CERTAIN COMMENTS ON THE APPLICATION OF THE METHOD OF AVERAGING TO THE STUDY OF THE ROTATIONAL MOTIONS OF A TRIAXIAL RIGID BODY

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Joseph J. F. Liu
Philip M. Fitzpatrick

Department of Mathematics
Auburn University
Auburn, Alabama

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1. Introduction

As indicated in our February 19, 1971, report, we have been interested during the past several years in describing the rotational motions of a rigid triaxial body about its center of mass, while the body is orbiting about, and influenced by, a second (primary) body. In our February report we mentioned that we had been studying the averaging technique as a preliminary step to using it in treating the variational equations for the trixial rigid body problem cited above. There we gave our interpretation of the theory which lies behind the method of averaging (which has been developed by Bogoliubov and Mitropolsky [1], as described in the recent studies and applications by Kyner [2] and Morrison [3]). In this report we describe some of the results we have obtained in applying the averaging method to the triaxial rigid body problem.

If the perturbing torque is known, the first-order, (in a small parameter), secular solutions associated with the canonical variables ($\alpha; \beta$) can frequently be obtained by applying the method of averaging to a dynamical system of the form

$$\dot{\alpha}_k = \frac{\partial H_1}{\partial \beta_k} \quad (k=1,2,\ldots,r)$$

$$\dot{\beta}_k = -\frac{\partial H_1}{\partial \alpha_k} \tag{1.1}$$

where H_1 is the perturbing Hamiltonian.

The discussion in this report is based on several assumptions. It is assumed that the rigid body moves in an elliptic orbit about an attracting point mass M. (Actually in carrying out the details, it is convenient to view M as moving about the rigid body). Although the orbital plane is assumed to have a constant nonzero inclination angle $\theta°$, it is to precess at a constant rate $\dot{\Omega}$. It is further assumed that the rotational kinetic energy of the rigid body is large compared to the effects of the perturbing gravitational potential and that the rotational rate of the body with respect to its center of mass is large when it is compared to both the orbital and orbital precession rates. These assumptions are satisfied if the angular velocity of rotation is greater than 1 deg./sec.

We can view the averaging procedure as being composed of two steps. In the first step, we introduce a transformation of variables to replace our starting dynamical system with an intermediate set of averaged differential equations in which (to first-order in a small parameter) the fast variables have been eliminated. In the second step the averaged differential equations are integrated, either numerically or (if possible) analytically, to obtain first-order secular solutions. Hitzl and Breakwell [4] have obtained a first-order, secular description of the motion of a gravity perturbed triaxial body in a non-precessing, elliptic orbit by applying the method of averaging to a dynamical system of the form (1.1). In Section 5 of this report we carry out the first step of the averaging procedure and derive the averaged differential equations for the extended problem of a triaxial body in a precessing, elliptic orbit. The development is carried out to the point that the averaged differential equations are in a form which can readily be integrated if it is so desired. These averaged differential equations will reduce to the corresponding equations of motions in [4] if (i) $\dot{\Omega} = 0$ and (ii) if two of the fast varying Euler angles which are used as canonical variables in [4] are transformed to the counterpart variables in our development. The second step of the averaging procedure will not be carried out for the canonical variables because we give, in later reports, first-order secular solutions for an alternative set of noncanonical variables. The formulation of the averaged differential equations of the motion in terms of the canonical variables will, however, be discussed thoroughly. For this reason, an explicit expression for the gravity-gradient potential energy V is derived in Section 2 of this report as a preliminary step in formulating the averaged variational equations.

The Hamilton-Jacobi method of canonical transformations is used in [5] to obtain an exact solution of the unperturbed problem of rotational motion. Since this solution will be used as a basis to establish the equations of motion for the perturbed problem discussed later, the necessary definitions and essential results for the unperturbed problem will first be summarized briefly. For further detail see [5].

2. Coordinate systems

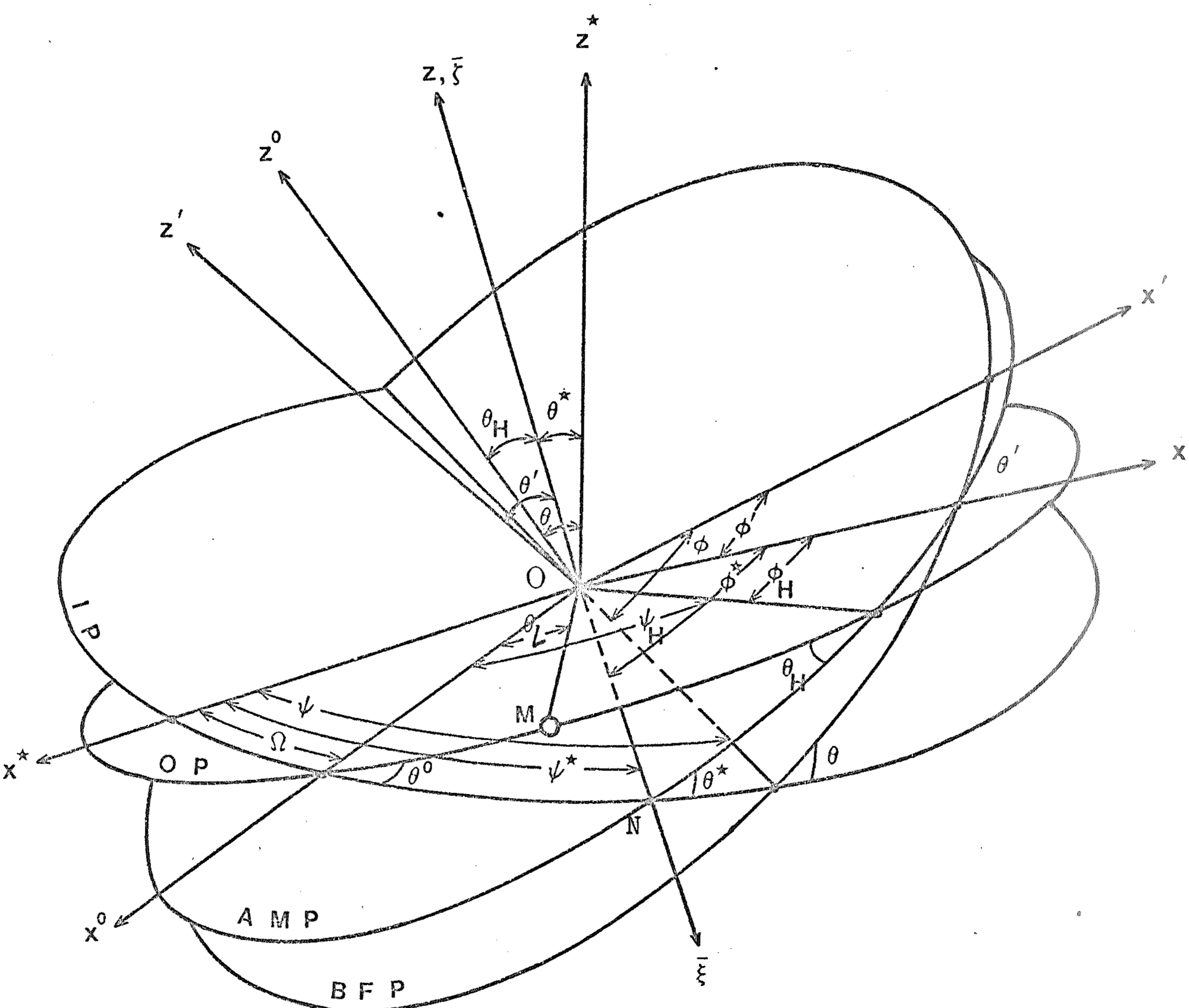
As shown in Figure 2.1, let 0 represent the center of mass of the body. Choose a rectangular coordinate system $0\bar{\xi}\bar{\eta}\bar{\zeta}$ so that the $\bar{\zeta}$-axis lies along the angular momentum vector $\underline{h}$, positive in the sense of $\underline{h}$. Consider a plane, perpendicular to the $\bar{\zeta}$-axis, which contains the center of mass. This plane intersects the x^*y^*-plane of the space-fixed, but otherwise arbitrary, rectangular frame $Ox^*y^*z^*$ in a line of nodes ON. The $\bar{\xi}$-axis is chosen to lie along the line of nodes, its positive sense being arbitrarily chosen. Then the $\bar{\eta}$-axis (not shown in Figure 2.1) is chosen to form a right-handed system. The Euler angles between the inertial system $Ox^*y^*z^*$ and the system $0\bar{\xi}\bar{\eta}\bar{\zeta}$ are ψ^*, θ^* and ϕ^*.

Let $Ox'y'z'$ be the body-fixed, rectangular, principal system and let ϕ^*, θ', ϕ' represent the Euler angles relating the $Ox'y'z'$ and $O\bar{\xi}\bar{\eta}\bar{\zeta}$ systems. We will refer to the $x'y'$-plane as the body-fixed plane. The Euler angles relating the body-fixed system $Ox'y'z'$ and the space-fixed system $Ox^*y^*z^*$ will be designated by ψ, θ, and ϕ.

Let $Ox^\circ y^\circ z^\circ$ be the orbital system. The z°-axis is taken positive in the sense of the angular momentum of the attracting point mass. We will refer to the $x^\circ y^\circ$-plane as the orbital plane. The Euler angles between the orbital system $Ox^\circ y^\circ z^\circ$ and space-fixed system Ox y z are Ω and θ°.

The system Oxyz is referred to as the angular momentum system. The z-axis is taken to coincide with the $\bar{\zeta}$-axis and the x-axis is the intersection of the angular momentum plane, i.e., the xy-plane, and the body-fixed plane. Then the y-axis is chosen to form a right-handed system. Let ψ^*, θ^*, ϕ^* be the Euler angles relating the inertial system $Ox^*y^*z^*$ and the angular momentum system Oxyz and let ψ_H, θ_H, ϕ_H be the Euler angles which relate the orbital system $Ox^\circ y^\circ z^\circ$ and the angular momentum system Oxyz.

Referring to Figure 2.1, we note that the orientation of the rigid body with respect to the $Ox^*y^*z^*$ system can be explicitly determined by one of the three sets of



BFP : body-fixed plane
AMP : angular momentum plane
OP : orbital plane
IP : inertial plane

Figure 2.1

angles (ψ, θ, ϕ), (ψ^*, θ^*, ϕ^*, θ', ϕ') or (Ω, θ°, ψ_H, θ_H, ϕ_H, θ', ϕ'), where Ω is the right ascension of the line of nodes and θ° is the orbital inclination angle.

The sets of vectors ($\underline{i}^*$, $\underline{j}^*$, $\underline{k}^*$), ($\underline{i}^\circ$, $\underline{j}^\circ$, $\underline{k}^\circ$), ($\underline{i}$, $\underline{j}$, $\underline{k}$) and ($\underline{i}'$, $\underline{j}'$, $\underline{k}'$) are the sets of unit vectors of the inertial system $Ox^*y^*z^*$, orbital system, $Ox^\circ y^\circ z^\circ$, angular momentum system Oxyz and body-fixed system $Ox'y'z'$, respectively.

3. The Unperturbed Problem

It has been shown in [5] that if the set of angles (ψ^*, θ^*, ϕ^*, θ', ϕ') and the magnitude h of the angular momentum are used, the six independent quantities comprised of (ψ^*, θ^*, ϕ') and their congugate momenta (p_{ψ^*}, p_{θ^*}, $p_{\phi'}$) will be sufficient to describe the unperturbed rotational motion of a body about its center of mass referenced to the inertial system $Ox^*y^*z^*$. When the unperturbed Hamiltonian H_o is expressed in terms of these variables, the associated Hamilton-Jacobi partial differential equation is separable. A canonical transformation from the variables (ψ^*, θ^*, ϕ', p_{ψ^*}, p_{θ^*}, $p_{\phi'}$) to a new set of canonical quantities (α_1, α_2, α_3, β_1, β_2, β_3) may thus be obtained. The new quantities are constant for the unperturbed motion. The explicit equations of transformation are

$$\phi^{*} + \beta_2 = M(\phi'), \quad \text{(a)}$$

$$t - \beta_1 = L(\phi'), \quad \text{(b)}$$

$$\beta_3 = -\psi^{*}, \quad \text{(c)}$$

$$p_{\phi^{*}} = \alpha_2 = h, \quad \text{(d)}$$

$$p_{\phi'} = \left[C\left(\frac{a' + b' \sin^2\phi'}{c' + d' \sin^2\phi'}\right)\right]^{1/2}, = h \cos\theta' \quad \text{(e)} \qquad (3.1)$$

$$p_{\psi^{*}} = \alpha_3, = h \cos\theta^{*} \quad \text{(f)}$$

where t represents the time and

$$L(\phi') = AB\sqrt{C}\, I_2(\phi'), \quad \text{(a)}$$

$$M(\phi') = -\sqrt{C}\, \alpha_2 I_3(\phi'), \quad \text{(b)}$$

$$I_2(\phi') = \int_{\phi_0'}^{\phi'} \frac{d\phi'}{\left[(a'+b'\sin^2\phi')(c'+d'\sin^2\phi')\right]^{1/2}}, \quad \text{(c)}$$

$$(3.2)$$

$$I_3(\phi') = \int_{\phi_0'}^{\phi'} \frac{[(A-B)\sin^2\phi' - A]\,d\phi'}{\left[(a'+b'\sin^2\phi')(c'+d'\sin^2\phi')\right]^{1/2}}, \quad \text{(d)}$$

$$a' = A(2B\alpha_1 - \alpha_2^2), \quad \text{(e)}$$

$$b' = \alpha_2^2 (A-B), \quad \text{(f)}$$

$$c' = A(B-C), \quad \text{(g)}$$

$$d' = C(A-B) \quad \text{(h)}$$

and A, B, C are the principal moments of inertia of the body. In writing equations (3.2), we assume that $0 < \theta' < \pi/2$ and also that $A > B > C$. The quantity a' may be positive, negative, or zero. We assume that a' is a nonnegative quantity.

The physical meanings of the canonical constants can be identified. The constant α_1 is the kinetic energy of the rotating body or the unperturbed Hamiltonian function H_o. The constant α_2 is the magnitude of the angular momentum vector $\underline{h}$ while α_3 is the magnitude p_{ψ^*} of the projection of $\underline{h}$ in the z^*-direction. The constant β_1 is an epoch time corresponding to a value $\phi' = \phi'_o$, β_2 is a reference value of ϕ^* when $\phi' = \phi'_o$ and β_3 is equal to $-\psi^*$.

Equations (3.2) have been inverted to express the independent variables $\psi^*, \theta^*, \phi', p_{\psi^*}, p_{\theta^*}, p_{\phi'}$ in terms of the canonical constants $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ and the time t. The details of the somewhat lengthy procedure appear in [5]. The results are

$$\tan\phi' = -\frac{1}{g}\frac{\text{cn } u}{\text{sn } u}, \qquad \text{(a)}$$

$$\phi^* = -\beta_2 + \frac{h}{B}(t-\beta_1) + \frac{\pi}{2K}\,[u\Lambda_o(\beta, k) - \Omega_5], \quad (k < -\gamma^2 < \infty)$$

$$= -\beta_2 + \frac{h}{B}\left[1 - \frac{(A-B)(k')^2}{A(\gamma^2 - k^2)}\right](t-\beta_1) \qquad \text{(b)}$$

$$- \frac{\pi}{2K}\,[u\Lambda_o(\bar{\psi}, k) + \Omega_1], \quad (0 < -\gamma^2 < k)$$

$$\psi^* = -\beta_3 \, , \qquad \text{(c)}$$

(3.3)

$$p_{\phi'} = h \cos\theta' = \left[\frac{C(2A\alpha_1 - h^2)}{A - C}\right]^{1/2} \text{dn } u \, , \qquad \text{(d)}$$

$$p_{\phi^*} = h = \alpha_2 \, , \qquad \text{(e)}$$

$$p_{\psi^*} = h \cos\theta^* = \alpha_3 \, , \qquad \text{(f)}$$

where

$$g = (1 - n_2^2)^{-1/2} = \left[\frac{B(A-C)}{A(B-C)}\right]^{1/2}, \qquad \text{(a)}$$

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}} \, , \qquad \text{(b)}$$

(3.4)

$$k = \left(\frac{n_1^2 - n_2^2}{1 - n_2^2}\right)^{1/2} = \left[\frac{(A-B)(h^2 - 2C\alpha_1)}{(B-C)(2A\alpha_1 - h^2)}\right]^{1/2}, \qquad \text{(c)}$$

$$k' = (1 - k^2)^{1/2} \, , \qquad \text{(d)}$$

periodic functions of the argument v with a period 2π. The functions Λ_o, Ω_5, and Ω_1 have the following definitions:[+]

$$\Lambda_o(\beta,k) = \frac{2}{\pi}[EF(\beta,k') + KE(\beta,k') - KF(\beta,k')], \qquad (a)$$

$$\Omega_5 = u + \frac{2K}{\pi}\tan^{-1}\left\{\frac{2\sum_1^\infty(-1)^{m+1}q^{m^2}\sin(2mv)\sinh[2m(p-w)]}{1+2\sum_1^\infty(-1)^m q^{m^2}\cos(2mv)\cosh[2m(p-w)]}\right\}, \qquad (b)$$

(3.5)

$$\Omega_1 = \frac{2K}{\pi}\tan^{-1}\left\{\frac{2\sum_1^\infty(-1)^{m+1}q^{m^2}\sin(2mv)\sinh(2mw)}{1+2\sum_1^\infty(-1)^m q^{m^2}\cos(2mv)\cosh(2mw)}\right\}, \qquad (c)$$

where

$$F(\beta,k') = \int_0^\beta \frac{d\theta}{(1-(k')^2\sin^2\theta)^{1/2}}, \qquad (a)$$

$$E(\beta,k') = \int_0^\beta (1-(k')^2\sin^2\theta)^{1/2}\, d\theta, \qquad (b)$$

(3.6)

[+] The definitions, basic information and the applications of elliptic integrals are all referenced to Byrd and Friedman [6].

$$E = E(\tfrac{\pi}{2},k) \;,\quad K = F(\tfrac{\pi}{2},k) \;,\quad K' = F(\tfrac{\pi}{2},k') \tag{c}$$

(3.6)

$$p = \frac{\pi K'}{2K} \;,\quad w^{*} = \frac{\pi F(\beta, k')}{2K} \;,\quad q = e^{-2p} \;. \tag{d}$$

We note that during the inverting procedure, we have taken $\phi'_o = -\pi/2$.

Equations (3.3) represent the exact solution to the unperturbed problem of rotational motion of a triaxial body. From elementary spherical trigonometry the variables $(\psi^{*}, \theta^{*}, \phi', p_{\psi^*}, p_{\theta^*}, p_{\phi'})$ can be expressed in terms of the variables $(\psi, \theta, \phi, p_{\psi}, p_{\theta}, p_{\phi})$ which relate the body-fixed and the space-fixed systems. The explicit equations of transformation, which appear in [5], are repeated here for convenience of the reader. They are

$$c_{\theta} = c_{\theta'} \, c_{\theta^*} - s_{\theta'} \, s_{\theta^*} \, c_{\phi^*} \tag{a}$$

$$s_{\theta} = (1 - c^2_{\theta})^{1/2} \tag{b}$$

$$s_{\theta^*} \, s_{\theta} \, c_{(\psi - \psi^*)} = c_{\theta'} - c_{\theta^*} \, c_{\theta} \tag{c}$$

$$s_{\theta} \, s_{(\psi - \psi^*)} = s_{\phi^*} \, s_{\theta'} \tag{d}$$

(3.7)

$$s_{\theta'} \, s_{\theta} \, c_{(\phi - \phi')} = c_{\theta^*} - c_{\theta'} \, c_{\theta} \tag{e}$$

$$s_{\theta} \, s_{(\phi - \phi')} = s_{\theta^*} \, s_{\phi^*} \tag{f}$$

$$p_{\theta} = - p_{\phi^*} \, s_{\theta'} \, s_{(\phi - \phi')} \tag{g}$$

$$p_{\psi} = h \, c_{\theta^*} \tag{h}$$

$$p_{\phi} = p_{\phi'} \tag{i}$$

where $c_\alpha = \cos\alpha$, $s_\alpha = \sin\alpha$.

In a later section of this report we will use the method of averaging to study the first-order, secular solution for gravity-induced perturbations of an orbiting triaxial body. For the problem of rotational motions, if the orbital plane is assumed to precess at a constant rate $\dot{\Omega}$, with constant inclination θ°, an alternative set of variables (ψ_H, θ_H, ϕ_H, θ', ϕ', h) has been found to be convenient, since the Euler angles ψ_H, ϕ_H, θ_H, relate directly to the orbital plane $x^\circ y^\circ$, rather than to the inertial plane $x^* y^*$. The geometric relations

$$c_{\theta_H} = c_{\theta^\circ} c_{\theta^*} - s_{\theta^\circ} s_{\theta^*} c_{(\psi^* - \Omega)} \qquad \text{(a)}$$

$$s_{\theta^*} s_{(\psi^* - \Omega)} \cot \psi_H = -s_{\theta^\circ} c_{\theta^*} - c_{\theta^\circ} s_{\theta^*} c_{(\psi^* - \Omega)} \qquad \text{(b)} \qquad (3.8)$$

$$s_{\theta^\circ} s_{(\psi^* - \Omega)} \cot_{(\phi^* - \phi_H)} = c_{\theta^\circ} s_{\theta^*} - s_{\theta^\circ} c_{\theta^*} c_{(\psi^* - \Omega)} \qquad \text{(c)}$$

are readily deduced from elementary spherical trigonometry. Equations (3.8) relate (ψ^*, θ^*, ϕ^*, θ', ϕ') to (Ω, θ°, ψ_H, θ_H, ϕ_H, θ', ϕ'). We will have to show later that the equations of motion for the latter set of variables has the proper form for application of the method of averaging. For this reason, we would like to express the trigonometric functions of the fast variables ϕ_H, θ', ϕ' in terms of periodic functions of the two new fast variables

v and v^* with period of 2π. One of these variables $v = \pi u/2K$ has already been defined. The second variable v^*, which is also a linear function of time t, will be defined in the equations to follow. Using the Fourier series expressions for the Jacobian elliptic functions sn u, cn u, dn u given in [6] and equations (3.3), we can show, if $k \neq 0$, that

$$s_{\phi'} = -\frac{2\pi}{gkK}\left[(c_v)q^{1/2} + (-c_v + c_{3v} + 2B_o^2 \quad c_v^2)q^{3/2}\right] + 0(q^{5/2}),$$

$$c_{\phi'} = \frac{2\pi}{kK}\left[(s_v)q^{1/2} + (s_v + s_{3v} + 2B_o^2 \; s_v c_v^2) \; q^{3/2}\right] + 0(q^{5/2}),$$

$$s_{\theta'} = \left(\frac{k^2}{n_1^2}\right)^{1/2}\left[1 - (2B_o^2 \; c_v^2)q + (4B_o^2 \; c_v^2 - 4B_o^2 \; c_v c_{3v} - 2B_o^4 \; c_v^4)q^2\right]$$

$$-0(q^3),$$

$$c_{\theta'} = \left(\frac{n_2^2}{n_1^2}\right)^{1/2}\left(\frac{\pi}{2K}\right)\left[1 + (4c_{2v})q + (4c_{4v})q^2\right] + 0(q^3), \qquad (3.9)$$

$$s_{\theta'}^{-1} = \left(\frac{n_1^2}{k^2}\right)^{1/2}\left[1 + (2B_o^2 \quad c_v^2)\; q + \right.$$

$$\left. + (-4B_o^2 \; c_v^2 + 4B_o^2 \; c_v c_{3v} + 6B_o^4 \; c_v^4) \; q^2\right] + 0(q^3),$$

$$s_{\phi_H} = s_{v^*} - (I_1 s_{2v} c_{v^*})q - (I_2 s_{4v} c_{v^*} + \frac{1}{2} I_1^2 \; s_{2v}^2 s_{v^*})q^2 + 0(q^3),$$

$$c\phi_H = c_{v^*} + (I_1 s_{2v} s_{v^*})q + (I_2 s_{4v} s_{v^*} - \frac{1}{2} I_1^2 s_{2v}^2 c_{v^*})q^2 + O(q^3) ,$$

$$\phi_H = v^* - (I_1 s_{2v})q - (I_2 s_{4v}) q^2 + O(q^3),$$

where

$$B_o^2 = \frac{n_2^2\pi^2}{k^2K^2} ,$$

$$v^* = \omega^* t - \alpha^* ,$$

$$\omega^* = \frac{h}{B} + \frac{\lambda\pi}{2K}\left[\Lambda_o(\beta, k) - 1\right] , \quad k < -\gamma^2 < \infty$$

$$= \frac{h}{B} - \frac{h}{B}\frac{(A-B)(k')^2}{A(\gamma^2 - k^2)} - \frac{\lambda\pi}{2K}\Lambda_o(\bar\psi, k), \quad 0 < -\gamma^2 < k$$

$$\alpha^* = -\beta_2 - \left[\frac{h}{B} + \frac{\lambda\pi}{2K}(\Lambda_o(\beta, k) - 1\right]\beta_1 + \text{constant}, \quad 0 < - \quad ^2 < k$$

$$= -\beta_2 - \left[\frac{h}{B} - \frac{h}{B}\frac{(A-B)k'^2}{A(\gamma^2 - k^2)} - \frac{\lambda\pi}{2K}\right]\beta_1 + \text{constant}, \quad 0 < -\gamma^2 < k \tag{3.10}$$

$$I_1 = \frac{\pi}{K}\sinh\left[2(p - w^*)\right], \quad k < -\gamma^2 < \infty$$

$$= 2 \sinh 2w^*, \quad 0 < -\gamma^2 < k$$

$$I_2 = \frac{\pi}{2K}\sinh\left[4(p - w^*)\right], \quad k < -\gamma^2 < \infty$$

$$= \sinh 4w^*, \quad 0 < -\gamma^2 < k .$$

4. Gravity-gradient Potential Energy

As shown in Figure 4.1, the origin O of the inertial system $Ox^*y^*z^*$ is located at the attracting point mass M and the origin O' of the body-fixed system $Ox'y'z'$ is located at the center of the mass of the orbiting rigid body m. The position vector $\underline{R}_o$ is drawn from M to the center of mass of the orbiting body; $\underline{R}$ represents the position vector of a differential mass dm of the body m referenced to O^*; $\underline{r}$ represents the position vector of the same differential mass referenced to O' and $\underline{\rho} = \underline{R}_o / R_o$, where $R_o = |\underline{R}_o|$. The function V which represents the potential energy of the mass m, due to its presence in the gravity field arising from M, can be written

$$V = -GM \iiint_m \frac{dm}{R}, \qquad (4.1)$$

where G is the gravitational constamt, $R = |\underline{R}|$, and the gravity field is the negative of the gradient of V. An explicit expression for the potential energy V for a body of unit mass is given in [5, equation(14.4.15), p.351]. This expression, which is valid to terms of the order of $\left(\frac{r}{R_o}\right)^2$, ($r = |\underline{r}|$), may, for a small body of mass m, be written in the form

$$V = -\frac{GMm}{R_o} + \frac{3GM}{2R_o^3}\left(\ell_1^2 A + \ell_2^2 B + \ell_3^2 C\right) - \frac{GM}{2R_o^3}(A + B + C). \qquad (4.2)$$

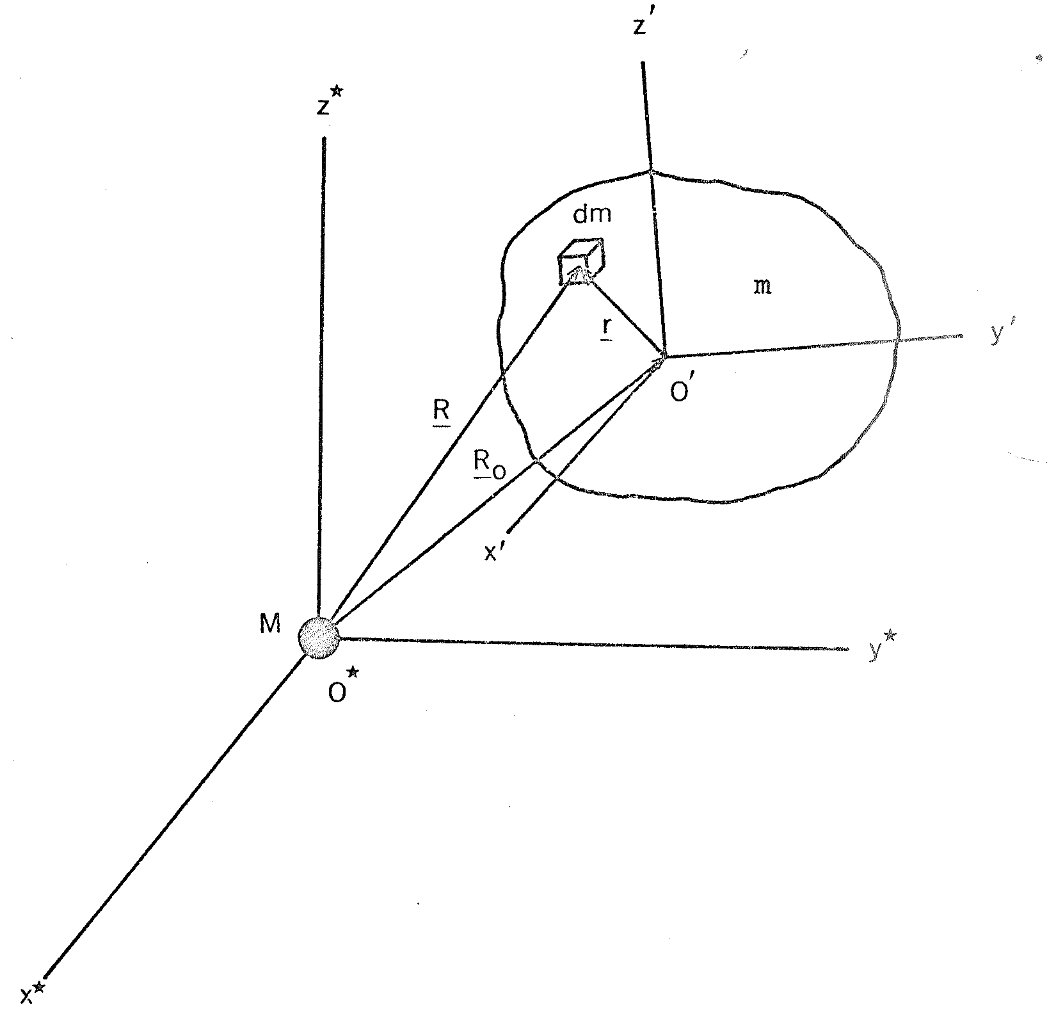


Figure 4.1

In (4.2) ℓ_1, ℓ_2 , and ℓ_3 are the direction cosines of the position vector $\underline{R}_o$ with respect to the body-fixed system Ox'y'z'.

For purpose of the later analysis, we would like to express V in terms of the fast variables ϕ_H, v and the argument of latitude θ_L. The equations of transformations from the body-fixed system Ox'y'z' to the angular momentum syatem Oxyz and from the orbital system Ox°y°z° to the angular momentum system are, respectively

$$\underline{a} = T' \; \underline{a}' \; , \qquad \text{(a)}$$
$$\underline{a} = T^{\circ} \; \underline{a}^{\circ} \; , \qquad \text{(b)} \qquad (4.3)$$

where tho elemcnts of the matrices

$$T' = \begin{bmatrix} c_{\phi'} & -s_{\phi'} & 0 \\ s_{\phi'} c_{\theta'} & c_{\phi'} c_{\theta'} & -s_{\theta'} \\ s_{\phi'} s_{\theta'} & c_{\phi'} s_{\theta'} & c_{\theta'} \end{bmatrix} , \qquad \text{(a)}$$

(4.4)

$$T^{\circ} = \begin{bmatrix} c_{\psi_H} c_{\phi_H} - s_{\psi_H} s_{\phi_H} c_{\theta_H} & s_{\psi_H} c_{\phi_H} + c_{\psi_H} s_{\phi_H} c_{\theta_H} & s_{\phi_H} s_{\theta_H} \\ -c_{\psi_H} s_{\phi_H} - s_{\psi_H} c_{\phi_H} c_{\theta_H} & -s_{\psi_H} s_{\phi_H} + c_{\psi_H} c_{\phi_H} c_{\theta_H} & c_{\phi_H} s_{\theta_H} \\ s_{\psi_H} s_{\theta_H} & -c_{\psi_H} s_{\theta_H} & c_{\theta_H} \end{bmatrix} , \qquad \text{(b)}$$

are readily deduced from the geometry of the rotations. In (4.3) $\underline{a}$ is any vector referenced to the angular momentum system Oxyz. The quantities $\underline{a}'$ and $\underline{a}^{\circ}$ represent the same vector referenced to the body-fixed system $Ox'y'z'$ and the orbital system $Ox^{\circ}y^{\circ}z^{\circ}$, respectively. By using the fact that

$$\rho = c_{\theta_L}\, \underline{i}^{\circ} + s_{\theta_L}\, \underline{j}^{\circ} \tag{4.5}$$

and the equations of transformations (4.4), we find, after some tedious manipulation, that

$$\begin{aligned}
&V(R_o,\ \psi_H - \theta_L,\ \theta_H,\ \phi_H(v^{*}),\ \theta'(v),\ \phi'(v)) \\
&= -\frac{GMm}{R_o} - \frac{1}{2}\frac{GM}{R_o^{3}}(A + B + C) + \\
&+\frac{3}{2}\frac{GM}{R_o^{3}}\Big\{ s^{2}_{(\psi_H-\theta_L)}\Big[(Ac^{2}_{\phi'} + Bs^{2}_{\phi'})\, s^{2}_{\phi_H} c^{2}_{\theta_H} + \frac{1}{2}(A-B)\, s_{2\phi'} c_{\theta'} s_{2\phi_H} c_{\theta_H} \\
&\qquad - \frac{1}{2}(A-B)\, s_{2\phi'} s_{\theta'} s_{2\phi_H} s_{2\theta_H} \\
&\qquad + \Big((As^{2}_{\phi'} + Bc^{2}_{\phi'})\, c^{2}_{\theta'} + Cs^{2}_{\theta'}\Big)\, c^{2}_{\phi_H} c^{2}_{\theta_H} \\
&\qquad - \frac{1}{2}(As^{2}_{\phi'} + Bc^{2}_{\phi'} - C)\, s_{2\theta'} c_{\phi_H} s_{2\theta_H} \\
&\qquad + \Big((As^{2}_{\phi'} + Bc^{2}_{\phi'})\, s^{2}_{\theta'} + Cc^{2}_{\theta'}\Big)\, s^{2}_{\theta_H}\Big] +
\end{aligned} \tag{4.6}$$

$$+c^2_{(\psi_H-\theta_L)}\Big[(Ac^2_{\phi'}+Bs^2_{\phi'})c^2_{\phi_H} -\frac{1}{2}(A-B)\; s_{2\phi'}c_{\theta'}s_{2\phi_H} +$$

$$+\left((As^2_{\phi'}+Bc^2_{\phi'})c^2_{\theta'}+Cs^2_{\theta'}\right)s^2_{\phi_H}\Big]+ \qquad (4.6)$$

$$+\frac{1}{2}s_{2(\psi_H-\theta_L)}\Big[-(Ac^2_{\phi'}+Bs^2_{\phi'})s_{2\phi_H}c_{\theta_H}+\frac{1}{2}(A-B)s_{2\phi'}c_{\phi_H}s_{\theta_H}s_{\theta'}$$

$$+(A-B)s_{2\phi'}c_{\theta'}(1-2c^2_{\phi_H})\;c_{\theta_H}$$

$$+\left((As^2_{\phi'}+Bc^2_{\phi'})c^2_{\theta'}+Cs^2_{\theta'}\right)s_{2\phi_H}c_{\theta_H}$$

$$-\left(As^2_{\phi'}+Bc^2_{\phi'}-C\right)s_{2\theta'}s_{\phi_H}s_{\theta_H}\Big]\Big\}$$

Since, from the unperturbed solution, the α_k's and β_k's are related to the Euler angles ψ_H, θ_H, ϕ_H, θ', ϕ', we can view V as a function of α_k and β_k, $k = 1,2,3$.

5. Equations of Motion for Canonical Variables

Since the perturbing gravity-gradient potential energy V is conservative, the differential equations

$$\dot{\alpha}_k = -\frac{\partial V(\alpha_k;\beta_k)}{\partial \beta_k}, \qquad k=1,2,3 \qquad (a)$$

$$\dot{\beta}_k = \frac{\partial V(\alpha_k;\beta_k)}{\partial \alpha_k}, \qquad k=1,2,3 \qquad (b) \qquad (5.1)$$

can be used to study the variations of α_k and β_k.

In equations (5.1), $\partial V/\partial\alpha_k$, $\partial V/\beta_k$, k=1,2,3, are continuous functions of α_k, β_k. Since certain of the canonical variables α_k and β_k are expressible in terms of θ_L, ϕ_H, θ', and ϕ', we can also view $\partial V/\partial\alpha_k$ and $\partial V/\partial\beta_k$ as containing explicitly these latter variables. In this sense $\partial V/\partial\alpha_k$ and $\partial V/\partial\beta_k$ have the same functional dependence as V in (4.6). Also they are periodic functions of time of each of θ_L, ϕ_H, θ', ϕ' with period 2π. We note also that the magnitude of these partial derivatives is of the order of ϵ as compared with the unperturbed values of α_k and β_k. Here ϵ is defined as the ratio of the perturbing potential energy V to the rotational kinetic energy T.

Through the well-known relations from orbital analysis [5]

$$\theta_L = \omega_o + f$$

$$= \omega_o + \overline{M} + (2e - \frac{1}{4} e^3) s_{\overline{M}} + \frac{5}{4} e^2 s_{2\overline{M}} + \frac{13}{12} e^3 s_{3\overline{M}} + O(e^4) \quad \text{(a)}$$

(5.2)

$$R_o = \frac{a(1 - e^2)}{1 - e\, c_f}, \quad \text{(b)}$$

and equations (3.10), the potential energy V can be also viewed as containing the variables $\overline{M}$, v^*, and v. In (5.2) ω_o represents the angle between the x^o-axis and the axis from the origin O through the perigee of the orbit, f and $\overline{M}$ represent the true and mean anomalies, respectively,

$e(0 < e < 1)$ and a represent the eccentricity and semi-major axis, respectively, of the orbit. The partial derivatives $\partial V/\partial \alpha_k$, and $\partial V/\partial \beta_k$ are therefore continuous functions in $\bar{M}$, v^*, and v and periodic functions in each of these three variables with period 2π.

We will choose $\bar{M}$, v^*, and v as the fast variables. All the fast variables in the potential energy function V can be expressed explicitly in terms of $\bar{M}$, v^* and v and the time derivatives $\dot{\bar{M}}$, $\dot{v}^*$ and $\dot{v}$ have the same mathematical form, to the first-order, as equations (1.1(b)) of our February 19, 1971, report. (Further references to equations in the February report will be prefixed by the letters F.R.) In order to identify them as the fast variables, we will designate $\bar{M}$, v^* and v by y_1, y_2 and y_3, respectively. Their time derivatives are given to the first-order by the equations

$$\dot{y}_1 = \dot{\bar{M}} = n, \qquad \text{(a)}$$

$$\dot{y}_2 = \dot{v}^* = \omega^*, \qquad \text{(b)} \qquad (5.3)$$

$$\dot{y}_3 = \dot{v} = \frac{\lambda \pi}{2K} . \qquad \text{(c)}$$

where n is the mean motion in the orbit.

So that the form of our equations will be consistent with the form of equations [F.R.(1.1(b))], we introduce the definition

$$V' = \frac{1}{\epsilon} V \qquad (5.4)$$

and rewrite equations (5.1) as

$$\dot{\alpha}_k = -\epsilon \frac{\partial V'(\alpha_k;\beta_k)}{\partial \beta_k}, \quad k=1,2,3 \qquad \text{(a)}$$

$$(5.5)$$

$$\dot{\beta}_k = -\epsilon \frac{\partial V'(\alpha_k;\beta_k)}{\partial \alpha_k}, \quad k=1,2,3 \qquad \text{(b)}$$

The equations of motion, specified by (5.5) for the six slow variables α_k and β_k and by (5.3) for the three fast variables $\overline{M}$, v^* and v, are of the form [F.R.(1.1)] when M = 6 and N = 3 and hence are in the proper form to apply the method of averaging as outlined in our February 19, 1971, report. We will carry out the method of averaging up to the point where we replace the six first order differential equations for $\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3, \dot{\beta}_1, \dot{\beta}_2, \dot{\beta}_3$ by the six transformed differential equations corresponding to [F.R.(1.3(a))]. Although we do not carry out the details of the integration, the transformed differential system is in a form where it can readily be integrated to obtain a first-order secular solution for the (α_k, β_k), k=1,2,3.

We assume that the nonresonance condition [F.R.(2.3)] is satisfied and we introduce the transformation [F.R.(1.2)]

$$\alpha_k = \bar{\alpha}_k + \epsilon P_k(\bar{\alpha}_k, \bar{\beta}_k, y_k) \qquad k=1,2,3 \qquad (a)$$

(5.6)

$$\beta_k = \bar{\beta}_k + \epsilon Q_k(\bar{\alpha}_k, \bar{\beta}_k, y_k) \qquad k=1,2,3 \qquad (b)$$

where

$$\bar{\alpha}_k = \bar{\alpha}_k(a, e, \omega_o, \psi_H, \theta_H), \quad \bar{\beta}_k = \bar{\beta}_k(a, e, \omega_o, \psi_H, \theta_H). \quad (5.6)'$$

If we consider only the secular solutions for the slow variables, equations [F.R.(1.3a)] become

$$\dot{\bar{\alpha}}_k = \epsilon U_k(\bar{\alpha}_k, \bar{\beta}_k), \qquad k=1,2,3 \qquad (a)$$

(5.7)

$$\dot{\bar{\beta}}_k = \epsilon V_k(\bar{\alpha}_k, \bar{\beta}_k), \qquad k=1,2,3 \qquad (b)$$

for suitable functions U_k and V_k. If we use [F.R.(1.13)] and [F.R.(1.14)], equations (5.7) become

$$\dot{\bar{\alpha}}_k = \epsilon \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial V'(\bar{\alpha}_k, \bar{\beta}_k, y_k)}{\partial \beta_k} dy_1 dy_2 dy_3, \qquad (a)$$

(5.8)

$$\dot{\bar{\beta}}_k = -\epsilon \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial V'(\bar{\alpha}_k, \bar{\beta}_k, y_k)}{\partial \alpha_k} dy_1 dy_2 dy_3, \qquad (b)$$

where k=1,2,3.

The remainder of our effort in this section is aimed at expressing the right hand sides of (5.8) in more compact form. Recalling (4.6), we define V'_e and V_e by the equations

$$V'_e = \frac{1}{\epsilon} V_e , \qquad (a)$$

$$V_e(\psi_H, \theta_H, R_o, \overline{M}, v^*, v) = V + \frac{GMm}{R_o} + \frac{1}{2}\frac{GM}{R_o^3}(A+B+C) \qquad (b) \qquad (5.9)$$

We note that the last two terms on the right hand side of (5.9(b)) are independent of the α_k's and β_k's and therefore (5.8) cam be written in the form

$$\overset{\circ}{\bar{\alpha}}_k = \epsilon \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial V'_e(\bar{\alpha}_k, \bar{\beta}_k, y_k)}{\partial \beta_k} dy_1 dy_2 dy_3, k=1,2,3 \qquad (a)$$

(5.10)

$$\overset{\circ}{\bar{\beta}}_k = -\epsilon \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial V'_e(\bar{\alpha}_k, \bar{\beta}_k, y_k)}{\partial \alpha_k} dy_1 dy_2 dy_3, \quad k=1,2,3 \qquad (b)$$

or simply from relation (5.9(a))

$$\dot{\bar{\alpha}}_k = -\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial V_e(\bar{\alpha}_k, \bar{\beta}_k, y_k)}{\partial \beta_k}\, dy_1 dy_2 dy_3, k=1,2,3 \quad \text{(a)}$$

(5.11)

$$\dot{\bar{\beta}}_k = \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial V_e(\bar{\alpha}_k, \bar{\beta}_k, y_k)}{\partial \alpha_k}\, dy_1 dy_2 dy_3 \;. \quad \text{(b)}$$

Since the α_k's and β_k's are slow variables they may, to the first order, be treated as constant parameters in the integrands. Thus, using Leibnitz' rule, the order of integration and differentiation may be interchanged and we have

$$\dot{\bar{\alpha}}_k = -\frac{\partial}{\partial \beta_k}\left[\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} V_e(\bar{\alpha}_k, \bar{\beta}_k, y_k)\, dy_1 dy_2 dy_3\right] \quad \text{(a)}$$

(5.12)

$$\dot{\bar{\beta}}_k = \frac{\partial}{\partial \alpha_k}\left[\left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} V_e(\bar{\alpha}_k, \bar{\beta}_k, y_k)\, dy_1 dy_2 dy_3\right]. \quad \text{(b)}$$

If we introduce the notation

$$\overline{V}_e(a, e, \omega_o, \psi_H, \theta_H) = \overline{V}_e(\overline{\alpha}_k, \overline{\beta}_k)$$

$$= \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} V_e \, dy_1 dy_2 dy_3 \qquad (5.13)$$

equations (5.12) become

$$\dot{\overline{\alpha}}_k = \frac{\partial \overline{V}_e(\overline{\alpha}_k, \overline{\beta}_k)}{\partial \beta_k}, \quad k=1,2,3 \qquad (a)$$

$$(5.14)$$

$$\dot{\overline{\beta}}_k = -\frac{\partial \overline{V}_e(\overline{\alpha}_k, \overline{\beta}_k)}{\partial \alpha_k}, \quad k=1,2,3 \qquad (b)$$

The true anomaly may be introduced as a variable of integration to replace $\overline{M}$ through the use of the well-known two body orbit relation [5]

$$d\overline{M} = \frac{R_o^2}{a^2(1-e^2)^{1/2}} df \qquad (5.15)$$

and relation (5.2(b)). Again, a, e, ω_o, ψ_H, θ_H may be treated as constant parameters within the integrands. If equations (5.15) and (5.2(b)) are used, we can write

$$\int_0^{2\pi} V_e dy_1 = \int_0^{2\pi} \frac{GM}{a(1-e^2)} \cdot \frac{1+c_f}{a^2(1-e^2)^{1/2}} V_e^* df$$

$$= \frac{n^2}{(1-e^2)^{3/2}} \left[\int_0^{2\pi} V_e^* df + \int_0^{2\pi} V_e^* c_f \, df \right], \qquad (5.16)$$

where $V_e^* = R_o^3 V_e/GM$. If we examine (4.6), we note that, in forming $\int_0^{2\pi} V_e dy_1$, integrals of the three types

$$\int_0^{2\pi} s^2_{(\psi_H - \theta_L)} c_f \, df, \quad \int_0^{2\pi} c^2_{(\psi_H - \theta_L)} c_f \, df, \quad \int_0^{2\pi} s_{2(\psi_H - \theta_L)} c_f \, df$$

appear. We note that $\psi_H - \theta_L = \psi_H - \omega_o - f$ from (5.2). Since ψ_H and ω_o are both slow variables, to first order, the three types of integrals will vanish because their integrands are odd functions which contain either cos f or sin f . We can thus conclude that

$$\int_0^{2\pi} V_e^* c_f \, df = 0$$

and write

$$\int_0^{2\pi} V_e \, dy_1 = \frac{n^2}{(1 - e^2)^{3/2}} \int_0^{2\pi} V_e^{*} df \; . \qquad (5.17)$$

To the first order, we have $d(\psi_H - \theta_L) = - df$, so that we can write*

$$\int_0^{2\pi} V_e dy_1 = \frac{n^2}{(1 - e^2)^{3/2}} \int_{(\psi_H - \omega_o)-2\pi}^{(\psi_H - \omega_o)} V_e^{*} d(\psi_H - \theta_L) \qquad (5.18)$$

Since for α real, definite integrals of the form

$$\int_{\alpha}^{\alpha+2\pi} s^2(\psi_H - \theta_L) \, d(\psi_H - \theta_L), \quad \int_{\alpha}^{\alpha+2\pi} c^2(\psi_H - \theta_L) \, d(\psi_H - \theta_L),$$

and $\int_{\alpha}^{\alpha+2\pi} s_2(\psi_H - \theta_L) \, d(\psi_H - \theta_L)$ have the same value

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* Relation (5.18) can also be obtained by direct substitution of (5.2) into equation (5.17) with no restriction as to order.

over any interval of length 2π equation (5.18) becomes

$$\int_0^{2\pi} V_e dy_1 = \frac{n^2}{(1 - e^2)^{3/2}} \int_0^{2\pi} V_e^* d(\psi_H - \theta_L) . \tag{5.19}$$

In carrying out the second integration (i.e., over $y_2 = v^*$) as indicated in (5.13), integrals of the types

$$\int_0^{2\pi} s\phi_H \, dv^*, \quad \int_0^{2\pi} c\phi_H \, dv^*, \quad \int_0^{2\pi} s^2\phi_H \, dv^* \text{ and } \int_0^{2\pi} c^2\phi_H \, dv^*$$

appear. We note from equations (3.9) that

$$\phi_H = v^* + v_r^*(v) \quad , \tag{5.20}$$

where $v_r^*(v) = -(I_1 \, s_{2v})q - (I_2 \, s_{4v})q^2 + 0(q^3)$ is treated as a constant during the integration over y_2.

Thus $\int_0^{2\pi} s\phi_H \, dv^*$ and $\int_0^{2\pi} c\phi_H \, dv^*$ both vanish and

$$\int_0^{2\pi} s^2\phi_H \, dv^* = \int_0^{2\pi} s^2\phi_H \, d\phi_H, \quad \int_0^{2\pi} c^2\phi_H \, dv^* = \int_0^{2\pi} c^2\phi_H \, d\phi_H .$$

We can then write

$$\int_0^{2\pi}\left[\int_0^{2\pi} v_e^{*}\, d(\psi_H - \theta_L)\right] dy_2 = \int_0^{2\pi}\left[\int_0^{2\pi} v_e^{*} d(\psi_H - \theta_L)\right] d\phi_H . \tag{5.21}$$

Using (5.19) and (5.21), we rewrite (5.13) in the form

$$\overline{V}_e = \frac{n^2}{(1 - e^2)^{3/2}} \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} V_e^{*} d(\psi_H - \theta_L)\, d\phi_H\, dv . \tag{5.22}$$

Treating all slow variables as constant, we can carry out, to first order, the integration with respect to $(\psi_H - \theta_L)$ and ϕ_H. It is clear from (4.6) that only trigonometric functions of the relevant arguments appear in the integrands. Thus the procedure is quite straightforward and we find that

$$\overline{V}_e = \frac{3n^2}{(1 - e^2)^{3/2}} \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \left[\frac{1}{48}(A+B-2C)(1- 3c^2_{\theta_H})(1- 3c^2_{\theta'}) - \frac{1}{16}(A-B)(1- 3c^2_{\theta_H}) c_{2\phi'}\, s^2_{\theta'} + \frac{1}{2}(A+B+C)\right] dv . \tag{5.23}$$

The last term in the integrand of (5.23) gives rise to a term which we label Q_v. Since A + B + C does not depend upon the α_k's and β_k's, Q_v will contribute nothing in forming the variational equations (5.14). Setting $\overline{V}_e^{**} = \overline{V}_e - Q_v$, we write (noting that $v = \pi u/2K$)

$$\overline{V}_e^{**} = \frac{3n^2(1-3c^2_{\theta_H})}{(1-e^2)^{3/2}} \frac{1}{4K} \int_0^{4K} \left[\frac{1}{48}(A+B-2C)(1-3c^2_{\theta'}) + \right. \tag{5.24}$$

$$\left. - \frac{1}{16}(A-B)\, c_{2\phi'}\, s^2_{\theta'} \right] du.$$

Using the integration formulas in [6] (integrals (314.02) and (310.02), respectively), we find that

$$\frac{1}{4K} \int_0^{4K} c^2_{\theta'}\, du = \frac{1}{4K} \int_0^{4K} dn^2 u\, du = e_3^2 \frac{E}{K}, \tag{5.25}$$

$$\frac{1}{4K} \int_0^{4K} c^2_{\phi'}\, s^2_{\theta'}\, du = \frac{1}{4K}\left(\frac{k^2}{n_1^2}\right) \int_0^{4K} sn^2 u\, du = \frac{e_2^2}{k^2}\left(1 - \frac{E}{K}\right), \tag{5.26}$$

where

$$e_2^2 = \frac{k^2}{n_1^2} = \frac{B(h^2 - 2C\alpha_1)}{h^2(B - C)} , \qquad \text{(a)}$$

$$(5.27)$$

$$e_3^2 = \frac{n_2^2}{n_1^2} = \frac{C(2A\alpha_1 - h^2)}{h^2(A - C)} , \qquad \text{(b)}$$

and hence

$$\overline{V}_e^{**} = \frac{3n^2(1 - 3c^2_{\theta_H})}{8(1 - e^2)^{3/2}} \left\{ (A+B-2C)\left(\frac{1}{6} - 3e_3^2 \frac{E}{K}\right) \right.$$

$$(5.28)$$

$$\left. - (A-B) \left[\left(\frac{e_2^2}{k^2} - \frac{1}{2}\right) + \left(\frac{e_3^2}{2} \quad 1\right) \frac{E}{K} \right] \right\} .$$

Noting (see [5]) the following relations

$$k = k(\alpha_1, \alpha_2), \qquad n_1 = n_1(\alpha_1, \alpha_2),$$

$$E = E(\alpha_1, \alpha_2), \qquad K = K(\alpha_1, \alpha_2), \qquad (5.29)$$

$$c_{\theta_H} = \frac{\alpha_3}{\alpha_2} c_{\theta_\circ} + \left(\frac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2}\right)^{1/2} s_{\theta_\circ} c_{(\beta_3 + \Omega)} ,$$

we see that $\overline{V}_e^{**}$ (and $\overline{V}_e$ also) depends upon only four of the canonical variables, in the form

$$\bar{V}_e^{**} = \bar{V}_e^{**}(\alpha_1, \alpha_2, \alpha_3, \beta_3). \qquad (5.30)$$

The differential equations which describe the first-order, secular variations of the α_k and β_k which appear in (5.1) are

$$\dot{\alpha}_1 = \dot{\alpha}_2 = 0, \qquad (a)$$

$$\dot{\alpha}_3 = \frac{\partial \bar{V}_e^{**}}{\partial \beta_3}, \qquad (b)$$

$$\dot{\beta}_1 = -\frac{\partial \bar{V}_e^{**}}{\partial \alpha_1}, \qquad (c) \qquad (5.31)$$

$$\dot{\beta}_2 = -\frac{\partial \bar{V}_e^{**}}{\partial \alpha_2}, \qquad (d)$$

$$\dot{\beta}_3 = -\frac{\partial \bar{V}_e^{**}}{\partial \alpha_3}, \qquad (e)$$

For the special case of a circular orbit, we have $e = 0$ and $n^2 = GM/a_o^3$, where a_o represents the radius of the circular orbit. Equation (5.28) then takes the form

$$\overline{V}_e^{**} = \frac{3GM}{8a_o^3} \left\{ (A+B-2C)\left(\frac{1}{6} - 3e_3^2 \frac{E}{K}\right) + \right. \tag{5.32}$$

$$\left. - (A-B) \left[\left(\frac{e_2^2}{k^2} - \frac{1}{2}\right) + \left(\frac{e_3^2}{2} - 1\right) \frac{E}{K} \right] \right\} .$$

If we enter (5.31) with $\overline{V}_e^{**}$ as given in (5.32), then equations (5.31) give the associated equations of motion for the dynamical system (5.1).

For the special case, $\overset{\circ}{\theta} = \overset{\circ}{\Omega} = 0$, i.e., the orbital plane is not precessing, we find from (5.29) that

$$c_{\theta_H} = c_{\theta^*} = \frac{\alpha_3}{\alpha_2} , \tag{5.33}$$

and it follows that $\overset{\circ}{\alpha}_1 = \overset{\circ}{\alpha}_2 = \overset{\circ}{\alpha}_3 = 0$ and $\beta_1, \beta_2, \beta_3$ have secular variations. Under these circumstances, it can be seen readily that the differential equations (5.31) reduce to the corresponding forms given by Hitzl and Breakwell in [4]. Since the complete first-order, secular solutions, for the alternative variables ($\psi_H, \theta_H, \phi_H, \theta', \phi'$, h), of the triaxial problem associated with a precessing orbit will be given in later reports, the corresponding first-order secular solutions for the α_k and β_k will not be given here. Equations (5.31) are, however, in a form in which they can be integrated readily. If the reader is further interested in the secular solutions for the canonical variables, he is referenced to [4] for the first-order secular solutions associated with a tri-axial body moving in an elliptic, nonprecessing orbit. The canonical variables used in [4] are slightly different then the canonical variables appearing in this report. One set of canonical variables can readily be obtained from the other by a simple transformation.

For a uniaxial body, we have $k = 0$, $\operatorname{dn} u = 1$. Equation (3.3(d)) reduces to

$$c_{\theta'} = \left[\frac{C(2A\alpha_1 - h^2)}{h^2(A - C)}\right]^{1/2}, \qquad (5.34)$$

hence θ' is a function of α_1 and α_2 only. Similarly, $\operatorname{sn} u = \sin u$, $\operatorname{cn} u = \cos u$ and from relations (3.2(a)) and (3.4(a)), we find that

$$\phi' = \phi'_o + u = -\frac{\pi}{2} + u. \qquad (5.35)$$

Therefore, for the axisymmetric case, equation (5.24) takes the form

$$\bar{V}_e^{**} = \frac{n^2(A - C)}{8(1 - e^2)^{3/2}}(1 - 3c^2_{\theta_H})(1 - 3c^2_{\theta'}). \qquad (5.36)$$

Equation (5.36) is to be used in connection with equation (5.31) in forming the variational equations for the motion of a uniaxial rigid body.

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