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ORBIT PERTURBATION THEORY USING HARMONIC OSCILLATOR SYSTEMS

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

(Principal Adviser)

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(Applied Mechanics)

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ABSTRACT

Unperturbed two-body, or Keplerian motion is transformed from the time domain into the domains of two unique three dimensional vector harmonic oscillator systems. One harmonic oscillator system is fully regularized and hence valid for all orbits including the rectilinear class up to and including periapsis passage. The other system is fully as general except that the solution becomes unbounded at periapsis passage of rectilinear orbits. The natural frequencies of the oscillator systems are related to certain Keplerian orbit scalar constants, while the independent variables are related to well-known orbit angular measurements, or anomalies. The solutions to both systems are universally applicable functionally to all types of orbits (elliptic, parabolic, hyperbolic, and rectilinear).

Perturbed two-body motion is then presented in the framework of perturbed harmonic oscillators. Nonlinear and linearized Encke perturbation equations, in vector form, are developed for both perturbed orbit oscillator systems, and the linearized vector perturbation equations are demonstrated to be directly solvable by quadrature in the domains of the respective oscillator systems.

The linearized analysis of general rectilinear orbits perturbed either by an external body or by the second spherical harmonic (J_2) is presented. Albebraic expressions are developed which represent the perturbation state vector referenced to the reference rectilinear orbit.

An analysis of near-parabolic transfer trajectories between the moon and the cislunar libration point L_1 is presented. Approximate formulae are developed which represent the velocity requirements at the moon and L_1 for passage in either direction on either side of the moon.

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INTRODUCTION

The physical basis for celestial mechanics was established more than 300 years ago with the publication of Kepler's three laws of planetary motion, based on the observations of Tycho Brahe. Subsequently, Newton established the mathematical foundation of modern celestial mechanics by postulating the law of gravitational attraction between two bodies as the force of attraction being proportional to the product of the masses of the two bodies and inversely proportional to the square of the distance separating them. In vector form, this famous mathematical relationship appears as

$$\ddot{\bar{r}} = -\frac{\mu}{r^3}\bar{r} \tag{1-1}$$

In spite of its apparent simplicity, it is a singular nonlinear differential equation of the motion whose scalar components are represented by coupled second order differential equations. First integrals of (1-1) are readily obtainable by elementary means, and the well-known transformation of the dependent variable r to $u = 1/r$ results in the differential equation

$$u'' + u = \frac{1}{p} \tag{1-2}$$

where $(\cdot)'$ denotes differentiation with respect to the true anomaly f . Although it may appear trivial at this point, the vector equation which affords the complete vector solution to (1-1) in conjunction with (1-2) is

$$\hat{r}'' + \hat{r} = 0 \tag{1-3}$$

where $\hat{r}$ is the unit radius vector.

The system of equations (1-2) and (1-3) is the introduction to the representation of Keplerian motion as an harmonic oscillator system and is referred to herein as the "classical" orbit oscillator system. Thus it would appear that the undesirable mathematical characteristics of (1-1) do not appreciably affect the development of a meaningful solution.

There exist two regions of mathematical interest associated with the general solution to (1-1); these regions are related to near-parabolic and near-rectilinear orbits. Rectilinear orbits may be elliptic, parabolic, or hyperbolic and are characterized by the particle encountering the singularity of (1-1) at periapsis. Geometrically, they are straight lines connecting the orbit foci.

Representation of the solution to (1-1) in terms of the true anomaly has the advantage that there is a natural smooth transition between the elliptic and hyperbolic regions of motion. However, the description of motion would appear (at this time) to break down for rectilinear orbits if the true anomaly is regarded as an angular measurement. This disadvantage becomes distinct in the analysis of perturbed Keplerian motion in which the instantaneous, or osculating, conic passes through the rectilinear region of motion. It will be subsequently shown that, for pure rectilinear orbits, an "effective" true anomaly related to the particle speed results in a valid solution.

If the solution to the unperturbed two-body problem is represented in terms of the eccentric or hyperbolic anomaly as opposed to the true anomaly, the resulting solution is then compatible with the simplest and most convenient form of Kepler's time equation. Furthermore, the solution is directly valid for rectilinear orbits (geometrically, an "eccentric circle" may still be circumscribed about a rectilinear ellipse). However, the penalty is the lack of a functionally smooth transition between the mathematical representation of elliptic and hyperbolic motion. This disadvantage would manifest itself in the analysis of perturbed Keplerian motion in which the osculating conic passes through the parabolic region of motion.

History of Universal Orbit Formulae

The difficulty of analysis of the continuous transition region between the elliptic and hyperbolic regions of Keplerian motion has led to the universal orbit formulae of numerous investigators such as Stumpff, Sperling, Herrick, Battin, etc. The formulation of the universal orbit solution is obtained by a generalization of the Keplerian solution to the various regions of motion and effectively describes the general orbit by means of an energy parameter and the appropriate "universal" anomaly (e.g., eccentric anomaly for eccentric orbits). The resulting solution functionally affords the necessary smooth transition between the various regions of motion and, in addition, is capable of handling the rectilinear boundary of the motion up to and including periapsis passage at the center of attraction. The derivation of the universal orbit solution was subsequently obtained by Pitkin [1] from a regularized form of Eq. (1-1), yielding a differential equation as the basis of the universal solution, but also establishing the relation between the universal orbit formulae and the mathematical process of regularization.

History of Regularization

Regularization may be described as a procedure for removing the singularities of a mathematical expression by means of, say, a change of variables. As referred to in this research, we will be concerned with the regularization of the differential equation (1-1). Although the mathematical objective is to obtain a well-behaved solution for a particle which passes through the singularity, it will be shown that a computational advantage is obtained for particles which pass near the singularity.

The history of regularization of the classical two-body problem is usually traced to Levi-Civita (1906). The regularization that bears his name is a relatively complicated procedure which transforms the time and coordinates of the two-dimensional two-body problem into a three dimensional space. This is accomplished by means of a theory of conformal mapping and representation of the two-dimensional position vector as a complex number. The Levi-Civita transformation has the important property that it is capable of removing the singularity at more than one center of attraction and is thus referred to as a "global" regularization. Szebehely makes extensive use of this theory in reference [2].

Recently, Kustaanheimo and Stiefel [3] succeeded in generalizing the Levi-Civita transformation to three dimensions by means of the generalization of complex numbers to spinors, thus effectively transforming the three-dimensional Cartesian space problem into a four-dimensional space with the appropriate time transformation. This method is referred to as the K-S transformation.

Local Regularization

A regularization procedure which eliminates the singularity at only one center of attraction is referred to as a "local" regularization. Application of the K-S transformation to the problem of local regularization reduces the equations of motion in four-space to a second order, linear, constant coefficient set for the unperturbed Keplerian problem.

Pitkin [1] uses only the time transformation of the Levi-Civita transformation (which he refers to as the Sundman transformation) to obtain a regularized set of a vector and scalar equation of motion. From this he eventually obtains the universal orbit formulae. In his derivation, one of the integrals of the Keplerian problem (energy) is used to simplify the resulting scalar orbit equation, which then becomes that governing the behavior of a simple harmonic oscillator. The regularization appearing in this research introduces a vector integral of the inverse square two-body problem that has the effect of simplifying Pitkin's complicated vector equation to that of a vector harmonic oscillator in three dimensions. The resulting three-dimensional regularization of the inverse-square two-body problem thus achieves all the advantages of the K-S transformation applied to local regularization without the transformation of the coordinates. More importantly, it supplies a simple orbit oscillator system analogous to the classical oscillator system but with the modified eccentric (hyperbolic) anomaly as the independent variable.

Contributions of this Research

The aforementioned regularization of the inverse-square two-body problem, obtained first by Burdet [4] and subsequently and independently by the author, yields the differential equations of a vector harmonic oscillator which leads directly to a modified form of universal orbit formulae. Following Burdet's terminology, this system is referred to as the "central oscillator" system. Since the central oscillator system arises directly from regularization, it is automatically valid for rectilinear orbits, up to and including periapsis passage at the center of attraction. The solution to the central oscillator system is identical to the orbit description using the eccentric or hyperbolic anomaly, but in a more general form.

At this point, the term "universal" must be given a precise meaning. This term will be used to designate a solution which may be expressed functionally in a form applicable to orbits of arbitrary energy and angular momentum (including zero for either or both) and including the time domain singularity of periapsis passage for rectilinearity. In spite of the obvious singularity of the term $1/p$ in (1-2), it will be shown that the classical oscillator system also admits a well defined solution to the general rectilinear orbit, where the independent variable is related to velocity and is unbounded only at periapsis passage. Thus the classical oscillator system might be regarded as quasi-universal. During the course of this research, however, Burdet [5] introduced the author to a subtle modification of the classical oscillator system and referred to as the "focal" oscillator, which uses a modified true anomaly as the independent variable.

This system will also be shown to be quasi-universal and admits the identical solution of the classical oscillator. However, it differs in one major respect in that it contains no terms which are unbounded for rectilinear orbits. The same will be shown to be true of the central oscillator system. Thus a perturbation analysis will not be concerned with perturbations of unbounded terms, and therein lies the primary reason for use of the focal and central oscillators in this research.

The primary purpose of this research is to establish a method of general perturbations based on the presentation of the unperturbed two-body problem as two distinct harmonic oscillator systems. The perturbed two-body equations of motion are developed for both systems and nonlinear Encke perturbation equations are then developed, along with comments on the numerical computation aspects.

The linearized Encke perturbation equations are developed for both oscillator systems and demonstrated to be directly solvable by quadrature in the domains of the respective systems for a large class of perturbing forces. The general solutions are presented along with a delineation of regions of applicability with respect to the structure of the perturbing force.

The solution to the problem of perturbations of a circular orbit due to the second spherical harmonic (J_2) term is obtained using the perturbed harmonic oscillator system solutions and is compared to a solution obtained from the Euler-Hill equations. Perturbations of a general rectilinear orbit are obtained for two forms of perturbing force: 1) the aforementioned J_2 oblateness term and 2) a fixed external perturbing body. Finally, an analytical treatment of near-parabolic trajectories between the moon and the cislunar libration point L_1 is presented, using the perturbed central oscillator system. The results are presented as approximate analytic formulae for velocity requirements at L_1 or the moon for transfer between L_1 and the moon.

Chapter II

THE UNPERTURBED TWO-BODY PROBLEM

The motion of a particle about an attracting primary mass whose force of attraction varies inversly as the square of the distance between the two bodies is governed by

$$\ddot{\bar{r}} = -\frac{\mu}{r^3}\bar{r} \qquad (1\text{-}1)$$

where $\mu = Gm$, m = mass of the attracting body, and G is the universal gravitational constant. It is advantageous at this point to review the elementary derivation of the first integrals of (1-1).

Conservation of Energy

By constructing the scalar product of (1-1) with the vector $\dot{\bar{r}}$ we obtain

$$\frac{d}{dt}\left(\frac{\dot{\bar{r}} \cdot \dot{\bar{r}}}{2} - \frac{\mu}{r}\right) = 0 \qquad (2\text{-}1)$$

resulting in the energy integral

$$\mathcal{E} = \frac{\dot{\bar{r}} \cdot \dot{\bar{r}}}{2} - \frac{\mu}{r} = \text{constant} \qquad (2\text{-}2)$$

It can be shown that, for motion in an inverse square field, $\mathcal{E} = -\mu/2a$, where a is the semimajor axis of the resulting conic section. For the purpose of this analysis, a modified energy parameter α is defined as the reciprocal of the semimajor axis, or

$$\alpha \triangleq \frac{1}{a} = -\frac{2\mathcal{E}}{\mu} = \frac{2}{r} - \frac{\dot{\bar{r}} \cdot \dot{\bar{r}}}{\mu} \qquad (2\text{-}3)$$

The use of an energy-related parameter in the description of an orbit eventually affords the desired smooth transition between the elliptic and hyperbolic regions of motion, recalling that

ellipse	$a > 0 \ , \ \mathcal{E} < 0 \ , \ \alpha > 0$
parabola	$a = \infty \ , \ \mathcal{E} \ , \ \alpha = 0$
hyperbola	$a < 0 \ , \ \mathcal{E} > 0 \ , \ \alpha < 0$

This description is also uniformly valid for the rectilinear class of orbits. An energy integral may be obtained for any conservative force field, not necessarily central.

Conservation of Angular Momentum

A vector integral of the two-body problem, valid only for central force fields, is obtained by constructing the vector cross product of (1-1) with the position vector $\bar{r}$, which leads to

$$\frac{d}{dt}(\bar{r} \times \dot{\bar{r}}) = 0 \tag{2-4}$$

leading, in turn, to the law of conservation of angular momentum,

$$\bar{h} = \bar{r} \times \dot{\bar{r}} = \text{constant} \tag{2-5}$$

The invariance of the length of this vector is another statement of Kepler's second law, namely, that the particle sweeps out equal areas in equal times, while the invariance of the direction is an alternate statement of the law that Keplerian motion takes place in a fixed plane. For rectilinear motion $(\bar{h}=0)$, both arguments are still valid, although the orbit plane is indeterminate and the area is zero.

An alternate statement of Kepler's second law is

$$\frac{1}{2}\dot{A} = h = r^2 \frac{df}{dt} \tag{2-6}$$

where $\dot{A}$ is the rate at which the area of the orbit is swept by the position vector $\bar{r}$, and f is the true anomaly. The expression for the scalar angular momentum is used as an independent variable transformation equation to establish the following well-known differential equation

$$u'' + u = 1/p = \mu/h^2 \tag{2-7}$$

where $u = 1/r$, p is the semilatus rectum and $(\cdot)'$ denotes the derivative with respect to the true anomaly. The integral of (2-7) is the familiar conic section relation

$$r = \frac{p}{1 + \epsilon C_f} \tag{2-8}$$

where the abbreviated notation C_f is used to represent cos f (equivalently, S_f would be used to represent sin f).

The above transformation of variables from the time t to a new independent variable f is an example of the general method to be subsequently outlined which transforms both the unperturbed and perturbed two-body problems out of the time domain into more convenient domains of integration defined by new independent variables.

Eccentricity Vector

Another vector integral may be obtained which is unique to the inverse square central force field and is variously known as Laplace's first vector, Hamilton's ϵ vector, or the apsidal vector; throughout this research it will be referred to as the eccentricity vector.

From

$$\begin{aligned} d(\bar{r}/r)/dt &= [(\bar{r} \cdot \bar{r})\dot{\bar{r}} - (\bar{r} \cdot \dot{\bar{r}})\bar{r}]/r^3 \\ &= [\bar{r} \times \dot{\bar{r}}) \times \bar{r}]/r^3 \\ &= \bar{h} \times \bar{r}/r^3 \\ &= \ddot{\bar{r}} \times \bar{h}/\mu \end{aligned} \tag{2-9}$$

we obtain

$$\bar{r}/r = \dot{\bar{r}} \times \bar{h}/\mu - \bar{\epsilon} \tag{2-10}$$

where $\bar{\epsilon}$ is the vector constant of integration. Construction of the dot product of (2-10) and $\bar{r}$ and comparison with (2-8) indentifies the vector $\bar{\epsilon}$ as directed along periapsis with length equal to the numerical eccentricity of the conic section. It might also be noted that the eccentricity vector is well defined for rectilinear orbits, for which $\bar{h} = 0$ and $\bar{\epsilon} = - \hat{r}$, or $\epsilon = 1$, and also for circular orbits.

For the unperturbed problem, we are apparently confronted with seven constants of integration (one scalar and two vectors) for a sixth order system. However, it is obvious and may be verified that the $\bar{h}$ and $\bar{\epsilon}$ vectors are orthogonal, or

$$\bar{h} \cdot \bar{\epsilon} = 0 \tag{2-11}$$

Also it may be shown that

$$\alpha \bar{h} \cdot \bar{h} = \mu (1 - \bar{\epsilon} \cdot \bar{\epsilon}) \tag{2-12}$$

or

$$p = a(1 - \epsilon^2) \tag{2-12a}$$

Thus Eq. (2-11) and (2-12) represent two constraint equations or side conditions, and the quantities α, $\bar{h}$ and $\bar{\epsilon}$ effectively represent five constants of integration. The sixth constant of integration is the time of periapsis passage.

Hodograph of Keplerian Motion

By constructing the cross product of $\bar{\epsilon}$ and $\bar{h}$

$$\bar{\epsilon} \times \bar{h} = (\bar{v} \times \bar{h}) \times \bar{h}/\mu - \hat{r} \times \bar{h} \tag{2-13}$$

one finds that

$$\bar{v} = - \bar{\epsilon} \times \bar{h}/p - \hat{r} \times \bar{h}/p \tag{2-14}$$

Thus the velocity vector is the sum of two vectors of constant magnitude, one orthogonal to the $\bar{\epsilon}$ vector and one orthogonal to the instantaneous unit radius vector. This is the familiar hodograph of Keplerian motion; reference [7] presents diagrams of Eq. (2-14) for elliptic, parabolic, and hyperbolic motion.

Regularization and the Central Oscillator System

Having reviewed the fundamental integrals of the two-body problem, we are now in a position to proceed with a regularization of (1-1).

The procedure to be employed is a change in the independent variable from time to a new variable x , the defining relation being

$$\frac{dx}{dt} = \frac{\sqrt{\mu}}{r} \tag{2-15}$$

The variable x is occasionally referred to as the fictitious or artificial time variable; however, it attains a more significant meaning by noting the equations for the scalar radius and time expressed in terms of the eccentric anomaly E .

$$r = a(1 - \epsilon C_E) \tag{2-16}$$

$$t = (a^3/\mu)^{1/2} (E - \epsilon S_E) \tag{2-17}$$

Differentiation of (2-17) with respect to time and substitution of (2-16) leads to

$$\dot{E} = (\mu/a)^{1/2} /r \tag{2-18}$$

which may be compared to (2-15) to obtain

$$x = \sqrt{a} (E - E_o) . \tag{2-19}$$

for the ellipse, and equivalently

$$x = \sqrt{|a|} (F - F_o) \tag{2-20}$$

where F is the hyperbolic anomaly for the hyperbola, and where the variable x is assumed to vanish at the initial values E_o and F_o . A more general quadrature may be obtained by using the energy equation (2-3)

$$\alpha = \frac{2}{r} - \frac{1}{\mu} (\dot{r}^2 + r^2 \dot{\theta}^2) \tag{2-21}$$

and the angular momentum equation (2-6) to obtain

$$\dot{r}^2 = 2\mu/r - \mu p/r^2 - \mu\alpha \tag{2-22}$$

Substitution of (2-22) in (2-15) yields an integral form

$$x = \int_{t_o}^{t} \frac{\sqrt{\mu}\, dt}{r} = \int_{r_o}^{r} \frac{\sqrt{\mu}\, dr}{r\dot{r}} = \int_{r_o}^{r} \frac{dr}{(2r-p-\alpha r^2)^{1/2}} \tag{2-23}$$

For the general parabola (α=0)

$$x = (2r - p)^{1/2} - (2r_o - p)^{1/2} \tag{2-24}$$

and for the rectilinear parabola ($\alpha = p = 0$)

$$x = (2r)^{1/2} - (2r_o)^{1/2} \tag{2-25}$$

The relations (2-19) and (2-20) may be obtained from the more general quadrature of (2-23) for $\alpha \neq 0$.

An alternative form for the general parabola may be obtained by using (2-6) and (2-8) in (2-23), yielding

$$x = \int_{f_o}^{f} \frac{\sqrt{\mu}\, df}{r\dot{f}} = \int_{f_o}^{f} \frac{\sqrt{p}\, df}{1 + C_f} \tag{2-26}$$

which results in

$$x = \sqrt{p}\,[\tan (f/2) - \tan (f_o/2)] \tag{2-27}$$

Regularization of (1-1) is then accomplished by taking twice derivatives of $\bar{r}$

$$\bar{r}' = \frac{d}{dt}(\bar{r})\frac{dt}{dx} = \dot{\bar{r}}\,t' \tag{2-28}$$

$$\bar{r}'' = \ddot{\bar{r}}\,t'^2 + \dot{\bar{r}}\,t'' \tag{2-28a}$$

where

$$t'' = \frac{d}{dt}(t')t' = \bar{r} \cdot \dot{\bar{r}}/\mu \tag{2-29}$$

Substitution of (1-1) and (2-29) into (2-28a) results in

$$\bar{r}'' = (\bar{r} \cdot \dot{\bar{r}}/\mu)\,\dot{\bar{r}} - \hat{r} \tag{2-30}$$

From (2-10) we obtain

$$\bar{\epsilon} = \dot{\bar{r}} \times \bar{h}/\mu - \hat{r} \tag{2-31}$$

Substitution of (2-5) yields

$$\bar{\epsilon} = (\dot{\bar{r}} \cdot \dot{\bar{r}}/\mu)\bar{r} - (\bar{r} \cdot \dot{\bar{r}}/\mu)\,\dot{\bar{r}} - \hat{r} \tag{2-31a}$$

and the second term in the RHS of (2-31a) is recognized as the second term in (2-30); substitution yields

$$\bar{r}'' + (2/r - \dot{\bar{r}} \cdot \dot{\bar{r}}/\mu)\,\bar{r} = -\bar{\epsilon} \tag{2-32}$$

The scalar factor of $\bar{r}$ is recognized as the modified energy α , thus establishing a regularized vector orbit equation

$$\bar{r}'' + \alpha\,\bar{r} = -\bar{\epsilon} \tag{2-33}$$

The resulting equation is not only nonsingular (as expected), but also a linear, constant coefficient vector equation. Since α is a scalar constant, the components of $\bar{r}$ are governed by uncoupled second-order differential equations. Moreover, both constants of (2-33) are orbit constants, and the type of conic (i.e., ellipse, parabola, hyperbola) directly corresponds to the type of (2-33) according to the signed value of α .

Constructing the dot product of (2-33) with $\bar{r}$, and noting that

$$r' = \bar{r}' \cdot \bar{r}/r \tag{2-34}$$

and

$$\begin{aligned} d(\bar{r}' \cdot \bar{r})/dx &= \bar{r}'' \cdot \bar{r} + \bar{r}' \cdot \bar{r}' \\ &= r'' r + r'^2 \end{aligned} \tag{2-34a}$$

we obtain a similar equation for the scalar radius

$$r'' + \alpha r = 1 \tag{2-35}$$

The system of equations (2-33) and (2 35), together with the inverse of the transformation equation (2-15) are summarized as

$$\bar{r}'' + \alpha \bar{r} = - \bar{\epsilon} \tag{2-36}$$

$$r'' + \alpha r = 1 \tag{2-36a}$$

$$t' = r/\sqrt{\mu} \tag{2-36b}$$

and comprise what is referred to as the central orbit oscillator system. It will be noted that the system is completely regular and well-defined for all types of orbits. Also, the necessary orbit constants are represented in bounded form (i.e., the term α as opposed to its reciprocal a).

The Focal Oscillator System

As stated previously, the defining equation (2-6) for the scalar angular momentum is used as an independent variable transformation equation to obtain (2-7) in the same sense (2-15) was used to obtain the central oscillator system. To review,

$$df/dt = h/r^2 , \quad \text{or} \quad t' = r^2/h \tag{2-6}$$

$$u' = \dot{u}\, t' \tag{2-37}$$

$$u'' = \ddot{u}\, t'^2 + \dot{u}\, t'' \tag{2-38}$$

where $u = 1/r$ and $(\cdot)'$ now denotes the derivative with respect to the true anomaly f . Also,

$$\dot{u} = - \bar{r} \cdot \dot{\bar{r}}/r^3 \tag{2-39}$$

$$\ddot{u} = - (\dot{\bar{r}} \cdot \dot{\bar{r}} + \bar{r} \cdot \ddot{\bar{r}})/r^3 + 3\, \bar{r} \cdot \dot{\bar{r}}/r^5 \tag{2-40}$$

and

$$\begin{aligned} t'' &= d(t')/dt \cdot t' \\ &= 2\, r^2\, \bar{r} \cdot \dot{\bar{r}}/h^2 \end{aligned} \tag{2-41}$$

Noting that

$$\bar{r} \cdot \ddot{\bar{r}} = - \mu/r \tag{2-42}$$

$$h^2 = r^2[\dot{\bar{r}} \cdot \dot{\bar{r}} - (\bar{r} \cdot \dot{\bar{r}}/r)^2] \tag{2-43}$$

we obtain

$$u'' = \frac{r^4}{h^2}\left[-\frac{\dot{\bar{r}}\cdot\dot{\bar{r}}}{r^3} + \frac{\mu}{r^4} + \frac{3(\bar{r}\cdot\dot{\bar{r}})^2}{r^5}\right] - \left(2\frac{r^2}{h^2}\bar{r}\cdot\dot{\bar{r}}\right)\left(\frac{\bar{r}\cdot\dot{\bar{r}}}{r^3}\right)$$

$$= \frac{\mu}{h^2} - \frac{r}{h^2}\left[\dot{\bar{r}}\cdot\dot{\bar{r}} - \left(\frac{\bar{r}\cdot\dot{\bar{r}}}{r}\right)^2\right] = \frac{\mu}{h^2} - u$$

(2-44)

The vector equation which is used in conjunction with this scalar equation may be obtained through the same transformation mechanics. From

$$\hat{r}' = \dot{\hat{r}}\,t' \tag{2-45}$$

$$\hat{r}'' = \ddot{\hat{r}}\,t'^2 + \dot{\hat{r}}\,t'' \tag{2-46}$$

and

$$\bar{r} = r\hat{r} \tag{2-47}$$

we obtain

$$\ddot{\bar{r}} = \ddot{r}\hat{r} + 2\,\dot{r}\dot{\hat{r}} + r\ddot{\hat{r}} = -\frac{\mu}{r^2}\hat{r} \tag{2-48}$$

where, noting that

$$r\ddot{r} + \dot{r}^2 = \bar{r}\cdot\ddot{\bar{r}} + \dot{\bar{r}}\cdot\dot{\bar{r}} = -\frac{\mu}{r} + \dot{\bar{r}}\cdot\dot{\bar{r}} \tag{2-49}$$

and

$$t'' = 2\,rr'/h = 2\,r^3\dot{r}/h^2 \tag{2-50}$$

we obtain

$$
\begin{aligned}
\hat{r}'' &= \frac{r^4}{h^2}\left(-\frac{\mu}{r^3}\,\hat{r} - \frac{\ddot{r}}{r}\,\hat{r} - \frac{2\dot{r}}{r}\,\dot{\hat{r}}\right) + \frac{2\,r^3\dot{r}}{h^2}\,\dot{\hat{r}} \\
&= \frac{r^4}{h^2}\left(-\frac{\mu}{r^3}\,\hat{r} + \frac{\mu}{r^3}\,\hat{r} - \frac{\dot{\bar{r}}\cdot\dot{\bar{r}}}{r^2}\,\hat{r} + \frac{\dot{r}^2}{r^2}\,\hat{r}\right) \\
&= \frac{r^4}{h^2}\left(-\frac{h^2}{r^4}\right)\hat{r} = -\,\hat{r}
\end{aligned}
\tag{2-51}
$$

It could be easily verified that

$$\hat{r} = \hat{e}\,C_f + (\hat{h} \times \hat{\epsilon})S_f \tag{2-52}$$

leads directly to (2-51); however, the more elaborate derivation using the independent variable transformation facilitates the inclusion of perturbing forces in the next chapter.

Thus the following analogous harmonic oscillator system is obtained for unperturbed Keplerian motion:

$$\hat{r}'' + \hat{r} = 0 \tag{2-53}$$

$$u'' + u = 1/p \tag{2-53a}$$

$$t' = r^2/h \tag{2-53b}$$

where the new independent variable is the true anomaly for all nonrectilinear orbits. Due to its origins, this system has been previously referred to as the classical oscillator system. For rectilinear orbits, the inverse of the time equation may be compared to Newton's equation along the rectilinear orbit ($\dot{v} = \pm\ \mu/r^2$) to obtain an expression for the independent variable analogous to the true anomaly as

$$f_{rect} = \pm\ hv/\mu \tag{2-54}$$

However, $h = 0$ for rectilinear orbits, and the classical oscillator system would not appear to be applicable for this class of orbits. It is, however, since a particular solution to (2-53a) is given by

$$u = (1 - C_f)/p + \text{constant} \tag{2-55}$$

and, by series expansion, using (2-54),

$$\begin{aligned} u &= \frac{1}{p}\left(\frac{h^2v^2}{2\mu}\right) + \text{constant} \\ &\doteq v^2/2\mu + \text{constant} \end{aligned} \tag{2-56}$$

where the constant is $\alpha/2$ from the energy integral (2-3).

The aforementioned algebraic difficulty may be avoided by modifying the transformation equation (2-6) to

$$dy/dt = \sqrt{\mu}/r^2 \tag{2-57}$$

where $y = f/\sqrt{p}$ for nonrectilinear orbits. Taking the independent variable to vanish at the initial value f_0 results in

$$y = (f - f_0)/\sqrt{p} \tag{2-58}$$

for nonrectilinear orbits. For rectilinear orbits, using (2-3) and (2-22) in (2-57),

$$y = \int \frac{\sqrt{\mu}\ dt}{r^2} = \int \frac{dr}{r^2 \dot{r}} = \pm \int \frac{dr}{r\sqrt{2r - \alpha r^2}} = \pm \frac{v - v_o}{\sqrt{\mu}} \qquad (2\text{-}59)$$

where the + sign is taken for the particle approaching the singularity.

Using the $(\cdot)'$ notation to denote derivatives with respect to the independent variable y, the resulting system of equations of unperturbed Keplerian motion may be obtained as

$$\hat{r}'' + p\,\hat{r} = 0 \qquad (2\text{-}60)$$

$$u'' + p\,u = 1 \qquad (2\text{-}60a)$$

$$t' = r^2/\sqrt{\mu} \qquad (2\text{-}60b)$$

and is referred to as the focal oscillator system. Interestingly, the natural frequency of this system is related to the angular momentum, whereas the natural frequency of the central oscillator system is related to energy.

The solution to the focal oscillator system is identical to that of the classical oscillator system; the advantage of the focal oscillator system is that a perturbation analysis of the focal oscillator will be concerned with variations of the bounded quantity p, whereas a perturbation analysis of the classical oscillator system would involve variations of the unbounded term $1/p$.

Solution to Central Oscillator System

The general solution to the central oscillator system (2-36) may be expressed by

$$\begin{pmatrix} \bar{r}(x) \\ \\ \bar{r}'(x) \end{pmatrix} = \begin{pmatrix} U_1(x) & U_2(x) & -U_3(x) \\ \\ U_0(x) & U_1(x) & -U_2(x) \end{pmatrix} \begin{pmatrix} \bar{r}(0) \\ \\ \bar{r}'(0) \\ \bar{\epsilon} \end{pmatrix} \tag{2-61}$$

$$r(x) = U_1(x)r(0) + U_2(x)r'(0) + U_3(x) \tag{2-61a}$$

$$\sqrt{\mu}(t-t_0) = U_2(x)r(0) + U_3(x)r'(0) + U_4(x) \tag{2-61b}$$

where

$$\begin{aligned} U_0(x) &= -\sqrt{\alpha}\,\sin\sqrt{\alpha}\,x \\ U_1(x) &= \cos\sqrt{\alpha}\,x \\ U_2(x) &= \sqrt{a}\,\sin\sqrt{\alpha}\,x \\ U_3(x) &= a\,(1-\cos\sqrt{\alpha}\,x) \\ U_4(x) &= a\,(x-\sqrt{a}\,\sin\sqrt{\alpha}\,x) \end{aligned} \tag{2-62}$$

and where the U_j have been defined such that

$$\frac{\partial}{\partial x} U_{j+1}(x) = U_j(x) \quad , \quad j = 0, \dots 3 \tag{2-63}$$

The U_j are a form of universal functions, or variables, in that they are valid for all type of orbits. This is directly obvious for elliptic orbits ($\alpha > 0$), and can be verified for hyperbolic orbits by noting that the circular functions convert to hyperbolic functions for α (and a) < 0 . To render the solution universal in the computational sense for arbitrary values of α the U_j may be represented by their series expansions

$$\begin{aligned}
U_0(x) &= -\alpha x + \alpha^2 x^3/3! - \ldots \\
U_1(x) &= 1 - \alpha x^2/2! + \alpha^2 x^4/4! - \ldots \\
U_2(x) &= x - \alpha x^3/3! + \alpha^2 x^5/5! - \ldots \\
U_3(x) &= x^2/2! - \alpha x^4/4! + \alpha^2 x^6/6! - \ldots \\
U_4(x) &= x^3/3! - \alpha x^5/5! + \alpha^2 x^7/7! - \ldots
\end{aligned} \tag{2-64}$$

It is important to observe that the universal functions may be used in their circular function form of Eq. (2-62) for analytical manuipulation, such as differentiation and integration, for arbitrary values of α ; the result of such manipulation may then be converted to hyperbolic functions for hyperbolic motion or be represented by the more general series expansions of (2-64). This precludes the necessity of developing a special table of derivatives and integrals of say, the series functions (2-64).

The universal solution (2-61) differs from the better known universal orbit formulae (such as that of reference [6]) in that the final and initial state vector are presented in the regularized domain, where the regularized velocity $\bar{r}'$ [or $\bar{r}'(0)$] approaches zero as the particle approaches the center of attraction whereas the physical, or time domain, velocity becomes unbounded. The final state vector is related to the initial state vector and the constant eccentricity vector by a state transition matrix of scalar quantities which is explicitly free of either initial or terminal values of either the scalar radius or time. The convenience of analytical manipulations in the domain of the oscillator systems (here the regularized domain of the central oscillator) is carried forth in the perturbation analyses.

Solution in Time Domain

The universal orbit solution (2-61) may be transformed to the time domain by noting from (2-15) that

$$\bar{r}' = \dot{\bar{r}}/\dot{x} = \dot{\bar{r}}r/\sqrt{\mu} \tag{2-65}$$

and from (2-34)

$$r' = \bar{r} \cdot \bar{r}'/r = \bar{r} \cdot \dot{\bar{r}}/\sqrt{\mu} \tag{2-66}$$

resulting in

$$\begin{pmatrix} \bar{r}(x) \\ \dot{\bar{r}}(x) \end{pmatrix} \begin{pmatrix} U_1(x) & \frac{r(0)}{\sqrt{\mu}}\,U_2(x) & -\,U_3(x) \\ \frac{\sqrt{\mu}}{r(x)}\,U_0(x) & \frac{r(0)}{r(x)}\,U_1(x) & -\,\frac{\sqrt{\mu}}{r(x)}\,U_2(x) \end{pmatrix} \begin{pmatrix} \bar{r}(0) \\ \bar{r}'(0) \\ \bar{\epsilon} \end{pmatrix}$$

(2-67)

and

$$r(x) = U_1(x)\, r(0) + U_2(x) \frac{\overline{r}(0) \cdot \dot{\overline{r}}(0)}{\sqrt{\mu}} + U_3(x) \tag{2-67a}$$

$$\sqrt{\mu}\,(t-t_o) = U_2(x)\, r(0) + U_3(x) \frac{\overline{r}(0) \cdot \dot{\overline{r}}(0)}{\sqrt{\mu}} + U_4(x) \tag{2-67b}$$

This manipulation necessarily introduces values of the scalar radius in the state transiton matrix of (2-67), and possibly singularities at $r(x) = 0$. It demonstrates the ease of operating entirely in the regularized domain with the well-behaved regularized velocities, and transforming to the time domain (if at all necessary) only as a final step.

Modification of Solution

It is instructive to investigate the substitutions required to modify the foregoing universal orbit solution to a more familiar form, for instance, the solution presented in reference [6], which is based on an alternate set of universal functions $S(\alpha x^2)$ and $C(\alpha x^2)$, defined by

$$S(\alpha x^2) = 1/3! - \alpha x^2/5! + \alpha^2 x^4/7! - \ldots \tag{2-68}$$

$$C(\alpha x^2) = 1/2! - \alpha x^2/4! + \alpha^2 x^6/6! - \ldots \tag{2-68a}$$

In terms of the U_j of (2-64), these functions appear as

$$S(\alpha x^2) = (U_o + \alpha x)/\alpha^2 x^2 = (x - U_2)/\alpha x^3 = U_4/x^3$$

$$C(\alpha x^2) = (1 - U_1)/\alpha x^2 = U_3/x^2$$

(2-69)

or inversely,

$$U_o = -\alpha x + \alpha^2 x^3 S(\alpha x^2)$$
$$U_1 = 1 - \alpha x^2 C(\alpha x^2)$$
$$U_2 = x - \alpha x^3 S(\alpha x^2)$$
$$U_3 = x^2 C(\alpha x^2)$$
$$U_4 = x^3 S(\alpha x^2)$$

(2-70)

Substitution of (2-70) into (2-67a) and (2-67b) directly yields the universal time equation of reference [6],

$$\sqrt{\mu}\,(t-t_o) = \frac{\bar{r}(0) \cdot \dot{\bar{r}}(0)}{\sqrt{\mu}} x^2 C(\alpha x^2) + [1 - \alpha r(0)]x^3 S(\alpha x^2) + r(0)x$$

(2-71)

and the universal scalar radius equation

$$r(x) = \sqrt{\mu}\,\frac{dt}{dx} = \frac{\bar{r}(0) \cdot \dot{\bar{r}}(0)}{\sqrt{\mu}} [x - \alpha x^3 S(\alpha x^2)] + [1 - \alpha r(0)]x^2 C(\alpha x^2) + r(0)$$

(2-72)

Evaluating (2-31a) at the initial conditons results in

$$\bar{\epsilon} = \frac{\dot{\bar{r}}(0) \cdot \dot{\bar{r}}(0)}{\mu} \bar{r}(0) - \frac{\bar{r}(0) \cdot \dot{\bar{r}}(0)}{\mu} \dot{\bar{r}}(0) - \frac{\bar{r}(0)}{r(0)} \quad (2\text{-}73)$$

which, when substituted along with (2-70) into (2-67), results in

$$\begin{bmatrix} \bar{r}(x) \\ \dot{\bar{r}}(x) \end{bmatrix} \begin{bmatrix} 1 - \dfrac{x^2}{r(0)} C(\alpha x^2) & \dfrac{\bar{r}(0) \cdot \dot{\bar{r}}(0)}{\mu} x^2 C(\alpha x^2) + \dfrac{r(0)}{\sqrt{\mu}} [x - \alpha x^3 S(\alpha x^2)] \\ - \dfrac{\sqrt{\mu}}{r(x) r(0)} [x - \alpha x^3 S(\alpha x^2)] & 1 - \dfrac{x^2}{r(x)} C(\alpha x^2) \end{bmatrix} \begin{bmatrix} \bar{r}(0) \\ \dot{\bar{r}}(0) \end{bmatrix}$$

(2-74)

By substituting from the universal time equation (2-71) into the upper right element of the above matrix, we may obtain a further modification

$$\begin{bmatrix} \bar{r}(x) \\ \dot{\bar{r}}(x) \end{bmatrix} \begin{bmatrix} 1 - \dfrac{x^2}{r(0)} C(\alpha x^2) & t - \dfrac{x^3}{\sqrt{\mu}} S(\alpha x^2) \\ - \dfrac{\sqrt{\mu}}{r(x) r(0)} [x - \alpha x^3 S(\alpha x^2)] & 1 - \dfrac{x^2}{r(x)} C(\alpha x^2) \end{bmatrix} \begin{bmatrix} \bar{r}(0) \\ \dot{\bar{r}}(0) \end{bmatrix}$$

(2-75)

which is the form presented in reference [6].

The particular form of the solution and/or form of universal functions to be used is a matter of choice. The purpose of the foregoing was to demonstrate a development of the universal orbit formulation from the regularized central orbit oscillator system equations to a well-known solution which is expressed in the time domain. Some of the advantages of the basic solution in the regularized domain (Eq. (2-61)) are the explicit dependence of the vector solution on a constant vector integral of the two-body problem, namely, the eccentricity vector, and the explicit independence of the state transition matrix on either the instantaneous scalar radius or time.

Initial Conditions at Periapsis

The general solution (2-61) to the central oscillator system may be particularized by referring the initial conditions to periapsis and expressing the solution in the orbit coordinate system defined by $(\hat{\epsilon}\ ,\ \hat{h}\times\hat{\epsilon}\ ,\ \hat{h})$. In this frame, using (2-62) and

$$\begin{aligned}\bar{r}_p &= (r_p\ ,\ 0\ ,\ 0\ )^T\\ \bar{r}_p' &= (0\ ,\ \sqrt{p}\ ,\ 0)^T\\ \bar{\epsilon} &= (\epsilon\ ,\ 0\ ,\ 0)^T\end{aligned}$$

we obtain

$$\bar{r}(x) = [r_p - U_3(x)\ ,\ \sqrt{p}\ U_2(x)\ ,\ 0]^T \tag{2-76}$$

$$\bar{r}'(x) = [-\ U_2(x)\ ,\ \sqrt{p}\ U_1(x)\ ,\ 0]^T \tag{2-76a}$$

Noting that

$$r_p' = \hat{r}_p \cdot \bar{r}_p' = 0\ ,$$

we obtain

$$\begin{aligned}r(x) &= r_p\ U_1(x) + U_3(x)\\ &= r_p + \epsilon\ U_3(x)\\ &= a[1 - \epsilon\ U_1(x)]\end{aligned} \tag{2-77}$$

and

$$\sqrt{\mu}\ (t-t_p) = r_p\, x + \epsilon\, U_4(x) \qquad (2\text{-}78)$$

In this form the independent variable is taken to vanish at periapsis. All the well-known expressions for the solution to the inverse square problem (for the ellipse, say, in terms of the eccentric anomaly) are directly obtainable.

Solution to Focal Oscillator System

The general solution to the focal oscillator system (2-60) is

$$\begin{pmatrix} \hat{r}(y) \\ \hat{r}'(y) \end{pmatrix} \quad \begin{pmatrix} V_1(y) & V_2(y) \\ V_0(y) & V_1(y) \end{pmatrix} \begin{pmatrix} \hat{r}(0) \\ \hat{r}'(0) \end{pmatrix} \qquad (2\text{-}79)$$

$$u(y) = V_1(y)\, u(0) + V_2(y)\, u'(0) + V_3(y) \qquad (2\text{-}79a)$$

where, following the pattern established in the definition of the universal functions U_j of the central oscillator, it is convenient to define similar functions

$$\begin{aligned} V_0(y) &= -\sqrt{p}\ \sin\sqrt{p}\, y \\ V_1(y) &= \cos\sqrt{p}\, y \\ V_2(y) &= \frac{1}{\sqrt{p}} \sin\sqrt{p}\, y \\ V_3(y) &= \frac{1}{p}\,(1 - \cos\sqrt{p}\, y) \end{aligned} \qquad (2\text{-}80)$$

for the focal oscillator system, noting that

$$\frac{\partial}{\partial y} V_{j+1}(y) = V_j(y), \quad j = 0,1,2 \qquad (2\text{-}81)$$

As stated before, this solution may be regarded as a quasi-universal solution in that it is applicable to any type of orbit (elliptic through hyperbolic), which may not be surprising recalling the "universal" definition of the true anomaly. Unlike that of the central oscillator system the solution is valid as it stands for parabolic motion, but must be modified for rectilinear motion through series expansions of the V_j and the alternate definition of y for rectilinear orbits. The independent variable y is still unbounded at periapsis for rectilinear orbits, whereas x is well defined. The solution for rectilinear orbits is

$$\hat{r}(y) = \text{constant} \tag{2-82}$$

$$u(y) = u(0) + u'(0)\, y + y^2/2 \tag{2-82a}$$

where $y = \pm (v - v_o)/\sqrt{\mu}$. A general solution to the time equations for both the rectilinear orbit and the general orbit (2-79) is more conveniently obtained by referencing the solution to periapsis.

Regarding the mechanics of the solution (2-79) it will be recalled that $u = 1/r$ and hence

$$u' = -\frac{r'}{r^2} = -\frac{\bar{r} \cdot \bar{r}'}{r^2} = -\frac{\bar{r} \cdot \dot{\bar{r}}}{\sqrt{\mu}} \tag{2-83}$$

The position vector $\bar{r}$ is obtained from $\bar{r} = \hat{r}/u$, and from (2-9),

$$\hat{r}' = \dot{\hat{r}}\, t' = \frac{\bar{h} \times \hat{r}}{\sqrt{\mu}} = \sqrt{p}\, (\hat{h} \times \hat{r}) \tag{2-84}$$

resulting in

$$\hat{r}' \cdot \hat{r}' = p \tag{2-85}$$

The velocity vector $\dot{\bar{r}}$ is obtained from

$$\dot{\bar{r}} = \bar{r}' \dot{y} \tag{2-86}$$

where

$$\bar{r}' = r'\hat{r} + r\hat{r}' \tag{2-87}$$

and

$$r' = - r^2 u' \tag{2-83}$$

<u>Initial Conditions at Periapsis</u>

For the general orbit at periapsis, $\hat{r}_p'$ is orthogonal to $\hat{r}_p$ and the solution, expressed in $(\hat{\epsilon}, \hat{h} \times \hat{\epsilon}, \hat{h})$, is

$$\hat{r} = (C_f , S_f , 0)^T. \tag{2-88}$$

for $\hat{r}_p = (1 , 0 , 0)^T$ and $\hat{r}_p' = \sqrt{p}\ (0, 1, 0)^T$. Since

$$u_p' = - \bar{r}_p \cdot \bar{r}_p'/r_p^2 = 0$$

we obtain

$$\begin{aligned} U_p &= \frac{1}{r_p} C_f + \frac{1}{p} (1 - C_f) \\ &= \frac{1 + \epsilon C_f}{p} \end{aligned} \tag{2-89}$$

The solution is identical to the classical orbit oscillator system (Eq. (2-8)). Hence, the time equation for the focal oscillator is identical to the standard forms obtained from the classical system.

For instance, for parabolic motion

$$t - t_p = \frac{1}{2}\sqrt{\frac{p^3}{\mu}}\,[\tan\,(f/2) + \frac{1}{3}\tan^3\,(f/2)] \qquad (2\text{-}90)$$

For nonparabolic nonrectilinear motion, it is convenient to appeal to the identities

$$\tan\frac{E}{2} = \sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan\frac{f}{2} \qquad (\epsilon < 1) \qquad (2\text{-}91)$$

$$\tanh\frac{F}{2} = \sqrt{\frac{1-\epsilon}{1+\epsilon}}\tan\frac{f}{2} \qquad (\epsilon > 1) \qquad (2\text{-}91a)$$

and use the resulting values for E(or F) in the universal time equation (Kepler's equation).

For rectilinear orbits, $\hat{r}(y)$ constant (as before) and

$$u = y^2/2 + \hat{r}/2 \text{ (Energy)}$$

where

$$y = \pm\, v/\sqrt{\mu} = \pm\sqrt{2u - \alpha}$$

and $y_0 = \infty$ at perapsis. For $\alpha = 0$,

$$t - t_p = \frac{4}{\sqrt{\mu}}\int_{\infty}^{y}\frac{d\sigma}{\sigma^4} = \frac{(2r)^{3/2}}{6\sqrt{\mu}} \qquad (2\text{-}92)$$

which agrees with the corresponding result obtained using the central oscillator (Eq. (2-61) for $r(0) = r'(0) = 0$). For $\alpha \neq 0$, it is most straightforward to obtain time from the central oscillator, or Kepler, time equation by obtaining u , hence r and x as a a function of y .

Chapter III

THE PERTURBED TWO-BODY PROBLEM

The perturbed two-body problem is defined in the time domain by the differential equation

$$\ddot{\bar{r}} = -\frac{\mu}{r^3}\bar{r} + \bar{f} \tag{3-1}$$

where $\bar{f} = \bar{f}(\bar{r}, \dot{\bar{r}}, \bar{t})$ is the force per unit mass of the particle.

The differential equation for the modified energy is directly obtained by differentiation of (2-3) and use of (3-1) to obtain

$$\dot{\alpha} = -\frac{2}{\mu}\dot{\bar{r}} \cdot \bar{f} \tag{3-2}$$

In like manner, from (2-5),

$$\dot{\bar{h}} = \bar{r} \times \bar{f} \tag{3-3}$$

and from (2-10)

$$\dot{\bar{\epsilon}} = \frac{1}{\mu}[\bar{f} \times \bar{h} + \dot{\bar{r}} \times (\bar{r} \times \bar{f})] \tag{3-4}$$

or, alternatively,

$$\dot{\bar{\epsilon}} = \frac{1}{\mu}[2\bar{r}\dot{\bar{r}} - \dot{\bar{r}}\bar{r} - \dot{\bar{r}} \cdot \bar{r}\, I]\,\bar{f} \tag{3-4a}$$

Perturbed Central Oscillator System

Repeating the derivation of the central oscillator system (2-36) using (3-1) leads directly to the perturbed central oscillator system

$$\bar{r}'' + \alpha(x)\bar{r} = -\bar{\epsilon}(x) + \frac{r^2}{\mu}\bar{f} \tag{3-5}$$

$$r'' + \alpha(x) r = 1 + \frac{r}{\mu} \bar{f} \cdot \bar{r} \tag{3-5a}$$

$$t' = \frac{r}{\sqrt{\mu}} \tag{3-5b}$$

where $\alpha(x)$ and $\bar{\epsilon}(x)$ are now varying orbit parameters whose differential equations are given by modified forms of (3-2) and (3-4) as

$$\alpha' = -\frac{2}{\mu} \bar{r}' \cdot \bar{f} \tag{3-6}$$

$$\bar{\epsilon}' = \frac{1}{\mu} [2\bar{r}\bar{r}' - \bar{r}'\bar{r} - \bar{r}' \cdot \bar{r} \, I] \, \bar{f} \tag{3-6a}$$

where

$$\bar{f} = \bar{f}(\bar{r}, \bar{r}'/t', t) \tag{3-6b}$$

The above system of equations (3-5) and (3-6) is regular and unaffected by the sign or numerical value of the energy parameter $\alpha(x)$. The system may be solved by numerical techniques directly as a thirteenth order system, although the instantaneous values of $\alpha(x)$ and $\bar{\epsilon}(x)$ may be calculated directly from the osculating element formulas (2-3) and (2-10), reducing the system order to nine. The one exception would be if the perturbed particle should pass through the singularity; at this point, $\alpha(x)$ is indeterminate due to the unboundedness of both terms on the RHS of (2-3). Thus it is necessary to appeal to the regularity and continuity of the differential equation (3-6) through the singularity. The eccentricity vector is still theoretically well defined from (2-10), since $\bar{h}(x) = 0$ if $r(x) = 0$ and hence $\bar{\epsilon}(x) = -\hat{r}(x)$. In a subsequent discussion concerning numerical computation, however, it will be shown that the direct computation of all orbital elements from osculating conic formulae is numerically inaccurate near the singularity and it is preferable to use the corresponding differential equations for the elements.

The order of the system may be further reduced by two by dispensing with the scalar radius equation (3-5a) and using

$$r(x) = [\bar{r}(x) \cdot \bar{r}(x)]^{1/2} \tag{3-7}$$

resulting in a total system order of seven. The scalar radius is used not only in the integrand of the time equation but also in the transformation of velocities in the regularized domain to physical velocities through

$$\dot{\bar{r}} = \bar{r}' \dot{x} \tag{3-8}$$

Perturbed Focal Oscillator System

Repeating the derivation of the focal oscillator system (2-60) using (3-1) results in the perturbed focal oscillator system

$$\hat{r}'' + p(y)\hat{r} = \frac{r^3}{\mu} [\bar{f} - (\bar{f}\cdot\hat{r})\hat{r}] \tag{3-9}$$

$$u'' + p(y)u = 1 - \frac{r^2}{\mu} \bar{f}\cdot\hat{r} \tag{3-9a}$$

$$t' = \frac{r^2}{\sqrt{\mu}} \tag{3-9b}$$

where $p(y)$ is now a varying orbit parameter whose differential equation is related to the perturbing force by

$$p' = \frac{2}{\mu} r^3 \hat{r}' \cdot \bar{f} \tag{3-10}$$

The latter expression is obtained by differentiating (2-85) and substituting (3-9), noting that $\hat{r} \cdot \hat{r}' = 0$. In a numerical computation the instantaneous value of $p(y)$ is obtained directly from (2-85), resulting in a total system order of nine. Also it is necessary to retain the unit vector nature of $\hat{r}$ at each step in the computation; this could be accomplished by a normalization. Note that the scalar equation of the system is not redundant as is its counterpart in the central oscillator system.

Numerical Computation

Burdet [7] investigates the numerical integration of both oscillator systems as presented, and demonstrates the numerical stability and adaptability of the systems in comparison to computation in the time domain. One of the results of his numerical experiments is that a greater degree of accuracy is achieved by numerical integration of the differential equation for the orbital elements $\alpha, \bar{h}$ (or p) and $\bar{\epsilon}$, as opposed to direct computation using the osculating element formulae. The numerical difficulty with the energy parameter α near periapsis arises from the unboundedness of both terms on the RHS of (2-3) and may result in the subtraction of large quantities. A different numerical problem near periapsis may occur in the computation of the angular momentum vector $\bar{h}$, to which $\bar{\epsilon}$ and p are related. The problem would be associated with the cross product nature of $\bar{h}$, which is obtained from the product of a very large quantity (velocity) and a very small quantity (the component of the radius vector orthogonal to $\dot{\bar{r}}$). The author's own experience with the evaluation of these quantities has led to the same conclusion. Burdet also advises against using any constraint relation (such as the unit vector nature of $\hat{r}$ or the orthogonality of $\hat{r}$ and $\hat{r}'$) to reduce the order of the system. In the interest of accuracy, he advocates solving the highest order system prior to imposing constraints. Burdet also compares the numerical integration of unperturbed focal and central oscillators with various time step size regulators, such as

$$\Delta t = r^n \Delta x \tag{3-11}$$

and automatic time-step size regulators defined during the process of integration which use some error criteria to either halve or double the time-step size. His results generally indicate lower numerical errors over a large number of revolutions of unperturbed circular orbits using either the oscillator systems or the time-step size regulators corresponding to the time transformation equations of the oscillators. He claims the accuracy increase is more pronounced for noncircular orbits.

For noncircular orbits, Burdet claims greater accuracy for the focal oscillator than for the central oscillator near periapsis, and just the reverse at apoapsis. Hence he proposes a mixed numerical procedure which uses the perturbed focal oscillator system near periapsis and the perturbed central oscillator near apoapsis. The reason for the increased accuracy of the focal oscillator near periapsis may be explained by examining a rectilinear orbit; the focal oscillator itself is not a regular system, since the corresponding independent variable is proportional to speed (Eq. (2-59)) and is unbounded at the singularity (periapsis), as is the reciprocal radius u. Therefore, fixed step sizes in the independent variable correspond to fixed increments in velocity, and the numerical procedure could theoretically take an unbounded number of steps reaching periapsis (which is equivalent to evaluating the unbounded quantity u). For the more realistic case of finite periapsis radius, the perturbed focal oscillator would then be expected to result in greater accuracy near periapsis, and an automatic step size regulator might be inclined to increase the step size near periapsis, just as it would decrease the step size in time if time were the independent variable.

The relation between the two oscillators and step size regulation is apparent from (3-11), where $n = 0$ corresponds to the time domain, and $n = 1,2$ correspond to the central and focal oscillators respectively. The lack of ambiguity of the central oscillator or its equivalent transformation equation used as a time-step size regulator would appear to make it the best general choice for numerical analysis. Use of the perturbed central oscillator system with fixed step size increments in the independent variable x results in a smooth numerical integration of the system state vector and a natural propagation of the system time at all regions of the orbit.

Cowell's Method

The equations of motion of the perturbed two-body problem have been presented in the time domain by (3-1) and in the domains of the central and focal oscillators in what is classically known as the Cowell form. This form of numerical computation is characterized by the determination of the total perturbed system state vector and generally requires a high degree of numerical precision. A much greater degree of numerical accuracy for the same precision is obtained by representing the total perturbed system state vector as the sum of a reference unperturbed conic state vector and a perturbation state vector, if the perturbation itself is small. Thus the major portion of the total perturbed state vector is obtainable from an analytic solution to the Kepler problem, and the method attains its greatest degree of utility when the particle is near periapsis. This method of representation of the perturbed two-body problem is referred to as Encke's method.

Encke's Method

The Encke perturbation differential equations of motion describe the difference in the perturbed system and the unperturbed reference system at the same instant of time, or possibly for the same value of the particular independent variable of the perturbation equations. The Encke perturbation equations, or variational equations, in the time domain are obtained in reference [6]. Linearization of these equations results in a time-varying system, which does not offer significant advantages over the nonlinear equations themselves. One important exception to the above is the linearized Encke equations in the time domain referenced to an unperturbed circular orbit; expressed in the rotating coordinate system which rotates with the particle (or at the same angular velocity as the particle about the primary body), the Encke perturbation equations reduce to a linear constant coefficient system known as the Euler-Hill equations. The extension to the noncircular case, however, reintroduces time-varying terms in the form of the angular velocity and angular acceleration of the reference coordinate system.

In the next chapter it will be shown that the representation of the perturbed two-body problem as perturbed harmonic oscillator systems leads to linearized variational equations which, although not constant coefficient systems of equations, are integrable in the domains of the oscillator systems (that is to say, using the independent variable and associated state vector representations of the oscillator systems). By their vector nature, the variational equations may be expressed in any nonrotating coordinate system and are valid for any value of eccentricity of the unperturbed reference conic.

Chapter IV

VARIATIONAL EQUATIONS

The development of Encke perturbation equations, or variational equations, is based on a vector identity of the form

$$\bar{r} = \bar{r}^o + \delta\bar{r} \tag{4-1}$$

where $\bar{r}$ represents the perturbed system radius vector, $\bar{r}^o$ the unperturbed system radius vector, and $\delta\bar{r}$ the vector difference of the two systems at the same value of the independent variable. The term $\delta\bar{r}$ is referred to as the "fixed x" variation, (e.g., for the central oscillator system) or perturbation in $\bar{r}^o$ due to the general perturbing force vector $\bar{f}(\bar{r},\bar{r}',t)$. This is contrasted to the "fixed t" variation, which compares the two systems at the same instant of the time. Since the preceding chapters have established the representation of perturbed Keplerian motion as two unique perturbed harmonic oscillator systems, we are led naturally into the establishment of fixed x (or fixed y) variational equations using the new independent variables x and y of the central and focal oscillators respectively. Since the derivations for both systems are essentially the same, the variational equations of the central oscillator will be examined in detail and the corresponding results for the focal oscillator will be presented without elaboration.

Central Oscillator System

From the vector identity (4-1) a corresponding scalar identity may be obtained for the scalar variation δr . Constructing the dot product of (4-1) with itself,

$$\bar{r} \cdot \bar{r} = r^2 = r^{o^2} + 2\bar{r}^o \cdot \delta\bar{r} + \delta\bar{r} \cdot \delta\bar{r} \tag{4-2}$$

where r and r^o are the perturbed and unperturbed scalar radii. Defining the scalar equivalent to (4-1) as

$$r = r^o + \delta r \tag{4-3}$$

we obtain

$$\delta r = (r^{o^2} + 2\bar{r}^o \cdot \delta\bar{r} + \delta\bar{r} \cdot \delta\bar{r})^{1/2} - r^o \tag{4-4}$$

Since (4-4) involves the differencing of nearly equal quantities, a more convenient form for computation is

$$\delta r = \frac{2\bar{r}^o \cdot \delta\bar{r} + \delta\bar{r} \cdot \delta\bar{r}}{(r^{o^2} + 2\bar{r}^o \cdot \delta\bar{r} + \delta\bar{r} \cdot \delta\bar{r})^{1/2} + r^o} \tag{4-4a}$$

Hence, for the central oscillator system only, a variational equation for the scalar radius is redundant; regardless of this, the scalar radius variational equation will be included in the central oscillator description for the sake of completeness and possible computational convenience. As will be subsequently shown, the scalar radius variation is required to obtain the variation in time along the perturbed system (since time is now regarded as a dependent variable of the system).

With regard to the future development of linearized variational equations, it might be noted that, to first order,

$$\delta r = \hat{r}^o \cdot \delta\bar{r} \tag{4-5}$$

which identifies the first order scalar variation as the radial component of the vector variation.

Time Variation

Since time is now regarded as a dependent variable of the motion, it may be greater or less along the perturbed system than along the unperturbed system. Defining the perturbed system time as

$$t = t^o + \delta t \tag{4-6}$$

along with the transformation equation (3-5b)

$$t' = \frac{r}{\sqrt{\mu}} ,$$

substitution of (4-3) along with

$$t^{o\prime} = \frac{r^o}{\sqrt{\mu}} \tag{4-7}$$

leads to

$$\delta t' = \frac{\delta r}{\sqrt{\mu}} \tag{4-8}$$

Variation of Orbit Elements

To facilitate the establishment of the system variational equations of motion, it is necessary to obtain the variations of the orbit elements α and $\bar{\epsilon}$ from their nominal or unperturbed values. The variations in α and $\bar{\epsilon}$ are represented as

$$\delta\alpha(x) = \alpha(x) - \alpha^o \tag{4-9}$$

$$\delta\bar{\epsilon}(x) = \bar{\epsilon}(x) - \bar{\epsilon}^o \tag{4-9a}$$

A direct calculation of the variations $\delta\alpha$ and $\delta\bar{\epsilon}$ using the osculating conic formulas (2-3) and (2-10) would involve the subtraction of sizeable quantities and violates the spirit of a variational treatment. Alternatively, the differential equations for the variations would be appropriate and are identical to (3-6);

$$\delta\alpha' = \alpha' = -\frac{2}{\mu}\bar{r}' \cdot \bar{f} \tag{4-10}$$

$$\delta\bar{\epsilon}' = \bar{\epsilon}' = \frac{1}{\mu}[2\bar{r}\bar{r}' - \bar{r}'\bar{r} - \bar{r}' \cdot \bar{r}I]\,\bar{f} \tag{4-10a}$$

The initial values $\delta\alpha(0)$ and $\delta\bar{\epsilon}(0)$ to be used in solving (4-10) are nonlinear functions of the initial value of the perturbation state vector $\delta\bar{r}(0)$ and $\delta\bar{r}'(0)$ and must be obtained from (4-9), however.

Nonlinear Variational Equations

The fixed x variational equations may be obtained directly by differencing the perturbed and unperturbed systems through the aforementioned identities, resulting in

$$\delta\bar{r}'' + (\alpha^o + \delta\alpha)\,\delta\bar{r} + \delta\alpha\,\bar{r}^o + \delta\bar{\epsilon} = \frac{r^2}{\mu}\bar{f} \tag{4-11}$$

$$\delta r'' + (\alpha^o + \delta\alpha)\,\delta r + \delta\alpha\, r^o = \frac{r}{\mu}\bar{f} \cdot \bar{r} \tag{4-11a}$$

$$\delta t' = \frac{\delta r}{\sqrt{\mu}} \tag{4-11b}$$

where

$$\bar{f} = \bar{f}(\bar{r}, \bar{r}', t) \tag{4-11c}$$

The total system, consisting of (4-11) and the differential equations for the orbital elements (4-10), may be evaluated directly as a thirteenth order system. The order of the system may be reduced by two by dispensing with the redundant equation (4-11a) and computing δr from (4-4).

Regularity of Variational Equations

Inspection of the variational equations (4-10) and (4-11) reveals them to be entirely free of any singularities at r, r^o, or $\delta r = 0$ (except, of course, when $\bar{f}$ has singular nature). The result is a well-behaved system of differential equations up to and including periapsis passage at the singularity. In addition, there is a natural smooth transition between elliptic and hyperbolic motion, including rectilinear motion. Since the theory is cast entirely in the regularized domain, using the well-behaved regularized velocities and variations, transformation to the possibly unbounded physical velocities is accomplished only as an end result through the unbounded transformation equation.

Another advantage in use of the central oscillator system is the relatively simple explicit dependence of the unperturbed system conic state vector and time on the independent variable x. In the time domain, given some particular value of t, it is necessary to resort to some iteration technique to establish x and hence the reference conic state vector.

This brings us to another interesting feature of the central oscillator system, namely, the "automatic" step size regulation feature of the central oscillator transformation equation (2-15). If one chooses to use variational equations in the time domain, a variable step size in time may be generated through a fixed step size in x through

$$\Delta t = t(x + \Delta x) - t(x) \tag{4-12}$$

which reduces to

$$\lim_{dx \to 0} dt = \frac{r^{o}}{\sqrt{\mu}} dx \tag{4-13}$$

Thus the reference conic state vector is obtained from the fixed-step incremented variable x while the numerical integration of the time-based variational equations proceeds using the variable step size in time. This step size variation in time tends to decrease the step size near the singularity and effectively smooths the numerical integration about periapsis.

Linearized Variational Equations

A set of linearized variational equations may be obtained directly from the nonlinear variational set. The nonlinearity due to the structure of the perturbing force is removed by expanding the perturbing force in a Taylor series about the unperturbed system and retaining only the first term; this is equivalent to simply evaluating $\bar{f}$ along the unperturbed system. (The first order terms of the Taylor series expansion of the forcing function could also be retained in some cases and still result in a linear system, although quite complicated.)

Retaining only the first order term in the differential equations for the orbit elements $\delta\alpha$ and $\delta\bar{\epsilon}$ results in

$$\delta\alpha' = -\frac{2}{\mu}\bar{r}^{o\prime} \cdot \bar{f} \tag{4-14}$$

$$\delta\bar{\epsilon}' = \frac{1}{\mu}\left[2\bar{r}^{o}\bar{r}^{o\prime} - \bar{r}^{o\prime}\bar{r}^{o} - \bar{r}^{o\prime} \cdot \bar{r}^{o}\, I\right]\bar{f} \tag{4-14a}$$

where

$$\bar{f} = \bar{f}(\bar{r}^{o}, \bar{r}^{o\prime}, t^{o})$$

The linearized variational equations may be written down by inspection as

$$\delta\bar{r}'' + \alpha^{o}\delta\bar{r} + \delta\alpha\,\bar{r}^{o} + \delta\bar{\epsilon} = \frac{r^{o^2}}{\mu}\bar{f} \tag{4-15}$$

$$\delta r'' + \alpha^{o}\delta r + \delta\alpha\, r^{o} = \frac{r^{o}}{\mu}\bar{r}^{o} \cdot \bar{f} \tag{4-15a}$$

$$\delta t' = \frac{\delta r}{\sqrt{\mu}} \tag{4-15b}$$

The resulting linearized variational equations are nonsingular, as expected, and involve functions of parameters of the unperturbed system, all of which may be expressed as polynomials in the independent variable x . Moreover, the structure of the resulting integrands are of a particularly simple form for a specific class of perturbing forces, to be discussed later.

Solution to Linearized Equations

The vector variational equation (4-15) may be written along with the auxiliary equations (4-14) as

$$\frac{d}{dx}\begin{pmatrix} \delta\bar{r} \\ \delta\bar{r}' \\ \delta\bar{\epsilon} \\ \delta\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\alpha & 0 & -1 & -\bar{r} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \delta\bar{r} \\ \delta\bar{r}' \\ \delta\bar{\epsilon} \\ \delta\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ r^2\bar{f}/\mu \\ [2\bar{r}\ \bar{r}'-\bar{r}'\ \bar{r}-\bar{r}'\cdot\bar{r}I]\bar{f}/\mu \\ -2\bar{r}'\cdot\bar{f}/\mu \end{pmatrix} \tag{4-16}$$

where the superscript 0 has been removed from the representation of the unperturbed system parameters. Expressing (4-16) in the abbreviated form

$$\frac{d}{dx}\begin{pmatrix} \delta\bar{r} \\ \delta\bar{r}' \\ \delta\bar{\epsilon} \\ \delta\alpha \end{pmatrix} = F_v[\bar{r}]\begin{pmatrix} \delta\bar{r} \\ \delta\bar{r}' \\ \delta\bar{\epsilon} \\ \delta\alpha \end{pmatrix} + \bar{g}_v[\bar{r},\bar{r}',\bar{f}] \tag{4-17}$$

the general solution to this linear, variable-coefficient system is given by

$$\begin{pmatrix} \delta\bar{r}(x) \\ \delta\bar{r}'(x) \\ \delta\bar{\epsilon}(x) \\ \delta\alpha(x) \end{pmatrix} = \Phi[(x-0),\bar{r}(0),\bar{r}'(0)] \begin{pmatrix} \delta\bar{r}_o \\ \delta\bar{r}'_o \\ \delta\bar{\epsilon}_o \\ \delta\alpha_o \end{pmatrix} + \int_0^x \Phi[(x-\sigma),\bar{r}(\sigma),\bar{r}'(\sigma)]\bar{g}_v[\sigma]d\sigma \tag{4-18}$$

where

$$\Phi[(x-z),\bar{r}(z),\bar{r}'(z)] = \begin{pmatrix} U_1(x-z) & U_2(x-z) & -U_3(x-z) & \phi_{r\alpha}[(x-z),\bar{r}(z),\bar{r}'(z)] \\ U_0(x-z) & U_1(x-z) & -U_2(x-z) & \phi_{r'\alpha}[(x-z),\bar{r}(z),\bar{r}'(z)] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4-19}$$

and

$$\begin{aligned} \phi_{r\alpha}[(x-z),\bar{r}(z),\bar{r}'(z)] &= -\bar{r}(z)\,\frac{x-z}{2}\,U_2(x-z) \\ &\quad + \bar{r}'(z)\,\frac{a}{2}[(x-z)\,U_1(x-z) - U_2(x-z)] \\ &\quad - a\bar{\epsilon}\left[\frac{x-z}{2}\,U_2(x-z) - U_3(x-z)\right] \end{aligned} \tag{4-20}$$

$$\phi_{r'\alpha}[(x-z),\bar{r}(z),\bar{r}'(z)] = \frac{\partial}{\partial x}\,\phi_{r\alpha}[(x-z),\bar{r}(z),\bar{r}'(z)] \tag{4-21}$$

The fixed x scalar perturbation equation (4-15a) may similarly be written as

$$\frac{d}{dx}\begin{pmatrix}\delta r\\ \delta r'\\ \delta\alpha\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ -\alpha & 0 & -r\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\delta r\\ \delta r'\\ \delta\alpha\end{pmatrix} + \begin{pmatrix}0\\ r[\bar{r}\cdot\bar{f}]/\mu\\ -2\bar{r}'\cdot\bar{f}/\mu\end{pmatrix} \tag{4-22}$$

or

$$\frac{d}{dx}\begin{pmatrix}\delta r\\ \delta r'\\ \delta\alpha\end{pmatrix} = F_s[r]\begin{pmatrix}\delta r\\ \delta r'\\ \delta\alpha\end{pmatrix} + \bar{g}_s[\bar{r},\bar{r}',\bar{f}] \tag{4-23}$$

with the resulting solution

$$\begin{pmatrix}\delta r(x)\\ \delta r'(x)\\ \delta\alpha(x)\end{pmatrix} = \Psi[(x-0),r(0),r'(0)]\begin{pmatrix}\delta r_o\\ \delta r_o'\\ \delta\alpha_o\end{pmatrix} + \int_0^x \Psi[(x-\sigma),r(\sigma),r'(\sigma)]\bar{g}_s[\sigma]d\sigma \tag{4-24}$$

where

$$\Psi[(x-z),r(z),r'(z)] = \begin{pmatrix}U_1(x-z) & U_2(x-z) & \psi_{r\alpha}[(x-z),r(z),r'(z)]\\ U_0(x-z) & U_1(x-z) & \psi_{r'\alpha}[(x-z),r(z),r'(z)]\\ 0 & 0 & 1\end{pmatrix} \tag{4-25}$$

and

$$\psi_{r\alpha}[(x-z),r(z),r'(z)] = -r(z)\,\frac{x-z}{2}\,U_2(x-z)$$

$$+\,r'(z)\,\frac{a}{2}\,[(x-z)U_1(x-z) - U_2(x-z)]$$

$$+\,a\left[\frac{x-z}{2}\,U_2(x-z) - U_3(x-z)\right] \qquad (4\text{-}26)$$

$$\psi_{r'\alpha}[(x-z),r(z),r'(z)] = \frac{\partial}{\partial x}\,\psi_{r\alpha}[(x-z),r(z),r'(z)] \qquad (4\text{-}27)$$

As stated before, the scalar variation δr may be obtained directly from (4-5); it is included here primarily for the sake of completeness.

Relation Between Fixed x and Fixed t Variations

The solution to the fixed nonlinear or linearized variational equations, including the time variation δt, is sufficient to establish the state vector of the perturbed system at the mutually common value of the independent variable x. It may be desirable, however, or even necessary to obtain a comparison of the perturbed and unperturbed systems at the same time, hence requiring evaluation of the corresponding fixed t variations. For example, in the next chapter, several examples are analyzed using both the focal and central oscillator systems. For comparison purposes, the fixed x variations of the central oscillator and the fixed y variations of the focal oscillator are converted to a common variation, namely, the fixed t variations. It would be possible to directly convert the fixed x variations to fixed y variations and vice versa; however, this would involve a more detailed discussion of the relationship between the independent variables.

The perturbed system radius vector $\bar{r}$ at some value of x and corresponding time $t^o + \delta t$ has been given as

$$\bar{r} = \bar{r}^o + \delta \bar{r} \tag{4-1}$$

Expanding $\bar{r}$ in a Taylor series about t^o in powers of δt results in the perturbed system radius vector at time t^o, which is $\bar{r}^o$ plus the fixed t variation $\Delta\bar{r}$, or

$$\bar{r} + \frac{d\bar{r}}{dt}(-\delta t) + \frac{1}{2}\frac{d^2\bar{r}}{dt^2}(-\delta t)^2 + \ldots = \bar{r}^o + \Delta\bar{r} \tag{4-28}$$

Therefore

$$\Delta\bar{r} = \delta\bar{r} - \frac{d\bar{r}}{dt}\delta t + \frac{1}{2}\frac{d^2\bar{r}}{dt^2}\delta t^2 - \ldots \tag{4-29}$$

where

$$\frac{d\bar{r}}{dt} = \frac{\bar{r}'}{t'} = \frac{\sqrt{\mu}\,\bar{r}'}{r} \tag{4-30}$$

Note that δt measures perturbed system time minus unperturbed system time from (4-6), while the Taylor series expansion (4-28) is in powers of unperturbed time minus perturbed time.

It should be apparent from (4-29) that while the fixed x variations and the time variation of the central oscillator are at all times well behaved, the fixed t variations may possibly be unbounded or ill-behaved for passage of the perturbed system arbitrarily near the singularity. Thus the advantages that were obtained by expressing the unperturbed two-body problem in the regularized domain as opposed to the time domain are reflected in the behavior of the corresponding variations. The unboundedness of the fixed t variations occurs only near $r = 0$, since only unbounded derivatives are involved.

The fixed t variation (4-29) may be approximated to first order by

$$\Delta\bar{r} = \delta\bar{r} - \bar{r}^{o\prime} \frac{\sqrt{\mu}}{r^o} \delta t \tag{4-31}$$

substitution of (4-8) for $\delta t'$ results in

$$\Delta\bar{r} = \delta\bar{r} + \bar{r}^{o\prime} \delta x \tag{4-32}$$

and, equivalently,

$$\Delta r = \delta r + r^{o\prime} \delta x \tag{4-32a}$$

where

$$\delta x = -\frac{1}{r^o} \int_{x_o}^{x} \delta r(\sigma) d\sigma = -\frac{\delta t}{t'} \tag{4-33}$$

The quantity δx represents the first order variation in the independent variable x along the perturbed system corresponding to the time variation δt .

Focal Oscillator System

An entirely analogous procedure is used to obtain the fixed y variational equations for the perturbed focal oscillator system. Since the objective of a variational treatment is to obtain variations which may be added directly to the unperturbed system parameters, the derivation must necessarily begin with the definitions

$$\hat{r} = \hat{r}^o + \delta\hat{r} \tag{4-34}$$

$$u = u^{o} + \delta u \tag{4-34a}$$

The relation between the unit vectors and variations introduced for the central oscillator are depicted in Figure 4.1, where

$$\bar{r} = r\hat{r} = r\hat{r}^{o} + r\,\delta\hat{r} = \frac{\hat{r}^{o} + \delta\hat{r}}{u^{o} + \delta u} \tag{4-35}$$

$$\delta\bar{r} = r\hat{r} - \bar{r}^{o} = r(\hat{r}^{o} + \delta\hat{r}) - \bar{r}^{o} \tag{4-36}$$

($\delta\bar{r}$ is a fixed y variation in this case).

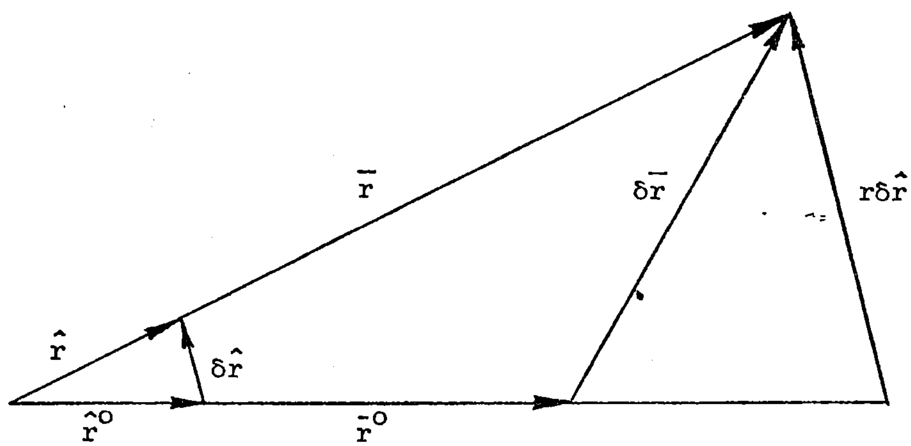

FIGURE 4.1. RELATION BETWEEN VECTOR VARIATIONS OF FOCAL OSCILLATOR SYSTEM.

The results of the focal oscillator derivation are listed in order corresponding to the order established in the derivation of the variational equations of the perturbed central oscillator system.

$$\delta t' = \frac{1}{\sqrt{\mu}}\left(\frac{1}{u^2} - \frac{1}{u^{o2}}\right) \tag{4-37}$$

$$\delta p' = p' = \frac{2}{\mu} r^3\, \hat{r}' \cdot \bar{f} \tag{4-37a}$$

$$\delta\hat{r}'' + (p^o + \delta p)\,\delta\hat{r} + \delta p\,\hat{r}^o = \frac{r^3}{\mu}[\bar{f} - (\bar{f}\cdot\hat{r})\hat{r}] \qquad (4\text{-}37b)$$

$$\delta u'' + (p^o + \delta p)\,\delta u + \delta p\,u^o = -\frac{r^2}{\mu}\bar{f}\cdot\hat{r} \qquad (4\text{-}37c)$$

where

$$r = \frac{1}{u}$$

$$\bar{f} = \bar{f}(\bar{r},\bar{r}',t)$$

Numerical computation of the foregoing nonlinear variational equations presents essentially the same difficulties noted for the perturbed focal oscillator in Cowell form. One of these is the unit vector nature of $\hat{r}(= \hat{r}^o + \delta\hat{r}$, where $\hat{r}^o$ is also a unit vector); as before, a normalization procedure would be appropriate. Also, it will be recalled from Chapter III that the computation of u cannot be accomplished in a straightforward manner if u is unbounded. The same is true in the variational system if u^o or u is unbounded; in either case, δu is unbounded, although this is not directly apparent from the differential equation (4-37c). This might be explained as follows: if u^o is unbounded, the argument of Chapter III would be applicable, since the effective variable of integration is proportional to unperturbed system speed. However, if the perturbed system particle intersects the attracting mass $(u \to \infty)$, this implies "instantaneous" rectilinearity, or $p = 0 = p^o + \delta p$. Recalling that the semi-latus rectum is positive semidefinite and ignoring $\bar{f}$, (4-37c) may be expressed as

$$\delta u'' - p\delta u = 0 \tag{4-38}$$

The solution to (4-38) is at least of exponential order relative to u^o for finite p .

The foregoing discussion is probably of little practical interest except for numerical analysis of near-pathological orbits.

Linearized Variational Equations

$$\delta t' = -\frac{2\,\delta u}{\sqrt{\mu}\,{u^o}^3} \tag{4-39}$$

$$\delta p' = \frac{2}{\mu}\,{r^o}^3 \hat{r}^{o\prime} \cdot \bar{f} \tag{4-39a}$$

$$\delta\hat{r}'' + p^o\delta\hat{r} + \delta p\hat{r}^o = \frac{{r^o}^3}{\mu}\left[\bar{f} - (\bar{f}\cdot\hat{r}^o)\hat{r}^o\right] \tag{4-39b}$$

$$\delta u'' + p^o\delta u + \delta p\, u^o = -\frac{{r^o}^2}{\mu}\,\bar{f}\cdot\hat{r}^o \tag{4-39c}$$

where

$$\bar{f} = \bar{f}(\bar{r}^o, \bar{r}^{o\prime}, t^o)$$

Special attention is directed to (4-39) for $\delta t'$, the time variation along the perturbed focal oscillator. Comparison with the corresponding equation (4-15b) of the central oscillator indicates the expression for $\delta t'$ of the focal oscillator system contains a dependent variable of the unperturbed system in the denominator of the integrand. It will subsequently be shown that this fact renders the integrand more complicated in one basic sense than any other integrand associated with this research.

Conversion of the linearized fixed y variations to fixed t variations is obtained through

$$\Delta\hat{r} = \delta\hat{r} + \hat{r}^{o\prime}\ \delta y \tag{4-40}$$

$$\Delta u = \delta u + u^{o\prime}\ \delta y \tag{4-40a}$$

where

$$\begin{aligned} \delta y &= -\ \delta t/t' \\ &= 2u^{o^2} \int_{y_o}^{y} \delta u(\sigma)/u^{o^3}(\sigma)\ d\sigma \end{aligned} \tag{4-41}$$

Solution to Linearized Equations

Entirely analogous to the solution presented for the linearized central oscillator, the solution to linearized fixed y variational equations may be represented by the state transition matrices

$$\Phi[(y-z),\hat{r}(z),\hat{r}'(z)] = \begin{pmatrix} V_1(y-z) & V_2(y-z) & \phi_{rp}[(y-z),\hat{r}(z),\hat{r}'(z)] \\ V_0(y-z) & V_1(y-z) & \phi_{r'p}[(y-z),\hat{r}(z),\hat{r}'(z)] \\ 0 & 0 & 1 \end{pmatrix} \tag{4-42}$$

where

$$\begin{aligned} \phi_{rp}[(y-z),\hat{r}(z),\hat{r}'(z)] &= -\ \hat{r}(z)\ \frac{y-z}{2}\ V_2(y-z) \\ &\quad + \hat{r}'(z)\ \frac{1}{2p}\ [(y-z)\ V_1(y-z)\ -\ V_2(y-z)]\ , \\ \phi_{r'p}[(y-z),\hat{r}(z),\hat{r}'(z)] &= \frac{\partial}{\partial y}\ \phi_{rp}[(y-z),\hat{r}(z),\hat{r}'(z)] \end{aligned} \tag{4-43}$$

and

$$\Psi[(y-z),u(z),u'(z)] = \begin{pmatrix} V_1(y-z) & V_2(y-z) & \psi_{up}[(y-z),u(z),u'(z)] \\ V_0(y-z) & V_1(y-z) & \psi_{u'p}[(y-z),u(z),u'(z)] \\ 0 & 0 & 1 \end{pmatrix} \tag{4-44}$$

where

$$\psi_{up} = -u(z)\,\frac{y-z}{2}\,V_2(y-z) + u'(z)\,\frac{1}{2p}\left[(y-z)\,V_1(y-z) - V_2(y-z)\right] + \frac{1}{p}\left[\frac{y-z}{2}\,V_2(y-z) - V_3(y-z)\right] ,$$

$$\psi_{u'p}[(y-z),u(z),u'(z)] = \frac{\partial}{\partial y}\,\psi_{up}[(y-z),u(z),u'(z)] \tag{4-45}$$

<u>Relation Between Linearized Variations</u>

The expressions (4-35) and (4-36) identify the nonlinear relations between the variations in $\bar{r}$, $\hat{r}$, r , and u . Linearized relations (fixed x,y or t) are obtained by

$$\delta u = \delta(1/r) = -\delta r/r^{o^2} = -\bar{r}^o \cdot \delta\bar{r}/r^{o^3} \tag{4-46}$$

$$\delta\hat{r} = \delta(u\bar{r}) = \delta u\bar{r}^o + u^o\delta\bar{r} \tag{4-46a}$$

$$\delta\bar{r} = \delta(r\hat{r}) = \delta r\hat{r}^o + r^o\delta\hat{r} \text{ , etc.} \tag{4-46b}$$

The relation between the variations of the rates is obtained by differentiation. The linearized relations between variations of the orbital elements and variations in the perturbation state vector $\delta\bar{r}$ and $\delta\bar{v}$ are

$$\delta\bar{h} = \delta\bar{r} \times \bar{v}^o + \bar{r}^o \times \delta\bar{v} \tag{4-47}$$

$$\delta p = \frac{2}{\mu}\bar{h} \cdot \delta\bar{h} \tag{4-47a}$$

$$\delta\alpha = -2\frac{\bar{r}^o \cdot \delta\bar{r}}{r^{o3}} - 2\frac{\bar{v}^o \cdot \delta\bar{v}}{\mu} \tag{4-47b}$$

$$\delta\bar{\epsilon} = \frac{\delta\bar{v} \times \bar{h}}{\mu} + \frac{\bar{v} \times \delta\bar{h}}{\mu} - \delta\hat{r} \tag{4-47c}$$

Note that velocity $\bar{v}$ is related to the regularized velocity $\bar{r}'$ by (e.g., the central oscillator)

$$\bar{v} = \bar{r}'\dot{x} = \bar{r}'\sqrt{\mu}/r \tag{4-48}$$

Thus

$$\delta\bar{v} = \frac{\sqrt{\mu}}{r^{o2}}(r^o\delta\bar{r}' - \bar{r}^{o\prime}\delta r) \tag{4-49}$$

Therefore, using the regularized velocities of the central oscillator would yield, for example,

$$\delta\alpha = -\frac{2\bar{r}^o\cdot\delta\bar{r}}{r^{o3}} - \frac{2\bar{r}^{o\prime}\cdot\delta\bar{r}'}{r^{o2}} + \frac{2(\bar{r}^{o\prime}\cdot\bar{r}^{o\prime})(\bar{r}^o\cdot\delta\bar{r})}{r^{o4}} \tag{4-50}$$

$$\delta\bar{\epsilon} = \left[\frac{1}{r^{o3}}(\bar{r}^o\bar{r}^o - \bar{r}^o\cdot\bar{r}^o I) + (\bar{r}^{o\prime}\bar{r}^{o\prime} - \bar{r}^{o\prime}\cdot\bar{r}^{o\prime} I)\left(\frac{2}{r^{o4}}\bar{r}^o\bar{r}^o - \frac{I}{r^{o2}}\right)\right]\delta\bar{r}$$
$$+\left[\frac{1}{r^{o2}}(2\bar{r}^o\bar{r}^{o\prime} - \bar{r}^{o\prime}\bar{r}^o - \bar{r}^{o\prime}\cdot\bar{r}^o I)\right]\delta\bar{r}' \tag{4-50a}$$

Expressions such as these are also needed to evaluate the initial values for $\delta\alpha$, δp and $\delta\bar{\epsilon}$ in the linearized equations; in a later chapter, (4-50) will be used to modify the system solution (4-18) to the central oscillator to simplify quadrature.

Choice of System: Numerical

The question may arise as to which representation, the perturbed focal or central oscillator, is to be used in the numerical analysis of some particular problem in celestial mechanics. From a numerical standpoint, the nonlinear central oscillator system would appear to be the best choice, due to the nonsingularity of the equations, and also due to the natural time step size regulation feature. It will be recalled that, in either Cowell or Encke form, the transformation equation of the central oscillator may be used to vary the step size in time relative to a fixed step size in x for close passage to the primary body. The effect of this step size regulation will be to decrease the step size in time and effectively "smooth" the numerical integration which uses time as the independent variable. Use of the central oscillator directly with the fixed step size in x offers the same numerical advantages. Modifications to this method would be the use of the focal oscillator system near periapsis or the original time domain differential equations near apoapsis. Either modification would result in finer integration steps in their respective regions and resulting higher accuracy. The primary emphasis of this research has, however, been on the analytical solution to the linearized variational equations.

Choice of System: Analytic

From an analytic standpoint the linearized variational equations are directly integrable as Poisson series in unique and distinct regions. A Poisson series is defined herein as a series in which each term of the series contains only positive powers of circular functions of the independent variable. The definition of these regions is found in the power series expansion of the perturbing force $\bar{f}$ and is best demonstrated by comparing the linearized scalar variational equations of the two systems:

$$\delta r'' + \alpha^o \delta r + \delta\alpha \, r^o = \frac{{r^o}^2}{\mu} \bar{f} \cdot \hat{r} \qquad (4\text{-}51)$$

$$\delta u'' + p^o \delta u + \delta p \, u^o = - \frac{{r^o}^2}{\mu} \bar{f} \cdot \hat{r} \qquad (4\text{-}51a)$$

Since r^o in the central oscillator system is represented by a Poisson term in x, the expansion of the perturbing force $\bar{f}$ may contain powers of the scalar radius ≥ -2 and still retain the overall Poisson series form. The resulting integrand is then integrable to almost any degree of complexity. Conversely, if the perturbing force is expandable as a convergent series in descending powers of the scalar radius ≥ -2, the RHS of (4-51a) will be in the form of a Poisson series of circular functions of the independent variable y of the focal oscillator, since $\frac{1}{r^o} = u^o$ in this system and is a Poisson type term

$$\left(= \frac{1 + \epsilon C_{\sqrt{p}y}}{p} \right)$$

The same argument is true for the vector variational equations, and both the scalar and vector variations are obtainable as Poisson series. As stated before, this convenient delineation does not extend to the time correction δt ; since δr of the central oscillator is in the form of a Poisson series, the integral for δt is straightforward, but the u_o^3 denominator in the integrand for δt of the focal oscillator disrupts the established pattern. Another advantageous aspect of the central oscillator system is that a time varying force in the form of a power series in time may be included in the analysis in a straightforward algebraic manner without disrupting the Poisson series nature of the integrands. Of the two types of perturbing force considered in the next chapter, the type of $\bar{f}$ amenable to analysis by the perturbed focal oscillator system is also time invariant by nature (i.e., the J_2 term of the earth's potential expansion), while the type associated with the central oscillator (perturbation by external attracting body) is time-varying by nature (although the time-varying nature is not included in the particular analysis of Chapter V).

Chapter V

ANALYSIS OF PERTURBED CIRCULAR AND RECTILINEAR ORBITS

The foregoing chapter has outlined a linearized perturbation theory based on the representation of perturbed Keplerian motion as perturbed harmonic oscillators. In this chapter, several applications of the perturbation theory are presented. The first example is the perturbation of a nominal circular orbit due to the second spherical harmonic (J_2) of the expansion of the primary body potential. The solution is obtained using both harmonic oscillator systems and is compared to the solution obtained from the Euler-Hill perturbation equations.

The second example considered is the general rectilinear orbit perturbed by the J_2 oblateness term, while the third example considers the perturbation of a general rectilinear orbit by a fixed external perturbing body. It is interesting to note that although the central oscillator system may appear to be the natural system for the analysis of perturbed rectilinear orbits, the perturbed focal oscillator system is simpler since not only the frequency $p = 0$ but also the first order variation in p can be shown to be equal to zero for the general reference rectilinear orbit. Since the frequency α of the central oscillator system is zero for parabolic motion, the rectilinear parabola is used as the reference orbit for the analysis of both types of perturbing forces using both perturbed oscillator systems. The extension to the nonparabolic rectilinear reference orbit is then obtained using one of the systems, the choice being dictated by the particular structure of the perturbing force.

Perturbing Force due to Oblateness (J_2)

The perturbing force due to the second spherical harmonic term, or oblateness term of the expansion of the potential of a central body may be expressed in vector form as

$$\bar{f} = -\frac{3\mu J_2 R_s^2}{2r^4}\left[\hat{r} + 2(\hat{r}\cdot\hat{n})\hat{n} - 5(\hat{r}\cdot\hat{n})^2\,\hat{r}\right] \tag{5-1}$$

where $\hat{n}$ is a unit vector directed along the polar axis of symmetry, $\hat{r}$ is the unit radius vector, and the remaining terms have their standard meaning (see reference 10).

Since $\bar{\epsilon} = 0$ for the reference circular orbit, some liberty exists in the establishment of the reference coordinate system. Referring to Figure 5.1,

$$\hat{n} = (0,\ s_i,\ c_i)^T \tag{5-2}$$

$$\hat{r} = (c_\theta, s_\theta,\ 0)^T \tag{5-2a}$$

where the argument of latitude θ (= $\omega + f$, where ω = argument of periapsis) is measured from the line of nodes.

Euler-Hill Equations

The Euler-Hill equations are the linearized Encke perturbation equations in the time domain referenced to a circular orbit and expressed in the rotating coordinate system defined by $\hat{r}, \hat{h}\times\hat{r}, \hat{h}$.

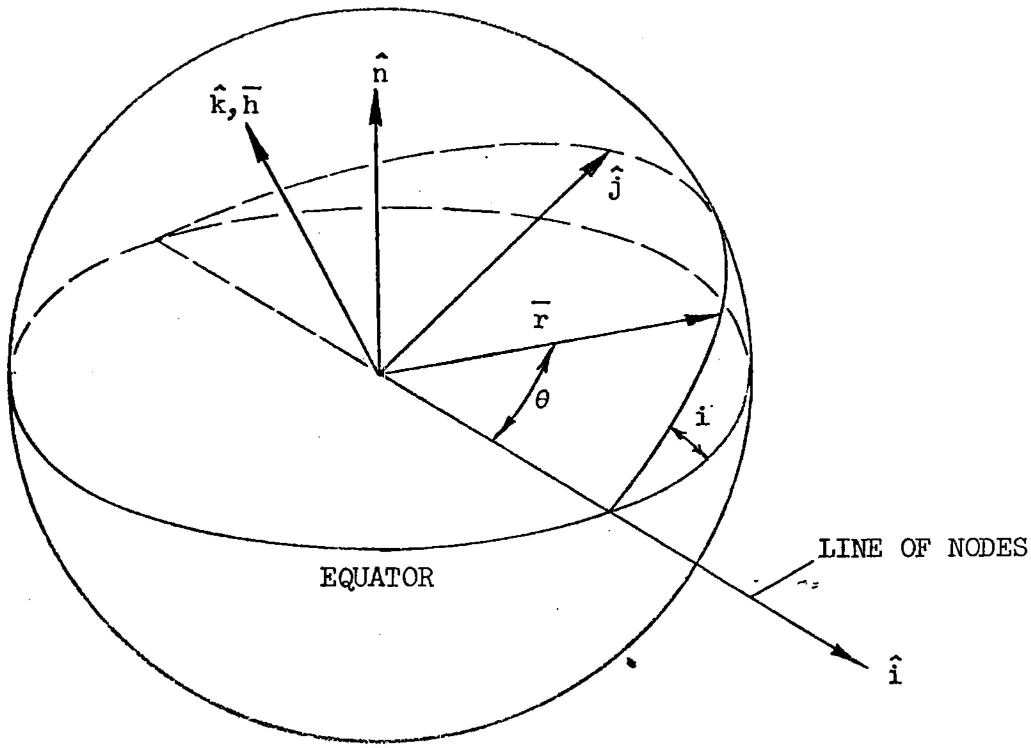

FIGURE 5.1. VECTOR GEOMETRY FOR REFERENCE CIRCULAR ORBITS.

To review, in a nonrotating vector space

$$\frac{d^2}{dt^2}(\Delta\overline{r}) = \frac{\mu}{a^3}[3\hat{r}\hat{r} - I]\,\Delta\overline{r} + \overline{f} \qquad (5\text{-}3)$$

and the dyad product $\hat{r}\hat{r}$ is time-varying, since $\hat{r} = (C_{nt}, S_{nt}, 0)^T$ in a typical coordinate system defined by $(\hat{i}, \hat{j}, \hat{k})$ of Figure 5.1. In the rotating coordinate system, however, denoting derivatives in this system by the over-dot,

$$\ddot{\Delta\bar{r}} + \bar{\omega} \times (\bar{\omega} \times \Delta\bar{r}) + 2\bar{\omega} \times \dot{\Delta\bar{r}} = \frac{\mu}{a^3} [3\hat{r}\hat{r} - I]\, \Delta\bar{r} + \bar{f} \quad (5\text{-}4)$$

$$\bar{\omega} = (0,0,n)^T \quad , \quad \hat{r} = (1,0,0)^T$$

and defining $\Delta\bar{r} = (x,y,z)^T$ results in

$$\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= f_r \\ \ddot{y} + 2n\dot{x} \qquad\quad &= f_\theta \\ \ddot{z} \qquad\quad + n^2 z &= f_z \end{aligned} \qquad (5\text{-}5)$$

where $n^2 = \mu/a^3$, so that the resulting system is linear and constant-coefficient. In the rotating system,

$$f_r = -\frac{3\mu J_2 R_s^2}{2a^4} (1 - 3S_i^2 S_\theta^2) \qquad (5\text{-}6)$$

$$f_\theta = -\frac{3\mu J_2 R_s^2}{2a^4}\, 2\, S_i^2 S_\theta C_\theta \qquad (5\text{-}6a)$$

$$f_z = -\frac{3\mu J_2 R_s^2}{2a^4}\, 2\, S_i C_i S_\theta \qquad (5\text{-}6b)$$

where $\theta = nt$.

The solution may be represented by

$$\Delta\bar{r}(t) = \Phi_{rr_o}(t,t_o)\Delta\bar{r}(t_o) + \Phi_{r\dot{r}_o}(t,t_o)\dot{\Delta\bar{r}}(t_o) + \int_{t_o}^{t} \Phi_{r\dot{r}_o}(t,\tau)\bar{f}(\tau)d\tau$$

(5-7)

where

$$\Phi_{rr_o}(t,t_o) = \begin{pmatrix} -3C_{n(t-t_o)}+4 & 0 & 0 \\ 6\left[S_{n(t-t_o)} - n(t-t_o)\right] & 1 & 0 \\ 0 & 0 & C_{n(t-t_o)} \end{pmatrix} \tag{5-8}$$

and

$$\Phi_{r\dot{r}_o}(t,t_o) = \begin{pmatrix} \frac{1}{n} S_{n(t-t_o)} & \frac{2}{n}\left(1 - C_{n(t-t_o)}\right) & 0 \\ \frac{2}{n}\left[C_{n(t-t_o)} - 1\right] & -3(t-t_o) + \frac{4}{n} S_{n(t-t_o)} & 0 \\ 0 & 0 & \frac{1}{n} S_{n(t-t_o)} \end{pmatrix} \tag{5-8a}$$

The initial values of the perturbation state vector $(\overline{\Delta r}, \dot{\overline{\Delta r}})$ in (5-7) are assumed to be zero and the particular solution to the convolution integral of (5-7) is

$$x = -\frac{3\, J_2 R_s^2}{2a}\left\{1 - C_\theta + S_i^2\left[\frac{2}{3}\,(C_\theta - 1) + \frac{1}{3}\, S_\theta^2\right]\right\} \tag{5-9}$$

$$y = -\frac{3\, J_2 R_s^2}{2a}\left\{-2\theta + 2S_\theta + S_i^2\left[\frac{3}{2}\,\theta - \frac{4}{3}\, S_\theta - \frac{1}{6}\, S_\theta C_\theta\right]\right\} \tag{5-9a}$$

$$z = \frac{3\, J_2 R_s^2}{2a}\,(\theta C_\theta - S_\theta) \tag{5-9b}$$

and represents the position variation as a function of time; with regard to the discussions of the foregoing chapters, this solution also represents a "fixed time" variation.

For purposes of comparison with subsequent solutions obtained from the perturbed harmonic oscillator systems, the solution (5-9) is expressed in the nonrotating $\hat{i}\hat{j}\hat{k}$ reference frame of Figure 5.1 by

$$\Delta\bar{r} = \begin{pmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5-10}$$

and results in

$$\Delta\bar{r}_x = -\frac{3 J_2 R_s^2}{2a} \Bigg\{ 2\theta S_\theta + C_\theta - 1 - S_\theta^2 + S_i^2 \left[-\frac{3\theta S_\theta}{2} + \frac{2}{3}(1 - C_\theta) + \frac{2}{3} S_\theta^2 + \frac{1}{2} S_\theta^2 C_\theta \right] \Bigg\} \tag{5-11}$$

$$\Delta\bar{r}_y = -\frac{3 J_2 R_s^2}{2a} \Bigg\{ S_\theta + S_\theta C_\theta - 2\theta C_\theta + S_i^2 \left[\frac{3}{2}\theta C_\theta - \frac{2}{3} S_\theta C_\theta - \frac{1}{3} S_\theta - \frac{1}{2} S_\theta C_\theta^2 \right] \Bigg\} \tag{5-11a}$$

$$\Delta\bar{r}_z = z = \frac{3 J_2 R_s^2}{2a} (\theta C_\theta - S_\theta) \tag{5-11b}$$

Perturbed Harmonic Oscillators

The Euler-Hill perturbation equations of the foregoing section were reduced to a linear constant-coefficient set of equations by expressing the differential equations in the rotation coordinate system defined by $\hat{r}, \hat{h} \times \hat{r}, \hat{h}$. Their application is limited to circular reference orbits for which $\dot{\bar{\omega}} = 0$. The perturbed harmonic oscillator systems are linear constant coefficient systems of equations in any nonrotating coordinate system and remain applicable for non-circular reference orbits. For the example problem being investigated, the integrals of the oscillator systems are simplified somewhat by the simplicity of the parameters of the reference circular orbit. However, even disregarding the difficulty of the integral expressions, an additional subtle penalty is exacted by use of the perturbed oscillator systems, that penalty being the time corrections to the resulting solutions necessary to obtain the time variation or to convert to fixed time variations in the state vector. The "fixed t" variations are only required in this instance for comparison purposes. Although for the circular reference orbit the unperturbed independent variables of all three systems are effectively identical, the perturbed system independent variables of the oscillator systems differ from their unperturbed counterparts (and from each other, of course); the resulting time corrections are then based on these differences.

In the nonrotating $\hat{i}\hat{j}\hat{k}$ coordinate system of Figure 5.1, the perturbing force $\bar{f}$ due to the J_2 oblateness term (Eq. (5-1)), is expressed as

$$\bar{f} = -\frac{3\mu J_2 R_s^2}{2a^4} \begin{pmatrix} C_\theta - 5\, S_i^2 S_\theta^2 C_\theta \\ S_\theta + 2\, S_i^2 S_\theta - 5\, S_i^2 S_\theta^3 \\ 2\, S_i C_i S_\theta \end{pmatrix} \qquad (5\text{-}12)$$

Solution to Perturbed Central Oscillator System

The vector perturbation equation for the perturbed central oscillator system is given by Eq. (4-15)

$$\delta\bar{r}'' + \alpha\,\delta\bar{r} + \delta\alpha\bar{r} + \delta\bar{\epsilon} = \frac{r^2}{\mu}\bar{f}$$

with the corresponding particular solution indicated by Eq. (4-18). Substitution of the perturbing force $\bar{f}$ due to oblateness and solution of the resulting integrals leads to the fixed x variation

$$\delta\bar{r} = \frac{3\,J_2R_s^2}{2a}\begin{pmatrix} 1 - C_\theta - \theta S_\theta + S_i^2\left[\theta S_\theta + \frac{2}{3}(C_\theta - 1) - \frac{2}{3}S_\theta^2 C_\theta\right] \\ -S_\theta + \theta C_\theta + S_i^2\left[-\theta C_\theta + S_\theta - \frac{2}{3}S_\theta^3\right] \\ S_iC_i(\theta C_\theta - S_\theta) \end{pmatrix} \qquad (5\text{-}13)$$

The time correction to the fixed x variation $\delta\bar{r}$, necessary to convert the result to a fixed time variation $\Delta\bar{r}$, may be obtained by first evaluating the fixed x scalar variation δr through (4-5),

$$\delta r = \hat{r}\cdot\delta\bar{r}$$

where $\hat{r} = (C_\theta, S_\theta, 0)^T$, resulting in

$$\delta r = \frac{3\,J_2R_s^2}{2a}\left\{C_\theta - 1 + S_i^2\left[\frac{2}{3}(1 - C_\theta) - \frac{1}{3}S_\theta^2\right]\right\} \qquad (5\text{-}14)$$

which is equal to (5-9). The time correction is then obtained through (4-33)

$$\delta x = -\,\delta t/t'$$

where

$$\delta t' = \delta r/\sqrt{\mu}$$

resulting in

$$\delta x = -\frac{3\, J_2 R_s^2}{2a^{3/2}}\left[S_\theta - \theta + S_i^2\left(\frac{\theta}{2} + \frac{S_\theta C_\theta}{6} - \frac{2}{3}\, S_\theta\right)\right] \qquad (5\text{-}15)$$

The fixed t variation is then obtained through (4-32)

$$\Delta\bar{r} = \delta\bar{r} + \bar{r}'\,\delta x$$

and results in the same solution (5-11) obtained from the Euler-Hill equations.

Solution to Perturbed Focal Oscillator System

The scalar perturbation equation for the perturbed focal oscillator system is given by (4-39c)

$$\delta u'' + p\,\delta u + \delta p\, u = -\frac{r^2}{\mu}\,\bar{f}\cdot\hat{r}$$

and the variation in p is given by (4-39a)

$$\delta p' = \frac{2}{\mu}\, r^3\, \hat{r}'\cdot\bar{f}$$

Substitution of (5-1) and (5-2) in the general solution (4-44) for the fixed y variation in u results in

$$\delta u = -\frac{3\, J_2 R_s^2}{2a^3}\left\{C_\theta - 1 + S_i^2\left[\frac{2}{3}\,(1 - C_\theta) - \frac{1}{3}\, S_\theta^2\right]\right\} \qquad (5\text{-}16)$$

which is equal to $-\delta r/a^2$ from (5-14), or $-x/a^2$ from (5-9).

The time correction term δy is then directly obtainable from (4-41)

$$\delta y = -\frac{1}{t'}\,\delta t$$

$$\delta t' = -2\,\delta u/\sqrt{\mu}\,u^3$$

resulting in

$$\delta y = -\frac{3\,J_2 R_s^2}{a^{5/2}}\left[S_\theta - \theta + S_i^2\left(\frac{\theta}{2} + \frac{S_\theta C_\theta}{6} - \frac{2}{3}\,S_\theta\right)\right] \tag{5-17}$$

which equals $2\delta x/a$ (from (5-15)) for the circular reference orbit. The vector perturbed focal oscillator system is given by (4-39b)

$$\delta\hat{r}'' + p\,\delta\hat{r} + \delta p\,\hat{r} = \frac{r^3}{\mu}\,[\overline{f} - (\overline{f}\cdot\hat{r})\hat{r}]$$

with the resulting solution

$$\delta\hat{r} = \frac{3\,J_2 R_s^2}{2a^2}\begin{pmatrix} S_i^2\left(\dfrac{\theta S_\theta}{2} - \dfrac{S_\theta^2 C_\theta}{2}\right) \\ S_i^2\left(-\dfrac{\theta C_\theta}{2} + \dfrac{S_\theta C_\theta^2}{2}\right) \\ S_i C_i(\theta C_\theta - S_\theta) \end{pmatrix} \tag{5-18}$$

The total fixed t variation in position is then obtainable from

$$\begin{aligned} \Delta\overline{r} &= \delta\overline{r} + \overline{r}'\,\delta y \\ &= a\delta\hat{r} - a^2\delta u\,\hat{r} + a\hat{r}'\,\delta y \end{aligned} \tag{5-19}$$

Substitution of (5-16), (5-17), and (5-18) in (5-19) yields a solution for $\Delta\bar{r}$ identical to (5-11).

The results of the foregoing analysis have demonstrated the application, for comparison purposes, of the harmonic oscillator perturbation theory to a problem which is solved directly in the time domain. The solutions to both oscillator systems were corrected to yield the fixed t variations between the perturbed orbit and the unperturbed reference orbit. The fixed t variations were introduced only for the purpose of comparing the three solutions. In a subsequent section, the fixed t variations are further used to compare the solutions to the perturbed harmonic oscillator systems for the problem of perturbed rectilinear orbits.

Extension to Noncircular Reference Orbit

The extension of the theory to the noncircular reference orbit is direct, using either the focal or central perturbed oscillator system. The major difficulty lies in the analytical quadrature of the resulting expressions, which are more complex due to the introduction of the orbit eccentricity. For the case of the perturbing force associated with the oblateness term, or for any force expandable in descending powers of the scalar radius, the perturbed focal oscillator system yields Poisson series as the integrand for $\delta\hat{r}$ and δu . However, the time correction is not of the Poisson series form due to the u^3 term in the denominator of $\delta t'$ of (4-39).

A Lunar Problem

Another problem is the perturbation of a satellite due to the potential of an external attracting mass; the problem is termed the "lunar" problem since a classical problem is cast in the framework of the motion of the moon, perturbed by the sun, about the earth.

Idealizing the sun as stationary, a similar analysis of perturbed circular orbits could be accomplished using the Euler-Hill equations or either perturbed harmonic oscillator system. The distinctive feature of the lunar problem is that the perturbing force is expandable as a convergent series in ascending powers of the scalar radius. Thus the extension to noncircular orbits is direct using the perturbed central oscillator system. Unlike the focal oscillator system, the perturbed central oscillator system yields Poisson series integrands not only for the variation $\delta\bar{r}$ but also for the required time correction integral necessary to convert to fixed t variations. Moreover, the time-varying effect of the motion of the perturbing body may also be included by expansion of $\bar{f}$ in powers of t ; time is a relatively simple function (Kepler's or the universal time equation) of the independent variable x , and additional terms of the integrands would also be of Poisson series form.

Perturbed Rectilinear Orbits

Although perturbed nonrectilinear orbits have been studied from the time of Lagrange, the subject of perturbed rectilinear orbits has apparently received little attention. This may be due not only to the relatively awkward description of rectilinear orbits in the more conventional time domain but also more likely to the fact that such orbits were a rare occurrence in astronomy.

The advent of modern space travel, however, suggests numerous examples for possible rectilinear orbits, such as sounding rockets or lunar ascent/descent vehicles. Regarding lunar operations, it should be recalled that the earth (and even the sun) exert a much greater perturbative force relative to the lunar gravity at the surface than, say, the moon's and sun's effect relative to earth gravity at the earth surface.

Due to the regularized feature of the central oscillator system, the perturbations of "complete" mathematical rectilinear orbits (here defined as orbits in which the particle starts at or passes through periapsis) may be analyzed using the theory of the perturbed central oscillator system. The focal oscillator would be inapplicable since its independent variable is unbounded at periapsis. However, both Keplerian oscillator systems may be used to investigate rectilinear orbits which do not involve periapsis passage.

In the following section, two forms of perturbing forces are investigated: 1) perturbations due to the oblateness term, and 2) perturbations due to the attraction of a fixed external body. The variation of the perturbed orbit is first obtained relative to an unperturbed reference rectilinear parabola as the solution to both harmonic oscillator systems; the extension to the nonparabolic reference orbit is then obtained as the solution to the oscillator system most appropriate to the particular perturbing force. Since the analysis of the rectilinear parabolic orbit is for the purpose of comparison of the two systems, the fixed t variations are obtained.

Rectilinear Orbits Perturbed by J_2 Spherical Harmonic

Referring to Figure 5.2, the reference rectilinear orbit relative to the oblate central body may be described by its colatitude angle λ ; thus

$$\hat{n} = (-S_\lambda, C_\lambda, 0)^T \tag{5-20}$$

$$\hat{r} = (-1, 0, 0)^T \tag{5-20a}$$

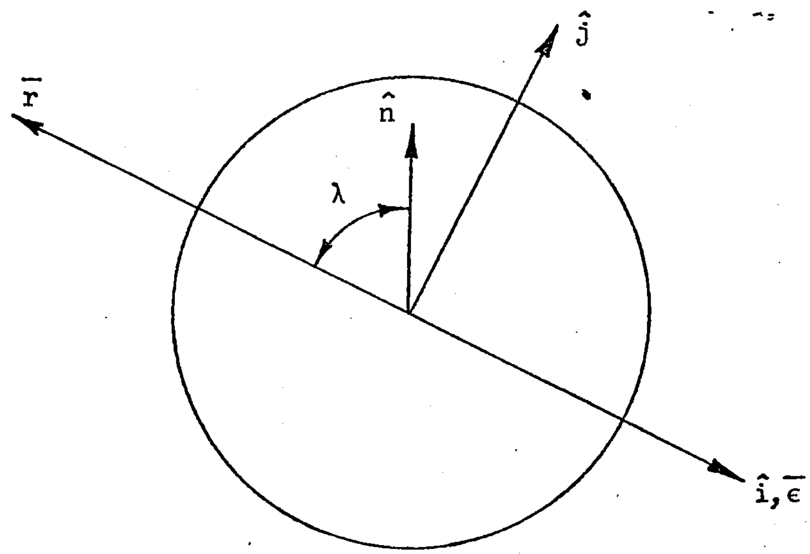

FIGURE 5.2. VECTOR GEOMETRY FOR REFERENCE RECTILINEAR ORBIT RELATIVE TO OBLATE CENTRAL BODY.

This particular orientation of the x-y coordinate system has been chosen to be compatible with the natural coordinate system associated with the central oscillator. In this system, from Equation (5-1),

$$\bar{f} = \frac{3\mu J_2 R_s^2}{2r^4} \begin{pmatrix} 1 - 3S_\lambda^2 \\ 2S_\lambda C_\lambda \\ 0 \end{pmatrix} \tag{5-21}$$

Solution to Perturbed Focal Oscillator

As noted in Chapter III, the independent variable of the focal oscillator for the general rectilinear orbit is given (for the particle progressing outward from the singularity) as

$$y = \frac{v_o - v}{\sqrt{\mu}} \tag{5-22}$$

where y is taken to vanish at the initial velocity v_o ; for this definition of y ,

$$u = y^2/2 + u_o' y + u_o \tag{5-23}$$

$$u' = y + u_o' \tag{5-23a}$$

Alternatively, y may be defined as

$$y = -\frac{v}{\sqrt{\mu}} \tag{5-24}$$

$$y_o = -\frac{v_o}{\sqrt{\mu}} \tag{5-24a}$$

and, from the last part of Chapter II,

$$u = \frac{1}{2}(y^2 + \alpha) \qquad \text{(Energy)} \tag{5-25}$$

$$u' = y = -\sqrt{2u - \alpha} \tag{5-25a}$$

for this definition of y. This form is more convenient for the quadratures involved in the following section.

All rectilinear orbits are characterized by zero angular momentum $(p = 0)$; thus the variational equations of the focal oscillator system are initially simplified. In addition, the first order variation for p also vanishes. This may be observed directly from the defining equation (4-39a) by noting that $\hat{r}'$ is identically zero for rectilinear orbits (i.e., the rate of change of the unit radius vector is zero). Therefore, the variational equations for δu and $\delta\hat{r}$ of the focal oscillator system reduce to the relatively simple equations

$$\delta u'' = -\frac{r^2}{\mu}\bar{f} \cdot \hat{r} \tag{5-26}$$

$$\delta\hat{r}'' = \frac{r^3}{\mu}[\bar{f} - (\bar{f} \cdot \hat{r})\hat{r}] \tag{5-26a}$$

Using (5-21), (5-26) may be expressed as

$$\delta u'' = \frac{3\, J_2 R_s^2 (1 - 3S_\lambda^2)}{2}\, u^2 \tag{5-27}$$

Integrating between the limits of y_o and y (or u_o and u , noting that $dy = -du/\sqrt{2u})$, the resulting solution is

$$\delta u = \frac{3\,J_2R_s^2(1 - 3S_\lambda^2)}{2}\left[\frac{u^3}{15} - \frac{2\,u^{1/2}u_o^{5/2}}{5} + \frac{u_o^3}{3}\right] \qquad (5\text{-}28)$$

The time correction term δy may then be obtained from (4-41)

$$\delta y = 2u^2 \int_{y_o}^{y} \frac{\delta u(\sigma)}{u(\sigma)^3}\, d\sigma$$

which results in

$$\delta y = \frac{3\,J_2R_s^2(1 - 3S_\lambda^2)}{15\sqrt{2}}\, u^2 \left[\frac{2\,u_o^3}{u^{5/2}} - \frac{3\,u_o^{5/2}}{u^2} + 3\,u_o^{1/2} - 2\,u^{1/2}\right] \qquad (5\text{-}29)$$

The relative magnitude of the time correction may be obtained by comparing δu, say, and the corresponding time correction $u'\delta y$ in the expression for Δu (Eq. (4-40a)). From (5-28) and (5-29), and noting that $u' = y = -\sqrt{2u}$,

$$\delta u = \frac{3\,J_2R_s^2(1 - 3S_\lambda^2)}{2r^3}\left[\frac{1}{15} - \frac{2}{5}\left(\frac{r}{r_o}\right)^{5/2} + \left(\frac{r}{r_o}\right)^3\right] \qquad (5\text{-}30)$$

$$u'\delta y = \frac{3\,J_2R_s^2(1 - 3S_\lambda^2)}{2r^3}\left[\frac{4}{15} - \frac{2}{5}\left(\frac{r}{r_o}\right)^{1/2} + \frac{2}{5}\left(\frac{r}{r_o}\right)^{5/2} - \frac{4}{15}\left(\frac{r}{r_o}\right)^3\right] \qquad (5\text{-}30a)$$

and it can be seen that for $r \gg r_o$, the two terms are not only of the same order but have the opposite sign. The predominant $(r/r_o)^3$ terms are of opposite sign, and the second most predominant terms are equal and opposite. The vector variational equation (5-26a) reduces to

$$\delta\hat{r}_x'' = 0 \tag{5-31}$$

$$\delta\hat{r}_y'' = 3\, J_2 R_s^2\, S_\lambda C_\lambda\, u \tag{5-31a}$$

resulting in

$$\delta\hat{r}_y = 3\, J_2 R_s^2\, S_\lambda C_\lambda \left[\frac{u^2}{6} - \frac{2\, u^{1/2} u_o^{3/2}}{3} + \frac{u_o^2}{2}\right] \tag{5-32}$$

The total vector perturbation $\Delta\bar{r}$ is then given by

$$\Delta\bar{r} = r(\delta\hat{r} + \hat{r}'\delta y) - \frac{1}{u^2}(\delta u + u'\delta y)\hat{r} \tag{5-33}$$

Using (5-29), (5-30), and (5-32), and recalling that $\hat{r}' = (0,0)^T$ and $\hat{r} = (-1,0)^T$, we obtain

$$\Delta\bar{r} = \frac{3\, J_2 R_s^2}{2 r_o} \begin{pmatrix} (1 - 3S_\lambda^2)\left[\frac{1}{15}\left(\frac{r}{r_o}\right)^2 - \frac{2}{5}\left(\frac{r_o}{r}\right)^{1/2} + \frac{1}{3}\left(\frac{r_o}{r}\right)\right] \\ \\ S_\lambda C_\lambda \left[\frac{r}{r_o} - \frac{4}{3}\left(\frac{r}{r_o}\right)^{1/2} + \frac{1}{3}\left(\frac{r_o}{r}\right)\right] \end{pmatrix} \tag{5-34}$$

<u>Solution to Perturbed Central Oscillator</u>

The vector variational equation for the perturbed central oscillator is given by (4-15)

$$\delta\bar{r}'' + \alpha\,\delta\bar{r} + \delta\alpha\,\bar{r} + \delta\bar{\epsilon} = \frac{r^2}{\mu}\bar{f}$$

Unlike the variational equations of the focal oscillator, this equation does not simplify at all for general reference rectilinear orbits; for a reference rectilinear parabola, at least the second term vanishes. Noting this fact, and carrying out the quadratures for $\delta\alpha$ and $\delta\bar{\epsilon}$ in advance, (4-15) may be reduced to a trivial double integration for the reference rectilinear parabola.

The variational equation for α is

$$\delta\alpha' = -\frac{2}{\mu}\bar{r}' \cdot \bar{f}$$

or

$$\delta\alpha = 3\, J_2 R_s^2 (1 - 3S_\lambda^2) \int_{x_o}^{x_f} \frac{r' dx}{r^4} \tag{5-35}$$

$$= 3\, J_2 R_s^2 (1 - 3S_\lambda^2) \int_{r_o}^{r_f} \frac{dr}{r^4} \tag{5-35a}$$

which leads to

$$\delta\alpha = -3\, J_2 R_s^2 (1 - 3S_\lambda^2) \left[\frac{1}{3r^3} - \frac{1}{3r_o^3}\right] \tag{5-36}$$

which is valid for arbitrary α . The variational equation for $\delta\bar{\epsilon}$ is

$$\delta\bar{\epsilon}' = \frac{1}{\mu}[2\bar{r}\bar{r}' - \bar{r}'\bar{r} - \bar{r}' \cdot \bar{r}I]\,\bar{f}$$

which reduces to

$$\delta\bar{\epsilon}_x' = 0 \tag{5-37}$$

$$\delta\bar{\epsilon}_y' = -3\, J_2 R_s^2\, S_\lambda C_\lambda \frac{r'}{r^3} \tag{5-37a}$$

and

$$\delta\bar{\epsilon}_y = -3\, J_2 R_s^2\, S_\lambda C_\lambda \int_{r_o}^{r_f} \frac{dr}{r^3} \tag{5-38}$$

$$= 3\, J_2 R_s^2\, S_\lambda C_\lambda \left(\frac{1}{2r^2} - \frac{1}{2r_o^2}\right) \tag{5-38a}$$

which is also valid for arbitrary α . Combining these results leads to

$$\delta\bar{r}'' = \frac{3\, J_2 R_s^2}{2} \begin{pmatrix} (1 - 3\, S_\lambda^2) \left[\dfrac{1}{3r^2} + \dfrac{2r}{3r_o^3}\right] \\ \\ S_\lambda C_\lambda \left[\dfrac{1}{r^2} + \dfrac{1}{r_0^2}\right] \end{pmatrix} \tag{5-39}$$

Defining $r = x^2/2$ (correspondingly, $r_o = x_o^2/2$) and integrating between the limits of x_o and x (or r_o and r) results in

$$\delta\bar{r} = \frac{3\, J_2 R_s^2}{2r_o} \begin{pmatrix} (1 - 3S_\lambda^2)\left[\dfrac{1}{9}\left(\dfrac{r}{r_o}\right)^2 - \dfrac{2}{9}\left(\dfrac{r}{r_o}\right)^{1/2} + \dfrac{1}{9}\left(\dfrac{r_o}{r}\right)\right] \\ \\ S_\lambda C_\lambda \left[\dfrac{r}{r_o} - \dfrac{4}{3}\left(\dfrac{r}{r_o}\right)^{1/2} + \dfrac{1}{3}\left(\dfrac{r_o}{r}\right)\right] \end{pmatrix}$$

(5-40)

To obtain the time correction δx , the scalar perturbation δr is first obtained from (4-5) as

$$\delta r = \hat{r} \cdot \delta \bar{r}$$

and is simply the negative of the component of $\delta \bar{r}$; δx is then given by (4-33),

$$\delta x = -\frac{1}{r} \int_{x_o}^{x_f} \delta r(x)\, dx$$

$$= -\frac{1}{r} \int_{r_o}^{r} \delta r \frac{dr}{\sqrt{2r}} \tag{5-41}$$

$$= \frac{3\, J_2 R_s^2 (1 - 3S_\lambda^2)}{18\sqrt{2}\; r\, r_o} \left[\frac{2}{5} \frac{r^{5/2}}{r_o^2} - 2 \frac{r}{r_o^{1/2}} + \frac{18}{5} r_o^{1/2} - 2 \frac{r_o}{r^{1/2}} \right] \tag{5-41a}$$

Once again, a comparison is made of the time correction term to the fixed x variation of, say, δr

where $r' = \sqrt{2r}$ and from (4-32a)

$$\Delta r = \delta r + r' \delta x$$

From (5-40) and (5-41),

$$\delta r = -\frac{3J_2 R_s^2 (1 - 3S_\lambda^2)}{2r_o} \left[\frac{1}{9} \left(\frac{r}{r_o} \right)^2 - \frac{2}{9} \left(\frac{r}{r_o} \right)^{1/2} + \frac{1}{9} \left(\frac{r_o}{r} \right) \right] \tag{5-42}$$

$$r' \delta x = -\frac{3J_2 R_s^2 (1 - 3S_\lambda^2)}{2r_o} \left[-\frac{2}{45} \left(\frac{r}{r_o} \right)^2 + \frac{2}{9} \left(\frac{r}{r_o} \right)^{1/2} - \frac{2}{5} \left(\frac{r_o}{r} \right)^{1/2} + \frac{2}{9} \left(\frac{r_o}{r} \right) \right]$$

(5-42a)

and it is noted that the second most predominant terms are equal and opposite, as was also noted in a similar comparison of (5-30), while the predominant terms are of the same order and opposite sign.

The total position variation $\Delta\bar{r}$ is then given by (4-32)

$$\Delta\bar{r} = \delta\bar{r} + \bar{r}'\delta x$$

where

$$\bar{r}' = (-\sqrt{2r}, 0)^T$$

Combining (5-40) and (5-41) through (4-32) results in

$$\Delta\bar{r} = \frac{3\, J_2 R_s^2}{2r_o} \begin{pmatrix} (1 - 3S_\lambda^2)\left[\dfrac{1}{15}\left(\dfrac{r}{r_o}\right)^2 - \dfrac{2}{5}\left(\dfrac{r_o}{r}\right)^{1/2} + \dfrac{1}{3}\left(\dfrac{r_o}{r}\right)\right] \\ \\ S_\lambda C_\lambda \left[\dfrac{r}{r_o} - \dfrac{4}{3}\left(\dfrac{r}{r_o}\right)^{1/2} + \dfrac{1}{3}\left(\dfrac{r_o}{r}\right)\right] \end{pmatrix} \tag{5-43}$$

which is identical to the solution (5-34) obtained from the perturbed focal oscillator system.

<u>Nonparabolic Reference Rectilinear Orbit</u>

The extension to nonparabolic reference rectilinear orbits using the central oscillator system may be indicated by inspection of the differential equation (4-15), where the additional term $\alpha\,\delta\bar{r}$ appears on the LHS and $\bar{r}$ would be expressed in the more general form involving the universal functions U_j . The U_j would necessarily appear in the denominator of the integrand and a direct quadrature would appear to be somewhat tedious, if not impossible.

The focal oscillator equations (4-39b) and (4-39c) are still relatively tractable however, for nonparabolic reference rectilinear orbits, and furthermore the general expression for u

$$u = \frac{y^2}{2} + \frac{\alpha}{2} \tag{5-44}$$

renders straightforward quadratures for δu and $\delta\hat{r}$. Accordingly,

$$\delta u = \frac{3\, J_2 R_s^2 (1 - 3S_\lambda^2)}{2} \left[\frac{1}{4}\left(\frac{y^6}{5\cdot 6} - \frac{y\, y_o^5}{5} + \frac{y_o^6}{6} \right) + \frac{\alpha}{2}\left(\frac{y^4}{3\cdot 4} - \frac{y\, y_o^3}{3} + \frac{y_o^4}{4} \right) + \frac{\alpha^2}{4}\left(\frac{y^2}{2} - y\, y_o + \frac{y_o^2}{2} \right) \right] \tag{5-45}$$

$$\delta\hat{r}_x = 0 \tag{5-45a}$$

$$\delta\hat{r}_y = 3\, J_2 R_s^2 S_\lambda C_\lambda \left[\frac{1}{2}\left(\frac{y^4}{3\cdot 4} - \frac{y\, y_o^3}{3} + \frac{y_o^4}{4} \right) + \frac{\alpha}{2}\left(\frac{y^2}{2} - y\, y_o + \frac{y_o^2}{2} \right) \right] \tag{5-45b}$$

where

$$y = -\frac{v}{\sqrt{\mu}} = -\sqrt{2u - \alpha}$$

Unfortunately, but as expected, the time correction δy is more difficult to obtain.

Using (4-41) and the solution for δu, the necessary quadratures may be expressed in the form

$$B(n,p) = \int_{y_o}^{y} \frac{\sigma^n \, d\sigma}{(\sigma^2 + \alpha)^p} \tag{5-46}$$

Restricting the solution to $\alpha \neq 0$, the integrals may be evaluated through recursive integration by parts where

$$B(n,p) = \frac{1}{2}\left[\frac{-\sigma^{n-1}}{(p-1)(\sigma^2+\alpha)^{p-1}}\Bigg|_{y_o}^{y} + \int_{y_o}^{y} \frac{(n-1)\,\sigma^{n-2}\, d\sigma}{(p-1)(\sigma^2+\alpha)^{p-1}}\right]_{p\neq 1} \tag{5-47}$$

By this manipulation, B and other integrals eventually may be reduced to standard forms, although the term α occasionally appears in a denominator, thus rendering the solution invalid for $\alpha = 0$.

For $\alpha = 0$ or α sufficiently small, the necessary integrals for δy may be represented in series form ($p = 3$ in (5-47)) by

$$B(n,3) = \sum_{m=0} \frac{(-1)^m \,(m+2)!\, \alpha^m \,\sigma^{n-(2m+5)}}{2!\; m!\; [n - (2m+5)]}\Bigg|_{y_o}^{y} \tag{5-48}$$

where convergence is assured for $r/a < 1$.

Rectilinear Orbits Perturbed By External Body

The orientation of an unperturbed rectilinear orbit relative to an external perturbing body (defined by μ_E at some position vector $\overline{R}$) is depicted in Figure 5.3; for the purpose of this analysis the vector $\overline{R}$ is assumed to be fixed.

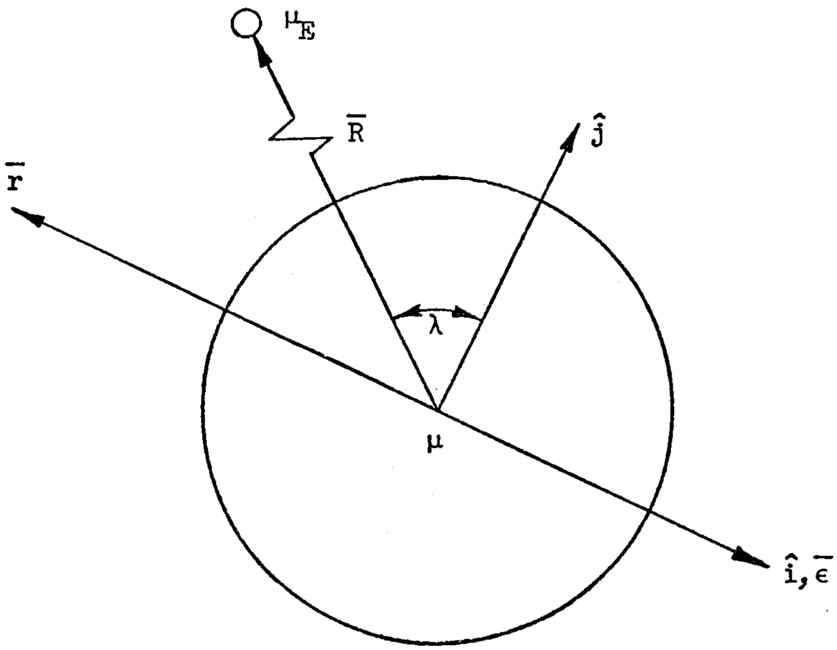

FIGURE 5.3. VECTOR GEOMETRY FOR REFERENCE RECTILINEAR ORBITS RELATIVE TO FIXED EXTERNAL BODY.

The attraction of the perturbing body is given by

$$\bar{f} = \mu_E \left[\frac{\bar{R} - \bar{r}}{|\bar{R} - \bar{r}|^3} - \frac{\bar{R}}{R^3} \right] \tag{5-49}$$

For $R \gg r$, (5-49) may be expanded as a convergent series in ascending powers of r/R; retaining only the first term in the series as the approximation to $\bar{f}$ results in

$$\bar{f} = -\frac{\mu_E r}{R^3} \begin{pmatrix} 3\, S_\lambda^2 - 1 \\ -3\, S_\lambda C_\lambda \\ 0 \end{pmatrix} \tag{5-50}$$

The fixed time variation in the position vector $\delta\bar{r}$ is obtained for the reference rectilinear parabola using both perturbed Keplerian oscillator systems, and the solution for the general rectilinear orbit is then obtained using the central oscillator system.

Solution to Perturbed Focal Oscillator

The same simplifying observation noted in the previous section on the perturbations due to the J_2 spherical harmonic also apply to the analysis of perturbations due to the perturbing force of (5-50) along a reference rectilinear parabola. The results are presented without elaboration as

$$\delta u = -\left(\frac{\mu_E}{\mu}\right)\frac{(3S_\lambda^2 - 1)}{R^3}\left[-\frac{r^2}{10} - \frac{2}{5}\frac{r_o^{5/2}}{r^{1/2}} + \frac{r_o^2}{2}\right] \tag{5-51}$$

$$\delta\hat{r} = \begin{pmatrix} 0 \\ \left(\frac{\mu_E}{\mu}\right)\frac{S_\lambda C_\lambda}{R^3}\left[\frac{r^3}{7} + \frac{6}{7}\frac{r_o^{7/2}}{r^{1/2}} - r_o^3\right] \end{pmatrix} \tag{5-51a}$$

The time correction δy is

$$\delta y = -\left(\frac{\mu_E}{\mu}\right)\frac{(3S_\lambda^2 - 1)}{R^3}\frac{2u^2}{\sqrt{2}}\left[\frac{1}{45u^{9/2}} + \frac{1}{5u_o^{5/2}u^2} - \frac{1}{45u_o^{9/2}} - \frac{1}{5u_o^2u^{5/2}}\right] \tag{5-52}$$

The total variation in position is then given by (4-32), which reduces to

$$\Delta\bar{r} = -\left(\frac{\mu_E}{\mu}\right)\left(\frac{r_o}{R}\right)^3 r_o \begin{pmatrix} (3S_\lambda^2-1)\left[\frac{1}{18}\left(\frac{r}{r_o}\right)^4 - \frac{1}{10}\left(\frac{r}{r_o}\right)^2 + \frac{2}{45}\left(\frac{r_o}{r}\right)^{1/2}\right] \\ \\ -\,3S_\lambda C_\lambda\left[\frac{1}{21}\left(\frac{r}{r_o}\right)^4 - \frac{1}{3}\left(\frac{r}{r_o}\right) + \frac{2}{7}\left(\frac{r}{r_o}\right)^{1/2}\right] \end{pmatrix} \tag{5-53}$$

<u>Solution to Perturbed Central Oscillator</u>

The solution to the perturbed central oscillator system for the perturbing force $\bar{f}$ due to the fixed external body is obtained in the same manner as that for the J_2 perturbation force of the previous section. The variation in the energy parameter α is obtained from the solution to (4-14a), resulting in

$$\delta\alpha = -\left(\frac{\mu_E}{\mu}\right)\frac{(3S_\lambda^2 - 1)}{R^3}\,[r^2 - r_o^2] \tag{5-54}$$

and the variation in the eccentricity vector is

$$\delta\bar{\epsilon} = \begin{pmatrix} 0 \\ \\ -\left(\frac{\mu_E}{\mu}\right)\frac{S_\lambda C_\lambda}{R^3}\,[r^3 - r_o^3] \end{pmatrix} \tag{5-55}$$

As with the corresponding results obtained for the J_2 perturbation, these results are valid for arbitrary value of α of the unperturbed reference conic. The differential equation for the fixed x variation in the position vector for the general reference rectilinear orbit is obtained by substituting (5-50), (5-54) and (5-55) in (4-15) to obtain

$$\delta\bar{r}'' + \alpha\,\delta\bar{r} = \begin{pmatrix} -\left(\dfrac{\mu_E}{\mu}\right)\dfrac{(3S_\lambda^2 - 1)}{R^3}\,[2r^3 - r_o^2\,r] \\ \\ \left(\dfrac{\mu_E}{\mu}\right)\dfrac{S_\lambda C_\lambda}{R^3}\,[4r^3 - r_o^3] \end{pmatrix} \tag{5-56}$$

For the reference rectilinear parabola $(\alpha = 0)$, the scalar radius r may be expressed as $r = x^2/2$ (also $r_o^2 = x_o^2/2$), thus defining the independent variable to vanish at periapsis rather than at the initial conditions (this redefinition is done only for convenience in evaluating the integral expressions). The resulting fixed x variation is

$$\delta\bar{r} = \begin{pmatrix} -\left(\dfrac{\mu_E}{\mu}\right)\dfrac{(3S_\lambda^2 - 1)}{R^3}\left[\dfrac{r^4}{14} - \dfrac{r^2 r_o^2}{6} + \dfrac{2\,r^{1/2} r_o^{7/2}}{21}\right] \\ \\ \left(\dfrac{\mu_E}{\mu}\right)\dfrac{S_\lambda C_\lambda}{R^3}\left[\dfrac{r^4}{7} - r r_o^3 + \dfrac{6\,r^{1/2} r_o^{7/2}}{7}\right] \end{pmatrix} \tag{5-57}$$

The time correction δx is obtained directly from (4-33), noting that $\delta r = \hat{r} \cdot \delta\bar{r} = -\delta\bar{r}_x$ and is

$$\delta x = -\frac{1}{r}\left[\left(\frac{\mu_E}{\mu}\right)\frac{(3S_\lambda^2 - 1)}{\sqrt{2}\,R^3}\left(\frac{r^{9/2}}{63} - \frac{r^{5/2}r_o^2}{15} + \frac{2\,rr_o^{7/2}}{21} - \frac{2\,r_o^{9/2}}{45}\right)\right] \tag{5-58}$$

The resulting fixed t variation is then obtained by combining (5-57) and (5-58) through (4-32); the resulting solution for the fixed time variation $\Delta\bar{r}$ is identical to (5-53).

Solution for Nonparabolic Reference Rectilinear Orbit

In the previous section the general expression for the scalar radius is

$$r = a(1 - C_{\sqrt{\alpha}x}) \; . \tag{5-59}$$

Thus the convolution integrand resulting from (5-56) will appear as a Poisson series in the transcendental functions $\cos(\sqrt{\alpha}x)$ and $\sin(\sqrt{\alpha}x)$ and the general solution to the problem may be obtained relatively easily. The solution in integral form is

$$\delta\bar{r} = \int_{x_o}^{x} \sqrt{a}\, S_{\sqrt{\alpha}(x-\sigma)} \begin{pmatrix} -\left(\frac{\mu_E}{\mu}\right)\frac{(3S_\lambda^2 - 1)}{R^3}\,[2r(\sigma)^3 - r_o^2 r(\sigma)] \\ o \\ \left(\frac{\mu_E}{\mu}\right)\frac{S_\lambda C_\lambda}{R^3}\,[4r(\sigma)^3 - r_o^3] \end{pmatrix} d\sigma \tag{5-60}$$

and is most easily obtained by evaluating the various component terms separately; thus

$$I_0(x) = \int_{x_o}^{x} \sqrt{a}\; S_{\sqrt{\alpha}(x-\sigma)} d\sigma = a\left[1 - S_{\sqrt{\alpha}\,x} S_{\sqrt{\alpha}\,x_o} - C_{\sqrt{\alpha}x} C_{\sqrt{\alpha}\,x_o}\right] \tag{5-61}$$

$$I_1(x) = \int_{x_o}^{x} \sqrt{a}\; S_{\sqrt{\alpha}(x-\sigma)} r(\sigma) d\sigma = a^2\left[1 - \tfrac{1}{2}\sqrt{\alpha}\,x\, S_{\sqrt{\alpha}\,x}\right.$$

$$+ S_{\sqrt{\alpha}\,x}\left(\sqrt{\alpha}\,x_o - S_{\sqrt{\alpha}\,x_o} + \tfrac{1}{2} S_{\sqrt{\alpha}\,x_o} C_{\sqrt{\alpha}\,x_o}\right)$$

$$\left. - C_{\sqrt{\alpha}\,x}\left(C_{\sqrt{\alpha}\,x_o} + \tfrac{1}{2} S^2_{\sqrt{\alpha}\,x_o}\right)\right] \tag{5-61a}$$

and

$$I_3(x) = \int_{x_o}^{x} \sqrt{a}\; S_{\sqrt{\alpha}(x-\sigma)}\; r^3(\sigma) d\sigma$$

$$= a^4\left\{S_{\sqrt{\alpha}\,x}\left[-\tfrac{15}{8}\sqrt{\alpha}\,\sigma + 4\, S_{\sqrt{\alpha}\,\sigma} - \tfrac{17}{8} S_{\sqrt{\alpha}\,\sigma} C_{\sqrt{\alpha}\,\sigma} - S^3_{\sqrt{\alpha}\,\sigma} + \tfrac{1}{4} S^3_{\sqrt{\alpha}\,\sigma} C_{\sqrt{\alpha}\,\sigma}\right]_{x_o}^{x}\right.$$

$$\left. - C_{\sqrt{\alpha}\,x}\left[\tfrac{1}{4}\left(1 - C_{\sqrt{\alpha}\,\sigma}\right)\right]_{x_o}^{x}\right\} \tag{5-61b}$$

Expressing the lower limit of (5-61b) as

$$L.L. = S_{\sqrt{\alpha}\,x} K_s(x_o) - C_{\sqrt{\alpha}\,x} K_c(x_o) \tag{5-62}$$

$I_3(x)$ may be simplified to

$$I_3(x) = a^4\Big[-\frac{15}{8}\sqrt{\alpha}\,x\,S_{\sqrt{\alpha}\,x} + 2(1 - C_{\sqrt{\alpha}\,x}) - \frac{1}{8}S^2_{\sqrt{\alpha}\,x}C_{\sqrt{\alpha}\,x}$$
$$+ S^2_{\sqrt{\alpha}\,x} + S_{\sqrt{\alpha}\,x}K_s(x_o) - C_{\sqrt{\alpha}\,x}K_c(x_o)\Big] \qquad (5\text{-}63)$$

The resulting fixed x variation $\delta\bar{r}$ may then be expressed as

$$\delta\bar{r} = \begin{pmatrix} -\left(\frac{\mu_E}{\mu}\right)\frac{(3S_\lambda^2 - 1)}{R^3}[2I_3(x) - r_o^2\,I_1(x)] \\ \\ \left(\frac{\mu_E}{\mu}\right)\frac{S_\lambda C_\lambda}{R^3}[4I_3(x) - r_o^3\,I_0(x)] \end{pmatrix}. \qquad (5\text{-}64)$$

The time correction is obtained through (4-33), where δr is the negative of the x component of $\delta\bar{r}$, resulting in

$$\delta x = \frac{1}{r}\left(\frac{\mu_E}{\mu}\right)\frac{(3S_\lambda^2 - 1)}{R^3}\left[2\int_{x_o}^{x} I_3(\sigma)d\sigma - r_o^2\int_{x_o}^{x} I_1(\sigma)d\sigma\right] \qquad (5\text{-}65)$$

where

$$\int_{x_o}^{x} I_1(\sigma)d\sigma = \Big[\sqrt{\alpha}\,x + \frac{1}{2}\sqrt{\alpha}\,x\,C_{\sqrt{\alpha}\,x} + C_{\sqrt{\alpha}\,x}\left(-\frac{1}{2}\sqrt{\alpha}\,x_o + S_{\sqrt{\alpha}\,x_o} - \frac{1}{2}S_{\sqrt{\alpha}\,x_o}C_{\sqrt{\alpha}\,x_o}\right)$$
$$- S_{\sqrt{\alpha}\,x}\left(\frac{1}{2} + C_{\sqrt{\alpha}\,x_o} + \frac{1}{2}S^2_{\sqrt{\alpha}\,x_o}\right) - \sqrt{\alpha}\,x_o + S_{\sqrt{\alpha}\,x_o}\Big]\,a^{5/2}$$

$$(5\text{-}66)$$

$$\int_{x_o}^{x} I_3(\sigma)d\sigma = \Bigg\{ \frac{5}{2}\sqrt{\alpha}\,x - S_{\sqrt{\alpha}\,x}\left[\frac{31}{8} + K_c(x_o)\right] - C_{\sqrt{\alpha}\,x} K_s(x_o)$$

$$+ \frac{15}{8}\sqrt{\alpha}\,x\, C_{\sqrt{\alpha}\,x} - \frac{1}{2} S_{\sqrt{\alpha}\,x} C_{\sqrt{\alpha}\,x} - \frac{1}{24} S^3_{\sqrt{\alpha}\,x} - \frac{5}{2}\sqrt{\alpha}\,x_o$$

$$+ \frac{5}{2} S_{\sqrt{\alpha}\,x_o} C_{\sqrt{\alpha}\,x_o} - \frac{15}{4}\sqrt{\alpha}\,x_o\, C_{\sqrt{\alpha}\,x_o} + \frac{15}{4} S_{\sqrt{\alpha}\,x_o} - \frac{5}{12} S^3_{\sqrt{\alpha}\,x_o} \Bigg\} a^{9/2}$$

(5-66a)

Noting that $\bar{r}' = (\sqrt{a}\, S_{\sqrt{\alpha}\,x},\ 0)$, the results may be combined through (4-32)

$$\Delta\bar{r} = \delta\bar{r} + \bar{r}'\,\delta x$$

to yield the total fixed t variation; the time correction enters only in the x component, which may be expressed as

$$\Delta\bar{r}_x = \left(\frac{\mu_E}{\mu}\right)\frac{(3S_\lambda^2 - 1)}{R^3}\left[r_o^2\left(I_1 - \frac{\sqrt{a}\, S_{\sqrt{\alpha}\,x}}{r}\int_{x_o}^{x} I_1(\sigma)d\sigma\right) - 2\left(I_3 - \frac{\sqrt{a}\, S_{\sqrt{\alpha}\,x}}{r}\int_{x_o}^{x} I_3(\sigma)d\sigma\right)\right] \qquad (5\text{-}67)$$

The above expression is not expanded further due to the lack of any apparent simplification of the results.

It would be expected that a series expansion of (5-67) and (5-64) in ascending powers of α would yield the solution for the reference rectilinear parabola (5-34) as the α^o term. This equivalence may be practically demonstrated on only the simplest of examples, since it will be noted that the resulting α^o term is obtained from the fourth term of the expansion; all preceding terms, which are negative powers of α (positive powers of a), will be expected to vanish.

Thus the solution to the problem of the perturbation of a rectilinear orbit due to an external body has been obtained for both the parabolic and nonparabolic reference rectilinear orbits.

It may be obvious that more extensive analyses could be effectively performed using algebraic computer techniques; such techniques are used extensively in the analysis of similar problems in celestial mechanics, based on Hamiltonian theory.

Chapter VI

ANALYSIS OF NEAR-PARABOLIC LUNAR TRAJECTORIES BETWEEN L_1 AND MOON

The advent of extensive manned exploration of the lunar surface has generated interest in the use of the cislunar libration point L_1 as the possible location of a staging space station. The (relatively) fixed location of the libration point L_1 in the rotating earth-moon space at approximately 15 per cent of the earth-moon distance from the moon presents the distinct advantage of no time constraint on passage between L_1 and the moon or communication with the visible portion of the lunar surface. The feasibility of and the stationkeeping requirements for such a space station, or libration point satellite, have been recently investigated by Farquhar [9] and others. In this chapter, the analytical theory developed for the perturbed central oscillator is applied to the analysis of near-parabolic lunar trajectories between the moon and L_1.

The particular family of lunar trajectories investigated are earth-perturbed, near-parabolic lunar orbits with perilune at the lunar surface. Although the theory may be extended to the elliptic or hyperbolic class of orbits, the choice of such would necessarily involve some considerations of available flight time vs available fuel and are beyond the scope of this analysis. Also, it might be noted that minimum energy transfer trajectories between the moon and L_1 are necessarily near-parabolic. The assumption of perilune at the lunar surface is justified by the patching of the L_1 - moon trajectory to a low lunar circular orbit, resulting in a more efficient overall coverage of the lunar surface, as opposed to a direct ascent/descent modus operandi. The analysis considers trajectories passing both in front of and behind the moon and both to and from L_1.

The application of the foregoing theory to the L_1-moon trajectory analysis is depicted in Figure 6.1, where $\bar{r}$ represents an unperturbed parabola with perilune at the lunar surface and $\delta\bar{r}$ represents the combined perturbation position vector due to both the perturbing force and variations in the tangential perilune velocity δv_y, directed along $\hat{j}_o$.

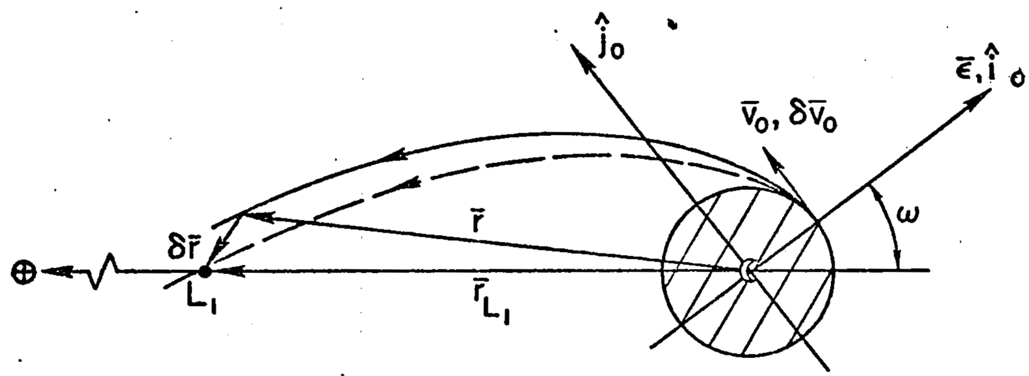

FIGURE 6.1 VECTOR GEOMETRY FOR REFERENCE PARABOLA RELATIVE TO EARTH-MOON LINE

The reference parabola is defined by the solutions (2-61) for $\alpha = 0$,

$$\bar{r} = (p/2 - x^2/2 \ , \sqrt{p}\ x)^T \tag{6-1}$$

$$\bar{r}' = (-x, \sqrt{p})^T \tag{6-1a}$$

$$r = p/2 + x^2/2 \tag{6-1b}$$

$$t = px/2 + x^3/6 \tag{6-1c}$$

Since $p = 2r_o$ for a parabola, solutions obtained in terms of x will be expressed through (6-1b) in the nondimensional form

$$\lambda = x/\sqrt{p} = (r/r_o - 1)^{1/2} \tag{6-2}$$

The state transition matrices (4-19) and (4-25) are simplified considerably by assuming the reference orbit to be a parabola, for which $\alpha = 0$. Thus

$$\Phi[(x-z),\bar{r}(z),\bar{r}'(z)] = \begin{pmatrix} 1 & x-z & -\frac{(x-z)^2}{2!} & -\bar{r}(z)\frac{(x-z)^2}{2!} - \bar{r}'(z)\frac{(x-z)^3}{3!} + \bar{\epsilon}\frac{(x-z)^4}{4!} \\ 0 & 1 & -(x-z) & -\bar{r}(z)\,(x-z) - \bar{r}'(z)\frac{(x-z)^2}{2!} + \bar{\epsilon}\frac{(x-z)^3}{3!} \\ & & & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(6-3)

and

$$\Psi[(x-z),r(z),r'(z)] = \begin{pmatrix} 1 & x-z & -r(z)\frac{(x-z)^2}{2!} - r'(z)\frac{(x-z)^3}{3!} - \frac{(x-z)^4}{4!} \\ 0 & 1 & -r(z)(x-z) - r'(z)\frac{(x-z)^2}{2!} - \frac{(x-z)^3}{3!} \\ 0 & 0 & 1 \end{pmatrix} \quad (6\text{-}4)$$

L_1-Moon Trajectories

Since the analysis is considering only variations in the initial (perilune) velocity vector (i.e., $\delta\bar{r}_o = 0$), expressed in the regularized domain as

$$\delta\bar{r}_o' = \delta\bar{v}_o/\dot{x}_o \quad (6\text{-}5)$$

the terms $\delta\alpha_o$ and $\delta\bar{\epsilon}_o$ in (4-50) and (4-50a) may be expressed simply as

$$\delta\alpha_o = -2\bar{r}_o' \cdot \delta\bar{r}_o'/r_o^2 \quad (6\text{-}6)$$

$$\delta\bar{\epsilon}_o = [2\bar{r}_o\bar{r}_o' - \bar{r}_o'\bar{r}_o - \bar{r}_o' \cdot \bar{r}_o I]\delta\bar{r}_o'/r_o^2 \quad (6\text{-}6a)$$

Substitution of (6-3) and (6-6) into (4-18) results in a somewhat simpler form for the integration of $\delta\bar{r}$, (recalling that $\bar{r}'' = -\bar{\epsilon}$ for the parabola)

$$\begin{aligned} \delta\bar{r} = & \{[r_o^2\, I]x + [\bar{r}_o'\bar{r}_o + \bar{r}_o' \cdot \bar{r}_o\, I]x^2/2! \\ & + [\bar{r}_o'\bar{r}_o']2x^3/3! + [\bar{r}_o''\bar{r}_o']2x^4/4!\}\delta\bar{r}/r_o^2 \\ & \int_0^x \{[r^2(\sigma)\, I](x-\sigma) + [\bar{r}'(\sigma)\,\bar{r}(\sigma) + \bar{r}'(\sigma) \cdot \bar{r}(\sigma)\, I](x-\sigma)^2/2! \\ & + [\bar{r}'(\sigma)\,\bar{r}'(\sigma)]2\,(x-\sigma)^3/3! + [\bar{r}''(\sigma)\,\bar{r}'(\sigma)]2(x-\sigma)^4/4!\}\bar{f}[\bar{r}(\sigma)]\,\sigma/\mu_m \end{aligned} \quad (6\text{-}7)$$

The attraction of the earth is given by

$$\bar{f}(\bar{r}) = \mu_e \left(\frac{\bar{R}-\bar{r}}{|\bar{R}-\bar{r}|^3} - \frac{\bar{R}}{R^3} \right) \tag{6-8}$$

Expanding (6-8) in ascending powers of $\bar{r}$ and assuming the earth to be fixed along the final orientation of the earth-L_1-moon line in the nonrotating lunar space (see Figure 6.1), the attraction of the earth $\bar{f}(\bar{r})$ is approximated to first order in $\bar{r}$ by

$$\bar{f}(\bar{r}) = \frac{\mu_e}{2R^3} \begin{pmatrix} 3\cos 2\omega + 1 & -3\sin 2\omega \\ -3\sin 2\omega & -3\cos 2\omega + 1 \end{pmatrix} \bar{r} \tag{6-9}$$

Using (6-2) and

$$\delta\bar{r}_o' = [0 \;,\; \delta r_y']^T \tag{6-10}$$

where

$$\delta r_y' = \delta v_y / \dot{x}_o \tag{6-11}$$

the homogeneous part of (6-7) reduces to

$$\delta\bar{r}_{(h)} = [\varphi_{xv}(\lambda), \varphi_{yv}(\lambda)]^T \delta r_y' \tag{6-12}$$

where

$$\varphi_{xv}(\lambda) = -\sqrt{p}\lambda^4/3 \tag{6-13}$$

$$\varphi_{yv}(\lambda) = \sqrt{p}\lambda(4\lambda^3 + 3)/3 \tag{6-13a}$$

Equivalently, changes in the final fixed x regularized velocity vector are related to $\delta r'_y$ by

$$\delta\bar{r}_{(H)}(\lambda) = [\varphi'_{xv}(\lambda), \varphi'_{yv}(\lambda)]^T \delta r'_y \tag{6-14}$$

where

$$\varphi'_{xv}(\lambda) = -4\lambda^3/3 \tag{6-15}$$

$$\varphi'_{yv}(\lambda) = 4\lambda^3 + 1 \quad . \tag{6-15a}$$

Using (6-1), the integrand of (6-7) may be expanded in ascending powers of $\sqrt{p}$ up to and including $\theta(p^3)$, and the integral may then be expressed in descending powers of the nondimensional parameter λ. Retaining only the highest power of λ (since $x \gg \sqrt{p}$ in vicinity of L_1), the particular solution to (6-7) is

$$\frac{\delta\bar{r}_{(F)}(\lambda)}{r_o} = \left(\frac{\mu_e}{\mu_m}\right)\left(\frac{r_o}{R}\right)^3 \begin{Bmatrix} -\lambda^8[3\cos 2\omega + 1]/28 \\ \lambda^8 \sin 2\omega/14 \end{Bmatrix} \tag{6-16}$$

and

$$\frac{\delta\bar{r}'_{(F)}(\lambda)}{\sqrt{p}} = \left(\frac{\mu_e}{\mu_m}\right)\left(\frac{r_o}{R}\right)^3 \begin{Bmatrix} -\lambda^7[3\cos 2\omega + 1]/7 \\ \lambda^7[2\sin 2\omega]/7 \end{Bmatrix} \tag{6-17}$$

The homogeneous solution to the scalar perturbation equation (4-24) is simplified by noting that for

$$\delta\bar{r}_o = \delta r_o = 0 \tag{6-18}$$

$$\bar{r}_o \cdot \bar{r}'_o = 0 \tag{6-18a}$$

$$\bar{r}_o \cdot \delta\bar{r}'_o = 0 \tag{6-18b}$$

we obtain

$$r_o' = \bar{r}_o \cdot \bar{r}_o'/r_o = 0 \tag{6-19}$$

$$\delta r_o' = (\bar{r}_o' \cdot \delta\bar{r}_o + \bar{r}_o \cdot \delta\bar{r}_o')/r_o - [(\bar{r}_o \cdot \delta\bar{r}_o)(\bar{r}_o \cdot \bar{r}_o')]/r_o^3 = 0 \tag{6-19a}$$

Thus, using (6-4) we obtain

$$\delta r_{(H)}(\lambda) = -p^2(\lambda^2/4 + \lambda^4/4!)\delta\alpha_o \tag{6-20}$$

where

$$\delta\alpha_o = -2\bar{r}_o \cdot \delta\bar{r}_o'/r_o^2 \tag{6-21}$$

$$= -2\sqrt{p}\,\delta r_y'/r_o^2 \tag{6-21a}$$

and

$$\delta r_{(H)}(\lambda) = -p\sqrt{p}(\lambda/2 + \lambda^3/3!)\delta\alpha_o \tag{6-22}$$

The particular solution to (4-24), retaining only the term of the highest power of λ, is given by

$$\frac{\delta r_{(F)}(\lambda)}{r_o} = \left(\frac{\mu_e}{\mu_m}\right)\left(\frac{r_o}{R}\right)^3 (3\cos 2\omega + 1)\lambda^8/28 \tag{6-23}$$

The difference in time between the perturbed and unperturbed systems is then obtained from (4-15b)

$$\delta t' = (\delta r_{(H)} + \delta r_{(F)})/\sqrt{\mu_m} \tag{6-24}$$

$$\delta t = \frac{1}{\sqrt{\mu_m}}\left\{p\left(\frac{2\lambda^3}{3} + \frac{2\lambda^5}{30}\right)\delta r_y' + \frac{p\sqrt{p}}{2}\left(\frac{\mu_e}{\mu_m}\right)\left(\frac{r_o}{R}\right)^3 (3\cos 2\omega + 1)\frac{\lambda^9}{252}\right\} \tag{6-25}$$

Referring now to Figure 6.1,

$$\bar{r}_{L_1}(x) = \bar{r}(x) + \delta\bar{r}_{(H)}(x) + \delta\bar{r}_{(F)}(x) \tag{6-26}$$

Using the expansion

$$x = x_{L_1} + \Delta x \tag{6-27}$$

and

$$\bar{r}(x) = \bar{r}(x_{L_1}) + \bar{r}'(x_{L_1})\Delta x \tag{6-28}$$

where

$$x_{L_1} = (2r_{L_1} - p)^{1/2} \tag{6-29}$$

$$= \sqrt{p}\,\lambda_{L_1} \tag{6-29a}$$

and the approximations

$$\delta\bar{r}_{(H)}(\lambda) \simeq \delta\bar{r}_{(H)}(\lambda_{L_1}) \tag{6-30}$$

$$\delta\bar{r}_{(F)}(\lambda) \simeq \delta\bar{r}_{(F)}(\lambda_{L_1}) \quad , \tag{6-30a}$$

Equation (6-26) becomes

$$\begin{pmatrix} -r_{L_1} \cos\omega \\ \\ r_{L_1} \sin\omega \end{pmatrix} = \begin{pmatrix} r_o - x_{L_1}^2/2 \\ \\ \sqrt{p}\,x_{L_1} \end{pmatrix} + \begin{pmatrix} -x_{L_1} \\ \\ \sqrt{p} \end{pmatrix} \Delta x + \delta\bar{r}_F + \begin{pmatrix} \varphi_{xv}(x_{L_1}) \\ \\ \varphi_{yv}(x_{L_1}) \end{pmatrix} \delta r_y' \tag{6-31}$$

and the terms $\delta r_y'$ and Δx are obtainable from

$$\begin{pmatrix} \delta r_y' \\ \\ \Delta x \end{pmatrix} = \frac{1}{\sqrt{p}\ \ \varphi_{xv}(x_{L_1}) + x_{L_1}\varphi_{yv}(x_{L_1})} \begin{pmatrix} \sqrt{p} & x_{L_1} \\ \\ -\varphi_{yv}(x_{L_1}) & \varphi_{xv}(x_{L_1}) \end{pmatrix} \times$$

$$\left\{ \begin{pmatrix} -r_{L_1} \cos\omega \\ \\ r_{L_1} \sin\omega \end{pmatrix} - \begin{pmatrix} r_o - \dfrac{x_{L_1}^2}{2} \\ \\ \sqrt{p}\ x_{L_1} \end{pmatrix} - \delta \bar{r}_{(F)} \right\} \tag{6-32}$$

From (6-16), (6-32), and noting that

$$\sqrt{p}\ \varphi_{xv}(x_{L_1}) + x_{L_1}\ \varphi_{yv}(x_{L_1}) = p\lambda_{L_1}^2 (\lambda_{L_1}^2 + 1) = 2r_{L_1}\lambda_{L_1}^2 \ , \tag{6-33}$$

we obtain an approximate relation for the variation in perilune velocity as a function of the orientation angle ω

$$\frac{\delta r_y'}{\sqrt{p}} = \frac{\delta v_y}{v_y} = \frac{\lambda_{L_1} \sin\omega - (\cos\omega + 1)}{2\lambda_{L_1}^2} + \frac{\left(\dfrac{\mu_e}{\mu_m}\right)\left(\dfrac{r_o}{R}\right)^3\left(\dfrac{r_o}{r_{L_1}}\right)\left[\dfrac{\lambda_{L_1}^8}{28}(3\cos 2\omega + 1) - \dfrac{\lambda_{L_1}^9}{14}\sin 2\omega\right]}{2\lambda_{L_1}^2} \tag{6-34}$$

The term Δx may be expressed as

$$\frac{\Delta x}{\sqrt{p}} = \delta\lambda \frac{(4\lambda_{L_1}^2 + 3)[r_{L_1}(\cos\omega + 1) + p + \delta\bar{r}_{F_x}]}{6\lambda_{L_1} r_{L_1}} - \frac{\lambda_{L_1}^3 [r_{L_1} \sin\omega - p\lambda_{L_1} - \delta\bar{r}_{F_y}]}{{}^6\lambda_{L_1} r_{L_1}} \tag{6-35}$$

The resultant velocity vector of arrival at L_1 in the nonrotating space is then

$$\bar{v}_{L_1} = \bar{r}'_{L_1} \dot{x}_{L_1} = \bar{r}'_{L_1}\sqrt{\mu_m}/r_{L_1} \tag{6-36}$$

where

$$\bar{r}'_{L_1}(x_{L_1}) = \bar{r}'(x_{L_1}) + \bar{r}''(x_{L_1})\Delta x + \delta\bar{r}'_{(H)}(x_{L_1}) + \delta\bar{r}'_{(F)}(x_{L_1}) \tag{6-37}$$

The time of arrival at L_1 is obtained from

$$t_{L_1} = t(x_{L_1}) + \delta t(x_{L_1}) + \Delta x(x_{L_1})/\dot{x}_{L_1} \tag{6-38}$$

where δt is obtained from (6-25)

All parameters have been normalized to the dimensions of the earth-moon system, namely,

Length: $R = 1$

Mass: $\mu_e + \mu_m = 1$

Time: $n^2 = 1 = (\mu_e + \mu_m)/R^3$

Normalized velocities are then related to un-normalized velocities through

$$v_{\text{un-norm.}} = [(\mu_e + \mu_m)/R]^{1/2} \, v_{\text{norm.}} \qquad (6\text{-}39)$$

and a unit of time is equal to $P_m/2\pi$, or approximately 4 1/3 days. The value for μ_m was taken to be 0.0121507, corresponding to $\mu_m/\gamma_e \simeq 81.3$, with $r_{L_1} = 0.151$ and $r_o = .00427$ (1740 km) [reference 9]. All calculations were made for the idealized planar earth-moon system (i.e., both bodies in circular orbits about the barycenter).

The exact nonlinear perturbation equations were numerically integrated for lunar orbits from the moon to L_1 , passing both in front of and behind the moon. Due to the reflection property of earth-moon trajectories in the earth-moon line of the rotating frame (exact), the velocity vector of departure (arrival) in the rotating frame for passage behind the moon is the reflection of the velocity vector of arrival (departure) for passage in front of the moon. With regard to the analytical theory, the simplifying assumption of the fixed earth renders the departure and arrival velocity vectors identical, for passage either in front of or behind the moon.

Figure 6.2 presents the variation in δv_p as a function of ω , δv_p being related to parabolic velocity v_p at perilune through (6-34). The variation in total transit time, (6-38), as a function of ω , is shown in Figure 6.3. It might be noted that the lunar longitude at the time of perilune passage must necessarily be obtained from

$$\text{Longitude} = \pi - \omega - t_{L_1}(\omega)$$

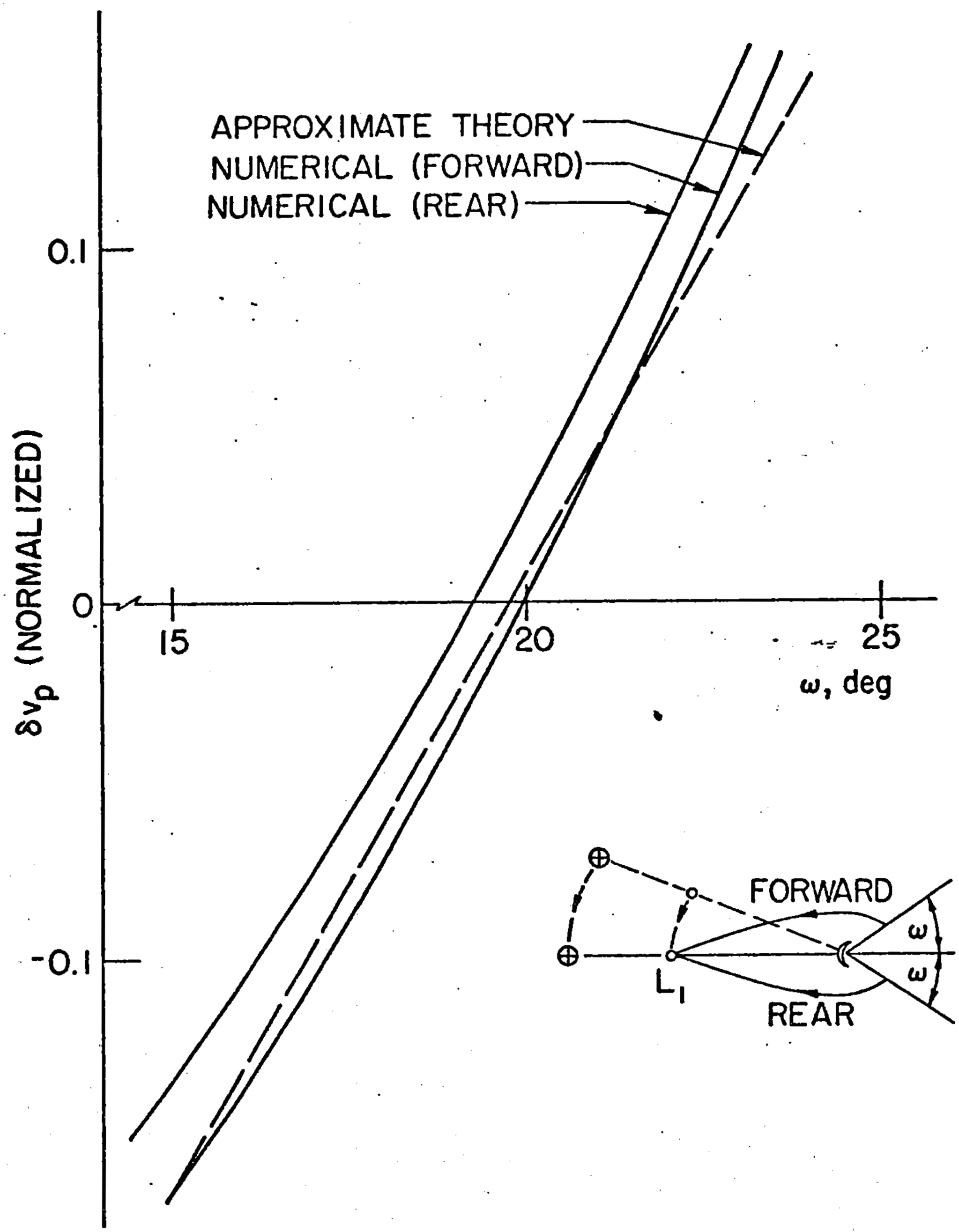

FIGURE 6.2. VARIATION PARABOLIC VELOCITY AT PERILUNE VS ORIENTATION ANGLE ω .

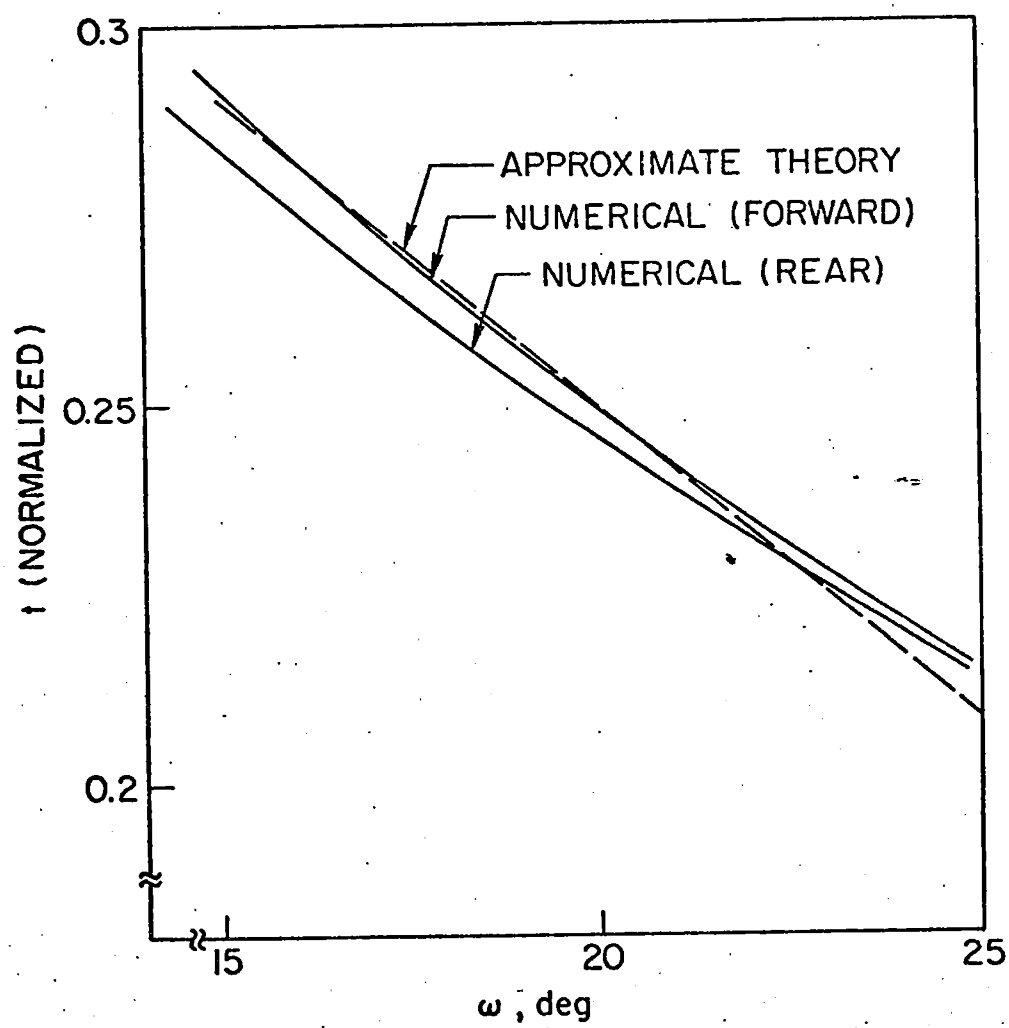

FIGURE 6.3. TRANSIT TIME BETWEEN PERILUNE AND L_1 VS ORIENTATION ANGLE ω .

The velocity vector of arrival/departure in the rotating earth-moon space is

$$\bar{v}_{(R)L_1} = \pm [C_{R/o}\,\bar{v}_{(o)L_1} + \bar{n} \times \bar{r}_{(R)L_1}] \qquad (6\text{-}40)$$

+ = arrival

- = departure

where

$$C_{R/o} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \pm\sin\omega & \pm\cos\omega \end{pmatrix}$$

+ = passage in front of moon

- = passage behind moon

(6-41)

$$\bar{n} = \hat{k}_R = \hat{k}_o \qquad (6\text{-}42)$$

$$\bar{r}_{(R)L_1} = r_{L_1}\,\hat{i}_R \qquad (6\text{-}42a)$$

The arrival velocity component v_x along $\hat{i}_R$ is given in Figure 6.4 and the lateral arrival velocity component v_y is shown in Figure 6.5 for passage in front of and behind the moon. The differences in the numerical integrations may be viewed (in the nonrotating frame) as the difference in geometry of the earth attraction relative to the lunar orbit, while in the rotating frame simply reflect the fact that passage in front of the moon takes advantage of the moon's motion about L_1. The degree of agreement between the approximate theory and either of the numerically integrated data would seem to depend on the validity of the approximate assumed geometry relative to either of the exact geometries of the earth attraction.

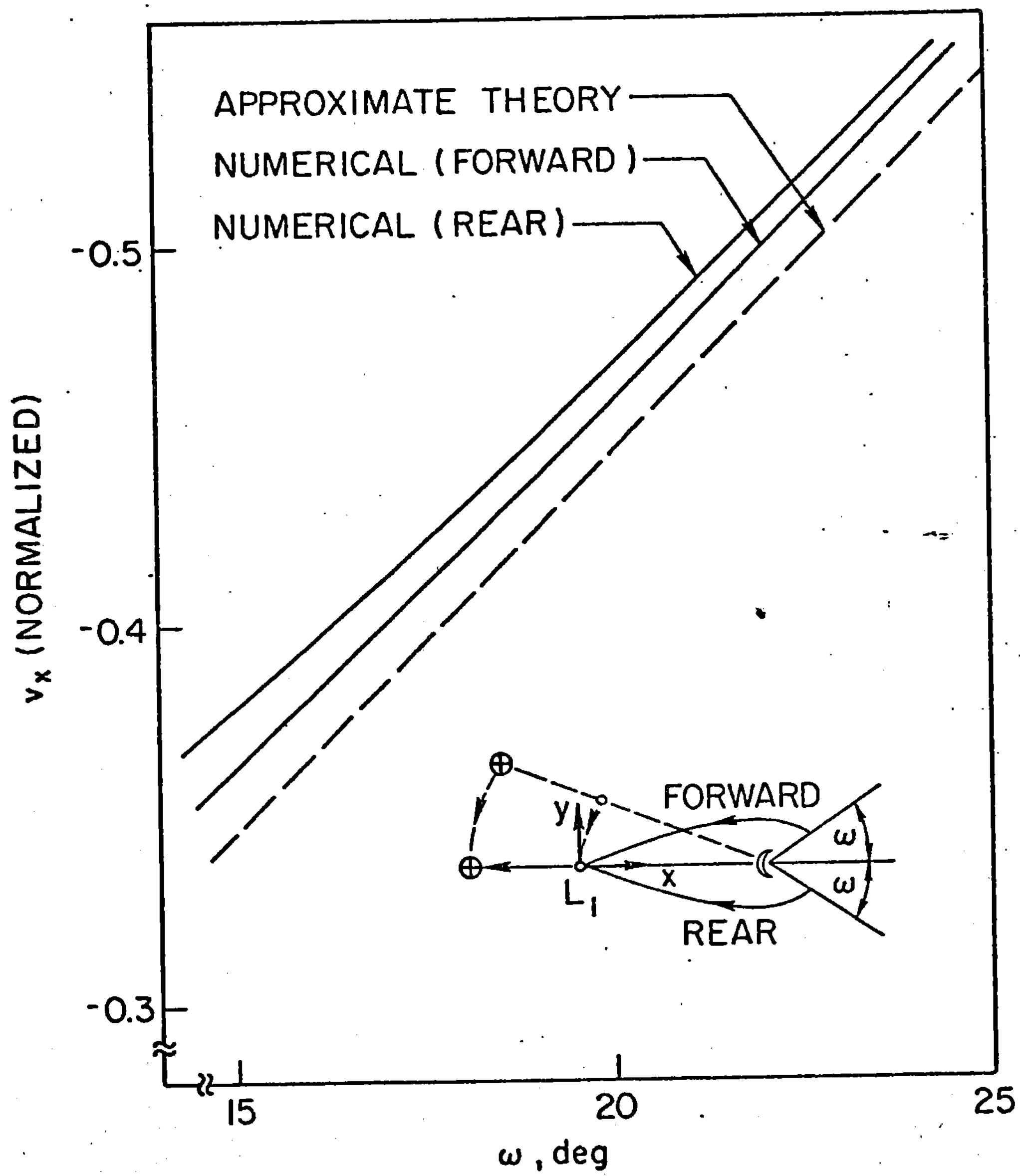

FIGURE 6.4. AXIAL VELOCITY COMPONENT V_x AT L_1 VS ORIENTATION ANGLE ω .

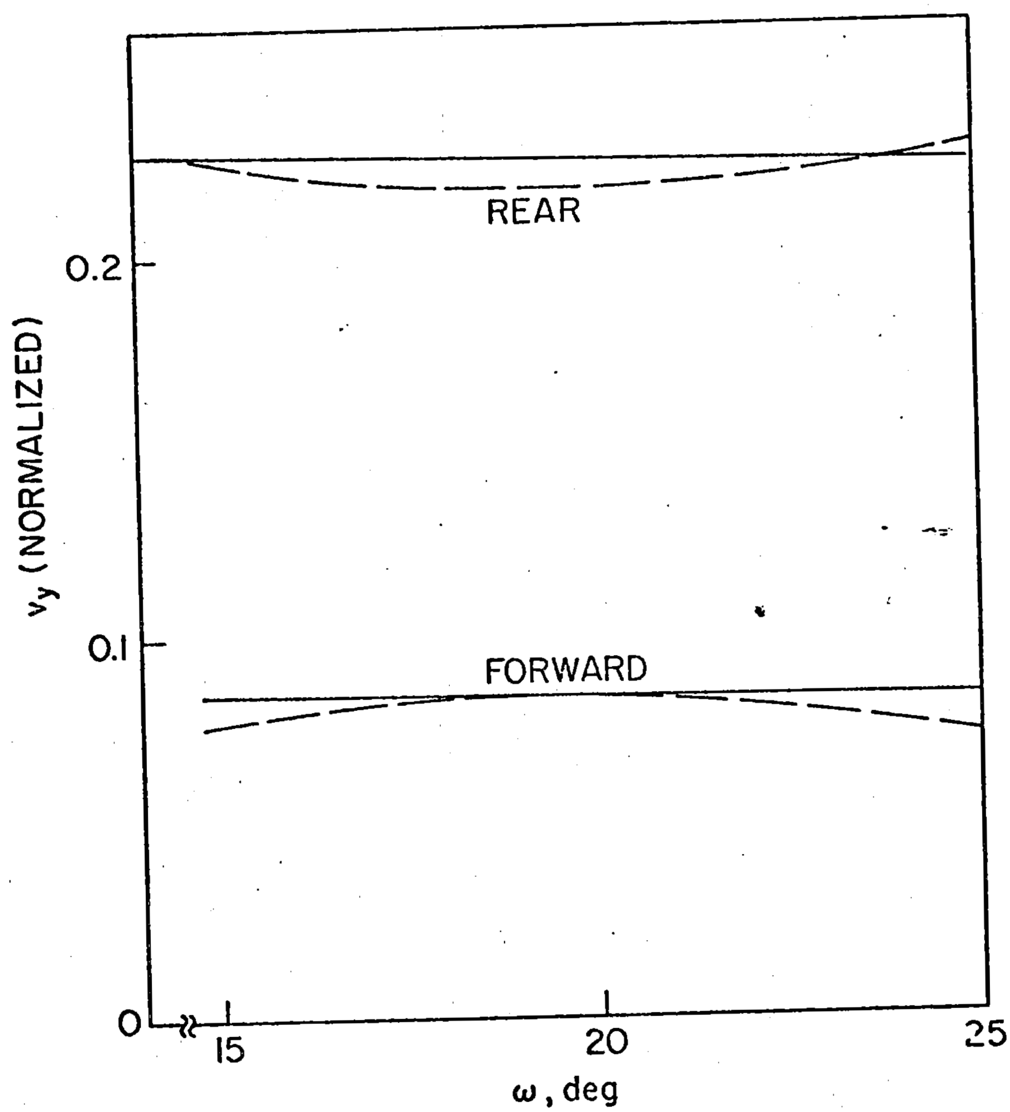

FIGURE 6.5. LATERAL VELOCITY COMPONENT V_y AT L_1 VS ORIENTATION ANGLE ω .

Chapter VII

SUMMARY AND CONCLUSIONS

Unperturbed Keplerian motion has been represented through a transformation of independent variables as either of two harmonic oscillator systems, where the new independent variables are related to the true and eccentric/hyperbolic anomalies. The harmonic oscillator systems are referred to respectively as the focal and central oscillator systems. The central oscillator system is distinguished by the fact that it effectively represents a local regularization of the two-body problem and is hence valid for rectilinear orbits up to and including periapsis passage at the singularity. The focal oscillator is fully as general except that the solution is unbounded at the singularity. The natural frequencies of the central and focal oscillators are related to the orbit energy and angular momentum respectively. The solutions to both oscillator systems are presented in universal forms which are directly applicable functionally to all types of orbits.

The differential equations governing perturbed motion were then obtained as perturbed harmonic oscillator systems, and nonlinear variational equations are developed in vector form for both systems. Several computational aspects of use of either the variational (Encke) equations or the total perturbed system (Cowell) equations of motion are discussed. The most important feature of the perturbed central oscillator system, in either the Cowell or Encke form, is that it is a regularized system and yields a well-behaved numerical solution regardless of proximity to the singularity. The advantage of use of the perturbed focal oscillator system in a numerical computation is in the vicinity of periapsis, where a degree of accuracy greater than that of the central oscillator results.

Linearized variational equations have been developed in vector form for both harmonic oscillator systems. They are solvable by quadrature in the domains of the new independent variables. The general solutions to the linearized variational equations have been presented as state transition matrices involving the universal functions obtained for the unperturbed solutions. The Poisson series nature of the integrands suggests the use of each oscillator system for particular forms of the perturbing force. Examples of each type have been given to illustrate the selection procedure.

The linearized variational theory has been demonstrated in the analysis of a perturbed circular orbit and perturbed rectilinear orbits. The solution to the perturbed circular orbit was obtained from the Euler-Hill equations, and identical results were obtained using the solutions to both perturbed harmonic oscillator systems. The direct extension to noncircular reference orbits has been indicated. The analysis of perturbed rectilinear orbits was accomplished using the theory developed for both oscillator systems for two distinct types of perturbing force. The two types of perturbing force considered were the attraction of a fixed external body and the attraction of the second spherical harmonic (J_2) term of the expansion of the primary body potential. Each perturbing force has been analyzed using both systems for a reference rectilinear parabola for verification.

The extension to the nonparabolic reference rectilinear orbit has been obtained using the oscillator system appropriate to the particular perturbing force.

As a further demonstration of the theory, near-parabolic transfer trajectories between the moon and the cislunar libration point L_1 have been analyzed using the results of the linearized theory of the central oscillator system. The results have been presented as algebraic expressions relating the variation in parabolic velocity for perilune at the lunar surface necessary for rendezvous at L_1 as a function of perilune orientation. The results may be interpreted as the velocity requirements at L_1 or the moon for passage in either direction and on either side of the moon. The analytical results agree quite closely with the numerical evaluation of corresponding nonlinear equations of motion. Retaining only the first term of the analytical expressions accounts for 75 - 90 percent of the nonlinear perturbation values.

Several areas of investigation exist in application of both the nonlinear and linearized theory to problems in celestial mechanics. Among these would be other forms of perturbing force and numerous theoretical and practical applications. The Poisson series nature of the integrands of the linearized variational theory also suggests extension to higher order perturbation theory using algebraic computer operations.

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