CRITERIA FOR, AND EXTRAPOLATION IN OVERSTRESS MODELS*

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ABSTRACT

An overstress life test is a life test in which the factors (stresses) which induce failures in a structure or a component are allowed to assume values much above their nominal or useage values. This has an effect of reducing the test times, an obvious economic advantage. The objective in such life tests is to be able to make an inference about the failure behavior of the device at use conditions.

Overstress life tests are feasible both from a practical and an analytical point of view if one chooses an appropriate model relating the failure behavior with the stresses, and given such an appropriate model, one has well developed analytical techniques to yield good extrapolations. This presentation mainly addresses itself to the latter question although it briefly hints at the criteria for the selection of an appropriate test model and references pertinent literature to assist the reader in his choice. To the knowledge of this author, there do not exist satisfactory techniques for the design and the analysis of overstress life tests, and hence, the significance of the results presented here. There is no claim made to the optimality of the results.

Next, this paper points out some accelerated life test models, currently used in practice, and briefly

discusses criteria for their selection. The general procedure is applied to each one of the specific models and a summary of the results is presented, keeping the mathematical details to a minimum.

The results of each model comprise of a formula which gives an estimate of the mean life at use conditions environment and confidence limits for the mean life. The significance of the results are their direct useability in the practice of reliability.

CRITERIA FOR, AND EXTRAPOLATIONS IN OVERSTRESS TEST MODELS

1. Introduction

In many situations it is impossible or undesirable to perform a life test on components under the environment in which they will be used. This is true in our space-related industries where testing must be simulated at laboratories here on earth. It is also a fact of life in economic enterprises where the duration of the test may determine the costs. In such cases, the shorter the test time, the more economical the procedure. In view of these considerations, accelerated life testing takes on great value.

In this paper, as is ascertained by Pieruschka [5], it is assumed that the type of the life-time distribution is not changed with the introduction of greater stresses, but that the parameters of the distribution may vary. For example, if the life-time distribution is normal with mean μ, and standard deviation σ, at use conditions, then it remains normal as we introduce more stresses but the overstressed life-time distribution may have mean μ_2 and standard deviation σ_2. This assumption is not unreasonable when you consider the difficulties encountered if a quantum change in behavior were to occur at a certain level of stress.

The general procedure used here is to express the parameters of the life-time distribution in terms of the environmental stresses. Perhaps the mean varies proportionally with the temperature or the standard deviation varies with the sum of the squares of the voltage and pressure, etc. We need know only the form of the relation between the parameters and the environmental stresses, for example linear, quadiatic, exponential and so on. For illustrative purposes, suppose that the failure distribution has two parameters α and β and the environment consists of only two stresses, say s_1 and s_2. Then one may have, for example

$$\alpha = a_0 + a_1 s_1 + a_2 s_2$$

$$\beta = b_0 + b_1 \log s_1 + \exp(b_2 s_2) \ .$$

One can obtain point estimates and confidence intervals (interval estimates) for the a_i and b_i as well as point and interval estimates of α and β for fixed values of s_1 and s_2 which are of interest. A joint prediction region for α and β can also be obtained. Having obtained the point estimates of α and β, one can plug these in the life-time distribution and study its properties under various values of the stresses. The procedure is to run life tests at different combinations of values of s_1 and s_2 and to obtain point estimates of α and β. If the distributions of the estimates of α and β are known, one can use the method of maximum likelihood to obtain the estimates of the a_i and b_i.

Some well known results about the exponential distribution which are useful in the subsequent development of the text are presented here. Details about these results can be found in [2].

Let $f(t;\ \lambda_i) = \lambda_i \exp(-\lambda_i t)$ be the time to failure distribution of a device when it is subjected to the constant application of a single stress V_i. λ_i is the hazard rate, and $\Theta_i = 1/\lambda_i$ is the mean time to failure.

If n_i items are put on test under an environment V_i and if the test is terminated when r_i items fail noting the times to failure $t_{1i}, t_{2i}, \ldots t_{r_i i}$, then the maximum likelihood estimator of Θ_i is given by

$$\hat{\Theta}_i = \left[\sum_{j=1}^{r_i} t_{ji} + (n_i - r_i)\ t_{r_i i}\right] \Big/ r_i \tag{1}$$

Let $\hat{\lambda}_i = 1/\hat{\Theta}_i$ be an estimate of the hazard rate λ_i. The mean and variance of $\hat{\lambda}_i$ can be easily computed [5] as,

$$E[\hat{\lambda}_i] = \lambda_i r_i/(r_i - 1) \qquad r_i > 1$$

$$VAR[\hat{\lambda}_i] = \lambda_i^2 r_i^2/(r_i - 1)^2\ (r_i - 2) \qquad r_i > 2$$

Thus, $\hat{\lambda}_i$ is a biased estimator of λ_i, but if r_i is large, then $E[\hat{\lambda}_i] \doteq \lambda_i$ and $VAR[\hat{\lambda}_i] \doteq \lambda_i/r_i$.
Hence, a large number of failures at each stress level is desirable.

2. <u>How to Conduct the Accelerated Life Test</u>

Suppose that V_i, i = 1, 2, ... , k are the k values of V_i at which it is desired to conduct an accelerated life test. These k values should be sufficiently high so that each item can be tested until failure. To ensure that there is no correlation among the tests conducted at the different values of the V_i, it is necessary to have some randomization scheme. A reasonable procedure is to arrange the V_i's according to a table of random numbers, and then to conduct the life tests according to this random sequence. Now having observed the values of r_i, n_i and t_{ij} corresponding to each V_i, one can calculate estimates of Θ_i from the formula (1) in the preceding paragraphs. Now let us see how to use these estimates to determine point and interval estimates of the mean time to failure at use conditions stress under different assumptions about the relationship between the hazard rate and the stress.

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*A model of the form $\lambda i = A + BVi^P$ has been conjectured in the literature by Adams [1], and could be reduced to the linear model if A and P were known.

3. The Linear-Stress Failure Model

Suppose that the device under consideration is subjected to the constant application of a single stress V_i and that its failure distribution is given by $f(t;\ \lambda_i) = \lambda_i \exp(-\lambda_i t_i)$. Furthermore, it is assumed that $\lambda_i = BV_i$ where B is an unknown parameter. One needs to estimate B and subsequently obtain point and interval estimates of λ_μ, the hazard rate at some use condition stress $V\mu$.

A model of the form $\lambda_i = A + BV_i^P$ has been conjectured in the literature by Adams [1], and could be reduced to the linear model if A and P were known. Based on accelerated life tests conducted a k different values of V_i, the maximum likelihood estimate of B is $\hat{B} = \sum_{i=1}^{k} r_i \Big/ \sum_{i=1}^{k} r_i V_i \hat{\theta}_i$ where V_i is the stress level, r_i is the number of failures noted at each V_i and $\hat{\theta}_i$ is calculated as before [6].

The exact small sample distribution of this estimator of B is an inverted gamma. It is easy to verify [6] that $E[\hat{B}] = BJ/J-1$ and $VAR[\hat{B}] = B^2J^2/(J-1)^2(J-2)$, where $J = \Sigma r_i$. If J is large, $E[\hat{B}] = B$ and $VAR[\hat{B}] = B^2/\Sigma r_i$. Note that the variance of the estimator is independent of the choice of the V_i. Confidence limits for B can be obtained by observing that $\frac{2JB}{\hat{B}}$ follows the chi-squared distribution with 2J degrees of freedom.

Now let the estimated hazard rate be denoted by $\Lambda_i = \hat{B}V_i$. For large values of J, $E[\Lambda_i] = \hat{B}V_i$ and $VAR[\Lambda_i] = \Lambda_i^2/J$ for all i, i=1, 2, ..., k.

For prediction purposes, $\Lambda_\mu = \hat{B}\, V_\mu$ is an unbiased estimator of λ_μ at some use condition stress V_μ. It can be verified [6] that $\frac{2J\lambda_\mu}{\Lambda_\mu}$ follows the chi-squared distribution with $(2J^2+2J)$ degrees of freedom and hence a γ $100(1-\alpha)$ % perdiction interval for λ_μ is given by:

$$P_r\left\{X^2_{1-\alpha/2}(2J(J+1))\,\frac{\Lambda_\mu}{2J} \leq X_\mu \leq X^2_{\alpha/2}(2J(J+1))\,\frac{\Lambda_\mu}{2J}\right\} = 1 - \alpha$$

Note: $X^2_{\alpha/2}(2J(J+1))$ represents the point of a X^2 distribution with 2J(J+1) degrees of freedom which cuts off an area $\alpha/2$ from the right hand tail of the distribution. This value can be found in almost any book of tables relating to statistics. This concludes estimation and inference under the linear-stress failure model.

4. The Power Rule Model

The Power Rule Model is a second model frequently used in practice. Its application to the accelerated life testing of paper capacitors has been discussed by Levenback, G. J. [5]. The Model can be written as $\Theta_i = \frac{C}{V_i^P}$ where C is a constant of proportionality and needs to be determined.

In order to obtain estimators of P and C that are asumptotically independent, the Power Rule Model has to be amended slightly, without changing its basic character, as $\Theta_i = \frac{C.}{(V_i/\dot{V})^P}$ where $\dot{V} = \Pi_{i=1}^{k}(V_i)^{r_i/\Sigma r_i}$, i.e., it is the weighted geometric mean of the V_i's. The estimator of P, $\hat{P}$ must satisfy the equation

$$\Sigma r_i\hat{\Theta}_i\left[\frac{V_i}{\dot{V}}\right]^P \log\left[\frac{V_i}{\dot{V}}\right] = 0 . \qquad (2)$$

Since it is non-linear, its solution has to be numerically obtained. The Newton-Raphson method (Hildebrand, F. [3]) was tried on this equation for various sets of data that were simulated on a computer. In all instances, it was found that the solution converged to a unique value in 5 to 10 iterations of the method.

After P has been obtained $\hat{C}$ is given by

$$\hat{C} = \frac{\Sigma r_i \hat{\Theta}_i \left(\frac{V_i}{\dot{V}}\right)^{\hat{P}}}{r_i} \quad . \tag{3}$$

$$VAR[\hat{P}] = \sigma_P^2 = \left[k \; \Sigma r_i \left[\log\left(\frac{V_i}{\dot{V}}\right)\right]^2\right]^{-1}$$

$$VAR[\hat{C}] = \sigma_C^2 = \hat{C}^2 (k \; \Sigma r_i)^{-1}$$

and

$$Cov[\hat{C},\hat{P}] = 0 \; .$$

The standard large sample theory ensures that these estimators are asymptotically approximately normal. To ascertain the goodness of this approximation, the shape of the relative likelihood function arising from the actual sample should be examined. If the shape of the relative likelihood function from the actual sample is skew, then the likelihood is considered to be non-normal and further sampling should be done. On the contrary, if the shape of the relative likelihood function is symmetric, then one can proceed with an application of the large sample approximation.

The Relative Likelihood Functions of P and C.

The relative likelihood function of P is given by [7]

$$R_M(P^*) = \frac{1}{J} \prod_{i=1}^{k} [\Gamma(r_i)]^{-1} \exp -\left[\frac{r_i\hat{\Theta}_i}{\hat{C}(P^*)} \left[\frac{V_i}{\dot{V}}\right]^{P^*}\right] \left[\frac{r_i}{\hat{C}(P^*)} \left[\frac{V_i}{\dot{V}}\right]^{P^*}\right]^{r_i} (\hat{\Theta}_i)^{r_i-1}$$

where

$$J = \prod_{i=1}^{k} \frac{1}{\Gamma(r_i)} \exp -\left[\frac{r_i\Theta_i}{\hat{C}}\left[\frac{V_i}{\dot{V}}\right]^{\hat{P}}\right]\left[\frac{r_i}{\hat{C}}\left[\frac{V_i}{\dot{V}}\right]^{\hat{P}}\right]^{r_i} (\hat{\Theta}_i)^{r_i-1}$$

and

$$\hat{C}(P^*) = \sum_{i=1}^{k} r_i\hat{\Theta}_i \left(\frac{V_i}{\dot{V}}\right)^{P^*} \Bigg/ \sum_{i=1}^{k} r_i \ .$$

A plot of $R_M(P^*)$ for different values of P^* would present the relative likelihood of P arising from the actual sample. Since the likelihood of P is a function of k, r_i and V_i the user must check in each problem to see if his parameters have suitable values to ensure a symmetric relative likelihood.

Similarly the relative likelihood function of C is [7]

$$R_M(C^*) = \frac{1}{J} \prod_{i=1}^{k} \frac{1}{\Gamma(r_i)} \exp -\left(\frac{r_i\hat{\Theta}_i}{C^*} \left(\frac{V_i}{\dot{V}}\right)^{\hat{P}(C^*)}\right) \left(\frac{r_i}{C^*} \left(\frac{V_i}{\dot{V}}\right)^{\hat{P}(C^*)}\right)^{r_i} \hat{\Theta}_i^{\,r_i-1}$$

where $\hat{P}(C^*)$ is a solution of $\sum_{i=1}^{k} r_i\hat{\Theta}_i \ln\left(\frac{V_i}{\dot{V}}\right)\left(\frac{V_i}{\dot{V}}\right)^{\hat{P}(C^*)} = 0$.

Now an unbiased estimator of Θ_μ^2 is given by reference [7]

$$\hat{\Theta}_\mu = \hat{C}\left(\frac{V_\mu}{\dot{V}}\right)^{-\hat{P}} \exp\left\{-\frac{1}{2}\sigma_P^2\left(\log\left(\frac{V_\mu}{\dot{V}}\right)^2\right)\right\}$$

$$VAR[\hat{\Theta}_\mu] = \left(\frac{V_\mu}{\dot{V}}\right)^{-2P}\left[(\sigma_C^2+\hat{C}^2)\exp\left\{\sigma_P^2\left(\log\left(\frac{V_\mu}{\dot{V}}\right)\right)^2\right\} - \hat{C}^2\right]$$

The distribution of $\hat{\Theta}_\mu$ cannot be analytically obtained. In view of this it is difficult to obtain confidence limits for Θ_μ and hence the following alternative approach is taken.

The relative likelihood of Θ_μ is given by [7]

$$R_M(\Theta_\mu) = \exp\left[-\frac{1}{2}\left(\frac{\overline{\Theta}_\mu-\Theta_\mu}{\sigma_C J_\mu^{-\hat{P}}}\right)^2\right] \quad \text{where } \overline{\Theta}_\mu = \hat{C}\, J^{-\hat{P}}.$$

This function can be used as a measure of plausibility for the unknown parameter. For example, all values of the parameter which have at least 10% relative likelihood can be considered as fairly plausible. Values of the unknown parameter outside this range are fairly implausible since there exist values of the unknown parameter for which the observations are at least ten times more probable

The methods discussed before will be illustrated here via an example. The data for this example was generated on a computer.

$C = 1000$, and $P = 3$ were arbitrarily chosen and 5 values of θ_i were generated using the Power Rule Model $\theta_i = CV_i^{-P}$, $i = 1, \ldots 5$. Corresponding to each value of V_i, the n_i and r_i were chosen as shown in the table below. The $\hat{\theta}_i$ were obtained using equation (7), and a point estimator of P, $\hat{P}$ was obtained by solving equation (2) using the Newton-Raphson method; $\hat{P}$ was found to be 3.09. A value of $\hat{C} = .038$ was next obtained from equation (3). It is to be remarked here, that the amended model $\theta_i = C(V_i|\dot{V})^{-P}$ does not effect the value of P used in the model $_i = CV_i^{-P}$, but the C does get effected, as is reflected by the estimate of C, $\hat{C}$. Next, unbiased estimators of $\tilde{\theta}_i$ were obtained using $\tilde{\theta}_i = C(V_i|\dot{V})^{-\hat{P}} \exp\{-\frac{1}{2}\sigma_P^2 \; \log(V_i|\dot{V})\}^2$.

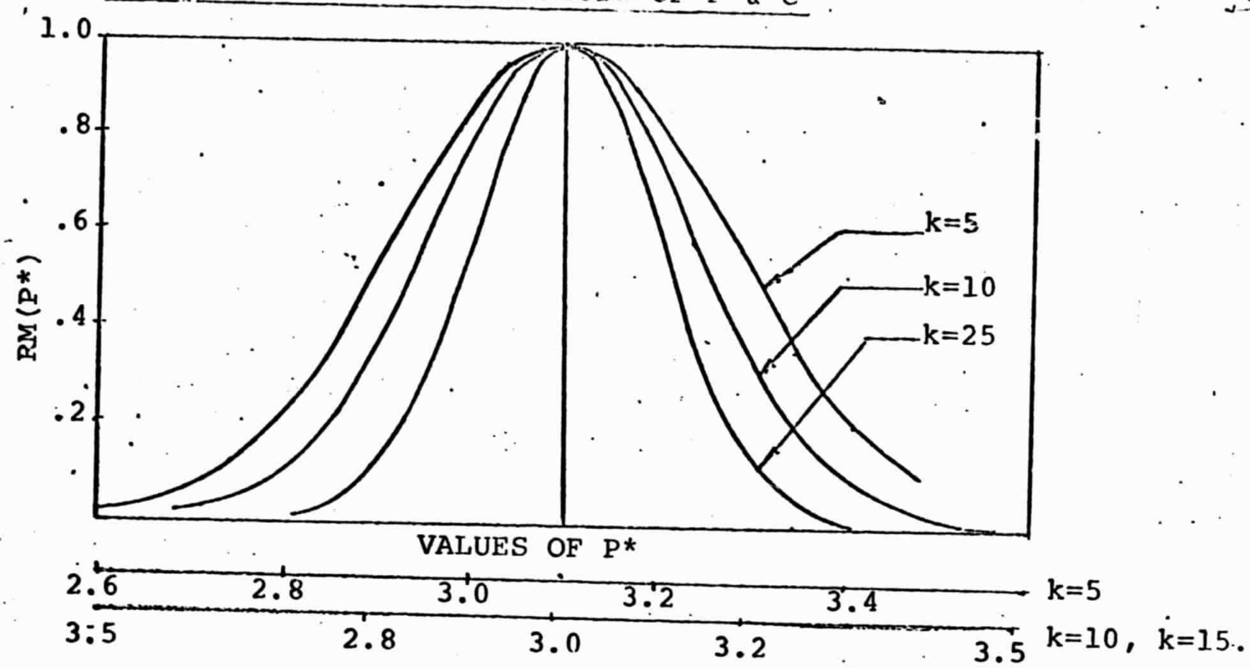
The above procedure was repeated for $k = 10$, and $k = 25$, but for brevity the following table portrays the results for $k = 5$ only.

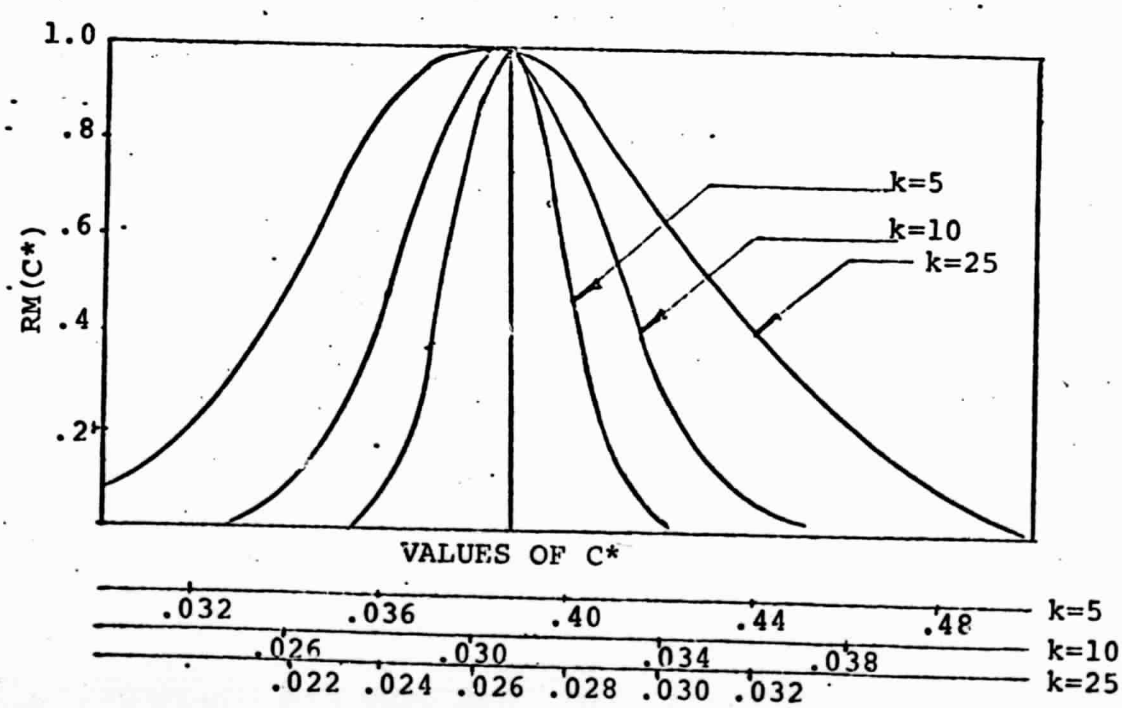
V_i	n_i	r_i	Θ_i	$\hat{\Theta}_i$	Θ_i
10	30	15	1.000	1.308	1.041
20	30	15	0.125	0.078	0.123
30	30	20	0.037	0.030	0.035
40	30	25	0.016	0.017	0.014
50	30	25	0.008	0.008	0.007

To assert the goodness of the large sample approximation of P and C, the actual relative likelihoods of P and C, $R_M(P^*)$ and $R_M(C^*)$ respectively, were plotted for k = 5, 10 and 25. These are portrayed in the graphs below. From these graphs, it appears that as k increases, their symmetry is enhanced, and that their spread narrows. These graphs assert the goodness of the normality approximation on $\hat{P}$ and $\hat{C}$ respectively.

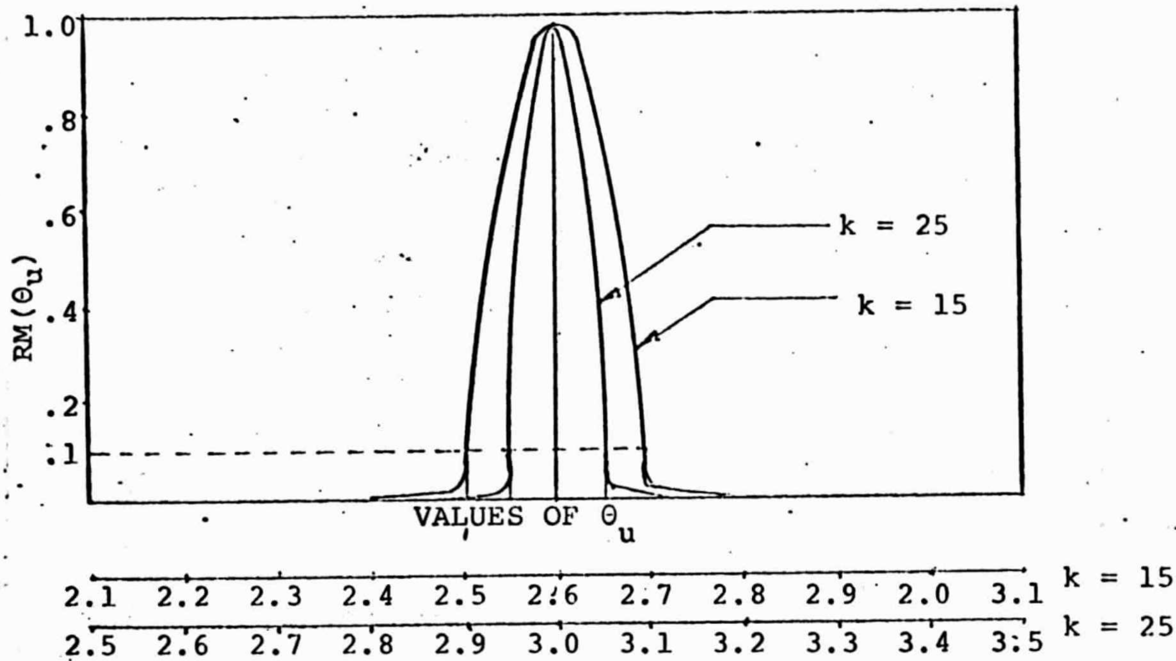
A plausibility interval for Θ_μ can be obtained from a plot of the relative likelihood function of Θ_μ. Such a plot for $V_\mu = 7$ is shown when k = 15 and when k = 25. These plots give (2.5, 2.7) and (2.95, 3.05) respectively as values of Θ_μ which have a 10% relative likelihood.

PLOT OF RELATIVE LIKELIHOODS OF P & C





PLOT OF RELATIVE LIKELIHOOD OF Θ_u



5. The Arrehenius Model

Next we turn to the Arrhenius Reaction Rate Model. The applicability of the Arrhenius Model in accelerated life testing has been discussed by several authors; see for example Thomas, R. [5].

A range of stress is prescribed and an exponential model for failure times for all values of the stress within this range is assumed. The two parameter exponential distribution is $f(t;\ \lambda_i,\gamma_i) = \lambda_i e^{-\lambda_i(t-\gamma_i)}$. The exponential scale parameter λ is assumed to equal the exponent of $A - B/V$, where V is a thermal stress and A and B are unknown parameters. For the purposes of this paper, it is assumed that the exponential location parameter γ equals $\alpha - \beta V$, where α and β are unknown parameters. The parameters A, B, α, and β have to be estimated by conducting life tests at k accelerated values of V_i each of which are sufficiently high to induce failures.

The maximum likelihood estimators of A and B, $\hat{A}$ and $\hat{B}$ are given by a solution of the following equations

$$\sum r_i - \sum r_i/\hat{\lambda}_i \exp(A-B(V_C^{-1}-\overline{V})) = 0$$

and

$$\sum \frac{r_i}{\hat{\lambda}_i} (V_i^{-1}-\overline{V}) \exp(A-B(V_i^{-1}-\overline{V})) = 0$$

where

$$\overline{V} = \sum \frac{r_i}{V_i} \Big/ \sum r_i \quad .$$

The above equations being non-linear, their solution can be numerically obtained by using the Newton-Raphson method [3]. For various sets of data that were simulated on a computer it was found that $\hat{A}$ and $\hat{B}$ could be obtained in a few iterations of the method [8].

$$VAR[\hat{A}] = \sigma_a = (k \sum r_i)^{-1}$$

$$VAR[\hat{B}] = \sigma_b = (k \sum r_i (V_C^{-1}-\overline{V})^2)^{-1}$$

and

$$Cov(\hat{A},\hat{B}) = 0 \quad .$$

$\Lambda_\mu = \exp[\hat{A}-\hat{B}(V_\mu^{-1}-\bar{V})]$ is the maximum likelihood extimator of λ_μ at use conditions stress V_μ. An unbiased estimator of λ_μ is given by [8]

$$\Delta_\mu = \Lambda_\mu \exp[-\frac{1}{2}(\alpha_a^2+J_\mu^2\sigma_b^2)]$$

where

$$J_\mu = V_\mu^{-1} - \bar{V} .$$

The estimators of α and β are

$$\hat{\alpha} = \frac{\sum \hat{\lambda}\ t_{1i}}{\sum \hat{\lambda}_{\not{c}i}} \quad \text{and} \quad \hat{\beta} = -\frac{\sum \hat{\lambda}_i (V_i-\dot{V})\ t_{1i}}{\sum \hat{\lambda}_i^* (V_i-\dot{V})^2}$$

$$\text{VAR}[\hat{\alpha}] = \frac{k}{(n \sum \hat{\lambda}_i)^2}$$

$$\text{VAR}[\hat{\beta}] = \frac{\sum (V_i-\dot{V})^2/n^2}{\sum \hat{\lambda}_i (V_i-\dot{V})^2 \ ^2}$$

In the light of the above results it follows that $\hat{\gamma}_\mu = \hat{\alpha} - \hat{\beta}(V_\mu-\dot{V})$ is an unbiased estimator of γ_μ at use stress. An unbiased estimator of Θ_μ is therefore given by

$$\hat{\Theta}_\mu = \hat{\gamma}_\mu + \frac{1}{\Delta_\mu} \exp\{-(\alpha_a^2 + J_\mu^2 \sigma_b^2),$$

with

$$\mathrm{VAR}[\hat{\Theta}_\mu] = S_1^2 + \frac{1}{\lambda_\mu{}^2} \{\exp(\alpha_a^2 + J_\mu^2 \sigma_b^2) - 1\}$$

where

$$S_1^2 = \mathrm{VAR}[\hat{\alpha}] + (V_\mu - \dot{V})^2\ \mathrm{VAR}[\hat{\beta}]\ .$$

As before a plausibility interval for Θ_μ can be obtained from a plot of the maximum relative likelihood function of Θ_μ.

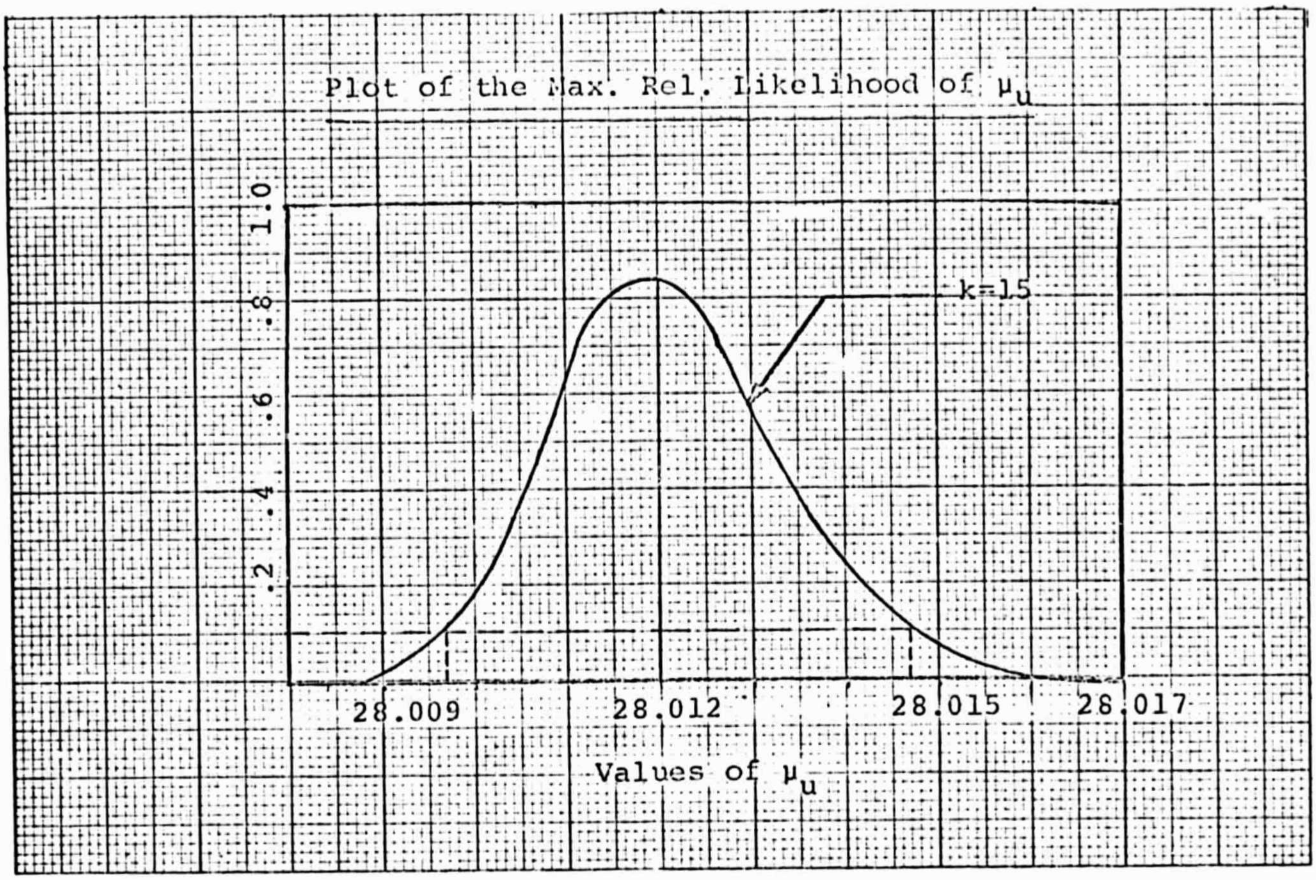
The maximum relative likelihood function of Θ_μ is

$$R_M(\hat{\Theta}_\mu) = \hat{\gamma}_\mu + \frac{1}{\Delta_\mu} \exp - \frac{1}{2} \quad \frac{\log \Delta_\mu + \log(\Theta_\mu - \hat{\gamma}_\mu) + \tau}{S_2}^{2}$$

where

$$\tau = \frac{\sigma_a^2 + J_u^2 \sigma_b^2}{2} \text{ and } S_2 = \sqrt{2\tau}$$

Such a plot for A = 5, B = 6, α = 35, β = 1, V_μ = 7 and k = 15 is shown. This plot gives 28.014 and 23.009 as values of Θ_μ which have a 10% relative likelihood.


Plot of the Max. Rel. Likelihood of μ_u
1.0
.8
.6
.4
.2
k=15
28.009
28.012
28.015
28.017
Values of μ_u

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