are given by:

$$\varphi_{jk}(r,\theta) = \sin(j\theta)\,J_j(S_{jk}r)$$

$$\Psi_{jk}(r,\theta) = \cos(j\theta)\,J_j(S_{jk}r) \qquad (8)$$

The chosen weighting functions must correspond to the terms that appear in the assumed series solution; that is Eq. (6).

Performing the spatial integrations indicated in Eqs. (7) yields the following system of nonlinear, ordinary differential equations:

$$\frac{d^2A_{jk}}{dt^2} + S_{jk}^2 A_{jk} + K\frac{dA_{jk}}{dt} + K_\tau \frac{d}{dt}\left[A_{jk}(t-\bar{\tau})\right]$$

$$+ \sum_{m,n}\sum_{\mu,\nu}\left\{C_1(m,n;\ \mu,\nu;\ j,k)A_{mn}\frac{dB_{\mu\nu}}{dt} + C_2(m,n;\ \mu,\nu;\ j,k)B_{mn}\frac{dA_{\mu\nu}}{dt}\right\} = 0 \quad (9)$$

$$\frac{d^2B_{jk}}{dt^2} + S_{jk}^2 B_{jk} + K\frac{dB_{jk}}{dt} + K_\tau \frac{d}{dt}\left[B_{jk}(t-\bar{\tau})\right]$$

$$+ \sum_{m,n}\sum_{\mu,\nu}\left\{C_3(m,n;\ \mu,\nu;\ j,k)A_{mn}\frac{dA_{\mu\nu}}{dt} + C_4(m,n;\ \mu,\nu;\ j,k)B_{mn}\frac{dB_{\mu\nu}}{dt}\right\} = 0 \quad (10)$$

where

$$K = \frac{\gamma\bar{u}_e}{2(L/D)}\left(1 + \frac{\gamma-1}{2\gamma} - n\right)$$

$$K_\tau = \frac{\gamma\bar{u}_e}{2(L/D)}\,n \qquad (11)$$

There is a set of equations (i.e., Eqs. (9) and (10)) corresponding to each value of j and k included in the series expansion, $\tilde{\Phi}$. Since the coefficients given by Eqs. (11) depend on the steady state velocity at the nozzle entrance, the knowledge of the function $\bar{u}(z)$ is not necessary. The coefficients of the nonlinear terms (i.e., C_1 through C_4) are determined by evaluating the various integrals of trigonometric and Bessel functions that arise from the spatial integrations indicated in Eqs. (7). These integrals and expressions for the coefficients are given in Appendix A.

The unstable behavior of an engine is determined by specifying the form of the initial disturbance and then following the subsequent behavior of the individual modes by numerically integrating Eqs. (9) and (10). Once the time-dependence of the individual modes is known, the velocity potential, $\tilde{\Phi}$, is calculated from Eq. (6). The pressure perturbation at any location within the chamber is related to $\tilde{\Phi}$ by the following second-order momentum equation (see Refs. 3 and 7):

$$p'(r,\theta,t) = -\gamma\left[\tilde{\Phi}_t + \frac{1}{2}\left(\tilde{\Phi}_r^2 + \frac{1}{r^2}\tilde{\Phi}_\theta^2 - \tilde{\Phi}_t^2\right)\right] \tag{12}$$

In summary, the theory presented in this section represents a two-stage simplification of the original problem. In the first stage the problem has been reduced to the solution of a single nonlinear, partial differential equation (i.e., Eq. (5)). In the second stage the solution was expanded in a series of acoustic modes with time dependent coefficients and the Method of Weighted Residuals was used to replace the solution of the nonlinear partial differential equation with the solution of a system of nonlinear, ordinary differential equations (i.e., Eqs. (9) and (10)). Typical numerical solutions of these equations will be presented and discussed in the following section.

RESULTS AND DISCUSSION

General Considerations

The applicability of an analytical model to the solution of a physical problem depends upon the ability of the analytical model to produce results similar to those observed in reality. As mentioned in the Introduction, most liquid-rocket motor combustion instabilities are characterized by nonlinear wave shapes which, after a transient period, reach a maximum amplitude (i.e., a limit-cycle amplitude) that characterizes their subsequent behavior. This instability can develop in the following two ways: (1) a motor can become spontaneously unstable from the noise generated by the combustion process (i.e., linear instability) or (2) a linearly stable motor can be driven into instability by the introduction of a finite amplitude disturbance (i.e., triggered instability). Consequently the objectives of this section are to demonstrate that this analytical development can:

1. Produce nonlinear wave shapes similar to those observed during the unstable operation of a rocket motor.
2. Predict the transient behavior and limit-cycle amplitude of a linearly unstable rocket motor.
3. Predict the threshold disturbance amplitude required to trigger a linearly stable motor into unstable operation and then determine the resulting limit-cycle amplitude.

Once it has been established that the analytical treatment can properly describe the nonlinear characteristics of unstable liquid-rocket motors, data will be presented to illustrate to the rocket designer how various motor parameters, such as chamber Mach number, $\bar{u}_e$, and chamber length-to-diameter ration, L/D, can affect the stability characteristics of a motor. Because of the numerous combinations of motor parameters used in actual liquid-rocket motors, an attempt to present a comprehensive parametric study is impractical. Instead, typical analytical predictions will be presented which illustrate the results that can be expected.

All stability predictions were obtained with a three-mode series expansion. The three-mode expansion includes the following acoustic modes: the first tangential (1T), the second tangential (2T), and the first radial (1R) modes. Stability predictions obtained with three types of initial disturbances will be presented; that is, a pure spinning 1T disturbance, a pure standing 1T disturbance, and a pure 1R disturbance.

Nonlinear Behavior

To determine the shape of the nonlinear pressure waveforms, calculations were made for a standing first tangential disturbance. The time dependence of the individual series terms (i.e., $A_{mn}(t)$ and $B_{mn}(t)$) at the limit cycle is shown in Fig. (2). These functions strongly resemble sinusoids. The frequency of the series term corresponding to the 1T mode (i.e., $A_{11}(t)$) differs only by a few percent from the corresponding pure acoustic frequency, while the 1R (i.e., $B_{01}(t)$) and 2T (i.e., $B_{21}(t)$) terms oscillate at twice the 1T acoustic frequency.

The three modes shown in Fig. (2) are combined (i.e., see Eq. (12)) to

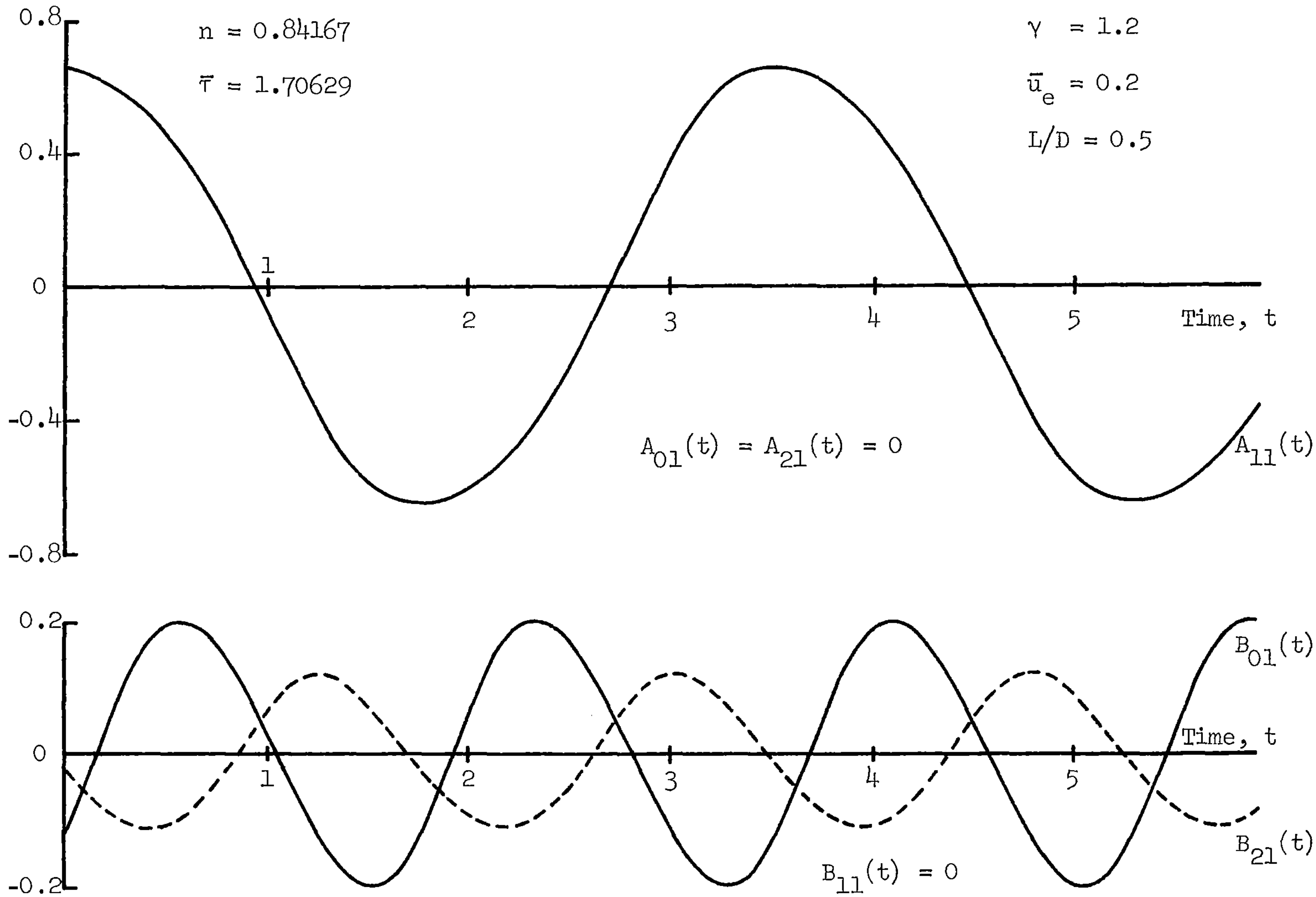


Figure 2. Limit-cycle behavior of series terms for standing first tangential mode instability.

obtain the nonlinear pressure waveform. Typical pressure waveforms determined at the chamber wall are shown in Fig. (3) for a standing 1T disturbance. The effect of the higher harmonics shown in Fig. (2) is to distort the pressure waveform into a characteristic nonlinear shape which exhibits sharp peaks and shallow minima. An oscillation of twice the fundamental frequency appears at the location of the acoustic pressure node, $\theta = 0^\circ$, of the 1T series term. Waveforms similar to the 90° curve in Fig. (3) were calculated for spinning oscillations; however in this case the pressure amplitude did not vary with angular position, and the waveforms differed only in phase.

Limit-cycle amplitudes of linearly unstable motors were computed by specifying an initial disturbance and continuing the step-by-step integration of Eqs. (9) and (10) until a periodic solution was obtained; that is, the amplitude of the pressure oscillation remained essentially constant. Such behavior is illustrated in Fig. (4). As observed in a linearly unstable rocket motor, a small disturbance is shown to grow to a limit-cycle amplitude. The computation time required to approach the limit-cycle amplitude corresponds to about 40 seconds on the Univac 1108 computer.

Such results can be combined to produce a limit-cycle amplitude map, which is of importance to rocket motor designers. Such a map is shown in Fig. (5) for a 1T spinning disturbance where lines of constant pressure amplitude and lines of constant frequency are plotted on an $(n,\bar{\tau})$ plane. The region in which limit-cycle amplitudes were found is bounded below by the neutral stability limit (line of zero amplitude) for the 1T mode. A similar stability map is shown in Fig. (6) for a standing 1T disturbance.

Limit-cycle amplitude maps similar to Figs. (5) and (6) could, in principle, be used to determine the operating $(n,\bar{\tau})$ values of a spontaneously unstable motor from test data. These figures cannot be used if the motor was triggered into unstable operation. The values of n and $\bar{\tau}$ for a linearly unstable motor can be obtained from the aforementioned stability maps by using the following procedure. Suppose a rocket engine is spontaneously unstable with respect to the 1T spinning mode and the maximum peak-to-peak chamber pressure amplitude and frequency are measured. Lines of constant amplitude and constant frequency corresponding to the measured values can be plotted on Fig. (5) by interpolation. The point of intersection of these lines determines

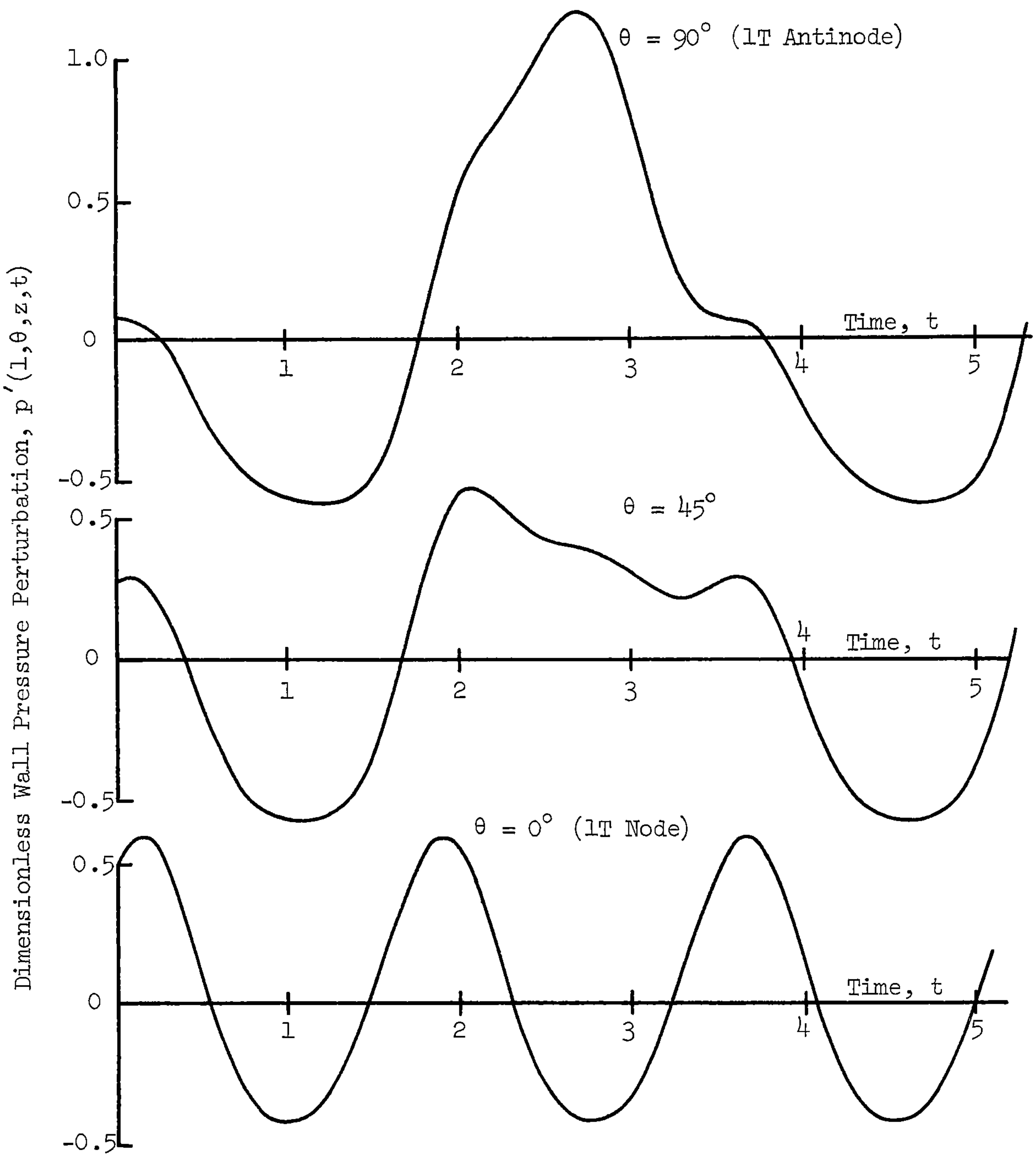


Figure 3. Nonlinear pressure waveforms for the standing first tangential mode.

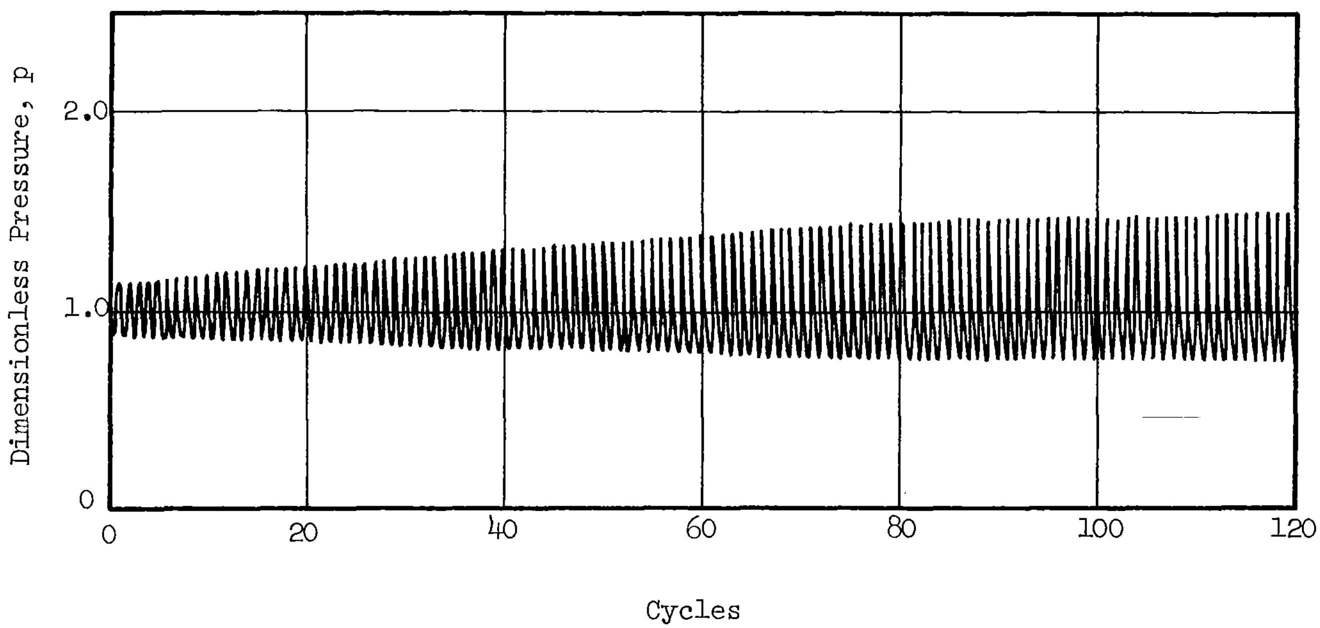


Figure 4. Time history of a spinning first tangential mode.

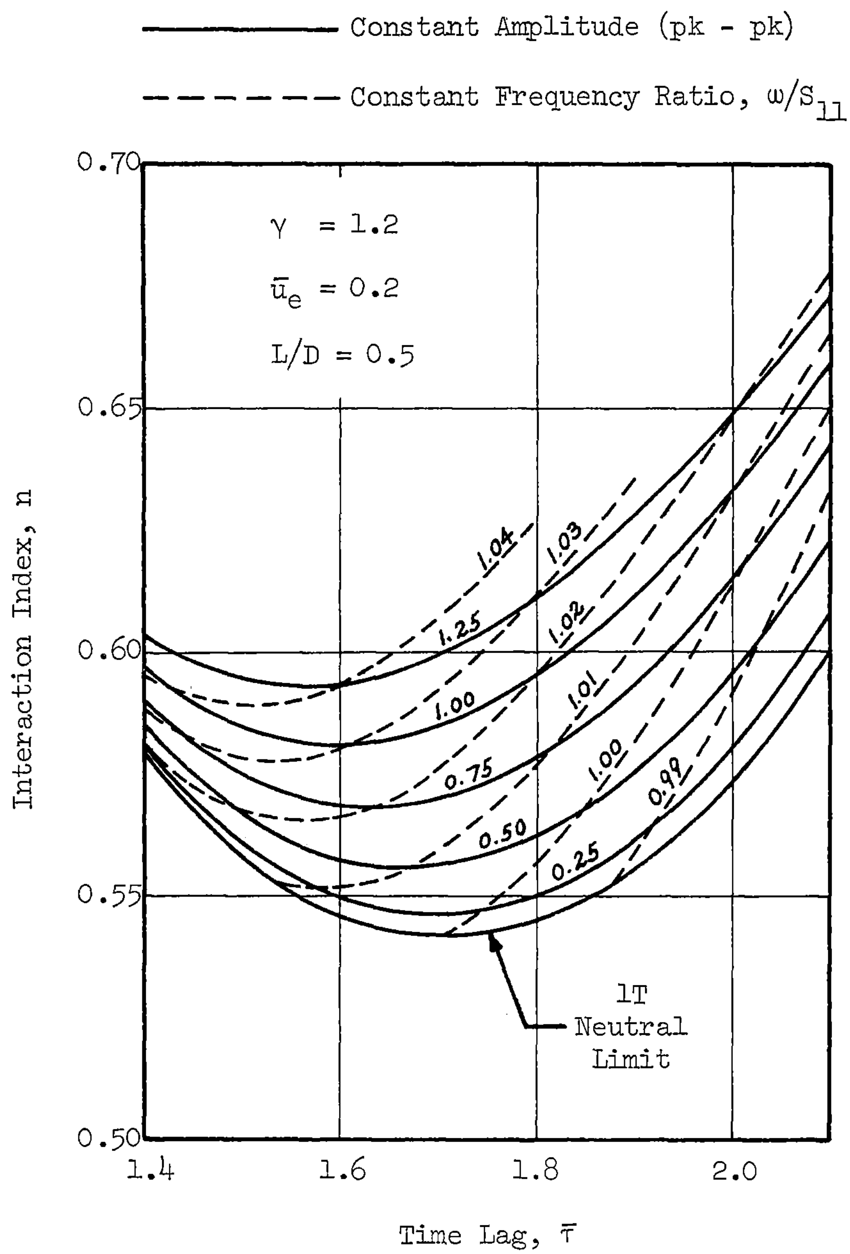


Figure 5. Limit-cycle amplitude map for a spinning mode.

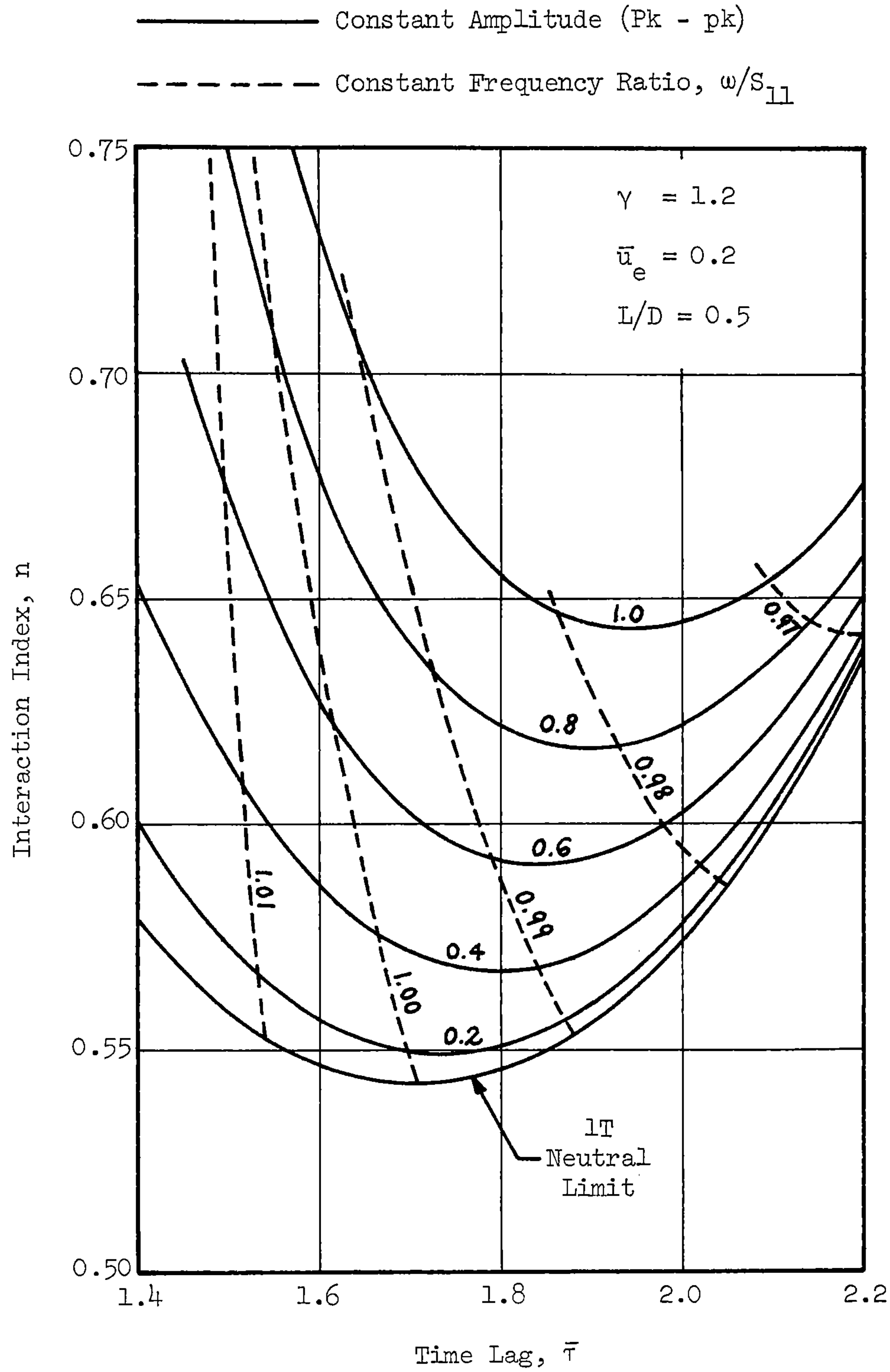


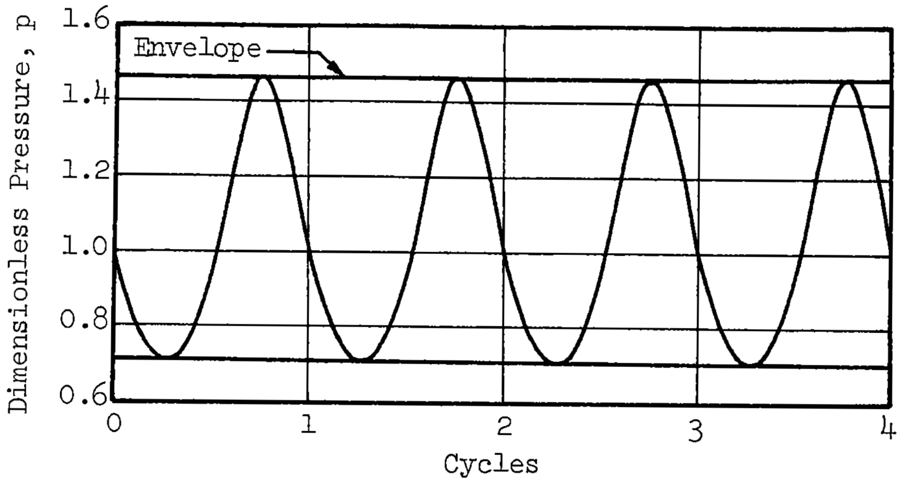
Figure 6. Limit-cycle amplitude map for a standing mode.

the n and $\bar{\tau}$ values for the unstable engine.

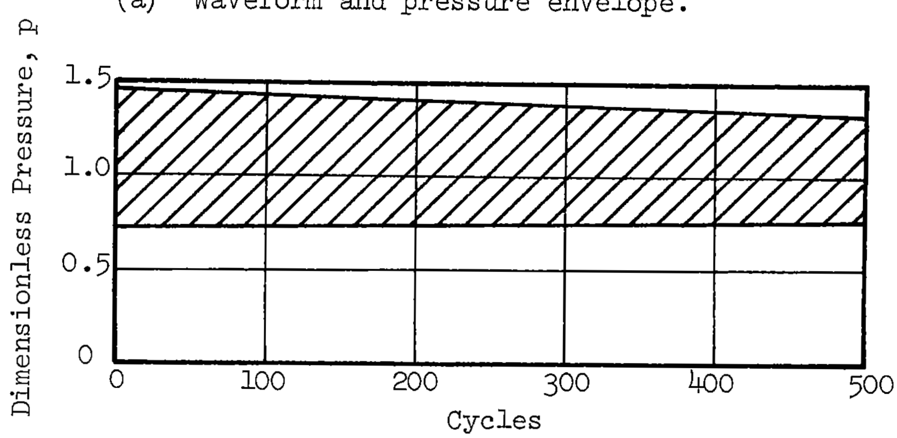
An example of a triggering limit is shown in Fig. (7) for a 1R disturbance. The upper plot of Fig. (7) shows the wave shape and wave envelope of an initial disturbance of small amplitude, whereas the wave envelope presented in the middle plot shows the manner in which this disturbance decays to zero amplitude. However, for the same operating conditions, an initial disturbance of somewhat larger amplitude (i.e., see lower plot), was found to grow. Interpolation between the amplitudes of these two initial disturbances yields the threshold disturbance amplitude required to trigger a linearly stable motor into unstable operation. The numerical procedure for calculating the triggering amplitude is described in Appendix B.

A triggering map for 1R disturbances is shown on the $(n,\bar{\tau})$ plane in Fig. (8). Above the uppermost solid curve (the neutral stability limit) the motor is linearly unstable and limit-cycle amplitudes were found. The lower solid curve is the nonlinear stability limit; below this curve all disturbances decayed. Between these curves is the region of nonlinear instability where small amplitude disturbances decay and large ones grow. In this nonlinearly unstable region, curves of constant triggering amplitude are shown. For values of $\bar{\tau}$ less than about 0.6 the linear and nonlinear stability limits coincide and triggering of 1R instability cannot occur. Here limit-cycle amplitudes were calculated for points in the linearly unstable region.

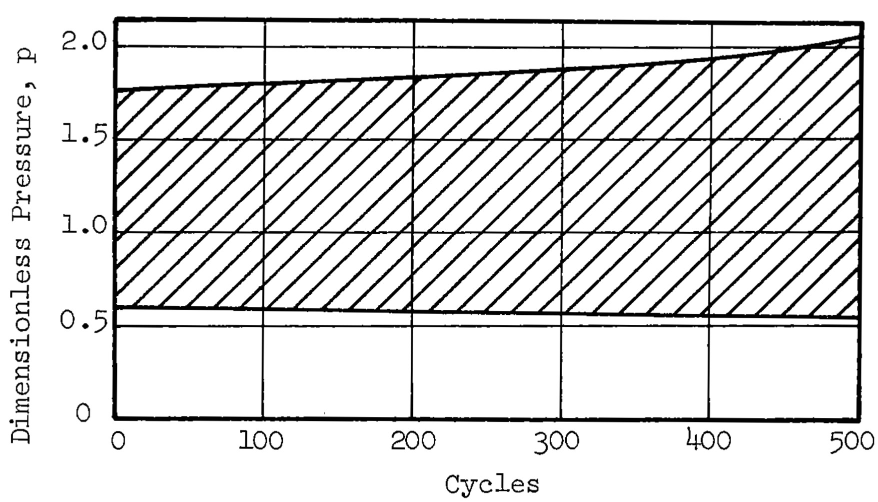
Several limitations that are imposed upon the results presented herein should be pointed out. First, these results were computed with the aid of a theory in which terms of third order (i.e., $O(\epsilon^3)$) or higher were neglected. Consequently the accuracy of the predicted limit-cycle amplitudes and triggering limits with amplitudes above a certain limit (say $p' = 0.5$) are open to question. Second, the $O(\epsilon^2)$ theory is unable to predict triggering for 1T disturbances, thus a triggering map for the 1T mode, similar to Fig. (8), cannot be generated. It can be shown, however, that triggering for 1T disturbances can be described when the $O(\epsilon^3)$ terms are retained in the analysis[3,10]. Finally, the limitations mentioned above are also the causes of the inability of the $O(\epsilon^2)$ theory to predict the limit-cycle amplitudes attained by triggered 1R mode instabilities.



(a) Waveform and pressure envelope.



(b) Initial disturbance below triggering limit.



(c) Initial disturbance above triggering limit.

Figure 7. Time histories of initial first radial mode disturbances near a triggering limit.

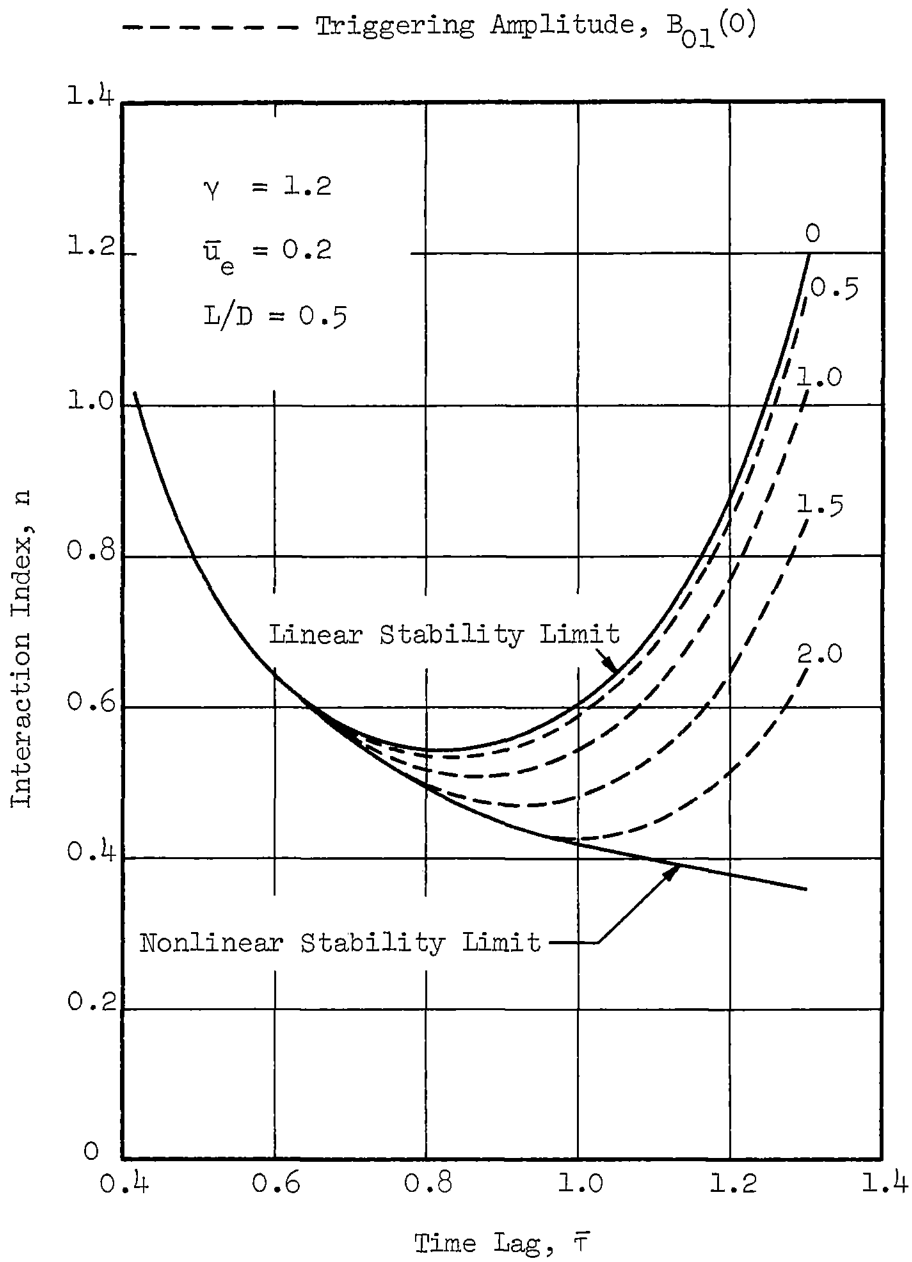


Figure 8. Nonlinear stability map for first radial mode disturbances.

Effect of Motor Parameters

Using the three-mode series, numerical calculations were made to determine the effect of the motor operating parameters $\bar{u}_e$ and L/D upon the final oscillation attained in a linearly unstable motor. Figure (9) shows the variation of limit-cycle amplitude with $\bar{u}_e$ for a fixed value of L/D for a standing 1T mode. These curves show that, for most cases, an increase in the steady state velocity, $\bar{u}_e$, results in a larger limit-cycle amplitude. For a fixed value of $\bar{u}_e$, Fig. (10) shows that the pressure amplitude decreases with increasing length.

SUMMARY AND CONCLUSIONS

An analytical technique has been developed for the solution of nonlinear combustion instability problems that are frequently associated with liquid-propellant rocket motors. Comparison of the analytical nonlinear results with the nonlinear results obtained during rocket motor instabilities show good qualitative agreement. The analysis can produce the nonlinear wave shape generally observed during unstable motor operation, the limit-cycle amplitude and frequency that characteristically typify unstable motor operation, and a threshold amplitude required to trigger a linearly stable motor into unstable operation.

The technique uses the Method of Weighted Residuals to reduce the solution of the original system of partial differential conservation equations to the solution of a system of ordinary differential equations. These ordinary differential equations describe the transient behavior of the amplitudes of the combustion chamber acoustic modes. Following the computed transient behavior of the chamber's mode-amplitudes provides the investigator with a description of the nonlinear phenomenon under consideration.

Calculated results establish the relationship that exists between the resulting instability (i.e., waveform, final amplitude, and final frequency), the combustion parameters, and the chamber Mach number and length-to-diameter ratio. These results indicate that the amplitude of a limit-cycle oscillation in a linearly unstable motor increases with increasing sensitivity of the

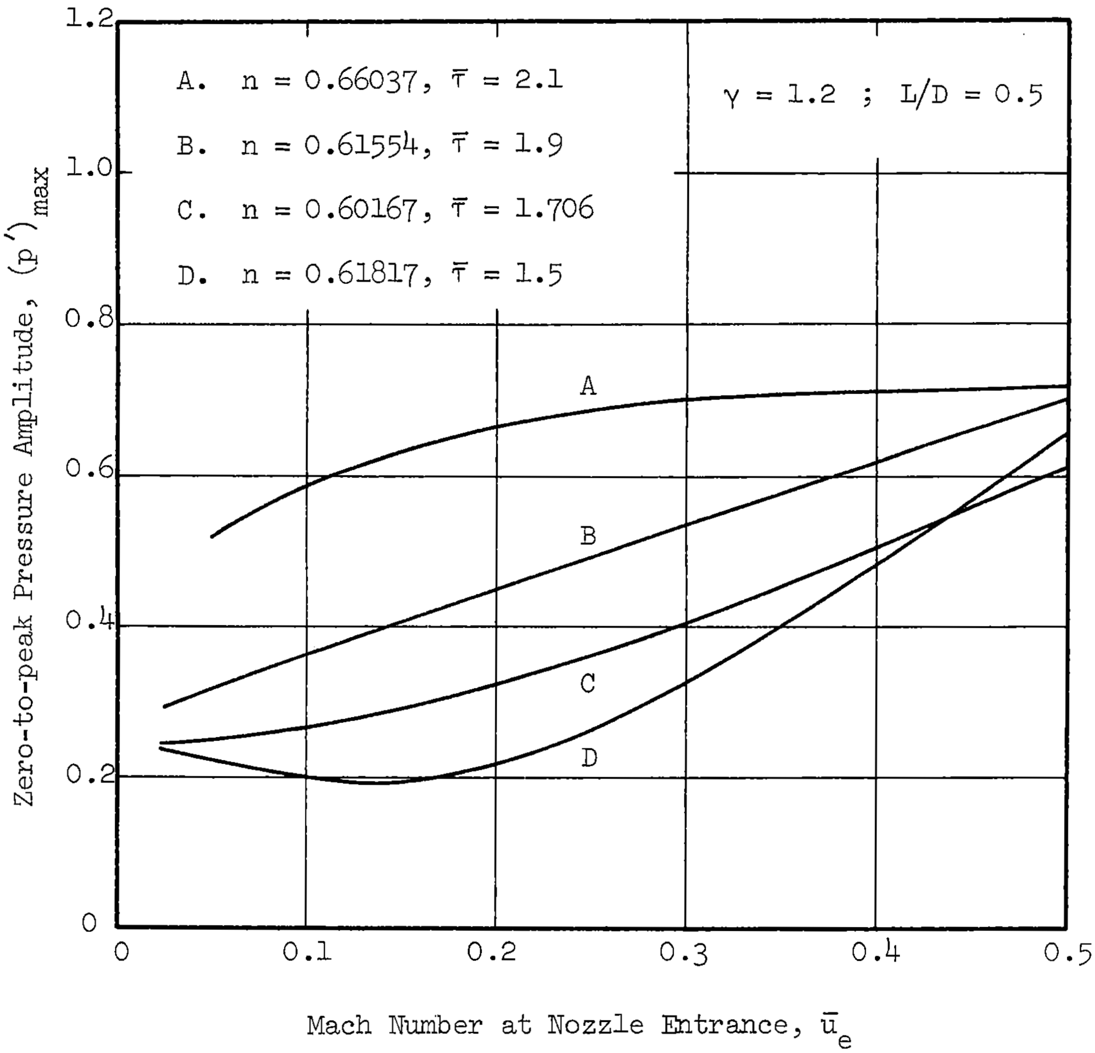


Figure 9. Effect of chamber Mach number on limit-cycle amplitude for a standing mode.

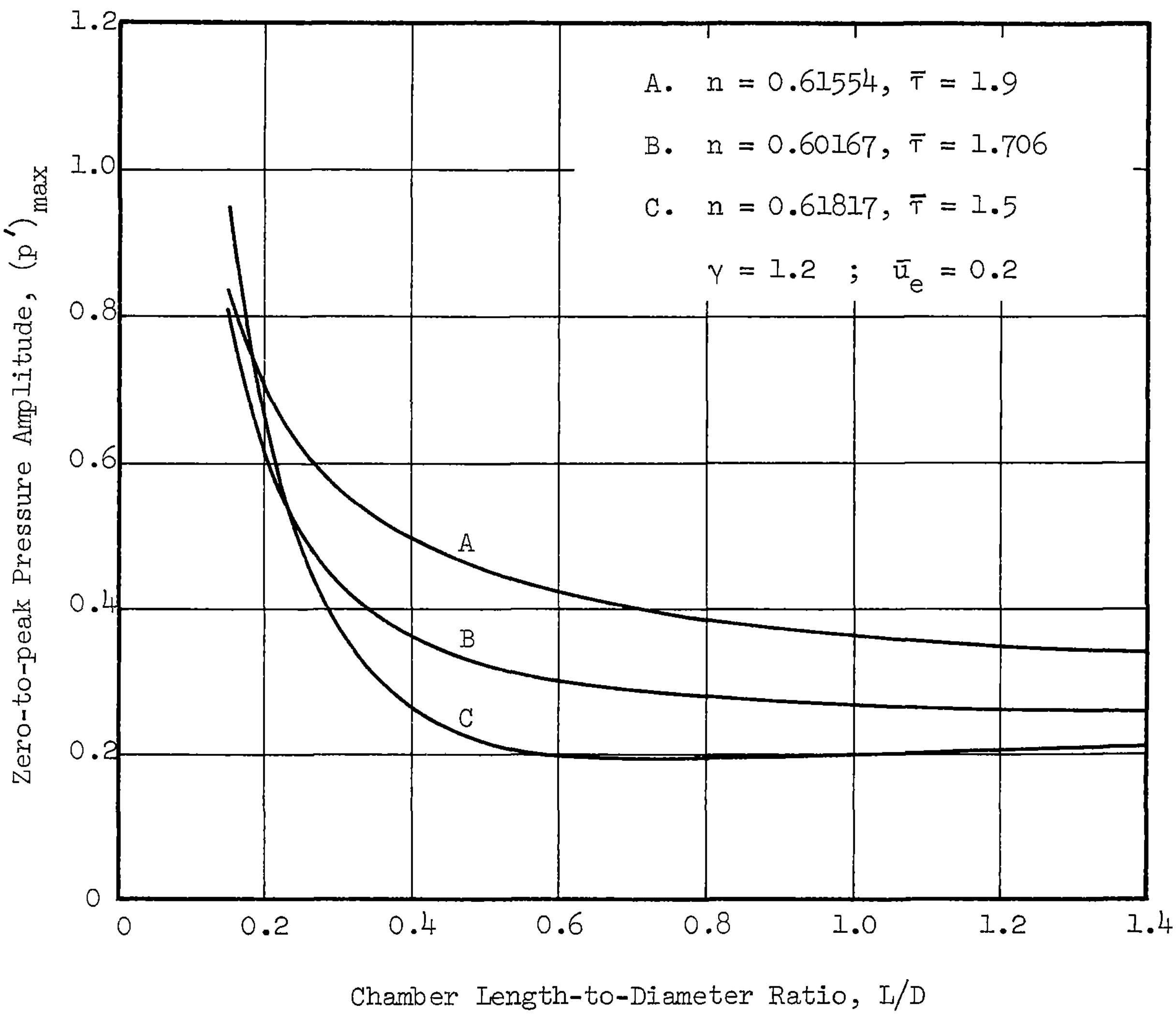


Figure 10. Effect of chamber length-to-diameter ratio on limit-cycle amplitude for a standing mode.

combustion process to pressure oscillations. The frequency of oscillation is always within a few percent of the frequency of the acoustic first tangential mode. For most cases an increase in the steady state velocity or a decrease in the chamber length-to-diameter ratio results in a larger amplitude oscillation. Calculated pressure waveforms exhibit sharp peaks and shallow minima.

Both listings and instructions for the use of the computer programs that were developed during the course of these investigations are provided in the appendices of this report.

APPENDIX A

PROGRAM NLCOEF: TO DETERMINE THE COEFFICIENTS OF THE NONLINEAR TERMS

Statement of the Problem

Program NLCOEF calculates the coefficients of the nonlinear terms which appear in Eqs. (9) and (10). These coefficients are required as input for Program LIMCYC which integrates this system of equations. The coefficients that are required depend on the choice of terms to be included in the series solution for $\tilde{\Phi}$ (see Eq. (6)), therefore this information must be provided as input to Program NLCOEF. The output of Program NLCOEF is punched onto cards for input to Program LIMCYC.

The coefficients to be calculated are functions of various integrals of trigonometric and Bessel functions and are given by the following expressions:

$$C_1(m,n;\ \mu,\nu;\ j,k) = \frac{1}{\pi}\frac{2S_{jk}^2}{\left[S_{jk}^2 - j^2\right]J_j^2\left(S_{jk}\right)} \times$$

$$\times \Big\{2\Big[S_{mn}S_{\mu\nu}I_{css}(\mu,m,j)\,Ji_3(m,n;\ \mu,\nu;\ j,k) -$$

$$- m\mu I_{css}(m,\mu,j)\,Ji_2(m,n;\ \mu,\nu;\ j,k)\Big] -$$

$$- (\gamma-1)S_{mn}^2 I_{css}(\mu,m,j)\,Ji_1(m,n;\ \mu,\nu;\ j,k)\Big\} \tag{A1a}$$

$$C_2(m,n;\ \mu,\nu;\ j,k) = \frac{1}{\pi}\frac{2S_{jk}^2}{\left[S_{jk}^2 - j^2\right]J_j^2\left(S_{jk}\right)} \times$$

$$\times \Big\{2\Big[S_{mn}S_{\mu\nu}I_{css}(m,\mu,j)\,Ji_3(m,n;\ \mu,\nu;\ j,k) -$$

$$- m\mu I_{css}(\mu,m,j)\, Ji_2(m,n;\ \mu,\nu;\ j,k)\Big] -$$

$$- (\gamma-1) S_{mn}^2 I_{css}(m,\mu,j)\, Ji_1(m,n;\ \mu,\nu;\ j,k)\Big\} \qquad \text{(A1b)}$$

$$C_3(m,n;\ \mu,\nu;\ j,k) = \frac{1}{N_j \pi} \frac{2S_{jk}^2}{\left[S_{jk}^2 - j^2\right] J_j^2\left(S_{jk}\right)} \times$$

$$\times \Big\{2\Big[S_{mn} S_{\mu\nu} I_{css}(j,m,\mu)\, Ji_3(m,n;\ \mu,\nu;\ j,k) +$$

$$+ m\mu I_{ccc}(m,\mu,j)\, Ji_2(m,n;\ \mu,\nu;\ j,k)\Big] -$$

$$- (\gamma-1) S_{mn}^2 I_{css}(j,m,\mu)\, Ji_1(m,n;\ \mu,\nu;\ j,k)\Big\} \qquad \text{(A1c)}$$

$$C_4(m,n;\ \mu,\nu;\ j,k) = \frac{1}{N_j \pi} \frac{2S_{jk}^2}{\left[S_{jk}^2 - j^2\right] J_j^2\left(S_{jk}\right)} \times$$

$$\times \Big\{2\Big[S_{mn} S_{\mu\nu} I_{ccc}(m,\mu,j)\, Ji_3(m,n;\ \mu,\nu;\ j,k) +$$

$$+ m\mu I_{css}(j,m,\mu)\, Ji_2(m,n;\ \mu,\nu;\ j,k)\Big] -$$

$$- (\gamma-1) S_{mn}^2 I_{ccc}(m,\mu,j)\, Ji_1(m,n;\ \mu,\nu;\ j,k)\Big\} \qquad \text{(A1d)}$$

where the number N_j is defined as follows:

$$N_j = 1 \quad \text{for} \quad j \neq 0$$

$$N_j = 2 \quad \text{for} \quad j = 0$$

The integrals appearing in Eqs. (A1) are defined as follows:

$$
\begin{aligned}
I_{ccc}(m,\mu,j) &= \int_0^{2\pi} \cos m\theta \cos \mu\theta \cos j\theta d\theta \\
I_{css}(m,\mu,j) &= \int_0^{2\pi} \cos m\theta \sin \mu\theta \sin j\theta d\theta \\
I_{css}(\mu,m,j) &= \int_0^{2\pi} \cos \mu\theta \sin m\theta \sin j\theta d\theta \\
I_{css}(j,m,\mu) &= \int_0^{2\pi} \cos j\theta \sin m\theta \sin \mu\theta d\theta
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
Ji_1(m,n;\ \mu,\nu;\ j,k) &= \int_0^1 J_m(S_{mn}r)\, J_\mu(S_{\mu\nu}r)\, J_j(S_{jk}r)\, r dr \\
Ji_2(m,n;\ \mu,\nu;\ j,k) &= \int_0^1 J_m(S_{mn}r)\, J_\mu(S_{\mu\nu}r)\, J_j(S_{jk}r)\, \frac{1}{r} dr \\
Ji_3(m,n;\ \mu,\nu;\ j,k) &= \int_0^1 J_m'(S_{mn}r)\, J_\mu'(S_{\mu\nu}r)\, J_j(S_{jk}r)\, r dr
\end{aligned}
\tag{A3}
$$

where m, μ, and j are tangential mode numbers and n, ν, and k are radial mode numbers and all are non-negative integers. Each of the coefficients given by Eqs. (A1) multiplies a term of the form $F_{mn} \frac{dG_{\mu\nu}}{dt}$ in Eqs. (9) and (10) where F and G are the appropriate mode-amplitude functions, either A(t) or B(t). Thus j is the tangential mode number of the equation in which a particular coefficient appears, and k is the corresponding radial mode number; m is the tangential mode number of the factor which is not differentiated (i.e., F_{mn}), and n is the radial mode number of this factor; while μ is the tangential mode number of the differentiated factor (i.e., $\frac{dG_{\mu\nu}}{dt}$), and ν is the radial mode number of the differentiated factor.

Structure of the Numerical Solution

A flow chart for Program NLCOEF is shown in Fig. (A-1). Subroutine

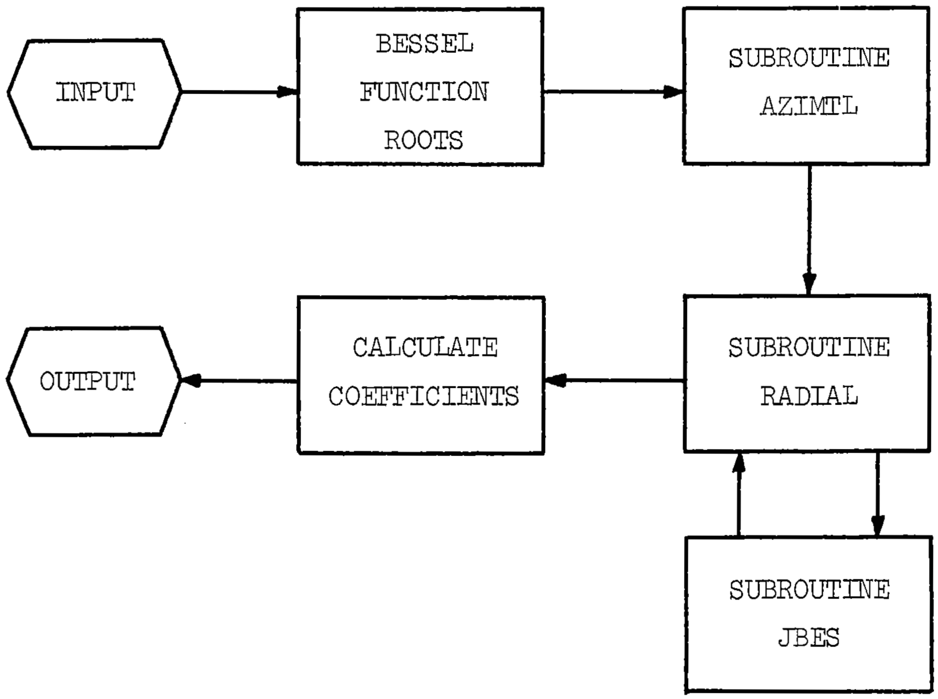


Figure A-1. Flow Chart.

AZIMTL calculates the azimuthal integrals given by Eqs. (A2). Subroutine RADIAL computes the radial integrals given by Eqs. (A3). Subroutine JBES calculates the values of the Bessel functions appearing in the integrands of Eqs. (A3). Program NLCOEF computes the coefficients according to Eqs. (A1).

Input Data

The input data consists of the ratio of specific heats and information indicating which modes are included in the approximate series expansion.

The coefficients given by Eqs. (A1) are of four types (i.e., C_1, C_2, C_3, and C_4) and require six indices (i.e., m, n, μ, ν, j, and k). These are reduced to one type of coefficient described by three indices as follows. The terms to be included in the series expansion are arranged in some order and numbered consecutively beginning with one. The order of this sequence is not important except when both the "A" functions and the "B" functions in Eq. (6) are included; in this case these quantities should occur in pairs with the "A" function first. If only one type of function is used (standing waves only) it must be a "B" function. Each term in the series is then identified by its position in this sequence given by the integer variable J. The nature of each term is specified by the three integers M(J), N(J), and NAB(J), and each term is given a three character name CNAME(J). In this manner the coefficients of the nonlinear terms in Eqs. (9) and (10) are identified by the integers J associated with the three modes involved rather than the corresponding azimuthal and radial mode numbers.

The following comments pertain to the detailed description of the input. The location number refers to columns of the card. Three formats are used for input: "A" indicates alphanumeric characters, "I" indicates integers, and "F" indicates real numbers with a decimal point. For the "I" and "F" formats the values are placed in fields of five locations, and the numbers must be placed in the rightmost locations of the allocated field.

Card	Location	Type	Input Item	Comments
1	1-72	A	TITLE	Title of case.
2	1-5	F	GAMMA	Ratio of specific heats.
	6-10	I	JMAX	Number of terms in series expansion.
3 through 2 + JMAX	1-5	I	J	Order of series term in sequence (identification number).
	6-10	I	M(J)	Tangential mode number ($0 \le M(J) \le 8$).
	11-15	I	N(J)	Radial mode number ($1 \le N(J) \le 5$).
	16-20	I	NAB(J)	For "A" function NAB(J) = 0. For "B" function NAB(J) = 1.

Card	Location	Type	Input Item	Comments
	21-22	Blank		
	23-25	A	CNAME(J)	Name of term (3 characters).

The proper input for program NLCOEF will be illustrated with the following example. Suppose the velocity potential Φ is expressed in terms of the first tangential (1T), the second tangential (2T), and the first radial (1R) modes. It is also desired to investigate instability of the spinning type, therefore both "A" functions and "B" functions are included in the series. However for the 1R mode ($m = 0$) there is no corresponding "A" function, therefore the resulting series will contain five terms. A sample input for this case is given below:

Table A-1. Sample Input

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
T	H	R	E	E		M	O	D	E	S			1	R	,	1	T	,	2	T				F	I	V	E		T	E	R	M	S						
		1	.	2					5																														
				1					0					1					1			B	0	1															
				2					1					1					0			A	1	1															
				3					1					1					1			B	1	1															
				4					2					1					0			A	2	1															
				5					2					1					1			B	2	1															

Roots of Bessel Functions

Roots of the Bessel functions S_{mn} which give zero slope at $r = 1$ and the associated values $J_m(S_{mn})$ are needed for these calculations. These values were taken from Ref. (11) for m = 0,1,...8 and n = 1,2,...5; they are automatically put into the program by means of a DATA statement, which is an integral part of the program.

Azimuthal Integrals

The azimuthal integrals are easily evaluated analytically. For most values of m, μ, and j they are zero. The nonzero integrals are given as follows:

$$I_{ccc}(m,\mu,j) = \pi/2 \quad \text{for} \quad j = m + \mu,\ m = \mu + j,\ \text{or}\ \mu = m + j$$

$$I_{css}(m,\mu,j) = \pi/2 \quad \text{for} \quad \mu = m + j\ \text{or}\ j = m + \mu$$

$$I_{css}(m,\mu,j) = -\pi/2 \quad \text{for} \quad m = \mu + j$$

$$I_{css}(\mu,m,j) = \pi/2 \quad \text{for} \quad m = \mu + j\ \text{or}\ j = m + \mu \tag{A4}$$

$$I_{css}(\mu,m,j) = -\pi/2 \quad \text{for} \quad \mu = m + j$$

$$I_{css}(j,m,\mu) = \pi/2 \quad \text{for} \quad m = j + \mu\ \text{or}\ \mu = j + m$$

$$I_{css}(j,m,\mu) = -\pi/2 \quad \text{for} \quad j = m + \mu$$

for m, μ, and j nonzero. If any one of the indices is zero (corresponding to a radial mode) the following values are obtained:

$$I_{ccc}(0,0,0) = 2\pi$$

$$I_{ccc}(m,\mu,0) = I_{ccc}(m,0,j) = I_{ccc}(0,\mu,j) = \pi \tag{A5}$$

$$I_{css}(0,\mu,j) = I_{css}(0,m,j) = I_{css}(0,m,\mu) = \pi$$

only if $m = \mu$, $m = j$, or $\mu = j$.

Subroutine AZIMTL is essentially a series of logical tests to determine if the indices m, μ, and j satisfy any of the conditions of Eqs. (A4) and (A5). If any of these conditions is satisfied the appropriate value is assigned to the output variable, otherwise the value zero is assigned.

Radial Integrals

Subroutine RADIAL evaluates the radial integrals given by Eqs. (A3). These integrals are computed numerically using Simpson's Rule with 100 subdivisions. In calculating the integrands the derivatives of the Bessel functions are given by:

$$J_m'\left(S_{mn}r\right) = \frac{1}{2}\left[J_{m-1}\left(S_{mn}r\right) - J_{m+1}\left(S_{mn}r\right)\right] \quad \text{for} \quad m = 1,2,3,\ldots$$
$$J_0'\left(S_{mn}r\right) = -J_1\left(S_{mn}r\right) \tag{A6}$$

The integrand of Ji_2 is indeterminate at the lower limit of integration. However a limit exists, denoted by L, which vanishes with the following exceptions:

$$L = S_{mn}/2 \quad \text{for} \quad m = 1,\ \mu = j = 0$$
$$L = S_{\mu\nu}/2 \quad \text{for} \quad \mu = 1,\ m = j = 0 \tag{A7}$$
$$L = S_{jk}/2 \quad \text{for} \quad j = 1,\ m = \mu = 0$$

All of the calculations in Subroutine RADIAL are carried out in double precision arithmetic. The results are given as a single precision number.

Bessel Function Subroutine

Subroutine JBES computes double precision Bessel functions for non-negative orders and arguments. A description of this subroutine and a program listing are given in Chapter 23 of Ref. (12).

Output Data

Program NLCOEF produces both printed output and punched cards. This output consists of three principal sections.

(1) The ratio of specific heats, GAMMA; the number of terms in the series expansion of Φ, JMAX; and the number of nonzero nonlinear coefficients generated, NLMAX are given.

(2) A restatement of the input parameters J, M(J), N(J), NAB(J) and CNAME(J) which describe the terms in the series expansion of Φ is given. Two additional parameters needed by Program LIMCYC are also given: S_{mn}, the dimensionless frequency of the mode (nth nonzero root of $J_m'(x) = 0$), denoted by S(J) and $J_m(S_{mn})$, the associated value of the Bessel function, denoted by SJ(J). In the punched version of this output the radial mode number N(J) is omitted as it is not needed by LIMCYC.

(3) The nonlinear coefficients are given. Each coefficient is identified by the integers I, J, and K as follows: I is the identification number of the term in the series expansion whose linear behavior is controlled by the equation in which the coefficient appears (corresponding to j and k in Eqs. (9) and (10)), J is the identification number of the factor which is not differentiated (indices m and n in Eqs. (9) and (10)), and K is the identification number of the time derivative factor (indices μ and ν in Eqs. (9) and (10)). Following these integers the value of the coefficient, C(I, J, K) is given to five decimal places.

A sample output for the five term series used in the sample input is given in Tables (A-2) and (A-3).

Table A-2. Sample Output

THREE MODES 1R,1T,2T FIVE TERMS

GAMMA = 1.20 JMAX = 5 NLMAX = 25

J	M	N	NAB	SMN	JM(SMN)	NAME
1	0	1	1	3.83171	-.40276	B01
2	1	1	0	1.84118	.58187	A11
3	1	1	1	1.84118	.58187	B11
4	2	1	0	3.05424	.48650	A21
5	2	1	1	3.05424	.48650	B21

Table A-3. Sample Output

I	J	K	C(I,J,K)
1	1	1	4.13771
1	2	2	1.04231
1	3	3	1.04231
1	4	4	-.20839
1	5	5	-.20839
2	1	2	-1.93938
2	2	1	-2.31228
2	2	5	-1.71871
2	3	4	1.71871
2	4	3	1.48273
2	5	2	-1.48273
3	1	3	-1.93938
3	2	4	1.71871
3	3	1	-2.31228
3	3	5	1.71871
3	4	2	1.48273
3	5	3	1.48273
4	1	4	-2.78489
4	2	3	-1.13183
4	3	2	-1.13183
4	4	1	-3.03876
5	1	5	-2.78489
5	2	2	1.13183
5	3	3	-1.13183
5	5	1	-3.03876

FORTRAN Listing

```
C      ************* PROGRAM NLCOEF **********************************
C
C           THIS PROGRAM COMPUTES THE NONLINEAR COEFFICIENTS WHICH
C      APPEAR IN THE DIFFERENTIAL EQUATIONS WHICH GOVERN THE
C      MODE-AMPLITUDE FUNCTIONS.   THESE COEFFICIENTS ARE PUNCHED
C      ONTO CARDS FOR INPUT INTO PROGRAM LIMCYC.
C
C      THE FOLLOWING INPUTS ARE REQUIRED:
C      THE TITLE OF THE CASE.
C      GAMMA IS THE RATIO OF SPECIFIC HEATS.
C      JMAX IS THE NUMBER OF MODE-AMPLITUDE FUNCTIONS IN THE ASSUMED
C      SERIES SOLUTION.   JMAX MUST NOT EXCEED 20.
C      EACH MODE-AMPLITUDE IS ASSIGNED AN INTEGER J.
C      THE MODE IS SPECIFIED BY THE INDICES M(J) AND N(J).
C      M(J) IS THE AZIMUTHAL MODE NUMBER AND MUST NOT EXCEED 8.
C      N(J) IS THE RADIAL MODE NUMBER AND MUST NOT EXCEED 5.
C      THE INTEGER NAB(J) IS ASSIGNED AS FOLLOWS:
C         NAB(J) = 0  A-FUNCTION (COEFFICIENT OF SIN(M*THETA))
C         NAB(J) = 1  B-FUNCTION (COEFFICIENT OF COS(M*THETA))
C      CNAME(J)  IS A THREE-CHARACTER NAME
C
C
      DIMENSION     COEF(3), TERM(3), S(20), SJ(20), CNAME(20),
     1              M(20), N(20), NAB(20), TITLE(72),
     2              RJROOT(10,5), RJVAL(10,5), C(20,20,20)
C
      PI = 3.1415927
C     INPUT ROOTS OF BESSEL FUNCTIONS
      DATA ((RJROOT(I,J), J = 1,5), I = 1,9)/
     1   3.83171,  7.01559, 10.17347, 13.32369, 16.47063,
     2   1.84118,  5.33144,  8.53632, 11.70600, 14.86359,
     3   3.05424,  6.70613,  9.96947, 13.17037, 16.34752,
     4   4.20119,  8.01524, 11.34592, 14.58585, 17.78875,
     5   5.31755,  9.28240, 12.68191, 15.96411, 19.19603,
     6   6.41562, 10.51986, 13.98719, 17.31284, 20.57551,
     7   7.50127, 11.73494, 15.26818, 18.63744, 21.93172,
     8   8.57784, 12.93239, 16.52937, 19.94185, 23.26805,
     9   9.64742, 14.11552, 17.77401, 21.22906, 24.58720/
      DATA ((RJVAL(I,J), J = 1,5), I = 1,9)/
     1  -0.40276,  0.30012, -0.24970,  0.21836, -0.19647,
     2   0.58187, -0.34613,  0.27330, -0.23330,  0.20701,
     3   0.48650, -0.31353,  0.25474, -0.22088,  0.19794,
     4   0.43439, -0.29116,  0.24074, -0.21097,  0.19042,
     5   0.39965, -0.27438,  0.22959, -0.20276,  0.18403,
     6   0.37409, -0.26109,  0.22039, -0.19580,  0.17849,
     7   0.35414, -0.25017,  0.21261, -0.18978,  0.17363,
     8   0.33793, -0.24096,  0.20588, -0.18449,  0.16929,
     9   0.32438, -0.23303,  0.19998, -0.17979,  0.16539/
C
C     INPUT DATA
      READ (5,5002)  (TITLE(I), I = 1, 72)
      READ (5, 5000) GAMMA, JMAX
      DO 10 I = 1, JMAX
      READ (5, 5001) J, M(J), N(J), NAB(J), CNAME(J)
   10 CONTINUE
C
```

```
      DO 40 J = 1, JMAX
      MM = M(J) + 1
      NN = N(J)
      S(J) = RJROOT(MM,NN)
      SJ(J) = RJVAL(MM,NN)
   40 CONTINUE
C
C     ZERO COEFFICIENT ARRAY
      DO 20 I = 1, JMAX
      DO 20 J = 1, JMAX
      DO 20 K = 1, JMAX
      C(I,J,K) = 0.0
   20 CONTINUE
C
C     COMPUTE NONZERO NONLINEAR COEFFICIENTS
C
      NLMAX = 0
      DO 30 I = 1, JMAX
C     COMPUTE NORMALIZING FACTOR
      RJ = 1.0
      IF (M(I) .EQ. 0) RJ = 2.0
      SSQ = S(I) * S(I)
      SQJ = M(I) * M(I)
      SJSQ = SJ(I) * SJ(I)
      FACTOR = (2.0 * SSQ)/((SSQ - SQJ) * SJSQ * PI * RJ)
      DO 30 J = 1, JMAX
      DO 30 K = 1, JMAX
C     TEST FOR ZERO VALUES
      IF ((NAB(I) .EQ. 0) .AND. (NAB(J) .EQ. NAB(K)))  GO TO 30
      IF ((NAB(I) .EQ. 1) .AND. (NAB(J) .NE. NAB(K)))  GO TO 30
C     COMPUTE COEF(NT)
      C2 = M(J) * M(K)
      COEF(1) = 2.0 * S(J) * S(K)
      COEF(2) = 2.0 * C2
      COEF(3) = (GAMMA - 1.0) * S(J) * S(J)
C     COMPUTE ARGUMENTS FOR RADIAL INTEGRALS
      L1 = M(J)
      L2 = M(K)
      L3 = M(I)
      A1 = S(J)
      A2 = S(K)
      A3 = S(I)
C     COMPUTE TERMS
      DO 35 NT = 1, 3
C     ASSIGN ARGUMENTS FOR AZIMUTHAL INTEGRALS
      NOPT = 2
      IF (NAB(I) .EQ. 0) GO TO 101
      IF (NAB(I) .EQ. 1) GO TO 103
  101 NC = M(I)
      NA = M(K)
      NB = M(J)
      IF ((NAB(J) .EQ. 0) .AND. (NT .EQ. 2)) GO TO 102
      IF ((NAB(J) .EQ. 1) .AND. (NT .NE. 2)) GO TO 102
      GO TO 104
  102 NA = M(J)
      NB = M(K)
      GO TO 104
  103 NA = M(I)
      NB = M(J)
      NC = M(K)
```

```
      IF ((NAB(J) .EQ. 0) .AND. (NT .EQ. 2))  NOPT = 1
      IF ((NAB(J) .EQ. 1) .AND. (NT .NE. 2))  NOPT = 1
  104 CONTINUE
C     COMPUTE AZIMUTHAL INTEGRAL
      CALL AZIMTL(NOPT,NA,NB,NC,TANINT)
      IF (TANINT) 110, 115, 110
  115 TERM(NT) = 0.0
      GO TO 35
C     COMPUTE RADIAL INTEGRALS
  110 NOPT = 4 - NT
      CALL RADIAL(NOPT,L1,L2,L3,A1,A2,A3,RADINT)
C     COMPUTE TERM(NT)
      TERM(NT) = COEF(NT) * TANINT * RADINT
      IF ((NAB(I) .EQ. 1) .AND. (NT .EQ. 2)) TERM(NT) = -TERM(NT)
   35 CONTINUE
C     COMPUTE COEFFICIENT
      C(I,J,K) = FACTOR * (TERM(1) - TERM(2) - TERM(3))
      IF (C(I,J,K)) 31, 30, 31
   31 NLMAX = NLMAX + 1
   30 CONTINUE
C
C     OUTPUT OF RESULTS
      WRITE (6,6000)
      WRITE (6,6006) (TITLE(I), I = 1, 72)
      PUNCH 5002 (TITLE(I), I = 1, 72)
      WRITE (6,6001)  GAMMA, JMAX, NLMAX
      PUNCH 7000  GAMMA, JMAX, NLMAX
C
      WRITE (6, 6002)
      DO 70 J = 1, JMAX
      WRITE (6,6003) J, M(J), N(J), NAB(J), S(J), SJ(J), CNAME(J)
      PUNCH  7001  J, M(J), NAB(J), S(J), SJ(J), CNAME(J)
   70 CONTINUE
C
      WRITE (6,6000)
      WRITE (6,6004)
      DO 75 I = 1, JMAX
      DO 75 J = 1, JMAX
      DO 75 K = 1, JMAX
      IF (C(I,J,K)) 71, 75, 71
   71 WRITE (6, 6005) I, J, K, C(I,J,K)
      PUNCH  7002  I, J, K, C(I,J,K)
   75 CONTINUE
C
 5000 FORMAT (F5.0,I5)
 5001 FORMAT (4I5,2X,A3)
 5002 FORMAT (72A1)
 6000 FORMAT (1H1)
 6001 FORMAT (5X,8HGAMMA = ,F5.2,5X,8HJMAX   = ,I2,5X,9HNLMAX   = ,I3//)
 6002 FORMAT (6X,17HJ    M     N    NAB,7X,3HSMN,5X,7HJM(SMN),6X,4HNAME/)
 6003 FORMAT (2X,4I5,3X,2F10.5,6X,A3)
 6004 FORMAT (6X,25HI    J    K       C(I,J,K)/)
 6005 FORMAT (2X,3I5,F13.5)
 6006 FORMAT (5X,72A1//)
 7000 FORMAT (F5.2,2I5)
 7001 FORMAT (3I5,2F10.5,7X,A3)
 7002 FORMAT (3I5,F10.5)
      END
```

```
      SUBROUTINE AZIMTL(NOPT,L,M,N,RESULT)
C
C     THIS SUBROUTINE COMPUTES THE INTEGRAL OVER THE INTERVAL
C     (0,2*PI) OF THE FOLLOWING PRODUCT OF SINES AND COSINES:
C
C     NOPT = 1  COS(L*THETA) * COS(M*THETA) * COS(N*THETA)
C
C     NOPT = 2  COS(L*THETA) * SIN(M*THETA) * SIN(N*THETA)
C
C     WHERE L, M, AND N ARE NON-NEGATIVE INTEGERS.
C
      RESULT = 0.0
      PI = 3.1415927
      IF ((L .NE. 0) .AND. (M .NE. 0) .AND. (N .NE. 0))  GO TO 101
      GO TO 103
  101 LM = L + M
      LN = L + N
      MN = M + N
      IF ((N .EQ. LM) .OR. (M .EQ. LN))  RESULT = PI/2.0
      IF (L .EQ. MN)  GO TO 102
      GO TO 104
  102 IF (NOPT .EQ. 1)  RESULT = PI/2.0
      IF (NOPT .EQ. 2)  RESULT = -PI/2.0
      GO TO 104
  103 IF ((L .EQ. 0) .AND. (M .EQ. 0) .AND. (N .EQ. 0))  GO TO 105
      IF ((NOPT .EQ. 1) .AND. (N .EQ. 0) .AND. (L .EQ. M)) RESULT = PI
      IF ((NOPT .EQ. 1) .AND. (M .EQ. 0) .AND. (L .EQ. N)) RESULT = PI
      IF ((L .EQ. 0) .AND. (M .EQ. N))  RESULT = PI
      GO TO 104
  105 IF (NOPT .EQ. 1)  RESULT = 2.0 * PI
  104 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE RADIAL(NOPT,L,M,N,A,B,C,RESULT)
C
C     THIS SUBROUTINE CALCULATES THE INTEGRAL OVER THE INTERVAL
C     (0,1) OF THE FOLLOWING PRODUCTS OF THREE BESSEL FUNCTIONS:
C
C     NOPT = 1  JL(A*R) * JM(B*R) * JN(C*R) * R
C
C     NOPT = 2  JL(A*R) * JM(B*R) * JN(C*R)/R
C
C     NOPT = 3  JPL(A*R) * JPM(B*R) * JN(C*R) * R
C
C     JL  IS THE BESSEL FUNCTION OF FIRST KIND OF ORDER L
C     JPL IS THE DERIVATIVE OF JL WITH RESPECT TO ITS ARGUMENT
C     L, M, N  ARE NON-NEGATIVE INTEGERS
C     A, B, C  ARE REAL NUMBERS
C
      DIMENSION  FUNCT(200)
      DOUBLE PRECISION  DN, DH, DSTEP, DR, ARG1, ARG2, ARG3,
     1                  BES1, BES2, BES3, BESH, BESL, PROD,
     2                  FUNCT, BESLIM, S1, S2, S3
C
      NN = 100
      DN = NN
      DH = 1.0/DN
      NP1 = NN + 1
C
      DO 10 I = 1, NP1
      DSTEP = I - 1
      DR = DH * DSTEP
      ARG1 = A * DR
      ARG2 = B * DR
      ARG3 = C * DR
C
      CALL JBES(N,ARG3,BES3,$500)
      IF (NOPT .EQ. 3)  GO TO 101
      CALL JBES(L,ARG1,BES1,$500)
      CALL JBES(M,ARG2,BES2,$500)
      GO TO 102
  101 IF (L .EQ. 0)  GO TO 103
      CALL JBES(L+1,ARG1,BESH,$500)
      CALL JBES(L-1,ARG1,BESL,$500)
      BES1 = (BESL - BESH)/2.0
      GO TO 104
  103 CALL JBES(1,ARG1,BES1,$500)
      BES1 = -BES1
  104 IF (M .EQ. 0)  GO TO 105
      CALL JBES(M+1,ARG2,BESH,$500)
      CALL JBES(M-1,ARG2,BESL,$500)
      BES2 = (BESL - BESH)/2.0
      GO TO 102
  105 CALL JBES(1,ARG2,BES2,$500)
      BES2 = -BES2
  102 PROD = BES1 * BES2 * BES3
C
      IF (NOPT .EQ. 2)  GO TO 110
      FUNCT(I) = PROD * DR
      GO TO 10
  110 IF (I .EQ. 1)  GO TO 111
      FUNCT(I) = PROD/DR
      GO TO 10
```

```
  111 BESLIM = 0.0
      IF ((L.EQ.1) .AND. (M.EQ.0) .AND. (N.EQ.0))  BESLIM = A/2.0
      IF ((L.EQ.0) .AND. (M.EQ.1) .AND. (N.EQ.0))  BESLIM = B/2.0
      IF ((L.EQ.0) .AND. (M.EQ.0) .AND. (N.EQ.1))  BESLIM = C/2.0
      FUNCT(I) = BESLIM
   10 CONTINUE
C
      NM1 = NN - 1
      S1 = FUNCT(1) + FUNCT(NP1)
      S2 = 0.0
      S3 = 0.0
      DO 20 I = 2, NN, 2
      S2 = S2 + FUNCT(I)
   20 CONTINUE
      DO 30 I = 3, NM1, 2
      S3 = S3 + FUNCT(I)
   30 CONTINUE
      RESULT = DH * (S1 + 4.0*S2 + 2.0*S3)/3.0
      GO TO 501
  500 WRITE (6, 6000)
 6000 FORMAT (1H1,10HERROR JBES)
  501 CONTINUE
      RETURN
      END
```

APPENDIX B

PROGRAM LIMCYC: A USER'S MANUAL

General Description

Using the second order theory described in this report Program LIMCYC calculates the nonlinear stability characteristics of a cylindrical combustion chamber. For given values of the operating parameters (i.e., n, $\bar{\tau}$, γ, $\bar{u}_e$, and L/D), a given series expansion, and a given initial disturbance Program LIMCYC integrates Eqs. (9) and (10) to obtain the time behavior of the unknown mode-amplitude functions (i.e., A_{jk} and B_{jk}). From this information a time history of the pressure oscillation is determined. The program determines: (1) the final amplitude of the pressure oscillation attained in a linearly unstable engine (i.e., limit-cycle amplitude) and (2) the threshold amplitude above which a finite amplitude disturbance can "trigger" instability in a linearly stable engine (i.e., triggering limits). In addition the shape of the nonlinear pressure waveforms and the frequency of oscillation is calculated. A flow chart for this program is given in Fig. (B-1).

Integration of the Differential Equations

For purposes of numerical integration Eqs. (9) and (10) are written as an equivalent system of first order differential equations as follows:

$$\frac{dA_{jk}}{dt} = A'_{jk}$$

$$\frac{dA'_{jk}}{dt} = - S^2_{jk}A_{jk} - KA'_{jk} - K_{\tau}A'_{jk}(t - \bar{\tau})$$

$$- \sum_{m,n} \sum_{\mu,\nu} \left\{C_1(m,n;\mu,\nu;j,k)A_{mn}B'_{\mu\nu} + C_2(m,n;\mu,\nu;j,k)B_{mn}A'_{\mu\nu}\right\} \qquad \text{(B1)}$$

$$\frac{dB_{jk}}{dt} = B'_{jk}$$

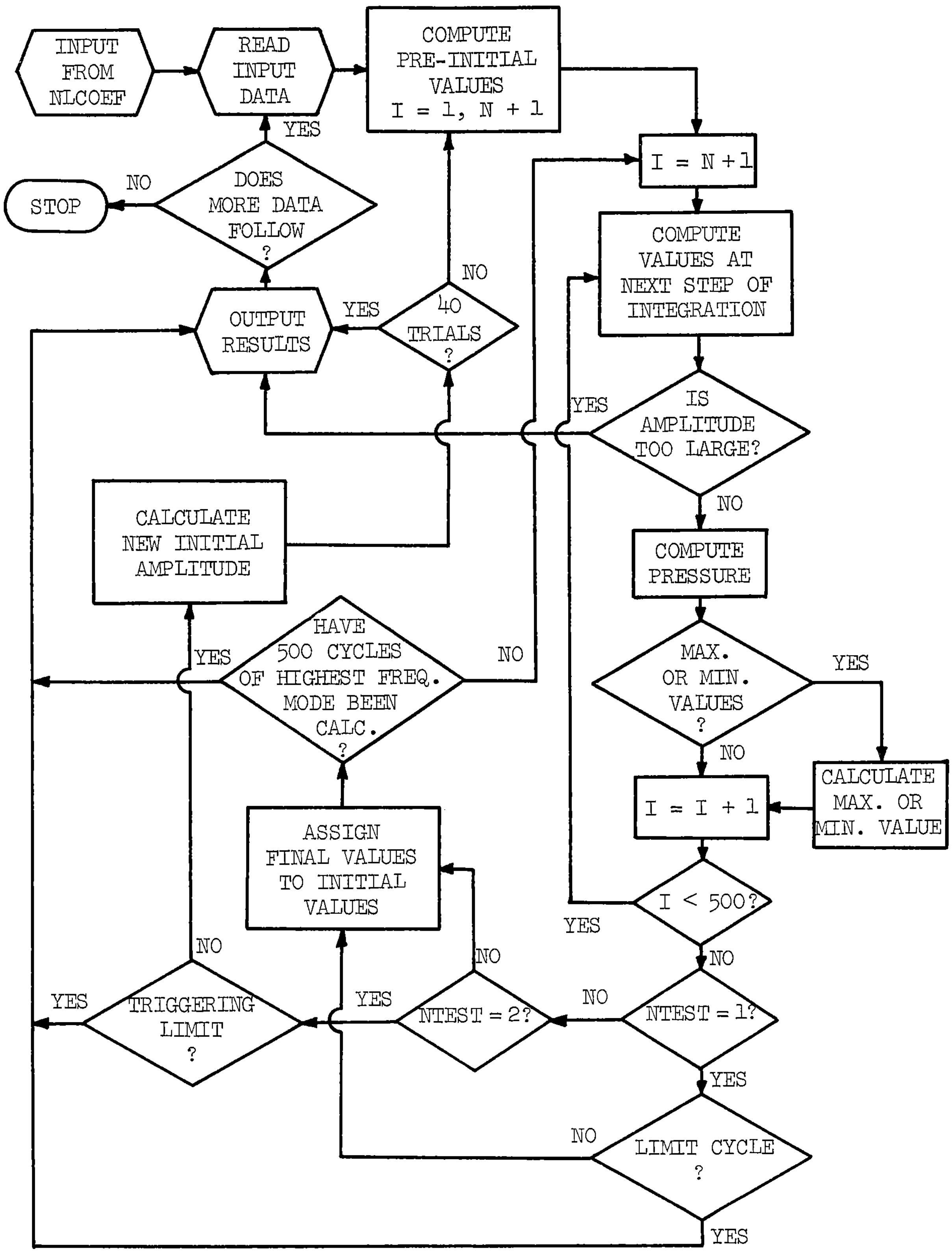


Figure B-1. Flow Chart

$$\frac{dB'_{jk}}{dt} = - S^2_{jk}B_{jk} - KB'_{jk} - K_{\tau}B'_{jk}(t - \bar{\tau})$$

$$- \sum_{m,n} \sum_{\mu,\nu} \Big\{ C_3(m,n;\mu,\nu;j,k) A_{mn}A'_{\mu\nu} + C_4(m,n;\mu,\nu;j,k) B_{mn}B'_{\mu\nu} \qquad \text{(B2)}$$

where the dependent variables are now A_{jk}, A'_{jk}, B_{jk}, and B'_{jk}. These equations are solved numerically using the fourth order Runge-Kutta method. Due to the presence of retarded variables in Eqs. (B1) and (B2) the formulas (see Ref. 13) used in the Runge-Kutta method must be slightly modified.

The appropriate formulas for applying the Runge-Kutta method to problems involving a time-delay are readily obtained by considering a single equation of the following form:

$$\frac{dx}{dt} = f(x,t) + g[x(t - \bar{\tau})] \qquad \text{(B3)}$$

Noting that at any step of the integration the value of $x(t - \bar{\tau})$ has already been determined from previous steps the function g can be considered to be a known function of time $g(t)$.

Since $x(t)$ is computed only at discrete points $x_n(t_n)$ it is desired that the retarded variable $x(t_n - \bar{\tau})$ will coincide with such previously computed points. This can be accomplished by choosing the step-size h such that it divides the time-lag $\bar{\tau}$ into k equal increments. Thus $\bar{\tau} = kh$ and the Runge-Kutta formulas which apply to Eq. (B3) can now be written as:

$$x_{n+1} = x_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = h\left\{f(x_n,t_n) + g(x_{n-k})\right\} \qquad \text{(B4)}$$

$$k_2 = h\left\{f(x_n + k_1/2,\ t_n + h/2) + g(x_{n-k+\frac{1}{2}})\right\}$$

$$k_3 = h\left\{f(x_n + k_2/2,\ t_n + h/2) + g(x_{n-k+\frac{1}{2}})\right\}$$

$$k_4 = h\left\{f(x_n + k_3,\ t_n + h) + g(x_{n-k+1})\right\} \qquad (B4)$$

Equations (B4) are readily extended to handle the system of equations given by Eqs. (B1) and (B2). It is seen from Eqs. (B4) that k values of the dependent variables prior to the initial values are needed to start the integration.

Although the initial wave shape can be an arbitrary function of time, it is assumed that initially the mode-amplitudes are sinusoidal functions of time oscillating with the natural frequency S_{jk}. Thus each mode-amplitude function is expressed in the following form:

$$A_{jk}(t) = C_{jk}\sin(S_{jk}t) + D_{jk}\cos(S_{jk}t)$$

$$A'_{jk}(t) = S_{jk}\left[C_{jk}\cos(S_{jk}t) - D_{jk}\sin(S_{jk}t)\right] \qquad (-\bar{\tau} \leq t \leq 0) \quad (B5)$$

and similar expressions hold for $B_{jk}(t)$ and $B'_{jk}(t)$.

Input Data

A precise definition of the input data required to run the computer program is given below. This input data consists of two parts: (1) the parameters and coefficients generated by Program NLCOEF, and (2) the data describing the cases to be run (see Fig. (B-1)). For each input case the following information must be provided: (1) the combustion parameters n and $\bar{\tau}$, the motor parameters $\bar{u}_e$ and L/D; (2) a series of control numbers; and (3) information describing the initial disturbance.

For each input case two control numbers NTEST and ITYPE must be specified. The task to be performed by Program LIMCYC is specified by NTEST (see Fig. (B-1)). If NTEST = 1 the program searches for a limit-cycle amplitude, while if NTEST = 2 the program tests for a triggering limit. If NTEST = 3 the transient behavior (growth or decay) of the pressure oscillation is determined.

The integer ITYPE specifies the form of the initial disturbance. For ITYPE = 1 the initial disturbance is a single standing mode described by:

$$\left.\begin{aligned} A_{jk}(t) &= 0 \\ B_{jk}(t) &= A\cos(S_{jk}t) \end{aligned}\right\} \quad (-\bar{\tau} \le t \le 0) \qquad \text{(B6)}$$

If ITYPE = 2 the initial disturbance is a spinning oscillation given by:

$$\left.\begin{aligned} A_{jk}(t) &= A\sin(S_{jk}t) \\ B_{jk}(t) &= A\cos(S_{jk}t) \end{aligned}\right\} \quad (-\bar{\tau} \le t \le 0) \qquad \text{(B7)}$$

In the above two cases only the mode initially present and its amplitude, A, are specified, and the initial amplitudes of all of the other modes included in the series expansion are zero. If ITYPE = 3 the initial disturbance is described by Eqs. (B5), and the amplitudes C_{jk} and D_{jk} must be specified for each of the modes present in the initial disturbance.

The data describing the cases to be run immediately follows the coefficient deck generated by Program NLCOEF. The following comments pertain to the detailed description of this input. The location number refers to the columns of the card. Three formats are used for input: "A" indicates alphanumeric characters, "I" indicates integers, and "F" indicates real numbers with a decimal point. For the "I" formats the values are placed in fields of five locations, while a field of ten locations is used with the "F" formats. In either case the numbers must be placed in the rightmost locations of the allocated field.

Card	Location	Type	Input Item	Comments
1	1-72	A	TITLE	Title of case.
2	1-10	F	EN	The interaction index, n (EN≥0).
	11-20	F	TAU	The dimensionless steady state value of the time-lag, $\bar{\tau}$ (TAU>0).
	21-30	F	UE	Steady state Mach number at nozzle entrance, $\bar{u}_e$ (UE≥0).
	31-40	F	RLD	Chamber length-to-diameter ratio, L/D (RLD>0).
3	1-5	I	NTEST	Control number which specifies task to be performed (NTEST = 1, 2, 3 only).
	6-10	I	ITYPE	Control number which specifies type of initial disturbance (ITYPE = 1, 2, 3 only).
	11-20	F	TQUIT	Time interval for which step-by-step output of pressure waveforms is desired (TQUIT≥0).

If ITYPE = 1 or ITYPE = 2 (single mode initial disturbance):

Card	Location	Type	Input Item	Comments
4	1-5	I	MODE	The identification number of the "B" function corresponding to the mode initially present (see Appendix A).
	6-15	F	AMPL	Amplitude of the initial disturbance, A (see Eqs. (B6) and (B7)).

End of input for ITYPE = 1 or ITYPE = 2.

If ITYPE = 3 (multi-mode initial disturbance):

Card	Location	Type	Input Item	Comments
4	1-5	I	MODE	Identification number of the "principal" series term (i.e., the function upon which the tests for limit cycles are performed).
	6-10	I	NTERMS	Number of series terms necessary to describe the initial disturbance.
5 (NTERMS cards)	1-5	I	J	Identification number of the series term (1≤J≤NTERMS).

Card	Location	Type	Input Item	Comments
	6-15	F	AS(J)	Amplitude of sine component, C_{jk} in Eqs. (B5).
	16-25	F	AC(J)	Amplitude of cosine component, D_{jk} in Eqs. (B5).

End of input for ITYPE = 3.

If NTEST = 3 an additional card is needed:

6	1-5	I	LSTCYC	Output begins after LSTCYC cycles of the principal series term (LSTCYC≥0).

The proper input for Program LIMCYC will be illustrated with the following example. Assuming that the velocity potential Φ is expressed in terms of the 1R, 1T, and 2T modes*, it is desired to investigate the nonlinear behavior of a linearly unstable engine ($n = 0.60167$, $\bar{\tau} = 1.70629$, $\bar{u}_e = 0.2$, $L/D = 0.5$) for various types of initial disturbances. Sample input for three cases will be given: (1) a 1T standing disturbance is initially present, (2) a 1T spinning disturbance is initially present, and (3) the initial disturbance consists of a radial mode of amplitude 0.2, a spinning 1T mode of amplitude 0.5, and a 2T spinning mode of amplitude 0.2. In the first two cases a limit-cycle amplitude is sought, while in the last case only the transient behavior is desired. In each case the principal series term is $B_{11}(t)$, from the sample output of Program NLCOEF it is seen that MODE = 3. For the transient case it is specified that the output begins after 100 cycles of $B_{11}(t)$.

To run the cases described above the data deck must be assembled as follows. The first item is the coefficient deck produced by Program NLCOEF, in this example it contains the information given in the sample output for NLCOEF shown in Appendix A. Since these coefficients depend only on the series expansion and γ, but not upon the combustion parameters n, $\bar{\tau}$, $\bar{u}_e$, and L/D, they need only be computed once. The coefficient deck is followed by the data for the cases to be run as shown in the sample input below:

---

* This is the same series expansion used to illustrate Program NLCOEF.

Table B-1. Sample Input

```
CASE 1:  LIMIT-CYCLE AMPLITUDE, 1T STANDING MODE.
   0.60167   1.70629       0.2       0.5
    1    1      10.0
    3       0.3
CASE 2:  LIMIT-CYCLE AMPLITUDE, 1T SPINNING MODE.
   0.60167   1.70629       0.2       0.5
    1    2      10.0
    3       0.5
CASE 3:  TRANSIENT, ALL MODES INITIALLY PRESENT.
   0.60167   1.70629       0.2       0.5
    3    3      10.0
    3    5
    1       0.0       0.2
    2       0.5       0.0
    3       0.0       0.5
    4       0.2       0.0
    5       0.0       0.2
  100
END OF RUN
       0.0       0.0       0.0       0.0
```

The data shown above ends with "signal cards" to indicate that no more data follows.

Calculation of Pressure

From the calculated time dependence of the series terms Program LIMCYC computes the dimensionless pressure perturbation with the aid of Eqs. (6) and (12). The pressure is calculated at three angular positions along the periphery of the chamber (i.e., $r = 1$, $\theta = 0, 45^\circ, 90^\circ$).

Maximum and Minimum Values

In order to determine the triggering limits and limit-cycle amplitudes it is necessary to follow the growth or decay of the amplitudes of the series terms and the pressure perturbation. The amplitude of the series terms is defined as the average of the maximum and minimum values occurring during one cycle. For the pressure, maximum, minimum, and peak-to-peak values are calculated. Since the variables are calculated only at discrete points, the maximum and minimum values are computed using a three-point interpolation scheme.

Calculation of Limit-Cycle Amplitude

A limit-cycle amplitude is calculated by specifying an initial disturbance and continuing the step-by-step integration of Eqs. (B1) and (B2) until a periodic solution is obtained; that is, the amplitude of the oscillation remains essentially constant. The test for convergence to a limit cycle is performed upon a single series term, usually the most important term in the series, in the following manner. After the first 500 integration steps, usually about 10 cycles for the 1T mode, the amplitude of the principal series term A_1 is compared with its initial amplitude A_0. If the change in amplitude $|A_1-A_0|$ is greater than the maximum permissible change ϵ, the calculations are continued and the change in amplitude during the next 500 integration steps is calculated. The process is repeated until $|A_k-A_{k-1}| < \epsilon$ at which point the computation is terminated. A value of $\epsilon = 0.0002$ is used in Program LIMCYC which gives sufficient accuracy for most cases.

Test for Triggering Limit

If a triggering limit exists, initial disturbances with an amplitude slightly greater than the threshold amplitude will grow while slightly smaller initial disturbances will decay. Therefore the triggering amplitude is calculated numerically by an iterative process in which the initial amplitude is varied until a solution is obtained which does not grow or decay appreciably during an initial time interval of about 20 cycles. The test for a triggering limit is also performed upon only one term in the series expansion.

A detailed description of the iteration process will now be given. An initial disturbance must be specified for the first trial. The initial conditions for all subsequent trials are obtained by multiplying the initial disturbance for the first trial by a variable factor C_k where k is the number of the trial. For each trial the amplitude of the principal mode is computed after 500 integration steps (A_1) and after 1000 integration steps (A_2) and the difference $\Delta A_k = A_2 - A_1$ is used to determine the growth or decay of this mode. The value of C_{k+1} for the next trial is calculated from the following formula:

$$C_{k+1} = C_k - \frac{\Delta A_k}{|\Delta A_k|} \delta_k \tag{B8}$$

where

$$\begin{aligned} \delta_k &= \delta_{k-1} \quad \text{if} \quad (\Delta A_k)(\Delta A_{k-1}) > 0 \\ & \qquad\qquad\qquad\qquad (k \geq 2) \\ \delta_k &= \delta_{k-1}/2 \quad \text{if} \quad (\Delta A_k)(\Delta A_{k-1}) < 0 \end{aligned} \tag{B9}$$

and the initial values are $C_1 = 1.0$ and $\delta_1 = 0.1$. Thus Eq. (B8) shows that if the oscillation decays (i.e., $\Delta A_k < 0$) the initial amplitude was too small and the factor C_k is increased by the increment δ_k, while if $\Delta A_k > 0$ the factor C_k is decreased by the same amount. In order to converge upon the desired triggering limit, the increment δ_k is halved, according to Eqs. (B9), each time the threshold is crossed (i.e., when ΔA_k and ΔA_{k-1} have opposite signs). The iterations are terminated when $\delta_k < 0.001$ and the last value of C_{k+1} computed is used to generate the desired solution. If no triggering limit exists the iterative process ends after 40 trials or when the initial amplitude of the trial solution vanishes (i.e., $C_k = 0$).

Output Data

The output data for Program LIMCYC consists of six sections as shown below.

Section 1 is a restatement of the input from Program NLCOEF. It includes the following information: (a) a title describing the series expansion employed, (b) the ratio of specific heats, GAMMA; the number of terms in the series, JMAX; and the number of nonzero nonlinear coefficients, NLMAX, (c) the parameters which describe and identify each term in the series expansion, and (d) the nonzero nonlinear coefficients.

Section 2 is a restatement of the input parameters for the case under investigation. It includes the following information: (a) the title of the case, (b) the operating conditions (i.e., interaction index, EN; time-lag, TAU; the ratio of specific heats, GAMMA; the steady flow Mach number at the nozzle entrance, UE; and the chamber length-to-diameter ratio, RLD), and (c) a statement of the initial disturbance assumed.

Section 3 is given for triggering limits (NTEST = 2). This section presents the results for each step in the iteration procedure. This includes the initial amplitude of the mode being tested, the amplitude after 500 steps, A_1, the amplitude after 1000 steps, A_2, and the growth or decay, $A_2 - A_1$.

Section 4 gives the characteristics of the limit cycle or triggering limit. For each series term the maximum and minimum values occurring during a cycle, the period of oscillation, and the frequency are given. For $\theta = 0, 45^\circ$, and 90° the maximum, minimum, and peak-to-peak values of the dimensionless wall pressure perturbation are given.

In Section 5 pressure vs. time waveforms for $\theta = 0, 45^\circ$, and 90° are given.

In Section 6 for transient runs (NTEST = 3) all extreme values (maxima and minima) of the principal series term and the pressure perturbation are given.

For each input case the output (Sections 2-6) appears in the order given above with the following exceptions: (1) if NTEST $\neq$ 2 Section 3 is omitted, and (2) if NTEST = 3 Section 5 precedes Section 4.

A description of the output according to the value of NTEST will now be given. For NTEST = 1 the program searches for a limit-cycle amplitude. In most cases the output begins (1) after the solution converges to a limit cycle or (2) after approximately 500 cycles of the highest frequency mode in the series have been computed. In the case of convergence the output consists only of Sections 2, 4, and 5. The data given in Section 4 pertains to the last few cycles calculated. The time at which either of conditions (1) or (2) is satisfied is taken as t = 0 and the calculations are continued to t = TQUIT to obtain the pressure waveforms given in Section 5. If convergence to a limit cycle is not obtained, a statement to that effect appears in Section 4 along with the number of cycles computed and the growth or decay rate. In the latter case all maxima and minima of the mode being tested are given in Section 6.

For NTEST = 2 the program searches for a triggering limit. If a triggering limit is found the output consists of Sections 2 through 6. The data given in Sections 4 and 5 corresponds to the initial few cycles of the oscillation obtained with the last trial value of the initial amplitude. The calculations are continued for several hundred cycles to obtain the data for Section 6; this data is included to determine how rapidly the solution diverges from the triggering limit. If no triggering limit exists the search ends when (1) forty trials have been made or (2) the initial amplitude approaches zero (which will occur in the linearly unstable region in the $(n,\bar{\tau})$ plane). In either case a message stating that no triggering limit was found is given in Section 3 along with the number of trials made and the initial amplitude for the last trial. Of course, if no triggering limit is found, Sections 4 and 5 are omitted, while Section 6 is given for the last trial if the initial amplitude is not zero.

If NTEST = 3 no test for limit cycles or triggering limits is made. Data output begins after LSTCYC cycles of the mode specified by MODE and ends several hundred cycles later. The amount of output is approximately 500 cycles of the highest frequency mode in the series expansion. For the pressure waveforms given in Section 5, t = 0 after LSTCYC cycles and output continues until t = TQUIT. The characteristics of the solution for the last few cycles calculated are given in Section 4, and all maxima and minima of the pressure and the principal mode-amplitude function are given in Section 6.

Under certain situations the amplitude of the solution increases very rapidly ("blows up") after only a few cycles. Therefore computations are terminated any time the amplitude of any of the series terms exceeds 20. A message is given stating the time at which the solution "blows up" and the initial amplitude of the principal mode. If NTEST $\neq$ 2 this is followed by maximum and minimum values of the principal mode (and pressure if NTEST = 3). For NTEST = 2 a solution "blow up" has the same effect as a normal growth; that is, a smaller value of the initial amplitude is taken for the next trial. If a "blow up" occurs on the last trial, however, the output is similar to that for NTEST = 1.

Sample Output

The following sample output illustrates the behavior of Program LIMCYC for different values of NTEST and ITYPE. The three cases given are the output generated from the sample input given previously. In these cases a three mode series expansion was used.

PROGRAM LIMCYC
SECOND ORDER NONLINEAR COMBUSTION INSTABILITY PROGRAM

THREE MODE SERIES: 1R, 1T, 2T   FIVE TERMS

INPUTS FROM PROGRAM NLCOEF

GAMMA = 1.20   JMAX = 5   NLMAX = 25

J	M	NAB	SMN	JM(SMN)	NAME
1	0	1	3.83171	-.40276	B01
2	1	0	1.84118	.58187	A11
3	1	1	1.84118	.58187	B11
4	2	0	3.05424	.48650	A21
5	2	1	3.05424	.48650	B21

I	J	K	C(I,J,K)
1	1	1	4.13771
1	2	2	1.04231
1	3	3	1.04231
1	4	4	-.20839
1	5	5	-.20839
2	1	2	-1.93938
2	2	1	-2.31228
2	2	5	-1.71871
2	3	4	1.71871
2	4	3	1.48273
2	5	2	-1.48273
3	1	3	-1.93938
3	2	4	1.71871
3	3	1	-2.31228
3	3	5	1.71871
3	4	2	1.48273
3	5	3	1.48273
4	1	4	-2.78489
4	2	3	-1.13183
4	3	2	-1.13183
4	4	1	-3.03876
5	1	5	-2.78489
5	2	2	1.13183
5	3	3	-1.13183
5	5	1	-3.03876

CASE 1: LIMIT-CYCLE AMPLITUDE, 1T STANDING MODE.

```
COMBUSTION PARAMETERS:  INTERACTION INDEX =  .60167          TIME-LAG = 1.70629
MOTOR PARAMETERS:                   GAMMA = 1.20000   EXIT MACH NUMBER =  .20000
                                                       LENGTH/DIAMETER =  .50000
```

CALCULATE LIMIT-CYCLE AMPLITUDE.

INITIAL CONDITIONS FOR -TAU < T < 0

SINGLE STANDING MODE INITIALLY PRESENT
(PRESSURE ANTINODE AT THETA = 0)

INITIAL AMPLITUDE OF B11 IS .30000

B11(T) = .30000 * COS(1.84118*T)


```
COMBUSTION PARAMETERS:  INTERACTION INDEX =  .60167          TIME-LAG = 1.70629
MOTOR PARAMETERS:                   GAMMA = 1.20000   EXIT MACH NUMBER =  .20000
                                                       LENGTH/DIAMETER =  .50000
```


LIMIT-CYCLE AMPLITUDE REACHED AFTER 81 CYCLES OF B11

J	MODE	MAXIMUM	MINIMUM	PERIOD	FREQUENCY
1	B01	.03208	-.03212	1.71548	3.66263
3	B11	.22972	-.22972	3.43102	1.83129
5	B21	.01288	-.01306	1.71548	3.66265

THETA (DEGREES)	WALL PRESSURE PERTURBATION MAXIMUM	MINIMUM	PEAK-TO-PEAK
.0	.32234	-.28146	.60380
45.0	.19553	-.23381	.42934
90.0	.06996	-.07518	.14515

COMBUSTION PARAMETERS: INTERACTION INDEX = .60167 TIME-LAG = 1.70629
MOTOR PARAMETERS: GAMMA = 1.20000 EXIT MACH NUMBER = .20000
LENGTH/DIAMETER = .50000

WALL PRESSURE WAVEFORMS

STEP	TIME	0 DEGREES	45 DEGREES	90 DEGREES
0	.00000	.13692	.09873	-.00552
1	.06563	.11115	.08831	.01171
2	.13125	.08595	.07730	.02829
3	.19688	.06129	.06534	.04322
4	.26251	.03697	.05201	.05554
5	.32813	.01273	.03693	.06442
6	.39376	-.01175	.01977	.06922
7	.45939	-.03670	.00039	.06956
8	.52501	-.06222	-.02118	.06538
9	.59064	-.08827	-.04468	.05695
10	.65627	-.11461	-.06962	.04483
11	.72189	-.14084	-.09532	.02985
12	.78752	-.16646	-.12098	.01301
13	.85314	-.19085	-.14573	-.00462
14	.91877	-.21341	-.16871	-.02195
15	.98440	-.23356	-.18914	-.03799
16	1.05002	-.25074	-.20632	-.05188
17	1.11565	-.26450	-.21969	-.06291
18	1.18128	-.27441	-.22880	-.07060
19	1.24690	-.28012	-.23333	-.07460
20	1.31253	-.28133	-.23304	-.07478
21	1.37816	-.27775	-.22778	-.07116
22	1.44378	-.26916	-.21751	-.06393
23	1.50941	-.25535	-.20227	-.05342
24	1.57504	-.23622	-.18224	-.04014
25	1.64066	-.21178	-.15774	-.02471
26	1.70629	-.18219	-.12926	-.00794
27	1.77192	-.14783	-.09749	.00931
28	1.83754	-.10931	-.06333	.02604
29	1.90317	-.06746	-.02780	.04126
30	1.96880	-.02333	.00792	.05400
31	2.03442	.02190	.04266	.06341
32	2.10005	.06697	.07530	.06880
33	2.16568	.11068	.10486	.06978
34	2.23130	.15192	.13058	.06623
35	2.29693	.18978	.15199	.05837
36	2.36256	.22355	.16890	.04673
37	2.42818	.25273	.18136	.03209
38	2.49381	.27699	.18966	.01545
39	2.55943	.29614	.19420	-.00214
40	2.62506	.31008	.19548	-.01957
41	2.69069	.31877	.19401	-.03585
42	2.75631	.32224	.19029	-.05008
43	2.82194	.32059	.18475	-.06155
44	2.88757	.31399	.17778	-.06973
45	2.95319	.30275	.16972	-.07427
46	3.01882	.28732	.16085	-.07499
47	3.08445	.26827	.15142	-.07189
48	3.15007	.24630	.14165	-.06515

CASE 2: LIMIT-CYCLE AMPLITUDE, 1T SPINNING MODE.

COMBUSTION PARAMETERS: INTERACTION INDEX = .60167 TIME-LAG = 1.70629
MOTOR PARAMETERS: GAMMA = 1.20000 EXIT MACH NUMBER = .20000
LENGTH/DIAMETER = .50000

CALCULATE LIMIT-CYCLE AMPLITUDE.

INITIAL CONDITIONS FOR -TAU < T < 0

SINGLE SPINNING MODE INITIALLY PRESENT
MOVING COUNTERCLOCKWISE (THETA INCREASING)

INITIAL AMPLITUDE OF A11 AND B11 IS .50000

A11(T) = .50000 * SIN(1.84118*T)

B11(T) = .50000 * COS(1.84118*T)

COMBUSTION PARAMETERS: INTERACTION INDEX = .60167 TIME-LAG = 1.70629
MOTOR PARAMETERS: GAMMA = 1.20000 EXIT MACH NUMBER = .20000
LENGTH/DIAMETER = .50000

LIMIT-CYCLE AMPLITUDE REACHED AFTER 84 CYCLES OF B11

J	MODE	MAXIMUM	MINIMUM	PERIOD	FREQUENCY
1	B01	.00000	.00000	1.72285	3.64697
2	A11	.42680	-.42680	3.29487	1.90696
3	B11	.42681	-.42680	3.29485	1.90697
4	A21	.07576	-.07577	1.64740	3.81400
5	B21	.07576	-.07577	1.64743	3.81392

THETA (DEGREES)	WALL PRESSURE PERTURBATION MAXIMUM	MINIMUM	PEAK-TO-PEAK
.0	.89914	-.38056	1.27970
45.0	.89915	-.38071	1.27986
90.0	.89912	-.38066	1.27978

COMBUSTION PARAMETERS: INTERACTION INDEX = .60167 TIME-LAG = 1.70629
MOTOR PARAMETERS: GAMMA = 1.20000 EXIT MACH NUMBER = .20000
LENGTH/DIAMETER = .50000

WALL PRESSURE WAVEFORMS

STEP	TIME	0 DEGREES	45 DEGREES	90 DEGREES
0	.00000	-.38028	-.34896	-.32151
1	.06563	-.37577	-.35457	-.32969
2	.13125	-.36486	-.36117	-.33469
3	.19688	-.34612	-.36812	-.33792
4	.26251	-.31811	-.37449	-.34057
5	.32813	-.27949	-.37910	-.34359
6	.39376	-.22910	-.38060	-.34763
7	.45939	-.16606	-.37759	-.35291
8	.52501	-.08997	-.36859	-.35928
9	.59064	-.00110	-.35215	-.36622
10	.65627	.09943	-.32683	-.37286
11	.72189	.20944	-.29127	-.37808
12	.78752	.32568	-.24423	-.38058
13	.85314	.44386	-.18475	-.37895
14	.91877	.55885	-.11226	-.37175
15	.98440	.66503	-.02684	-.35751
16	1.05002	.75672	.07064	-.33478
17	1.11565	.82872	.17832	-.30218
18	1.18128	.87679	.29323	-.25841
19	1.24690	.89810	.41135	-.20244
20	1.31253	.89151	.52777	-.13356
21	1.37816	.85770	.63695	-.05166
22	1.44378	.79906	.73319	.04264
23	1.50941	.71942	.81109	.14775
24	1.57504	.62370	.86611	.26102
25	1.64066	.51735	.89499	.37872
26	1.70629	.40587	.89611	.49614
27	1.77192	.29440	.86963	.60789
28	1.83754	.18730	.81751	.70826
29	1.90317	.08800	.74321	.79171
30	1.96880	-.00113	.65140	.85343
31	2.03442	-.07870	.54742	.88975
32	2.10005	-.14421	.43682	.89859
33	2.16568	-.19790	.32487	.87961
34	2.23130	-.24048	.21619	.83427
35	2.29693	-.27307	.11445	.76566
36	2.36256	-.29699	.02234	.67817
37	2.42818	-.31372	-.05850	.57699
38	2.49381	-.32478	-.12735	.46768
39	2.55943	-.33170	-.18424	.35561
40	2.62506	-.33594	-.22978	.24561
41	2.69069	-.33884	-.26500	.14164
42	2.75631	-.34152	-.29117	.04668
43	2.82194	-.34485	-.30974	-.03738
44	2.88757	-.34930	-.32221	-.10956
45	2.95319	-.35500	-.33012	-.16971
46	3.01882	-.36164	-.33495	-.21830
47	3.08445	-.36859	-.33810	-.25625
48	3.15007	-.37488	-.34075	-.28478

CASE 3: TRANSIENT, ALL MODES INITIALLY PRESENT.

```
COMBUSTION PARAMETERS:  INTERACTION INDEX =  .60167          TIME-LAG = 1.70629
MOTOR PARAMETERS:                   GAMMA = 1.20000   EXIT MACH NUMBER =  .20000
                                                      LENGTH/DIAMETER =  .50000
```

CALCULATE TRANSIENT BEHAVIOR.

INITIAL CONDITIONS FOR -TAU < T < 0

ARBITRARY COMBINATION OF MODES INITIALLY PRESENT

```
B01(T) =   .20000 * COS( 3.83171*T)

A11(T) =   .50000 * SIN( 1.84118*T)

B11(T) =   .50000 * COS( 1.84118*T)

A21(T) =   .20000 * SIN( 3.05424*T)

B21(T) =   .20000 * COS( 3.05424*T)
```

DATA OUTPUT BEGINS AFTER 100 CYCLES OF B11(T)

COMBUSTION PARAMETERS: INTERACTION INDEX = .60167 TIME-LAG = 1.70629
MOTOR PARAMETERS: GAMMA = 1.20000 EXIT MACH NUMBER = .20000
LENGTH/DIAMETER = .50000

WALL PRESSURE WAVEFORMS

STEP	TIME	0 DEGREES	45 DEGREES	90 DEGREES
0	.00000	-.17273	-.37803	-.35228
1	.06563	-.09794	-.36968	-.35856
2	.13125	-.01031	-.35403	-.36547
3	.19688	.08912	-.32965	-.37218
4	.26251	.19829	-.29515	-.37759
5	.32813	.31405	-.24929	-.38041
6	.39376	.43220	-.19106	-.37925
7	.45939	.54770	-.11988	-.37265
8	.52501	.65495	-.03573	-.35916
9	.59064	.74825	.06061	-.33733
10	.65627	.82235	.16736	-.30575
11	.72189	.87289	.28167	-.26314
12	.78752	.89690	.39964	-.20841
13	.85314	.89305	.51641	-.14083
14	.91877	.86184	.62651	-.06021
15	.98440	.80551	.72421	.03289
16	1.05002	.72777	.80410	.13701
17	1.11565	.63344	.86150	.24958
18	1.18128	.52793	.89303	.36699
19	1.24690	.41677	.89689	.48462
20	1.31253	.30514	.87309	.59713
21	1.37816	.19748	.82337	.69881
22	1.44378	.09733	.75110	.78412
23	1.50941	.00716	.66082	.84813
24	1.57504	-.07155	.55784	.88705
25	1.64066	-.13824	.44770	.89862
26	1.70629	-.19305	.33571	.88235
27	1.77192	-.23667	.22657	.83951
28	1.83754	-.27018	.12405	.77305
29	1.90317	-.29490	.03094	.68723
30	1.96880	-.31227	-.05103	.58720
31	2.03442	-.32382	-.12105	.47849
32	2.10005	-.33108	-.17908	.36651
33	2.16568	-.33553	-.22570	.25616
34	2.23130	-.33851	-.26187	.15150
35	2.29693	-.34118	-.28887	.05558
36	2.36256	-.34442	-.30812	-.02959
37	2.42818	-.34875	-.32113	-.10293
38	2.49381	-.35433	-.32942	-.16424
39	2.55943	-.36090	-.33450	-.21392
40	2.62506	-.36786	-.33777	-.25287
41	2.69069	-.37425	-.34042	-.28226
42	2.75631	-.37892	-.34343	-.30350
43	2.82194	-.38053	-.34743	-.31807
44	2.88757	-.37766	-.35267	-.32752
45	2.95319	-.36886	-.35902	-.33334
46	3.01882	-.35265	-.36595	-.33697
47	3.08445	-.32761	-.37260	-.33969
48	3.15007	-.29237	-.37788	-.34252

COMBUSTION PARAMETERS: INTERACTION INDEX = .60167 TIME-LAG = 1.70629
MOTOR PARAMETERS: GAMMA = 1.20000 EXIT MACH NUMBER = .20000
LENGTH/DIAMETER = .50000

AMPLITUDE OF B11 IS CHANGING BY -.000000 PER CYCLE AFTER 329 CYCLES

J	MODE	MAXIMUM	MINIMUM	PERIOD	FREQUENCY
1	B01	.00000	.00000	1.73775	3.61570
2	A11	.42661	-.42661	3.29497	1.90690
3	B11	.42661	-.42661	3.29495	1.90691
4	A21	.07570	-.07571	1.64746	3.81386
5	B21	.07570	-.07570	1.64751	3.81375

THETA (DEGREES)	WALL PRESSURE PERTURBATION MAXIMUM	MINIMUM	PEAK-TO-PEAK
.0	.89847	-.38043	1.27891
45.0	.89850	-.38058	1.27908
90.0	.89849	-.38053	1.27902

WALL PRESSURE PEAKS THETA = .0 DEGREES
459 VALUES COMPUTED

.8987189	-.3805723	.8986697	-.3805200	.8986860	-.3804727	.8986926	-.3806254
.8986614	-.3805958	.8986543	-.3805498	.8986177	-.3804981	.8986366	-.3804539
.8986433	-.3806105	.8986317	-.3805779	.8986054	-.3805306	.8985808	-.3804798
.8986008	-.3804391	.8986066	-.3805979	.8985938	-.3805623	.8985681	-.3805139
.8985550	-.3804643	.8985750	-.3804275	.8985791	-.3805866	.8985646	-.3805483
.8985120	-.3804991	.8985372	-.3804510	.8985562	-.3804186	.8985580	-.3805763
.8985419	-.3805354	.8984988	-.3804856	.8985253	-.3804395	.8985425	-.3805922
.8985416	-.3805665	.8985238	-.3805233	.8984909	-.3804734	.8985174	-.3804297
.8985324	-.3805866	.8985286	-.3805570	.8985092	-.3805118	.8984864	-.3804621
.8985124	-.3804213	.8985247	-.3805807	.8985179	-.3805476	.8984972	-.3805007
.8984848	-.3804517	.8985094	-.3804144	.8985186	-.3805745	.8985088	-.3805381
.8984873	-.3804900	.8984848	-.3804421	.8985076	-.3804089	.8985134	-.3805680
.8985009	-.3805286	.8984574	-.3804796	.8984862	-.3804333	.8985065	-.3804049
.8985089	-.3805610	.8984938	-.3805190	.8984596	-.3804696	.8984882	-.3804255
.8985058	-.3805816	.8985047	-.3805536	.8984874	-.3805094	.8984629	-.3804599
.8984906	-.3804185	.8985051	-.3805775	.8985006	-.3805458	.8984814	-.3804997
.8984669	-.3804507	.8984932	-.3804125	.8985043	-.3805728	.8984964	-.3805376
.8984759	-.3804900	.8984713	-.3804419	.8984955	-.3804076	.8985031	-.3805673
.8984921	-.3805290	.8984465	-.3804803	.8984758	-.3804337	.8984976	-.3804038
.8985015	-.3805612	.8984877	-.3805200	.8984510	-.3804708	.8984802	-.3804260

EXTREME VALUES OF B11(T)
OUTPUT STARTED AFTER 100 CYCLES STOPPED AFTER 329 CYCLES

.4266847	-.4266819	.4266789	-.4266761	.4266735	-.4266714	.4266700	-.4266695
.4266689	-.4266668	.4266646	-.4266622	.4266599	-.4266577	.4266558	-.4266544
.4266538	-.4266543	.4266532	-.4266515	.4266497	-.4266477	.4266458	-.4266441
.4266427	-.4266419	.4266421	-.4266428	.4266416	-.4266401	.4266385	-.4266369
.4266353	-.4266340	.4266331	-.4266329	.4266337	-.4266339	.4266329	-.4266316
.4266302	-.4266288	.4266275	-.4266266	.4266261	-.4266264	.4266279	-.4266273
.4266264	-.4266252	.4266240	-.4266228	.4266217	-.4266211	.4266211	-.4266219
.4266230	-.4266223	.4266215	-.4266204	.4266193	-.4266183	.4266175	-.4266171
.4266175	-.4266189	.4266193	-.4266186	.4266178	-.4266168	.4266158	-.4266149
.4266143	-.4266143	.4266151	-.4266169	.4266165	-.4266158	.4266150	-.4266140
.4266131	-.4266124	.4266121	-.4266124	.4266136	-.4266147	.4266143	-.4266136
.4266128	-.4266119	.4266111	-.4266105	.4266105	-.4266112	.4266129	-.4266131
.4266126	-.4266119	.4266111	-.4266103	.4266096	-.4266092	.4266094	-.4266105
.4266122	-.4266119	.4266113	-.4266106	.4266098	-.4266090	.4266085	-.4266083
.4266089	-.4266104	.4266113	-.4266109	.4266103	-.4266095	.4266087	-.4266081
.4266076	-.4266077	.4266086	-.4266106	.4266105	-.4266101	.4266095	-.4266087
.4266079	-.4266073	.4266071	-.4266075	.4266088	-.4266102	.4266100	-.4266094
.4266087	-.4266080	.4266073	-.4266068	.4266068	-.4266075	.4266092	-.4266098
.4266095	-.4266089	.4266081	-.4266074	.4266068	-.4266065	.4266067	-.4266077
.4266096	-.4266095	.4266090	-.4266083	.4266076	-.4266069	.4266064	-.4266062
.4266067	-.4266082	.4266094	-.4266091	.4266086	-.4266079	.4266071	-.4266065
.4266061	-.4266062	.4266070	-.4266089	.4266092	-.4266088	.4266082	-.4266075

FORTRAN Listing

```
C      ****************** PROGRAM LIMCYC ******************************
C
C         THIS PROGRAM CALCULATES THE NONLINEAR STABILITY CHARACTERISTICS
C      OF A CYLINDRICAL COMBUSTION CHAMBER WITH UNIFORM PROPELLANT
C      INJECTION, DISTRIBUTED COMBUSTION PROCESS, AND A QUASI-STEADY
C      NOZZLE.   THE COMBUSTION PROCESS IS DESCRIBED BY CROCCOS TIME-LAG
C      MODEL.  LIMIT-CYCLE AMPLITUDE, TRIGGERING LIMITS, AND THE
C      TRANSIENT BEHAVIOR ARE CALCULATED.
C
C      THE FOLLOWING INPUTS ARE REQUIRED:
C      (1)  THE DECK FROM PROGRAM NLCOEF.
C      (2)  THE DATA DECK.
C
C      THE DATA DECK CONSISTS OF THE FOLLOWING CARDS:
C
C      FIRST CARD:  PARAMETERS
C      EN  IS THE PRESSURE INTERACTION INDEX.
C      TAU IS THE SENSITIVE TIME-LAG.
C      UE IS THE MEAN FLOW VELOCITY AT THE NOZZLE ENTRANCE.
C      RLD  IS THE CHAMBER LENGTH-TO-DIAMETER RATIO.
C
C      SECOND CARD:  CONTROL NUMBERS
C      ACCORDING TO THE VALUE OF NTEST THE FOLLOWING
C      CALCULATIONS ARE MADE:
C      NTEST = 1  CALCULATE LIMIT-CYCLE AMPLITUDE.
C      NTEST = 2  CALCULATE TRIGGERING AMPLITUDE.
C      NTEST = 3  CALCULATE THE TRANSIENT BEHAVIOR.
C      THE FORM OF THE INITIAL CONDITIONS IS DETERMINED BY ITYPE.
C      ITYPE = 1  SINGLE STANDING MODE.
C      ITYPE = 2  SINGLE SPINNING TANGENTIAL MODE.
C      ITYPE = 3  ARBITRARY COMBINATION OF MODES.
C      TQUIT  IS  TIME INTERVAL FOR STEP BY STEP OUTPUT
C
C      THIRD CARD:  INITIAL AMPLITUDE OF SINUSOIDAL DISTURBANCE
C      IF ITYPE = 1 OR 2  (SINGLE MODE INITIALLY PRESENT):
C      MODE IDENTIFIES THE CORRESPONDING B-FUNCTION.
C      AMPL IS THE INITIAL AMPLITUDE OF THIS B-FUNCTION.
C      IF ITYPE = 3:
C      MODE IDENTIFIES THE PRINCIPAL SERIES TERM.
C      NTERMS IS THE NUMBER OF TERMS GIVEN INITIAL VALUES.
C      THIS CARD IS FOLLOWED BY NTERMS CARDS CONTAINING THE FOLLOWING
C      INFORMATION:
C      J IDENTIFIES THE SERIES TERM.
C      AS(J) IS THE AMPLITUDE OF THE SINE COMPONENT.
C      AC(J) IS THE AMPLITUDE OF THE COSINE COMPONENT.
C
C      IF NTEST = 3 AN ADDITIONAL CARD IS NEEDED.
C      LSTCYC:  DATA OUTPUT BEGINS AFTER LSTCYC CYCLES OF THE
C               PRINCIPAL SERIES TERM.
C
```

```
C     ******************************************************************
      DIMENSION  S(20), C(20,20,20), U(500,40), TI(500),
     1           T(20,4), Y(40), A(4), FZ(4,40), YP(40), UZ(40),
     2           CI(50), UMAX(20,1000), MAXNO(20), NCYC(20),
     3           UAVG(200), CF(3,40), PRESS(3,500), MAXP(3),
     4           PMAX(3,1000), TIME1(20), TIME2(20), THETA(3),
     5           CNAME(20), DELSGN(50), M(20), SJ(20), NAB(20),
     6           AS(20), AC(20), KCYC(200), ANGLE(3), TITLE(72)
      COMMON CP, CPTAU, IMAX, C, S, T
      EXTERNAL COMB
C
      L = 500
      NCIMAX = 40
      ERR = 0.0002
      SMALL = 0.000001
      PI = 3.1415927
C     ******************************************************************
C     INPUT DECK FROM PROGRAM NLCOEF
      WRITE (6,6000)
      WRITE (6,6001)
      READ (5,5000) (TITLE(I), I = 1, 72)
      WRITE (6,6038)  (TITLE(I), I = 1, 72)
      READ  (5,5001)  GAMMA, IMAX, KMAX
      WRITE (6,6009)  GAMMA, IMAX, KMAX
      KLO = IMAX + 1
      NU = 2 * IMAX
C     ******************************************************************
      WRITE (6,6002)
      DO 25 K = 1, IMAX
      READ  (5,5002) I, M(I), NAB(I), S(I), SJ(I), CNAME(I)
      WRITE (6,5002) I, M(I), NAB(I), S(I), SJ(I), CNAME(I)
   25 CONTINUE
C     ******************************************************************
C     FILL NONLINEAR COEFFICIENT ARRAY WITH ZEROES
      DO 30 I = 1, IMAX
      DO 30 J = 1, IMAX
      DO 30 K = 1, IMAX
      C(I,J,K) = 0.0
   30 CONTINUE
C     INPUT NONZERO NONLINEAR COEFFICIENTS
      WRITE (6,6003)
      LINE = IMAX + 20
      DO 35 K1 = 1, KMAX
      READ  (5,5003) I, J, K, C(I,J,K)
      WRITE (6,5003) I, J, K, C(I,J,K)
      LINE = LINE + 1
      IF (LINE .LT. 52)  GO TO 35
      WRITE (6,6000)
      WRITE (6,6003)
      LINE = 4
   35 CONTINUE
C     ******************************************************************
C     COMPUTE COEFFICIENTS FOR PRESSURE (THETA = 0, PI/4, PI/2)
C     COEFFICIENTS IN THE SERIES FOR THETA DERIVATIVE
      DO 36 NTHETA = 1, 3
      RTHETA = NTHETA - 1
      ANGLE(NTHETA) = RTHETA * 45.0
      THETA(NTHETA) = RTHETA * PI/4.0
```

```
      DO 36 J = 1, IMAX
      ARG = M(J) * THETA(NTHETA)
      FSIN = SIN(ARG)
      FCOS = COS(ARG)
      IF (NAB(J) .EQ. 0)  FCN = FCOS
      IF (NAB(J) .EQ. 1)  FCN = -FSIN
      CF(NTHETA,J) = M(J) * FCN * SJ(J)
C     COEFFICIENTS IN THE SERIES FOR THE TIME DERIVATIVE
      JP = J + IMAX
      IF (NAB(J) .EQ. 0)  FCN = FSIN
      IF (NAB(J) .EQ. 1)  FCN = FCOS
      CF(NTHETA,JP) = FCN * SJ(J)
   36 CONTINUE
C
C     ****************************************************************
C
  800 DO 204 K = 1, IMAX
      AS(K) = 0.0
      AC(K) = 0.0
  204 CONTINUE
C
C     INPUT DATA DECK
      WRITE(6, 6000)
      READ (5,5000) (TITLE(I), I = 1, 72)
      READ (5,5005)  EN, TAU, UE, RLD
      IF (EN) 801, 801, 802
  802 WRITE (6,6038)  (TITLE(I), I = 1, 72)
      WRITE (6,6004) EN, TAU, GAMMA, UE, RLD
      READ (5,5006)  NTEST, ITYPE, TQUIT
      IF (NTEST .EQ. 1)  WRITE (6,6022)
      IF (NTEST .EQ. 2)  WRITE (6,6023)
      IF (NTEST .EQ. 3)  WRITE (6,6024)
      WRITE (6,6025)
      IF (ITYPE .EQ. 3)  GO TO 201
      READ (5,5004)  MODE, AMPL
  201 GO TO (280,282,284), ITYPE
  280 AC(MODE) = AMPL
      WRITE (6,6026) CNAME(MODE), AMPL
      GO TO 202
  282 MODE1 = MODE - 1
      AS(MODE1) = AMPL
      AC(MODE)  = AMPL
      WRITE (6,6027) CNAME(MODE-1), CNAME(MODE), AMPL
      GO TO 202
  284 READ (5,5007)  MODE, NTERMS
      WRITE (6,6028)
      DO 203 K = 1, NTERMS
      READ (5,5008)  J, AS(J), AC(J)
  203 CONTINUE
  202 DO 209 J = 1, IMAX
      IF (AS(J)) 211, 210, 211
  211 IF (AC(J)) 215, 212, 215
  212 WRITE (6,6036) CNAME(J), AS(J), S(J)
      GO TO 209
  210 IF (AC(J)) 213, 209, 213
  213 WRITE (6,6037) CNAME(J), AC(J), S(J)
      GO TO 209
  215 WRITE (6,6029) CNAME(J), AS(J), S(J), AC(J), S(J)
  209 CONTINUE
```

```
      CI(1) = 1.0
      LSTCYC = 0
      IF (NTEST .EQ. 3)  READ (5, 5009)  LSTCYC
      IF (NTEST.EQ.3) WRITE (6,6030) LSTCYC, CNAME(MODE)
      WRITE (6,6000)
C
      NTEST1 = 0
      IF (NTEST .EQ. 2) NTEST1 = 1
C     ******************************************************************
C     COMPUTE LINEAR COEFFICIENTS
      ZE = 2.0 * RLD
      GUZ = GAMMA * UE/ZE
      G1 = (GAMMA - 1.0)/2.0
      GG = G1/GAMMA
      CP = GUZ * (1.0 + GG - EN)
      CPTAU = GUZ * EN
C     ******************************************************************
C     COMPUTE STEP SIZE
C     N IS THE NUMBER OF PARTITIONS OF THE TIME-LAG, THAT IS,
C     STEP SIZE = TIME-LAG/N
      N = 1.0 + TAU/0.068
      RN = N
      H = TAU/RN
      H6 = H/6.0
C     ******************************************************************
      JAY = 0
      LARGEU = 0
      NOTRIG = 0
      KCI = 0
      NCI = 1
      NQUIT = 0
      DELTA = 0.1
C     ******************************************************************
      WRITE (6,6004)  EN, TAU, GAMMA, UE, RLD
      LINE = 3
C
C     ******************************************************************
C
C     COMPUTE INITIAL AND PREINITIAL VALUES FROM
C     GIVEN INITIAL CONDITIONS
  505 NP1 = N + 1
      DO 206 J = 1, IMAX
      TIME2(J) = 0.0
  206 CONTINUE
      DO 45 I = 1, NP1
      NSTEP = I - NP1
      RSTEP = NSTEP
      TI(I) = RSTEP * H
      DO 40 J = 1, IMAX
      JP = J + IMAX
      ARG = S(J) * TI(I)
      FSIN = SIN(ARG)
      FCOS = COS(ARG)
      U(I,J)  = (AS(J)*FSIN + AC(J)*FCOS) * CI(NCI)
      U(I,JP) = (AS(J)*FCOS - AC(J)*FSIN) * S(J) * CI(NCI)
   40 CONTINUE
      DO 45 NTHETA = 1, 3
      SUMT = 0.0
      SUMTH = 0.0
```

```
      DO 270 J = 1, IMAX
      SUMTH = SUMTH + CF(NTHETA,J) * U(I,J)
      JP = J + IMAX
      SUMT = SUMT + CF(NTHETA,JP) * U(I,JP)
  270 CONTINUE
      STHSQ = SUMTH * SUMTH
      STSQ = SUMT * SUMT
      PRESS(NTHETA,I) = -GAMMA * (SUMT + (STHSQ - STSQ)*0.5)
   45 CONTINUE
      IF (AS(MODE)) 632, 633, 632
  632 FIRST = CI(NCI) * AS(MODE)
      GO TO 631
  633 FIRST = CI(NCI) * AC(MODE)
  631 NSTEP = 0
C     ****************************************************************
      DO 403 NTHETA = 1, 3
      MAXP(NTHETA) = 0
  403 CONTINUE
      DO 400 JJ = 1, IMAX
      MAXNO(JJ) = 0
      NCYC(JJ) = -1
  400 CONTINUE
      IF (CI(NCI) .LT. DELTA)  GO TO 750
      K = 0
C     ****************************************************************
  530 I = N + 1
C
C     COMPUTE U(I+1,J) FROM KNOWN VALUES OF U(I,J)
  515 NTRIG = 0
      NT = (I - NP1 + (L - NP1) * K)
      RNT = NT
      TT = RNT * H
      TI(I) = TT
      P = -0.5
      DO 60 J = 1, IMAX
      JP = J + IMAX
      T(J,1) = U(I-N,JP)
      T(J,4) = U(I-N+1,JP)
      PA = (P - 1.0) * P * 0.5
      PB = 1.0 - (P * P)
      PC = (P + 1.0) * P * 0.5
      T(J,2) = PA*U(I-N,JP) + PB*U(I-N+1,JP) + PC*U(I-N+2,JP)
      T(J,3) = T(J,2)
   60 CONTINUE
      DO 65 J = 1, NU
      Y(J) = U(I,J)
   65 CONTINUE
C     RUNGEKUTTA INTEGRATION OF SYSTEM OF D. E.
      A(1) = 0.0
      A(2) = 0.5
      A(3) = 0.5
      A(4) = 1.0
      CALL COMB(NU,1,Y,YP)
      DO 70 J = 1, NU
      FZ(1,J) = YP(J)
   70 CONTINUE
      DO 75 II = 2, 4
      DO 80 J = 1, NU
      UZ(J) = Y(J) + A(II) * H * FZ(II-1,J)
   80 CONTINUE
```

```
      CALL COMB(NU,II,UZ,YP)
      DO 85 J = 1, NU
      FZ(II,J) = YP(J)
   85 CONTINUE
   75 CONTINUE
      DO 90 J = 1, NU
      U(I+1,J) = Y(J) + (FZ(1,J)+2.0*(FZ(2,J)+FZ(3,J))+FZ(4,J))*H6
   90 CONTINUE
C
C     *****************************************************************
C
C     TEST SIZE OF U(I+1,J)
      DO 92 J = 1, IMAX
      ABSU = ABS(U(I+1,J))
      IF (LARGEU .GT. 0)  GO TO 92
      IF (ABSU .GE. 20.0)   LARGEU = J
   92 CONTINUE
      IF (LARGEU .EQ. 0) GO TO 605
      IF (NTEST .NE. 2)  GO TO 94
      DELSGN(NCI) = -1.0
      GO TO 731
   94 IF (NTEST1 .EQ. 1)  WRITE (6,6000)
      WRITE (6,6008) CNAME(LARGEU), TI(I), CNAME(MODE), FIRST
      IF  (NCYC(MODE) .LT. LSTCYC)  GO TO 800
      GO TO 750
C
C     *****************************************************************
C
C     COMPUTE PRESSURE PERTURBATION FOR THETA = 0, PI/4, PI/2
  605 DO 604 NTHETA = 1, 3
      SUMT = 0.0
      SUMTH = 0.0
      DO 570 J = 1, IMAX
      SUMTH = SUMTH + CF(NTHETA,J) * U(I+1,J)
      JP = J + IMAX
      SUMT = SUMT + CF(NTHETA,JP) * U(I+1,JP)
  570 CONTINUE
      STHSQ = SUMTH * SUMTH
      STSQ = SUMT * SUMT
      PRESS(NTHETA,I+1) = -GAMMA * (SUMT + (STHSQ - STSQ)*0.5)
  604 CONTINUE
C     *****************************************************************
C     DETERMINE MINIMUM AND MAXIMUM PRESSURE
      IF ((NTEST .EQ. 3) .AND. (NCYC(MODE).LT.LSTCYC)) GO TO 610
      DO 902 NTHETA = 1, 3
      P1 = PRESS(NTHETA,I-1)
      P2 = PRESS(NTHETA,I)
      P3 = PRESS(NTHETA,I+1)
      DPL = P3 - P2
      DPS = P2 - P1
      IF (DPL * DPS) 900, 900, 902
  900 PNUM = P1 - P3
      PDEN = 2.0 * (P1 + P3 - 2.0*P2)
      P = PNUM/PDEN
      PA = (P - 1.0) * P * 0.5
      PB = 1.0 - (P * P)
      PC = (P + 1.0) * P * 0.5
      PEXT = PA * P1  +  PB * P2  +  PC * P3
      MAXP(NTHETA) = MAXP(NTHETA) + 1
      MXP = MAXP(NTHETA)
```

```
      PMAX(NTHETA,MXP) = PEXT
  902 CONTINUE
C     ******************************************************************
C     DETERMINE WHERE DERIVATIVE CHANGES SIGN TO FIND EXTREMUM
C     COMPUTE EXTREMUM UMAX(MAXNO) FROM KNOWN VALUES OF U(I-1,2),
C     U(I,2), AND U(I+1,2) BY THREE POINT INTERPOLATION
  610 DO 511  J = 1, IMAX
      JP = J + IMAX
      IF ( U(I,JP) * U(I+1,JP) ) 500, 500, 615
  500 PDEN = U(I,JP) - U(I+1,JP)
      IF (PDEN) 501, 511, 501
  501 P = U(I,JP)/PDEN
      PA = (P - 1.0) * P * 0.5
      PB = 1.0 - (P * P)
      PC = (P + 1.0) * P * 0.5
      FOFPH = (PA*U(I-1,J)) + (PB*U(I,J)) + (PC*U(I+1,J))
      IF (FOFPH .LT. 0.0)  GO TO 518
      NCYC(J) = NCYC(J) + 1
      IF ((J .NE. MODE) .OR. (NOTRIG .EQ. 1))  GO TO 518
      IF ((NTEST.EQ.1) .AND. (NTEST1.EQ.1) .AND. (NCYC(MODE).EQ.2))
     1             NTRIG = 1
  518 IF ((NTEST .EQ. 3) .AND. (NCYC(MODE).LT.LSTCYC)) GO TO 511
      MAXNO(J) = MAXNO(J) + 1
      MAX = MAXNO(J)
      UMAX(J,MAX) = FOFPH
  615 IF (U(I,J) * U(I+1,J)) 513, 513, 511
  513 IF (U(I,JP))  511, 511, 512
  512 TIME1(J) = TIME2(J)
      P = U(I,J)/(U(I,J) - U(I+1,J))
      TIME2(J) = TI(I)  +  P * H
  511 CONTINUE
C     ******************************************************************
      IF ((NTEST .EQ. 3) .AND. (NCYC(MODE).EQ.LSTCYC)) NQUIT = 1
      IF (NQUIT .NE. 1) GO TO 514
C     OUTPUT STEP BY STEP INTEGRATION VALUES
      IF (LINE .EQ. 3)  WRITE (6,6005)
      RTI = NSTEP
      TIME = RTI * H
      WRITE (6,6006) NSTEP, TIME,  (PRESS(NTHETA,I), NTHETA = 1, 3)
      NSTEP = NSTEP + 1
      LINE = LINE + 1
      IF (LINE .LT. 52) GO TO 517
      LINE = 3
      WRITE(6, 6000)
C     ******************************************************************
  517 IF (TIME .GT. TQUIT)  NQUIT = 2
  514 IF (NQUIT .NE. 2) GO TO 510
      IF ((NTEST .EQ. 1) .AND. (NTEST1 .EQ. 0))  NQUIT = 3
      MAX = MAXNO(MODE)
      ABSU = ABS(UMAX(MODE,MAX))
C     ******************************************************************
  510 IF (NQUIT .EQ. 3) GO TO 750
      IF ((NTEST1 .EQ. 1) .AND. (ABSU .LE. SMALL))  NQUIT = 3
      I = I + 1
      IF (NTRIG .EQ. 1)  GO TO 707
      IF ( I .LT. L )  GO TO 515
C
```

```
C     ****************************************************************
C     TESTS FOR LIMIT CYCLES AND TRIGGERING LIMITS
  700 K = K + 1
      DO 711 J = 1, IMAX
      IF (MAXNO(J) .GT. 900)  JAY = J
  711 CONTINUE
      DO 713 NTHETA = 1, 3
      IF (MAXP(NTHETA) .GT. 900)  JAY = 1
  713 CONTINUE
      UTOT = 0.0
      DO 701 J = 0, 3
      JMAX = MAXNO(MODE) - J
      UTOT = UTOT + ABS(UMAX(MODE,JMAX))
  701 CONTINUE
      UAVG(K) = UTOT/4.0
      KCYC(K) = NCYC(MODE)
      IF (NTEST .EQ. 3) GO TO 739
      IF (K .EQ. 1) GO TO 702
C     ****************************************************************
C     TEST FOR TRIGGERING LIMITS
      IF (NTEST .EQ. 2)  GO TO 730
      GO TO 739
  730 UDIFF = UAVG(2) - UAVG(1)
      KCI = KCI + 1
      IF (UDIFF .GT. 0.0)  DELSGN(NCI) = -1.0
      IF (UDIFF .LT. 0.0)  DELSGN(NCI) =  1.0
  731 IF (NCI .EQ. 1) GO TO 733
      CHGSGN = DELSGN(NCI) * DELSGN(NCI-1)
      IF (CHGSGN .LT. 0.0)  DELTA = DELTA/2.0
      IF (DELTA .LT. 0.001) NTEST = 1
  733 CI(NCI + 1) = CI(NCI) + DELSGN(NCI) * DELTA
      IF (KCI.EQ.1) WRITE (6,6021) CNAME(MODE), KCYC(1), KCYC(2)
      IF (KCI .EQ. 0)  GO TO 736
      IF (LARGEU.EQ.0) WRITE (6,6015)
     1                 NCI, FIRST, UAVG(1), UAVG(2), UDIFF
      IF (LARGEU .GT. 0)  WRITE (6,6007) NCI, FIRST
      KCI = KCI + 1
  736 NCI = NCI + 1
      IF ((CI(NCI).GE.DELTA) .AND. (NCI.LT.NCIMAX)) GO TO 734
      NTEST = 1
      NOTRIG = 1
  734 LARGEU = 0
      GO TO 505
C     ****************************************************************
C     TEST FOR LIMIT-CYCLE AMPLITUDE
  739 CHANGE = UAVG(K) - UAVG(K-1)
      CYCLES = KCYC(K) - KCYC(K-1)
      GROWTH = CHANGE/CYCLES
      IF ((JAY.GT.0) .AND. (NTEST1.EQ.0))  GO TO 707
      IF ((MAXNO(MODE).GT.150) .AND. (NTEST1.EQ.1))  GO TO 750
      IF ((NTEST.GE.2) .OR. (NTEST1.EQ.1))  GO TO 702
      ABSCHG = ABS(CHANGE)
      IF ((ABSCHG.LE.ERR) .AND. (MAXNO(MODE).GE.120))  GO TO 707
      GO TO 702
C
```

```
C     OUTPUT DATA FOR LIMIT CYCLES.
  707 IF ((NTEST.LT.3) .AND. (NTEST1.EQ.0)) GO TO 708
      WRITE (6,6000)
      WRITE (6,6004)  EN, TAU, GAMMA, UE, RLD
  708 IF ((JAY.GT.0) .AND. (NTEST .EQ. 1))  WRITE (6,6020)
      IF (JAY .GT. 0)  WRITE (6,6019) CNAME(MODE), GROWTH, NCYC(MODE)
      IF ((JAY.EQ.0) .AND. (NTEST1.EQ.0))
     1                 WRITE (6,6031) NCYC(MODE), CNAME(MODE)
      IF (NTEST1 .EQ. 1)  WRITE (6,6014) NCI, CNAME(MODE), FIRST
      WRITE (6,6032)
      DO 761 J = 1, IMAX
      JTEST = 0
      JJ = MAXNO(J)
      FMAX = 0.0
      FMIN = 0.0
  760 PEAK = UMAX(J,JJ)
      IF (PEAK .GT. 0.0)  FMAX = PEAK
      IF (PEAK .LT. 0.0) FMIN =  PEAK
      JJ = JJ - 1
      JTEST = JTEST + 1
      IF (JTEST .LE. 1)  GO TO 760
      PERIOD = TIME2(J) - TIME1(J)
      IF (PERIOD) 761, 761, 762
  762 FREQ = 2.0 * PI/PERIOD
      WRITE (6,6033)  J, CNAME(J), FMAX, FMIN, PERIOD, FREQ
  761 CONTINUE
      WRITE (6,6034)
      DO 763 NTHETA = 1, 3
      JTEST = 0
      JJ = MAXP(NTHETA)
      JJMIN = MAXP(NTHETA) - 25
      FMAX = 0.0
      FMIN = 0.0
  764 IF (JTEST .NE. 0)  PEAK = PMAX(NTHETA,JJ)
  767 IF (PMAX(NTHETA,JJ) * PMAX(NTHETA,JJ-1))  768, 768, 769
  769 IF (JTEST .EQ. 0)  GO TO 765
      PEAK1 = PMAX(NTHETA,JJ-1)
      ABSPK = ABS(PEAK)
      ABSPK1 = ABS(PEAK1)
      IF (ABSPK1 .GT. ABSPK)  PEAK = PEAK1
      JJ = JJ - 1
      GO TO 767
  768 JTEST = JTEST + 1
      IF (JTEST .EQ. 1)  GO TO 765
      IF (PEAK .GT. 0.0)  FMAX = PEAK
      IF (PEAK .LT. 0.0)  FMIN = PEAK
  765 IF (JJ .LE. JJMIN)  GO TO 763
      JJ = JJ - 1
      IF (JTEST .LE. 2)  GO TO 764
      PKTOPK = FMAX - FMIN
      WRITE (6,6035)  ANGLE(NTHETA), FMAX, FMIN, PKTOPK
  763 CONTINUE
      IF (NTEST .EQ. 3)  GO TO 703
      WRITE (6,6000)
      WRITE (6,6004) EN, TAU, GAMMA, UE, RLD
C
  703 NQUIT = NTEST
      IF (NTRIG .EQ. 1)  GO TO 515
```

```
C     ****************************************************************
C     REASSIGN DEPENDENT VARIABLE ARRAYS
  702 DO 95 I = 1, NP1
      DO 704 NTHETA = 1, 3
      PRESS(NTHETA,I) = PRESS(NTHETA,L - NP1 + I)
  704 CONTINUE
      DO 95 J = 1, NU
      U(I,J) = U(L - NP1 + I,J)
   95 CONTINUE
      GO TO 530
C
C     OUTPUT OF PRESSURE AMPLITUDES
  750 IF (NOTRIG .EQ. 0)  GO TO 754
      WRITE (6,6010) NCI, CNAME(MODE), FIRST
      IF (CI(NCI) .LT. DELTA) GO TO 800
      WRITE (6,6000)
  754 IF (NTEST .NE. 3)  GO TO 779
      DO 751 NTHETA = 1, 3
      WRITE(6, 6000)
      WRITE (6,6016) ANGLE(NTHETA), MAXP(NTHETA)
      MAX = MAXP(NTHETA)
      LINE = 1
      DO 751 JST = 1, MAX, 8
      LST = JST
      JSTOP = JST + 7
      IF (JSTOP .GT. MAX)  JSTOP = MAX
      WRITE (6,6013) (PMAX(NTHETA,J), J = LST, JSTOP)
      LINE = LINE + 1
      IF (LINE .LT. 25)  GO TO 751
      LINE = 0
      WRITE(6, 6000)
  751 CONTINUE
C
  779 IF (LARGEU .GT. 0)  GO TO 777
      IF ((NTEST.EQ.1).AND.(NTEST1.EQ.0).AND.(JAY.EQ.0))  GO TO 800
C     OUTPUT AMPLITUDES OF SERIES TERMS
  777 IF ((LARGEU.EQ.0 .AND. NOTRIG.EQ.0) .OR. NTEST.EQ.3)
     1            WRITE (6,6000)
      WRITE (6,6017)  CNAME(MODE), LSTCYC, NCYC(MODE)
      MAX = MAXNO(MODE)
      LINE = 3
      IF (LARGEU .GT. 0)  LINE = 11
      DO 776 JST = 1, MAX, 8
      LST = JST
      JSTOP = JST + 7
      IF (JSTOP .GT. MAX)  JSTOP = MAX
      WRITE (6,6013) (UMAX(MODE,J), J = LST, JSTOP)
      LINE = LINE + 1
      IF (LINE .LT. 25)  GO TO 776
      LINE = 0
      WRITE(6, 6000)
  776 CONTINUE
      GO TO 800
  801 CONTINUE
C
```

```
C      *****************************************************************
C      FORMAT SPECIFICATIONS
C
 5000 FORMAT (72A1)
 5001 FORMAT (F5.2,2I5)
 5002 FORMAT (3I5,2F10.5,7X,A3)
 5003 FORMAT (3I5,F10.5)
 5004 FORMAT (I5,F10.5)
 5005 FORMAT (4F10.0)
 5006 FORMAT (2I5,F10.0)
 5007 FORMAT (2I5)
 5008 FORMAT (I5,2F10.0)
 5009 FORMAT (I5)
 6000 FORMAT (1H1)
 6001 FORMAT (2X,14HPROGRAM LIMCYC/2X,22HSECOND ORDER NONLINEAR,
     1        31H COMBUSTION INSTABILITY PROGRAM///)
 6002 FORMAT (4X,12HJ    M    NAB,4X,3HSMN,5X,7HJM(SMN),6X,4HNAME/)
 6003 FORMAT (1H ,//4X,22HI     J     K    C(I,J,K)/)
 6004 FORMAT (2X,44HCOMBUSTION PARAMETERS:  INTERACTION INDEX = ,F7.5,
     1        12X,11HTIME-LAG = ,F7.5/2X,17HMOTOR PARAMETERS:,19X,
     2        8HGAMMA = ,F7.5,23H    EXIT MACH NUMBER = ,F7.5,
     3        22H    LENGTH/DIAMETER = ,F7.5//)
 6005 FORMAT (37X,23HWALL PRESSURE WAVEFORMS/2X,5H STEP,9X,4HTIME,
     1        9X,9H0 DEGREES,5X,10H45 DEGREES,5X,10H90 DEGREES/)
 6006 FORMAT (2X,I5,4F15.5)
 6007 FORMAT (6X,I3,F16.6,50X,24HAMPLITUDE LIMIT EXCEEDED)
 6008 FORMAT (2X///2X,13HAMPLITUDE OF ,A3,23H(T) EXCEEDED 20 AT T = ,
     1        F8.5/2X,21HINITIAL AMPLITUDE OF ,A3,8H(T) WAS ,F8.5//)
 6009 FORMAT (2X,26HINPUTS FROM PROGRAM NLCOEF//
     1        2X,8HGAMMA = ,F5.2,5X,8HJMAX  = ,I2,5X,9HNLMAX  = ,I3//)
 6010 FORMAT (2X//2X,38HFAILED TO FIND TRIGGERING LIMIT AFTER ,
     1        I5,7H TRIALS/2X,21HINITIAL AMPLITUDE OF ,A3,
     2        20H(T) ON LAST TRY WAS ,F8.5//)
 6013 FORMAT (1H0,7X,8F13.7)
 6014 FORMAT (2X,29HTRIGGERING LIMIT FOUND AFTER ,I5,7H TRIALS/
     1        2X,21HINITIAL AMPLITUDE OF ,A3,8H(T) WAS ,F8.5//)
 6015 FORMAT (6X,I3,1X,4F15.6)
 6016 FORMAT (11X,30HWALL PRESSURE PEAKS   THETA = ,F4.1,8H DEGREES/
     1        11X,I3,17H  VALUES COMPUTED/)
 6017 FORMAT (11X,18HEXTREME VALUES OF ,A3,3H(T)/
     1        11X,21HOUTPUT STARTED AFTER ,I5,7H CYCLES,
     2        19H     STOPPED AFTER ,I5,7H CYCLES//)
 6019 FORMAT (2X,13HAMPLITUDE OF ,A3,16H IS CHANGING BY ,F9.6,
     1        17H PER CYCLE AFTER ,I5,7H CYCLES//)
 6020 FORMAT (2X,39HFAILED TO ATTAIN LIMIT-CYCLE AMPLITUDE.)
 6021 FORMAT (17X,7HINITIAL,7X,9HAMPLITUDE,6X,9HAMPLITUDE/16X,
     1        9HAMPLITUDE,9X,5HAFTER,10X,5HAFTER,7X,9HCHANGE IN/6X,
     2        5HTRIAL,5X,3HOF ,A3,3H(T),6X,I2,7H CYCLES,6X,
     3        I2,7H CYCLES,6X,9HAMPLITUDE/)
 6022 FORMAT (2X,32HCALCULATE LIMIT-CYCLE AMPLITUDE.//)
 6023 FORMAT (2X,27HCALCULATE TRIGGERING LIMIT.//)
 6024 FORMAT (2X,29HCALCULATE TRANSIENT BEHAVIOR.//)
 6025 FORMAT (2X,36HINITIAL CONDITIONS FOR  -TAU < T < 0//)
 6026 FORMAT (2X,38HSINGLE STANDING MODE INITIALLY PRESENT/
     1        2X,32H(PRESSURE ANTINODE AT THETA = 0)//
     2        2X,21HINITIAL AMPLITUDE OF ,A3,4H IS ,F8.5//)
 6027 FORMAT (2X,38HSINGLE SPINNING MODE INITIALLY PRESENT/
     1        2X,42HMOVING COUNTERCLOCKWISE (THETA INCREASING)//
     2        2X,21HINITIAL AMPLITUDE OF ,A3,5H AND ,A3,4H IS ,F8.5//)
```

```
6028 FORMAT (2X,48HARBITRARY COMBINATION OF MODES INITIALLY PRESENT//)
6029 FORMAT (2X,A3,6H(T) = ,F8.5,7H * SIN(,F8.5,8H*T)  +  ,
    1        F8.5,7H * COS(,F8.5,3H*T)/)
6030 FORMAT (2X//2X,25HDATA OUTPUT BEGINS AFTER ,I5,11H CYCLES OF ,
    1        A3,3H(T))
6031 FORMAT (2X,37HLIMIT-CYCLE AMPLITUDE REACHED AFTER   ,
    1        I5,12H  CYCLES OF ,A3//)
6032 FORMAT (2X,23H  J    MODE      MAXIMUM,8X,7HMINIMUM,8X,
    1        6HPERIOD,8X,9HFREQUENCY/)
6033 FORMAT (2X,I3,4X,A3,F13.5,3F15.5)
6034 FORMAT (2X///4X,5HTHETA,15X,26HWALL PRESSURE PERTURBATION/
    1        2X,9H(DEGREES),7X,7HMAXIMUM,8X,7HMINIMUM,6X,
    2        12HPEAK-TO-PEAK//)
6035 FORMAT (F8.1,2X,3F15.5)
6036 FORMAT (2X,A3,6H(T) = ,F8.5,7H * SIN(,F8.5,3H*T)/)
6037 FORMAT (2X,A3,6H(T) = ,F8.5,7H * COS(,F8.5,3H*T)/)
6038 FORMAT (2X,72A1///)
     END


     SUBROUTINE COMB(NU,II,U,UP)
     DIMENSION U(NU), UP(NU)
     DIMENSION  S(20), T(20,4), C(20,20,20)
     COMMON CP, CPTAU, IMAX, C, S, T
C
     DO 10 I = 1, IMAX
     IP = I + IMAX
     SSQ = S(I) * S(I)
     RV = T(I,II)
     UP(I) = U(IP)
     SNL = 0.0
     DO 20 J = 1, IMAX
     DO 20 K = 1, IMAX
     COEF = C(I,J,K)
     KP = K + IMAX
     SNL = SNL + (COEF * U(J) * U(KP))
  20 CONTINUE
     UP(IP) =  -(SSQ*U(I) + CP * U(IP) + CPTAU * RV + SNL)
  10 CONTINUE
     RETURN
     END
```

REFERENCES

1. Hefner, R. J., "Diagnosis of High Frequency Combustion Stability Characteristics from Pressure Measurements," Third ICRPG Combustion Conference CPIA Publication No. 138, Vol. I, October 1966.

2. Crocco, L. and Cheng, S. I., Theory of Combustion Instability in Liquid Propellant Rocket Motors, AGARD Monograph No. 8, Butterworths Scientific Pub., Ltd., London, 1956, pp. 1-24.

3. Powell, E. A., "Nonlinear Combustion Instability in Liquid Propellant Rocket Engines," Ph.D. Thesis, Georgia Institute of Technology, GITAER 70-6, September, 1970.

4. Powell, E. A. and Zinn, B. T., "Stable Limit Cycles and Triggering Limits of the First Radial Mode in Unstable Liquid Rockets," Proceedings of the Thirteenth Israel Annual Conference on Aviation and Astronautics, 1971.

5. Maslen, S. H. and Moore, F. K., "On Strong Transverse Waves Without Shocks in a Circular Cylinder," Journal of Aeronautical Sciences, Vol. 23, No. 6, 1956, pp. 583-593.

6. Crocco, L. and Sirignano, W. A., "Effect of the Transverse Velocity Component on the Nonlinear Behavior of Short Nozzles," AIAA Journal, Vol. 4, No. 8, August 1966, pp. 1428-1430.

7. Zinn, B. T. and Powell, E. A., "Nonlinear Combustion Instability in Liquid-Propellant Rocket Engines," Thirteenth Symposium (International) on Combustion, Salt Lake City, Utah, August, 1970.

8. Finlayson, B. A. and Scriven, L. E., "The Method of Weighted Residuals -- A Review," Applied Mechanics Reviews, Vol. 19, No. 9, September 1966, pp. 735-744.

9. Ames, W. F., Nonlinear Partial Differential Equations in Engineering, Academic Press, New York, 1965, pp. 243-262.

10. Powell, E. A. and Zinn, B. T., "A Single Mode Approximation in the Solution of Nonlinear Combustion Instability Problems," AIAA Paper No. 71-208, January, 1971.

11. Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, Dover Publications, Inc., New York, 1965, p. 411.

12. McCracken, D. D., FORTRAN with Engineering Applications, John Wiley and Sons, Inc., New York, 1967, pp. 146-153.

13. Conte, S. D., Elementary Numerical Analysis, McGraw-Hill Book Company, New York, 1965, pp. 223-224.