Final Report of the

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August 31, 1971

Part II

INFLUENCE OF BANDWIDTH RESTRICTION ON THE SIGNAL-TO-NOISE PERFORMANCE OF A MODULATED PCM/NRZ SIGNAL

Prepared

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CR-115160

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ABSTRACT

The effects of bandlimiting on the performance of various digital transmission systems corrupted by additive white Gaussian noise are analyzed using two methods, the averaging method and the series expansion method. The results from both methods agree.

The performance of an ideal bandlimited NRZ (Non-Return-to-Zero) baseband transmission system is examined using correlation detection and sampling. The explicit expression for the degradation of the signal and the intersymbol interference as a function of system parameters is derived. The average, lower bounds and upper bounds of the probabilities of bit-error are computed for both detectors. It is shown that the correlation detector performs better than the sample detector for $BT \geq 0.6$ and worse for $BT=0.5$.

A Split-Phase baseband system is also analyzed following the same steps used for analyzing the NRZ system. It is shown that a Split-Phase baseband system requires less than twice as much bandwidth as the NRZ system to have the same probability of bit-error for the same value of signal-to-noise ratio using the correlation detector.

An NRZ baseband system using Gaussian filters is also analyzed employing correlation detection. It is found that the system introduces more intersymbol interference and performs poorly compared to the ideal bandlimited NRZ system.

The effects of bandlimiting on the performance of modulation the Phase-Shift-Keying (PSK) System, the Amplitude-Shift-Keying (ASK) System, and the Frequency-Shift-Keying (FSK) System are analyzed assuming a correlation receiver and using ideal filters as well as correlation detection. The explicit expression for the degradation of the signal and the intersymbol interference as a function of bandwidths of the filters, signal-to-noise ratio and carrier frequencies is given. It is found that the aliasing effect can be neglected if the carrier frequency is more than three times the bit rate. It is also found that PSK requires 3 db less on an average power basis than ASK. If the spacing between the two carrier tones in FSK is less than three times the bit rate, FSK shows a better performance than that of ASK. The optimum setting of the tone spacing of FSK is shown to be equal to the bit rate. However, PSK always gives the best performance. Thus for a coherent system, PSK should always be used.

Finally, a tapped-delay-line (TDL) filter is introduced at the receiver of the NRZ baseband system in conjunction with the correlation detector as an intersymbol eliminator. On an average probability of bit-error basis, and using only three taps, it is demonstrated that the performance of this system is near optimum.

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CHAPTER I

INTRODUCTION

Today the advances in the fields of digital computers and electronic circuits have resulted in an enhanced interest in communication systems which transfer binary data from one location to another.

The data communication systems can generally be considered as consisting of three basic blocks, the transmitter, the channel and the receiver. The transmitter has the task of assigning an electrical waveform to each possible sequence of digits received from the information source. The electrical waveform is then passed through the channel, which may typically be a wire link, a satellite link, a microwave system, or a radio link. In passage through the channel the transmitted waveform is invariably corrupted by unwanted, random signals known as noise. Because of these random signals, the received waveform does not correspond exactly to any of the possible transmitted waveforms. Nevertheless, the receiver must make a decision as to which of the sequence of digits is most likely to have given rise to the particular received waveform.

For binary communication systems, the most popularly used today, the possible electrical waveforms consist of two, one is used for a binary "one" and the other is used for a binary "zero." Throughout this dissertation only the binary systems will be considered. In passing through the channel, these signals are corrupted by additive white Gaussian noise.

The measure of the performance of a digital communication system is the bit error probability at the output of the receiver bit detector. The bit detector which achieves the lowest possible error probability for a given signal-to-noise ratio is generally considered optimum. For ideal (infinite system bandwidth) binary communications over an additive Gaussian noise channel, the optimum bit detector is a correlation detector which turns out to be a matched filter.

In practice, the restriction of the system bandwidth is inevitable. Transmitter filtering, bandlimited channel, or receiver filtering are the usual sources. Bandwidth limiting will not only cause the energy loss of the desired signal, but more importantly will introduce interference. This interference consists of intersymbol interference (signal waveforms smearing in time) and intermodulation interference (aliasing effect). The performance of the optimum linear bit detector then will be degraded.

The primary concern of this dissertation will be to systematically analyze the effects of bandlimiting on the performance of various baseband transmission systems as well as modulation systems.

In Chapter II the optimum receiver structure in the case of infinite bandwidth and Gaussian noise for a minimum probability of error performance criterion will be derived.

In Chapter III the performance of an ideal bandlimited NRZ (Non-Return-to-Zero) baseband transmission system will be examined very closely. First, the explicit expressions for the degradation of the signal and intersymbol interference will be derived as a function of system parameters. Second, the average probability of bit error will be computed by using the averaging method. This method makes an assumption that the intersymbol interference is limited to a finite number of symbols preceding and following the symbol under detection. The conditional error probabilities are computed for each of the truncated pulse sequences and then averaged with respect to the probability of occurrence of those sequences. Third, an analytical expression for the probability of error based on the series expansion of the characteristic functions of the intersymbol interference and Gaussian noise will be introduced. This expression can be divided into two terms, one term corresponds to detecting the degraded signal itself, and the other corresponds to the influence of the intersymbol interference. The methods discussed in this chapter then will be generalized to any data transmission system.

In Chapter IV the results of Chapter III will be applied to three practical baseband systems: (1) Split-Phase using a correlation detector, (2) NRZ using a filter and sample detector, (3) NRZ (Gaussian filtering) using a correlation detector. In each case the explicit expressions for the degraded signal and intersymbol interference will be presented. The probability of bit error is also determined and calculated.

In Chapter V the results of Chapter III will be applied to three practical modulation systems, Phase Shift Keying (PSK), Amplitude Shift Keying (ASK) and Frequency Shift Keying (FSK). The explicit expressions for the intermodulation interference will be derived. The probability of error will be computed for each case.

In Chapter VI a modified Tapped-Delay-Line (TDL) filter will be proposed to alleviate the intersymbol interference for the bandlimited NRZ baseband system. The results then will be generalized for any system.

In Chapter VII some conclusions are drawn. Some recommendations for future research studies are also put forth.

CHAPTER II

OPTIMUM DETECTION OF BINARY SIGNALS IN THE PRESENCE OF WHITE GAUSSIAN NOISE

2.1 Formulation of the Optimum Solution

The binary message is assumed to be carried by either of two signals $s_1(t)$ (corresponding to information "1") and $s_0(t)$ (corresponding to information "0") of arbitrary and different shape over an ideal channel (infinite bandwidth) with additive white Gaussian noise n(t) with zero mean and spectral density $N_0/2$ (two sided) as shown in Figure 2.1.

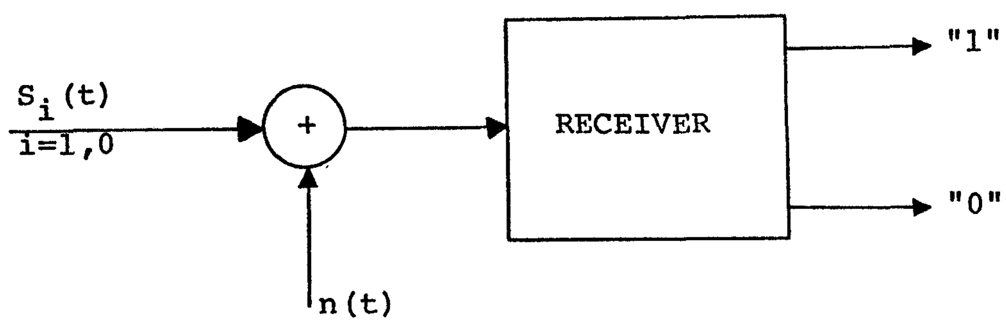

Figure 2.1 A Binary Transmission System

The basic problem of detection then is to find a receiver to distinguish between either of these two wave shapes $s_1(t)$ and $s_0(t)$, each defined over the bit interval T sec in length in an optimum way to minimize the probability of error.

This can be formulated as a statistical hypothesis testing problem, i.e., test the hypothesis H_1 that $s_1(t)$ was transmitted versus the hypothesis H_0 that $s_0(t)$ was transmitted. Since the performance criterion is taken to be the minimum average probability of error, Bayes' solution with equal cost yields the optimum decision rule [8]. This solution leads to the likelihood ratio test.

Given a sequence of random variables $x_1, x_2, \ldots x_n$ the likelihood ratio test is formed by finding the ratio of the conditional joint probability density function of $x_1, x_2, \ldots x_n$, given hypothesis H_1 or H_0. This ratio is compared to a threshold d and the decision H_1 rendered if the likelihood ratio is greater than d and H_0 otherwise. The likelihood ratio can be expressed as

$$\frac{f(x_1, x_2, \ldots x_n \mid H_1)}{f(x_1, x_2, \ldots x_n \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} d \qquad (2.1)$$

The optimum value of d under the equal cost assumption can be given by the ratio of the a priori probability P_0 and P_1 of occurrence of H_1 and H_0, respectively. Thus we have

$$\frac{f(x_1, x_2, \ldots x_n \mid H_1)}{f(x_1, x_2, \ldots x_n \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0}{P_1} \qquad (2.2)$$

It is readily seen that any monotonic function will yield the decision and hence the test is usually implemented in the form of logarithm of the likelihood ratio and the threshold

$$\ln \frac{f(x_1,x_2,\ldots x_n \mid H_1)}{f(x_1,x_2,\ldots x_n \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{P_0}{P_1} \tag{2.3}$$

2.2 Determination of the Likelihood Ratio

The input to the receiver under either hypothesis can be written as

$$z(t) = \begin{cases} s_1(t) + n(t) & 0<t<T, \quad : H_1 \quad (2.4) \\ s_0(t) + n(t) & 0<t<T, \quad : H_0 \quad (2.5) \end{cases}$$

Since both signals $s_1(t)$ and $s_0(t)$ are defined over the same interval $0<t<T$, we can expand each into an orthonormal series which has the form:

$$s_i(t) = \sum_{k=1}^{\infty} s_{ki}\, q_k(t) \qquad i = 1,0 \tag{2.6}$$

with the coefficients given by

$$s_{ki} = \int_o^T s_i(t)\, q_k(t)\, dt \qquad i = 1,0 \tag{2.7}$$

and the orthonormality condition implies:

$$\int_o^T q_i(t)\, q_j(t)\, dt = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.8}$$

The coefficients can be referred as the generalized Fourier coefficients.

The noise n(t) can also be expanded into the orthonormal series:

$$n(t) = \sum_{k=1}^{\infty} n_k \; q_k(t) \tag{2.9}$$

with

$$n_k = \int_o^T n(t) \; q_k(t) \; dt \tag{2.10}$$

Since n(t) is Gaussian distributed, the Fourier coefficients n_k are also Gaussian distributed. By proper choice of the orthonormal functions the coefficients can be made uncorrelated and hence statistically independent. The condition for statistically independent coefficients in the case of Gaussian noise is given by the solution of the integral equation [9], [24]

$$\int_o^T R(t-s) \; q_k(t) \; dt = \sigma_k^2 \; q_k(s) \tag{2.11}$$

Here $R(\tau)$ is the autocorrelation function of the noise n(t), and σ_k^2 is the ensemble average of n_k^2 or the variance of n_k.

Under the assumption that n(t) is white Gaussian noise with two sided spectral density $N_0/2$, we have

$$R(\tau) = (N_0/2)\,\delta(\tau) \tag{2.12}$$

Thus from Equation (2.11), we have

$$\sigma_k^2 = N_0/2 \tag{2.13}$$

The probability density function of n_k then is given by

$$f(n_k) = \frac{e^{-\frac{n_k^2}{N_0}}}{\sqrt{N_0 \pi}} \tag{2.14}$$

Therefore, instead of dealing with the continuous time functions $s_1(t)$, $s_0(t)$ and $n(t)$ defined over $0<t<T$, we can now represent each by its Fourier coefficients so that the Bayes' likelihood ratio test (Equation (2.3)) can be applied.

Recall that

$$z(t) = s_i(t) + n(t) \qquad i = 1,0 \;, \qquad 0<t<T$$

we can also expand $z(t)$ into a generalized Fourier series

$$z(t) = \sum_{k=1}^{\infty} z_k \, q_k(t)$$

$$= \sum_{k=1}^{\infty} s_{ki} \, q_k(t) + \sum_{k=1}^{\infty} n_k \, q_k(t) \tag{2.15}$$

Comparing the coefficients of $q_k(t)$, we obtain

$$z_k = s_{ki} + n_k \qquad i = 1,0 \tag{2.16}$$

Since the noise coefficients n_k are Gaussian distributed independent random variables with zero mean, the coefficients z_k are also Gaussian distributed and independent with mean at

s_{ki}. Therefore the density function of z_k can then be given by

$$f(z_k) = \frac{e^{\frac{-(z_k - s_{ki})^2}{N_0}}}{\sqrt{N_0 \pi}} \qquad i = 1,0 \qquad (2.17)$$

Now we can apply Bayes' likelihood ratio test. The receiver measures z(t) over the interval 0<t<T and from this measurement generates the generalized Fourier coefficients z_k. It then performs the test

$$\ln \frac{f(\vec{z} \mid H_1)}{f(\vec{z} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{P_0}{P_1} \qquad (2.18)$$

where

$$\vec{z} = [z_1, z_2, z_3 \ldots]$$

But

$$f(\vec{z} \mid H_1) = \prod_{k=1}^{\infty} f(z_k \mid H_1) = \prod_{k=1}^{\infty} \frac{e^{\frac{-(z_k - s_{k1})^2}{N_0}}}{\sqrt{N_0 \pi}} \qquad (2.19)$$

and

$$f(\vec{z} \mid H_0) = \prod_{k=1}^{\infty} f(z_k \mid H_0) = \prod_{k=1}^{\infty} \frac{e^{\frac{-(z_k - s_{k0})^2}{N_0}}}{\sqrt{N_0 \pi}} \qquad (2.20)$$

Substituting into Equation (2.18), we have

$$\sum_{k=1}^{\infty} (z_k - s_{k0})^2 - \sum_{k=1}^{\infty} (z_k - s_{k1})^2 \underset{H_0}{\overset{H_1}{\gtrless}} N_0 \ln \frac{P_0}{P_1} \qquad (2.21)$$

Recall that

$$s_i(t) = \sum_{k=1}^{\infty} s_{ki}\, q_k(t)$$

and

$$z(t) = \sum_{k=1}^{\infty} z_k\, q_k(t) \quad ,$$

then

$$z(t) - s_i(t) = \sum_{k=1}^{\infty} (z_k - s_{ki})\, q_k(t) \tag{2.22}$$

Squaring and averaging in time

$$\int_o^T (z(t) - s_i(t))^2\, dt = \int_o^T \left(\sum_{k=1}^{\infty} (z_k - s_{ki})\, q_k(t) \cdot \sum_{m=1}^{\infty} (z_m - s_{mi})\, q_m(t)\right) dt \tag{2.23}$$

Using the orthonormal property (Equation (2.8)), Equation (2.23) can be simplified as

$$\int_o^T (z(t) - s_i(t))^2\, dt = \sum_{k=1}^{\infty} (z_k - s_{ki})^2 \tag{2.24}$$

Substituting into Equation (2.21), the desired likelihood ratio test becomes

$$\int_o^T (z(t) - s_0(t))^2\, dt - \int_o^T (z(t) - s_1(t))^2\, dt \underset{H_0}{\overset{H_1}{\gtrless}} N_0 \ln \frac{P_0}{P_1} \tag{2.25}$$

For equal a priori probabilities, we then decide $s_1(t)$ was transmitted with smallest probability of being in error, if the difference between $z(t)$ and the known wave shape $s_1(t)$ is smaller than the corresponding difference between $z(t)$ and $s_0(t)$ in the mean-square sense and decide $s_0(t)$ otherwise.

2.3 Structure of the Optimum Receiver

The mathematical structure of the optimum receiver is completely specified by Equation (2.25). In this section, this equation will be used to yield a model for the optimum receiver.

Under the assumption of equal cost and equal a priori probabilities of occurrence of the signalling states, Equation (2.25) can be reduced to

$$\int_0^T z(t)[s_1(t) - s_0(t)]\, dt \underset{H_0}{\overset{H_1}{\gtrless}} 1/2\ (E_1 - E_0) \qquad (2.26)$$

where $E_1 = \int_0^T s_1^2(t)\, dt$ and $E_0 = \int_0^T s_0^2(t)\, dt$, represent the energy in the signals $s_1(t)$ and $s_0(t)$ respectively.

The optimum receiver structure now can be realized by a multiplier cascaded with an integrator (memoryless correlation detector) in series with a threshold device as shown in Figure 2.2. The message will be decided to be a "1", if the signal plus noise at the output of the integrator samples at $t=T$ is larger than the threshold $d = (E_1 - E_0)/2$, and a "0" otherwise.

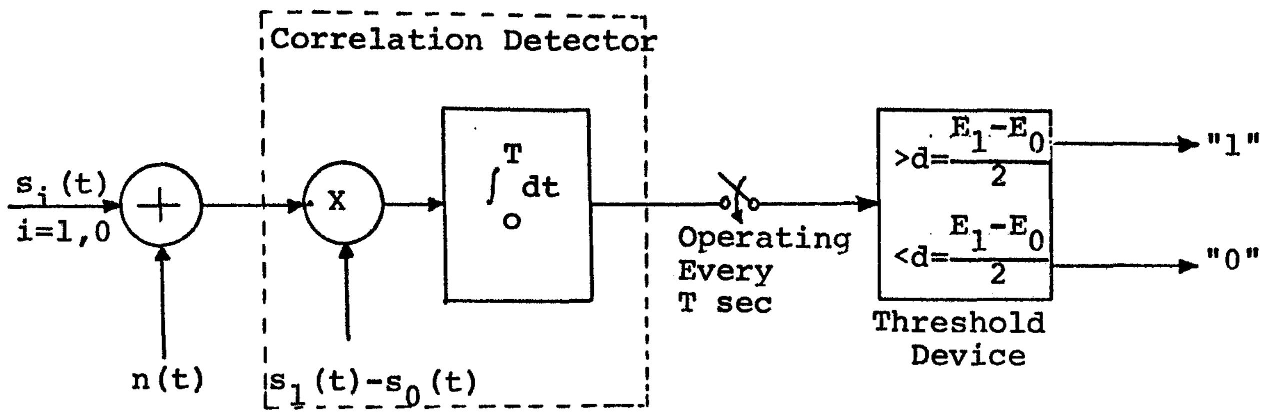

Figure 2.2 Optimum Receiver Structure

The dotted block can also be viewed as a matched filter with an impulse response $h(t) = s_1(T-t) - s_0(T-t)$ [11]. A matched filter of this sort is called an integrate-and-dump circuit [32], [42]. It is well known that the matched filter will give maximum signal-to-noise ratio at the output. Thus for white Gaussian noise, both maximization of the signal-to-noise ratio and minimization of the probability of error lead to the same optimum receiver structure.

2.4 Probability of Error

The optimum procedure for distinguishing between two known signals $s_1(t)$ and $s_0(t)$ has been discussed. Now the probability that an error will be made in such a decision process will be derived.

The output of the integrator at t=T due to both signal and noise is given by

$$y = \int_0^T z(t)(s_1(t) - s_0(t))\,dt$$
$$= w_i + n_1 \tag{2.27}$$

where

$$w_i = \int_0^T s_i(t)(s_1(t) - s_0(t))dt \qquad i = 1,0 \tag{2.28}$$

and

$$n_1 = \int_0^T n(t)(s_1(t) - s_0(t))\,dt \tag{2.29}$$

Clearly, n_1 is still Gaussian distributed with mean zero and variance $\sigma_1{}^2$, where

$$\sigma_1^2 = E(n_1{}^2) = E(\int_0^T n(\tau)(s_1(\tau)-s_0(\tau))\,d\tau \int_0^T n(t)(s_1(t)-s_0(t))dt)$$
$$= \int_0^T \int_0^T E(n(\tau)n(t))(s_1(\tau)-s_0(t))(s_1(t)-s_0(t))d\tau\,dt \tag{2.30}$$

But for white Gaussian noise as assumed here,

$$E(n(\tau)n(t)) = N_0/2\ \delta(\tau-t) \tag{2.31}$$

Thus Equation (2.30) becomes

$$\sigma_1{}^2 = N_0/2 \int_0^T (s_1(t)-s_0(t))^2\,dt \tag{2.32}$$

Now the output of the integrator can be described by two Gaussian distributions with mean values at w_1 and w_0 and variance σ_1^2, where one dsitribution is for a "1" decision and the other for a "0" decision. The probability that an error will be made can be written as

$$P_e = P(y>d|H_0)\ P_0 + P(y<d|H_1)\ P_1 \tag{2.33}$$

With $P_0 = P_1 = 1/2$ we have

$$P_e = \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}\sigma_1} \int_d^{\infty} e^{\frac{-(x-w_0)^2}{2\sigma_1^2}} dx + \frac{1}{\sqrt{2\pi}\ \sigma_1} \int_{-\infty}^{d} e^{\frac{-(x-w_1)^2}{2\sigma_1^2}} dx\right) \tag{2.34}$$

Changing variables and simplifying, we obtain

$$P_e = 1/2\ \left(1/2\ (1-\mathrm{erf}(z_0)) + 1/2\ (1-\mathrm{erf}(z_1))\right) \tag{2.35}$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_o^x e^{-u^2} du$, the error function

and

$$z_0 = \frac{d-w_0}{\sqrt{2}\ \sigma_1} \quad , \quad z_1 = \frac{w_1-d}{\sqrt{2}\ \sigma_1}$$

Using Equation (2.28), Equation (2.30) and the fact that $d = (E_1-E_0)/2$, we have

$$z_0 = z_1 = \frac{E_1+E_0-2\int_o^T s_1(t)\ s_0(t)\ dt}{4N_0} \tag{2.36}$$

Finally, with $E=(E_1 + E_0)/2$, the average energy per bit, the probability of error is

$$P_e = 1/2 \ (1-\mathrm{erf}(\sqrt{\frac{E}{N_0} \propto})) \tag{2.37}$$

where

$$\propto = 1/2 - \frac{\int_0^T s_1(t) \ s_0(t) \ dt}{E_1 + E_0} \tag{2.38}$$

Thus with the correlation detection or matched filtering incorporated in the decision process, it is the signal energy, rather than the signal wave shape, that determines the probability of error. However, the exact knowledge of $s_1(t)$ and $s_0(t)$ are required at the receiver end.

It is very important in this connection to note that a large portion of the theoretical analysis in communications, such as the analysis in this chapter and those following, is based upon the assumption of perfect bit synchronization (perfect knowledge of the time of arrival of the individual symbol waveform). Techniques for achieving and maintaining synchronization are an important part of the communication science [10], [45]. However, it appears to be a practical truism that synchronization per se can be maintained well under the conditions where the channel is already useless as a communication link because of high error rate. Hence, except where particularly specified otherwise, we assume perfect synchronization in the receiving process.

2.5 Examples and Applications

Case 1. Antipodal signals

When $s_1(t) = -s_0(t)$, the signals are called antipodal signals. For this case, α is equal to one and the probability of error is minimum and is given by

$$P_e = 1/2\,(1-\mathrm{erf}(\sqrt{\frac{E}{N_0}})) \tag{2.39}$$

The NRZ (Non-Return-to-Zero), Split-Phase and PSK (Phase-Shift-Keying) signals are some examples of such signal sets.

A. NRZ

$s_1(t) = -s_0(t) = A \qquad 0<t<T$

$E = E_1 = E_0 = A^2T$

$d = 0$

B. Split-Phase

$s_1(t) = -s_0(t) = A \qquad 0\leq t\ T/2$

$s_1(t) = -s_0(t) = -A \qquad T/2\leq t<T$

$E = E_1 = E_0 = A^2T$

$d = 0$

C. PSK

$s_1(t) = -s_0(t) = A\cos(2\pi f_c t) \qquad 0<t<T$

$E = E_0 = E_1 = A^2T/2$

$d = 0$

Case 2. Orthogonal signals

When $\int_0^T s_1(t)\, s_0(t)\, dt = 0$, the signals are called orthogonal signals. For this case $\alpha = 1/2$ and the probability of error is given by

$$P_e = 1/2(1-\mathrm{erf}(\sqrt{\frac{E}{2N_0}})) \tag{2.40}$$

The On-Off binary signals, ASK (Amplitude Shift Keying) and FSK (Frequency Shift Keying) signals are some examples of such signal sets.

A. On-Off binary signals

$s_1(t) = A$, $s_0(t) = 0$, $\quad 0<t<T$

$E_1 = A^2T$, $E_0 = 0$, $E = A^2T/2$

$d = \frac{A^2T}{2}$

B. ASK

$s_1(t) = A\cos(2\pi f_c t)$, $s_0(t) = 0$, $\quad 0<t<T$

$E_1 = A^2T/2$, $E_0 = 0$, $E = A^2T/4$

$d = \frac{A^2T}{4}$

C. FSK

$s_1(t) = A\cos(2\pi f_{c1} t)$, $s_0(t) = A\cos(2\pi f_{c0} t)$ $\quad 0<t<T$

$E = E_1 = E_0 = A^2T/2$

$d = A^2T/2$

It follows from Equations (2.39) and (2.40) that antipodal signals require 3dB less than the orthogonal signal on an average power basis to have the same probability of error.

CHAPTER III

THE PERFORMANCE OF A BANDLIMITED BASEBAND TRANSMISSION SYSTEM IN THE PRESENCE OF GAUSSIAN NOISE

3.1 Introduction

The performance of digital transmission systems in the presence of white Gaussian noise is conveniently expressed by the bit-error probability. In Chapter II, it was seen that the optimum detector which achieves the lowest bit-error probability for a given signal-to-noise ratio (SNR) can be realized by a memoryless correlation detector if the system bandwidth is infinite.

In practice, the restriction of the system bandwidth is inevitable. Transmission filtering, channel bandlimiting or receiver filtering usually cause the restriction of bandwidth. Bandwidth limiting will not only cause degradation of the desired signal (energy loss), but more importantly will introduce intersymbol interference (overlapping in time of successive signals). The performance of the optimum detector discussed in Chapter II then will be degraded. For high signal-to-noise channel, the intersymbol interference becomes the determining factor in the design of the higher speed data transmission system. Intersymbol interference can be minimized by careful shaping of the transmitted signal and equalization of the channel [4], [6], [13], [18], [26], However, it may not be possible to eliminate the intersymbol interference completely, and a measure of the degradation would be extremely useful.

The primary objective of this dissertation is to systematically analyze the intersymbol interference and its effect on the performance of various bandlimited digital transmission systems in terms of the bit-error probability. In this chapter, the explicit expressions for the intersymbol interference as a function of system bandwidth and bit position for a bandlimited NRZ baseband system using a correlation detector (an integrate-and-dump circuit) will be presented. The detector performance in terms of bit-error probabilities caused by the degradation of the signal and intersymbol interference will be determined and calculated separately. The basic approach developed for the analysis of this particular system will be used to analyze various transmission systems considered in the later chapters.

3.2 The Baseband Model

The bandlimited baseband transmission system can be modeled as shown in Figure 3.1. Here $\sum_{n=-\infty}^{\infty} a_n(t)$ is the random NRZ signal with amplitude A or -A, and bit duration equal to T. n(t) is additive white Gaussian noise with zero mean and

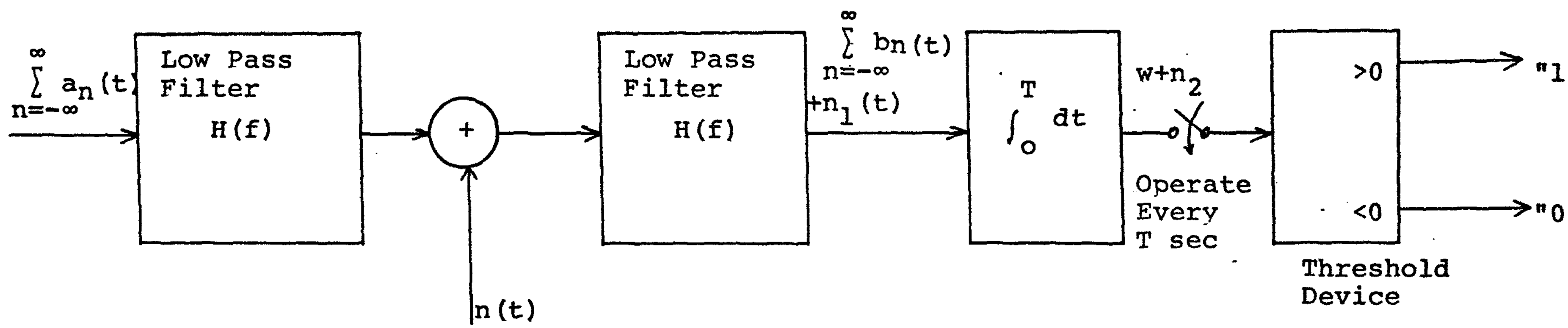

Figure 3.1 A Bandlimited NRZ Baseband Transmission Model

spectral density $N_0/2$. The ideal lowpass filter H(f) has a transfer function equal to one for $-B<f<B$ and zero elsewhere. The receiver consists of an ideal lowpass filter in series with a correlation detector. Since both lowpass filters are the same, the first lowpass filter can be removed as far as the signals are concerned, and the results will be the same.

3.3 Intersymbol Interference for NRZ Signal

The n^{th} bit of information can be represented by

$$a_n(t) = \begin{cases} A_n & NT<t<(N+1)T \\ 0 & \text{elsewhere} \end{cases} \tag{3.1}$$

where $A_n = A$ or $-A$

The response of the lowpass filter due to the n^{th} bit is

$$b_n(t) = \int_{-B}^{B} \left(\int_{-\infty}^{\infty} a_n(x) e^{-j2\pi fx}\, dx\right) e^{j2\pi ft}\, df$$

$$= \int_{-B}^{B} \left(\int_{nT}^{(n+1)T} A_n\, e^{-j2\pi fx}\, dx\right) e^{j2\pi ft}\, df$$

$$= \int_{-B}^{B} A_n T \left(\frac{\sin \pi fT}{\pi fT}\right) e^{-j\pi fT(1+2n)}\, df \tag{3.2}$$

The integrator output $C_n(T)$ sampled at $t=T$ due to n^{th} bit alone can be found to be

$$C_n(T) = \int_c^T b_n(t)dt = \int_0^T \int_{-B}^{B} A_n T\left(\frac{\sin\pi fT}{\pi fT}\right) e^{-j\pi fT(1+2n)} e^{j2\pi ft} df\, dt$$

$$= \int_{-B}^{B} A_n T\left(\frac{\sin\pi fT}{\pi fT}\right) e^{-j\pi fT(1+2n)} T\frac{\sin\pi fT}{\pi fT} e^{j\pi fT} df$$

$$= \int_{-B}^{B} A_n T^2 \frac{\sin^2\pi fT}{(\pi fT)^2} e^{-j2n\pi fT} df \qquad (3.3)$$

Changing variables and simplifying, $C_n(T)$ becomes

$$C_n(T) = A_n T\ J(BT,n) \qquad (3.4)$$

where

$$J(BT,n) = \frac{2}{\pi}\int_0^{\pi BT} \frac{\sin^2 x}{x^2} \cos 2nx\, dx \quad , \qquad (3.5)$$

an even function of n.

Notice that

$$J(BT,0) = \frac{2}{\pi}\int_0^{\pi BT} \frac{\sin^2 x}{x^2} dx \qquad (3.6)$$

can be simplified in terms of elementary functions and the tabulated sine integral function, i.e.,

$$J(BT,0) = \frac{2}{\pi}\left(\int_0^{2\pi BT} \frac{\sin x}{x} dx - \frac{\sin^2\pi BT}{\pi BT}\right) \qquad (3.7)$$

Also $J(BT,n)$ can be evaluated in terms of $J(BT,0)$ as

$$J(BT,n) = \frac{n+1}{2} J[(n+1)BT,0] - nJ(nBT,0) + \frac{n-1}{2} J[(n-1)BT,0] \qquad (3.8)$$

The output of the integrator sampled at T due to an infinite bit train is

$$W = \sum_{n=-\infty}^{\infty} C_n(T)$$

$$= A_0T\ J(BT,0) + \sum_{n=1}^{\infty} (A_n+A_{-n})T\ J(BT,n) \qquad (3.9)$$

The first term is the desired signal and the second term is the intersymbol interference caused by bandlimiting the signal. Notice that $J(BT,n)$ is less than or equal to one for any n and BT. Thus $J(BT,0)$ represents the degradation of the signal and $J(BT,n)$ represents the effect of intersymbol interference on the bit under detection. As $B\to\infty$ $J(BT,0)\to 1, J(BT,n)\to 0$, $W\to A_0T$ as expected.

The influence of the adjacent bits can now be easily calculated. Table 3.1 shows some values of $J(BT,n)$ for various bandwidths and bit positions.

The output of the integrator samples at t=T due to both signal and noise can be given by

$$y = W + n_2$$

$$= A_0T\ J(BT,0) + \sum_{n=1}^{\infty} (A_n+A_{-n})TJ(BT,n)+n_2 \qquad (3.10)$$

where $n_2 = \int_0^T n_1(t)dt$ and $n_1(t)$ is the output of the lowpass

Table 3.1

Some Values of J(BT,n)

BT	J(BT,0)	J(BT,1)	J(BT,2)	J(BT,3)	J(BT,4)	J(BT,5)
0.5	0.7737	0.1291	-0.0222	0.0094	-0.0052	0.0033
0.6	0.8393	0.0673	0.0292	-0.0271	0.0152	-0.0028
0.7	0.8776	0.0441	0.0204	0.0030	-0.0107	0.0020
0.8	0.8960	0.0433	0.0033	0.0054	0.0031	-0.0012
0.9	0.9021	0.0464	0.0007	0.0001	0.0003	0.0005
1.0	0.9028	0.0471	0.0011	0.0002	0.0001	0.0000
1.2	0.9066	0.0493	0.0002	-0.0024	-0.0017	0.0003
1.5	0.9311	0.0353	-0.0113	0.0004	-0.0002	0.0001
2.5	0.9592	0.0206	-0.0003	0.0001	-0.0000	0.0000

filter due to the noise n(t) alone.

The probability that an error will be made can be given by

$$P_e = P(A_0=A)P(y<0|A_0=A)+p(A_0=-A)P(y>0|A_0=-A) \quad (3.11)$$

The evaluation of P_e represents a long-standing challenge in digital communication problems. The main source of difficulty is the fact that, with the exception of a few special cases, the probability distribution of the intersymbol interference is typically highly complex and irregular. Using the convolution method [27], [38] to obtain the probability density function of the intersymbol interference and noise is very difficult. Approximation of this distribution by a simpler function may lead to gross misinterpretation.

For all practical bandlimited transmission systems, one can assume that intersymbol interference is limited to a finite number of symbols preceding and following the symbol under detection. The conditional error probabilities are computed for each of the truncated pulse sequences and then averaged with respect to the probability of occurrence of these sequences [1], [17], [36], [37], [39], [40].

Using the basic property of the characteristic function as suggested in [5], a new method, called the series expansion method, is developed to obtain the explicit expression for the bit-error probability P_e. P_e is divided into two terms, one corresponds to detecting the signal itself (in the absence of intersymbol interference) and another corresponds to the influence of the intersymbol interference.

In the following two sections, the averaging method and series expansion method will be examined and compared.

3.4 Bit-Error Probability--Averaging Method

Recall that the output of the integrator samples at $t=T$ is given by

$$\begin{aligned} y &= W + n_2 \\ &= A_0 T\ J(BT,0) + \sum_{n=1}^{\infty} (A_n + A_{-n}) T\ J(BT,n) + n_2 \end{aligned}$$

Thus the output of the integrator can be described by a Gaussian distributed function with mean at W and variance ${\sigma_2}^2$. ${\sigma_2}^2$ can be obtained as

$$\begin{aligned} \sigma_2^2 &= E[{n_2}^2] \\ &= E[\int_0^T n_1(\tau)d\tau \int_0^T n_1(t)dt] \\ &= \int_0^T \int_0^T E(n_1(\tau)n(t))d\tau\ dt \end{aligned} \tag{3.12}$$

Notice that $E[n_1(\tau)\ n_1(t)]$ is the covariance of $n_1(t)$ and is given by [25]

$$E[n_1(\tau)\ n_1(t)] = N_0 B \frac{\sin[2\pi B(\tau-t)]}{2\pi B(\tau-t)} \tag{3.13}$$

The expression for $\sigma_2{}^2$ can then be simplified as

$$\sigma_2{}^2 = \frac{N_0T}{2} \cdot \frac{2}{\pi} \int_0^{\pi BT} \frac{\sin^2 x}{x^2}\, dx = \frac{N_0T}{2} J(BT,0) \tag{3.14}$$

Let the effects of the intersymbol interference on the bit under detection be confined to N preceding and N subsequent bits. There is a total of $2 \cdot 2^{2N}$ different adjacent bit patterns around the bit under detection, which can be numbered in such a way that the first 2^{2N} patterns around a "1" ($A_0 = A$) and the second 2^{2N} patterns around a "0" ($A_0=-A$). Denote P_{ei} as the probability that the center bit is detected in error given that the i^{th} pattern is transmitted. Since each pattern will occur with the same probability the average bit-error probability P_e can be given by

$$P_e = \frac{1}{2^{2N+1}} \sum_{i=1}^{2N+1} P_{ei} \tag{3.15}$$

Since the noise n_2 is Gaussian, Equation (3.15) can be rewritten in a more explicit form

$$P_e = \frac{1}{2 \cdot 2^{2N}} \sum_{i=1}^{2^{2N}} \frac{1}{\sqrt{2\pi}\sigma_2} \int_{-\infty}^{o} e^{\frac{-(x-W_i)^2}{2\sigma_2^2}}\, dx$$

$$+ \frac{1}{2 \cdot 2^{2N}} \sum_{i=2^{2N}+1}^{2^{2N+1}} \frac{1}{\sqrt{2\pi}\sigma_2} \int_{o}^{\infty} e^{\frac{-(x-W_i)^2}{2\sigma_2{}^2}}\, dx \tag{3.16}$$

where W_i is the value of W for the i^{th} pattern and is equal to $ATJ(BT,0) + \sum_{n=1}^{\infty} (A_n+A_{-n})TJ(BT,n)$ for $i \leq 2^{2N}$ and $-ATJ(BT,0)+ \sum_{n=1}^{\infty} (A_n+A_{-n})TJ(BT,n)$ for $i \geq 2^{2N} + 1$. Since the probability that a "1" in the middle of a particular pattern is erroneously detected is the same as that for a "0" in the middle of the complement of this pattern, the second term in Equation (3.16) is equal to the first term. Thus we have

$$P_e = \frac{1}{2^{2N}} \sum_{i=1}^{2^{2N}} \frac{1}{\sqrt{2\pi}\sigma_2} \int_{-\infty}^{0} e^{\frac{-(x-W_i)^2}{2\sigma_2^2}} dx \tag{3.17}$$

Therefore it is sufficient to compute the bit-error probability by only examining the patterns around the "1" bit.

For the system considered here, it can be seen that $|J(BT,n)| << J(BT,0)$ when $n>5$ (see Table 3.1). Thus we can confine the effects of the intersymbol interference to the five preceding and five following bits on the bit under detection. From Equation (3.17), we have

$$P_{ei} = \frac{1}{\sqrt{2\pi}\sigma_2} \int_{-\infty}^{0} e^{\frac{-(x-W_i)}{2\sigma_2^2}} dx$$

where

$$W_i = AT[J(BT,0) + \sum_{n=1}^{5} \frac{(A_n+A_{-n})}{A} J(BT,n)]$$

Let $u=\frac{x-W_i}{\sqrt{2}\sigma_2}$, P_{ei} becomes

$$P_{ei} = \frac{1}{\sqrt{\pi}} \int_{z_i}^{\infty} e^{-u^2} du \tag{3.18}$$

$$= \frac{1}{2} [1-\mathrm{erf}(z_i)]$$

where

$$z_i = \frac{W_i}{\sqrt{2}\sigma_2} = \frac{AT[J(BT,0)+\sum_{n=1}^{5} \frac{A_n+A_{-n}}{A} J(BT,n)]}{\sqrt{2} \sqrt{\frac{N_0T}{2} J(BT,0)}}$$

$$= \sqrt{\frac{E}{N_0} D_i^2(BT)}, \tag{3.19}$$

$E = A^2T$, the energy per bit

and

$$D_i^2(BT) = \frac{[J(BT,0)+\sum_{n=1}^{5} (\frac{A_n+A_{-n}}{A}) J(BT,n)]^2}{J(BT,0)} \tag{3.20}$$

Thus $D_i^2(BT)$, a function of the bandwidth-bit duration product and bit patterns can be considered as the degradation of signal-to-noise ratio. This quantity is easily calculated. The probability of bit error for a particular pattern can also be viewed as the shaded area under the curve $\frac{e^{-u^2}}{\sqrt{\pi}}$ from Z_i to ∞ as shown in Figure 3.2. Z_i is bounded by Z_W, the pattern giving minimum value of $D_i^2(BT)$, and Z_B, the pattern giving maximum value of $D_i^2(BT)$. Z_0 corresponds to the pattern such that the net effect of intersymbol interference is zero, i.e.

$$D_i^2(BT) = J(BT,0)$$

It can be shown that

$$Z_0 = \frac{Z_W+Z_B}{2}$$

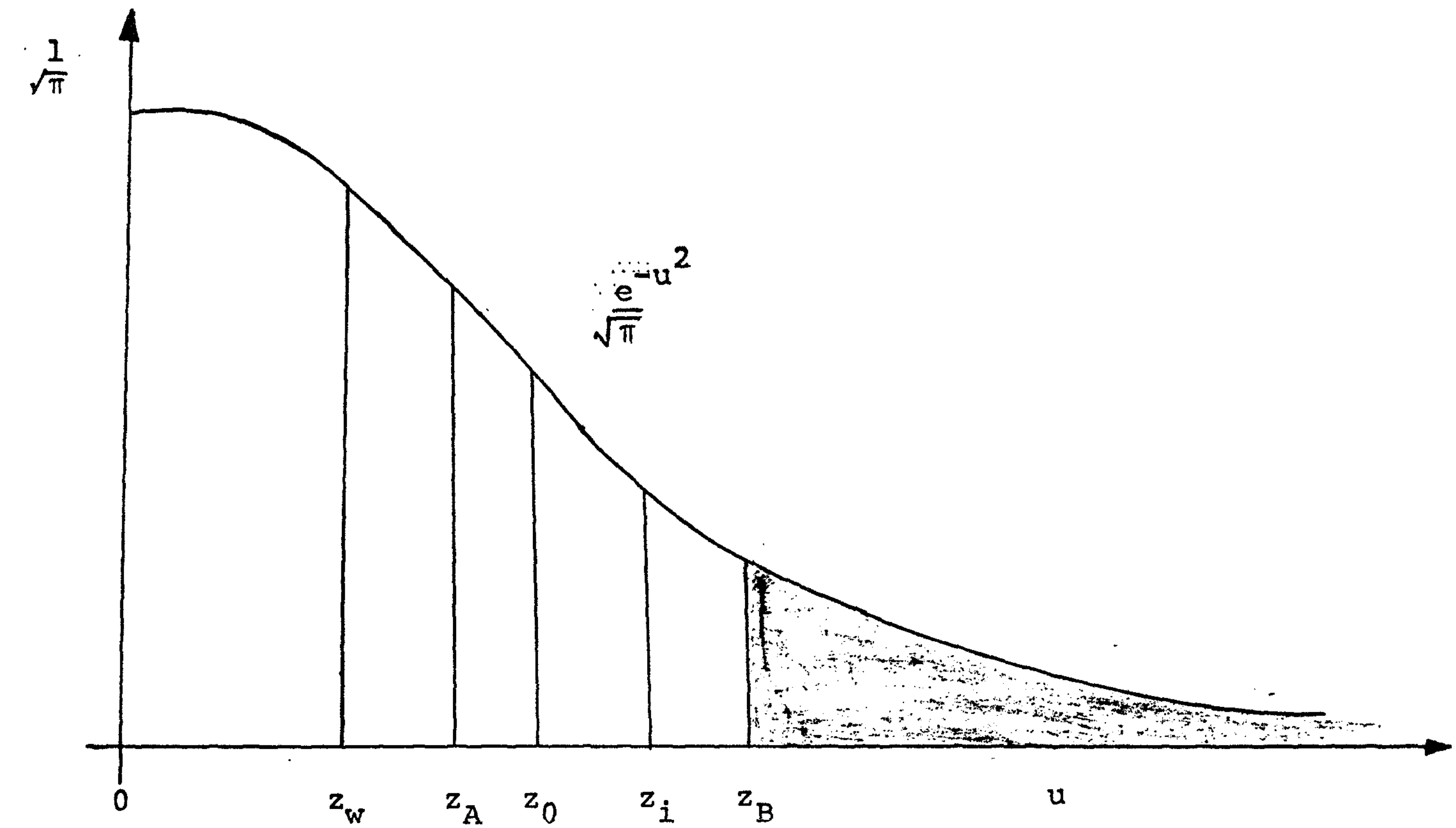

Figure 3.2 Probability of Bit-Error for the NRZ Baseband System

and all the values of Z_i 's are symmetric about Z_0.

There are 1024 different patterns. Thus the bit-error probability P_e of a random NRZ signal for a particular bandwidth is given by

$$P_e = \frac{1}{1024} \sum_{i=1}^{1024} P_{ei}$$

$$= \frac{1}{1024} \sum_{i=1}^{1024} 1/2[1-\text{erf}(Z_i)] \qquad (3.21)$$

or

$$P_e = 1/2[1-\text{erf}(Z_A)] \qquad (3.22)$$

where Z_A is bounded between Z_w and Z_0 (see Figure 3.2) and P_e is the area under the curve of $\frac{e^{-u^2}}{\sqrt{\pi}}$ from Z_A to ∞. But Z_A cannot be determined analytically, because intersymbol interference is not Gaussian distributed. We can only find P_e by Equation (3.21) and then Z_A can be found numerically. Z_A will be different for different P_e. As $B \to \infty$, $Z_i \to \sqrt{\frac{E}{N_0}}$, $P_e \to 1/2[1-\text{erf}(\sqrt{\frac{E}{N_0}})]$, the optimum case presented in Chapter II.

The probability of bit-error, P_e, for a random NRZ signal as a function of signal-to-noise ratio $(\frac{E}{N_0})$ and bandwidth-bit duration product BT with the intersymbol interference confined to the nearest 10 bits is shown in Figure 3.3.

The upper bound of P_e, P_{emax} can be obtained by finding the worst pattern which gives minimum value of $D_i^2(BT)$. This

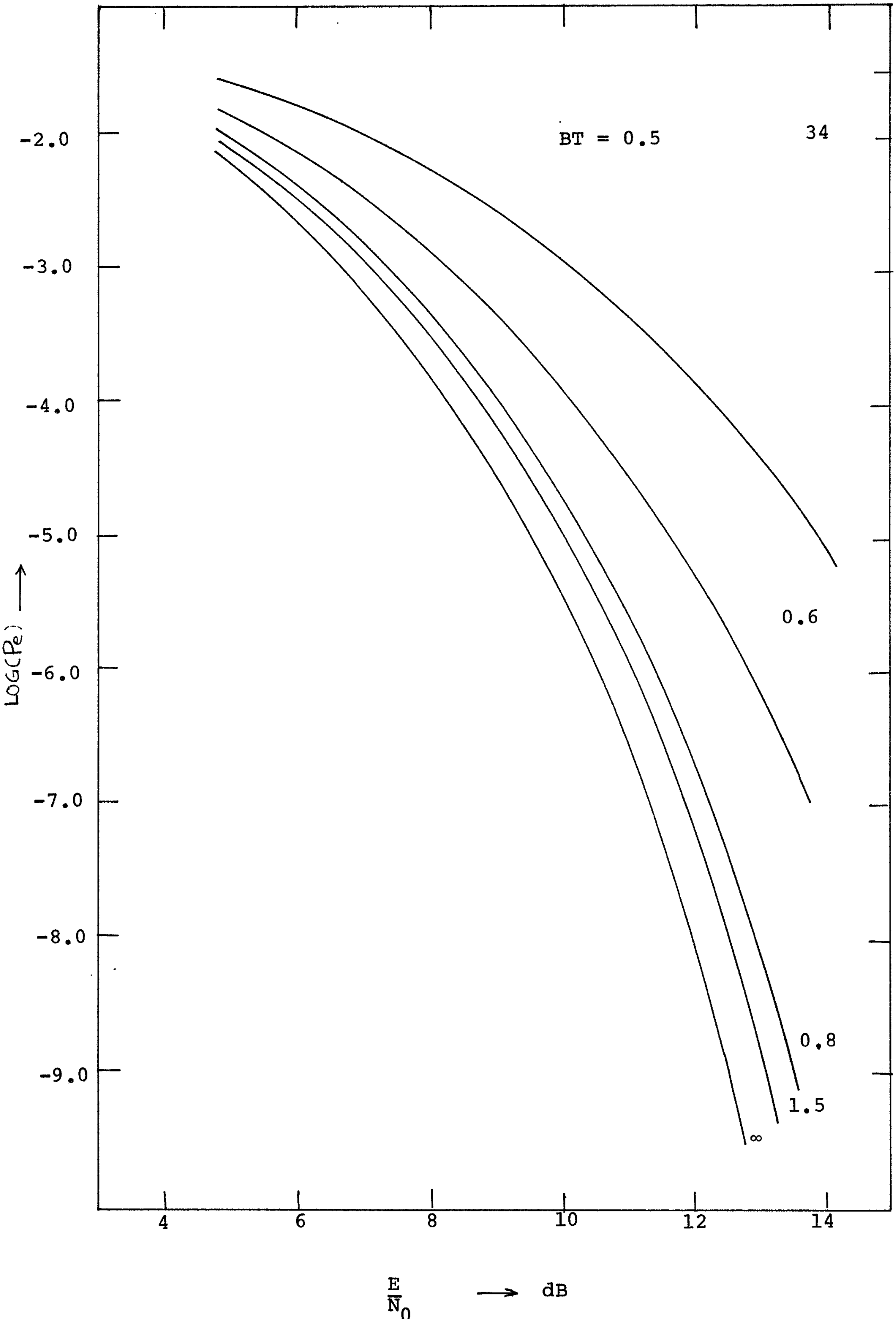

Figure 3.3 Probability of Bit-Error vs $\frac{E}{N_0}$ for the NRZ Baseband System

can be accomplished by choosing $A_n = A_{-n} = -A_0$ if $J(BT,n)>0$, and $A_n = A_{-n} = A_0$ if $J(BT,n)<0$. (See Table 3.1). The upper bound thus can be expressed as

$$P_{emax} = 1/2[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}\frac{[J(BT,0)+\sum_{n=1}^{\infty} 2|J(BT,n)|]^2}{J(BT,0)}})] \quad (3.23)$$

The lower bound of P_e, P_{emin} can be obtained by choosing $A_n = -A_n$ for all n, which gives $D_i^2(BT) = J(BT,0)$. The lower bound thus can be expressed as

$$P_{emin} = 1/2[1-\mathrm{erf}(\sqrt{\frac{E}{N_0} J(BT,0)})] \quad (3.24)$$

Table 3.2 lists P_{emin}, P_{emax} and the corresponding values of Z_0, Z_w, and P_e as a function of signal-to-noise ratio E/N_0 for BT equal to one.

Martinides and Reijns [17] studied the same system using the averaging method. The explicit expression for the intersymbol interference was not determined. The problem was analyzed by using a 40 bit periodic sequence instead of random sequences. Also they only considered the effects of four nearest bits (N=2), which will introduce considerable truncation error by ignoring the influence of intersymbol interference beyond N = 2 especially when $BT \leq 1$. That is why the results presented in this section are significantly different from Martinides' for $BT \leq 1$.

Table 3.2

Values of P_{emin}, P_{emax}, P_e vs $\frac{E}{N_o}$ for NRZ Baseband system with BT=1.0

$\frac{E}{N_o}$ (in dB)	$\text{Log}(P_{emin})$	Z_o	$\text{Log}(P_e)$	$\text{Log}(P_{emax})$	Z_w
0.0	-1.048	0.950	-1.043	-0.938	0.848
4.8	-2.001	1.646	-1.963	-1.723	1.468
7.0	-2.876	2.125	-2.777	-2.435	1.896
8.5	-3.724	2.514	-3.542	-3.121	2.243
9.5	-4.556	2.850	-4.273	-3.793	2.543
10.5	-5.380	3.151	-4.981	-4.456	2.812
11.1	-6.197	3.426	-5.672	-5.113	3.057
11.8	-7.013	3.680	-6.351	-5.767	3.284
12.3	-7.822	3.917	-7.020	-6.415	3.496
12.8	-8.629	4.142	-7.683	-7.064	3.700
13.2	-9.433	4.354	-8.340	-7.709	3.885
13.6	-10.236	4.557	-8.993	-8.352	4.066
14.0	-11.037	4.751	-9.642	-8.993	4.239

Although the averaging method gives the approximation of average bit-error probability, the main disadvantage is that the computational effort becomes prohibitive as N becomes large.

In the past some efforts have been made to obtain the upper bounds on the average bit-error probability. Hartman [7] analyzed the bandlimited PSK system by finding the worst case probability of error as Equation (3.23) indicated. To use this method to predict the error probability is, in some cases, exceedingly pessimistic and may lead to gross over design of the system. On the other hand, ignoring the intersymbol interference to predict the error probability such as Equation (3.24) indicated is sometimes too optimistic especially for high signal-to-noise channel (see Table 3.2).

Some improved bounds have been proposed recently. Saltzburg [28] separated the intersymbol interference terms into two sets, one set containing terms which are treated as a degradation of the signal and the other set containing terms which increase the effective noise power. The chief attribute of this approach is mathematical utility. However, as a theoretical tool it suffers from one drawback, the determination of the optimum set is an arduous task.

Lugannani [16] obtained an upper bound by using the Chernoff inequality. The expression for this bound is rather complicated in appearence compared with that of Saltzburg's. But it is relatively easy to evaluate. The chief difficutly is that evaluating the parameters of this bound is a problem equal in magnitude to the problem of evaluating a large set of sequences by using the averaging method, and the method does not yield an analytic solution.

3.5 Bit-Error Probability--Series Expansion Method

Recall that the output of the integrator sampled at t=T due to both signal and noise is given by Equation (3.10)

$$y = A_0 T \; J(BT,0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_n T \; J(BT,n) + n_2$$

Divide both sides by AT, we obtain

$$X = Z_0 J_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n Z_n + N \qquad (3.25)$$

where

$$X = \frac{y}{AT}, \quad Z_n = \frac{A_n T}{AT} = \pm 1, \quad J_n = J(BT,n), \text{ and } N = \frac{n_2}{AT}$$

The variance $\sigma_N{}^2$ of N is $E[(\frac{n_2}{AT})^2]$, which can be evaluated as

$$\sigma_N{}^2 = \frac{1}{A^2T^2} \; E(n_2{}^2) = \frac{\sigma_2{}^2}{A^2T^2} \qquad (3.26)$$

The probability that an error will be made is given by Equation (3.11)

$$P_e = P(A_0=A) \; P(y<0\,|\,A_0=A) + P(A_0=-A) \; P(y>0\,|\,A_0=-A)$$

or equivalently can be given by

$$P_e = P(Z_0=1)P\ (X<0|Z_0=1)+P(Z_0=-1)P(X>0|Z_0=-1) \quad (3.27)$$

Let

$$S = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Z_nJ_n + N \quad (3.28)$$

and

$$X_n = Z_nJ_n \quad (3.29)$$

Since $P(Z_0=1)=P(A=A_0)=1/2$ and $P(Z_0=-1)=P(A=-A_0)=1/2$, from Equations (3.25) and (3.27) we have

$$\begin{aligned} P_e &= 1/2\ P(-J_0>S)\ +1/2\ P(S>J_0) \\ &= 1/2\ [1-P(-J_0\leq S\leq J_0)] \\ &= 1/2\ (1-Q_e) \end{aligned} \quad (3.30)$$

and $$Q_e = P(-J_0\leq S\leq J_0) \quad (3.31)$$

X_n is a random variable assuming values J_n and $-J_n$ with equal probability. Therefore the characteristic function $\Phi_{X_n}(w)$ of X_n is

$$\begin{aligned} \Phi_{X_n}(w) &= \int_{-\infty}^{\infty} [1/2\delta(X_n-J_n)+1/2\delta(X_n+J_n)]e^{jwX_n}\,dX_n \\ &= \cos(J_nw) \end{aligned} \quad (3.32)$$

The characteristic function of N can be obtained as [19]

$$\Phi_N(w) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{x^2}{2\sigma_N^2}} e^{jwx}\, dx \qquad (3.33)$$

$$= e^{-\frac{w}{2}\sigma_N^2}$$

Since X'_ns and N are all independent random variables, the characteristic function of S can be expressed as the product of the characteristic functions of X_n'S and N

$$\Phi_s(w) = \Phi_N(w) \cdot \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Phi_{X_n}(w)$$

$$= \Phi_N(w) \cdot \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \cos(J_n w) \qquad (3.34)$$

It is well known that the probability for a random variable r distributed between a and b can be evaluated in terms of its characteristic function [20]

$$P(a\leq r\leq b) = \int_{-\infty}^{\infty} \frac{e^{-jaw} - e^{-jbw}}{j2\pi w} \Phi_r(w)\, dw \qquad (3.35)$$

Thus Equation (3.31) can be evaluated as

$$Q_e = P(-J_0\leq S\leq J_0)$$

$$= \int_{-\infty}^{\infty} \frac{e^{jJ_0 w} - e^{-jJ_0 w}}{j2\pi w} \Phi_s(w)\, dw \qquad (3.36)$$

Since $\cos(J_n w)$ can be expanded into a power series of w, $\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \cos J_n w$ can also be put into a power series of w

$$\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \cos J_n w = 1 + \sum_{n=1}^{\infty} b_{2n} w^{2n} \tag{3.37}$$

The expression for b_{2n} will be derived in Appendix A. Thus from Equation (3.33), Equation (3.34), and Equation (3.37), we have

$$\Phi_s(w) = (1+\sum_{n=1}^{\infty} b_{2n} w^{2n})\, e^{\frac{-w^2}{2}\sigma_N^2} \tag{3.38}$$

Substituting Equation (3.38) into Equation (3.36), we obtain

$$Q_e = Q_{e1} + Q_{e2} \tag{3.39}$$

where

$$Q_{e1} = \int_{-\infty}^{\infty} \frac{e^{jJ_0 w} - e^{-jJ_0 w}}{j2\pi w}\, e^{\frac{-w^2}{2}\sigma_N^2}\, dw \tag{3.40}$$

and

$$Q_{e2} = \int_{-\infty}^{\infty} \frac{e^{jJ_0 w} - e^{-jJ_0 w}}{j2\pi w} \left(\sum_{n=1}^{\infty} b_{2n} w^{2n}\right) e^{\frac{-w^2}{2}\sigma_N^2}\, dw \tag{3.41}$$

It is readily recognized that Q_{e1} is the probability that the Gaussian random variable N lies between $-J_0$ and J_0. Thus we can evaluate Q_{e1} as

$$Q_{e1} = \int_{-J_0}^{J_0} \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{\frac{-x^2}{2\sigma_N^2}} dx \qquad (3.42)$$

with $u = \frac{x}{\sqrt{2}\sigma_N}$, we obtain

$$Q_{e1} = \frac{2}{\sqrt{2}\pi\sigma_N} \int_o^{\frac{J_0}{\sqrt{2}\sigma_N}} e^{-u^2} du = \mathrm{erf}\left(\frac{J_0}{\sqrt{2}\sigma_N}\right) \qquad (3.43)$$

Taking the summation sign out of the integral sign, Q_{e2} becomes

$$Q_{e2} = \sum_{n=1}^{\infty} 2b_{2n}(-1)^n\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-w^2}{2}\sigma_N^2}(-jw)^{2n-1}e^{-jJ_0w}\,dw\right]$$

$$= \sum_{n=1}^{\infty} 2b_{2n}(-1)^n\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}(-jw)^{2n-1}\,\Phi_N(w)\,e^{-jJ_0w}\,dw\right] \qquad (3.44)$$

The term inside the bracket is [21]

$$\frac{d^{2n-1}}{d\,J_0^{2n-1}}\left(\Phi_N(w)\right)$$

Thus Q_{e2} can be evaluated as

$$Q_{e2} = \sum_{n=1}^{\infty} 2b_{2n}(-1)^n \frac{d^{2n-1}}{d\,J_0^{2n-1}}\left(\frac{1}{\sqrt{2\pi}\sigma_N} e^{\frac{-J_0^2}{2\sigma_N^2}}\right)$$

$$(3.45)$$

Let

$$G_n = \frac{d^n}{dJ_0^{\,n}} \left(\frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{J_0^2}{2\sigma_N^2}} \right) , \tag{3.46}$$

a recurrence formula to evaluate G_n can be found

$$G_n = -\frac{J_0}{\sigma_N^2} G_{n-1} - \frac{n-1}{\sigma_N^2} G_{n-2} \tag{3.47}$$

Now Q_{e2} can be written as

$$Q_{e2} = \sum_{n=1}^{\infty} 2b_{2n}(-1)^n G_{2n-1} \tag{3.48}$$

Combining Equations (3.39), (3.43), (3.48), and (3.30), the probability of bit-error then can be given by

$$P_e = 1/2 \; (1-Q_e) = P_{e1} + P_{e2} \tag{3.49}$$

where

$$P_{e1} = 1/2 \; [1-\mathrm{erf}(\frac{J_0}{\sqrt{2}\sigma_N})] \tag{3.50}$$

and

$$P_{e2} = \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \tag{3.51}$$

Since $\sigma_N^{\,2} = \dfrac{\sigma_2^{\,2}}{A^2T^2}$ and $\sigma_2^{\,2} = \dfrac{N_0T}{2} J_0$ (see Equation 3.14) we have

$$\sigma_N^2 = \frac{J_0}{2} \cdot \frac{N_0}{A^2T}$$

$$= \frac{J_0}{2} \cdot \frac{1}{(\frac{E}{N_0})} \tag{3.52}$$

where $E=A^2T$ is the energy per bit for the infinite bandwidth. Thus

$$P_{e1} = 1/2\ [1-\mathrm{erf}(\sqrt{\frac{E}{N_0}\ J_0})] \tag{3.53}$$

Now we can recognize that P_{e1} is the probability of bit-error for the detection of a single bandlimited NRZ bit. Indeed, if we only consider the bit under detection itself, from Equations (3.9) and (3.10) we have

$$W = A_0TJ_0 \tag{3.54}$$

and

$$Y = A_0TJ_0 + n_2 \tag{3.55}$$

Since A_0 is equal to A or -A with the equal probability and n_2 is Gaussian noise with zero mean and variance, the output of the integrator due to both signal and noise can now be described by two Gaussian distributions with mean values at $\pm ATJ_0$, and variance σ_2^2 , where one distribution is for a logical "zero" decision, the probability that a "zero" or a "one" is erroneously detected can be given by

$$P_{es} = 1/2 \frac{1}{\sqrt{2\pi}\sigma_2} \int_{-\infty}^{0} e^{\frac{-(x-ATJ_0)^2}{2\sigma_2^2}} dx$$

$$+ 1/2 \frac{1}{\sqrt{2\pi}\sigma_2} \int_{0}^{\infty} e^{\frac{-(x+ATJ_0)^2}{2\sigma_2^2}} dx \qquad (3.56)$$

Changing variable and simplifying, P_{es} becomes

$$P_{es} = 1/2[1-\mathrm{erf}(\frac{ATJ_0}{\sqrt{2}\sigma_2})] \qquad (3.57)$$

but

$$\sigma_2 = \frac{N_0 T}{2} J_0$$

thus immediately we can see that

$$P_{es} = 1/2[1-\mathrm{erf}(\sqrt{\frac{E}{N_0} J_0})] \quad ,$$

which is identically equal to P_{e1}. Therefore the degradation of the signal itself caused by the restriction of bandwidth in terms of probability of bit-error can now be described by Equation (3.53).

Obviously the effect of the intersymbol interference on the bit under detection in terms of the probability of bit-error can now be illustrated by Equation(3.51) . Notice that

G_{2n-1} in Equation (3.51) can also be expressed as a function of signal-to-noise ratio E/N_0.

$$G_{2n-1} = -2\left(\frac{E}{N_0}\right)\left(G_{2n-2} + \frac{2n-2}{J_0} G_{2n-3}\right) \tag{3.58}$$

b_{2n} in Equation (3.53) can be expressed as the function of intersymbol interference terms, J_n's and can be evaluated in a recurrence form (see Appendix A)

$$b_{2n} = -\frac{1}{2n}\left(d_{2n-1} + \sum_{\ell=1}^{n-1} b_{2n-2\ell}\, d_{2\ell-1}\right) \tag{3.59}$$

where

$$d_{2\ell-1} = \frac{2^{2\ell}(2^{2\ell}-1)}{2\ell!} B_{2\ell} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{2\ell} \tag{3.60}$$

and $B_{2\ell}$ is the Bernoulli number. Generally speaking, $J_n^2 << J_1^2$ for $n>5$ (see Table 3.1), thus the coefficient b_{2n} can be calculated with negligible error by using only the terms from J_{-5} to J_5. In other words, the influence of intersymbol interference can be confined to the five preceding and five subsequent bits on the bit under detection without significant error. The resulting probability of bit-error P_e can be rewritten as

$$P_e = 1/2\left[1-\mathrm{erf}\left(\sqrt{\frac{E}{N_0}}\, J_0\right)\right] + \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \tag{3.61}$$

which can be evaluated for a given value of signal-to-noise ratio E/N_0 and bandwidth-bit duration BT if the series can be truncated with negligible error. It will be shown in Appendix B that the series can be truncated with negligible error provided that

$$\beta = \frac{\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^2}{\sigma_N^2} \leq 0.5 \qquad (3.62)$$

For the system considered here, the series converges rapidly. P_e can be evaluated accurately using only 10 terms in the series.

By confining the intersymbol interference to five bits and using ten terms in the series, the resulting P_e's exactly agree with those obtained using the averaging method of Section 3.4. Two completely different approaches yield the same answers! The computer time, however, is much less using the series expansion method. By extending intersymbol interference to more than 10 bits, and using more than 10 terms in the series of Equation (3.61) the resulting P_e does not change significantly. This verifies our previously assumptions.

Table 3.3 shows the values of P_e, P_{e1} and P_{e2} for various bandwidths and signal-to-noise ratio with the intersymbol interference confined to the 10 bits and the series truncated to

Table 3.3

Values of P_e, P_{e1} and P_{e2} vs $\frac{E}{N_0}$ for the NRZ Baseband System with BT = 0.5, 0.8 and 1.5

BT = 0.5

$\frac{E}{N_0}$ (dB)	$\mathrm{Log}(P_e)$	$\mathrm{Log}(P_{e1})$	$\mathrm{Log}(P_{e2})$
0	-0.933	-0.972	-1.999
4.77	-1.558	-1.807	-1.918
6.99	-2.012	-2.568	-2.153
8.45	-2.392	-3.302	-2.449
9.54	-2.735	-4.022	-2.758
10.00	-2.898	-4.378	-2.913
10.41	-3.057	-4.733	-3.066
11.14	-3.366	-5.438	-3.370
11.76	-3.667	-6.139	-3.668
12.30	-3.961	-6.836	-3.961
12.78	-4.248	-7.532	-4.248
13.22	-4.528	-8.225	-4.528
13.62	-4.804	-8.915	-4.805
13.98	-5.085	-9.604	-5.085

Table 3.3 (continued)

BT = 0.8

$\frac{E}{N_0}$ (dB)	$\mathrm{Log}(P_e)$	$\mathrm{Log}(P_{e1})$	$\mathrm{Log}(P_{e2})$
0	-1.040	-1.044	-3.026
4.77	-1.957	-1.991	-3.083
6.99	-2.773	-2.860	-3.514
8.45	-3.541	-3.702	-4.050
9.54	-4.277	-4.528	-4.633
10.00	-4.635	-4.938	-4.935
10.41	-4.989	-5.346	-5.241
11.14	-5.685	-6.158	-5.863
11.76	-6.367	-6.965	-6.493
12.30	-7.039	-7.771	-7.128
12.788	-7.702	-8.571	-7.766
13.22	-8.360	-9.370	-8.404
13.62	-9.012	-10.167	-9.044
13.98	-9.661	-10.963	-9.683

Table 3.3 (continued)

BT = 1.5

$\frac{E}{N_0}$ (dB)	Log (P_e)	Log (P_{e1})	Log (P_{e2})
0	-1.062	-1.065	-3.240
4.77	-2.021	-2.043	-3.330
6.99	-2.886	-2.943	-3.797
8.45	-3.709	-3.816	-4.372
9.54	-4.504	-4.673	-4.996
10.00	-4.894	-5.098	-5.320
10.41	-5.280	-5.522	-5.649
11.14	-6.039	-6.364	-6.319
11.76	-6.788	-7.003	-6.999
12.30	-7.526	-8.038	-7.686
12.788	-8.257	-8.869	-8.379
13.72	-8.981	-9.698	-9.074
13.62	-9.701	-10.576	-9.771
13.98	-10.417	-11.352	-10.121

10 terms. From Table 3.3, it can be seen that P_e is very close to P_{e1} when E/N_0 is low and almost dominated by P_{e2} when E/N_0 is high. This is expected, because the system is essentially noise limited for low signal-to-noise ratio and intersymbol interference limited for high signal-to-noise ratio [15].

For the infinite bandwidth case, J_0 is equal to one and b_{2n} is equal to zero. Then the probability of error is given by

$$P_e = 1/2[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}})] \tag{3.63}$$

The additional signal power needed to give the performance as an optimum detector described by Equation (3.63) for the detection of NRZ signals in the presence of white Gaussian noise and in a bandlimited channel using the correlation detector is tabluated in Table 3.4. Here S is the additional power in dB needed for the single pulse case in the absence of intersymbol interference. This can be given by 10 log (J_0) (comparing Equation (3.63) and Equation (3.53)). This table can be used as a design guide for the tradeoff between signal-to-noise ratio and system bandwidth.

For the system where Equation (3.62) cannot be maintained, we can make β sufficiently smaller by starting the summation from, for example m instead of 1. Once we choose m>1, the

Table 3.4

Additional Power needed for the Detection of NRZ Signals to give the same Performance as an Optimum Detector

	BT=	BT=0.5		0.6		0.8		1.0		2.5	
P_e	$(\frac{E}{N_0})$dB	S	A	S	A	S	A	S	A	S	A
10^{-2}	4.3	1.1	2.7	0.8	1.3	0.5	0.7	0.4	0.7	0.2	0.2
10^{-3}	6.8	1.1	3.6	0.8	1.6	0.5	0.7	0.4	0.7	0.2	0.2
10^{-4}	8.4	1.1	4.0	0.8	1.8	0.5	0.7	0.4	0.7	0.2	0.3
10^{-5}	9.6	1.1	4.4	0.8	2.1	0.5	0.8	0.4	0.8	0.2	0.3
10^{-6}	10.5	1.1	4.8	0.8	2.4	0.5	0.9	0.4	0.9	0.2	0.3
10^{-7}	11.3	1.1	5.1	0.8	2.6	0.5	1.0	0.4	1.0	0.2	0.3

S : Additional power needed for the single pulse case in dB in the abscent of intersymbol interference

A : Additional power needed for the average case in dB. Second column is the signal-to-noise ratio required for the unlimited bandwidth (optimum case) expression for the probability of bit-error, Equation (3.49), will also be changed. This change is trivial and the new expression can be immediately written as

$$P_e = \frac{1}{2^{2(m-1)}} \sum_{i=1}^{2^{2(m-1)}} 1/2[1-\mathrm{erf}(\frac{J_0(i)}{\sqrt{2}\sigma_N})]$$

$$+ \sum_{n=1}^{\infty} (-1)^{n+1}\, b_{2n}'\, G_{2n-1}(i)) \qquad (3.64)$$

where $J_0(i)$ is one of the combinations of $\pm J_{-(m-1)} \pm \cdots \pm J_{-1}$ $+J_0 \pm J_1 \cdots \pm J_{m-1}$, and

$$b_{2n}' = -\frac{1}{2n}\,(d_{2n-1}' + \sum_{\ell=1}^{n-1} b'_{2n-2\ell}\, d_{2\ell-1}') \qquad (3.65)$$

$$d_{2\ell-1}' = \frac{2^{2\ell}(2^{2\ell}-1)}{2\ell!}\; B_{2\ell}(\sum_{n=m}^{\infty} J_n^{2\ell} + \sum_{n=-m}^{-\infty} J_n^{2\ell}) \qquad (3.66)$$

$$G_{2n-1}(i) = -\frac{J_0(i)}{\sigma_N}\, G_{2n-2}(i) - \frac{2n-2}{J_0(i)}\, G_{2n-3}(i) \qquad (3.67)$$

also

$$\beta = (\sum_{n=m}^{\infty} J_n^2 + \sum_{n=-m}^{-\infty} J_n^2) / \sigma_N^2 \qquad (3.68)$$

For small m, the computation of Equation (3.64) does not require a long computer time. For all the practical systems, m=2 is sufficient to make $\beta \leq 0.5$.

3.6 Discussion of the Main Result

The averaging method and the series expansion method have been used for computing the average probability of bit-error. Both methods give the same results. However, for the cases where the intersymbol interference is not confined to a few symbols, the series expansion method is preferred.

The explicit expression Equation (3.49) for the probability of error by using the series expansion method is simple and the computation is easy to perform. Most importantly, the influence of the intersymbol interference on the detected signal in terms of the probability of bit-error can be determined analytically. Also all of the constants involved (J_0, b_{2n}'s, G_{2n}'s) can be obtained with only a knowledge of the system parameters. Equation (3.25) is the generalized expression for any received signal [1], [16], [18]. Thus Equation (3.49) can be applied to any linear time invariant data transmission system perturbed by the intersymbol interference and Gaussian noise.

For most practical transmission systems, the intersymbol interference can be confined to very few bits [1]. The averaging method can also be applied equally well as far as the average probability of bit-error is concerned. The upper bound P_{emax} and the lower bound P_{emin} (equal to P_{el} in the series expansion method) can also give useful information about the system performance.

In the next two chapters, both the series expansion method and averaging method will be used to analyze the performances of various baseband and modulation transmission systems.

CHAPTER IV

ANALYSIS OF SOME PRACTICAL BASEBAND SYSTEMS

4.1 Introduction

The effect of bandlimiting on the performance of an NRZ baseband transmission system using the correlation detector has been studied in Chapter III.

In this chapter, some practical baseband systems will be analyzed using the results of Chapter III.

(1) Bandlimited (ideal filtering) Split-Phase baseband system using a correlation detector.

(2) Bandlimited (ideal filtering) NRZ baseband systems using a sample detector.

(3) Bandlimited (Gaussian filtering) NRZ baseband system using a correlation detector.

The effects of intersymbol interference on the performance of the systems will be analyzed and the bit-error probability will be computed.

4.2 Bandlimited Split-Phase Baseband System

The bandlimited Split-Phase baseband transmission system in the presence of additive white Gaussian noise can be modeled as in Figure 4.1. Here $\sum_{n=-\infty}^{\infty} a_n(t)$ is the random Split-Phase signal with amplitude +A or -A, bit duration equal to T and n(t) is an additive Gaussian noise with zero

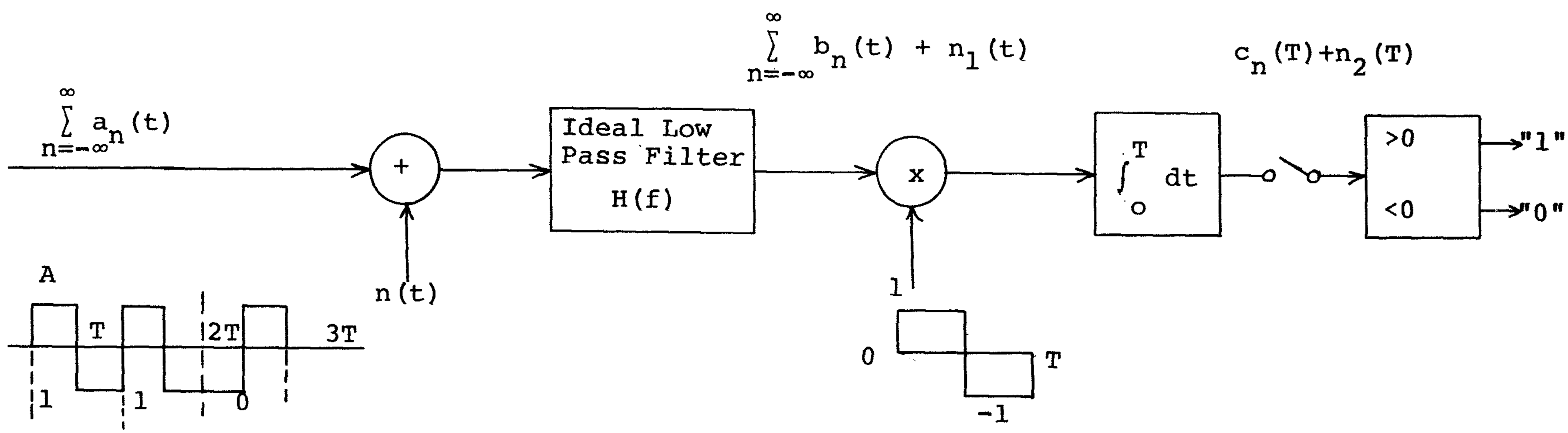

Figure 4.1 A Bandlimited Split-Phase Baseband Transmission System

mean and spectral density $N_0/2$. The ideal lowpass filter has a transfer function H(f) which is one for $-B \le f \le B$, and zero elsewhere.

The n^{th} bit of information can be represented by

$$A_n(t) = \begin{cases} A_n & NT < t \le NT + \frac{T}{2} \\ -A_n & NT + \frac{T}{2} < t \le (n+1)T \\ 0 & \text{elsewhere} \end{cases} \tag{4.1}$$

where $A_n = \pm A$ is the amplitude of the n^{th} pulse.

The response of the lowpass filter due to the n^{th} bit can be obtained as

$$b_n(t) = \int_{-B}^{B} \left(\int_{-\infty}^{\infty} a_n(t) e^{-j2\pi ft}\, dt\right) e^{j2\pi ft}\, df$$

$$= \frac{A_n T}{2} \int_{-B}^{B} \frac{\sin\frac{\pi fT}{2}}{\frac{\pi fT}{2}} e^{-j2\pi nfT} e^{-j\frac{\pi fT}{2}} (1-e^{-j\pi fT}) e^{j2\pi fT}\, df \tag{4.2}$$

The integrator output sampled at t=T due to n^{th} bit alone, is

$$C_n(T) = \int_{o}^{\frac{T}{2}} b_n(t)dt - \int_{\frac{T}{2}}^{T} b_n(t)dt$$

$$= \frac{A_n T}{2} \int_{o}^{\frac{T}{2}} \left(\int_{-B}^{B} \frac{\sin\frac{\pi fT}{2}}{\frac{\pi fT}{2}} e^{-j2\pi nfT} e^{-j\frac{\pi fT}{2}} (1-e^{-j\pi fT}) (1-e^{j\pi fT})\, e^{j2\pi ft}\, df\right) dt$$

$$= \frac{A_n T^2}{4} \int_{-B}^{B} \frac{\sin^2 \frac{\pi f T}{2}}{\left(\frac{\pi f T}{2}\right)^2} e^{-j2\pi n f T} (1-e^{-j\pi f T})(1-e^{j\pi f T})\, df$$

$$= \frac{A_n T^2}{4} \left[\int_{-B}^{B} \frac{4\sin^2 \frac{\pi f T}{2}}{\left(\frac{\pi f T}{2}\right)^2} e^{-j2\pi f n T}\, df - \int_{B}^{B} \frac{4\sin^2 \frac{\pi f T}{2}}{\left(\frac{\pi f T}{2}\right)^2} \cos^2 \frac{\pi f T}{2} e^{-j2n\pi f T}\, df \right] \quad (4.3)$$

(continued)

Changing variable and simplifying, $C_n(T)$ becomes

$$C_n(T) = A_n T F(BT,n) \quad (4.4)$$

where

$$F(BT,n) = \frac{4}{\pi} \int_0^{\frac{\pi BT}{2}} \frac{\sin^4 x}{x^2} \cos 4nx\, dx$$

$$= 2 \cdot \frac{2}{\pi} \int_0^{\frac{\pi BT}{2}} \frac{\sin^2 x}{x^2} \cos 4nx\, dx - \frac{2}{\pi} \int_0^{\pi BT} \frac{\sin^2 x}{x^2} \cos 2nx\, dx \quad (4.5)$$

Using Equation (3.5), $F(BT,n)$ can then be expressed as

$$F(BT,n) = 2 \cdot J\left(\frac{BT}{2}, 2n\right) - J(BT,n) \quad (4.6)$$

which can be evaluated in terms of $J(\frac{BT}{2}, 0)$ and $J(BT,0)$ (see Equation 3.8)).

The output of the integrator sampled at $t = T$ due to an and $n_1(t)$ is the output of the lowpass filter due to the noise $n(t)$ alone with covariance

$$P(t,\tau) = E[n_1(t)n_1(\tau)] = N_0 B \frac{\sin 2\pi B(t-\tau)}{2\pi B(t-\tau)} \tag{4.10}$$

Let

$$h(t) = \begin{cases} 1 & 0<t\leq\frac{T}{2} \\ -1 & \frac{T}{2}<t\leq T \end{cases} \tag{4.11}$$

then the variance of n_2 can be expressed as

$$\sigma_2{}^2 = E[n_2{}^2] = E[\int_o^T h(t)n_1(t)dt \int_o^T h(\tau)\ n_1(\tau)\ d\tau\ ]$$

$$= \int_o^T \int_o^T h(t)h(\tau)\ N_0 B \frac{\sin 2\pi B(t-\tau)}{2\pi B(t-\tau)}\ dt d\tau \tag{4.12}$$

This expression can be evaluated easily to give

$$\sigma_2{}^2 = \frac{N_0 T}{2} \cdot \frac{4}{\pi} \int_o^{\frac{\pi BT}{2}} \frac{\sin^4 x}{x^2}\ dx = \frac{N_0 T}{2} F(BT,0) \tag{4.13}$$

4.2.1 Probability of Bit-Error Using the Averaging Method

The output of the integrator due to both signal and noise is given by Equation (4.8)

$$y = W + n_2$$

$$= A_0 T\ F(BT,0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_n F(BT,n) + n_2 \tag{4.14}$$

infinite bit train is

$$W = \sum_{n=-\infty}^{\infty} A_n TF(BT,n)$$

$$= A_0T\ F(BT,0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_nT\ F(BT,n) \qquad (4.7)$$

The first term is the desired signal and the second term is the intersymbol interference caused by bandlimiting the signal. Thus F(BT,0) represents the degradation of the signal and F(BT,n) represents the effect of intersymbol interference on the bit under detection. As $B\to\infty$, $F(BT,n)\to 0$ and $W\to A_0T$ as expected. The influence of the adjacent bits can now be easily calculated. Table 4.1 shows some values of F(BT,n) for various bandwidths and bit positions.

The output of the integrator sampled at t = T due to both signal and noise can be given by

$$Y = W + n_2$$

$$= A_0TF(BT,0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_nTF(BT,n) + n_2 \qquad (4.8)$$

where

$$n_2 = \int_0^{\frac{T}{2}} n_1(t)dt - \int_{\frac{T}{2}}^{T} n_1(t)dt \qquad (4.9)$$

Table 4.1

Values of F(BT,n) for Various Bandwidths

BT	F(BT,0)	F(BT,1)	F(BT,2)	F(BT,3)	F(BT,4)	F(BT,5)
1.0	0.6446	-0.0910	-0.0116	-0.0048	-0.0026	-0.0017
1.4	0.8225	-0.0032	-0.0133	0.0074	-0.0026	-0.0011
2.2	0.8560	-0.0223	-0.0000	-0.0000	-0.0000	0.0000
2.6	0.8639	-0.0294	0.0043	-0.0024	0.0009	0.0002
3.0	0.8958	-0.0189	-0.0004	-0.0002	-0.0001	-0.0000
3.4	0.9200	-0.0884	-0.0023	0.0013	-0.0005	-0.0002
3.8	0.9250	-0.0123	-0.0000	0.0000	0.0000	-0.0000
4.2	0.9250	-0.0122	-0.0000	0.0000	-0.0000	-0.0000

The probability that a "1" in the middle of a particular pattern is detected to be a "0" is given by Equation (3.18)

$$P_{ei} = 1/2\ (1-\mathrm{erf}(z_i)) \tag{4.15}$$

where

$$\begin{aligned} z_i &= \frac{W_i}{\sqrt{2}\,\sigma_2} \\ &= \frac{AT[F(BT,0)+\sum_{n=1}^{\infty}(\frac{A_n+A_{-n}}{A})F(BT,n)]}{\sqrt{2}\ \sqrt{\frac{N_0T}{2}}\ F(BT,0)} \end{aligned} \tag{4.16}$$

and

$$D_i^{\,2}(BT) = \frac{[F(BT,0)+\sum_{n=1}^{\infty}(\frac{A_{-n}+A_n}{A})F(BT,n)]^2}{F(BT,0)} \tag{4.17}$$

$D_i^{\,2}(BT)$ can be considered as the degradation of signal-to-noise ratio (E/N_0) for a particular pattern. From Table 4.1, it can be seen that $F(BT,0) >> |F(BT,n)|$ when $n>5$. Thus the effect of intersymbol interference can be confined to the 10 bits nearest to the bit under detection. There is a total of 1024 different bit patterns. Then the probability of bit-error is given by

$$P_e = \frac{1}{1024}\sum_{i=1}^{1024} 1/2(1-\mathrm{erf}(\sqrt{\frac{E}{N_0}D_i^{\,2}(BT)}\,)) \tag{4.18}$$

The upper bound and the lower bound of the average probability of error can be given by Equation (3.23) and Equation (3.24) respectively

$$P_{emax} = 1/2[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}\,\frac{(F(BT,0)-2\sum_{n=1}^{\infty}|F(BT,n)|)^2}{F(BT,0)}})] \tag{4.19}$$

$$P_{emin} = 1/2[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}F(BT,0)})] \tag{4.20}$$

4.2.2 Probability of Bit-Error Using the Series Expansion Method

Dividing both sides of Equation (4.8) by AT, we obtain

$$X = Z_0J_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_nZ_n + N \tag{4.21}$$

where

$$X = \frac{Y}{AT}, \quad Z_n = \frac{A_nT}{AT} = \pm 1, \qquad J_n = F(BT,n)$$

and

$$N = \frac{n_2}{AT}$$

The variance of N, σ_N^2, can be evaluated as

$$\sigma_N^2 = \frac{\sigma_2^2}{A^2T^2} = \frac{J_0}{2}\,\frac{1}{(\frac{E}{N_0})} \tag{4.22}$$

Equation (4.21) is of the same form as Equation (3.25). The probability of bit-error is then given by Equation (3.49)

$$P = P_{e1} + P_{e2} \tag{4.23}$$

where

$$P_{e1} = \frac{1}{2} [1-\text{erf}(\sqrt{\frac{E}{N_0} J_0})] \quad , \tag{4.24}$$

and

$$P_{e2} = \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \; , \tag{4.25}$$

b_{2n} can be evaluated using Equation (3.59) and G_{2n-1} using Equation (3.58). From Table 4.1, the coefficient b_{2n} can be calculated with negligible error using only the terms from J_{-5} to J_5. For the system considered in this section, the series for P_{e2} converges rapidly and P_{e2} can be evaluated closely using only 10 terms in the series. By confining the intersymbol interference to the nearest 10 bits and using 10 terms in the series, the resulting P_e's, which are plotted as a function of signal-to-noise ratio and system bandwidth are shown in Figure 4.2. The results agree with those obtained using the averaging method. Table 4.2 shows the values of P_e, P_{e1} and P_{e2} for various bandwidths and signal-to-noise ratio. Table 4.3 shows the additional power needed to give the same performance as an optimum detector. As predicted, the probability of error is dominated by P_{e2} for high signal-to-noise ratio, and by P_{e1} for low signal-to-noise ratio.

The results obtained by both methods compared with those

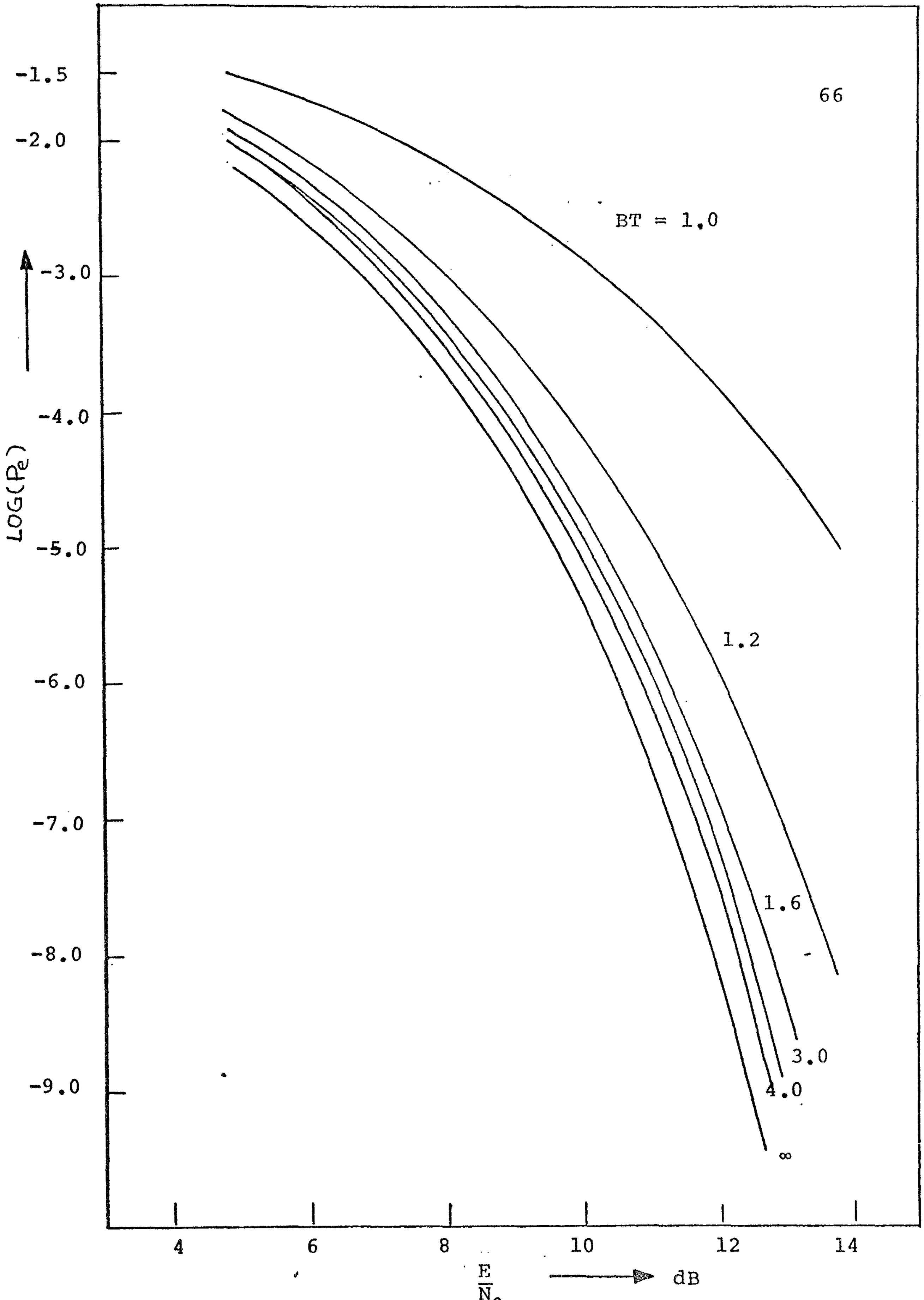

Figure 4.2 Probability of Bit-Error vs $\frac{E}{N_0}$ for the Split-Phase Baseband System

Table 4.2

Values of P_e, P_{e1} and P_{e2} vs $\frac{E}{N_0}$ for the Split-Phase Baseband System with BT = 1.0 and 1.2

BT = 1.0

$\frac{E}{N_0}$ (dB)	$\text{Log}(P_e)$	$\text{Log}(P_{e1})$	$\text{Log}(P_{e2})$
0	-0.872	-0.892	-2.208
4.77	-1.471	-1.609	-2.036
6.99	-1.932	-2.255	-2.212
8.45	-2.327	-2.875	-2.471
9.54	-2.683	-3.482	-2.758
10.41	-3.017	-4.081	-3.056
11.14	-3.337	-4.673	-3.357
11.76	-3.647	-5.261	-3.658
12.30	-3.951	-5.846	-3.956
12.79	-4.250	-6.428	-4.253
13.22	-4.545	-7.010	-4.547
13.62	-4.838	-7.588	-4.838
13.98	-5.127	-8.165	-5.128

Table 4.2 (Continue)

BT = 1.2

$\frac{E}{N_0}$ (dB)	Log(P_e)	Log(P_{e1})	Log(P_{e2})
0	-0.966	-0.971	-3.000
4.77	-1.774	-1.804	-2.950
6.99	-2.486	-2.564	-3.272
8.45	-3.151	-3.296	-3.698
9.54	-3.784	-4.015	-4.169
10.41	-4.392	-4.724	-4.664
11.14	-4.979	-5.428	-5.170
11.76	-5.548	-6.127	-5.681
12.30	-6.101	-6.822	-6.193
12.79	-6.642	-7.518	-6.704
13.22	-7.170	-8.208	-7.212
13.62	-7.687	-8.897	-7.715
13.98	-8.195	-9.585	-8.213

Table 4.3

Additional Power needed for the Detection of Split-Phase Signals to give the same Performance as an Optimum Detector

P_e	BT= ∞ $\frac{E}{N_o}$ (dB)	1.0 S	1.0 A	1.2 S	1.2 A	1.6 S	1.6 A	3.0 S	3.0 A
10^{-2}	4.3	1.9	2.9	1.1	1.4	0.7	0.8	0.4	0.6
10^{-3}	6.8	1.9	3.6	1.1	1.4	0.7	0.8	0.4	0.6
10^{-4}	8.4	1.9	4.0	1.1	1.4	0.7	0.8	0.4	0.6
10^{-5}	9.6	1.9	4.3	1.1	1.5	0.7	0.9	0.4	0.7
10^{-6}	10.5	1.9	4.7	1.1	1.6	0.7	1.0	0.4	0.7
10^{-7}	11.3	1.9	5.0	1.1	1.7	0.7	1.0	0.4	0.7

S : Additional power needed for the single pulse case in the absence of intersymbol interference.

A : Additional power needed for the average case.

Second colomn is the signal-to-noise ratio required for the infinite bandwidth (optimum case) obtained for the NRZ baseband system in Chapter III, demonstrate that the Split-Phase system requires about less than twice as much bandwidth as the NRZ baseband system to have the same probability of bit-error for the same values of signal-to-noise ratio.

4.3 Bandlimited NRZ Baseband System Using a Sample Detector

The system shown in Figure 4.3 is the same one analyzed in Chapter III except a sample detector is used instead of an integrator. A sample detector gives the value of the function at the sampling time.

The Fourier Transform of the output of the lowpass filter to the n^{th} bit is

$$B_n(f) = \int_{-\infty}^{\infty} a_n(t)\; e^{-j2\pi ft}\, dt \qquad -B \leq f \leq B$$

$$= \quad 0 \qquad\qquad \text{elsewhere} \qquad (4.25)$$

and the time response is

$$b_n(t) = \int_{-B}^{B} B_n(f) e^{j2\pi ft} df = A_n \frac{2}{\pi} \int_{0}^{\pi BT} \frac{\sin x}{x} \cos x\left(1-\frac{2t}{T}+2n\right) dx \qquad (4.26)$$

Figure 4.4 shows the plot of $b_n(t)$. It can be seen that the response extended from $-\infty$ to ∞ instead of being restricted from nT to $(n+1)T$. The response of the lowpass filter due to the infinite bit train can then be expressed as

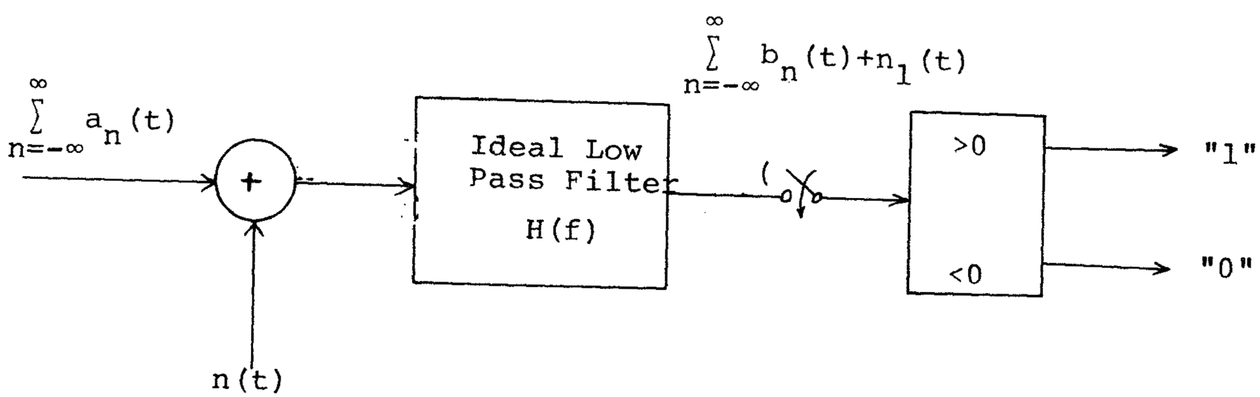

Figure 4.3 Filter and Sample Detector

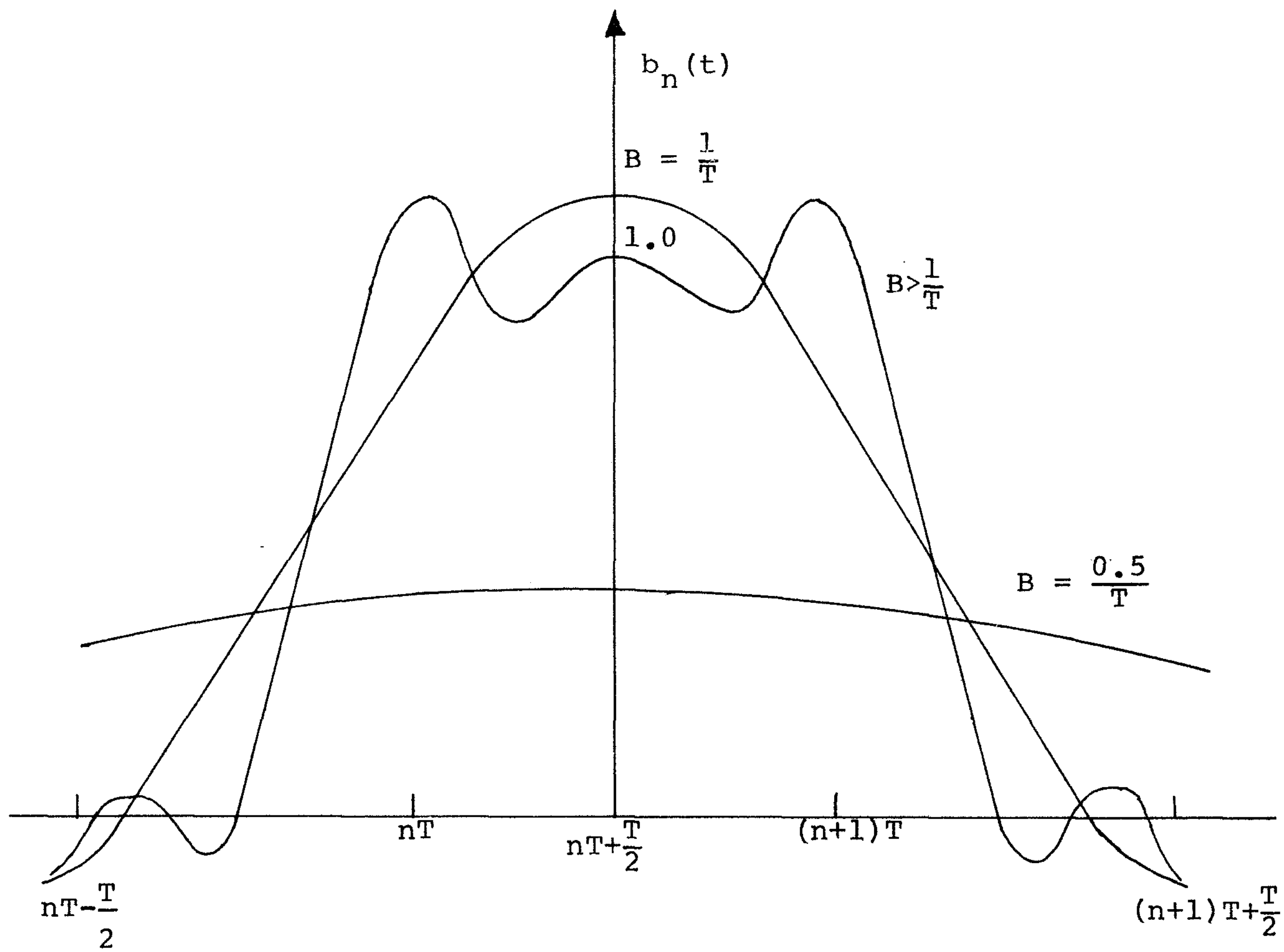

Figure 4.4 Response of Lowpass Filter to a Rectangular Pulse

$$W(t) = \sum_{n=-\infty}^{\infty} b_n(t)$$

$$= A_0 \frac{2}{\pi} \int_0^{\pi BT} \frac{\sin x}{x} \cos x(1-\frac{2t}{T})\, dx$$

$$+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_n \frac{2}{\pi} \int_0^{\pi BT} \frac{\sin x}{x} \cos x(1-\frac{2t}{T} + 2n)dx \qquad (4.27)$$

The first term is the desired signal and is peaked at $t=T/2$ for $BT\leq 1$ (see Figure 4.4) [29]. The second term is the intersymbol interference due to bandlimiting the signal. Thus sampled at $t=T/2$, the response can be simplified to give

$$W = A_0\, S(BT,0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_n\, S(BT,n) \qquad (4.28)$$

where

$$S(BT,n) = \frac{2}{\pi} \int_0^{\pi BT} \frac{\sin x}{x} \cos 2nx\, dx \qquad (4.29)$$

an even function of n, and

$$S(BT,0) = \frac{2}{\pi} S_i(\pi BT)$$

where

$$S_i(y) = \int_0^{Y} \frac{\sin x}{x}\, dx$$

the sine integral.

S(BT,n) can be evaluated in terms of S(BT,0)

$$S(BT,n) = 1/2\ S[(2n+1)BT,0] - 1/2\ S[(2n-1)BT,0] \quad (4.30)$$

Table 4.4 lists some values of S(BT,n).

The output of the lowpass filter sampled at t=T/2 due to both signal and noise can be given by

$$\begin{aligned} Y &= W + n_1 \\ &= A_0\ S(BT,0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_n\ S(BT,n) + n_1 \end{aligned} \quad (4.31)$$

where n_1 is the output of the sampler due to the noise n(t) alone. The covariance of $n_1(t)$ is given by Equation (4.10) and the variance of n_1 can be expressed as

$$\sigma_1^{\,2} = E[n_1^{\,2}] = N_0 B \quad (4.32)$$

4.3.1 Probability of Bit-Error Using the Averaging Method

The probability of bit-error for a particular pattern can be given by Equation (3.18)

$$P_{ei} = 1/2(1-\mathrm{erf}(Z_i)] \quad (4.33)$$

Table 4.4

Some Values of S(BT,n)

BT	S(BT,0)	S(BT,1)	S(BT,2)	S(BT,3)	S(BT,4)	S(BT,5)
0.5	0.873	0.0756	-0.0167	0.0073	-0.0041	0.0026
0.6	0.987	-0.0317	0.0713	-0.0534	0.0271	-0.0028
0.7	1.074	-0.0833	0.0487	0.0166	-0.0292	0.0028
0.8	1.134	-0.0847	-0.0075	0.0204	0.0163	-0.0027
0.9	1.168	-0.0660	-0.0197	-0.0085	-0.0020	0.0024
1.0	1.179	-0.0564	-0.0130	-0.0057	-0.0032	-0.0020

where

$$z_i = \frac{W_i}{\sqrt{2}\,\sigma_1} = \frac{A\left(S(BT,0) + \sum_{n=1}^{\infty} \frac{A_{-n}+A_n}{A} S(BT,n)\right)}{\sqrt{2} \cdot \sqrt{N_0 B}}$$

$$= \sqrt{\frac{E}{N_0} D_i^{\,2}(BT)} \qquad (4.34)$$

and

$$D_i^{\,2}(BT) = \frac{\left[S(BT,0) + \sum_{n=1}^{\infty} \frac{A_{-n}+A_n}{A} S(BT,n)\right]^2}{2BT} \qquad (4.35)$$

$D_i^{\,2}(BT)$ is the degradation of signal-to-noise ratio for a particular bit pattern.

From Table 4.4 it is clear that $|S(BT,n)| << S(BT,0)$ for $n>5$, thus the effects of the intersymbol interference can be confined to the 10 nearest bits on the bit under detection. There are 1024 different patterns. Thus the average probability of error can be evaluated as

$$P_e = \frac{1}{1024} \sum_{i=1}^{1024} 1/2\left(1-\mathrm{erf}\left(\sqrt{\frac{E}{N_0} D_i^{\,2}(BT)}\right)\right) \qquad (4.36)$$

The upper bound and lower bound of P_e can also be given by Equation (3.23) and Equation (3.24) respectively

$$P_{emax} = 1/2\left(1-\mathrm{erf}\left(\sqrt{\frac{E}{N_0} \frac{\left[S(BT,0) - 2\sum_{n=1}^{\infty} S(BT,n)\right]^2}{2BT}}\right)\right) \qquad (4.37)$$

$$P_{emin} = 1/2\left(1-\mathrm{erf}\left(\sqrt{\frac{E}{N_0} \frac{S(BT,0)^2}{2BT}}\right)\right) \qquad (4.38)$$

4.3.2 Probability of Bit-Error Using the Series Expansion Method

Dividing both sides of Equation (4.24) by A, we obtain

$$X = Z_0 J_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n Z_n + N \tag{4.39}$$

where $X = \frac{y}{A}$, $Z_n = \frac{A_n}{A} = \pm 1$, $J_n = S(BT,n)$ and

$$N = \frac{n_1}{A}$$

Equation (4.39) is of the same form as Equation (3.25). Therefore the probability of bit-error can now be given by Equation (3.49)

$$P_e = P_{e1} + P_{e2}$$

$$= 1/2[1-\text{erf}(\frac{J_0}{\sqrt{2}\sigma_N})] + \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \tag{4.40}$$

The variance of N can be evaluated as

$$\sigma_N^2 = \frac{\overline{n_1^2}}{A^2} = \frac{N_0 B}{A^2} = \frac{N_0 BT}{A^2 T} = \frac{N_0 BT}{A^2 T} = BT \frac{1}{(\frac{E}{N_0})} \tag{4.41}$$

Thus

$$P_{e1} = 1/2[1-\text{erf}(\sqrt{\frac{E}{N_0} (\frac{J_0^2}{2BT})})] , \tag{4.42}$$

and

$$P_{e2} = \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \quad , \tag{4.43}$$

b_{2n} can be evaluated using Equation (3.59). G_{2n-1} is given by Equation (3.47)

$$G_{2n-1} = - \frac{J_0}{\sigma_N^2} G_{2n-2} - \frac{2n-2}{\sigma_N^2} G_{2n-3}$$

$$= - \left(\frac{E}{N_0}\right) \left[\left(\frac{J_0}{BT}\right) G_{2n-2} - \frac{2n-2}{BT} G_{2n-3} \right] \tag{4.44}$$

From Table 4.4, the coefficient b_{2n} can be computed with insignificant error using only the terms from J_{-5} to J_5. For the system considered here, the series for P_{e2} converges rapidly so that P_{e2} can be computed accurately using only 10 terms in the series.

By confining the intersymbol interference to the nearest 10 bits the resulting P_e's are shown in Figure 4.5. Table 4.5 gives the values of P_e, P_{e1} and P_{e2} for various bandwidths and signal-to-noise ratios.

The resulting P_e's obtained by Equation (4.43) do agree with those obtained by the averaging method.

The bandwidth of the lowpass filter is limited to be less than 1/T. Because for B>1/T, the peak value of signal will not occur at t=T/2 (see Figure 4.4), and more noise is allowed through. [33]

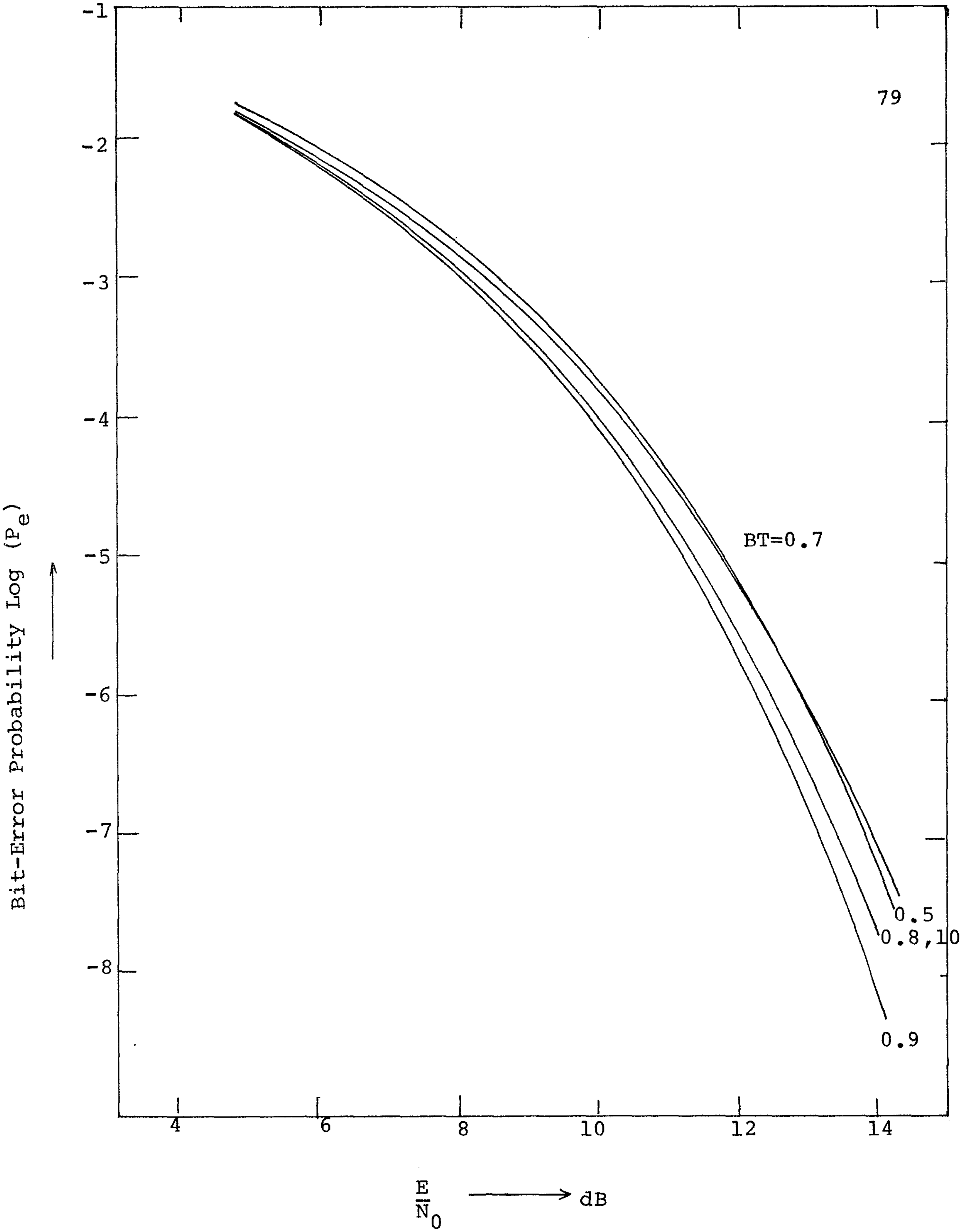

Figure 4.5 Probability of Bit-Error vs $\frac{E}{N_0}$ for the NRZ System Using the Filter and Sample Detector

Table 4.5

Values of P_e, P_{e1} and P_{e2} vs $\frac{E}{N_0}$ for the NRZ Baseband System Using Filter and Sample Detector with BT=0.7 and 0.9

BT = 0.7

$\frac{E}{N_o}$ (dB)	$\mathrm{Log}(P_e)$	$\mathrm{Log}(P_{e1})$	$\mathrm{Log}(P_{e2})$
0.00	-0.987	-1.002	-2.452
4.77	-1.775	-1.883	-2.433
6.99	-2.422	-2.698	-2.761
8.45	-2.992	-3.467	-3.169
9.54	-3.509	-4.231	-3.600
10.41	-3.987	-4.987	-4.033
11.14	-4.436	-5.736	-4.458
11.76	-4.861	-6.480	-4.872
12.30	-5.267	-7.223	-5.272
12.79	-5.658	-7.962	-5.661
13.22	-6.036	-8.698	-6.037
13.98	-6.761	-10.165	-6.761

Table 4.5 (continued)

BT = 0.9

$\frac{E}{N_o}$ (dB)	$\mathrm{Log}(P_e)$	$\mathrm{Log}(P_{e1})$	$\mathrm{Log}(P_{e2})$
0.00	-0.957	-0.962	-2.904
4.77	-1.747	-1.784	-2.842
6.99	-2.438	-2.531	-3.152
8.45	-3.080	-3.252	-3.567
9.54	-3.690	-3.958	-4.027
10.41	-4.276	-4.656	-4.510
11.14	-4.844	-5.347	-5.007
11.76	-5.397	-6.035	-5.510
12.30	-5.938	-6.718	-6.017
12.79	-6.471	-7.402	-6.525
13.22	-6.996	-8.081	-7.033
13.62	-7.515	-8.758	-7.540
13.98	-8.028	-9.434	-8.045

If a single pulse (in the absence of the intersymbol interference) is transmitted, $P_e = P_{e1}$ is minimum when BT=0.7, which gives the maximum value of $J_0^2/2BT$ equal to 0.82. This agrees with the predicition by Schwartz [30]. But from Table 4.5 and Figure 4.5, it can be seen that the probability of bit error is minimum when BT=0.9. Thus the intersymbol interference has a considerable effect on the detection of bandlimited signals using the filter and sample detector. The optimum bandwidth of the filter should thus be set to 0.9 of the bit rate of the transmitted NRZ signal if the filter and sample detector is used.

Comparing Figure 4.5 with Figure 3.2, it can be seen that the correlation detector is superior to the filter and sample detector for $BT \geq 0.6$. But the performance of the filter and sample detector is better than that of the correlation detector for BT is equal to 0.5.

4.4 Bandlimited (Gaussian Filtering) NRZ Baseband System Using the Correlation Detector

So far we have considered the ideal bandlimited channel for various baseband transmission systems. In this section, we intend to analyze the performance of the NRZ baseband system for a bandlimited channel with a filter whose transfer function G(f) is Gaussian as shown in Figure 4.6. The expression for

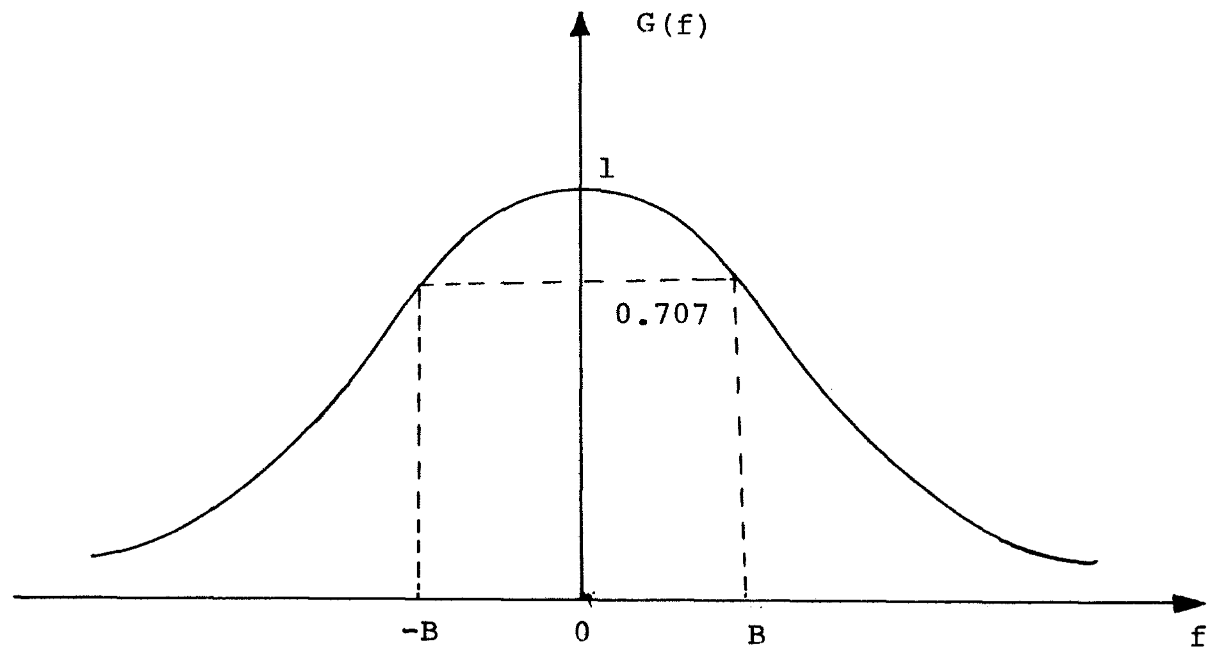

Figure 4.6 Characteristics of the Gaussian Filter

G(f) is [31]

$$G(f) = e^{-0.347(\frac{f}{B})^2} \tag{4.45}$$

where B is the 3dB bandwidth of the filter.

As we know from Chapter II, the optimum receiver for the detection of binary signals corrputed by additive white Gaussian noise can be obtained by using the matched filter. For a single NRZ pulse (i.e. in the absence of intersymbol interference) transmitted over this channel, the optimum receiver (matched filter) can be determined from the signal and channel characteristics [1]. The Fourier transform of a pulse with amplitude A and duration T is

$$F(f) = \int_0^T A\, e^{-j2\pi ft}dt = AT\,\frac{\sin\pi fT}{\pi fT}\,e^{-j\pi fT} \tag{4.46}$$

Thus the transfer function of the matched filter will be [12]

$$R(f) = K[F(f)G(f)e^{j2\pi fT}]^* \tag{4.47}$$

where K is an arbitrary real number and * indicates the conjugate. Substituting Equations (4.45) and (4.46) into Equation (4.47), R(f) becomes

$$R(f) = KAT\,\frac{\sin\pi fT}{\pi fT}\,e^{j\pi fT}\,e^{-0.347(\frac{f}{B})^2}\,e^{-j2\pi fT}$$

$$= (KA)(e^{-0.347(\frac{f}{B})^2})(T\frac{\sin\pi fT}{\pi fT}\,e^{-j\pi fT}) \tag{4.48}$$

KA can be chosen to be 1. We recognize that $T\frac{\sin\pi fT}{\pi fT}\, e^{-j\pi fT}$ is the transfer function of the correlation detector. Therefore the matched filter consists of the Gaussian filter cascaded with a correlation detector.

The primary objective in this section is to examine the effects of intersymbol interference on the performance of this optimum receiver using the basic principles developed in Chapter III. The total system can now be modeled as shown in Figure 4.7. Again $\sum_{n=-\infty}^{\infty} a_n(t)$ is the random NRZ signals with amplitude A or -A and bit duration T. n(t) is the Gaussian noise with spectral density $N_0/2$ (two sided).

Rewrite G(f) as

$$G(f) = e^{-\frac{\alpha^2(2\pi f)^2}{4}} \tag{4.49}$$

where

$$\alpha = \frac{\sqrt{0.347}}{\pi B} \tag{4.50}$$

The impulse response of the two Gaussian filters cascaded together can be obtained as

$$g(t) = \int_{-\infty}^{\infty} G(f)^2\, e^{j2\pi ft}\, df$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-\alpha^2 w^2}{2}}\, e^{jwt}\, dw \tag{4.51}$$

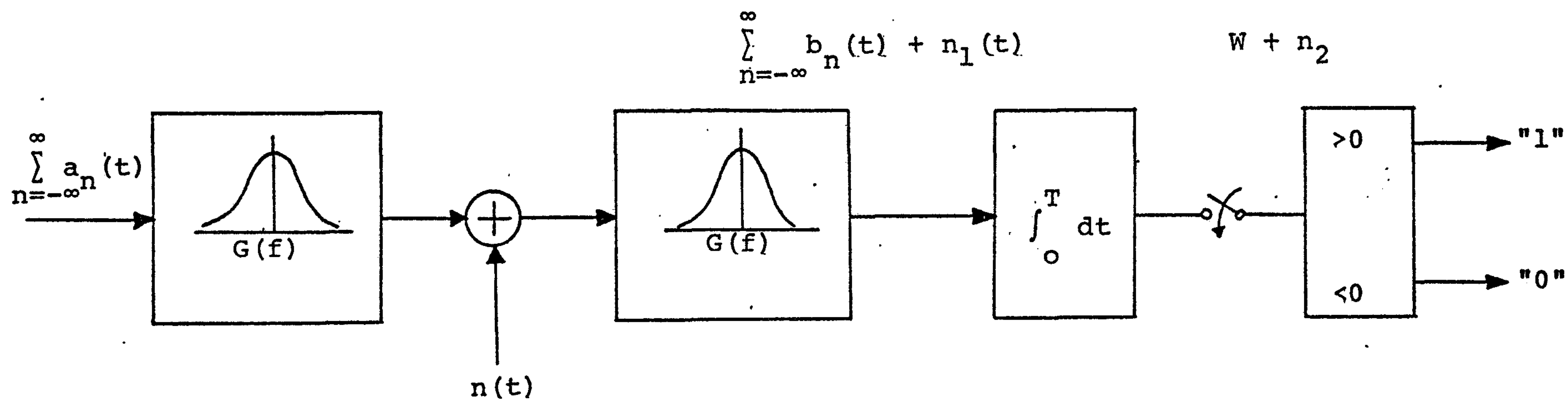

Figure 4.7 Bandlimited (Gaussian Filtering) NRZ Baseband System

where

$$w = 2\pi f$$

and [19]

$$g(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-\alpha^2 w^2}{2}} e^{-jwt}\, dw$$

$$= \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{t^2}{2\alpha^2}}$$

Thus

$$g(t) = \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{t^2}{2\alpha^2}} \qquad (4.53)$$

The output of the Gaussian filter in the receiver due to the n^{th} bit can be obtained as the convolution of $a_n(t)$ and $g(t)$

$$b_n(t) = \int_{-\infty}^{\infty} a_n(t-x)g(x)dx$$

$$= \int_{t-(n+1)T}^{t-nT} A_n \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{x^2}{2\alpha^2}} dx \qquad (4.54)$$

which can be simplified as

$$b_n(t) = \frac{A_n}{2} [\mathrm{erf}[\frac{(n+1)T-t}{\sqrt{2}\alpha}] - \mathrm{erf}(\frac{nT-t}{\sqrt{2}\alpha})] \qquad (4.55)$$

The response of the integrator sampled at t=T due to the n^{th} bit is

$$C_n(T) = \int_0^T b_n(t)dt$$

$$= \frac{A_n}{2} \int_0^T [\mathrm{erf}[\frac{(n+1)T-t}{\sqrt{2}\,\alpha}] - \mathrm{erf}(\frac{nT-t}{\sqrt{2}\,\alpha})]dt \tag{4.56}$$

With $\alpha = \frac{\sqrt{0.347}}{\pi B}$, changing variable and simplifying, $C_n(T)$ becomes

$$C_n(T) = A_n\, T\, E(BT,n) \tag{4.57}$$

where

$$E(BT,n) = \frac{n+1}{2} E[(n+1)BT,0]$$

$$- nE(nBT,0)$$

$$+ \frac{n-1}{2} E[(n-1)BT,0]$$

and

$$E(BT,0) = \mathrm{erf}(\frac{\pi BT}{\sqrt{2}\sqrt{0.347}}) - \sqrt{\frac{0.347}{\pi}}\ \frac{\sqrt{2}}{\pi BT}\ (1-e^{\frac{-(\pi BT)^2}{2\cdot 0.347}})$$

Thus the output of the integrator due to the infinite bit train is

$$W = \sum_{n=-\infty}^{\infty} C_n(T)$$

$$= A_0 T\, E(BT,0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_n T\, E(BT,n) \tag{4.60}$$

The first term is the desired signal and the second term is the intersymbol interference. As $B\to\infty$, $E(BT,0)\to 1$, $E(BT,n)\to 0$ and $W\to A_0T$ as expected. Table 4.6 shows some values of $E(BT,n)$.

The output of the integrator due to signal and noise can be described by

$$= A_0TE(BT,0) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A_nTE(BT,n) + n_2 \tag{4.61}$$

where

$$n_2 = \int_o^T n_1(t)\ dt \tag{4.62}$$

and $n_1(t)$ is the output of the Gaussian filter due to the noise $n(t)$ alone.

The variance of n_2, σ_2^2, can be shown to be (see Appendix C)

$$\sigma_2^2 = \frac{N_0T}{2}\left[\operatorname{erf}\left(\frac{\pi BT}{\sqrt{2}\sqrt{0.347}}\right) - \sqrt{\frac{0.347}{\pi}}\ \frac{\sqrt{2}}{\pi BT}\ \left(1-e^{\frac{-(\pi BT)^2}{2\cdot 0.347}}\right)\right]$$

$$= \frac{N_0T}{2}\ E(BT,0) \tag{4.63}$$

4.4.1 <u>Probability of Error Using the Averaging Method</u>

From Table 4.6, it can be seen that $E(BT,0) \gg |E(BT,n)|$ when $n>1$, thus the effect of intersymbol interference can just be confined to the two adjacent bits. The average probability

Table 4.6

Some Values of E(BT,n)

BT	E(BT,0)	E(BT,1)	E(BT,2)
0.5	0.7017	0.1487	0.0000
0.6	0.7508	0.1246	0.0000
0.7	0.7863	0.1069	0.0000
0.8	0.8130	0.0935	0.0000
0.9	0.8338	0.0831	0.0000
1.0	0.8504	0.0748	0.0000
1.2	0.8753	0.0623	0.0000
1.5	0.9003	0.0499	0.0000
2.0	0.9252	0.0374	0.0000
2.5	0.9402	0.0299	0.0000
3.0	0.9501	0.0249	0.0000

of bit-error now can be expressed as

$$P_e = 1/2(1-\mathrm{erf}(Z_1)) + 1/2(1-\mathrm{erf}(Z_2)$$

$$+ 1/2(1-\mathrm{erf}(Z_3)) + 1/2(1-\mathrm{erf}(Z_4) \tag{4.64}$$

where

$$Z_1 = \frac{W_1}{\sqrt{2}\sigma_2} = \frac{AT[E(BT,0)+(1+1)E(BT,1)]}{\sqrt{2} \cdot \sqrt{\frac{N_0 T}{2} E(BT,0)}}$$

$$= \sqrt{\frac{E}{N_0} \frac{(E(BT,0)+2E(BT,1))^2}{E(BT,0)}} \tag{4.65}$$

$$Z_2 = \frac{W_2}{\sqrt{2}\sigma_2} = \frac{AT[E(BT,0)+(1-1)E(BT,1)]}{\sqrt{2} \sqrt{\frac{N_0 T}{2} E(BT,0)}}$$

$$= \sqrt{\frac{E}{N_0} E(BT,0)} \tag{4.66}$$

$$Z_3 = \frac{W_3}{\sqrt{2}\sigma_2} = \frac{AT[E(BT,0)+(-1+1)E(BT,1)]}{\sqrt{2} \sqrt{\frac{N_0 T}{2} E(BT,0)}}$$

$$= \sqrt{\frac{E}{N_0} E(BT,0)} \tag{4.67}$$

and

$$Z_4 = \frac{W_4}{\sqrt{2}\sigma_2} = \frac{AT(E(BT,0)+(-1-1)E(BT,1))}{\sqrt{2}\sqrt{\frac{N_0T}{2}E(BT,0)}}$$

$$= \sqrt{\frac{E}{N_0} \frac{[E(BT,0)-2E(BT,1)]^2}{E(BT,0)}} \tag{4.68}$$

Figure 4.8 shows the plots of the probability of error as a function of signal-to-noise ratio for various band-widths.

4.4.2 Probability of Bit-Error Using the Series Expansion Method

Normalizing Equation (4.61) by dividing both sides by AT, we obtain

$$X = Z_0J_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_nZ_n + N \tag{4.69}$$

where $X = \frac{y}{AT}$, $A_n = \frac{A_nT}{AT} = \pm 1$, $J_n = E(BT,n)$ and $N = \frac{n_2}{AT}$

The variance of N can be given by

$$\sigma_N^2 = \frac{\sigma_2^2}{A^2T^2} = \frac{1}{2(\frac{E}{N_0})} J_0 \tag{4.70}$$

Again Equation (4.69) is of the same form as Equation (3.25). Therefore the probability of error can be given by Equation (3.50)

$$P_e = P_{e1} + P_{e2} \tag{4.71}$$

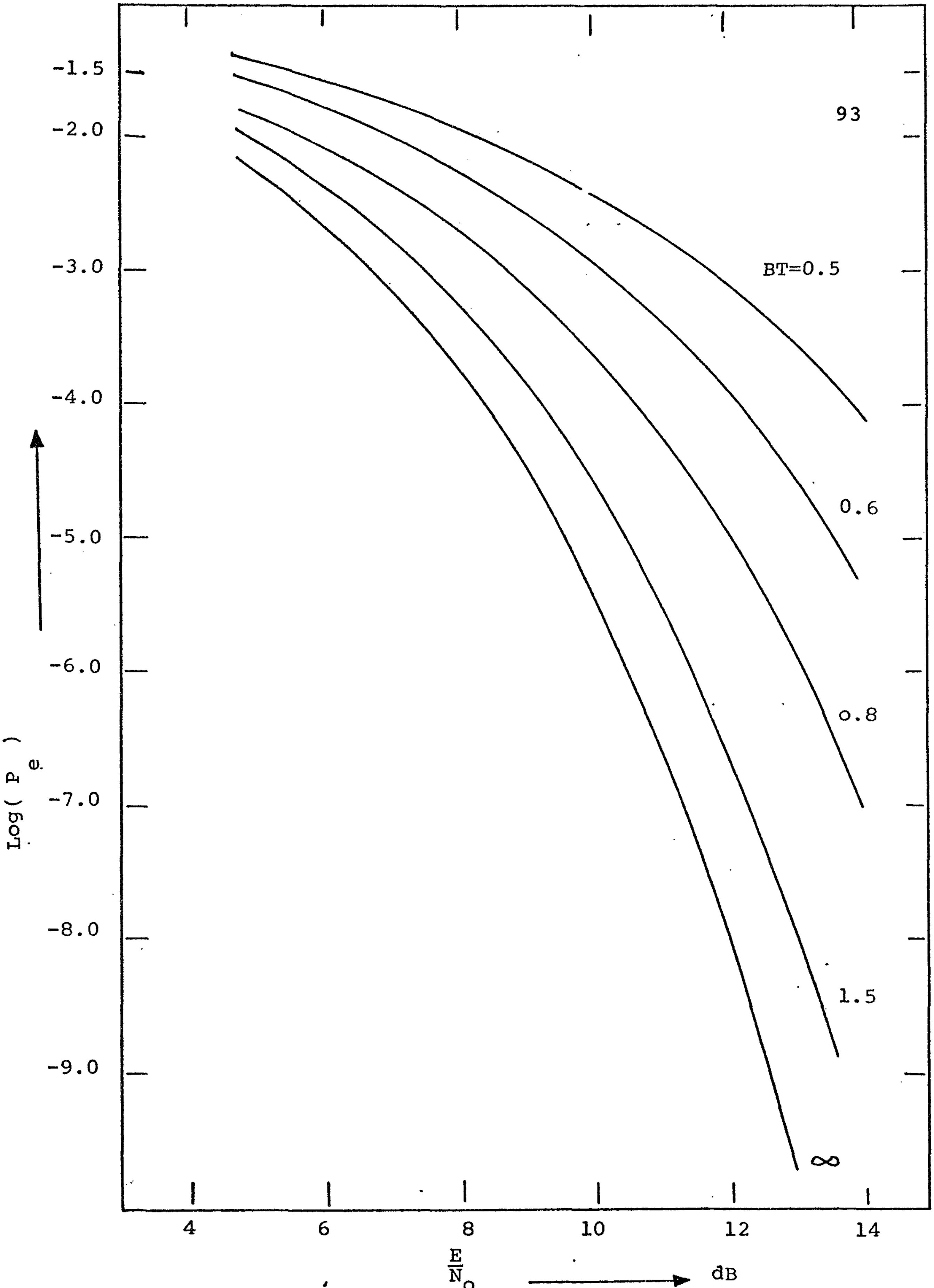

Figure 4.8 Probability of Error vs $\frac{E}{N_o}$ for the Gaussian Filtered NRZ Signals

where

$$P_{e1} = 1/2[1-\mathrm{erf}(\frac{J_0}{\sqrt{2}\sigma_N})]$$

$$= 1/2[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}}\, J_0\,)] \qquad , \qquad (4.72)$$

and

$$P_{e2} = \sum_{n=1}^{\infty} (-1)^{n+1}\, b_{2n}\, G_{2n-1} \qquad , \qquad (4.73)$$

b_{2n} can be evaluated using Equation (3.59) and G_{2n-1} using Equation (3.58). As $B\to\infty$, $b_{2n}\to 0$, and $P_e \to P_{e1} \to 1/2[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}})]$ as expected for the infinite bandwidth case. b_{2n} can be evaluated accurately using only the terms from J_{-1} to J_1. In other words, almost all the influence of intersymbol interference comes from the immediate adjacent bits. The series for P_{e2} can be truncated to 10 terms without introducing any significant error. The resulting P_e's agree with those found using the averaging method.

The performance of the system considered in this section is much worse than the systems described in Section 4.3 and Chapter III (comparing the curves in Figure 4.8 with those in Figure 4.5 and Figure 3.3). The reason for this is that the Gaussian bandlimited channel introduces more signal distortion than the ideal bandlimited channel does (see Table 4.5, Table 4.3 and Table 3.1).

CHAPTER V

ANALYSIS OF SOME PRACTICAL MODULATION SYSTEMS

5.1 Introduction

The effects of bandlimiting on the performance of some practical baseband systems have been analyzed in Chapter IV. Bandlimiting not only causes the loss of signal energy but also introduces intersymbol interference. It is the intersymbol interference that dominates the total system performance for high signal-to-noise ratio.

For the modulation systems, the restriction of bandwidth is usually caused by (1) premodulation filtering (2) postmodulation filtering (3) bandlimited channel (4) receiver bandpass filtering or IF filtering. In the case of (1), the performance of the system can be analyzed the same way as the baseband system [17]. But for the cases of (2), (3) and (4) additional signal distortion and interference will be introduced by the aliasing effect if the carrier frequency is not much greater than the bit rate [43].

In this chapter, the effects of the receiver IF filtering, the most common cause of bandlimiting for a modulation system, will be analyzed using the main results of Chapter III.

The performance of the three basic data modulation systems, which are almost exclusively used in practice, will be investigated:

(1) Phase Shift Keying or PSK

(2) Amplitude Shift Keying or ASK

(3) Frequency Shift Keying or FSK

The explicit expressions for the degradation of signal, intersymbol interference, and aliasing effect as functions of system bandwidth and carrier frequency will be determined first. The probability of bit-error for each case will then be computed.

5.2 Phase Shift Keying

The PSK coherent communication system can be modeled as shown in Figure 5.1 [14], [34], [46]. The PSK signals (see Chapter II) at the transmission end can be generated by amplitude modulating a carrier $\cos(w_c t)$ by a random NRZ signal with bit duration T, amplitude A or -A. n(t) is white Gaussian noise with zero mean and power spectral density $N_0/2$ (two sided). The receiver IF filtering can be modeled by using a rectangular bandpass filter centered at the carrier frequency f_c with bandwidth 2B, where B is defined as the equivalent baseband system bandwidth. The transfer function of the bandpass filter can be represented as

$$H_B(f) = \begin{cases} 1 & f_c-B\leq f\leq f_c+B \\ 1 & -f_c-B\leq f\leq -f_c+B \\ 0 & \text{elsewhere} \end{cases} \tag{5.1}$$

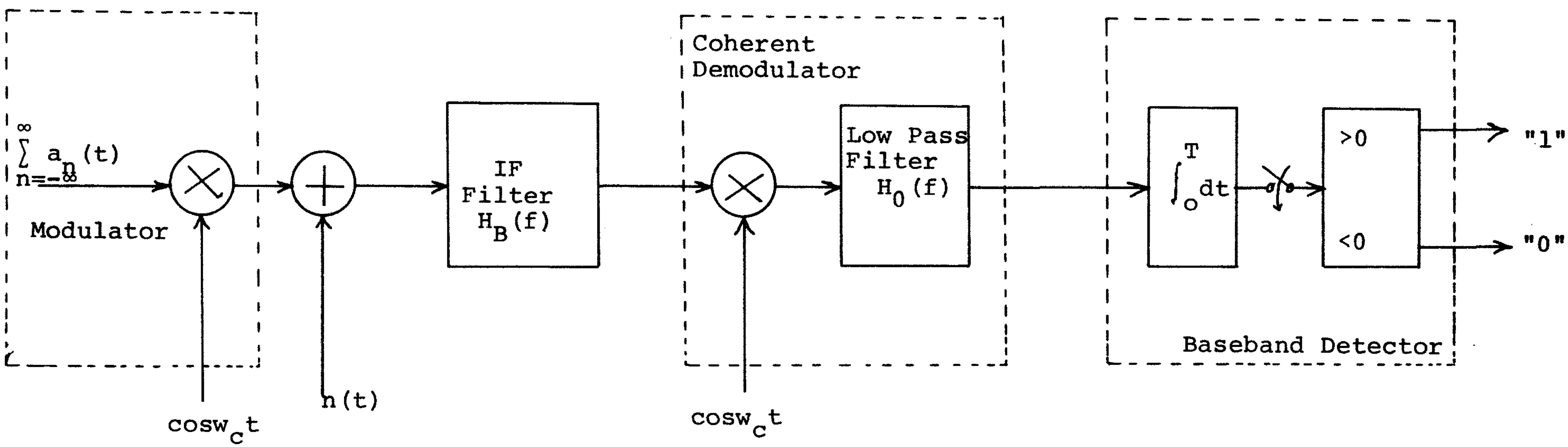

Figure 5.1 A Bandlimited PSK Model

The modulated signal plus noise at the output of the IF filter is demodulated by the coherent demodulator, which consists of a multiplier followed by a lowpass filter with bandwidth B. The demodulated baseband signal plus noise then is fed to the baseband detector. This detector, which consists of a correlation detector followed by a threshold device, is optimum if the system bandwidth is infinite. Since PSK signals are antipodal, the optimum threshold d is set to be zero (see Chapter II).

For practical consideration, the carrier frequency is assumed to be a multiple of the bit rate [3], [10], [47]. The communication model shown in Figure 5.1 can be replaced by an equivalent one (see Appendix D) as shown in Figure 5.2. Here

$$H(f)=\begin{cases}\frac{1}{2}\,T\,\frac{\sin\pi fT}{\pi(f-f_c)T}\,e^{-j\pi fT} & f_c-B\leq f\leq f_c+B\\ \frac{1}{2}\,T\,\frac{\sin\pi fT}{\pi(f+f_c)T}\,e^{-j\pi fT} & -f_c-B\leq f\leq -f_c+B\\ 0 & \text{elsewhere}\end{cases}\tag{5.2}$$

The n^{th} bit can be represented as

$$a_n(t) = \begin{cases} A_n & nT\leq t\leq (n+1)T\\ 0 & \text{elsewhere}\end{cases}\tag{5.3}$$

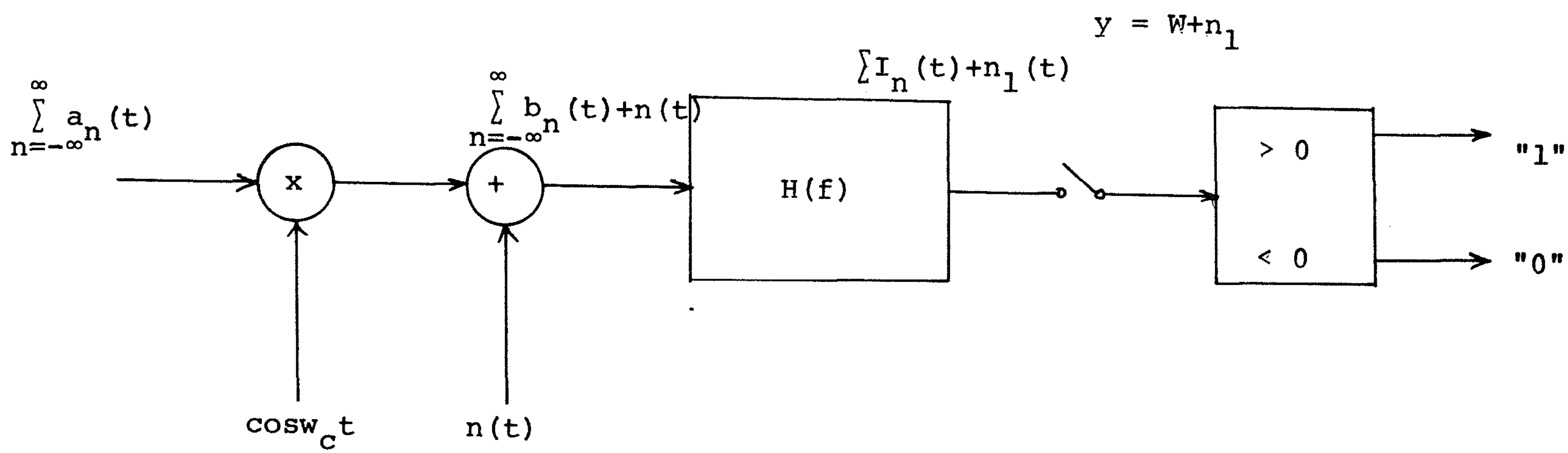

Figure 5.2 A Simplified PSK Model

where

$$A_n = \begin{cases} A & \text{for a "1"} \\ -A & \text{for a "0"} \end{cases} \tag{5.4}$$

Then the modulated carrier can be expressed as

$$b_n(t) = a_n(t)\ \cos\omega_c t \qquad nT \leq t \leq (n+1)T \tag{5.5}$$

The Fourier transform of $b_n(t)$ is

$$B_n(f) = \int_{nT}^{(n+1)T} A_n \cos 2\pi f_c t e^{-j2\pi ft} dt$$

$$= \frac{1}{2} A_n T \frac{\sin\pi(f-f_c)T}{\pi(f-f_c)T} e^{-j\pi(f-f_c)T(1+2n)}$$

$$+ \frac{1}{2} A_n T \frac{\sin\pi(f+f_c)T}{\pi(f+f_c)T} e^{-j\pi(f+f_c)T(1+2n)} \tag{5.6}$$

Since f_cT is an integer, we have

$$\sin\pi(f-f_c)T\ e^{-j\pi(f-f_c)T(1+2n)} = \sin\pi fT\ e^{-j\pi fT(1+2n)} \tag{5.7}$$

and

$$\sin\pi(f+f_c)T\ e^{-j\pi(f+f_c)T(1+2n)} = \sin\pi fT\ e^{-j\pi fT(1+2n)} \tag{5.8}$$

Thus

$$B_n(f) = \frac{1}{2} A_n T \frac{\sin\pi fT}{\pi(f-f_c)T} e^{-j\pi fT(1+2n)} + \frac{1}{2} A_n T \frac{\sin\pi fT}{\pi(f+f_c)T} e^{-j\pi fT(1+2n)}$$

$$= A_n \frac{fT\sin\pi fT}{\pi(f^2-f_c^2)T} e^{-j\pi fT(1+2n)} \tag{5.9}$$

The output $I_n(t)$ of the filter $H(f)$ in the time interval $0 \le t \le T$ due to the n^{th} bit can be determined to be

$$I_n(t) = \int_{-\infty}^{\infty} B_n(f)H(f)e^{j2\pi ft}\,df$$

$$= \int_{f_c-B}^{f_c+B} A_n T \frac{f\sin\pi fT}{\pi(f^2-f_c^2)T} e^{-j\pi fT(1+2n)} \cdot \frac{1}{2}T \frac{\sin\pi fT}{\pi(f-f_c)T} e^{-j\pi fT} e^{j2\pi ft}\,df$$

$$+ \int_{-f_c-B}^{-f_c+B} A_n T f \frac{\sin\pi fT}{\pi(f^2-f_c^2)T} e^{-j\pi fT(1+2n)} \cdot \frac{1}{2}T \frac{\sin\pi fT}{\pi(f+f_c)T} e^{-j\pi fT} \cdot e^{j2\pi ft}\,df$$

$$0 \le t \le T \tag{5.10}$$

Changing variables and simplifying, we obtain

$$I_n(t) = \frac{A_n T}{2}\left[\frac{2}{\pi}\int_o^{\pi BT} \frac{\sin^2 x}{x^2} \cos 2x(1+n-\frac{t}{T})\,dx\right.$$

$$\left. - \frac{2}{\pi}\int_o^{\pi BT} \frac{\sin^2 x}{(2\pi f_c T)^2 - x^2} \cos 2x(1+n-\frac{t}{T})\,dx\right] \qquad 0 \le t \le T \tag{5.11}$$

For t=T, we have

$$I_n(T) = \frac{A_n T}{2}\left[J(BT,n) - C(BT,f_c T,n)\right] \tag{5.12}$$

where

$$J(BT,n) = \frac{2}{\pi}\int_{0}^{\pi BT} \frac{\sin^2 x}{x^2} \cos 2nx \; dx \quad \text{(see Equation (3.5)}$$

and

$$C(BT,f_cT,n) = \frac{2}{\pi}\int_{0}^{\pi BT} \frac{\sin^2 x}{(2\pi f_cT)^2 - x^2} \cos 2nx \; dx \quad (5.13)$$

Both $J(BT,n)$ and $C(BT,f_cT,n)$ are even functions of n.

Also from Equation (3.7)

$$J(BT,0) = \frac{2}{\pi}\int_{0}^{\pi BT} \frac{\sin^2 x}{x^2} \, dx = \frac{2}{\pi}\left[S_i(2\pi BT) - \frac{\sin^2 \pi BT}{\pi BT}\right], \quad (5.14)$$

(5.15)

Similarly

$$C(BT,f_cT,0) = \frac{2}{\pi}\int_{0}^{\pi BT} \frac{\sin^2 x}{(2\pi f_cT)^2 - x^2} \, dx$$

which can be evaluated as (see Appendix E)

$$C(BT,f_cT_0) = \begin{cases} \dfrac{1}{4\pi^2 f_cT}\left[\ln \dfrac{2\pi f_cT + \pi BT}{2\pi f_cT - \pi BT} + C_i(|4\pi f_cT - 2\pi BT|) - C_i(4\pi f_cT + 2\pi BT)\right] & B \neq 2f_c \quad (5.16) \\ \dfrac{1}{4\pi^2 f_cT}\left[0.5772 + \ln(8\pi f_cT) - C_i(8\pi f_cT)\right] & B = 2f_c \quad (5.17) \end{cases}$$

where

$$C_i(y) = -\int_y^{\infty} \frac{\cos x}{x} \, dx$$

is a cosine integral, also a tabulated value. $C(BT,f_cT,n)$ can also be expressed in terms of the function $C(w,y,0)$

$$\begin{aligned} C(BT,f_cT,n) &= \frac{n+1}{2} C[(n+1)BT,(n+1)f_cT,0] \\ &- nC[nBT,nf_cT,0] \\ &+ \frac{n-1}{2} C[(n-1)BT,(n-1)f_cT,0] \end{aligned} \quad (5.18)$$

The signal at the output of the integrator sampled at t=T due to an infinite bit train can then be given by

$$\begin{aligned} W &= \sum_{n=-\infty}^{\infty} I_n \ (T) \\ &= \frac{A_0T}{2} [J(BT,0)-C(BT,f_cT,0)] \\ &+ \frac{AT}{2} \sum_{n=1}^{\infty} (\frac{A_n+A_{-n}}{A}) \ [J(BT,n)-C(BT,f_cT,n)] \end{aligned} \quad (5.19)$$

The first term is the desired signal and the second term is the interference on the signal under detection. Note that each $J(BT,n)$ and $C(BT,f_cT,n)$ is less than or equal to one for any n, BT, and f_cT. Also as $B\to\infty$, $J(BT,0)\to 1$, $J(BT,n)\to 0$ (for $n\neq 0$), $C(BT,f_cT,n)\to 0$ and $W\to A_0T/2$ as expected for an infinite bandwidth. $J(BT,n)$ represents the effect of intersymbol interference on the bit under detection and $C(BT,f_cT,n)$ represents the effect of aliasing. Also note when $f_c >> 1/T$, $C(BT, f_cT, n) \to 0$ as expected and the effect of aliasing is insignificant. Table 5.1 gives some values of $C(BT, f_cT, n)$.

The output of the integrator due to both signal and noise can now be given by

$$y = \sum_{n=-\infty}^{\infty} I_n(T) + n_1 \qquad = I_0(T) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} I_n(T) + n_1$$

$$= \frac{A_0 T}{2} \; [J(BT,0) - C(BT, f_cT, 0)]$$

$$+ \frac{AT}{2} \sum_{n=1}^{\infty} \frac{(A_n + A_{-n})}{A} [J(BT,n) - C(BT, f_cT, n)] + n_1 \qquad (5.20)$$

where n_1 is the response of the receiver to the channel noise $n(t)$.

The variance of the noise n_1 can be obtained as

$$\sigma_1^2 = \int_{-\infty}^{\infty} \frac{N_0}{2} \; H(f)^{\;2} \, df$$

$$= \int_{f_c - B}^{f_c + B} \frac{N_0}{2} \cdot \frac{T^2}{4} \cdot \frac{\sin^2 \pi f T}{\pi^2 (f - f_c)^2 T^2} \; df$$

$$+ \int_{-f_c - B}^{-f_c + B} \frac{N_0}{2} \cdot \frac{T^2}{4} \cdot \frac{\sin^2 \pi f T}{\pi^2 (f + f_c)^2 T^2} \, df \qquad (5,21)$$

Changing variables and simplifying, σ_1^2 becomes

$$\sigma_1^2 = \frac{N_0 T}{4} J(BT,0) \qquad (5.22)$$

Table 5.1

Some Values of $C(BT, f_c T, n)$

$f_c T = 1$

BT	$C(BT,f_cT,0)$	$C(BT,f_cT,1)$	$C(BT,f_cT,2)$	$C(BT,f_cT,3)$	$C(BT,f_cT,4)$	$C(BT,f_cT,5)$
0.5	0.0131	-0.0066	0.0001	-0.0000	0.0000	-0.0000
0.6	0.0184	-0.0116	0.0042	-0.0028	0.0013	-0.0001
0.8	0.0258	-0.0144	0.0003	0.0010	0.0007	-0.0002
0.9	0.0271	-0.0137	-0.0003	-0.0001	0.0001	0.0001
1.0	0.0273	-0.0135	-0.0001	-0.0000	-0.0000	-0.0000
1.2	0.0292	-0.0124	-0.0007	-0.0010	-0.0008	0.0002
1.5	0.0511	-0.0267	0.0016	-0.0006	0.0003	-0.0002
2.5	0.0575	-0.0299	0.0016	-0.0006	0.0003	-0.0002
			$f_c T = 2$			
0.5	0.0032	-0.0016	0.0000	0.0000	0.0000	0.0000
0.6	0.0044	-0.0028	0.0009	-0.0006	0.0003	-0.0000
0.8	0.0061	-0.0034	0.0001	0.0002	0.0001	-0.0000
0.9	0.0064	-0.0033	-0.0000	-0.0000	-0.0000	-0.0000
1.0	0.0064	-0.0032	-0.0000	-0.0000	-0.0000	-0.0000
1.2	0.0068	-0.0030	-0.0001	-0.0002	-0.0001	0.0000
1.5	0.0100	-0.0050	0.0000	-0.0000	0.0000	-0.0000
2.5	0.0187	-0.0094	0.0001	-0.0000	0.0000	0.0000
			$f_c T = 3$			
0.5	0.0014	-0.0007	0.0000	-0.0000	0.0000	-0.0000
0.6	0.0019	-0.0012	0.0004	-0.0003	0.0001	-0.0000
0.7	0.0027	-0.0015	0.0000	0.0001	0.0001	-0.0000
0.9	0.0028	-0.0014	0.0000	-0.0000	0.0000	0.0000
1.0	0.0028	-0.0014	-0.0000	-0.0000	-0.0000	-0.0000

Table 5.1 (continued)

BT	$C(BT,f_cT,0)$	$C(BT,f_cT,1)$	$C(BT,f_cT,2)$	$C(BT,f_cT,3)$	$C(BT,f_cT,4)$	$C(BT,f_cT,5)$
1.2	0.0029	-0.0013	-0.0000	-0.0001	-0.0001	0.0000
1.5	0.0043	-0.0022	0.0000	-0.0000	0.0000	-0.0000
2.5	0.0075	-0.0038	0.0000	-0.0000	0.0000	0.0000

5.2.1 Probability of Bit-Error Using the Averaging Method

Recall that the output of the integrator due to both signal and noise is given by Equation (5.20)

$$y = W + n_1$$
$$= \frac{A_0 T}{2}\,[J(BT,0) - C(BT,f_cT,0)]$$
$$+ \frac{AT}{2}\sum_{n=1}^{\infty}\frac{(A_n+A_{-n})}{A}\,[J(BT,n) - C(BT,f_cT,n)] + n_1 \tag{5.23}$$

Using Equation (3.18) and Equation (3.19), the probability of bit error for a particular bit pattern is

$$P_{ei} = \frac{1}{\sqrt{\pi}}\int_{Z_i}^{\infty} e^{-u^2}\,du = \frac{1}{2}\,[1-\mathrm{erf}(Z_i)] \tag{5.24}$$

where

$$Z_i = \frac{W_i}{\sqrt{2}\sigma_1}$$
$$= \frac{\frac{AT}{2}\{J(BT,0)-C(BT,f_cT,0)+\sum_{n=1}^{\infty}(\frac{A_n+A_{-n}}{A})[J(BT,n)-C(BT,f_cT,n)]\}}{\sqrt{2}\sqrt{\frac{N_0T}{4}J(BT,0)}}$$
$$= \sqrt{\frac{E}{N_0}D_i^{\,2}(BT,f_cT)} \tag{5.25}$$

and $E = A^2T/2$, the energy per bit for the PSK signal (see Chapter II),

and

$$D_i^2(BT,f_cT) = \frac{\{J(BT,0)-C(BT,f_cT,0)+\sum_{n=1}^{\infty}(\frac{A_n+A_{-n}}{A})[J(BT,n)-C(BT,f_cT,n)]\}^2}{J(BT,0)}$$

Thus D_i^2 (BT,f_cT) as a function of system bandwidth, carrier frequency and bit patterns can be considered as the degradation of signal-to-noise ratio and can be calculated easily.

From Table 3.1 and Table 5.1, it can be seen that $|J(BT,n) - C(BT,f_cT,n)| << [J(BT,0) - C(BT,f_cT,0)]$ when $n>5$. Thus the effects of the interference can be confined to the nearest 10 bits. There is a total of 1024 different patterns. Thus the of error is given by

$$P_e = \frac{1}{1024} \sum_{i=1}^{1024} P_{ei}$$

$$= \frac{1}{1024} \sum_{i=1}^{1024} \frac{1}{2} (1-\text{erf}(Z_i)) \qquad (5.27)$$

The upper bound P_{emax}, and lower bound P_{emin} can be obtained in a fashion similar to those in Chapter III (Equations (3.23) and (3.24))

$$P_{emax} = \frac{1}{2}[1-\text{erf}(\sqrt{\frac{E}{N_0} \frac{[J(BT,0)-CBT,f_cT,0)-\sum_{n=1}^{\infty} 2\ J(BT,n)-C(BT,f_cT,n)\ ]^2}{J(BT,0)}})] \qquad (5.28)$$

$$P_{emin} = \frac{1}{2}[1-\text{erf}(\sqrt{\frac{E}{N_0} \frac{[J(BT,0)-C(BT,f_cT,0)]^2}{J(BT,0)}})] \qquad (5.29)$$

5.2.2 Probability of Bit-Error Using the Series Expansion Method

Dividing both sides of Equation (5.20) by AT/2, we have

$$X = Z_0 J_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} Z_n J_n + N \tag{5.30}$$

where

$$X = \frac{Y}{AT/2} \quad , \; A_n = \frac{A_n T}{AT} = \pm 1, \; J_n = J(BT,n) - C(BT, f_c T, n)$$

and

$$N = \frac{n_1}{(\frac{AT}{2})}$$

The variance of N is

$$\sigma_N^2 = E[\frac{n_1^2}{(AT/2)^2}]$$

$$= \frac{4}{A^2 T^2} \quad \sigma_1^2$$

$$= \frac{4}{A^2 T^2} \cdot \frac{N_0 T}{4} \quad J(BT,0) = \frac{J(BT,0)}{2(\frac{E}{N_0})} \tag{5.31}$$

Equation (5.30) is of the same form as Equation (3.25). Thus the probability of bit-error is given by Equation (3.49)

$$P_e = P_{e1} + P_{e2} \tag{5.32}$$

where

$$P_{e1} = \frac{1}{2}\,[1-\mathrm{erf}(\frac{J_0}{\sqrt{2}\sigma_N})]$$

$$= \frac{1}{2}[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}\,\frac{(J(BT,0)-C(BT,f_cT,0))^2}{J(BT,0)}})] \quad (5.33)$$

and

$$P_{e2} = \sum_{n=1}^{\infty} (-1)^{n+1}\, b_{2n}\, G_{2n-1} \quad (5.34)$$

b_{2n} can be evaluated using Equation (3.59). G_{2n-1} can be given by Equation (3.47)

$$G_{2n-1} = -\frac{J_0}{\sigma_N^2}\, G_{2n-2} - \frac{2n-2}{\sigma_N^2}\, G_{2n-3}$$

$$= 2(\frac{E}{N_0})(\frac{J(BT,0)-C(BT,f_cT,0)}{J(BT,0)}\, G_{2n-2}$$

$$- \frac{2n-2}{J(BT,0)}\, G_{2n-3}) \quad (5.35)$$

From Table 3.1 and Table 5.1, the coefficient b_{2n} can be calculated accurately using only the terms from J_{-5} to J_5 (i.e. from $J(BT,-5) - C(BT,f_cT,-5)$ to $J(BT,5) - C(BT,f_cT,5)$). In other words, the effects of interference can be confined to the 10 nearest bits without significant error. The series for P_{e2} also converges rapidly so that it can be evaluated accurately using only 10 terms in the series. As $B\to\infty$, $b_{2n}\to 0$, $J(BT,0)\to 1$, $C(BT,f_cT,0)\to 0$ and $P_e = 1$, $P_{e1} = 1/2(1-\mathrm{erf}(\frac{E}{N_0}))$ as expected for the infinite bandwidth case (see Chapter II).

The resulting probability of error, P_e, for a random PSK signal as a function of signal-to-noise ratio E/N_0 and bandwidth-bit duration product BT for various carrier frequencies is shown in Figures 5.3, 5.4, 5.5, 5.6 and 5.7. The results agree with those found using the averaging method.

Table 5.2 lists P_e, P_{e1}, and P_{e2} as a function of signal-to-noise ratio E/N_0 for some values of BT and f_cT. It can be seen from Table 5.2 that, as predicted, P_e is close to P_{e1} when E/N_0 is small and close to P_{e2} when E/N_0 is large. Also as the bandwidth of IF filter becomes wider, the interference becomes less.

From Figures 5.3, 5.4, 5.5, 5.6, 5.7 and Table 5.1, it can be seen that for $f_c>3/T$, the effect of aliasing can be neglected, and the results are the same as obtained in Chapter III for the baseband NRZ system. In other words, the modulation has no influence on the detection of PSK signals for carrier frequencies greater than three times the bit rate. This is significant result, because it can serve as a guideline for the design of an aliasing free frequency division multiplexing (FDM) transmission system.

For BT=2.5, $f_cT>3$, the additional power needed to give the performance same as an optimum case (infinite bandwidth) is only 0.3 dB(see Table 3.1). This suggests that an IF bandwidth of five times the bit rate is wide enough to achieve the optimum results for the PSK system.

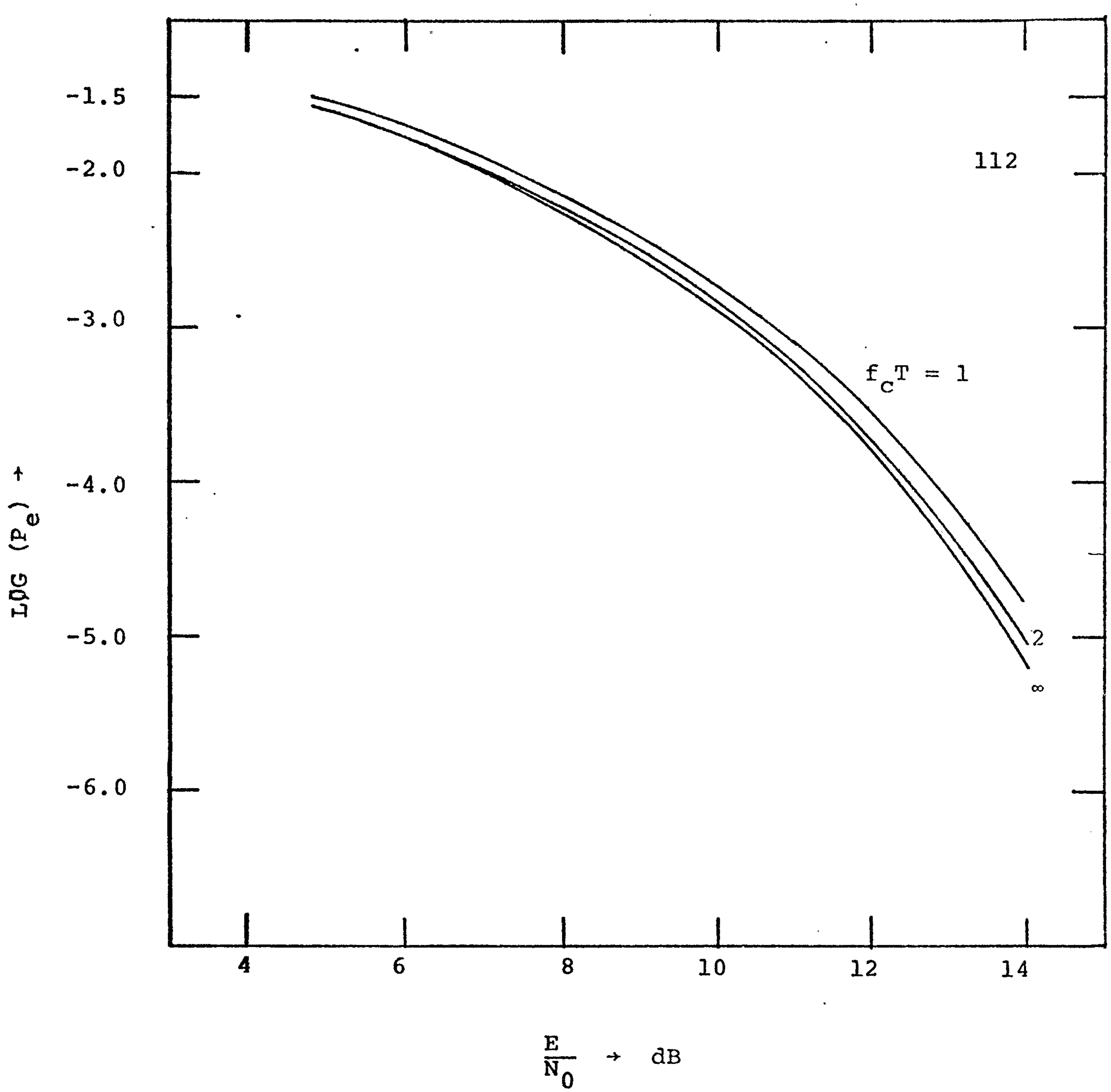

Figure 5.3- P_e vs $\frac{E}{N_0}$ for the PSK System with BT = 0.5

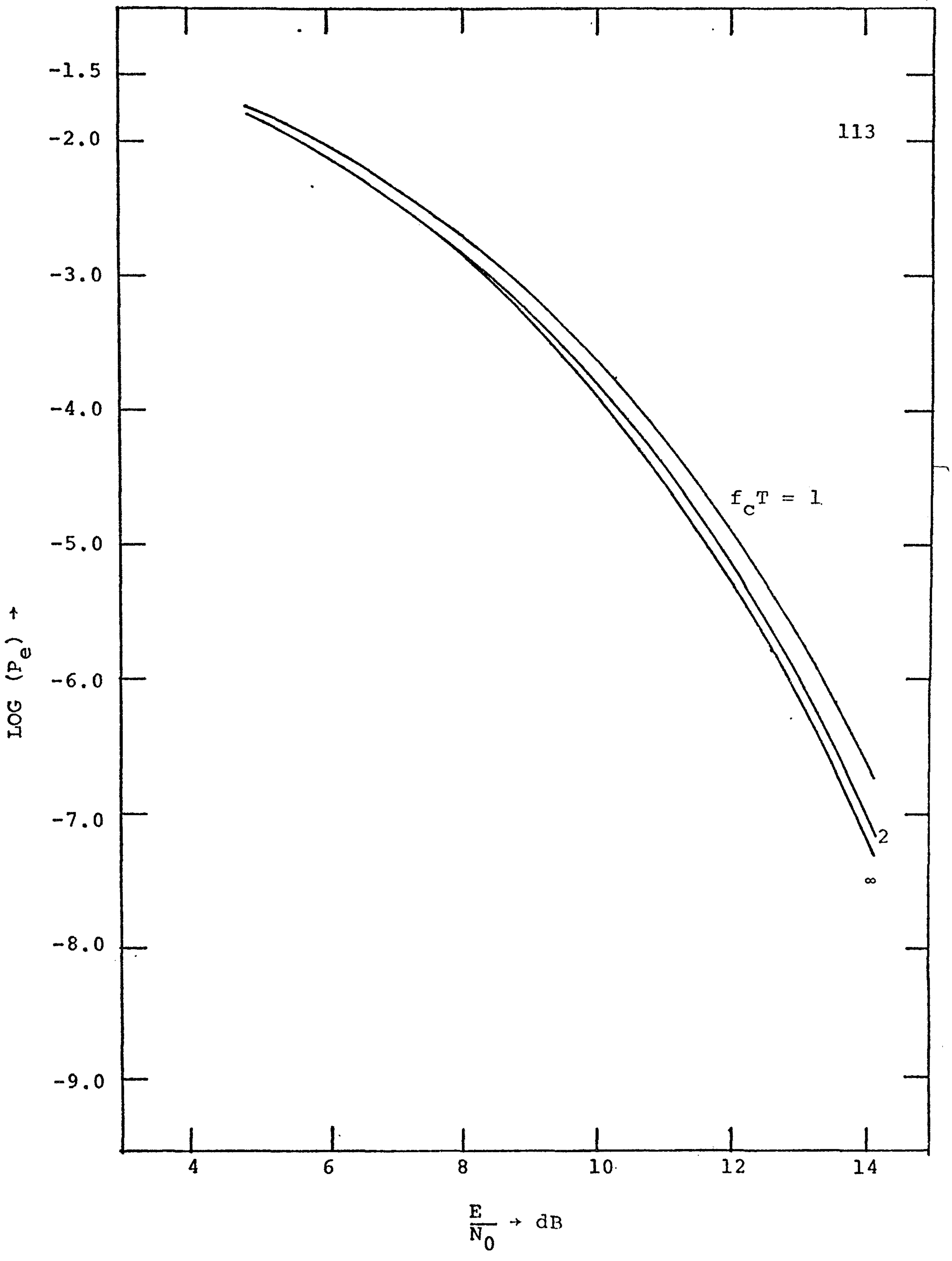

Figure 5.4 - P_c vs $\frac{E}{N_o}$ for the PSK System with BT = 0.6

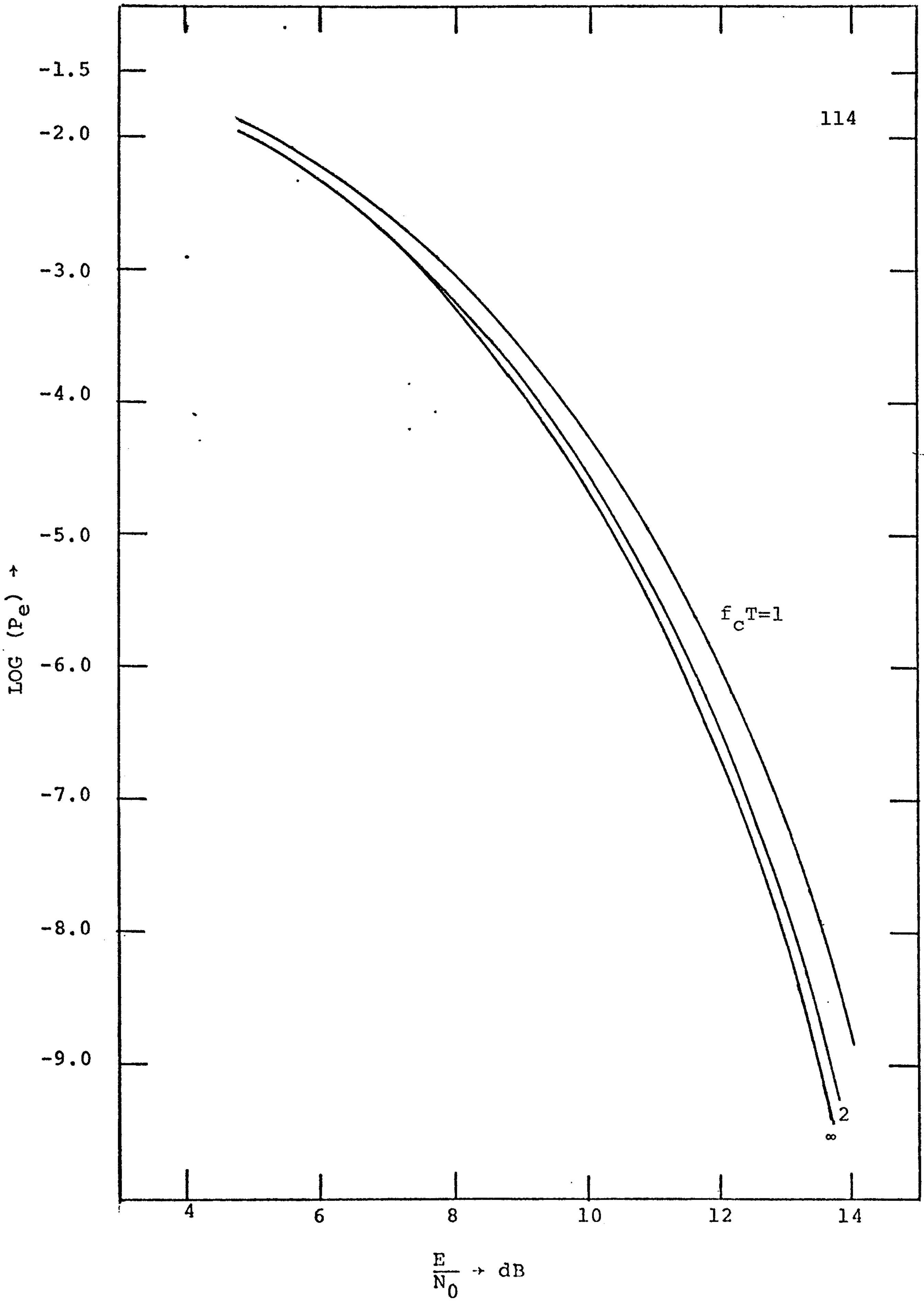

Figure 5.5 - P_e vs $\frac{E}{N_0}$ for the PSK System with BT=0.8

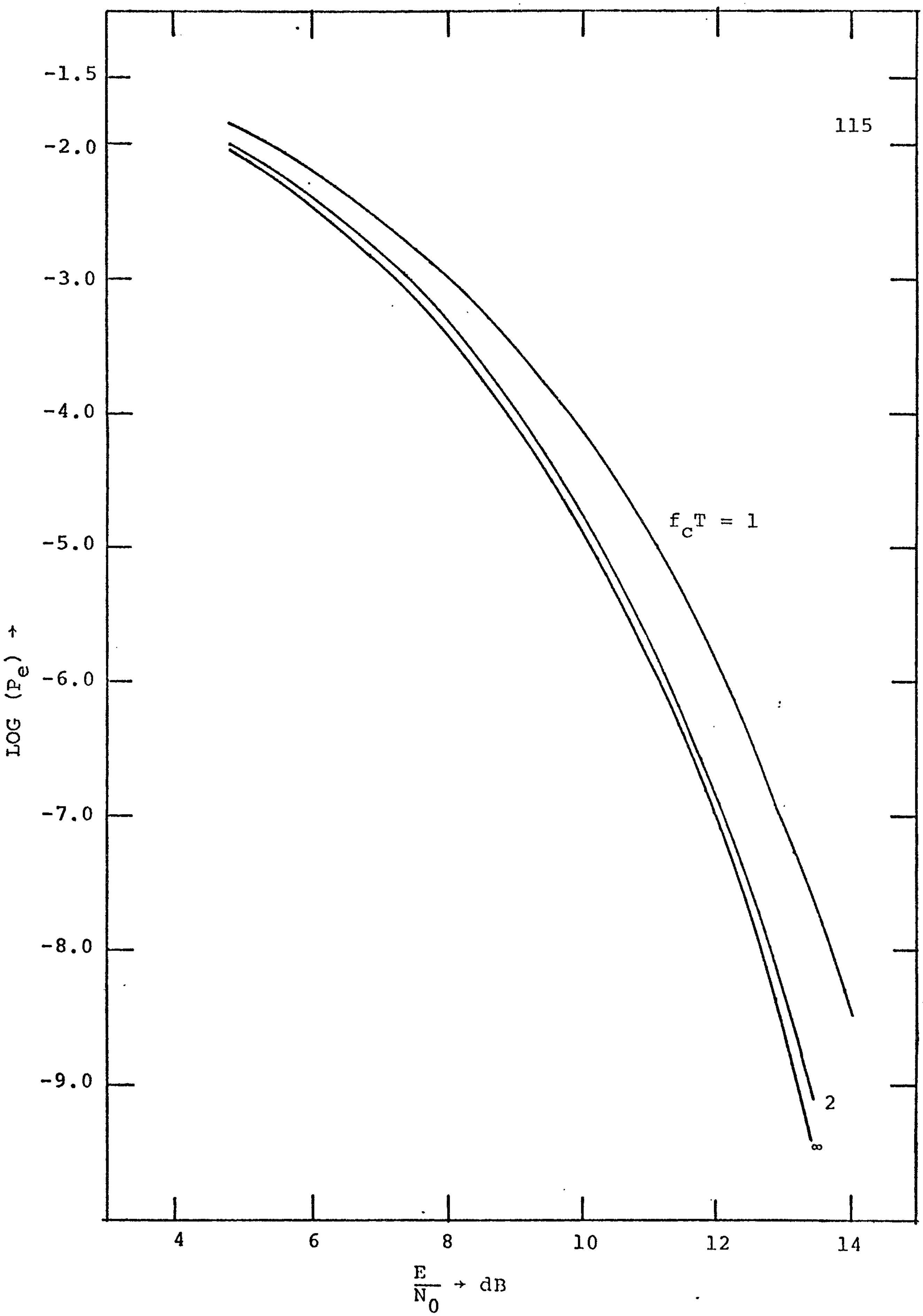

Figure 5.6 – P_e vs $\frac{E}{N_0}$ for the PSK System with BT=1.5

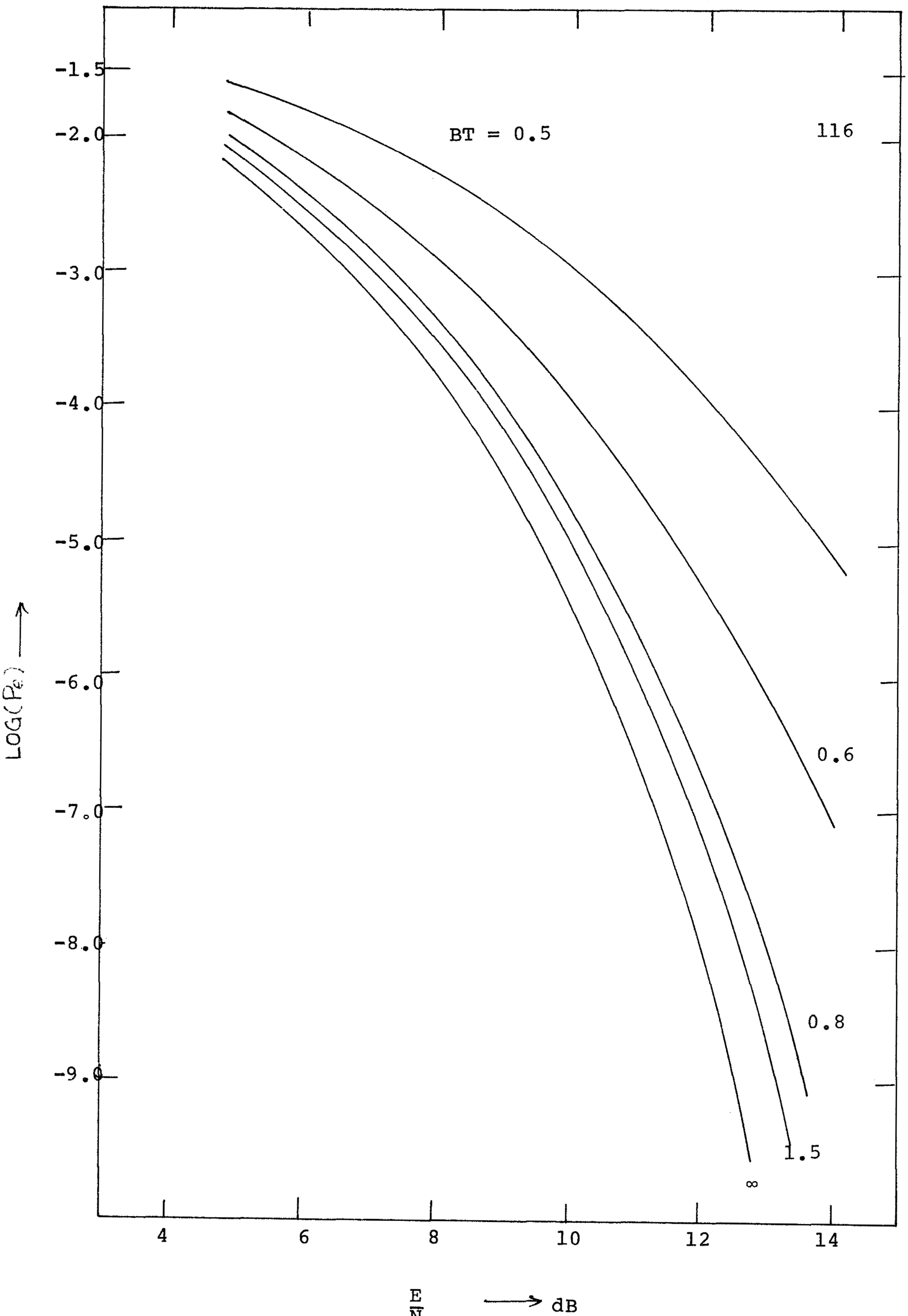

Figure 5.7 P_e vs $\frac{E}{N_0}$ for the PSK System with $f_cT = \infty$

Table 5.2

Values of P_e, P_{e1} and P_{e2} vs $\frac{E}{N_0}$ for the PSK System with BT=1.0 and f_cT = 1.0, 3.0

$f_cT = 1.0$ BT = 1.0

$\frac{E}{N_0}$ (dB)	$\text{Log}(P_e)$	$\text{Log}(P_{e1})$	$\text{Log}(P_{e2})$
0.00	-1.008	-1.016	-2.744
4.77	-1.862	-1.921	-2.757
6.99	-2.600	-2.748	-3.137
8.45	-3.280	-3.548	-3.617
9.54	-3.925	-4.334	-4.139
10.41	-4.544	-5.111	-4.682
11.14	-5.147	-5.882	-5.285
11.76	-5.737	-6.648	-5.794
12.30	-6.319	-7.413	-6.356
12.79	-6.895	-8.173	-6.919
13.22	-7.467	-8.931	-7.482
13.62	-8.035	-9.687	-8.045
13.98	-8.601	-10.441	-8.607

Table 5.2

(Continued)

$f_cT = 3.0$ BT = 1.0

$\frac{E}{N_o}$ (dB)	$\text{Log}(P_e)$	$\text{Log}(P_{e1})$	$\text{Log}(P_{e2})$
0.00	-1.039	-1.045	-2.945
4.77	-1.952	-1.993	-3.002
6.99	-2.759	-2.863	-3.432
8.45	-3.515	-3.705	-3.965
9.54	-4.238	-4.533	-4.544
10.41	-4.937	-5.352	-5.147
11.14	-5.618	-6.165	-5.764
11.76	-6.288	-6.973	-6.388
12.30	-6.948	-7.779	-7.017
12.79	-7.601	-8.581	-7.649
13.22	-8.248	-9.381	-8.282
13.62	-8.892	-10.179	-8.915
13.98	-9.533	-10.976	-9.549

5.3 Amplitude Shift Keying

The ASK system is the same as a PSK system shown in Figure 5.1 except that A_n is equal to A or zero instead of A or -A (see Chapter II). The threshold setting is AT/4 as will be shown later.

Equation (5.18) can be rewritten as

$$Y = \left(\frac{A_0'T}{4} + \frac{AT}{4}\right) [J(BT,0)-C(BT,f_cT,0)]$$

$$+ \frac{T}{2} \sum_{n=1}^{\infty} \left[\frac{(A_n'+A_{-n}')}{2} + A\right][J(BT,n)-C(BT,f_cT,n))] + n_1 \tag{5.36}$$

where

$$A_n' = A \text{ or } -A$$

Thus

$$Y = \frac{A_0'T}{4} [J(BT,0)-C(BT,f_cT,0)]$$

$$+ \frac{AT}{4} \sum_{n=1}^{\infty} \frac{A_n' + A_{-n}'}{A} [J(BT,n)-C(BT,f_cT,n))] \tag{5.37}$$

$$+ \frac{AT}{4} \sum_{n=-\infty}^{\infty} J(BT,n)$$

$$- \frac{AT}{4} \sum_{n=-\infty}^{\infty} C(BT,f_cT,n)$$

$$+ n_1 \tag{5.38}$$

But $\sum_{n=-\infty}^{\infty} J(BT,n) = 1$, $\sum_{n=-\infty}^{\infty} C(BT,f_cT,n) = 0$ (See Appendix E), we have

$$y = \frac{A_0'T}{4} [J(BT,0)-C(BT,f_cT,0)]$$

$$+ \frac{AT}{4} \sum_{n=1}^{\infty} \frac{A_n'+A_{-n}'}{A} [J(BT,n)-C(BT,f_cT,n)]$$

$$+ \frac{AT}{4} + n_1 \tag{5.39}$$

Thus the optimum threshold should be set at AT/4 (if the gain of the integrator is A, d will be $A^2T/4$, which agrees with the result of Chapter II). The decision error will occur whenever Y-AT/4 plus noise is greater than zero if a "0" is being sent and less than zero if a "1" is being sent. Let Y' = Y-AT/4, Equation (5.39) becomes

$$y' = \frac{A_0'T}{4} [J(BT,0)-C(BT,f_cT,0)]$$

$$+ \frac{AT}{4} \sum_{n=1}^{\infty} \frac{A_n' + A_{-n}'}{A} [J(BT,n)-C(BT,f_cT,n)]+ n_1 \tag{5.40}$$

Therefore we can treat the ASK system the same way as we treated the PSK system.

Dividing both sides of Equation (5.40) by AT/4, Equation (5.40) becomes

$$X = Z_0J_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Z_nJ_n + N \tag{5.41}$$

where $J_n = J(BT,n) - C(BT,f_cT,n)$, $Z_n = \pm 1$, and $N = n_1/(AT/4)$. Equation (5.41) is of the same form as Equation (5.30) except that the variance of N is

$$\sigma_N^2 = \frac{E[n_1^2]}{(\frac{AT}{4})^2}$$

$$= \frac{J(BT,0)}{2(\frac{E}{2N_0})} \tag{5.42}$$

where $E = A^2T/4$ is the average energy per bit for the ASK signals (see Chapter II).

By comparing Equation (5.42) with Equation (5.31), immediately it is clear that the ASK system requires twice as much energy to achieve the same performance as the PSK system.

This can also be verified using the averaging method. Using Equations (3.20) and (5.40), the probability of error for a particular bit pattern is

$$P_{ei} = \frac{1}{2}\ (1-\text{erf}(Z_i))$$

where

$$Z_i = \sqrt{\frac{\frac{A^2T}{4}\ \left(J(BT,0)-C(BT,f_cT,0)+\sum_{n=1}^{\infty} \frac{A_n'+A_n'}{A}\ J(BT,n)-C(BT,f_cT,n)\right)^2}{2N_0J(BT,0)}} \tag{5.43}$$

Since $\frac{A^2T}{4}$ is the average energy per bit, we have

$$Z_i = \sqrt{\frac{E}{2N_0}\ \ D_i^2(BT,f_cT)} \tag{5.44}$$

Compared with Equation (5.25), the ASK system indeed requires twice as much energy to achieve the same performance as the PSK system. Thus regardless of the restriction of the bandwidth, the ASK system always requires 3dB more power than the PSK system. The probabilities of bit-error plotted in Figures 5.3, 5.4, 5.5, 5.6 and 5.7 can all be used for the ASK system except that all the curves must be moved to the right by 3dB.

5.4 Frequency Shift Keying

The bandlimited FSK coherent communication system can be modeled as in Figure 5.8. $u_n(t)$ is the random NRZ signal with bit period T, and amplitude +1 or -1. Bandpass filter $B_1(f)$ is centered at one carrier frequency $f_0 + \Delta f$ with bandwidth 2B, and bandpass filter $B_0(f)$ is centered at the other carrier frequency $f_0-\Delta f$ with bandwidth 2B.

As in Section 5.2 the model can be replaced by an equivalent one as shown in Figure 5.9, where

$$H_1(f)=\begin{cases} \frac{1}{2}\, T\, \dfrac{\sin \pi f T}{\pi(f-f_0-\Delta f)T}\, e^{-j\pi f T} & f_0+\Delta f-B \leq f \leq f_0+\Delta f+B \\ \frac{1}{2}\, T\, \dfrac{\sin \pi f T}{\pi(f+f_0+\Delta f)T}\, e^{-j\pi f T} & -f_0-\Delta f-B \leq f \leq -f_0-\Delta f+B \\ 0 & \text{elsewhere} \end{cases} \tag{5.45}$$

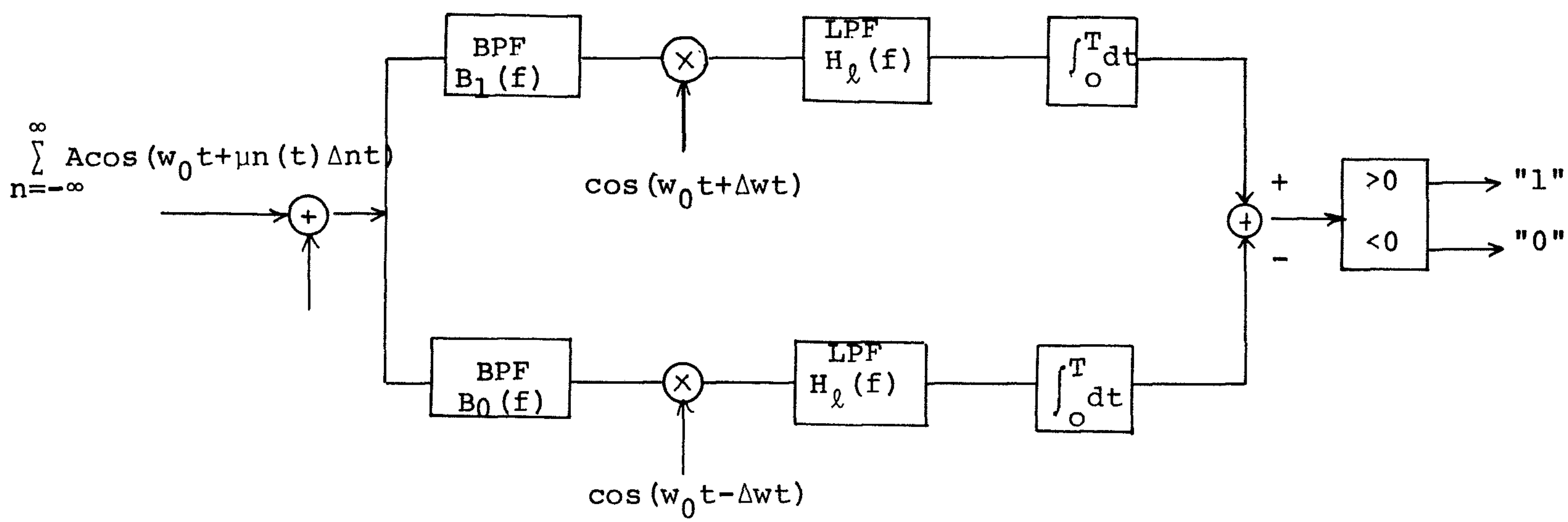

Figure 5.8 Bandlimited FSK Coherent Communication System

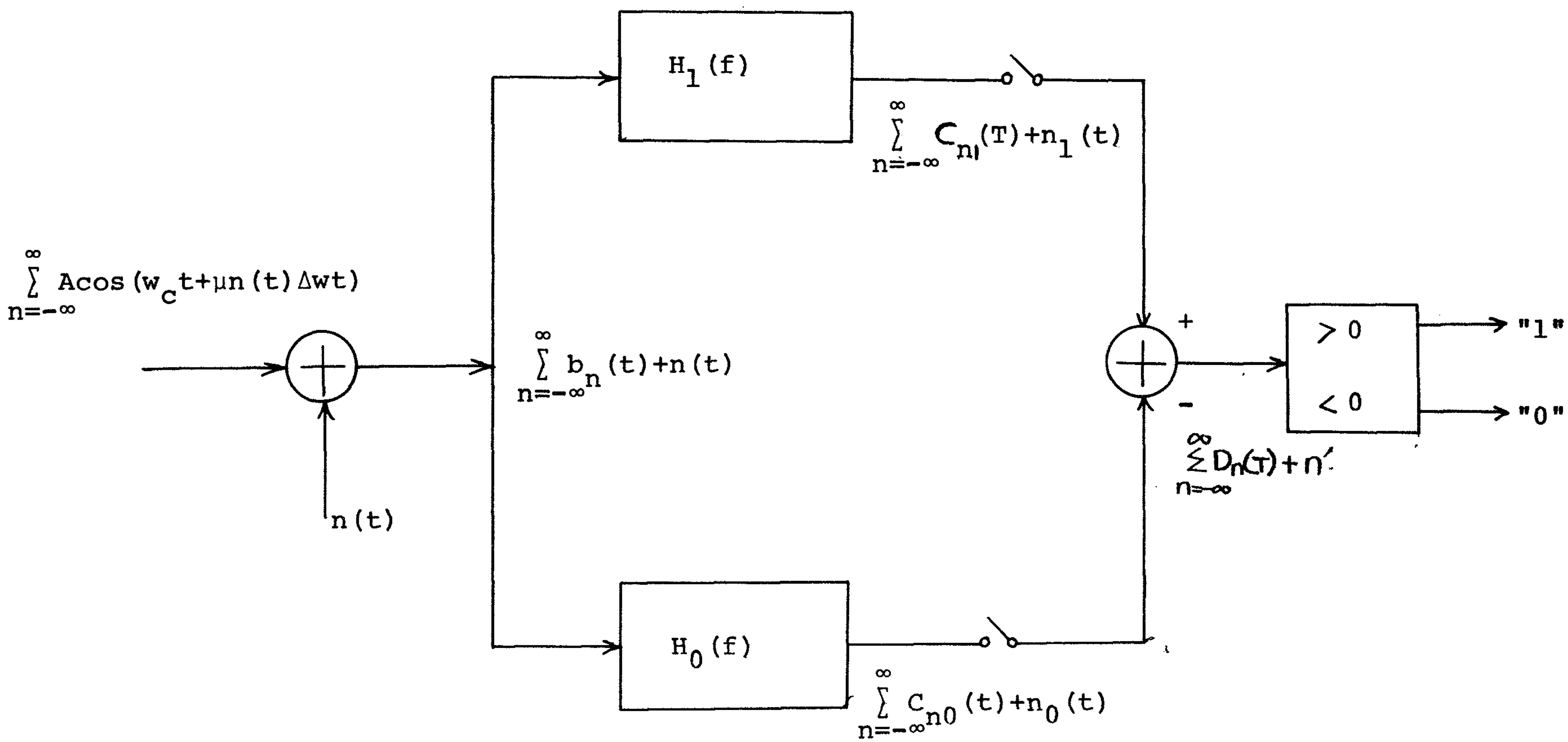

Figure 5.9 A Simplified FSK System

$$H_0(f)=\begin{cases}\frac{1}{2}\,T\,\dfrac{\sin\pi fT}{\pi(f-f_0+\Delta f)T}\,e^{-j\pi fT} & f_0-\Delta f-B\le f\le f_0-\Delta f+B \\ \frac{1}{2}\,T\,\dfrac{\sin\pi fT}{\pi(f+f_0-\Delta f)T}\,e^{-j\pi fT} & -f_0+\Delta f-B\le f\le f_0+\Delta f+B \\ 0 & \text{elsewhere}\end{cases} \tag{5.46}$$

The frequencies $f_0+\Delta f$ and $f_0-\Delta f$ for the two carrier tones are assumed to be the multiples of the bit rate (i.e. the signals are orthogonal) [34]. The n^{th} bit can be represented as

$$b_n(t) = A\cos(\omega_0 t+U_n\Delta\omega t) \qquad nT\le t\le(n+1)T$$

where

$$U_n = 1 \text{ or } -1$$

The Fourier transform of $b_n(t)$ can be written as

$$B_n(f) = \int_{nT}^{(n+1)T} A\cos(\omega_0 t+U_n\Delta\omega t)e^{-j2\pi ft}\,dt$$

$$= A_n fT\,\frac{\sin\pi fT}{\pi[f^2-(f_0+U_n\Delta f)^2]T}\,e^{-j\pi fT(1+2n)} \tag{5.47}$$

The output $C_{n1}(t)$ of the upper integrator due to the n^{th} bit can be determined to be

$$C_{n1}(t) = \int_{-\infty}^{\infty} B_n(f)\quad H_1(f)e^{j2\pi ft}\,df \tag{5.48}$$

The output $C_{n0}(t)$ of the lower integrator due to the n^{th} bit can also be determined to be

$$C_{n0}(t) = \int_{-\infty}^{\infty} B_n(f)H_0(f)e^{j2\pi ft}\,df \tag{5.49}$$

Thus the input to the decision device due to n^{th} bit sampled at t=T is

$$D_n(T) = C_{n1}(T) - C_{n0}(T)$$

$$= \int_{-\infty}^{\infty} B_n(f)[H_1(f) - H_0(f)]e^{j2\pi fT}\, df \tag{5.50}$$

Changing variables and simplifying, we have

$$D_n(T) = \frac{AT}{2} U_n\{J(BT,n) - C[BT,(f_0+U_n\Delta f)T,n] + C(BT,f_0T,n) + C(BT,\Delta fT,n)\} \tag{5.51}$$

The signal presented at the input of decision device due to the infinite bit train can then be expressed as

$$W = \sum_{n=-\infty}^{\infty} D_n(T) = D_0(T) + \sum_{n=-\infty}^{\infty} D_n(T)$$

$$= \frac{AT}{2} U_0\{J(BT,0) - C[BT,(f_0+U_0\Delta f)T,0] + C(BT,f_0T,0) + C(BT,\Delta fT,0)\}$$

$$+ \frac{AT}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} U_n\{J(BT,n) - C[BT,(f_0+U_n\Delta f)T,n] + C(BT,f_0T,n) + C(BT,\Delta fT,n)\} \tag{5.52}$$

Compared with Equation (5.19), it is apparent that in addition to the effects of aliasing on the bit under detection there exists signal crosstalk caused by the IF filtering which can be represented by $C(BT,f_0T,n) + C(BT,\Delta fT,n)$.

The noise present at the decision device can be written as

$$n' = n_1(T) - n_0(T)$$

$$= \int_{-\infty}^{\infty} n(Z)h_1(T-Z)dZ - \int_{-\infty}^{\infty} n(Z)h_0(T-Z)dZ \qquad (5.53)$$

where $h_1(t)$, $h_0(t)$ are the impulse responses of $H_1(f)$ and $H_0(t)$ respectively.

$n_1(T)$ and $n_0(T)$ are both Gaussian processes with zero mean and variance $\sigma_1^{\ 2}$ and $\sigma_0^{\ 2}$, where

$$\sigma_1^{\ 2} = \int_{\infty}^{\infty} \frac{N_0}{2} |H_1(f)|^2 \, df = \frac{N_0 T}{4} J(BT,0) \qquad (5.54)$$

and

$$\sigma_1^{\ 2} = \int_{\infty}^{\infty} \frac{N_0}{2} |H_0(f)|^2 \, df = \frac{N_0 T}{4} J(BT,0) \qquad (5.55)$$

Thus n' the difference of two Gaussian noise processes is still Gaussian process with zero mean and variance

$$\sigma^2 = \sigma_1^{\ 2} + \sigma_0^{\ 2} - 2E[n_1(T)n_0(T)] \qquad (5.56)$$

But $E[n_1(T)n_0(T)]$ can be written as [41]

$$E[n_1(T)n_0(T)] = \int_{-\infty}^{\infty} \frac{N_0}{2} \; H_1(f)H_0^*(f)df \qquad (5.57)$$

For $B>\Delta f$, we have

$$E[n_1(T)n_0(T)]$$

$$= \frac{N_0}{2}\int_{f_0+\Delta f-B}^{f_0-\Delta f+B} \frac{1}{2}\,T\,\frac{\sin\pi fT}{\pi(f-f_0-\Delta f)T}\,e^{-j\pi fT}\cdot\frac{1}{2}\,T\,\frac{\pi\sin\ fT}{\pi(f-f_0+\Delta f)T}\,e^{j\pi fT}\,df$$

$$+ \frac{N_0}{2}\int_{-f_0+\Delta f-B}^{-f_0-\Delta f+B} \frac{1}{2}\,T\,\frac{\sin\pi fT}{\pi(f+f_0+\Delta f)T}\,e^{-j\pi fT}\cdot\frac{1}{2}\,T\,\frac{\sin\pi fT}{\pi(f+f_0-\Delta f)T}\,e^{j\pi fT}\,df$$

(5.58)

Simplifying, we obtain

$$E[n_1(T)n_0(T)] = \frac{N_0T}{4}\,C[(B-\Delta f)T,\ \frac{\Delta fT}{2}\ ,\ 0] \qquad (5.59)$$

For $B\leq f$, we have

$$H_1(f)H_0^*(f) = 0 \qquad (5.60)$$

and

$$E[n_1(T)n_0(T)] = 0$$

Now we can express σ^2 as

$$\sigma^2 = \begin{cases} \dfrac{N_0T}{2}\,J(BT,0) - \dfrac{N_0T}{2}\,C[(B-\Delta f)T,\dfrac{\Delta fT}{2}\ ,\ \dfrac{\Delta fT}{2}\ ,\ 0] & B>\ \Delta f \\ \dfrac{N_0T}{2}\,J(BT,0) & B\leq\ \Delta f \end{cases} \qquad (5.61)$$

The input to the threshold device due to both signal and noise can now be given by

$$y = w + n'$$

$$= \frac{AT}{2} U_0 [J(BT,0) - C[BT,(f_0+U_0\Delta f)T,0] + C(BT,f_0T,0) + C(BT,\Delta fT,0)]$$

$$+ \frac{AT}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} U_n \; J(BT,n) - C[BT,(f_0+U_n\Delta f)T,n] + C(BT,f_0T,n) + C(BT,\Delta fT,n]$$

$$+ n' \qquad (5.62)$$

For carrier frequencies much greater than the bit rate, which is a practical assumption for the FSK systems [34], $C(BT,(f_0 \pm \Delta f)T,n)$ and $C(BT,f_0T,n)$ will approach zero. Then Equation (5.62) becomes

$$y = \frac{AT}{2} \quad U_0[J(BT,0) + C(BT,\Delta fT,0)]$$

$$+ \sum_{n=1}^{\infty} (U_n + U_{-n})[J(BT,n) + C(BT,\Delta fT,n)] \qquad +n' \qquad (5.63)$$

In the following, the probability of bit-error will be determined using the averaging method and the series expansion method based on Equation (5.63).

5.4.1 Probability of Bit-Error Using the Averaging Method

Using Equation (3.18) and Equation (5.63), the probability of bit-error for a particular bit pattern can be given by

$$P_{el} = \frac{1}{2} (1 - \mathrm{erf}(Z_i)) \qquad (5.64)$$

where

$$Z_i = \sqrt{\frac{E}{2N_0} D_i^2 (BT,\Delta fT)} \quad , \tag{5.65}$$

$E = A^2T/2$, the energy per bit for the FSK signals, and

$$D_i^2(BT,\Delta fT) = \frac{[J(BT,0)+C(BT,\Delta fT,0)+\sum_{n=1}^{\infty}(U_n+U_{-n})[J(BT,n)+C(BT,\Delta fT,n)]^2}{J(BT,0) - C[(B-\Delta f)T, \frac{\Delta fT}{2}, 0]} \tag{5.66}$$

The upper and lower bounds of the probabilities of bit-error can also be expressed as

$$P_{emax} = \frac{1}{2}[1-\mathrm{erf}(\sqrt{\frac{E}{2N_0} \frac{[J(BT,0)+C(BT,\Delta fT,0)+2\sum_{n=1}^{\infty}|J(BT,n)+C(BT,\Delta fT,n)|]^2}{J(BT,0) - C[(B-\Delta f)T, \frac{\Delta fT}{2}, 0]}})] \tag{5.67}$$

$$P_{emin} = \frac{1}{2}[1-\mathrm{erf}(\frac{E}{2N_0} \frac{[J(BT,0)+C(BT,\Delta fT,0)]^2}{J(BT,0)-C[(B-\Delta f)T,\frac{\Delta fT}{2},0]})] \tag{5.68}$$

From From Table 3.1 and Table 5.1, it can be seen that $|J(BT,n) + C(BT,\Delta fT,n)| << J(BT,0)+C(BT,\Delta fT,0)$ for $n>5$. Thus the effects of interference can be confined to the nearest 10 bits. The average probability of error now can be given by

$$P_e = \frac{1}{1024} \sum_{i=1}^{1024} \frac{1}{2} (1-\mathrm{erf}(Z_i)) \tag{5.69}$$

5.4.2 Probability of Bit-Error Using the Series Expansion Method

Dividing both sides of Equation (5.63) by AT/2, we obtain

$$X = Z_0 J_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Z_n J_n + N \tag{5.70}$$

where

$$Z_n = U_n = \pm 1, \; J_n = J(BT,n) + C(BT,\Delta fT,n),$$

and

$$N = \frac{n'}{(\frac{AT}{2})}$$

The variance of N is

$$\sigma_N^2 = \frac{E(n'^2)}{(\frac{A^2T^2}{4})} \tag{5.71}$$

Substituting Equation (5.61) into Equation (5.71), σ_N^2 becomes

$$\sigma_N^2 = \frac{\frac{N_0 T}{2}\{J(BT,0) - C[(B-\Delta f)T, \frac{\Delta fT}{2}, 0]\}}{\frac{A^2T^2}{4}}$$

$$= \frac{J(BT,0)-C[(B-\Delta f)T,\frac{\Delta fT}{2}, 0]}{2(\frac{E}{2N_0})} \tag{5.72}$$

Equation (5.72) is of the same form as Equation (3.27). Thus the probability of the bit-error is given by Equation (3.51)

$$P_e = P_{e1} + P_{e2} \tag{5.73}$$

where

$$P_{e1} = \frac{1}{2}\,[1-\mathrm{erf}(\frac{J_0}{\sqrt{2}\sigma_N})]$$

$$= \frac{1}{2}\,[1-\mathrm{erf}(\sqrt{\frac{E}{2N_0}\,\frac{[J(BT,0)+C(BT,\Delta fT,0)]^2}{J(BT,0)-C[(B-\Delta f)T,\frac{\Delta fT}{2},0]}})]$$

(5.74)

and

$$P_{e2} = \sum_{n=1}^{\infty} (-1)^{n+1}\, b_{2n} G_{2n-1} \tag{5.75}$$

b_{2n} can be evaluated using Equation (3.59). G_{2n-1} can be given by Equation (3.47)

$$G_{2n-1} = -\frac{J_0}{\sigma_N^2}\, G_{2n-2} - \frac{2n-2}{\sigma_N^2}\, G_{2n-3}$$

$$= -\frac{E}{N_0}\Big[\frac{J(BT,0)+C(BT,\Delta fT,0)}{J(BT,0)-C[(B-\Delta f)T,\frac{\Delta fT}{2},0]}\; G_{2n-2}$$

$$-\frac{2n-2}{J(BT,0)-C[(B-\Delta f)T,\frac{\Delta fT}{2},\,0]}\; G_{2n-3}]$$

(5.76)

From Table 5.1 and 5.2, it can be seen that $[J(BT,n)+C(BT,\Delta fT,n)]^2 << [J(BT,1)+C(BT,\Delta fT,1)]^2$ for $n>5$. Thus b_{2n} can be evaluated accurately using only the terms from J_{-5} to J_5. Therefore as for the averaging method the effects of the interference can be confined to the 10 nearest bits. Also the series for P_{e2} can be truncated to 10 terms with insignificant error. As $B\to\infty$, $J(BT,0)\to 1$, $C(BT,\Delta fT,0)\to 0$, $J(BT,n)\to 0$, $J(BT,\Delta fT,n)\to 0$, and $C((BT-\Delta fT),\ \Delta fT/2,\ 0)\to 0$, thus $b_{2n}\to 0$ and $P_e = P_{e1} = 1/2(1-\mathrm{erf}(\sqrt{\frac{E}{2N_0}}))$, the well known result as predicted in Chapter II for the infinite bandwidth FSK system.

Table 5.3 lists P_e, P_{e1}, and P_{e2} for some values of ΔfT and BT. Figures 5.10, 5.11, and 5.12 show the plots of the probability of bit-error for Δf equal to 0.5/T, 1.5/T, and 3.0/T respectively. Figure 5.13 shows the plots of the probability of bit-error as a function of ΔfT for E/N_0 equal to 10dB and 15 dB. The results obtained here agree with those obtained by the averaging method.

Comparing the results obtained by either method with those for the PSK system (Figure 5.7), it can be seen that for $\Delta f>3/T$, the performance of the FSK system is the same as that of the ASK system and is 3dB poorer than the PSK system on an average power basis. However, for $\Delta f\leq 3/T$, the performance of the FSK is better than that of the ASK. The reason for this is that the signal crosstalk $C(BT,\Delta fT,n)$ tends to reduce the

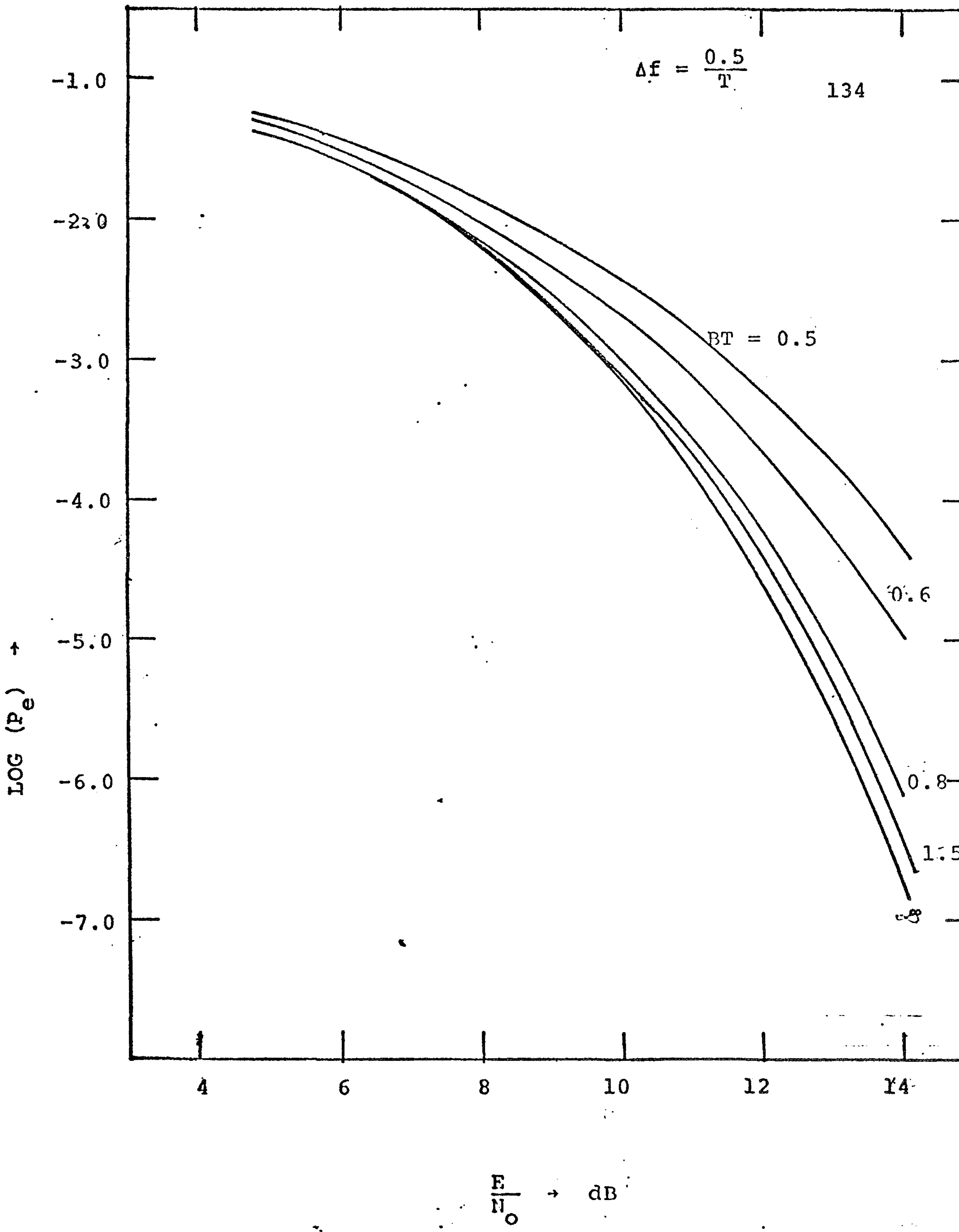

Figure 5.10 : P_e vs $\frac{E}{N_0}$ for the FSK System with $\Delta f = \frac{0.5}{T}$

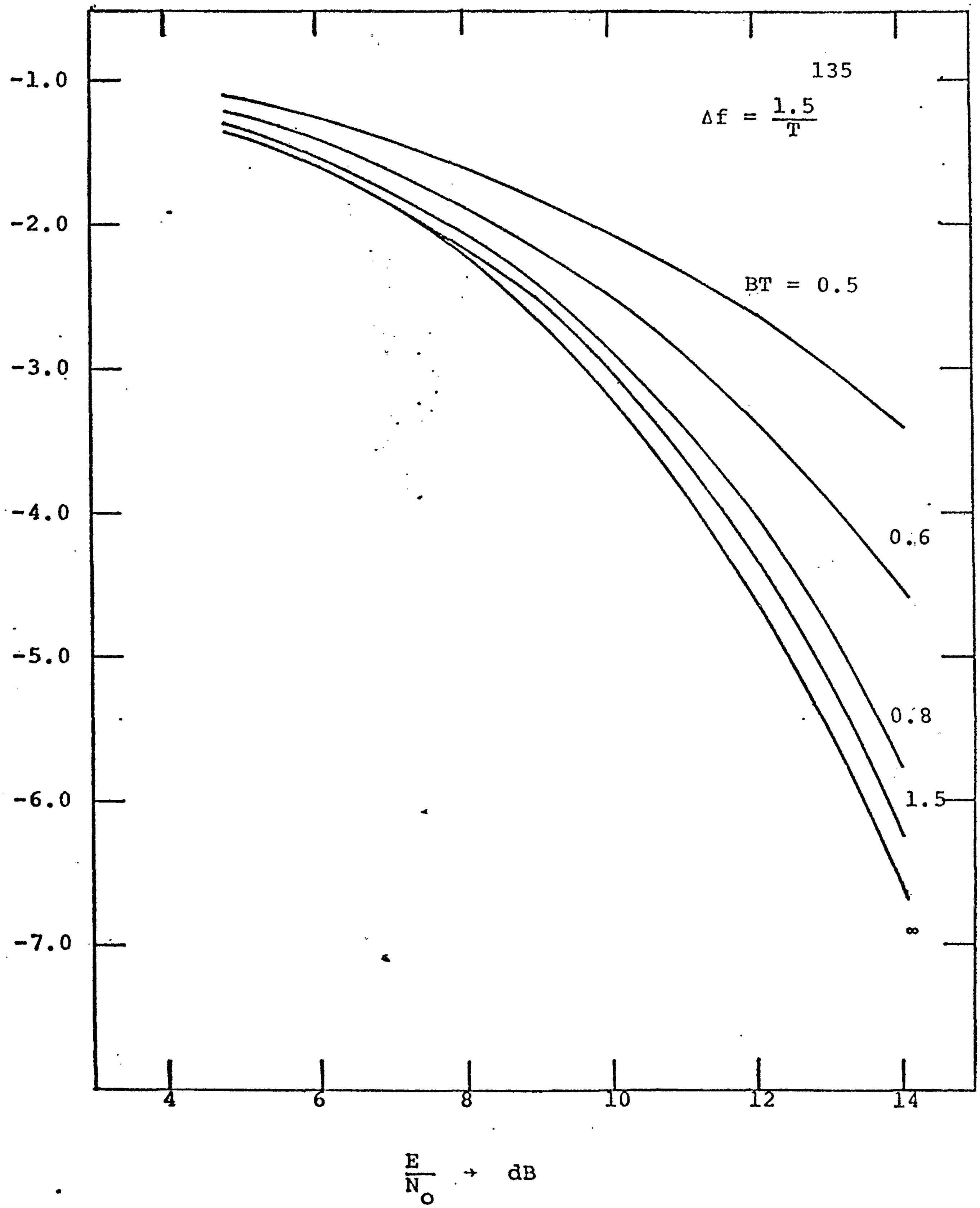

Figure 5.11 : P_e vs $\frac{E}{N_0}$ for the FSK System with $\Delta f = \frac{1.5}{T}$

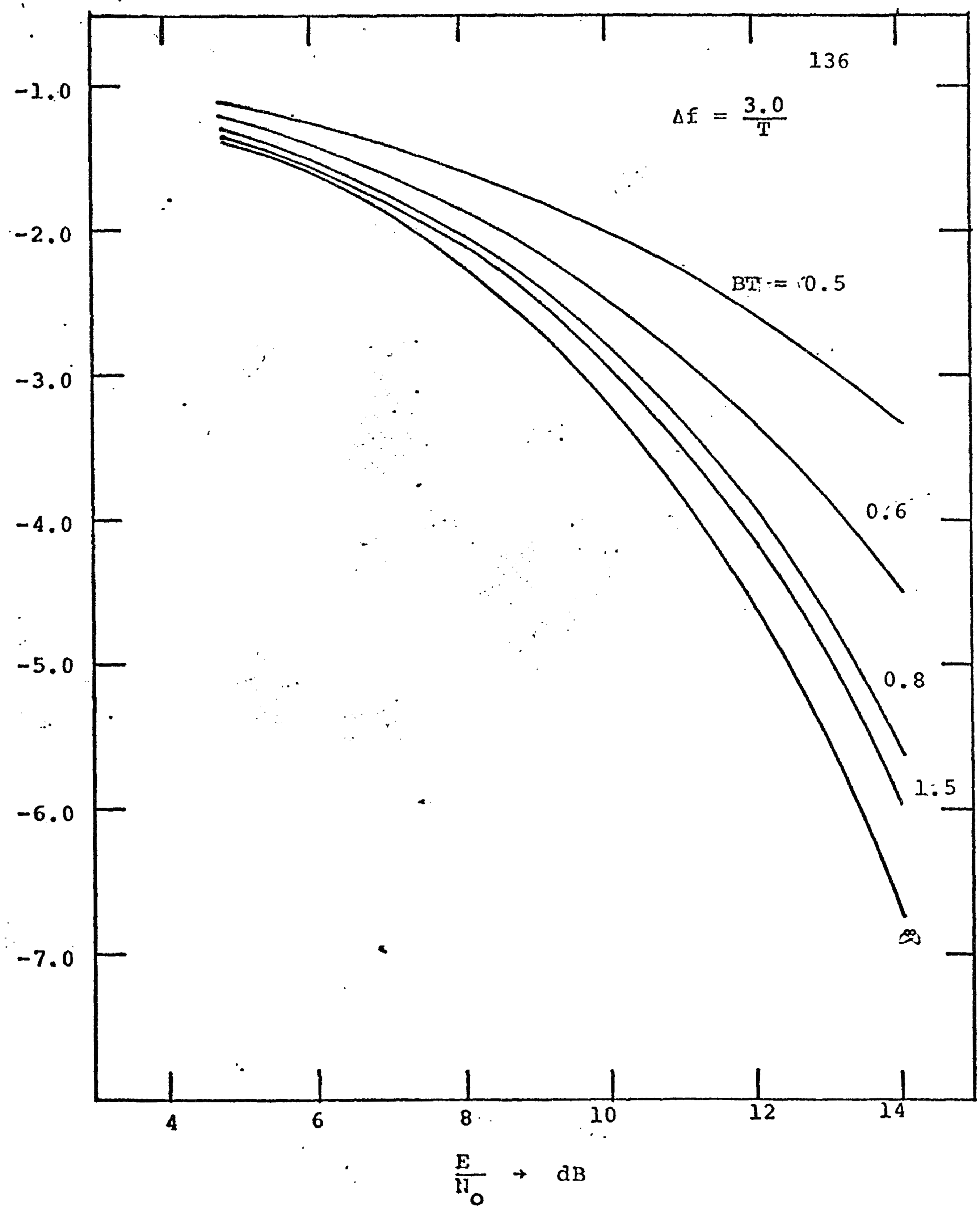

Figure 5.12: P_e vs $\frac{E}{N_0}$ for the FSK System with $\Delta f = \frac{3}{T}$

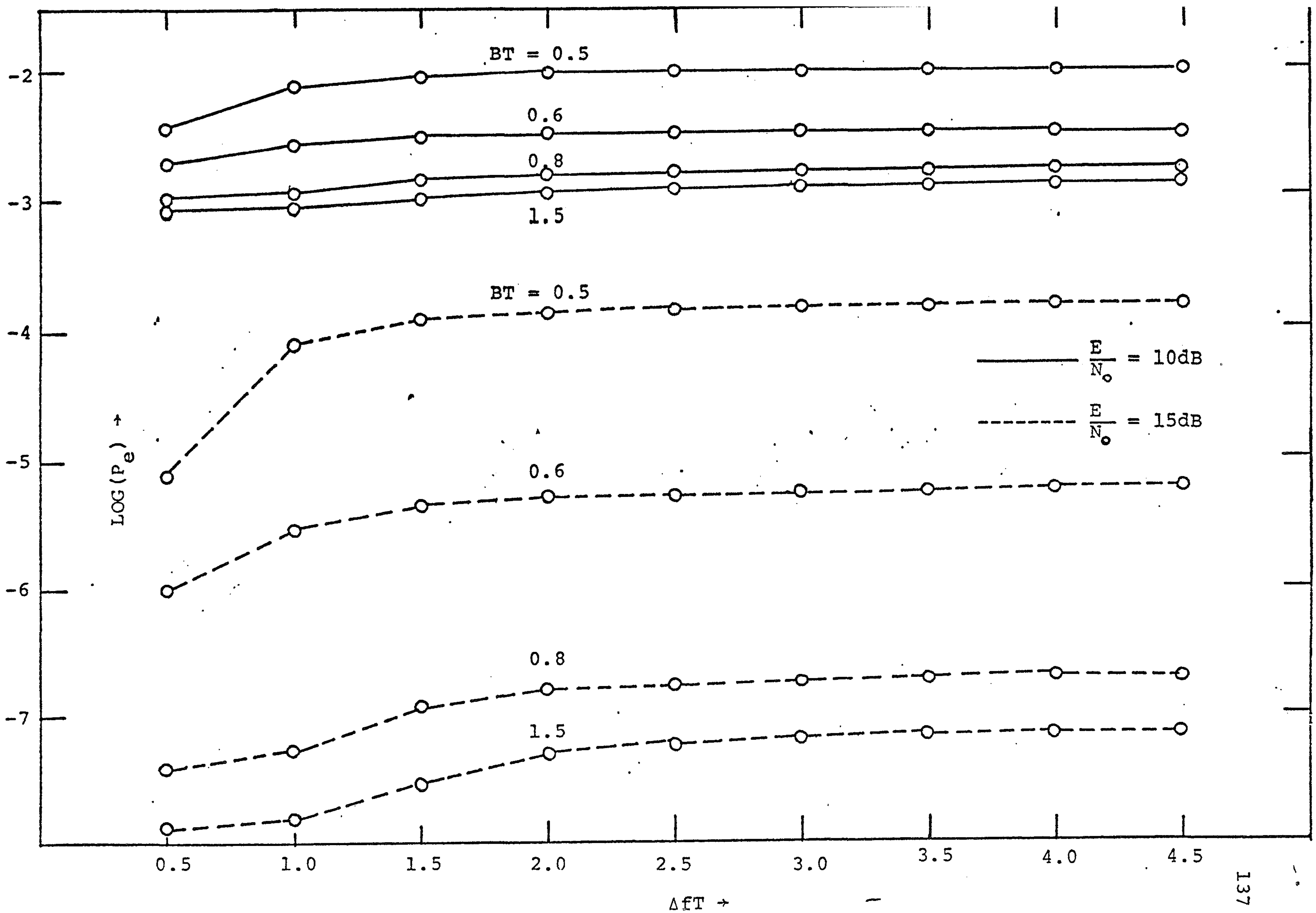

Figure 5.13: P_e vs BT for the FSK System with $\frac{E}{N_0}$ = w dB, and 15 dB

Table 5.3

Values of P_e, P_{e1} and P_{e2} vs $\frac{E}{N_0}$ for the FSK System with BT=1.0 and ΔfT=0.5, 3.0

ΔfT = 0.5 BT = 1.0

$\frac{E}{N_o}$ (dB)	Log(P_e)	Log(P_{e1})	Log(P_{e2})
0.00	-0.792	-0.792	-4.272
4.77	-1.362	-1.363	-3.981
6.99	-1.867	-1.870	-4.073
8.45	-2.348	-2.353	-4.278
9.54	-2.815	-2.823	-4.538
10.41	-3.723	-3.285	-4.831
11.14	-3.725	-3.742	-5.146
11.76	-4.272	-4.194	-5.475
12.30	-4.616	-4.644	-5.817
12.79	-5.056	-5.091	-6.167
13.22	-5.493	-5.535	-6.524
13.62	-5.928	-5.978	-6.886
13.98	-6.360	-6.420	-7.254

Table 5.3

(Continued)

$\Delta fT = 3.0 \quad BT = 1.0$

$\frac{E}{N_o}$ (dB)	Log(P_e)	Log(P_{e1})	Log(P_{e2})
0.00	-0.767	-0.769	-3.253
4.77	-1.296	-1.306	-2.932
6.99	-1.756	-1.782	-2.991
8.45	-2.185	-2.234	-3.161
9.54	-2.596	-2.673	-3.384
10.41	-2.993	-3.105	-3.638
11.14	-3.380	-3.531	-3.912
11.76	-3.758	-3.953	-4.199
12.30	-4.128	-4.372	-4.495
12.79	-4.492	-4.788	-4.799
13.22	-4.851	-5.202	-5.107
13.62	-5.206	-5.615	-5.420
13.98	-5.556	-6.026	-5.736

influence of the intersymbol interference $J(BT,n)$, while $C(BT, fT,0)$ tends to aid the signal strength $J(BT,0)$. Also the noise $n_1(T)$ and $n_2(T)$ are correlated, thus the variance of n' , σ^2, is reduced $(\frac{N_0T}{2}(J(BT,0)-C((B-\Delta f)T,\frac{\Delta fT}{2},N))$.

The optimum Δf is found to be equal to the half of the bit rate. In other words, to obtain the greatest discrimination, the optimum spacing between the two carrier tones is equal to the bit rate. Notice that for the case of infinite bandwidth the two carrier tones spacing does not affect the performance. Thus the IF filtering indeed has a great effect on the performance of the FSK system.

CHAPTER VI

EQUALIZATION OF INTERSYMBOL INTERFERENCE

6.1 Introduction

In previous chapters, the performance of various band-limited baseband and modulation systems has been analyzed in terms of the probability of bit-error. It has been shown that the intersymbol interference severely degrades the performance of these communication systems operating in a high signal-to-noise channel.

Currently, the demand of the high data transmission rate utilizing the high signal-to-noise channel such as telephone line as the communication link has resulted in an enhanced interest in alleviating the influence of the intersymbol interference. The well known optimum equalizer is a tapped-delay-line (TDL) filter [1]. The purpose of any TDL filter is to eliminate the intersymbol interference. For an unknown channel characteristics, some adaptive alogorithms using steepest-decent techniques for automatically adjusting the tap gains of TDL filter have been proposed [13], [18], [23]. Different performance indices were used in these works. Lucky [13] minimized the sum of the absolute values of intersymbol interference by adjusting tap gains, while others [18], [23], adjusted the tap gains to minimize the mean-square error (MSE) due to the combination of intersymbol interference and additive noise.

On the other hand, for a known channel characteristic the tap gains can be obtained by minimizing the probability of bit-error and solving a set of nonlinear equations [1]. However, in most cases, the nonlinear equations are not particularly tractable. The optimum gains are usually obtained by trial and error.

In this chapter, first, the receiver in the bandlimited NRZ baseband model shown in Figure 3.1 of the Chapter III will be proven to be the optimum detector for detecting a single NRZ pulse in the absence of intersymbol interference and then a new modified tap-delay-line filter in tandem with the receiver will be proposed for equalizing the intersymbol interference. The overall performance improvement will be determined in terms of bit-error probability. The principle developed for this particular system then will be generalized for any data transmission system using the linear detector.

6.2 Optimum Detection of a Single Bandlimited NRZ Signal

The bandlimited NRZ baseband communication system shown in Figure 3.1 is repeated in Figure 6.1. From Chapter II, it is known that the optimum receiver for the detections of binary signals corrupted by additive white Gaussian noise can

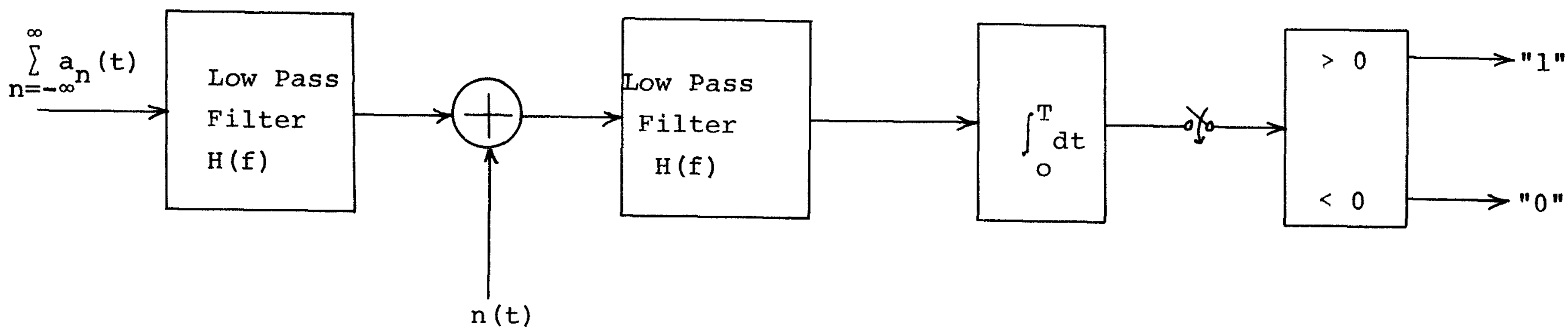

Figure 6.1 A Bandlimited NRZ Baseband System

be implemented using the matched filter. For a single NRZ pulse (i.e. in the absence of intersymbol interference) the optimum receiver can be determined from the signal and channel characteristics.

Now consider a single NRZ pulse with amplitude +A or -A and duration T. The Fourier transform of this pulse is given by

$$F(f) = \int_0^T A\, e^{-j2\pi ft}\, dt = AT\, \frac{\sin\pi fT}{\pi fT}\, e^{-j\pi fT} \tag{6.1}$$

Thus the transfer function of the matched filter will be (see Equation (4.48))

$$\begin{aligned} R(f) &= K[F(f)H(f)e^{-j2\pi fT}]^* \\ &= (KA)H(f)\cdot\left(T\frac{\sin\pi fT}{\pi fT}\, e^{-j\pi fT}\right) \end{aligned} \tag{6.2}$$

Choosing KA to be 1, the optimum receiver is readily recognized to be a lowpass filter followed by a correlation detector as shown in Figure 6.1. This model is the same as the one used in Chapter III.

The probability of error for detecting the single pulse is then given by Equation (3.53) or Equation (3.24)

$$P_{es} = \frac{1}{2}\left(1-\mathrm{erf}\left(\sqrt{\frac{E}{N_0}}\; J(BT,0)\right)\right) \tag{6.3}$$

Figure 6.2 shows the plots of P_{es} for various bandwidths. No other receiver can give better performance than the one shown in Figure 6.1 for detecting a single bandlimited NRZ pulse. However, from Chapter III, the performance of this receiver is severely degraded by the intersymbol interference especially for the high signal-to-noise channel. In the following, a modified TDL filter equalizer will be proposed to eliminate the intersymbol interference to achieve the minimum probability of error as given by Equation (6.3).

6.3 A Modified TDL Filter Equalizer

A modified TDL filter in tandem with the receiver (see Figure 6.1) is shown in Figure 6.2. The output of the integrator is sampled and normalized before being sent to the TDL filter. There are (2n+1) taps (C_{-N} to C_N) in the TDL filter. Thus the delay line spans (2n+1)T seconds, and in it there are stored the most recent (2n+1) samples. The operations performed on these (2n+1) samples are as follows (see Figure 6.3).

The sample stored at each tap (except the central one) is fed to a sign detector. If the sign is positive at the decision time (t=T), the corresponding gain element will subtract an amount J(BT,n) from the bit under detection (central element) and vice versa. The central bit then will

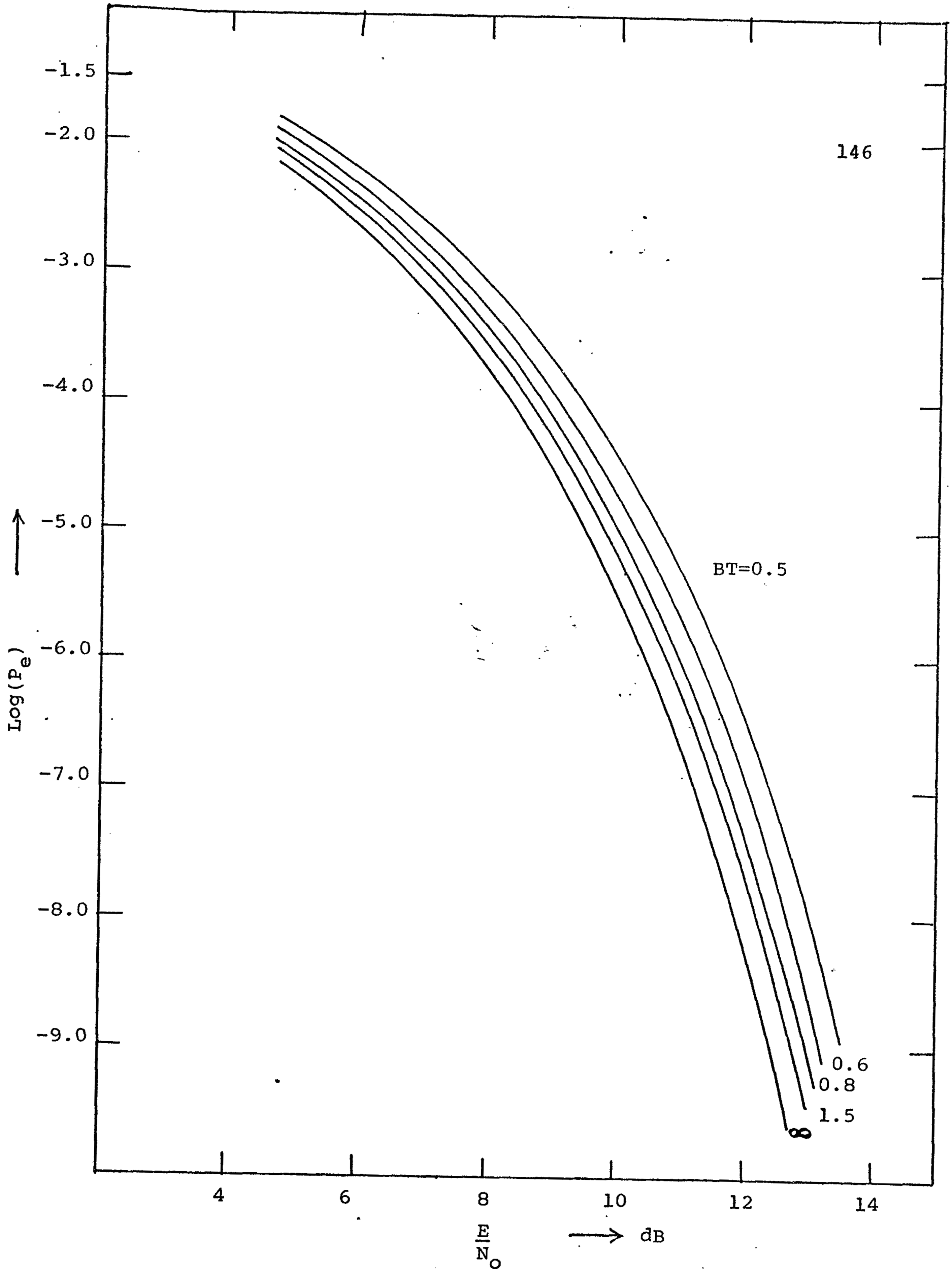

Figure 6.2 Probability of Bit Error for a Single Pulse

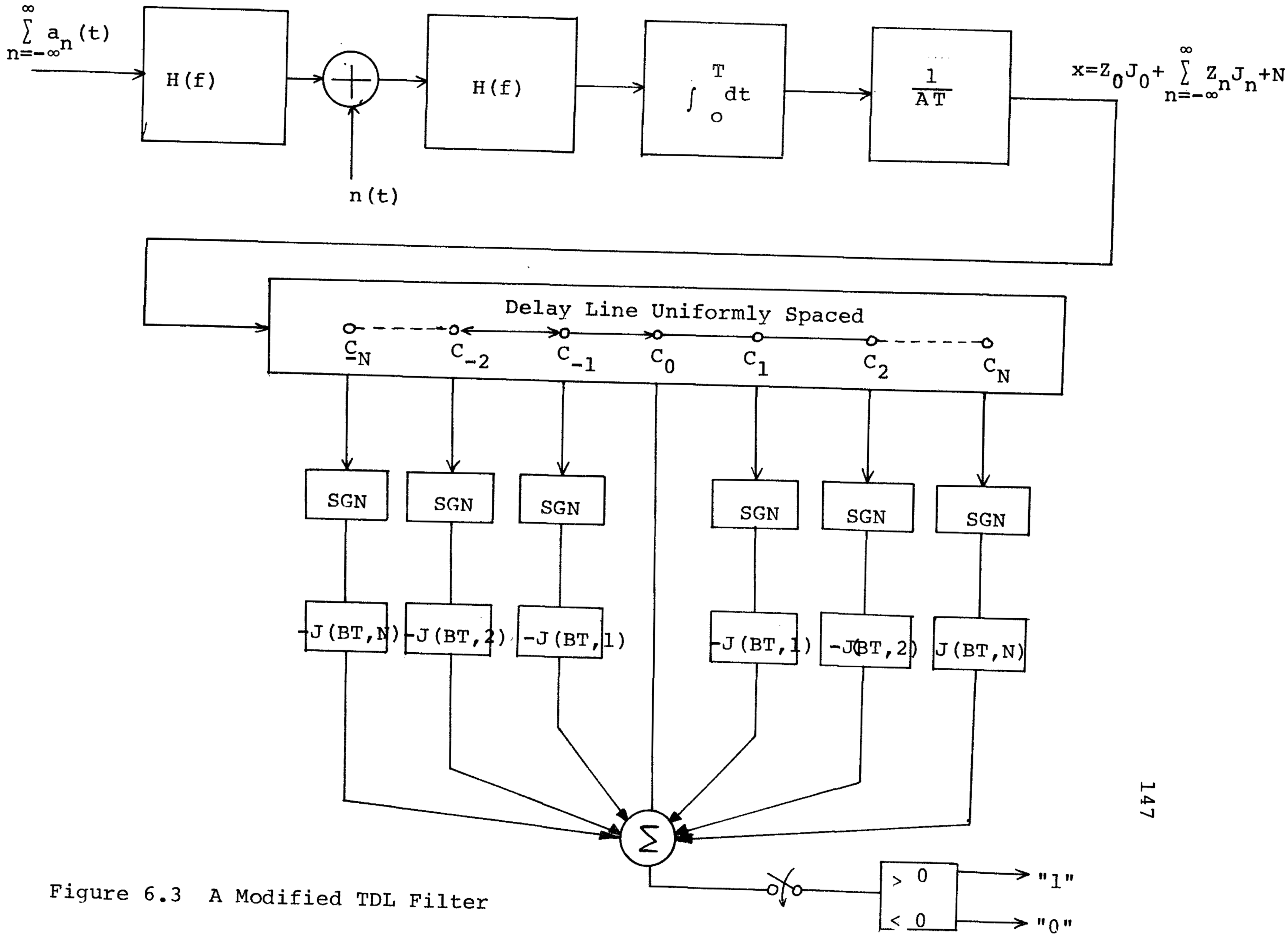

Figure 6.3 A Modified TDL Filter

be decided to be a "1" or "0" depending upon the resulting signal plus noise at the output of the summer greater than zero or not. After this decision is being made, the content of the samples stored at different taps is shifted to the right and detection is performed as before.

The output of central element can be expressed as

$$X = Z_0 J_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Z_n J_n + N \tag{6.4}$$

where

$$Z_n = \pm 1, \; J_n = J(BT,n)$$

The variance of the noise N is given by Equation (3.52)

$$\sigma_N^2 = \frac{J_0}{2(\frac{E}{N_0})} \tag{6.5}$$

In the following sections, the performance of this modified TDL filter will be analyzed by considering only three taps, namely C_{-1}, C_0 and C_{+1} using the averaging method and series expansion method.

6.4 Performance Analysis - Averaging Method

There are only four possibilities associated with the signs of taps C_{-1} and C_{+1}:

A. Both signs of C_{-1} and C_{+1} are correct.

B. Sign of C_{-1} is correct and C_{+1} is incorrect.

C. Sign of C_{-1} is incorrect and C_{+1} is correct.

D. Both signs of C_{-1} and C_{+1} are incorrect.

The probability P that the sign is incorrect for the taps C_{-1} and C_{+1} can be thought as the probability of bit-error for the receiver without using the TDL filter and is given by Equation (3.21) or Equation (3.61) in Chapter III. Thus the probability of Case A, P(Case A) can be written as $(1-P)^2$. Similarly the probabilities of Case B, Case C and Case D can be expressed as

$$P(\text{Case B}) = P(1-P)$$

$$P(\text{Case C}) = P(1-P)$$

$$P(\text{Case D}) = P^2$$

The probability of bit-error for a particular pattern can be obtained as follows.

<u>Case A</u>

The output of the summer is

$$X = Z_0 J_0 + \sum_{n=2}^{\infty} (Z_n + Z_{-n}) J_n + N \tag{6.6}$$

The probability of error for this particular pattern can be given by Equation (3.18)

$$P_{ei_A} = \frac{1}{2}[1-\text{erf}(\sqrt{\frac{E}{N_0}} \cdot \frac{J_0 + \sum_{n=2}^{\infty} (Z_n+Z_{-n}) J_n}{\sqrt{J_0}})] \tag{6.7}$$

Case B

The output of the summer is

$$X = A_0 J_0 + \sum_{n=2}^{\infty} (A_n + Z_{-n}) J_n + (2J_1) Z_1 + N \qquad (6.8)$$

and the probability of error B

$$P_{ei_B} = \frac{1}{2} [1 - \mathrm{erf}(\sqrt{\frac{E}{N_0}} \cdot \frac{J_0 + \sum_{n=2}^{\infty} (Z_n + Z_{-n}) J_n + (2J_1) Z_1}{\sqrt{J_0}})] \qquad (6.9)$$

Case C

The output of the summer is

$$X = Z_0 J_0 + \sum_{n=2}^{\infty} (Z_n + Z_{-n}) J_n + (2J_1) Z_{-1} + N \qquad (6.10)$$

and the probability of error is

$$P_{ei_C} = \frac{1}{2} [1 - \mathrm{erf}(\sqrt{\frac{E}{N_0}} \frac{J_0 + \sum_{n=2}^{\infty} (Z_n + Z_{-n}) J_n + (2J_1) Z_{-1}}{\sqrt{J_0}})] \qquad (6.11)$$

Case D

The output of the summer is

$$X = Z_0 J_0 + \sum_{n=2}^{\infty} (Z_n + Z_{-n}) J_n + 2(Z_1 + Z_{-1}) J_1 + N \qquad (6.12)$$

and the probability of error is

$$P_{ei_D} = \frac{1}{2}\,[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}} \cdot \frac{J_0 + \sum_{n=2}^{\infty}(Z_n+Z_{-n})J_n + 2(Z_1+Z_{-1})J_1}{\sqrt{J_0}})] \tag{6.13}$$

The average probability of bit-error for this particular bit pattern using only three taps can then be evaluated as

$$\begin{aligned} P_{ei} &= P(\text{Case A})\, P_{ei_A} + P(\text{Case B}) P_{ei_B} \\ &\quad + P(\text{Case C})\, P_{ei_C} + P(\text{Case D}) P_{ei_D} \\ &= (1-P)^2\, P_{ei_A} + P(1-P) P_{ei_B} + P(1-P) P_{ei_C} \\ &\quad + P^2 P_{ei_D} \end{aligned} \tag{6.14}$$

The average probability of bit-error for the intersymbol interference confined to the 10 nearest bits then can be computed as

$$P_e = \frac{1}{1024} \sum_{i=1}^{1024} P_{ei} \tag{6.15}$$

6.5 <u>Performance Analysis--Series Expansion Method</u>

The probability of bit-error can be obtained using the series expansion method by analyzing the same four cases of Section 6.3.

The probability of error for the Case A can be given by Equation (3.61)

$$P_A = \frac{1}{2}[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}\;J_0})] + \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \qquad (6.16)$$

except that $d_{2\ell-1}$ which is used to evaluate b_{2n} (see Equation (3.60)) is modified as

$$d_{2\ell-1} = \frac{2^{2\ell}(2^{2\ell}-1)}{2\ell!} B_{2\ell} \left(\sum_{n=-\infty}^{-2} J_n^{2\ell} + \sum_{n=2}^{\infty} J_n^{2\ell} \right) \qquad (6.17)$$

Similarly, for the Case B, the probability of error is

$$P_B = \frac{1}{2}[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}\;J_0})] + \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \qquad (6.18)$$

and $d_{2\ell-1}$ is changed to

$$d_{2\ell-1} = \frac{2^{2\ell}(2^{2\ell}-1)}{2\ell!} B_{2\ell} \left[\sum_{n=-\infty}^{-2} J_n^{2} + \sum_{n=2}^{\infty} J_n^{2\ell} + (2J_1)^{2\ell} \right] \qquad (6.19)$$

For the Case C, the probability of error is

$$P_C = \frac{1}{2}[1-\mathrm{erf}(\sqrt{\frac{E}{N_0}\;J_0})] + \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \qquad (6.20)$$

and $d_{2\ell-1}$ now becomes

$$d_{2\ell-1} = \frac{2^{2\ell}(2^{2\ell}-1)}{2\ell!} B_{2\ell} \left[\sum_{n=-\infty}^{-2} J_n^{2\ell} + \sum_{n=2}^{\infty} J_n^{2\ell} + (2J_{-1})^{2} \right] \qquad (6.21)$$

Finally, for the Case D, the probability of error is

$$P_D = \frac{1}{2}[1-\mathrm{erf}(\frac{E}{N_0}\;J_0)] + \sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \qquad (6.22)$$

and $d_{2\ell-1}$ is changed to

$$d_{2\ell-1} = \frac{2^{2\ell}(2^{2\ell}-1)}{2\ell!} B_{2\ell} \left[\sum_{n=-\infty}^{-2} J_n^{2\ell} + \sum_{n=2}^{\infty} J_n^{2\ell} + (2J_{-1})^{2\ell} + (2J_1)^{2\ell} \right] \qquad (6.23)$$

For the system considered in this chapter, J_n is equal to J_{-n}. Thus P_B is equal to P_C. The average probability of error for the detection of the central bit can now be computed as

$$P_e = (1-p)^2 \; P_A + 2P(1-p)P_B + P^2 P_D \tag{6.24}$$

By confining the intersymbol interference to the 10 nearest bits as for the averaging method, the resulting P_e is in agreement with that obtained using the averaging method and is shown in Figure 6.4 for various bandwidths (dashed lines). Table 6.1 lists P_{es}, P_e, P, $(1-p)^2 P_A$, $2P(1-p)P_B$ and $P^2 P_D$ for BT=0.6 and BT=0.8.

Comparing the results obtained using either method with those for the single pulse case (Figure 6.2), the effect of the intersymbol interference has almost been cancelled out. Table 6.1 gives the reason. Since P is much smaller than 1, P_e in Equation (6.24) tends to approach P_A, which corresponds to the case of the cancellation of J(BT,1). From Table 3.1, it can be seen that the most effect of the intersymbol interference comes from the immediate adjacent bits. Thus using only three taps, the performance of the modified receiver is almost near optimum and can be predicted using Equation (6.3).

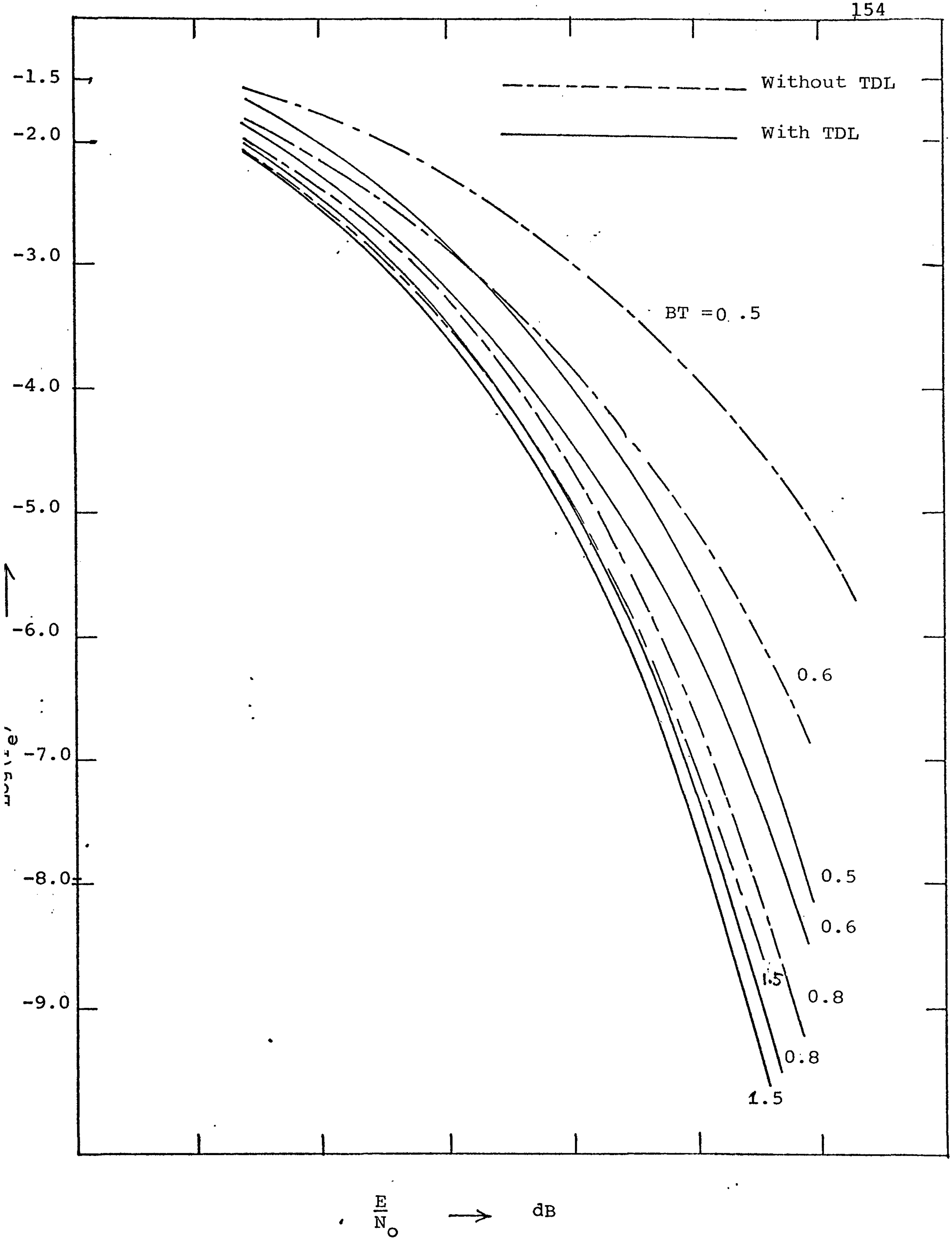

Figure 6.4 Performance of the Modified TDL Filter

Table 6.1

Values of P_{es}, P_e, P, $(1-P)^2 P_A$, $2P(1-P)P_B$ and $P^2 P_C$

vs $\frac{E}{N_o}$ for the modified NRZ Baseband Receiver with BT=0.6 and 0.8

BT = 0.6

$\frac{E}{N_o}$ (dB)	$\mathrm{Log}(P_{es})$	$\mathrm{Log}(P_e)$	$\mathrm{Log}(P)$	$\mathrm{Log}((1-P)^2 P_A)$	$\mathrm{Log}(2P(1-P)P_B)$	$\mathrm{Log}(P^2 P_C)$
0.00	-1.011	-1.003	-0.995	-1.098	-1.736	-2.957
4.77	-1.906	-1.866	-1.796	-1.883	-3.399	-5.215
6.99	-2.725	-2.628	-2.455	-2.632	-4.611	-7.004
8.45	-3.517	-3.337	-3.036	-3.339	-5.772	-8.556
9.54	-4.294	-4.010	-3.565	-4.010	-6.830	-9.960
10.41	-5.062	-4.654	-4.057	-4.654	-7.812	-11.262
11.14	-5.824	-5.274	-4.519	-5.274	-8.737	-12.487
11.76	-6.582	-5.874	-4.958	-5.874	-9.616	-13.652
12.30	-7.338	-6.457	-5.380	-6.457	-10.458	-14.796
12.79	-8.090	-7.024	-5.786	-7.024	-11.270	-15.846
13.22	-8.839	-7.579	-6.179	-7.579	-12.057	-16.889
13.93	-10.333	-8.654	-6.934	-8.654	-13.566	-18.891

Table 6.1 (continued)

BT = 0.8

$\frac{E}{N_o}$ (dB)	$Log(P_{es})$	$Log(P_e)$	$Log(P)$	$Log((1-P)^2 P_A)$	$Log(2P(1-P)P_B)$	$Log(P^2 P_C)$
0.00	-1.044	-1.043	-1.040	-1.127	-1.820	-3.106
4.77	-1.991	-1.989	-1.957	-2.000	-3.618	-5.784
6.99	-2.860	-2.857	-2.773	-2.858	-5.246	-8.112
8.45	-3.702	-3.695	-3.541	-3.696	-6.781	-10.273
9.54	-4.528	-4.519	-4.277	-4.519	-8.252	-12.330
10.41	-5.346	-5.332	-4.989	-5.332	-9.677	-14.316
11.14	-6.158	-6.138	-5.658	-6.138	-11.068	-16.252
11.76	-6.965	-6.938	-6.367	-6.938	-12.432	-18.152
12.30	-7.771	-7.737	-7.039	-7.737	-13.776	-20.026
12.79	-8.571	-8.529	-7.702	-8.529	-15.104	-21.879
13.22	-9.[illegible]0	-9.319	-8.360	-9.319	-16.419	-23.715
13.62	-10.167	-10.106	-9.012	-10.106	-17.723	-25.539
13.98	-10.963	-10.890	-9.661	-10.890	-19.020	-27.352

6.6 Discussion

The performance of the modified TDL filter equalizer has now been verified by both methods. The analysis procedure laid out in Sections 6.4 and 6.5 can be used to analyze the performance of the modified TDL filter with more than 3 taps. By using 5 taps, it is found that the performance does not improve significantly. In other words, the correction contributed by C_{-2} and C_{+2} has little effect on the detection of the central bit. Because $|J(BT,2)|$ is much smaller than $J(BT,0)$. Thus for all practical purposes, there is little point in using the C_2 and C_{-2} taps for the bandlimited NRZ transmission system.

The biggest advantage of this modified equalizer is that the gains can be obtained analytically and the performance is almost near the optimum case (single pulse correlation detection). As pointed out before, the detector output for any data transmission system with known channel characteristics can be given by Equation (3.25). Therefore the modified TDL filter developed in this chapter can be applied to any data system to alleviate the influence of intersymbol interference and thus speed up the data transmission rate.

In practice, the delay lines can be replaced by digital shift registers and all the gain element and summer can also be realized by the logic gates and flip-flops. Thus the operation of the modified TDL filter can be performed digitally with high speed.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

The effect of bandlimiting on the performance of various transmission systems corrupted by additive white Gaussian noise have been analyzed using two methods , the averaging method and the series method. The results from both methods agree.

First, the performance of an ideal bandlimited NRZ (Non-Return-to-Zero) baseband transmission system was examined using correlation detection and sampling. The explicit expression for the degradation of the signal and the intersymbol interference was derived as a function of system parameters, such as the bandwidth of the filter and signal-to-noise ratio. The average probabilities of bit-error were computed. It was shown that the correlation detector performs better than the sampler detector for $BT \geq .6$ and worse for $BT \leq .5$.

Second, a split-phase baseband system was analyzed following the same steps used for analyzing the NRZ system. It was shown that a split-phase baseband system requires about less than twice as much bandwidth as the NRZ system to have the same probability of bit-error for the same value of signal-to-noise ratio using the correlation detector.

Third, an NRZ baseband system using Gaussian filters was analyzed employing correlation detection. It was found that this system introduces more intersymbol interference and performs poorly compared to ideal bandlimited NRZ system.

Fourth, the effect of bandlimiting the modulated system, the Phase-Shift-Keying (PSK), the Amplitude-Shift-Keying (ASK), and the Frequency-Shift-Keying (FSK) have been analyzed assuming coherent receiver and using ideal filters as well as correlation detection. The explicit expression for the degradation of the signal and the intersymbol interference as a function of bandwidth of the filter, signal-to-noise ratio and carrier frequencies were given. It was found that the aliasing effected can be neglected if the carrier frequency is three times more than the bit rate. The PSK system requires 3 dB less on the average power basis than the ASK system regardless of the restriction of the bandwidth. If the spacing between two carrier tones in the FSK system is less than three times of the bit rate, the FSK system shows a better performance than that of the ASK system. The optimum setting of the tone spacing is shown to be equal to the bit rate. However, PSK system still gives the best performance. Thus for a coherent modulation transmission system, the PSK should always be favored.

Finally, a tapped-delay-line (TDL) filter has been introduced at the receiver of the NRZ baseband system in conjunction with the correlation detector as an intersymbol eliminator. On an average probability of bit-error basis, and using only three taps, it was demonstrated that the performance of this system is near optimum.

7.2 Recommendations

In this dissertation, the channel is modeled by a linear filter by an additive white Gaussian noise source. The first topic for future study suggested by this dissertation is an investigation of the effects of bandwidth restriction on the performance of a modulation system over a nature (not man-made) mutipath fading communication channel. Because of the random changeability which often accompanies this natural channel, mutipath is inevitable and is crucially dependent on the signal bandwidth. Mutipath not only introduces a Rayleigh fading envelop but also a uniformly distributed r-f (carrier) phase. The analysis of the performance of digital transmission system over a slow and nonselective Rayleigh fading channel in the absence intersymbol interference has been reported [35]. It will be very interesting to extend the analysis to the case of intersymbol interference. However, it is believed that the analysis will be highly complicated.

The coherent detection for the modulated signals was assumed in this dissertation. In the absence of the intersymbol interference, the analysis of the performance of the binary communication system with partially coherent reception has been obtained by Viterbi [44]. The second topic for the future study suggested by this dissertation is to look into the effect of intersymbol interference on the performance of this partially coherent system.

The receivers analyzed in this dissertation are the linear detectors which are optimum for the infinite system bandwidth. However, the performance of these detectors is degraded if the system bandwidth is restricted. Therefore, it is worthwhile to compare the performance of linear detector with that of nonlinear detector such as envelop detector for ASK and FM discriminator for FSK under the bandlimiting hypothesis. The third topic suggested by this dissertation for the future study is then the investigation of the influence of the bandlimiting on the performance of the envelop detector and FM discriminator.

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APPENDIX

APPENDIX A

THE SERIES EXPANSION OF $\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \cos(J_n w)$

Let $\quad F(w) = \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \cos(J_n w)$

then

$$F'(w) = \ldots\ldots \frac{J_{-1}\sin(J_{-1}w)}{\cos(J_{-1}w)} F(w) - \frac{J_1 \sin(J_1 w)}{\cos(J_1 w)} F(w) \ldots\ldots$$

$$= -F(w) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n \tan(J_n w) \tag{A.1}$$

Using power series, we have

$$J_n \tan(J_n w) = J_n\Big[J_n w + \frac{1}{3}(J_n w)^3 + \frac{2}{15}(J_n w)^5 + \ldots\ldots + \frac{2^{2\ell}(2^{2\ell}-1)}{2\ell\,!} B_{2\ell}\ (J_n w)^{2\ell-1} + \ldots\ldots\Big], \tag{A.2}$$

where $B_{2\ell}$ is the Bernoulli number.

Combining all terms, we obtain

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n \tan(J_n w) = \sum_{\ell=1}^{\infty} d_{2\ell-1}\ w^{2\ell-1} \tag{A.3}$$

where

$$d_{2\ell-1} = \frac{2^{2\ell}(2^{2\ell}-1)}{2\ell\,!} B_{2\ell} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2\ell} \tag{A.4}$$

Since cos(x) can only be expanded into a series of even power of x, we can let

$$F(w) = 1 + \sum_{n=1}^{\infty} b_{2n} w^{2n} \tag{A.5}$$

Thus

$$F'(w) = \sum_{n=1}^{\infty} 2n\, b_{2n} w^{2n-1} \tag{A.6}$$

Using Equation (A.1), Equation (A.3) and Equation (A.6), we have

$$\sum_{n=1}^{\infty} 2n\, b_{2n} w^{2n-1} = -(1 + \sum_{n=1}^{\infty} b_{2n} w^{2n}) \sum_{\ell=1}^{\infty} d_{2\ell-1} w^{2\ell-1} \tag{A.7}$$

Comparing the coefficient for w^{2n-1}, we obtain

$$2n\, b_{2n} = -(d_{2n-1} + \sum_{\ell=1}^{n-1} b_{2n-2\ell}\, d_{2\ell-1}) \tag{A.8}$$

Thus b_{2n} can be evaluated in a recurrsive formula,

$$b_{2n} = -\frac{1}{2n} (d_{2n-1} + \sum_{\ell=1}^{n-1} b_{2n-2\ell} d_{2\ell-1}) \tag{A.9}$$

Therefore b_{2n} is only the function of intersymbol interference $\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}$.

APPENDIX B

ANALYSIS OF THE CONVERGENCE OF THE SERIES

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1},$$

Analysis of the convergence of the series

$$\sum_{n=1} (-1)^{n+1} b_{2n} G_{2n-1} \tag{B.1}$$

The error E introduced by using only K terms can be expressed by

$$E = \sum_{n=K+1}^{\infty} (-1)^{n+1} b_{2n} G_{2n-1} \tag{B.2}$$

Thus

$$|E| \leq \sum_{n=K+1}^{\infty} |b_{2n}||G_{2n-1}| \tag{B.3}$$

From Equation (3.46), we know

$$|G_{2n-1}| = \left|\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2} \sigma_N^2} (-jw)^{2n-1} e^{-jJ_O w} dw\right| \tag{B.4}$$

Using Schwarze inequality, we have

$$|G_{2n-1}| \leq \left|\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2} \sigma_N^2} |(-jw)^{2n-1}||e^{-jJ_O w}| dw\right|$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2} \sigma_N^2} |w^{2n-1}| dw \tag{B.5}$$

But $e^{-\frac{w^2}{2}\sigma_N^{\;2}}\ |w^{2n-1}|$ is an even function, we have

$$|G_{2n-1}| \le \frac{1}{\pi} \int_0^\infty e^{-\frac{w^2}{2}\sigma_N^{\;2}}\ w^{2n-1}\, dw \tag{B.6}$$

Let $t = -\frac{w^2}{2}\sigma_N^{\;2}$, then $dt = w\sigma_N^{\;2}\, dw$

Thus

$$|G_{2n-1}| \le \frac{1}{\pi\ \sigma_N^{\;2}} \int_0^\infty \left(\frac{2}{\sigma_N^{\;2}}\right)^{n-1} e^{-t}\ t^{n-1}\, dt$$

$$= \frac{1}{\pi} \frac{2^{n-1}}{\sigma_N^{\;2n}}\ (n-1)\ ! \tag{B.7}$$

Note

$$\int_0^\infty e^{-t}\ t^{n-1}\, dt = \Gamma(n) = (n-1)\ ! \quad , \tag{B.8}$$

where $\Gamma(n)$ is the gamma function.

Next the desired bound for b_{2n} will be derived. Let us consider finite terms for $\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \cos(J_n w)$, namely from $n = -m$ to m; then we can write

$$\prod_{\substack{n=-m \\ n\neq 0}}^{m} \cos(J_n w) = \prod_{\substack{n=-m \\ n\neq 0}}^{m} \frac{e^{jJ_n w} + e^{-jJ_n w}}{2}$$

$$= \frac{1}{2^{2m}} \sum_{\ell=1}^{2^{2m}} e^{j\alpha_\ell w} \tag{B.9}$$

where α_ℓ is one of the combinations

$$\pm J_{-m} \pm \dots \pm J_{-1} \pm J_1 \pm \dots \pm J_m \quad . \tag{B.10}$$

Now $e^{j\alpha_\ell w}$ can be expanded into a power series of w

$$e^{j\alpha_\ell w} = \sum_{n=0}^{\infty} \frac{(j\alpha_\ell w)^n}{n!} \tag{B.11}$$

Thus

$$\prod_{\substack{n=-m \\ n\neq 0}}^{m} \cos J_n w = \frac{1}{2^{2m}} \sum_{n=0}^{\infty} \left[\sum_{\ell=1}^{2^{2m}} (j\alpha_\ell)^n \right] \frac{w^n}{n!} \tag{B.12}$$

We can also expand $\prod_{\substack{n=-m \\ n\neq 0}}^{m} \cos J_n w$ into a power series of w directly

$$\prod_{\substack{n=-m \\ n\neq 0}}^{m} \cos J_n w = 1 + \sum_{n=1}^{\infty} b_{2n} w^{2n} \tag{B.13}$$

Comparing Equation (B.12) with Equation (B.13), we obtain

$$\sum_{\ell=1}^{2^{2m}} (j\alpha_\ell)^n = 0 \qquad \text{for} \qquad n = \text{odd integer} \tag{B.14}$$

and

$$b_{2n} = \frac{1}{2^{2m}} \sum_{\ell=1}^{2^{2m}} (j\alpha_\ell)^{2n} \frac{1}{2n!} \tag{B.15}$$

Thus

$$|b_{2n}| = \frac{1}{2^{2m}} \sum_{\ell=1}^{2^{2m}} \alpha_\ell^{2n} \cdot \frac{1}{2n!} \tag{B.16}$$

But

$$\sum_{\ell=1}^{2^{2m}} \alpha_\ell^{2n} = \sum 2^{2m}(2n)! \; \frac{(J_{-m})^{2K_{-m}} \ldots . (J_m)^{2K_m}}{(2K_{-m})! \ldots \ldots . (2K_m)!} \tag{B.17}$$

where $\sum$ indicates all the combination of integers K_{-m} , K_{-m+1} ,.......K_1.....K_m under the constraint that $K_{-m} + K_{-m+1} + \ldots . + K_m = n$.

Substituting Equation (B.17) into Equation (B.16), we have

$$|b_{2n}| = \sum \frac{(J_{-m})^{2K_{-m}} \ldots .. (J_m)^{2K_m}}{(2K_{-m})! \ldots \ldots .. (2K_m)!} \tag{B.18}$$

It is very easy to see

$$\sum \frac{(J_{-m})^{2K_{-m}} \ldots . (J_m)^{2K_m}}{(2K_{-m})! \ldots \ldots . (2K_m)!} \le \frac{1}{2^n} \sum \frac{(J_{-m}{}^2)^{K_{-m}} \ldots . (J_m{}^2)^{K}}{K_{-m}! \ldots \ldots \ldots \ldots . K_m!} \tag{B.19}$$

But

$$\sum \frac{(J_{-m}{}^2)^{K_{-m}} \ldots . (J_m{}^2)^{K_m}}{(K_{-m})! \ldots \ldots \ldots . (K_m)!} \cdot n! = \Bigg(\sum_{\substack{n=-m \\ n\neq 0}}^{m} J_n{}^2 \Bigg)^n \tag{B.20}$$

Thus

$$|b_{2n}| \le \frac{1}{2^n} \frac{1}{n!}\Big(\sum_{\substack{n=-m \\ n\neq 0}}^{m} J_n^{\,2} \Big)^n \tag{B.21}$$

Let $m \to \infty$, we obtain the desired bound for b_{2n}

$$|b_{2n}| \le \frac{1}{2^n} \frac{1}{n!}\Big(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2} \Big)^n \tag{B.22}$$

Substituting Equation (B.22) and Equation (B.7) into Equation (B.1), we have

$$\begin{aligned}
|E| &\le \sum_{n=K=1}^{\infty} \frac{1}{\pi} \frac{2^{n-1}}{\sigma_N^{\,2n}} \frac{(n-1)!}{2^n n!} \Big(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2} \Big)^n \\
&= \sum_{n=K+1}^{\infty} \frac{1}{2\pi} \frac{1}{n} \left(\frac{\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}}{\sigma_N^{\,2}} \right)^n \\
&\le \frac{1}{2\pi(K+1)} \sum_{n=K+1}^{\infty} \left(\frac{\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}}{\sigma_N^{\,2}} \right)^n ,
\end{aligned} \tag{B.23}$$

but

$$\sum_{n=K+1}^{\infty} \left(\frac{\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}}{\sigma_N^{\,2}} \right)^n = \left(\frac{\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}}{\sigma_N^{\,2}} \right)^{K+1} \Bigg/ 1 - \frac{\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}}{\sigma_N^{\,2}} \tag{B.24}$$

Thus

$$|E| \le \frac{1}{2\pi(K+1)} \frac{\left(\dfrac{\sum\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}}{\sigma_N^{\,2}} \right)^{K+1}}{1 - \dfrac{\sum\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}}{\sigma_N^{\,2}}} \tag{B.25}$$

Let $\beta = \dfrac{\sum\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} J_n^{\,2}}{\sigma_N^{\,2}}$,

then for $\beta \le 0.5$, by suitable choice of K , P_{e2} can be evaluated accurately. For the system considered in this work, $K = 10$ is sufficiently enough to be used to calculate P_{e2} very closely. It is believed that there still exists a tighter bound than the one given by Equation (B.25).

APPENDIX C

EVALUATION OF THE VARIANCE OF THE OUTPUT OF GAUSSIAN - CORRELATION DETECTOR WITH THE GAUSSIAN NOISE INPUT

The transfer function of the Gaussian filter is given by Equation (4.34)

$$G(f) = e^{-0.347 \left(\frac{f}{B}\right)^2} \tag{C.1}$$

Let $\beta = \frac{\sqrt{0.347}}{\sqrt{2}\ \pi B}$, we have

$$G(f) = e^{-2\pi^2\beta^2 f^2} \quad \text{or} \quad G(w) = e^{-\frac{\beta^2 w^2}{2}} \tag{C.2}$$

Since the power spectrum of $n(t)$ is $N_o/2$, the power spectrum of the output $n_1(t)$ of the Gaussian filter can be given by

$$S_{n_1 n_1}(w) = \frac{N_o}{2} \, | \, G(w) \, |^2$$

$$= \frac{N_o}{2} \; e^{-\beta^2 w^2} \tag{C.3}$$

The autocorrelation of $n_1(t)$ can be obtained as the inverse of $S_{n_1 n_1}(w)$

$$R_{n_1n_1}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{n_1n_1}(w)\ e^{jw\tau}\ dw$$

$$= \frac{N_o}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\beta^2 w^2}\ e^{jw\tau}\ dw$$

$$= N_o \frac{1}{2\pi} \int_{0}^{\infty} e^{-\beta^2 w^2} \cos w\tau\, dw \qquad \text{(C.4)}$$

Changing variables, we obtain

$$R_{n_1n_1}(\tau) = N_o \int_{0}^{\infty} e^{-cx^2} \cos 2\, rx dx \qquad \text{(C.5)}$$

where

$$c = 4\pi^2\beta^2 \quad , \; r = \pi\tau$$

Equation (C.5) can be evaluated as [2]

$$R_{n_1n_1}(\tau) = N_o \cdot \frac{1}{2} \quad \frac{\pi}{c} \quad e^{-\frac{r^2}{c}}$$

$$= a\ e^{-b\tau^2} \qquad \text{(C.6)}$$

with

$$a = \frac{N_o}{4\sqrt{\pi}\ \beta} \quad , \quad b = \frac{1}{4\beta^2}$$

The transfer function of the integrator is given by

$$H_i(f) = T \frac{\sin \pi f T}{\pi f T} e^{-j\pi f T}$$

or

$$H_i(w) = T \frac{\sin \frac{wT}{2}}{\frac{wT}{2}} e^{-j\frac{wT}{2}} \qquad (C.7)$$

Then the power spectrum of n_2 is given by

$$S_{n_2 n_2}(w) = S_{n_1 n_1}(w) \, |H_i(w)|^2 \qquad (C.8)$$

The inverse of $H_i(w)^2$ is

$$h_i(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H_i(w)|^2 e^{jw} \, dw$$

$$= \frac{T^2}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{wT}{2}}{\frac{wT}{2}^2} e^{jw\tau} \, dw \quad , \qquad (C.9)$$

which is a triangle [22]

$$h_i(\tau) = T(1 - \frac{|\tau|}{T}) \quad - T \le \tau \le T$$

$$= 0 \qquad \text{elsewhere} \qquad (C.10)$$

The autocorrelation $R_{n_2 n_2}(\tau)$ is the inverse of $S_{n_2 n_2}(w)$, which can be obtained as the convolution of $R_{n_1 n_1}(\tau)$ and $h_i(\tau)$

$$R_{n_2 n_2}(\tau) = \int_{-\infty}^{\infty} R_{n_1 n_1}(\tau - \alpha)\, h_i(\alpha)\, d\alpha \qquad \text{(C.11)}$$

Now the variance of n_2 can be obtained as

$$\sigma_2^{\,2} = R_{n_2 n_2}(0) = \int_{-\infty}^{\infty} R_{n_1 n_1}(-\alpha) h_i(\alpha) d\alpha \qquad \text{(C.12)}$$

Substituting Equations (C.6) and (C.10) into Equation (C.11), $\sigma_2^{\,2}$ becomes

$$\sigma_2^{\,2} = \int_{-T}^{T} T\left(1 - \frac{|\alpha|}{T}\right) a e^{-b\alpha^2}\, d\alpha$$

$$= 2T\left[\frac{\sqrt{\pi}}{2}\,\frac{a}{b}\left(\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{b}\,T} e^{-x^2}\, dx\right) - \frac{a}{T}\,\frac{e^{-b\alpha^2}}{-2b}\Big|_0^T\right]$$

$$= T\left(\sqrt{\pi}\,\frac{a}{\sqrt{b}}\,\text{erf}\,(\sqrt{b}\,T) - \frac{a}{Tb}\,(1 - e^{-bT^2})\right) \qquad \text{(C.13)}$$

With $a = \dfrac{N_o}{4\sqrt{\pi}\,\beta}$, $b = \dfrac{1}{4\beta^2}$ and $\beta = \dfrac{0.347}{\sqrt{2}\,\pi\,\beta}$, we obtain the desired result

$$\sigma_2^{\,2} = \frac{N_o T}{2}\left[\text{erf}\,\left(\frac{T}{2\beta}\right) - \frac{2\beta}{T\sqrt{\pi}}\left(1 - e^{-\frac{T^2}{4\beta^2}}\right)\right]$$

$$= \frac{N_o T}{2} \ \mathrm{erf}\left[\left(\frac{\pi BT}{\sqrt{2\cdot 0347}}\right) - \sqrt{\frac{2\cdot 0347}{\pi}} \cdot \frac{1}{\pi BT}\left(1-e^{-\frac{(\pi BT)^2}{2\cdot 0347}}\right)\right]$$

$$= \frac{N_o T}{2} \ E(BT,\ 0) \qquad (C.14)$$

APPENDIX D

SIMPLIFICATION OF BANDLIMITED COHERENT PSK COMMUNICATION MODEL

The model of Figure 5.1 is shown in Figure A.1. Suppose the input to the dotted block is $r(t)$, and the corresponding Fourier transform function is $R(f)$, then the response of the lowpass filter $\ell(t)$ due to $r(t)$ can be expressed as

$$\ell(t) = \int_{-B}^{B} \left(\int_{-\infty}^{\infty} r(t)\cos 2\pi f_c t \; e^{-j2\pi ft} dt\right) e^{j2\pi ft} df \tag{D.1}$$

Simplifying, we obtain

$$\ell(t) = \int_{-B}^{B} \frac{1}{2} [R(f-f_c) + R(f + f_c)] e^{j2\pi ft} df \tag{D.2}$$

The output of the integrator sampled at $t = T$ due to $r(t)$ can now be obtained as

$$y(T) = \int_{0}^{T} \ell(t) dt \tag{D.3}$$

Substituting Equation (D.2) into Equation (D.3), we have

$$y(T) = \int_{0}^{T} \int_{-B}^{B} \frac{1}{2} [R(f - f_c) + R(f + f_c)] e^{j2\pi ft} df \; dt$$

$$= \int_{-B}^{B} \int_{0}^{T} \frac{1}{2} [R(f - f_c) + R(f + f_c)] e^{j2\pi ft} dt \; df$$

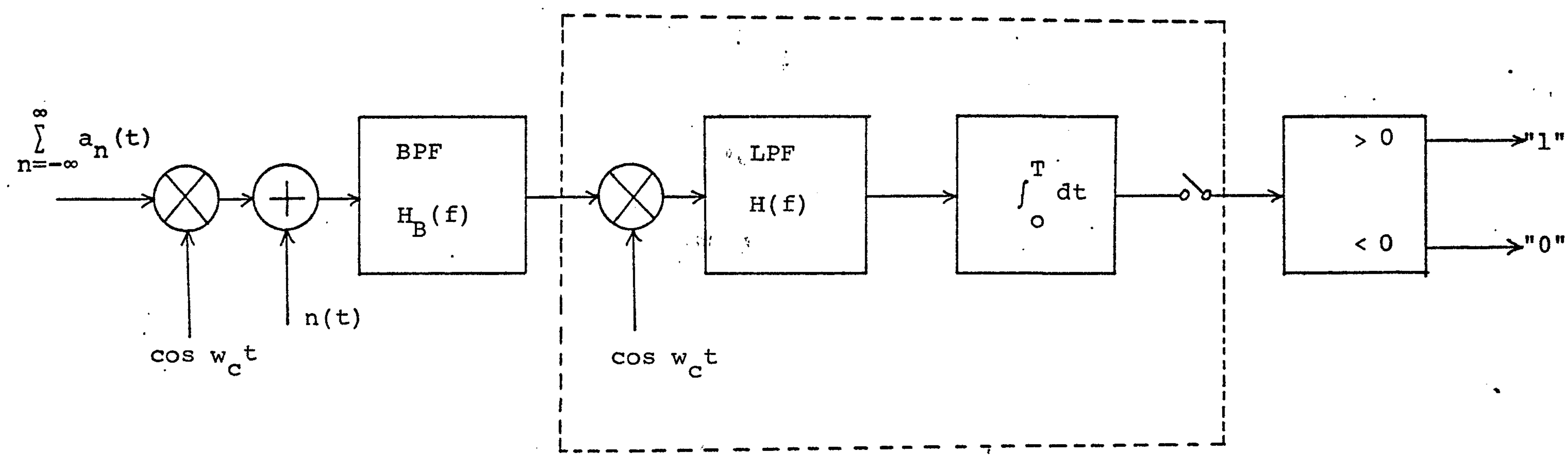

Figure A.1 Model of Figure 5.1

$$= \int_{-B}^{B} \frac{1}{2} [R(f - f_c) + R(f + f_c)] \cdot T \frac{\sin\pi fT}{\pi fT} e^{j\pi fT} df$$

$$= \int_{-B-f_c}^{B-f_c} \frac{1}{2} R(f) \ T \frac{\sin\pi(f + f_c)T}{\pi(f + f_c)T} e^{j\pi f_c T} e^{j\pi fT} df$$

$$+ \int_{-B+f_c}^{B+f_c} \frac{1}{2} R(f) \cdot \frac{T\sin\pi(f - f_c)T}{\pi(f - f_c)T} e^{-j\pi f_c T} e^{j\pi fT} df \qquad (D.4)$$

Since $f_c T$ = integer, we have

$$\sin(\pi(f - f_c)T)e^{-j\pi f_c T} = \sin\pi fT , \qquad (D.5)$$

$$\sin(\pi(f + f_c)T)e^{j\pi f_c T} = \sin\pi fT \qquad (D.6)$$

Thus the expression of y(T) can be written as

$$y(T) = \int_{-\infty}^{\infty} R(f)H'(f)e^{j2\pi fT} df \qquad (D.7)$$

Where

$$H'(f) = \begin{cases} \frac{1}{2} \frac{T\sin\pi fT}{\pi(f+f_c)T} e^{-j\pi fT} & -B-f_c \le f \le B-f_c \\ \\ \frac{1}{2} \frac{T\sin\pi fT}{\pi(f-f_c)T} e^{-j\pi fT} & -B+f_c \le f \le B+f_c \end{cases} \qquad (D.8)$$

Thus the dotted block can be replaced by a block whose transfer function is H'(f). Since the transfer function of the bandpass filter $H_B(f)$ is equal to one for $-B-f_c \leq f \leq B-f_c$ and $B+f_c \leq f \leq B+f_c$, and equal to zero elsewhere, we can combine these two blocks into a single one with the transfer function H(f), where H(f) can be expressed as

$$H(f) = \begin{cases} \frac{1}{2} T \frac{\sin \pi fT}{\pi (f-f_c)T} e^{-j\pi fT} & -B+f_c \leq f \leq B+f_c \quad \text{(D.10)} \\ \\ \frac{1}{2} T \cdot \frac{\sin \pi fT}{\pi (f+f_c)T} e^{-j\pi fT} & -B-f_c \leq f \leq B-f_c \quad \text{(D.11)} \end{cases}$$

The system model now can be reduced as shown in Figure A.2.

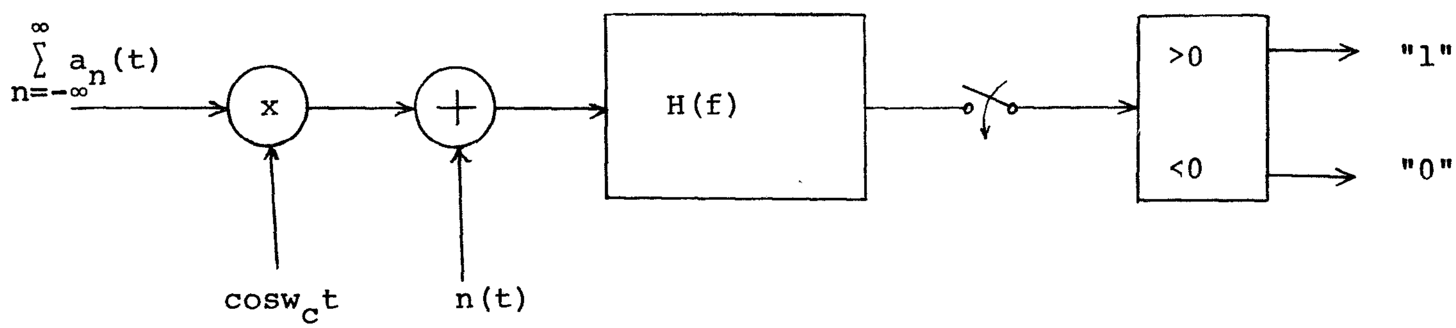

Figure A.2 The Simplified PSK Model

APPENDIX E

EVALUATIONS OF $\sum_{n=-\infty}^{\infty} J(BT,n)$ AND $\sum_{n=-\infty}^{\infty} C(BT,f_cT,n)$

We can write $\sum_{n=-K}^{K} J(BT,n)$ as

$$\sum_{n=-K}^{K} J(BT,n) = J(BT,0) + 2\sum_{n=1}^{K} J(BT,n) \tag{E.1}$$

Since $J(BT,n) = \frac{n+1}{2} J[(n+1)BT,0]-nJ(nBT,0)+\frac{n-1}{2}J[(n-1)BT,0]$ (E.2)

we obtain

$$\begin{aligned}\sum_{n=-K}^{K} J(BT,n) &= J(BT,0)+2\{J(2BT,0)-J(BT,0)\\
&+\tfrac{3}{2} J(3BT,0)-2J(2BT,0) + \tfrac{1}{2} J(BT,0)\\
&+\tfrac{4}{2} J(4BT,0)-3J(3BT,0) + \tfrac{2}{2} J(2BT,0)\\
&+\tfrac{5}{2} J(5BT,0)-4J(4BT,0) + \tfrac{3}{2} J(3BT,0)\\
&+ \ldots\\
&+\tfrac{K}{2}J(KBT,0)-(K-1)J[(K-1)BT,0]+\tfrac{K-2}{2}J[(K-2)BT,0]\\
&+\tfrac{K+1}{2}J[(K+1)BT,0]-KJ(KBT,0)+\tfrac{K-1}{2}J[(K-1)BT,0]\}\\
&= 2\{\tfrac{K}{2}J[(K+1)BT,0]-\tfrac{K}{2}J(KBT,0)+\tfrac{1}{2}J[(K+1)BT,0)]\}\end{aligned} \tag{E.3}$$

Now let $K\to\infty$, we have

$$\sum_{n=-\infty}^{\infty} J(BT,n) = \lim_{K\to\infty} J[(K+1)BT,0] = 1 \tag{E.4}$$

Similarly, we can write $\sum_{n=-K}^{K} C(BT,f_cT,n)$ as

$$\sum_{n=-K}^{K} C(BT,f_cT,n) = C(BT,f_cT,0)+2\sum_{n=1}^{K} C(BT,f_cT,n) \tag{E.5}$$

also

$$C(BT,f_cT,n) = \frac{n+1}{2} C[(n+1)BT,(n+1)f_cT,0]-nC(nBT,nf_cT,0)$$

$$+\frac{n-1}{2} C[(n-1)BT,(n-1)f_cT,0] \tag{E.6}$$

Thus

$$\sum_{n=-K}^{K} C(BT,f_cT,n) = 2\{\frac{K}{2}C[(K+1)BT,(K+1)f_cT,0]-\frac{K}{2} C(KBT,Kf_cT,0)$$

$$+ \frac{1}{2} C[(K+1)BT,(K+1)f_cT,0]\} \tag{E.7}$$

Let $K\to\infty$, we have

$$\sum_{n=-\infty}^{\infty} C(BT,f_cT,n) = \lim_{K\to\infty} C[(K+1)BT,(K+1)f_cT,0]$$

$$= 0 \tag{E.8}$$

APPENDIX F

EVALUATION OF $C(BT,f_cT,0)$

$$C(BT,f_cT,0) = \frac{2}{\pi}\int_0^{\pi BT} \frac{\sin^2 x}{(2\pi f_cT)^2 - x^2}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi BT} \frac{1}{4\pi f_cT}\left[\frac{\sin^2 x}{2\pi f_cT - x} + \frac{\sin^2 x}{2\pi f_cT + x}\right] dx \qquad \text{(F.1)}$$

Changing variables and simplifying, we have

$$C(BT,f_cT,0) = \frac{1}{2\pi^2 f_cT}\left[\int_{2\pi f_cT - \pi BT}^{2\pi f_cT} \frac{\sin^2 y}{y}\,dy + \int_{2\pi f_cT}^{2\pi f_cT + \pi BT} \frac{\sin^2 y}{y}\,dy\right]$$

$$= \frac{1}{2\pi^2 f_cT}\int_{2\pi f_cT - \pi BT}^{2\pi f_cT + \pi BT} \frac{\sin^2 y}{y}\,dy \qquad \text{(F.2)}$$

Since $\frac{\sin^2 y}{y}$ is an odd function, we have

$$C(BT,f_cT,0) = \frac{1}{2\pi^2 f_cT}\int_{|2\pi f_cT - \pi BT|}^{2\pi f_cT + \pi BT} \frac{\sin^2 y}{y}\,dy$$

$$= \frac{1}{4\pi^2 f_cT}\int_{|2\pi f_cT - \pi BT|}^{2\pi f_cT + \pi BT} \frac{1-\cos 2y}{y}\,dy \qquad \text{(F.3)}$$

For $B \neq 2fc$, $C(BT,f_cT,0)$ can be evaluated as

$$C(BT,f_cT,0) = \frac{1}{4\pi^2 f_cT}\left[l_n \frac{2\pi f_cT+\pi BT}{2\pi f_cT-\pi BT} - \int_{|4\pi f_cT-2\pi BT|}^{4\pi f_cT+2\pi B} \frac{\cos x}{x}\,dx\right]$$

$$= \frac{1}{4\pi^2 f_cT}\left[\ln\left|\frac{2\pi f_cT+\pi BT}{2\pi f_cT-\pi BT}\right|\right.$$

$$\left. + C_i(|4\pi f_cT-2\pi BT|)-C_i(4\pi f_cT+2\pi BT)\right] \qquad B\neq 2f_c \qquad (F.4)$$

where

$$C_i(y) = -\int_y^{\infty}\frac{\cos x}{x}\,dx\ , \qquad (F.5)$$

a cosine integral.

For $B = 2f_c$, we have

$$C(BT,f_cT,0) = \frac{1}{4\pi^2 f_cT}\int_0^{4\pi f_cT} \frac{1-\cos^2 y}{y}\,dy \qquad (F.6)$$

Changing variables, we obtain

$$C(BT,f_cT,0) = \frac{1}{4\pi^2 f_cT}\int_0^{8\pi f_cT} \frac{1-\cos x}{x}$$

$$= \frac{-1}{4\pi^2 f_cT}\int_0^{8\pi f_cT} \frac{\cos x-1}{x}\,dx \qquad (F.7)$$

But [2]

$$\int_0^{8\pi f_c T} \frac{\cos x - 1}{x}\, dx = C_i(8\pi f_c T) - \ln(8\pi f_c T) - 0.5772$$

Thus

$$C(BT, f_c T, 0) = \frac{1}{4\pi^2 f_c T}\,[0.5772 + \ln(8\pi f_c T) - C_i(8\pi f_c T)] \qquad B = 2f_c \tag{F.8}$$